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Automorphism groups of minimal real-analytic CR manifolds

Abstract. We show that the local automorphism group of a minimal real-analytic CR manifold M is a finite-dimensional Lie group if (and only if) M is holomorphically nondegenerate by constructing a jet parametrization.

1. Introduction

In 1907, Poincaré observed that it is extremely unlikely that one can find an analytic map between two given real hypersurfaces in $\mathbb{C}^2$; in his paper, he formulated the "problème local": Given two hypersurfaces, is it possible to determine all such maps? In this paper, we shall answer the analogous question for automorphisms of real-analytic CR manifolds; we establish a necessary and sufficient condition (known as holomorphic nondegeneracy) for the finite-dimensionality of the automorphism groups and furthermore give an algorithm to compute them. We note that this problem is markedly different from the analogous problem in the complex plane, where it is well known that two real-analytic arcs can always be mapped into each other by an analytic map, and the space of such maps is necessarily of infinite dimension.

The first positive answer to this question was given by Élie Cartan in 1932 [6, 7] for strictly pseudoconvex real-analytic hypersurfaces in $\mathbb{C}^2$, by applying his method of equivalence. This type of result, which establishes a complete list of invariants which can be used to solve the local equivalence problem, is today known for Levi-nondegenerate real-analytic hypersurfaces in $\mathbb{C}^N$, by the work of Tanaka [17] and Chern and Moser [8], as well as for minimal real-analytic hypersurfaces in $\mathbb{C}^2$ by the work of Kolář [14]. An important corollary to all of their works is that the local automorphism group of such a real-analytic hypersurface is a subgroup of the automorphism group of the hyperquadric (or, in Kolář's case, a more general model); these groups, however, are well known, and turn out to be finite-dimensional Lie groups.

The methods used to prove the results for Levi-nondegenerate real-analytic hypersurfaces do not carry over to degenerate cases, and Kolář's techniques only extend to very

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specific cases in higher dimensions [13]. However, Poincaré's observation was very general, and leads us to expect that even though we might not be able to give a complete list of invariants, there should not be "too many" maps; maybe we can even prove that the automorphism groups of more general real-analytic submanifolds are finite-dimensional Lie groups?

Some partial positive answers to these questions have been given over the last decade. The first results valid for Levi-degenerate real-analytic hypersurfaces, due to Baouendi, Ebenfelt, and Rothschild [2] and generalized to higher codimension by Zaitsev [18], gave not only a solution to the problem in the "finitely nondegenerate" case, but also a general way of attacking the problem through so-called jet parametrizations. In the context of automorphism groups of germs of real-analytic submanifold $M \subset \mathbb{C}^N$ at a distinguished point $p \in M$, a jet parametrization inverts the natural mapping $\mathrm{Aut}(M, p) \to G_p^k(\mathbb{C}^N)$ from the local automorphism group

$$\mathrm{Aut}(M, p) = \{H \colon (\mathbb{C}^N, p) \to (\mathbb{C}^N, p) \colon \det H'(p) \neq 0, \ H(M \cap U) \subset M$$
$$\text{for some neighbourhood } U \text{ of } p\}$$

of germs of biholomorphisms of $(\mathbb{C}^N, p)$ mapping M to itself endowed with its natural inductive limit topology, to the jet group of order k at the point p for some k. It thus provides a mapping Ψ defined near $\{p\} \times \mathcal{U}$ for some $\mathcal{U} \subset G_p^k(\mathbb{C}^N)$ with

$$\Psi(Z, j_p^k H) = H(Z), \quad H \in \mathrm{Aut}(M, p).$$

A jet parametrization which is in addition holomorphic in the first argument and real-analytic in the second shows that the image of the natural embedding of a neighbourhood of, say, the identity in $\mathrm{Aut}(M, p)$ as a neighbourhood of the identity in a subgroup of $G_p^k(\mathbb{C}^N)$ can be defined by real-analytic equations and thus endows $\mathrm{Aut}(M, p)$ with a Lie-group structure.

Jet parametrizations have turned out to be powerful tools. For minimal real-analytic hypersurfaces in $\mathbb{C}^2$, a jet parametrization was used by Ebenfelt, Zaitsev and the second author [10] to provide a solution of our question above; Mir and the second author [15] used it to prove that $\mathrm{Aut}(M, p)$ is a finite-dimensional Lie group for any real-analytic CR manifold M which is minimal at p and "essentially finite".

All of the results discussed so far provide sufficient conditions for $\mathrm{Aut}(M, p)$ to be finite-dimensional; in order to see what a necessary condition looks like, it is helpful to look at the Lie algebra of infinitesimal automorphisms. An *infinitesimal automorphism* of (M, p) is a germ of a holomorphic $(1, 0)$-vector field

$$X = \sum_{j=1}^{N} a_j(Z) \frac{\partial}{\partial Z_j},$$

vanishing at p, whose real part is tangent to M; we denote the space of infinitesimal automorphisms of (M, p) by $\mathfrak{hol}(M, p)$. Clearly, if X itself is tangent to M near p, then $X \in \mathfrak{hol}(M, p)$; we shall denote the subspace of all such X by $\mathfrak{ht}(M, p) \subset \mathfrak{hol}(M, p)$ and refer to them as *tangent holomorphic vector fields*.

It is easy to see that if $\mathfrak{ht}(M, p) \neq \{0\}$, then necessarily $\dim_{\mathbb{C}} \mathfrak{ht}(M, p) = \infty$ (after all, $\mathfrak{ht}(M, p)$ is a module over the ring of germs of holomorphic functions at 0), which in turn precludes that $\mathrm{Aut}(M, p)$ can be given the structure of a finite-dimensional Lie group. Thus, $\mathfrak{ht}(M, p) = \{0\}$ is clearly necessary for $\mathrm{Aut}(M, p)$ to be finite-dimensional. We say that a connected CR manifold M is *holomorphically nondegenerate* if $\mathfrak{ht}(M, p) = \{0\}$ for one (or, equivalently, for all) $p \in M$. Holomorphic nondegeneracy was introduced by Stanton [16] and shown to be sufficient for $\mathfrak{hol}(M, p)$ to be finite-dimensional by Baouendi, Ebenfelt, and Rothschild [3]. In this paper we show that actually $\mathrm{Aut}(M, p)$ is a finite-dimensional Lie group for all minimal holomorphically nondegenerate CR manifolds, thus showing that holomorphic nondegeneracy characterizes the finite-dimensionality of $\mathrm{Aut}(M, p)$:

Theorem 1. *Let M be a connected real-analytic CR manifold which is minimal at $p \in M$. Then the group $\mathrm{Aut}(M, p)$ of real-analytic CR automorphisms of (M, p) is a finite-dimensional Lie group if and only if M is holomorphically nondegenerate. Furthermore, in this case, $\mathrm{Aut}(M, p)$ can be identified with a real algebraic subgroup of the jet group $G_p^k(M)$ for some $k \in \mathbb{N}$.*

Let us recall that M is *minimal* at $p \in M$ if there is no proper CR submanifold $N \subset M$ containing p of the same CR dimension as M. We shall establish Theorem 1 by constructing a jet parametrization for automorphisms of a holomorphically nondegenerate minimal CR manifold. Before we state our jet parametrization theorem, let us note the following interesting corollary of Theorem 1.

Corollary 1. *Let M be a connected, minimal real-analytic CR manifold. If $\mathrm{Aut}(M, p)$ is a finite-dimensional Lie group for some $p \in M$, then $\mathrm{Aut}(M, q)$ is a finite-dimensional Lie group for all $q \in M$.*

Indeed, as we already noted above, a real-analytic CR manifold has the property that if $\mathfrak{ht}(M, p) = \{0\}$ for some $p \in M$, then $\mathfrak{ht}(M, q) = \{0\}$ for all q in the connected component of M containing p.

Let us now state our main parametrization theorem.

Theorem 2. *Let $(M, p) \subset (\mathbb{C}^N, p)$ be a germ of a generic real analytic submanifold of $\mathbb{C}^N$, which is minimal and holomorphically nondegenerate at p. Then there exists $k \in \mathbb{N}$, a neighbourhood $U \subset G_p^k(\mathbb{C}^N)$ of $j_p^k \mathrm{id}$, and a map $\Psi : \mathcal{U} \to \mathbb{C}^N$, defined on a neighbourhood $\mathcal{U}$ of $\{p\} \times U$ in $\mathbb{C}^N \times G_p^k(\mathbb{C}^N)$, holomorphic in the first variable and real-analytic in the second, such that*

$$H(Z) = \Psi(Z, j_p^k H) \quad \text{for all } H \in \mathrm{Aut}(M, p) \text{ with } j_p^k H \in U. \tag{1}$$

Furthermore, there exists a polynomial s with $s(j_p^k \mathrm{id}) \neq 0$ and for every $\alpha \in \mathbb{N}^N$ a polynomial p_α on $G_p^k(\mathbb{C}^N)$ and an integer r_α such that

$$\Psi(Z, \Lambda) = \sum_{\alpha \in \mathbb{N}^N} \frac{p_\alpha(\Lambda)}{s(\Lambda)^{r_\alpha}} Z^\alpha. \tag{2}$$

Furthermore, the integer $k = k(M, p)$ can be bounded by local biholomorphic invariants of (M, p) and can be chosen in such a way that it remains uniformly bounded on compact subsets of M.

Let us quickly review how a parametrization as in (1) lets us identify the image of $j_p^k \colon \mathrm{Aut}(M, p) \to G_p^k(\mathbb{C}^N)$. Using a real-analytic defining function $\rho(Z, \bar{Z})$ of M and a real-analytic parametrization $\mathbb{R}^{2N-d} \ni \xi \mapsto \varphi(\xi)$ of M, we see that $\Lambda \in j_p^k(\mathrm{Aut}(M, p))$ if and only if

$$0 = \rho\big(\Psi(\varphi(\xi), \Lambda), \overline{\Psi(\varphi(\xi), \Lambda)}\big) = \sum_\beta \frac{P_\beta(\Lambda)}{Q(\Lambda)^{d_\beta}} \xi^\beta,$$

and thus $j_p^k(\mathrm{Aut}(M, p))$ can be defined by the (real) polynomial equations $P_\beta(\Lambda) = 0$. This argument shows that Theorem 1 follows from Theorem 2, since it shows that a neighbourhood of the identity in $\mathrm{Aut}(M, p)$ is homeomorphic to a real-algebraic subset of a neighbourhood of the identity in $G_p^k(\mathbb{C}^N)$. It actually yields the following, somewhat more explicit theorem, which identifies $\mathrm{Aut}(M, p)$ with a subgroup of $G_p^k(\mathbb{C}^N)$:

Theorem 3. *Let M be the germ of a generic real-analytic submanifold of $\mathbb{C}^N$ at p. Then there exists an integer $k = k(p)$ such that the natural map $j_p^k \colon \mathrm{Aut}(M, p) \to G_p^k(\mathbb{C}^N)$ is an embedding which is a homeomorphism onto its image, which is a real-algebraic closed subgroup of $G_p^k(\mathbb{C}^N)$. Furthermore, the integer $k(p)$ can be chosen in such a way that it remains uniformly bounded on compact subsets of M.*

A typical application of a finite jet parametrization only uses that $H \in \mathrm{Aut}(M, p)$ is uniquely determined by $j_0^k H$. In our setting, this unique determination property is due to Baouendi, Mir and Rothschild [5], and in more generality to the first author [12] who both actually proved it for mappings which are not necessarily automorphisms. We now state a uniqueness property which follows from our theorem.

Corollary 2. *Let M be a real-analytic, holomorphically nondegenerate minimal CR manifold. Then for any compact subset $K \subset M$ there exists an integer $k = k(K)$ such that if H and $\tilde{H}$ are real-analytic CR automorphisms defined in a neighbourhood of $p \in M$ with $j_p^k H = j_p^k \tilde{H}$, then $H = \tilde{H}$.*

A further implication of our Theorem 2 is a global statement. For a general real-analytic CR manifold M, it is well known that for p outside a real-analytic subvariety V, the germ (M, p) is biholomorphically equivalent to $(\tilde{M} \times \mathbb{C}^s, 0)$ where $\tilde{M}$ is a germ of a holomorphically nondegenerate submanifold of $\mathbb{C}^{N-s}$ through 0. To be exact, the sections of the sheaf $\mathfrak{ht}(M)$ with stalks $\mathfrak{ht}(M, p)$ defined above define a foliation of M by holomorphic manifolds which we refer to as the *holomorphic foliation* of M. Any biholomorphism of M has to preserve leaves of this foliation. Let us denote by V the singular set of this foliation. At points $p \in M \setminus V$, in terms of the local product decomposition $(M, p) \cong (\tilde{M} \times \mathbb{C}^s, 0)$, any $H \in \mathrm{Aut}(\tilde{M} \times \mathbb{C}^s, 0)$ can be written as $H = (H_1, H_2)$ where $H_1 \in \mathrm{Aut}(\tilde{M}, 0)$ and H_2 is any map of the form $H_2(Z_1, Z_2)$ with $\det \frac{\partial H_2}{\partial Z_2}(0) \neq 0$. Of course, the set of these maps does not form a finite-dimensional Lie group, but we can

show that they form an infinite-dimensional Lie group, where we shall define a Lie group to be a group which is a locally convex manifold (actually, our groups are modelled on spaces of germs of power series) whose multiplication and inversion are analytic (in the usual infinite-dimensional sense, i.e. analytic along finite-dimensional subsets and continuous in the vector space topology). The groups which appear here will be subgroups of the group of germs of biholomorphisms of $\mathbb{C}^N$, which is a Lie group in this sense.

