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Structure of second-order symmetric Lorentzian manifolds

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Abstract. *Second-order symmetric Lorentzian spaces*, that is, Lorentzian manifolds with vanishing second derivative $\nabla\nabla R \equiv 0$ of the curvature tensor R, are characterized by several geometric properties, and explicitly presented. Locally, they are a product $M = M_1 \times M_2$ where each factor is uniquely determined as follows: M_2 is a Riemannian symmetric space and M_1 is either a constant-curvature Lorentzian space or a definite type of plane wave generalizing the Cahen–Wallach family. In the proper case (i.e., $\nabla R \neq 0$ at some point), the curvature tensor turns out to be described by some local affine function which characterizes a globally defined parallel lightlike direction. As a consequence, the corresponding global classification is obtained, namely: any complete second-order symmetric space admits as universal covering such a product $M_1 \times M_2$. From the technical point of view, a direct analysis of the second-symmetry partial differential equations is carried out leading to several results of independent interest on spaces with a parallel lightlike vector field, the so-called Brinkmann spaces.

Keywords. Second-order symmetric spaces, curvature conditions, Brinkmann spaces, Lorentzian symmetric spaces, plane waves, holonomy of Lorentzian manifolds

Contents

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1. Introduction

A venerable result in differential geometry (Nomizu and Ozeki [34], Tanno [43]) states that, for a Riemannian manifold (M, g), the vanishing of the rth covariant derivative of its curvature tensor R,

$$\nabla^r R \; (= \nabla \overset{(r)}{\cdots} \nabla R) \equiv 0, \qquad r \geq 2, \tag{1}$$

implies the vanishing of the first one, i.e., that (M, g) is *locally symmetric*. As a consequence, the standard generalization of Riemannian locally symmetric spaces are the semi-symmetric spaces, introduced by Cartan [13] and defined by the commutativity of the covariant derivatives applied to R:

$$R(X, Y)R := (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})R = 0 \qquad \text{for all vector fields } X, Y. \tag{2}$$

Their structure was determined by Szabó locally in [41] and globally later in [42]. However, when (M, g) is a Lorentzian manifold, the equality (1) does not imply $\nabla R = 0$. In fact, the irreducible character of the de Rham decomposition, which was an essential ingredient in the Riemannian result, fails for Lorentzian metrics. Thus, a ladder of logical generalizations of Lorentzian locally symmetric spaces is given by (1). We call these semi-Riemannian manifolds *rth-order symmetric* (or *rth-symmetric* for short) spaces.[1] The purpose of the present article is to determine the simplest of these new classes explicitly: the Lorentzian 2nd-symmetric spaces.[2]

The classification of Riemannian locally symmetric spaces has been known since Cartan's work [12] (see also [25, 7]), and the classification of Lorentzian simply-connected symmetric spaces was carried out by Cahen and Wallach in [10]. Extensions to other signatures and to non-simply-connected cases are also available: see Cahen and Parker [9], Neukirchner [33] and specially Kath and Olbrich [26, 27]. Lorentzian semi-symmetric spaces have also been studied in the literature; see for instance their classification in four dimensions [16, 22] and references therein. Nevertheless, prior to the paper [37] by one of the authors, the 2nd-symmetric spaces had not been studied systematically. As pointed out in this reference, simple examples of *proper rth-symmetric*—rth-symmetric but not $(r-1)$th-symmetric—Lorentzian spaces can be constructed within the class of

[1] rth-order symmetric spaces were introduced in [37] and termed r-symmetric for short. However, a different notion of *3-symmetric space*, introduced by Gray [20], was already available and somehow spread in the literature (for example, see the recent article [21]). Thus, we have preferred to use the ordinal (*3rd-symmetric*, say) to avoid any possible confusion.

[2] The results of the present paper were announced in particular at the Spanish Relativity Meetings celebrated in Bilbao, 7–11 September '09 (only the four dimensional case, see [5]) and in Granada, 6–10 September '10 where the general n-dimensional case was considered (see [6]).

n-dimensional plane waves (see Subsection 3.3 below). They constitute a straightforward generalization of the locally symmetric *Cahen–Wallach spaces* [10] and, as we will prove, they essentially exhaust the whole class of proper 2nd-symmetric Lorentzian spaces.