To discuss the structure of $\mathrm{Aut}(\tilde{M} \times \mathbb{C}^s, 0)$, we use the product decomposition $H = (H_1, H_2)$ where $H_1 \in \mathrm{Aut}(\tilde{M}, 0)$ and H_2 is any map of the form $H_2(Z_1, Z_2)$ with $\det \frac{\partial H_2}{\partial Z_2}(0) \neq 0$. Equivalently, H_2 is a germ of a holomorphic map $(\mathbb{C}^{N-s}, 0) \to \mathrm{Aut}(\mathbb{C}^s, 0)$. The set of all such germs, which we denote by $\mathcal{H}_s = \mathcal{H}((\mathbb{C}^{N-s}, 0), \mathrm{Aut}(\mathbb{C}^s, 0))$, inherits the structure of an infinite-dimensional Lie group from $\mathrm{Aut}(\mathbb{C}^s, 0)$. Furthermore, $\mathrm{Aut}(\mathbb{C}^{N-s}, 0)$ acts on $\mathcal{H}_s$ in an obvious way. The group of all germs of biholomorphisms of $\mathbb{C}^N$ of the form $(Z_1, Z_2) \mapsto (h_1(Z_1), h_2(Z_1, Z_2))$ is then exactly the semidirect product $\mathrm{Aut}(\mathbb{C}^{N-s}, 0) \ltimes \mathcal{H}_s$.

At a point where the holomorphic foliation is nonsingular, we can thus explicitly describe the infinite-dimensional part of the automorphism group of a general real-analytic CR manifold at all points $p \in M \setminus V$ where V is the real-analytic subvariety defined before.

Theorem 4. *Let M be a real-analytic, minimal CR manifold. Then there exists an integer s such that for the subvariety $V \subset M$ defined before, for all $p \in M \setminus V$ there exists an integer k such that $\mathrm{Aut}(M, p) = F \ltimes \mathcal{H}_s$ for a finite-dimensional Lie-group $F = F(p)$ with the property that $F(p)$ embeds into the jet group $G_p^k(\mathbb{C}^N)$ as a closed, real-algebraic subgroup. In particular, $\mathrm{Aut}(M, p)$ is an (infinite-dimensional) Lie group for every $p \in M \setminus V$.*

We now outline the proof of our main result. The main technical innovation of this paper is a new kind of parametrization theorem (Theorem 7) for solutions of singular systems of analytic equations. It replaces the corresponding result used in [15] for the situation studied here and allows us to "invert" a certain system of equations for our map associated to a holomorphically nondegenerate submanifold. This inversion is only valid for maps which are on a stratum where a certain vanishing order is constant; we relate this vanishing order to local invariants of (M, p). These invariants are introduced in §3 and their basic properties are investigated. This step, which is in essence based on mapping formulas from [12], allows us to construct jet parametrizations "along the Segre maps" (see Corollary 14); we thus obtain a parametrization of the form $H(S(x)) = \Psi(x, j_p^k H)$, where S is a holomorphic submersion.

In order to pass from this to a parametrization as in Theorem 2, we have to use another form of inversion. While a technique to overcome this problem is available in the literature (dubbed as the "doubling trick"), we have decided to attack this problem in a more direct way, and thus include the relevant theorem, which is another new result of this paper (Theorem 5): We introduce an inversion formula for the mapping induced by composition with the germ of a holomorphic map $A: (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ which is generically of full rank n on the spaces of germs of holomorphic functions. As a byproduct, we get a new

proof of a celebrated result due to Gabrielov [11]: for an $A: (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ which is generically of full rank n, if $f \circ A$ is convergent for some formal power series f, than f is necessarily convergent, or equivalently, every formal relation between analytic functions $(A_1, \ldots, A_n)$ is necessarily analytic (under the assumption that the rank of $A = (A_1, \ldots, A_n)$ is generically full). Our Theorem 5 allows us to write an inductive formula for f given $f \circ A$, and we establish the convergence by estimating this induction process directly. We note that this gives an elementary proof of Gabrielov's theorem, which we think is more accessible than the proofs which previously appeared in the literature (see Eakin and Harris [9]).

Let us note that our parametrization works both in the formal and in the analytic setting; and the convergent parametrization can be used to conclude that every formal automorphism between real-analytic CR manifolds which are minimal and holomorphically nondegenerate is convergent. However, these statements are known in greater generality from the work of Baouendi, Mir and Rothschild [5], so we shall refrain from stating them.

2. Some basics and notation

This section introduces some basic notation used throughout the paper. Noting that all of the problems considered here are of a local nature, we may (by considering an intrinsic complexification of M) assume that M is a formal or real analytic generic submanifold of $\mathbb{C}^N$, which is defined in *normal coordinates* $(z, w) \in (\mathbb{C}^n \times \mathbb{C}^d, 0)$ by $w = Q(z, \bar{z}, \bar{w})$. Saying that (z, w) are normal coordinates means that Q satisfies

$$Q(z, 0, \tau) = Q(0, \chi, \tau) = \tau, \quad Q(z, \chi, \bar{Q}(\chi, z, w)) \equiv w. \tag{3}$$

The first condition makes the coordinates *normal*, and the second one is referred to as the *reality condition*.

We study mappings (formal or holomorphic) $H: (\mathbb{C}^{n+d}, 0) \to (\mathbb{C}^{n+d}, 0)$ taking M into M', where M and M' are generic submanifolds of codimension d in $\mathbb{C}^{n+d}$ defined in normal coordinates as above by Q and Q', respectively. We will write $H: (M, 0) \to (M', 0)$ if H is such a map taking M into M'; this is equivalent to the following power series identity:

$$Q'\big(F(z, Q(z, \chi, \tau)), \bar{H}(\chi, \tau)\big) = G(z, Q(z, \chi, \tau)), \tag{4}$$

where $H = (F, G)$ is the splitting according to the normal coordinates (z', w'); we often refer to F as the *tangential part* of H and to G as the *transversal* part of H.

We are now ready to introduce one of the basic tools we will use, the *Segre mappings*; we have chosen a little variation of the standard notation which is more suitable for our purposes.

2.1. Iterated Segre mappings

Assume that we are given a generic manifold M as above defined by Q in normal coordinates (z, w).

For $j \geq 1$, we define the iterated Segre mappings $S^j : \mathbb{C}^{nj} \times \mathbb{C}^d \to \mathbb{C}^{n+d}$ iteratively by the following formulas, where $t \in \mathbb{C}^d$ and $x^k \in \mathbb{C}^n$ for all k:

$$
\begin{aligned}
S^1(x^1; t) &= (x^1, t), \\
S^{j+1}(x^1, \ldots, x^{j+1}; t) &= (x^1, Q(x^1, \bar{S}^j(x^2, \ldots, x^{j+1}; t))), \quad j \geq 1.
\end{aligned}
\tag{5}
$$

We also agree to write $S^0(t) = (0, t)$, which makes the second line of (5) valid for all $j \geq 0$. We denote by $U^j : \mathbb{C}^{nj} \times \mathbb{C}^d \to \mathbb{C}^d$ the second component of S^j, i.e.,

$$
S^j(x^1, \ldots, x^j; t) = (x^1, U^j(x^1, \ldots, x^j; t)).
$$

It will be useful to write

$$
x^{[j;k]} = (x^j, \ldots, x^k), \quad j \leq k.
$$

With this notation, the definition of the Segre maps (5) reads

$$
S^1(x^1; t) = (x^1, t), \quad S^{j+1}(x^{[1;j+1]}; t) = \big(x^1, Q(x^1, \bar{S}^j(x^{[2;j+1]}; t))\big).
$$

Remark 1. By setting $t = 0$ in S^j, that is, if we consider $S^j(x^{[1;j]}; 0)$, we get the usual iterated Segre maps. However, we need to take derivatives with respect to t before setting $t = 0$, which is why we will use a nonstandard notation.

We note the following two useful properties of the Segre maps S^j:

$$
S^{j+1}(x^2, x^{[1;j]}; t) = S^{j-1}(x^{[2;j]}; t), \quad j \geq 2, \tag{6}
$$

$$
S^2(x^1, x^2; \bar{U}^j(x^{[2;j+1]}; t)) = S^{j+1}(x^{[1;j+1]}; t), \quad j \geq 2, \tag{7}
$$

where the reality condition (3) has been used to prove (6).

The mapping condition (4) is expressed in terms of the Segre maps as the power series identity

$$
Q'\big(F(S^2(z, \chi; \tau)), \bar{H}(\chi, \tau)\big) = G(S^2(z, \chi; \tau)). \tag{8}
$$

By setting $z = x^1$, $\chi = x^2$ and $\tau = \bar{U}^{j-1}(x^2, \ldots, x^j; t)$ into (8) and using (7), we get

$$
Q'\big(F(S^j(x^{[1;j]}; t)), \bar{H}(\bar{S}^{j-1}(x^{[2;j]}; t))\big) = G(S^j(x^{[1;j]}; t)). \tag{9}
$$

If we instead substitute $z = x^2$, $\chi = x^1$, $\tau = \bar{U}^j(x^1, \ldots, x^j; t)$ and use (6) and (7), we get

$$
Q'\big(F(S^{j-1}(x^{[2;j]}; t)), \bar{H}(\bar{S}^j(x^{[1;j]}; t))\big) = G(S^{j-1}(x^{[2;j]}; t)).
$$

Another way of writing the mapping condition (9) is

$$
H(S^j(x^{[1;j]}; t)) \equiv S'^2\big(F(S^j(x^{[1;j]}; t)), \bar{H}(\bar{S}^{j-1}(x^{[2;j]}; t))\big). \tag{10}
$$

Let $q \geq 2$. By using (10) inductively, together with property (7), we obtain the mapping property of the Segre maps for maps $H : (M, 0) \to (M', 0)$, which we refer to as "$H \circ S^q = S'^q \circ H$", or more precisely:

Lemma 3. *Let $H \colon (M, 0) \to (M', 0)$ be a mapping. Then*

$$H(S^q(x^{[1;q]}; t)) = S'^q(y^{[1;q]}; u),$$

where

$$y^{2j+1} = F(S^{q-2j}(x^{[2j+1;q]}; t)), \quad y^{2j} = \bar{F}(S^{q-2j+1}(x^{[2j;q]}; t)),$$

$$u = \begin{cases} \bar{G}(x^q, t) & \text{for } q \text{ even,} \\ G(x^q, t) & \text{for } q \text{ odd.} \end{cases}$$

2.2. A lemma on derivatives

We will need the following lemma, which roughly tells us that if we know the transversal derivatives of a function along a Segre map, then we know all of its derivatives along that map.

Lemma 4. *For every $q \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}^d$, there exists a universal power series $p(x^{[1;q]}, t; \Lambda)$ (depending on q, α, β, and M, convergent if M is), polynomial in its last argument, such that for any power series $h(z, w)$, we have*

$$p\left(x^{[1;q]}, t; \frac{\partial^{|\gamma|}}{\partial(x^1)^\gamma} \frac{\partial^{|\delta|}}{\partial t^\delta} h(S^q(x^{[1;q]}; t))\right),$$

where $|\gamma| + |\delta| \le |\alpha| + |\beta|$.

Proof. This is an easy consequence of the chain rule, and the fact that $\frac{\partial U^j}{\partial t}$ is invertible near $0 \in \mathbb{C}^{jn+d}$, which follows since $\frac{\partial U^j}{\partial t}(0; 0) = I_{d \times d}$ by the normality of the coordinates. $\qquad \square$

2.3. Jet spaces

We will need the notion of the jet space along a submanifold. Given the germ of a holomorphic manifold $(X, 0) \subset (\mathbb{C}^r, 0)$, the jet space $J_X^k((\mathbb{C}^r, 0), \mathbb{C})$ is the quotient of the local ring $\mathcal{O}_r$ by the equivalence relation identifying $f \in \mathcal{O}_r$ with $g \in \mathcal{O}_r$ if

$$f - g \in I(X)^{k+1},$$

where

$$I(X) = \{f \in \mathcal{O}_r \colon f|_X = 0\}$$

denotes the ideal associated to X. The natural projection $\mathcal{O}_r \to J_X^k((\mathbb{C}^r, 0), \mathbb{C})$ will be denoted by $j_{X,0}^k$.

In suitable coordinates, we can always assume that X is given by the vanishing of the last, say, q coordinates. In this setting, if $\mathbb{C}^r = \mathbb{C}_x^p \times \mathbb{C}_t^q$ and we consider jets along $X = \{t = 0\}$, we will denote the corresponding jet space by $J_t^k(\mathbb{C}_x^p \times \mathbb{C}_t^q, 0)$, with the jet map $j_{t,0}^k$. We then have for this choice of coordinates a natural identification

$$J_t^k((\mathbb{C}_x^p \times \mathbb{C}_t^q, 0), \mathbb{C}) = (\mathcal{O}_p)^{N(k)},$$

where $N(k) = \sum_{j \leq k} \binom{q+j-1}{q-1}$ denotes the number of multiindices in q variables of order at most k, given by

$$j_{t,0}^k h = \left(h(x,0), \ldots, \frac{\partial^{|\alpha|} h}{\partial t^\alpha}(x,0), \ldots \right),$$

and we have to include all multiindices α with $|\alpha| \leq k$.

In a similar way we define the jet spaces $J_X^k((\mathbb{C}^r, 0), \mathbb{C}^n)$ and $J_X^k((\mathbb{C}^r, 0), (\mathbb{C}^n, 0))$.

We will use $\varphi(x,t)[(\alpha_1, \alpha_2)]$ to denote the coefficient of $x^{\alpha_1} t^{\alpha_2}$ in $\varphi(x,t)$, and for any multiindex $\alpha \in \mathbb{N}^{p+q} = \mathbb{N}^p \times \mathbb{N}^q$, α^1 will denote its projection on the first p entries, and α^2 will denote its projection on the last q entries. For brevity, we sometimes also write $\varphi(x,t)[\alpha] = \varphi_\alpha$.

We will be working with the following notion of an "initial term" of a power series $\lambda(x,t)$. We decompose λ into homogeneous terms with respect to x and t,

$$\lambda(x,t) = \sum_{j,k} \lambda_{j,k}(x,t), \quad \text{where} \quad \lambda_{j,k}(s_1 x, s_2 t) = s_1^j s_2^k \lambda_{j,k}(x,t)$$

and define the *type* $\mathrm{tp}\,\lambda$ of λ as the smallest multiindex (n_1, n_2) in the lexicographic ordering on $\mathbb{N}^2$ which satisfies $\lambda_{n_1,n_2}(x,t) \not\equiv 0$, i.e.

$$\lambda_{n_1,n_2} \not\equiv 0, \quad \text{and} \quad \lambda_{j,k}(x,t) = 0 \quad \text{if } k < n_2 \text{ or if } k = n_2 \text{ and } j < n_1. \quad (11)$$

With the notation introduced above, this just means that

$$\lambda[(\alpha^1, \alpha^2)] = 0 \quad \text{if } |\alpha^2| < n_2,$$
$$\lambda[(\alpha_0^1, \alpha_0^2)] \neq 0 \quad \text{for some } (\alpha_0^1, \alpha_0^2) \text{ with } |\alpha_0^1| = n_1, |\alpha_0^2| = n_2.$$

We write

$$\mathrm{it}\,\lambda(x,t) = \sum_{|\alpha^1| = n_1, \, |\alpha^2| = n_2} \lambda_{(\alpha^1, \alpha^2)} x^{\alpha^1} t^{\alpha^2}$$

and call this the *initial term* of $\lambda(x,t)$.

3. Invariants

Let $(M, 0)$ be a formal generic submanifold, given as usual in normal coordinates by $w = Q(z, \chi, \tau)$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d = \mathbb{C}^N$; to simplify notation, we write $\zeta = (\chi, \tau)$. In this section, we define an invariant associated to holomorphic nondegeneracy of $(M, 0)$, namely a descending chain of pairs of integers measuring the vanishing of the Segre map (descending in the sense of the lexicographic ordering in $\mathbb{N}^2$). Recall that holomorphic nondegeneracy of $(M, 0)$ is equivalent to the map

$$\zeta \mapsto (Q_{z^\alpha}^j(0, \zeta) \colon \alpha \in \mathbb{N}^n, 1 \leq j \leq d)$$

being of full rank over the field of meromorphic functions in ζ (see e.g. [4, Lemma 11.3.5]); again, equivalently, writing $\alpha = (\alpha^1, \ldots, \alpha^N)$ and $r = (r_1, \ldots, r_N)$, the func-

tions

$$\Delta(\alpha, r)(z, \zeta) = \begin{vmatrix} Q^{r_1}_{z^{\alpha^1};\zeta}(z, \zeta) \\ \vdots \\ Q^{r_N}_{z^{\alpha^N};\zeta}(z, \zeta) \end{vmatrix}, \quad \text{where} \quad Q^{r_j}_{z^{\alpha^j};\zeta} = \left(\frac{\partial Q^{r_j}_{z^{\alpha^j}}}{\partial \zeta_1}, \ldots, \frac{\partial Q^{r_j}_{z^{\alpha^j}}}{\partial \zeta_N} \right), \quad (12)$$

defined for $\alpha \in \mathbb{N}^{nN}$, $r \in \{1, \ldots, d\}^N$, are not all identically zero. To ease the notation, we define the index set J by

$$J = \mathbb{N}^{(n+d)n} \times \{1, \ldots, d\}^{n+d}.$$

Now, with the decomposition $\zeta = (\chi, \tau)$ we define the type of a power series $\lambda(\zeta)$, tp λ, as in (11). Using the lexicographic ordering on $\mathbb{N}^2$, we define

$$(n_1^1, n_2^1) = \min\{\text{tp} \, \Delta(\alpha, r)(0, \zeta) \colon (\alpha, r) \in J\}. \quad (13)$$

The higher order version of this invariant is defined for $q \geq 2$ and $(\alpha, r) \in J$ by the functions

$$\Delta^q(\alpha, r)(x^{[1;q]}; t) = \Delta(\alpha, r)(x^2, \bar{S}^q(x^{[1;q]}; t)) = \begin{vmatrix} Q^{r_1}_{z^{\alpha^1};\zeta}(x^2, \bar{S}^q(x^{[1;q]}; t)) \\ \vdots \\ Q^{r_N}_{z^{\alpha^N};\zeta}(x^2, \bar{S}^q(x^{[1;q]}; t)) \end{vmatrix};$$

again, by holomorphic nondegeneracy, these are not all zero, since

$$\Delta^q(\alpha, r)(x_1, 0, \ldots, 0; t) = \Delta(\alpha, r)(0, x_1, t),$$

and we have already observed that the latter are not all zero.

As above, we define the type of a power series $\lambda(x, t)$, tp λ, as in (11). Using the lexicographic ordering on $\mathbb{N}^2$, we let

$$(n_1^q, n_2^q) = \min\{\text{tp} \, \Delta^q(\alpha, r)(x^{[1;q]}; t) \colon (\alpha, r) \in J\}.$$

Let us show that the definition of (n_1^q, n_2^q) for $q \geq 1$ is indeed independent of the choice of normal coordinates. We write (n_1^q, n_2^q), $(n_1'^q, n_2'^q)$ for the numbers defined by (13) when using Q or Q' respectively.

By differentiating the mapping equation $Q'\big(F(z, Q(z, \zeta)), \bar{H}(\zeta)\big) = G(z, Q(z, \zeta))$ with respect to ζ we obtain

$$Q'_{\zeta'}\big(F(z, Q(z, \zeta)), \bar{H}(\zeta)\big)\bar{H}_\zeta(\zeta) = C_0(z, \zeta)Q_\zeta(z, \zeta), \quad (14)$$

where

$$C_0(z, \zeta) = G_w(z, Q(z, \zeta)) - Q'_{z'}\big(F(z, Q(z, \zeta)), \bar{H}(\zeta)\big)F_w(z, Q(z, \zeta)).$$

To continue, we first recall

Lemma 5. *Let $(M, 0)$, $(M', 0)$ be given in normal coordinates by $w = Q(z, \chi, \tau)$ and $w' = Q'(z', \chi', \tau')$, respectively, and assume that $H: (M, 0) \to (M', 0)$. Then for any $\alpha \in \mathbb{N}^n$ with $|\alpha| > 0$, there exists a polynomial P_α such that*

$$\left(\det \left(\frac{\partial F(z, Q(z, \chi, \tau))}{\partial z} \right) \right)^{2|\alpha|-1} Q'_{z'^\alpha}(F(z, Q(z, \zeta)), \bar{H}(\zeta))$$

$$= P_\alpha \left(\frac{\partial^{|\beta|}}{\partial z^\beta} H(z, Q(z, \zeta)) : |\beta| \le |\alpha| \right).$$

For a multi-index α with $|\alpha| > 0$, we deduce from Lemma 5 after differentiation with respect to ζ that

$$Q'_{z'^\alpha \zeta}\big(F(z, Q(z, \zeta)), \bar{H}(\zeta)\big) \bar{H}_\zeta(\zeta) = \sum_{|\beta| \le |\alpha|} C_{\alpha\beta}(z, \zeta) Q_{z^\beta \zeta}(z, \zeta), \tag{15}$$

where $C_{\alpha\beta}(z, \zeta)$ are some power series. From (14) and (15) we obtain, for any $(\alpha, r) \in J$,

$$\Delta'(\alpha, r)\big(F(z, Q(z, \zeta)), \bar{H}(\zeta)\big) \det \bar{H}_\zeta(\zeta)$$

$$= \sum_{\substack{(\beta, s) \in J \\ \max |\beta^j| \le \max |\alpha^j|}} c_{\alpha, r}^{\beta, s}(z, \zeta) \Delta(\beta, s)(z, \zeta), \tag{16}$$

where $c_{\alpha, r}^{\beta, s}(z, \zeta)$ are some power series.

Let us start with the case $q = 1$. By substituting $z = 0$ into (16), we obtain

$$\Delta'(\alpha, r)(F(0, \tau), \bar{H}(\zeta)) \det \bar{H}_\zeta(\zeta) = \sum_{\substack{(\beta, s) \in J \\ \max |\beta^j| \le \max |\alpha^j|}} c_{\alpha, r}^{\beta, s}(0, \zeta) \Delta^1(\beta, s)(\zeta). \tag{17}$$

By using the identity

$$\frac{\partial^{|\beta|}}{\partial t^\beta} \bigg|_{t=0} f(x, t, t) \equiv \sum_{\beta^1 + \beta^2 = \beta} \frac{\beta!}{\beta^1! \beta^2!} \frac{\partial^{|\beta|}}{\partial t^{1^{\beta^1}} \partial t^{2^{\beta^2}}} \bigg|_{\substack{t^1=0 \\ t^2=0}} f(x, t^1, t^2),$$

we get, for any γ,

$$\frac{\partial^{|\gamma|}}{\partial \tau^\gamma} \bigg|_{\tau=0} \Delta'(\alpha, r)(F(0, \tau), \bar{H}(\chi, \tau)) = \frac{\partial^{|\gamma|}}{\partial \tau^\gamma} \bigg|_{\tau=0} \Delta'(\alpha, r)(0, \bar{H}(\chi, \tau))$$

$$+ \sum_{\alpha'} \sum_{|\gamma'| < |\gamma|} e_{\alpha'\gamma'} \frac{\partial^{|\gamma'|}}{\partial \tau^{\gamma'}} \bigg|_{\tau=0} \Delta'(\alpha', r)(0, \bar{H}(\chi, \tau)), \tag{18}$$

where we have used that derivatives of $\Delta'(\alpha, r)(z', \zeta')$ with respect to z' are linear combinations of $\Delta'(\alpha', r)(z', \zeta')$ over (finitely) many α'. Since $\bar{H}(\zeta)$ is invertible and $\bar{G}(\chi, 0) \equiv 0$, we have $(n'^1_1, n'^1_2) = \min\{\text{tp}\, \Delta'(\alpha, r)(0, \bar{H}(\zeta)) : (\alpha, r) \in J\}$, so from (18) we obtain

$$\min\{\text{tp}\, \Delta'(\alpha, r)(F(0, \tau), \bar{H}(\chi, \tau)) : (\alpha, r) \in J\} = (n'^1_1, n'^1_2).$$

Hence, we see from (17), that $(n'^1_1, n'^1_2) \geq (n^1_1, n^1_2)$ lexicographically; and, by using the inverse mapping to H, we obtain the converse inequality, from which we conclude that $(n'^1_1, n'^1_2) = (n^1_1, n^1_2)$.

For $q \geq 2$, we substitute $z = x^2$ and $\zeta = \bar{S}^q(x^{[1;q]}; t)$ in (16) to obtain

$$\Delta'(\alpha, r)\big(F(S^{q-1}(x^{[2;q]}; t)), \bar{H}(\bar{S}^q(x^{[1;q]}; t))\big) \det \bar{H}_\zeta(\bar{S}^q(x^{[1;q]}; t))$$
$$= \sum_{\beta, s} \tilde{c}^{\beta, s}_{\alpha, r}(x^{[1;q]}, t) \Delta^q(\beta, s)(x^{[1;q]}; t). \quad (19)$$

Now Lemma 6 below gives

$$\min\big\{\text{tp } \Delta'(\alpha, r)\big(F(S^{q-1}(x^{[2;q]}; t)), \bar{H}(\bar{S}^q(x^{[1;q]}; t))\big): (\alpha, r) \in J\big\} = (n'^q_1, n'^q_2).$$

Hence, we see from (19) that also in this case $(n'^q_1, n'^q_2) \geq (n^q_1, n^q_2)$.

Again, we can repeat the argument with H replaced with its inverse, so we have $(n'^q_1, n'^q_2) = (n^q_1, n^q_2)$ for all q, and so (n^q_1, n^q_2) are indeed invariants.