It is worth pointing out that the 2nd-symmetric spaces are appealing from the viewpoint of the local group of symmetries of the manifold, because the condition $\nabla^2 R = 0$ can be expressed in terms of the infinitesimal holonomy algebra of the manifold (as this algebra is generated by the image of the curvature two-form and its first derivative; see, for example, [28, Th. 9.2, Ch. III]); in fact, the main result in [37] is a property of this holonomy group. With the help of this property and the well-established results on locally symmetric spaces by Cahen and Wallach, our proof will be completely self-contained, by solving the equations of 2nd-symmetry crudely.[3]

Specifically, the main result we will prove is:

Theorem 1.1. *An n-dimensional proper 2nd-symmetric Lorentzian space (M, g) is locally isometric to a direct product $(M_1 \times M_2, g_1 \oplus g_2)$ where (M_2, g_2) is a non-flat Riemannian symmetric space and (M_1, g_1) is a proper generalized Cahen–Wallach space of order 2, defined as $M_1 = \mathbb{R}^{d+2}$ $(d \geq 0)$ endowed with the metric*

$$g_1 = -2du\Big(dv + du \sum_{i,j=2}^{d+1} p_{ij}(u)x^i x^j\Big) + \sum_{i=2}^{d+1}(dx^i)^2,$$

where $(u, v, x^2, \ldots, x^{d+1})$ are the natural coordinates of $\mathbb{R}^{d+2}$ and each function p_{ij} is affine: $p_{ij}(u) = \alpha_{ij}u + \beta_{ij}$ for some $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$ with at least one of the α_{ij} non-vanishing, and $i, j = 2, \ldots, d+1$.

Moreover, if (M, g) is also geodesically complete and simply connected, then (M, g) is globally isometric to one such direct product.

Very roughly, the idea of the proof is the following. The starting point is a significant result obtained by one of the authors ([37, Theorem 4.2]): *any simply-connected Lorentzian proper 2nd-symmetric space (M, g) admits a parallel lightlike vector field K.* Lorentzian spaces with such a K were obtained by Brinkmann [8] and will be studied in Sec. 4, where local bases associated to what we call *Brinkmann charts* $\{u, v, x^i\}$ will be introduced. The fact that, for any such chart, the slices with constant u and v happen to be locally symmetric suggests a reduction of the equations for 2nd-symmetry. This reduction is carried out by exploiting the integrability conditions in full, and by applying some technical algebraic properties. Then, Eisenhart-type decompositions can be used to transform the original equations in (M, g) into the 2nd-symmetry equations of two simpler spacetimes from which the stated parts (M_1, g_1) and (M_2, g_2) emerge locally. Some geometric elaborations yield the global result. In fact, along the proof it will become apparent that (M, g) admits a globally defined parallel lightlike direction. Thereupon, the global requirements complete the proof easily.

[3] While the present paper was being finished, a different local approach fully based on holonomy groups (including landmarks such as the classification of all the holonomy groups in Lorentzian signature [30]) was developed in [1], [19]. By using it, the crucial result in [37, Theorem 4.2] is revisited in [1], and a partial version of Theorem 1.1 below follows from [1, 19].

Summing up, we will prove that the implicit symmetries in the equations of 2nd-symmetry turn out to be sufficient to solve them, and actually to find their general solution explicitly. Incidentally, different technical tools for some classes of partial differential equations, which may be of interest in its own right, will be developed.

1.1. Outline of the paper

This article is organized as follows. In Section 2, we fix all our conventions and explain the notation. We are specially careful with the latter, for we will use a powerful combination of intrinsic expressions and tensor-component computations.

In Section 3, we review some results on locally symmetric spaces (Subsection 3.1), and compare them with the known results on 2nd-symmetric ones (Subsection 3.2). We also describe the properties of the generalized Cahen–Wallach family of Lorentzian manifolds (Subsection 3.3). They will turn out to be the non-trivial part of the proper 2nd-symmetric spaces (Proposition 3.9). Judicious interpretations alongside some technical results (Corollary 3.8, Lemma 3.10) will imply that the global counterpart of Theorem 1.1 can be obtained from the local one. Accordingly, in the following sections we will work locally, except when otherwise stated explicitly.

Section 4 is devoted to a local study of Brinkmann spaces. It has a technical nature, but it may be of interest in its own right. In Subsection 4.1 we revisit the known procedure to find a *Brinkmann chart* $\{u, v, x^i\}$ associated to a *Brinkmann decomposition* $\{u, v\}$, so that the metric of the manifold is written as

$$g = -2du(dv + H(u, x^k)du + W_i(u, x^k)dx^i) + g_{ij}(u, x^k)dx^i dx^j, \quad k = 2, \dots, n-1 \tag{3}$$