Lemma 6. *Let $q \geq 2$. With the notation above, we have*

$$\text{it } \Delta'(\alpha, r)\big(F(S^{q-1}(x^{[2;q]}; t)), \bar{H}(\bar{S}^q(x^{[1;q]}; t))\big) = \text{it } \Delta'^q(\alpha, r)(y^{[1;q]}; u),$$

where

$$y^{2j+1} = \overline{F_z(0)} x^{2j+1}, \quad y^{2j} = F_z(0) x^{2j}, \quad u = \begin{cases} G_w(0)t & \text{for } q \text{ even,} \\ \overline{G_w(0)}t & \text{for } q \text{ odd.} \end{cases}$$

Proof. From Lemma 3, we see that

$$\Delta'(\alpha, r)\big(F(S^{q-1}(x^{[2;q]}; t)), \bar{H}(\bar{S}^q(x^{[1;q]}; t))\big) = \Delta'^q(\alpha, r)(y^{[1;q]}; u),$$

where

$$y^{2j+1} = \bar{F}(\bar{S}^{q-2j}(x^{[2j+1;q]}; t)), \quad y^{2j} = F(S^{q-2j+1}(x^{[2j;q]}; t)),$$
$$u = \begin{cases} G(x^q, t) & \text{for } q \text{ even.} \\ \bar{G}(x^q, t) & \text{for } q \text{ odd.} \end{cases}$$

The lemma now follows from the fact that $G(z, 0) \equiv 0$, and that the matrices $F_z(0)$ and $G_w(0)$ are invertible. $\quad\square$

In the course of establishing the invariance, we have actually proved the following stability property of (n^q_1, n^q_2) for $q \geq 1$ for which we will have use later. Here, we use the convention that $S^{q-1}(x^{[2;q]}; 0) = 0$ for $q = 1$.

Lemma 7. *Let* $(M, 0)$, $(M', 0)$ *be given in normal coordinates by* $w = Q(z, \chi, \tau)$ *and* $w' = Q'(z', \chi', \tau')$, *respectively, and assume that* $H: (M, 0) \to (M', 0)$. *Then for any choice of* $(\alpha, r) \in J$ *with* $\operatorname{tp} \Delta'^q(\alpha, r)(x^{[1;q]}; t) = (n_1^q, n_2^q)$ *we have* $\operatorname{tp} \Delta'(\alpha, r)\big(F(S^{q-1}(x^{[2;q]}; t)), \bar{H}(\bar{S}^q(x^{[1;q]}; t))\big) = (n_1^q, n_2^q)$, *and*

$$\operatorname{it}\big(\Delta'(\alpha, r)\big(F(S^{q-1}(x^{[2;q]}; t)), \bar{H}(\bar{S}^q(x^{[1;q]}; t))\big)\big) = (\operatorname{it} \Delta'^q(\alpha, r))(y^{[1;q]}, u),$$

where

$$y^{2j+1} = \overline{F_z(0)} x^{2j+1}, \quad y^{2j} = F_z(0) x^{2j}, \quad u = \begin{cases} G_w(0)t & \text{for } q \text{ even,} \\ \overline{G_w(0)}t & \text{for } q \text{ odd.} \end{cases}$$

4. A concrete version of Gabrielov's Theorem

One of the new tools which we use in this paper is the following concrete version and generalization of a theorem of Gabrielov [11]: Given a holomorphic map $A(z): (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$, where $m \geq n$, we can conclude that $g(w)$ is convergent if $(A^*g)(z) = g(A(z))$ is, provided that the "generic rank" of A is full. This last condition means that the rank of the Jacobian matrix $A'(z)$ is full over the field of fractions of $\mathbb{C}\{z\}$. Our Theorem 5 provides a linear inversion of the pullback map $A^*: \mathbb{C}[[w]] \to \mathbb{C}[[z]]$ which restricts to a map $\mathbb{C}\{w\} \to \mathbb{C}\{z\}$ if $A(z)$ is convergent. Furthermore, our inversion depends holomorphically on A as long as a certain vanishing order associated to A is constant.

For a formal map $A(z): (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ we denote by $\nu(A)$ the minimum order of vanishing of minors of size n of the Jacobian of A. To be more precise, we denote $D = \{(\delta_1, \ldots, \delta_n) \in \{1, \ldots, m\}^n: \delta_1 < \cdots < \delta_n\}$, and for $\delta \in D$ and a map $A \in \mathcal{H}$, we define

$$\delta(A) = \begin{vmatrix} \frac{\partial A_1}{\partial z_{\delta_1}} & \cdots & \frac{\partial A_1}{\partial z_{\delta_n}} \\ \vdots & & \vdots \\ \frac{\partial A_n}{\partial z_{\delta_1}} & \cdots & \frac{\partial A_n}{\partial z_{\delta_n}} \end{vmatrix}.$$

We can thus define

$$\nu(A) = \min_{\delta \in D} \operatorname{ord} \delta(A).$$

We write $\mathcal{H}(m, n)$ (resp. $\hat{\mathcal{H}}(m, n)$) for the space of holomorphic (resp. formal holomorphic) maps $(\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$. If the dimensions are clear, we simply write $\mathcal{H}$ or $\hat{\mathcal{H}}$. We can thus consider $\nu: \hat{\mathcal{H}} \to \mathbb{N}$; note that the maps of generic full rank are exactly the ones for which $\nu < \infty$.

Theorem 5. *For every* $s \geq 0$ *there exists a finite family of polynomials* $\Psi_1, \ldots, \Psi_{\ell(s)}$ *on* $J_0^{s+1}((\mathbb{C}^m, 0), (\mathbb{C}^n, 0))$ *and corresponding holomorphic functions*

$$\Phi_k(A, f): \hat{U}_k \times \mathbb{C}[[z]] \to \mathbb{C}[[w]], \quad 1 \leq k \leq \ell(s),$$

where $U_k = \{A \in \mathcal{H}: \Psi_k(j_0^{s+1}A) \neq 0\}$, $\hat{U}_k = \{A \in \hat{\mathcal{H}}: \Psi_k(j_0^{s+1}A) \neq 0\}$, *and* Φ_k *is linear in its second variable, such that*

$$\Phi_k(A, A^*g) = g \quad \text{for } A \in \hat{U}_k \text{ with } \nu(A) = s,$$

and

$$\bigcup_k \hat{U}_k \supset \{A \in \hat{\mathcal{H}} \colon \nu(A) = s\}, \quad \bigcup_k U_k \supset \{A \in \mathcal{H} \colon \nu(A) = s\}$$

Furthermore, if $A \in U_k$, the operator $\Phi_k(A, \cdot)$ restricts to a linear operator $\mathbb{C}\{w\} \to \mathbb{C}\{z\}$, and the map $\Phi_k \colon U_k \times \mathbb{C}\{z\} \to \mathbb{C}\{w\}$ is holomorphic, where $\mathcal{H}$, $\mathbb{C}\{w\}$, and $\mathbb{C}\{z\}$ are all equipped with their natural inductive limit topologies.

This theorem is actually a consequence of the following more explicit theorem. We work with the weighted lexicographic ordering on monomials, i.e.

$\alpha \prec \beta$ if $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and $\beta - \alpha$ is negative in its first nonvanishing component;

correspondingly, for a series $f(z) = \sum_\alpha f_\alpha z^\alpha$ we write

$$\text{in } f(z) = \min_\alpha \{f_\alpha \neq 0\}, \quad \text{it } f(z) = f_{\text{in } f} z^{\text{in } f}.$$

Theorem 6. *For every $\delta \in D$ and every $\delta_0 \in \mathbb{N}^m$, writing $q_0 = |\delta_0|$ and $\ell(\beta) = |\beta|(q_0+1)$, there exist polynomials P_β on $J_0^{\ell(\beta)}((\mathbb{C}^m, 0), (\mathbb{C}^n, 0)) \times J_0^{\ell(\beta)}((\mathbb{C}^m, 0), \mathbb{C})$ and a polynomial c on $J_0^{q_0+1}((\mathbb{C}^m, 0), (\mathbb{C}^n, 0))$ such that if $g(A(z)) = f(z)$ and in $\delta(A) = \delta_0$, then*

$$g(w) = \sum_\beta \frac{P_\beta(j_0^{\ell(\beta)} A, j_0^{\ell(\beta)} f)}{c(j_0^{q_0+1} A)^{2|\beta|-1}} w^\beta. \tag{20}$$

Actually, $c(j_0^{q_0+1} A)$ is given by

$$c(j_0^{q_0+1} A) = \frac{\partial^{|\delta_0|} \delta(A)}{\partial z^{\delta_0}}(0),$$

and thus $c(j_0^{q_0+1} A) \neq 0$ for all A with in $\delta(A) = \delta_0$. Furthermore, for any $M, R > 0$, there exist $N, S > 0$ such that if $|A_\alpha| \leq M R^{-|\alpha|}$ and $|f_\alpha| \leq M R^{-|\alpha|}$, then

$$|P_\beta(j_0^{\ell(\beta)} A, j_0^{\ell(\beta)} f)| \leq N S^{-|\beta|}. \tag{21}$$

We split the proof of Theorem 6 in two parts: In 4.1 we discuss how the P_β are constructed in the formal setting, and we present the estimates in the analytic setting in 4.2. Let us sketch how Theorem 5 follows from Theorem 6: For this, we choose finitely many linear maps $L_1, \ldots, L_{\ell^0(s)} \colon \mathbb{C}^m \to \mathbb{C}^m$ such that for any homogeneous polynomial $p \in \mathbb{C}[z_1, \ldots, z_m]$ of degree s there exists a k, $1 \leq k \leq \ell^0(s)$ with $\text{in}(p \circ L_k) = (s, 0, \ldots, 0)$.

We apply Theorem 6 for every $\delta \in D$ and $\delta_0 = (s, 0, \ldots, 0)$; let us call the corresponding polynomials P_β^δ and c_δ. We define

$$\Psi_{k,\delta} = \frac{\partial^s \delta(A \circ L_k)}{\partial z_1^s}(0) = c_\delta(j_0^{q_0+1}(A \circ L_k)).$$

Thus, for any $A \in \mathcal{H}$ with $\Psi_{k,\delta}(A) \neq 0$, we have

$$g(w) = \sum_{\beta} \frac{P_{\beta}^{\delta}\left(j_0^{\ell(\beta)}(A \circ L_k), j_0^{\ell(\beta)}(f \circ L_k)\right)}{c_{\delta}(j_0^{q_0+1}(A \circ L_k))^{2|\beta|-1}} w^{\beta} = \Phi_{k,\delta}(A, f)(w),$$

provided that $g(A(z)) = f(z)$. The holomorphy of $\Phi_{k,\delta}$ follows from the estimates (21) by the same arguments as presented in [15, §5], and Theorem 5 is proved.

4.1. Formal construction of the P_{β}

We assume that $\delta(A)(z) \not\equiv 0$, and write $R^{\delta}(z)$ for the $m \times n$ matrix comprised of the classical adjoint of the columns in $\partial A / \partial z$ corresponding to δ in the rows corresponding to δ, and 0 in the other rows; thus $A'(z) R^{\delta}(z) = \delta(A)(z) I_{n \times n}$. From now on, we will only consider functions f and g such that $f(0) = g(0) = 0$, and think about δ as fixed; we will write $R = R^{\delta}$ and $\lambda(z) = \delta(A)(z)$ for ease of notation.

Given a fixed nonnegative integer q_0, we define functions $T_{\alpha,\beta}(f, A)$ for $|\alpha| \geq 0$ and $|\beta| \geq 1$, where $f \in \mathbb{C}[[z]]$, $A \in \mathbb{C}[[z]]^n$ with $f(0) = 0$, $A(0) = 0$ by the following recursion formulas:

$$T_{\alpha,e_j}(f, A) = \sum_{\substack{\alpha^1+\alpha^2=\alpha \\ 1 \leq k \leq m}} \frac{\alpha!}{\alpha^1! \alpha^2!} \frac{\partial^{|\alpha|+1} f}{\partial z^{\alpha^1+e_k}}(0) \frac{\partial^{|\alpha^2|} R_j^k}{\partial z^{\alpha^2}}(0)$$

$$T_{\alpha,\beta+e_j}(f, A) = \sum_{\substack{(\alpha^1,\alpha^2,\alpha^3) \in I_{\alpha}^1 \\ 1 \leq k \leq m}} \frac{\alpha!}{\alpha^1! \alpha^2! \alpha^3!} \frac{\partial^{|\alpha^1|} \lambda}{\partial z^{\alpha^1}}(0) T_{\alpha^2+e_k,\beta}(f, A) \frac{\partial^{|\alpha^3|} R_j^k}{\partial z^{\alpha^3}}(0)$$

$$- (2|\beta| - 1) \sum_{\substack{(\alpha^1,\alpha^2,\alpha^3) \in I_{\alpha}^2 \\ 1 \leq k \leq m}} \frac{\alpha!}{\alpha^1! \alpha^2! \alpha^3!} \frac{\partial^{|\alpha^1|+1} \lambda}{\partial z^{\alpha^1+e_k}}(0) T_{\alpha^2,\beta}(f, A) \frac{\partial^{|\alpha^3|} R_j^k}{\partial z^{\alpha^3}}(0), \quad (22)$$

where I_{α}^1 and I_{α}^2 are index sets defined by

$$I_{\alpha}^1 = \{(\alpha^1, \alpha^2, \alpha^3): \alpha^1 + \alpha^2 + \alpha^3 = \alpha, \ |\alpha^1| \geq q_0, \ |\alpha^2| \geq (2|\beta| - 1)q_0 - 1\},$$
$$I_{\alpha}^2 = \{(\alpha^1, \alpha^2, \alpha^3): \alpha^1 + \alpha^2 + \alpha^3 = \alpha, \ |\alpha^1| \geq q_0 - 1, \ |\alpha^2| \geq (2|\beta| - 1)q_0\}.$$

We will use the following observation to determine the order $\ell(\beta)$ in Theorem 6; the proof is a straightforward induction on $|\beta|$.