(see Section 2 for notation). These yield an associated spacelike $(n-2)$-foliation $\overline{\mathcal{M}}$ characterized by constant values of u and v, and a not necessarily orthogonal timelike 2-foliation $\mathcal{U}$ generated by ∂_u, ∂_v, as well as other distributions of interest. For the sake of clarity, the relations among tensor fields on these distributions are briefly explained. In Subsection 4.2, three operators $\overline{\nabla}$, $\overline{d}$ and $\dot{}$ adapted to the foliations $\overline{\mathcal{M}}$ and $\mathcal{U}$ are introduced and related to the connection ∇, the exterior differential d and the geometry of the foliation $\overline{\mathcal{M}}$. In Subsection 4.3, computations *à la Cartan* on local vector-field bases introduced in (10) yield manageable expressions of the connection one-forms and the first two covariant derivatives of the curvature tensor R. To help with simplifying the resulting expressions, a fourth differential operator D_0 with a transverse nature, which complements the three previous ones, is introduced. The dependence of all the above mentioned objects on the Brinkmann chart and their behavior under changes of charts is emphasized and explicitly controlled. Then, a version of a classical Eisenhart theorem [17] adapted to our problem, which involves u-dependent metrics on the leaves of a foliation endowed with the intrinsic $\overline{\nabla}$ and the transverse D_0 derivatives, is provided in Theorem 4.29 of Subsection 4.4. This allows us to obtain sufficient conditions for the u-family of Riemannian metrics $\overline{g} = g_{ij}(u, x^k)dx^i dx^j$ in (3) to be simultaneously reducible.

In Section 5 we solve the equations of 2nd-symmetry on Brinkmann spaces in several steps. Firstly, we prove that the equations of 2nd-symmetry imply that the foliation $\overline{\mathcal{M}}$

must be locally symmetric for all Brinkmann charts $\{u, v, x^i\}$ associated to a fixed decomposition $\{u, v\}$, hence severe restrictions on the curvature must hold. Secondly, using some auxiliary algebraic results on vector spaces, those restrictions are explicitly determined in Propositions 5.6 and 5.7. Moreover, a two-covariant tensor field $\tilde{A}$ on $\overline{\mathcal{M}}$, associated to the Brinkmann decomposition $\{u, v\}$ only but not to the other coordinates of the Brinkmann chart, is defined in Corollary 5.2. All the information on the tensor field ∇R is codified in $\tilde{A}$ and the leaves of $\overline{\mathcal{M}}$. These results are summarized in Theorem 5.1 and its corollary of Subsection 5.1. Thirdly, a reorganization of the 2nd-symmetry equations into two independent blocks associated to different Brinkmann manifolds $(M^{[m]}, g^{[m]})$ with $m \in \{1, 2\}$ is proven in Subsection 5.2 (Theorem 5.16 and Proposition 5.18). This is achieved by applying our version of the Eisenhart theorem showing that the metric is suitably reducible to a Ricci-flat part and a non-Ricci-flat one. Actually, the former is flat as a consequence of a known result [2] and the operator $\tilde{A}$ lives only in this flat part (Proposition 5.12). Finally, the proof of the main result is completed in Subsection 5.3. The first space $(M^{[1]}, g^{[1]})$ is directly computable leading to the required generalized Cahen–Wallach expression of the part (M_1, g_1) in Theorem 1.1. The 2nd-symmetry equations for the second space $(M^{[2]}, g^{[2]})$ become equivalent to the equations for local symmetry so that $(M^{[2]}, g^{[2]})$ collapses to a locally symmetric Brinkmann space, therefore the Cahen–Wallach classification allows us to determine the Riemannian locally symmetric part (M_2, g_2) in Theorem 1.1.

To end this introduction, we would like to emphasize that our results open new questions and lines of interest. The first two are obvious, consisting in the study of proper rth-symmetric spaces with $r > 2$, and the study of 2nd-order symmetric spaces with metrics of index greater than 1. In both cases, our approach is not directly applicable and, in fact, it is not clear how these generalizations would affect even our starting point ([37, Theorem 4.2]). Moreover, recall that the resulting proper 2nd-symmetric spaces share the symmetries of plane waves (for explicit expressions including the more exotic Kerr–Schild symmetries, see [15]). So, an interesting question may be to find connections between the group of symmetries inherent a priori to rth-symmetry, and the actual symmetries of the proper rth-symmetric spaces eventually obtained. Lastly, the non-simply connected case and, in particular, the existence of compact quotients of plane waves, becomes also a natural problem in this setting (we thank Professor A. Zeghib, from ENS Lyon, for discussions stressing the importance of this question).