Lemma 8. *With the notation above, $T_{\alpha,\beta}(f, A)$ is a polynomial in $\frac{\partial^{|\gamma|} f}{\partial z^{\gamma}}(0)$ and $\frac{\partial^{|\gamma|} A}{\partial z^{\gamma}}(0)$ for $|\gamma| \leq |\alpha| - |\beta| q_0 + |\beta| + q_0$, which is linear in f. In particular for a δ_0 with $|\delta_0| = q_0$, $T_{(2|\beta|-1)\delta_0,\beta}(f, A)$ is a polynomial in $\frac{\partial^{|\gamma|} f}{\partial z^{\gamma}}(0)$ and $\frac{\partial^{|\gamma|} A}{\partial z^{\gamma}}(0)$ for $|\gamma| \leq |\beta|(q_0 + 1)$.*

We are now in a position to define the P_β by

$$P_\beta(j_0^{\ell(\beta)}A, j_0^{\ell(\beta)}f) = \frac{T_{(2|\beta|-1)\delta_0,\beta}(f, A)}{\beta!((2|\beta|-1)\delta_0)!},$$

and c by $c(j_0^{q_0+1}A) = \frac{\partial^{q_0}\lambda(z)}{\partial z_1^{q_0}}(0)$.

We now need to show that the P_β defined in that way satisfy (20).

For every multi-index β with $|\beta| > 0$, we let the function $\psi_\beta(z)$ be defined by

$$\lambda(z)^{2|\beta|-1}g_{w^\beta}(A(z)) = \psi_\beta(z). \tag{23}$$

First note that

$$\psi_{e_j}(z) = \sum_{k=1}^{m} f_{z_k}(z)R_j^k(z);$$

indeed, start by taking the derivative of the equation $g(A(z)) = f(z)$ with respect to z; this gives $g'(A(z))A'(z) = f'(z)$, which after application of $R(z)$ reads

$$\lambda(z)g'(A(z)) = f'(z)R(z).$$

If we compare the jth rows in the last equation, we obtain the claimed formula for ψ_{e_j}. Now we take a derivative of (23), after dividing by $\lambda(z)^{2|\beta|-1}$, and see that

$$\frac{\partial g_{w^\beta}}{\partial w}(A(z))A'(z) = \frac{\lambda(z)\frac{\partial \psi_\beta(z)}{\partial z} - (2|\beta|-1)\psi_\beta(z)\frac{\partial \lambda(z)}{\partial z}}{\lambda(z)^{2|\beta|}};$$

an application of R leads to

$$\lambda(z)g_{w^{\beta+e_j}}(A(z)) = \frac{1}{\lambda(z)^{2|\beta|}}\sum_{k=1}^{m}\left(\lambda(z)\frac{\partial \psi_\beta(z)}{\partial z_k} - (2|\beta|-1)\psi_\beta(z)\frac{\partial \lambda(z)}{\partial z_k}\right)R_j^k(z).$$

Thus the ψ_β satisfy the following recursion formulas:

$$\psi_{\beta+e_j}(z) = \sum_{k=1}^{m}\left(\lambda(z)\frac{\partial \psi_\beta(z)}{\partial z_k} - (2|\beta|-1)\psi_\beta(z)\frac{\partial \lambda(z)}{\partial z_k}\right)R_j^k(z).$$

It is now easy to see inductively that $\frac{\partial^{|\alpha|}\psi_\beta}{\partial z^\alpha}(0) = T_{\alpha,\beta}(f, A)$.

4.2. Convergence estimates

To prove the convergence properties, we show that if A is convergent, then the $\psi_\beta(z)$ defined in (23) satisfy an estimate of the form $|\psi_\beta(0)| \le M\beta!C^{|\beta|}$ for some constants M and C, by adapting a method introduced by the first author in his thesis. We start by introducing some notation and stating some lemmata which are needed in the proof of the main estimate in Proposition 11. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$ and a real number $N \ge 0$, we will use $(1 + \alpha)^N$ to denote $\prod_{j=1}^{m}(1 + \alpha_j)^N$.

Lemma 9. *Let $N > 1$ be a real number. Then there exists a constant $E(N) \leq 2^{N+1}\zeta(N)$ (where $\zeta(N)$ denotes the classical Riemann zeta function) such that for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_m)$,*

$$\sum_{\alpha^1 + \alpha^2 = \alpha} \frac{(1 + \alpha)^N}{(1 + \alpha^1)^N (1 + \alpha^2)^N} \leq E(N)^m. \tag{24}$$

Proof. Note that it is enough to treat the case $m = 1$; for $m > 1$, the estimate follows by rewriting the sum as a product of sums, and applying the estimate for $m = 1$:

$$\sum_{\alpha^1 + \alpha^2 = \alpha} \frac{(1 + \alpha)^N}{(1 + \alpha^1)^N (1 + \alpha^2)^N} = \prod_{j=1}^{n} \left(\sum_{\alpha_j^1 + \alpha_j^2 = \alpha_j} \frac{(1 + \alpha_j)^N}{(1 + \alpha_j^1)^N (1 + \alpha_j^2)^N} \right) \leq E(N)^m.$$

For $m = 1$, we estimate as follows:

$$\sum_{\alpha^1 + \alpha^2 = \alpha} \frac{(1 + \alpha)^N}{(1 + \alpha^1)^N (1 + \alpha^2)^N} \leq 2 \sum_{\ell=0}^{\lfloor \alpha/2 \rfloor} \frac{(1 + \alpha)^N}{(1 + \ell)^N (1 + \alpha - \ell)^N}$$

$$\leq 2 \sum_{\ell=0}^{\lfloor \alpha/2 \rfloor} \frac{(1 + \alpha)^N}{(1 + \ell)^N (1 + \alpha/2)^N} \leq 2^{N+1} \sum_{\ell=0}^{\lfloor \alpha/2 \rfloor} \frac{1}{(1 + \ell)^N} < 2^{N+1} \sum_{\ell=0}^{\infty} \frac{1}{(1 + \ell)^N} < \infty. \quad \square$$

We define a useful function $e(\alpha, k)$, where $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a multi-index and k is a nonnegative integer in the following way: Choose p, $1 \leq p \leq m$, such that $\alpha_p = \max_j \alpha_j$ and set

$$e(\alpha, k) = (\alpha_p + k)! \prod_{\substack{1 \leq j \leq m \\ j \neq p}} \alpha_j! \tag{25}$$

We will need the following two properties of $e(\alpha, k)$, the proof of which are left to the reader:

Lemma 10. *$e(\alpha, k)$, defined in (25), satisfies the following:*

$$e(\beta + e_j, k) \leq e(\beta, k + 1), \quad 1 \leq j \leq m,$$

$$\frac{\alpha! e(\alpha^1, k)}{\alpha^1!} \leq e(\alpha, k), \quad \alpha^1 \leq \alpha.$$

We can now state the main result of this section.

Proposition 11. *Let $N > 1$, and assume that*

$$\left| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(0) \right| \leq \alpha! \frac{M_1}{K^{|\alpha|}(1 + \alpha)^N}, \quad \left| \frac{\partial^{|\alpha|} h}{\partial z^\alpha}(0) \right| \leq \alpha! \frac{M_2}{K^{|\alpha|}(1 + \alpha)^N}, \tag{26}$$

where h denotes any of the functions $\lambda(z)$, $\lambda_{z_j}(z)$, or entries of the matrix $R(z)$. There exists a constant $C_1 = C_1(K, M_2)$ such that for all multi-indices $\alpha \in \mathbb{N}^m$ and $\beta \in \mathbb{N}^n$ with $|\beta| > 0$ we have

$$|T_{\alpha, \beta}(f, A)| \leq e(\alpha, |\beta|) \frac{M_1 C_1^{|\beta|}}{K^{|\alpha| + |\beta|}(1 + \alpha)^N}. \tag{27}$$

Proof. We will prove (27) by induction on $|\beta|$. First, we look at the case $|\beta| = 1$, so we estimate, using (26) and Lemma 10:

$$
\begin{aligned}
|T_{\alpha,e_j}(f, A)| &\leq \sum_{k=1}^{m} \sum_{\alpha^1+\alpha^2=\alpha} \frac{\alpha!}{\alpha^1!\alpha^2!} \frac{(\alpha^1+e_k)!M_1}{K^{|\alpha^1|+1}(1+\alpha^1+e_k)^N} \frac{(\alpha^2)!M_2}{K^{|\alpha^2|}(1+\alpha^2)^N} \\
&\leq \frac{M_1M_2}{K^{|\alpha|+1}} \sum_{k=1}^{m} \sum_{\alpha^1+\alpha^2=\alpha} \frac{\alpha!}{\alpha^1!} e(\alpha^1, 1) \frac{1}{(1+\alpha^1)^N} \frac{1}{(1+\alpha^2)^N} \\
&\leq \frac{mM_1M_2}{K^{|\alpha|+1}} e(\alpha, 1) \sum_{\alpha^1+\alpha^2=\alpha} \frac{1}{(1+\alpha^1)^N} \frac{1}{(1+\alpha^2)^N} \\
&\leq e(\alpha, 1) \frac{M_1 C_1}{K^{|\alpha|+1}(1+\alpha)^N},
\end{aligned}
$$

provided that $C_1 \geq mM_2E(N)^m$. We now turn to the induction step, and prove (27) holds with β replaced by $\beta + e_j$ for some j, provided that it holds for all β of lesser length. To estimate $T_{\alpha,\beta+e_j}$, let us write (22) as

$$
T_{\alpha,\beta+e_j}(f, A) = \sum_{\substack{\alpha^1+\alpha^2=\alpha \\ |\alpha^1| \geq 2|\beta|q_0-1 \\ 1 \leq k \leq m}} \frac{\alpha!}{\alpha^1!\alpha^2!} Q_{\alpha^1,\beta,k}(f, A) \frac{\partial^{|\alpha^2|}R_j^k}{\partial z^{\alpha^2}}(0), \qquad (28)
$$

where $Q_{\alpha,\beta,k}(f, A) = Q_{\alpha,\beta,k}^1(f, A) + Q_{\alpha,\beta,k}^2(f, A)$ with

$$
Q_{\alpha,\beta,k}^1(f, A) = \sum_{\substack{\alpha^1+\alpha^2=\alpha \\ |\alpha^1| \geq q_0 \\ |\alpha^2| \geq (2|\beta|-1)q_0-1}} \frac{\alpha!}{\alpha^1!\alpha^2!} \frac{\partial^{|\alpha^1|}\lambda}{\partial z^{\alpha^1}}(0) T_{\alpha^2+e_k,\beta}(f, A),
$$

$$
Q_{\alpha,\beta,k}^2(f, A) = -(2|\beta|-1) \sum_{\substack{\alpha^1+\alpha^2=\alpha \\ |\alpha^1| \geq q_0-1 \\ |\alpha^2| \geq (2|\beta|-1)q_0}} \frac{\alpha!}{\alpha^1!\alpha^2!} \frac{\partial^{|\alpha^1|+1}\lambda}{\partial z^{\alpha^1+e_k}}(0) T_{\alpha^2,\beta}(f, A).
$$

We will first establish estimates for $Q_{\alpha,\beta,k}^1$ and $Q_{\alpha,\beta,k}^2$. Again using (26), and Lemma 10 as at the induction start, we have

$$
\begin{aligned}
|Q_{\alpha,\beta,k}^1(f, A)| &\leq \sum_{\alpha^1+\alpha^2=\alpha} \frac{\alpha!}{\alpha^1!\alpha^2!} \frac{e(\alpha^1+e_k, |\beta|)M_1 C_1^{|\beta|}}{K^{|\alpha^1|+|\beta|+1}(1+\alpha^1+e_k)^N} \frac{\alpha^2!M_2}{K^{|\alpha^2|}(1+\alpha^2)^N} \\
&\leq e(\alpha, |\beta|+1) \frac{M_1 M_2 E(N)^m C_1^{|\beta|}}{K^{|\alpha|+|\beta|+1}(1+\alpha)^N}.
\end{aligned}
$$

Similarly, we see that

$$
|Q_{\alpha,\beta,k}^2(f, A)| \leq e(\alpha, |\beta|+1) \frac{2M_1 M_2 C_1^{|\beta|} E(N)^m}{K^{|\alpha|+|\beta|}(1+\alpha)^N},
$$

and combining this for $Q_{\alpha,\beta,k} = Q^1_{\alpha,\beta,k} + Q^2_{\alpha,\beta,k}$, we have

$$|Q_{\alpha,\beta,k}(f, A)| \leq e(\alpha, |\beta| + 1) \frac{M_1 M_2 C_1^{|\beta|} E(N)^m (1 + 2K)}{K^{|\alpha|+|\beta|+1}(1 + \alpha)^N}.$$

Finally, using (28) we estimate

$$
\begin{aligned}
|T_{\alpha,\beta+e_j}(f, A)| &\leq \sum_{k=1}^{m} \sum_{\alpha^1+\alpha^2+\alpha} \frac{\alpha!}{\alpha^1!} e(\alpha^1, |\beta| + 1) \frac{M_1 M_2^2 C_1^{|\beta|} E(N)^m (1 + 2K)}{K^{|\alpha|+|\beta|+1}(1 + \alpha^1)^N (1 + \alpha^2)^N} \\
&\leq e(\alpha, |\beta| + 1) \frac{M_1 M_2^2 (1 + 2K) C_1^{|\beta|} m E(N)^{2m}}{K^{|\alpha|+|\beta|+1}(1 + \alpha)^N} \\
&\leq e(\alpha, |\beta| + 1) \frac{M_1 C_1^{|\beta|+1}}{K^{|\alpha|+|\beta|+1}(1 + \alpha)^N}
\end{aligned}
$$

provided that $C_1 \geq M_2^2(1 + 2K)m E(N)^{2m}$. We thus obtain (27) with

$$C_1 \geq m M_2 E(N)^m \max(M_2 E(N)^m (1 + 2K), 1). \qquad \square$$

The proposition implies that

$$\frac{1}{((2|\beta| - 1)\delta_0)!\beta!} |T_{(2|\beta|-1)\delta_0,\beta}(f, A)| \leq \left(\frac{(n+1)^{2\delta_{0m}+1} C_1}{K^{2q_0+1}} \right)^{|\beta|} \cdot \frac{K^{q_0}}{(n+1)^{\delta_{0m}}} \cdot M_1,$$

where $\delta_{0m} = \max_j \delta_{0j}$, which in particular implies the last statement of Theorem 6, the proof of which is thus finished.

5. A parametrization theorem

As noted in the introduction, we need a parametrization theorem for solutions to a certain type of singular analytic equations. While these equations usually do not possess any solutions, if they do, the dependence of the solutions on the right hand side of the equations is analytic as well. More specifically, we have the following theorem. We refer the reader back to §2.3 for our notation for the jets appearing in its statement.

Theorem 7. *Let* $P : (\mathbb{C}^r, 0) \to (\mathbb{C}^r, 0)$ *be a holomorphic map, and assume that* $\lambda(y) = \det P_y(y) \not\equiv 0$. *Let* $h^0(x, t) : (\mathbb{C}^p_x \times \mathbb{C}^q_t, 0) \to (\mathbb{C}^r, 0)$ *be a holomorphic map, and write* $\lambda \circ h^0 = \lambda^0$, $g^0 = P \circ h^0$. *Furthermore assume that* $\lambda^0(x, t) \not\equiv 0$, *and write* $(n_1, n_2) = \text{tp}\,\lambda^0$ *and* $k(\ell) = \max(2n_2 - 1, n_2 + \ell)$. *Then there exists an integer* k_0, *a neighbourhood* $U^0 \subset J^{k_0}((\mathbb{C}^{p+q}, 0), (\mathbb{C}^r, 0))$ *of* $j_0^{k_0} h^0$, *and for every* $\ell \geq 0$ *a neighbourhood* $V_\ell^0 \subset J_t^{k(\ell)}((\mathbb{C}^p_x \times \mathbb{C}^q_t, 0), \mathbb{C}^r)$ *of* $j_{t,0}^{k(\ell)} g^0$ *and a holomorphic mapping* $\Phi_\ell : U^0 \times V_\ell^0 \supset W_\ell \to J_t^\ell((\mathbb{C}^p \times \mathbb{C}^q, 0), \mathbb{C}^r)$, *where* $U^0 \times V_\ell^0 \subset W_\ell$, *with the property that*

$$j_{t,0}^\ell h = \Phi_\ell\big(j_0^{k_0} h, j_{t,0}^{k(\ell)}(P \circ h)\big) \quad \text{whenever} \quad \text{tp}(\lambda \circ h) = (n_1, n_2).$$

Following the arguments given in [15, §5], we see that Theorem 7 is an immediate consequence of the following, more explicit theorem, whose proof is the main objective of this section.

Theorem 8. *Let $P: (\mathbb{C}^r, 0) \to (\mathbb{C}^r, 0)$ be a holomorphic map, and assume that $\lambda(y) = \det P_y(y) \not\equiv 0$. Let $h^0(x, t): (\mathbb{C}_x^p \times \mathbb{C}_t^q, 0) \to (\mathbb{C}^r, 0)$ be a holomorphic map, and write $\lambda \circ h^0 = \lambda^0$. Furthermore assume that $\lambda^0(x, t) \not\equiv 0$, and denote $(n_1, n_2) = \operatorname{tp} \lambda^0$. Then there exists an integer k_0, a polynomial $e: J^{k_0}((\mathbb{C}^{p+q}, 0), (\mathbb{C}^r, 0)) \to \mathbb{C}$ with $e(j_0^{k_0} h^0) \neq 0$, and for every $\alpha \in \mathbb{N}^{p+q}$ a set $N_\alpha \subset \mathbb{N}^{p+q}$, a polynomial $p_\alpha(Y, Z_\alpha)$, where Z_α is a variable in $\mathbb{C}^{r|N_\alpha|}$, and an integer d_α, with the property that for any formal power series $h(x, t) = \sum_\beta h_\beta x^{\beta^1} t^{\beta^2}$ with $h(0, 0) = 0$ and $g(x, t) = \sum_\beta g_\beta x^{\beta^1} t^{\beta^2}$ which satisfy*

$$\left.\frac{\partial^{|\gamma|}}{\partial t^\gamma}\right|_{t=0} P(h(x, t)) = \left.\frac{\partial^{|\gamma|}}{\partial t^\gamma}\right|_{t=0} g(x, t), \quad |\gamma^2| \le \max(2n_2 - 1, n_2 + |\alpha^2|), \quad (29)$$

and $\operatorname{tp}(\lambda \circ h) = (n_1, n_2)$, we have

$$h_\alpha = \frac{p_\alpha(j_0^{k_0} h, (g_\beta)_{\beta \in N_\alpha})}{e(j_0^{k_0} h)^{d_\alpha}}. \quad (30)$$

The sets N_α can be chosen to satisfy $N_\alpha \subset \{(\beta^1, \beta^2): |\beta^2| \le \max(|\alpha^2| + n_2, 2n_2 - 1), |\beta^1| \le a + b|\alpha^1|\}$ for some integers $a, b \in \mathbb{N}$.

Furthermore, the p_α can be chosen in such a way that for every multiradius $R \in \mathbb{R}_+^{p+q}$, $N > 0$, and every compact subset $L \subset J^{k_0}((\mathbb{C}^{p+q}, 0), (\mathbb{C}^r, 0)) \setminus \{e = 0\}$ there exists a multiradius $C \in \mathbb{R}_+^{p+q}$ and $M > 0$, depending on R, N, L and P, with the property that

$$\text{if} \quad |g_\beta| \le \frac{N}{R^\beta}, \; \beta \in N_\alpha, \quad \text{then} \quad \left|\frac{p_\alpha(\Lambda, (g_\beta)_{\beta \in N_\alpha})}{e(\Lambda)^{d_\alpha}}\right| \le \frac{M}{C^\alpha} \quad \text{for all } \Lambda \in L.$$

We note the following useful corollary, which can be deduced from either Theorem 8 or Theorem 7 by arguments analogous to the proof of Corollary 3.2 in [15]. We note that similar statements hold with X replaced by e.g. a smooth or a topological manifold, with the conclusions valid with the appropriate regularity.

Corollary 12. *Let X be a complex manifold, and $g(x, t, \omega)$ be a map defined on a neighbourhood of $(0, 0, \omega_0) \in \mathbb{C}^p \times \mathbb{C}^q \times X$. Let $P: (\mathbb{C}^r, 0) \to (\mathbb{C}^r, 0)$ be a holomorphic map, and assume that $\lambda(y) = \det P_y(y) \not\equiv 0$. Let $h^0(x, t): (\mathbb{C}_x^p \times \mathbb{C}_t^q, 0) \to (\mathbb{C}^r, 0)$, write $\lambda \circ h^0 = \lambda^0$ and $g^0 = P \circ h^0$, and assume that $g^0(x, t) = g(x, t, \omega_0)$. Furthermore assume that $\lambda^0(x, t) \not\equiv 0$. Then there exists an integer k_0, a neighbourhood $U^0 \subset J^{k_0}((\mathbb{C}^{p+q}, 0), (\mathbb{C}^r, 0))$ of $j_0^{k_0} h^0$, and for every $\ell \ge 0$ a neighbourhood $V_\ell^0 \subset X$ of ω_0 and a holomorphic mapping $\Phi_\ell: U^0 \times V_\ell^0 \supset W_\ell \to J_t^\ell((\mathbb{C}^p \times \mathbb{C}^q, 0), \mathbb{C}^r)$, where $(n_1, n_2) = \operatorname{tp} \lambda^0$, which satisfies*

$$j_{t,0}^\ell h = \Phi_\ell(j_0^{k_0} h, \omega) \quad \text{if } j_{t,0}^{k(\ell)}(P \circ h) = j_{t,0}^{k(\ell)}(g(x, t, \omega)) \text{ and } \operatorname{tp}(\lambda \circ h) = (n_1, n_2).$$

Here we again set $k(\ell) = \max(2n_2 - 1, n_2 + \ell)$.

Proof of Theorem 8. By assumption, $\lambda^0 = \det \frac{\partial P}{\partial y}(h^0(x,t)) \not\equiv 0$, and we recall that we write $(n_1, n_2) = \mathrm{tp}\, \lambda^0$. We write $A(y)$ for the classical adjoint of $\partial P / \partial y$, that is, $A(y)$ is the $r \times r$-matrix satisfying $\frac{\partial P}{\partial y}(y) A(y) = A(y) \frac{\partial P}{\partial y}(y) = \lambda(y) I_{r \times r}$.

We first make a linear change of coordinates in x and a linear change of coordinates in t to ensure that

$$e_0 := \frac{1}{n_1! n_2!} \frac{\partial^{n_1 + n_2} \lambda^0}{\partial x_1^{n_1} \partial t_1^{n_2}}(0) \neq 0,$$

and write $\delta^1 = (n_1, 0, \ldots, 0) \in \mathbb{N}^p$, $\delta^2 = (n_2, 0, \ldots, 0) \in \mathbb{N}^q$ and $\delta = (\delta^1, \delta^2) \in \mathbb{N}^{p+q}$. Similarly, if h is any map with $\mathrm{tp}\, \lambda \circ h = \mathrm{tp}\, \lambda^0$, we write

$$e := \frac{1}{n_1! n_2!} \frac{\partial^{n_1 + n_2} (\lambda \circ h)}{\partial x_1^{n_1} \partial t_1^{n_2}}(0); \tag{31}$$

e is a polynomial in the $n_1 + n_2$-jet of h at 0.

For the remainder of the proof, we recall that we use the following notation. For any multiindex $\alpha \in \mathbb{N}^{p+q} = \mathbb{N}^p \times \mathbb{N}^q$, α^1 denotes its projection on the first p entries, and α^2 its projection on the last q entries. We write $\varphi(x,t)[\alpha]$ for the coefficient of $x^{\alpha^1} t^{\alpha^2}$ in $\varphi(x,t)$. For brevity, we sometimes also write $\varphi(x,t)[\alpha] = \varphi_\alpha$.

We first define a weight $\omega_k(\alpha) = k_1 |\alpha^1| + k_2 |\alpha^2|$ by choosing $k = (k_1, k_2)$, where k_1 and k_2 are positive integers. Let $p_0 = \omega_k(\delta)$, and assume k_2 / k_1 is chosen large enough so that $|\alpha^2| < n_2 + 1$ whenever $\omega_k(\alpha) \leq p_0$. Note that if $\lambda^0[\alpha] \neq 0$ and $\omega_k(\alpha) \leq p_0$ then $(|\alpha^1|, |\alpha^2|) = (n_1, n_2)$. We also introduce a monomial order $\prec$ by

$$\alpha \prec \beta \;:\Leftrightarrow\; \begin{cases} \omega_k(\alpha) < \omega_k(\beta), \text{ or} \\ \omega_k(\alpha) = \omega_k(\beta), \text{ and } \alpha^1 <_{\mathrm{lex}} \beta^1, \text{ or} \\ \omega_k(\alpha) = \omega_k(\beta), \text{ and } \alpha^1 = \beta^1, \text{ and } \alpha^2 <_{\mathrm{lex}} \beta^2. \end{cases}$$

By Taylor's Theorem, we can rewrite the equation

$$P(h(x,t)) = g(x,t)$$

for any decomposition of $h(x,t)$ as the sum of two terms, say $h = T_1 + T_2$, as

$$P(T_1) + P_y(T_1) T_2 + R(T_1, T_2) = g(x,t),$$

where $R(T_1, T_2)$ is of order at least 2 in T_2. After multiplying by $A(T_1)$, we obtain

$$\lambda(T_1) T_2 = A(T_1)(g(x,t) - P(T_1) - R(T_1, T_2)). \tag{32}$$

We are going to exploit (32) iteratively, and start with the formal statement (i.e. we prove the existence of the polynomials p_α). For any given multiindex γ with $\omega_k(\gamma) > p_0$, we let $\ell = \max(|\gamma^2|, n_2 - 1)$ and write $h = T_\gamma h + S_\gamma h$, where $T_\gamma h = T^0 h + T_\gamma^1 h$ and

$$(T^0 h)(x,t) = \sum_{\omega_k(\alpha) \leq p_0} h_\alpha x^{\alpha^1} t^{\alpha^2}, \qquad (T_\gamma^1 h)(x,t) = \sum_{\substack{\omega_k(\alpha) > p_0,\, \alpha \prec \gamma \\ |\alpha^2| \leq \ell}} h_\alpha x^{\alpha^1} t^{\alpha^2}.$$

If h satisfies the hypotheses of the theorem, we substitute $T_1 = T_\gamma h$, $T_2 = S_\gamma h$ in (32) and compute the coefficient of $x^{\gamma^1 + \delta^1} t^{\gamma^2 + \delta^2}$. On the left hand side, we obtain

$$(\lambda(T_\gamma h) S_\gamma h)[\gamma + \delta] = \sum_{\alpha + \beta = \gamma + \delta} \lambda(T_\gamma h)[\alpha]\, S_{\gamma, \ell} h[\beta] = \lambda(T_\gamma h)[\delta]\, S_\gamma h[\gamma] = e h_\gamma.$$

Since $\ell \geq n_2 - 1$ and $\omega_k(\gamma) > \omega_k(\delta)$, the right hand side simplifies to $A(T_\gamma h)(g(x, t) - P(T_\gamma h))$, which is a polynomial in the coefficients g_α of $g(x, t)$ with $\alpha \preceq \gamma + \delta$, $|\alpha^2| \leq \ell + n_2$, as well as in h_α, where $\alpha \prec \gamma$, and $\alpha^2 \leq \ell$.

We define $k_0 = \max\{|\alpha^1| + |\alpha^2| : \omega_k(\alpha) \leq p_0\}$. Note that we can regard $e = e((h_\alpha)_{\omega_k(\alpha) \leq p_0})$ as a polynomial on $J_0^{k_0}((\mathbb{C}_x^p \times \mathbb{C}_t^q, 0), \mathbb{C}^r)$. Now a simple induction shows that we can define polynomials $p_\gamma(Y, Z)$ in the variables $Y = (Y_\alpha)_{\omega_k(\alpha) \leq p_0}$ and $Z = (Z_\beta)_{\beta \preceq \gamma + \delta, |\beta^2| \leq \ell + n_2}$ such that with some numbers d_γ for any map $h(x, t)$ satisfying the hypotheses of the theorem we have

$$h_\gamma = \frac{p_\gamma((h_\alpha)_{\omega_k(\alpha) \leq p_0}, (g_\beta)_{\beta \preceq \gamma + \delta, |\beta^2| \leq \ell + n_2})}{e((h_\alpha)_{\omega_k(\alpha) \leq p_0})^{d_\gamma}}, \tag{33}$$

as required.

In order to carry out the estimates, we need to be a bit more precise about how we define the polynomials p_γ. To do that, we will use the splitting $T_\gamma h = T^0 h + T_\gamma^1 h$. We rewrite the right hand side of (32) (with R removed, as it does not contribute any terms) by setting $T_1 = Y_1 + Y_2$:

$$A(T_1)(g(x, t) - P(T_1)) = A(Y_1 + Y_2)(g(x, t) - P(Y_1 + Y_2))$$

$$= A(Y_1)(g(x, t) - P(Y_1)) + \sum_{j,k} \frac{\partial A_j}{\partial Y^k}(Y_1)(g^j(x, t) - P^j(Y_1)) Y_2^k$$

$$- \lambda(Y_1) Y_2 + R_2(x, t, Y_1, Y_2), \tag{34}$$

where R_2 is at least of order 2 in Y_2; we have written $Y = (Y^1, \ldots, Y^r)$, and denoted the jth column of A by A_j; g^j, P^j are the components of g and P, respectively. We will replace Y_1 and Y_2 by $(T^0 h)(x, t)$ and $(T_\gamma^1 h)(x, t)$, respectively.

To simplify notation, we write

$$\tilde{A}_k(x, t) = \sum_j \frac{\partial A_j}{\partial Y^k}((T^0 h)(x, t))\big(g^j(x, t) - P^j((T^0 h)(x, t))\big),$$

$$\tilde{J}(x, t) = \sum_j A_j((T^0 h)(x, t))\big(g^j(x, t) - P^j((T^0 h)(x, t))\big), \tag{35}$$

$$R_2(x, t, Y_1, Y_2) = \sum_{j,k} R_{2,j,k}(x, t, Y_1, Y_2) Y_2^j Y_2^k.$$

Using this notation, we define the polynomials p_γ satisfying (33) by the recursion formula

$$e \cdot h_\gamma = \tilde{J}(x,t)[\gamma + \delta] + \sum_k \sum_{\substack{\alpha+\beta=\gamma+\delta \\ \omega_k(\beta) \geq p_0+1}} (T_\gamma^1 h^k)(x,t)[\alpha] \tilde{A}_k(x,t)[\beta]$$

$$- \sum_{\substack{\alpha+\beta=\gamma+\delta \\ \omega_k(\beta) \geq p_0}} (T_\gamma^1 h)(x,t)[\alpha] \lambda((T^0 h)(x,t))[\beta] + R_2(x,t,(T^0 h)(x,t),(T_\gamma^1 h)(x,t))[\gamma+\delta].$$

(36)

Next we turn to the convergence estimates. For this part we will no longer assume that h satisfies (29) and $\mathrm{tp}(\lambda \circ h) = (n_1, n_2)$, but we assume that h satisfies the recursion formula (36). We assume the following convergence estimates for some $R \in \mathbb{R}_+^{p+q}$, $S \in \mathbb{R}_+^r$, and some $N > 1$:

$$\|\tilde{A}_k(x,t)[\alpha]\|_\infty \leq \frac{M_2}{R^\alpha (1+\alpha)^N}, \qquad |\alpha^2| \leq n_2 + \ell,$$

$$\|\tilde{J}(x,t)[\alpha]\|_\infty \leq \frac{M_2}{R^\alpha (1+\alpha)^N}, \qquad |\alpha^2| \leq n_2 + \ell,$$

$$|\lambda(T^0 h(x,t))[\alpha]| \leq \frac{M_2}{R^\alpha (1+\alpha)^N}, \qquad |\alpha^2| \leq n_2 + \ell,$$

$$\|R_{2,j,k}(x,t,T^0 h(x,t),Y)[x^{\alpha^1} t^{\alpha^2} Y^\eta]\|_\infty \leq \frac{M_2}{R^\alpha S^\eta (1+\alpha)^N}, \quad |\alpha^2| \leq n_2 + \ell.$$

We also write

$$N_0 = \sum_{\substack{\omega_k(\alpha)=p_0 \\ \alpha > \delta}} |\lambda(T^0 h(x,t))[\alpha]|$$

We define e from (31). We will regard N_0 and e as functions on $J_0^{k_0}((\mathbb{C}_x^p \times \mathbb{C}_t^q, 0), \mathbb{C}^r)$. In order to have a common notation, we will assume that we choose the neighbourhood U^0 such that e and N_0 remain bounded above, and e remains bounded below on it, and we write

$$\mathbf{e} = \min_{\Lambda \in U^0} |e(\Lambda)|, \qquad \mathbf{N_0} = \max_{\Lambda \in U^0} |N_0(\Lambda)|.$$

We assume that by induction we already have

$$\|T_\gamma^1 h(x,t))[\alpha]\|_\infty \leq \frac{M}{C^\alpha (1+\alpha)^N} \quad \text{for } \alpha \prec \gamma, \ |\alpha^2| \leq \ell. \tag{37}$$

We can assume that the multiradius R is of the form

$$R = (\underbrace{R_0^{k_1}, \ldots, R_0^{k_1}}_{p \text{ entries}}, \underbrace{R_0^{k_2}, \ldots, R_0^{k_2}}_{q \text{ entries}})$$

for some $R_0 > 0$, and we will choose C of the form

$$C = (C_0^{k_1}, \underbrace{C_0^{k_1}/K, \ldots, C_0^{k_1}/K}_{p-1 \text{ entries}}, C_0^{k_2}, \underbrace{C_0^{k_2}/K, \ldots, C_0^{k_2}/K}_{q-1 \text{ entries}}),$$

where C_0 and K are real numbers satisfying $0 < C_0 < R_0$ and $K \geq 1$. We note that by our choice of ordering it then follows that

$$\frac{C^\alpha}{C^\delta} \leq \frac{1}{K} \quad \text{for } \omega_k(\alpha) = \omega_k(\delta), \ \delta \prec \alpha.$$

We also note that

$$\frac{1}{C^\delta} \frac{C^\beta}{R^\beta} = \frac{1}{C_0^{p_0}} \frac{C^\beta}{R^\beta} \leq \frac{C_0}{R_0^{p_0+1}} \quad \text{if } \omega_k(\beta) > p_0. \tag{38}$$

We furthermore choose M in (37) to be of the form

$$M = M_1 C_0^{p_0}.$$

Now, we will successively restrict the constants K, M_1, and C_0 by requiring that they satisfy the following inequalities (E denotes the constant given by Lemma 9 for N):

$$\frac{(1 + |\delta|)^{(p+q)N} \mathbf{N}_0}{K} \leq \frac{\mathbf{e}}{4}, \qquad C_0 \frac{M_2}{M_1} \frac{1}{R_0^{2p_0+1}} \leq \frac{\mathbf{e}}{4},$$

$$r^2 M_1 M_2 E^{2(p+q)} \leq \frac{\mathbf{e}}{4}, \qquad \frac{(r+1)C_0 M_2 E^{p+q}}{R_0^{p_0+1}} \leq \frac{\mathbf{e}}{4}, \tag{39}$$

and

$$\max_j \frac{C_j}{R_j} + \sum_{|\eta|>0} \frac{(C_0^{p_0} M_1 E^{p+q})^{|\eta|}}{S^\eta} \leq 1. \tag{40}$$

Let us now start estimating terms in (36). We start with

$$\|\tilde{J}(x, t)[\gamma + \delta]\|_\infty \leq \frac{M_2}{R^{\gamma+\delta}(1 + \gamma + \delta)^N} = \frac{M}{C^\gamma (1 + \gamma + \delta)^N} \frac{M_2}{M_1 R_0^{p_0}} \frac{C^\gamma}{C_0^{p_0} R^\gamma}$$

$$\leq \frac{M}{C^\gamma (1 + \gamma)^N} \frac{C_0 M_2}{M_1 R_0^{2p_0+1}} \leq \frac{\mathbf{e}}{4} \frac{M}{C^\gamma (1 + \gamma)^N}, \tag{41}$$

where we have used (39) and (38).

We continue with $\sum_{\alpha+\beta=\gamma+\delta, \ \omega_k(\beta) \geq p_0} (T_\gamma^1 h)(x, t)[\alpha] \lambda((T^0 h)(x, t))[\beta]$, in which we are going to treat the sums over β with $\omega_k(\beta) = p_0$ and with $\omega_k(\beta) > p_0$ differently. We have

$$\sum_{\substack{\alpha+\beta=\gamma+\delta \\ \omega_k(\beta)=p_0,\, \delta \prec \beta}} \|(T_\gamma^1 h)(x,t)[\alpha]\|_\infty \, |\lambda((T^0 h)(x,t))[\beta]|$$

$$\leq \sum \frac{M}{C^\alpha(1+\alpha)^N} |\lambda((T^0 h)(x,t))[\beta]|$$

$$\leq \frac{M}{C^\gamma(1+\gamma)^N} \sum \frac{C^\gamma}{C^\alpha} \frac{(1+\gamma)^N}{(1+\alpha)^N} |\lambda((T^0 h)(x,t))[\beta]|$$

$$\leq \frac{M(1+|\delta|)^{(p+q)N}}{C^\gamma(1+\gamma)^N} \sum \frac{C^\beta}{C^\delta} |\lambda((T^0 h)(x,t))[\beta]| \leq \frac{M(1+|\delta|)^{(p+q)N}\mathbf{N_0}}{KC^\gamma(1+\gamma)^N}$$

$$\leq \frac{\mathbf{e}}{4} \frac{M}{C^\gamma(1+\gamma)^N} \tag{42}$$

for terms with $\omega_k(\beta) = p_0$, where we have used (39) in the last line. For the other terms, we use (38) and estimate

$$\sum_{\substack{\alpha+\beta=\gamma+\delta \\ \omega_k(\beta)>p_0}} \|(T_\gamma^1 h)(x,t)[\alpha]\|_\infty \, |\lambda((T^0 h)(x,t))[\beta]| \leq \sum \frac{M}{C^\alpha(1+\alpha)^N} \frac{M_2}{R^\beta(1+\beta)^N}$$

$$\leq \frac{MM_2}{C^\gamma} \sum \frac{1}{C^\delta} \frac{C^\beta}{R^\beta} \frac{1}{(1+\alpha)^N(1+\beta)^N} \leq \frac{M}{C^\gamma(1+\gamma)^N} \frac{C_0 M_2 E^{p+q}}{R_0^{p_0+1}}. \tag{43}$$

We next turn to terms arising from the second term on the right hand side of (36). We can estimate these terms as in (43) by

$$\sum_k \sum_{\substack{\alpha+\beta=\gamma+\delta \\ \omega_k(\beta)\geq p_0+1}} |(T_\gamma^1 h^k)(x,t)[\alpha]| \, \|\tilde{A}_k(x,t)[\beta]\|_\infty$$

$$\leq r \sum \frac{M}{C^\alpha(1+\alpha)^N} \frac{M_2}{R^\beta(1+\beta)^N} \leq \frac{M}{C^\gamma(1+\gamma)^N} \frac{rC_0 M_2 E^{p+q}}{R_0^{p_0+1}}.$$

If we combine this with the terms from (43), we see that we can estimate them together using (39) by

$$\frac{M}{C^\gamma(1+\gamma)^N} \frac{(r+1)C_0 M_2 E^{p+q}}{R_0^{p_0+1}} \leq \frac{\mathbf{e}}{4} \frac{M}{C^\gamma(1+\gamma)^N}. \tag{44}$$

Now let us turn to the remaining term in (36) containing R_2. We write

$$R_{2,j,k}(x,t,T^0 h(x,t)),Y) = \tilde{R}_{2,j,k}(x,t,Y)$$

and claim that

$$\|\tilde{R}_{2,j,k}(x,t,T_\gamma^1 h(x,t))[\alpha]\|_\infty \leq \frac{M_2}{C^\alpha(1+\alpha)^N}.$$

To prove the claim, we first note that it holds for $\alpha = 0$, so we assume $|\alpha| > 0$. We estimate

$$\|\tilde{R}_{2,j,k}(x,t,T_\gamma^1 h(x,t))[\alpha]\|_\infty = \Big\| \sum_{\substack{\nu+\sigma=\alpha \\ \nu+\eta>0}} \tilde{R}_{2,j,k}(x,t,Y)[x^{\nu^1}t^{\nu^2}Y^\eta](T_\gamma^1 h(x,t))^\eta[\sigma] \Big\|_\infty$$

$$\leq \frac{M_2}{R^\alpha(1+\alpha)^N} + \sum_{\substack{\nu+\sigma=\alpha \\ |\eta|>0}} \frac{M_2}{R^\nu S^\eta(1+\nu)^N} \frac{M^{|\eta|}E^{(p+q)(|\eta|-1)}}{(1+\sigma)^N C^\sigma}.$$

Now, since $C_j \leq R_j$ and $|\alpha| > 0$, we have

$$\frac{M_2}{R^\alpha(1+\alpha)^N} \leq \frac{M_2}{C^\alpha(1+\alpha)^N} \max_j \frac{C_j}{R_j}.$$

Again using that $C_j \leq R_j$, we get

$$\sum_{\substack{\nu+\sigma=\alpha \\ |\eta|>0}} \frac{M_2}{R^\nu S^\eta(1+\nu)^N} \frac{M^{|\eta|}E^{(p+q)(|\eta|-1)}}{(1+\sigma)^N C^\sigma} \leq \frac{M_2}{C^\alpha(1+\alpha)^N} \sum_{|\eta|>0} \frac{(C_0^{p_0}M_1 E^{p+q})^{|\eta|}}{S^\eta}.$$

Hence the claim follows from the assumption (40). We continue to estimate the remainder by

$$\|\tilde{R}_{2,j,k}(x,t,T_\gamma^1 h(x,t))(T_\gamma^1 h)_j(T_\gamma^1 h)_k[\gamma+\delta]\|_\infty \leq \frac{M_2 M^2 E^{2(p+q)}}{C^{\gamma+\delta}(1+\gamma+\delta)^N}$$

$$\leq \frac{M}{C^\gamma(1+\gamma)^N} M_1 M_2 E^{2(p+q)}. \quad (45)$$

So from (45) and (39), we have

$$\Big\| \sum_{j,k} \tilde{R}_{2,j,k}(T_\gamma^1 h)_j(T_\gamma^1 h)_k[\gamma+\delta] \Big\|_\infty \leq \frac{M}{C^\gamma(1+\gamma)^N} r^2 M_1 M_2 E^{2(p+q)}$$

$$\leq \frac{\mathbf{e}}{4} \frac{M}{C^\gamma(1+\gamma)^N}. \quad (46)$$

We can now combine (41), (42), (44) and (46); we see that with our choices, also

$$|eh_\gamma| \leq \mathbf{e} \frac{M}{C^\gamma(1+\gamma)^N},$$

which ends the proof of the theorem. $\square$

6. Proof of Theorem 2

After having discussed all the necessary tools, we can finally turn to the proof of Theorem 2. We start with the following proposition, which acts as the induction step in the construction of the parametrization.

Proposition 13. *Let $(M, 0)$ be a generic real-analytic submanifold of $\mathbb{C}^N$ which is holomorphically nondegenerate. Then for any $q \geq 1$, there exists a sequence $\{m_q(\ell)\}_{\ell \geq 0}$ of integers with the following property: If for some $\ell \geq 0$, there exists a mapping Φ defined on some open neighbourhood U of $\{0\} \times V \subset \mathbb{C}^{(q-1)n} \times J_0^k((\mathbb{C}^N, 0), (\mathbb{C}^N, 0))$, taking values in $J_t^{m_q(\ell)}((\mathbb{C}^{(q-1)n} \times \mathbb{C}^d, 0), (\mathbb{C}^N, 0))$, holomorphic in the first variable and real-analytic in the second, where V is an open neighbourhood of j_0^k id, such that*

$$j_{t,0}^{m_q(\ell)} H \circ S^{q-1}(x^{[2;q]}, t) = \Phi(x^{[2;q]}, j_0^k H),$$

for all $H \in \mathrm{Aut}(M, 0)$ with $j_0^k H \in V$, then there exists an integer $\tilde{k}$ and a mapping

$$\Psi \colon \mathbb{C}^N \times J_0^{\tilde{k}}((\mathbb{C}^N, 0), (\mathbb{C}^N, 0)) \supset \tilde{U} \to \mathbb{C}^N$$

defined in an open neighbourhood $\tilde{U}$ of $\{0\} \times \tilde{V}$, where $\tilde{V}$ is an open neighbourhood of $j_0^{\tilde{k}}$ id, holomorphic in the first variable and real-analytic in the second, which satisfies

$$j_{t,0}^\ell H \circ S^q(x^{[1;q]}, t) = \Psi(x^{[1;q]}, j_0^k H) \quad \text{for all } H \in \mathrm{Aut}(M, 0) \text{ with } j_0^k H \in \tilde{V}.$$

Proof. Let us choose normal coordinates for $(M, 0)$ as usual, and choose $(\alpha, r) \in J$ with tp $\Delta^q(\alpha, r) = (n_1^q, n_2^q)$. Let us write

$$\mathcal{Q}(z, \zeta) = (Q_{\alpha^1}^{r_1}(z, \zeta), \dots, Q_{\alpha^N}^{r_N}(z, \zeta)),$$

and recall that we define $|\alpha| = \max |\alpha^j|$. Lemma 5 and an application of the chain rule provide us with a mapping P_1, polynomial in its last argument, such that we can write

$$\mathcal{Q}\big(F(z, Q(z, \zeta)), \bar{H}(\zeta)\big) = \frac{P_1\big(z, \zeta; H_{z^\gamma w^\delta}(z, Q(z, \zeta)) \colon |\gamma| + |\delta| \leq |\alpha|\big)}{\big(\det\big(\frac{\partial F(z, Q(z, \zeta))}{\partial z}\big)\big)^{2|\alpha|-1}}. \qquad (47)$$

Let us write $R(z, \zeta; F_z(z, \zeta), F_w(z, \zeta))$ for the denominator in (47). In (47), we replace ζ by $\bar{S}^q(x^{[1;q]}; t)$ and z by x^2 to obtain

$$\mathcal{Q}\big(F(S^{q-1}(x^{[2;q]}; t)), \bar{H}(\bar{S}^q(x^{[1;q]}; t))\big)$$
$$= \frac{P_2\big(x^{[1;q]}, t; H_{z^\gamma w^\delta}(S^{q-1}(x^{[2;q]}; t)) \colon |\gamma| + |\delta| \leq |\alpha|\big)}{R\big(x^{[1;q]}, t; F_z(S^{q-1}(x^{[2;q]}; t)), F_w(S^{q-1}(x^{[2;q]}; t))\big)}. \qquad (48)$$

If we choose $m_q(\ell) \geq |\alpha|$, we can use the assumption and Lemma 4 applied to the representation of $H \circ S^{q-1} = \Phi$ in order to replace the right hand side of (48) by a fraction $\mathcal{R}(x^{[1;q]}, t; j_0^k H) = P(x^{[1;q]}, t; j_0^k H)/R(x^{[1;q]}, t; j_0^k H)$ satisfying

$$j_{t,0}^{m_q(\ell)-|\alpha|} \mathcal{Q}\big(F(S^{q-1}(x^{[2;q]}; t)), \bar{H}(\bar{S}^q(x^{[1;q]}; t))\big) = j_{t,0}^{m_q(\ell)-|\alpha|} \mathcal{R}(x^{[1;q]}, t; j_0^k H),$$

where both $P(x, t; \Lambda)$ and $R(x, t; \Lambda)$ are convergent in a neighbourhood of $\{0\} \times J_0^k((\mathbb{C}^N, 0), (\mathbb{C}^N, 0))$, and $R(0, 0; \Lambda) \neq 0$ if Λ is invertible. We now consider the system of equations

$$H \circ S^{q-1} = \Phi, \quad \mathcal{Q}(F \circ S^{q-1}, \bar{H} \circ \bar{S}^q) = \mathcal{R},$$

and we would like to apply Corollary 12 to it. The system is of the form considered there, with $P(y_1, y_2, y_3) = (y_1, y_2, \mathcal{Q}(y_1, y_3))$, so that $\lambda(y) = \Delta(\alpha, r)(y_1, y_3)$, with Δ defined in (12). Corresponding to $H = \mathrm{id}$, we set $h^0(x, t) = (S^{q-1}(x^{[2;q]}, t), \bar{S}^q(x^{[1;q]}, t))$, and we have $\lambda^0(x, t) = \lambda \circ h^0(x, t) = \Delta^q(\alpha, r)(x^{[1;q]}; t)$, and $\mathrm{tp}\,\lambda^0 = (n_1^q, n_2^q)$. We now use Corollary 12 with $X = J_0^k((\mathbb{C}^N, 0), (\mathbb{C}^N, 0))$, setting $g(x, t, \omega) = (\Phi(x, t, \omega), \mathcal{R}(x, t, \omega))$, $\omega_0 = j_0^k\,\mathrm{id}$. It supplies us with an integer k_0, a neighbourhood V_ℓ^0 of $j_0^{k_0}\,\mathrm{id}$, and a holomorphic function Φ_ℓ with $j_{t,0}^\ell h = \Phi_\ell(j_0^{k_0} h, \omega)$ if $h = (h_1, h_2, h_3)$ satisfies $j_{t,0}^{k(\ell)} P(h_1, h_2, h_3) = j_{t,0}^{k(\ell)} g(x, t, \omega)$. Hence, if we set $m_q(\ell) = |\alpha| + k(\ell)$ and use Lemma 7, we see that $(h_1, h_2, h_3) = (F \circ S^{q-1}, G \circ S^{q-1}, \bar{H} \circ \bar{S}^q)$ satisfies all the assumptions of Corollary 12, and we have, with $\Phi_\ell = (\Phi_\ell^1, \Phi_\ell^2, \Phi_\ell^3)$,

$$j_{t,0}^\ell \bar{H} \circ \bar{S}^q = \Phi_\ell^3(j_0^{k_0} \bar{H} \circ \bar{S}^q, j_0^k H),$$

and hence we have the required function Ψ with $\tilde{k} = \max(k, k_0)$ after substituting $j_0^{k_0} \bar{H} \circ \bar{S}^q = \varphi(j_0^{k_0} \bar{H})$ and taking the complex conjugate of the last equation. $\qquad\square$

Corollary 14. *Let $(M, 0)$ be a generic real-analytic submanifold of $\mathbb{C}^N$ which is holomorphically nondegenerate. Then for any $q \geq 1$, there exists an integer $k(q)$ and a map Ψ^q defined on an open neighbourhood of $\{0\} \times V_q \subset \mathbb{C}^{qn} \times J_0^{k(q)}((\mathbb{C}^N, 0), (\mathbb{C}^N, 0))$ where V_q is an open neighbourhood of $j_0^{k(q)}\,\mathrm{id}$, valued in $\mathbb{C}^N$, holomorphic in the first variable and real-analytic in the second, such that*

$$H \circ S^q(x^{[1;q]}, 0) = \Psi^q(x^{[1;q]}, j_0^{k(q)} H) \quad \textit{for } H \in \mathrm{Aut}(M, 0) \textit{ with } j_0^{k(q)} H \in V_q.$$

Furthermore, there exists a polynomial s defined on $J_0^{k(q)}((\mathbb{C}^N, 0), (\mathbb{C}^N, 0))$ with $s(j_0^{k(q)}\,\mathrm{id}) \neq 0$, integers r_α and polynomials p_α (also defined on $J_0^{k(q)}((\mathbb{C}^N, 0), (\mathbb{C}^N, 0)))$ defined for $\alpha \in \mathbb{N}^{nq}$ such that

$$\Psi^q(x, \omega) = \sum_{\alpha \in \mathbb{N}^{nq}} \frac{p_\alpha(\omega)}{s(\omega)^{r_\alpha}} x^\alpha.$$

Proof. The corollary follows in a straightforward manner from Proposition 13 after noting that $H(0, t)$ satisfies the assumptions for $q = 1$ trivially for any order; in particular, for the order $m_1(m_2(\ldots m_q(0) \ldots))$. Now we can apply the proposition inductively, ending with an integer $k(q)$ as claimed.

In order to check the second part of the corollary, we refer the reader to the specific form of the solution used in the proposition which is constructed in Theorem 8; that is, we use (30) to obtain this result. $\qquad\square$

Proof of Theorem 2. If $(M, 0)$ is minimal, the Baouendi–Ebenfelt–Rothschild criterion [1] ensures that for some $q \leq d + 1$, the Segre map $S^q(x^{[1;d]}; 0)$ is generically of full rank. We can thus apply Theorem 5 to the parametrization of H along S^q constructed in Corollary 14. The result now follows immediately. $\qquad\square$

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