2. Notation and conventions

M will denote a (connected) n-dimensional manifold. For simplicity, it will be implicitly assumed to be differentiable of class C^k with $k = \infty$, but one only needs $k = r + 3$ for rth-order symmetric spaces, i.e., $k = 5$ for most of the paper. Accordingly, all objects will be assumed to be as differentiable as necessary depending on k. For Lorentzian metrics, our convention on the signature is $(-, +, \ldots, +)$. Indices written in Greek small letters $\alpha, \beta, \lambda, \ldots$ will run from 0 to $n - 1$, while those in Latin small letters starting from i $(i, j, k, \ldots)$ will run from 2 to $n - 1$ and the usual *summation convention* is used.

A chart on the manifold will be indicated simply with its coordinate functions $\{x^\alpha\}$. When working on a Brinkmann space the coordinates x^0 and x^1 will also be written as u and v, respectively, according to (3). For $f \in C^\infty(M)$ its partial derivatives are denoted by

$$\dot{f} \equiv \frac{\partial f}{\partial u} \quad \text{and} \quad f_{,i} \equiv \frac{\partial f}{\partial x^i}. \tag{4}$$

Most of our computations will be local, in an appropriate neighborhood U of any point $p \in M$, but we will not specify the neighborhood —as we have already done with the charts. In general, we will use the notation as if $U = M$ except if there were some possibility of confusion.

Let TM and T^*M be the tangent and the cotangent bundles of M, respectively, and $\pi : TM \to M$ the natural projection. An *l-distribution* on M will be regarded as an l-subbundle E of TM and it will be involutive if $E = T\mathcal{F}$ for some foliation $\mathcal{F}$ of M. In this case, the bundle of all the s-covariant and r-contravariant tensors on $T\mathcal{F}$ is denoted by $T^r_s\mathcal{F}$. The space of sections of a fiber bundle $\pi_E : E \to M$ will be written $\Gamma(E)$, unless the base is not evident, in which case we use $\Gamma(M, E)$. In the case of sections of s-form bundles the notation will be simplified: $\Lambda^s E$ denotes the space of all s-forms.

To write tensor equations in components, some local vector field basis $\{V_\alpha\}$ on TM, or on some of its subbundles, plus its dual basis $\{\zeta^\alpha\}$ for T^*M are used. Of course, these bases are not necessarily holonomic, i.e., associated to specific coordinates $\{x^\alpha\}$, for which we use the standard notation $\{\partial_\alpha\}$, $\{dx^\alpha\}$. We will follow typical notation for covariant derivatives and their components as, for example, in [36, pp. 30–35]. Notice that we denote the components of R as defined in (2) by $R^\alpha{}_{\beta\lambda\mu}$, which agrees with [24, 40] but differs from [36] where the same is written as $R_{\lambda\mu\beta}{}^\alpha$ (therefore, $R_{\alpha\beta\lambda\mu}$ differs in sign).

Sometimes the *abstract index notation* is also used (see [44] for more information). The *symmetrization* (respectively *antisymmetrization*) of a tensor field T is denoted by round (respectively square) brackets that enclose the indices to be symmetrized (respectively antisymmetrized). For example, $T_{(\alpha\beta)\lambda} = (T_{\alpha\beta\lambda} + T_{\beta\alpha\lambda})/2$ and $T_{[\alpha|\beta|\lambda]} = (T_{\alpha\beta\lambda} - T_{\lambda\beta\alpha})/2$. Furthermore, we write $2dx^\alpha dx^\beta = dx^\alpha \otimes dx^\beta + dx^\beta \otimes dx^\alpha$. For the *wedge product*, the convention is $\beta^1 \wedge \cdots \wedge \beta^m = \sum_{\sigma \in S_m} (-1)^{[\sigma]} \beta^{\sigma(1)} \otimes \cdots \otimes \beta^{\sigma(m)}$, where β^i are one-forms, $i = 1, \ldots, m$, and S_m denotes the set of all permutations of $\{1, \ldots, m\}$. The metric isomorphism $\flat : TM \to T^*M$, $\vec{v} \mapsto g(\vec{v}, \cdot)$, and its inverse $\sharp : T^*M \to TM$, are written in components or abstract index notation so that $X_\alpha := (X^\flat)_\alpha = g_{\alpha\beta} X^\beta$ and $\tau^\alpha := (\tau^\sharp)^\alpha = g^{\alpha\beta}\tau_\beta$ for all $X \in \Gamma(TM)$ and $\tau \in \Gamma(T^*M)$.

3. Locally symmetric versus 2nd-symmetric semi-Riemannian manifolds

3.1. Generalities on local symmetry

Let (M, g) be a semi-Riemannian manifold and $p \in M$. The *local geodesic symmetry s_p* with respect to p is the diffeomorphism $s_p : N_p \to N_p$, defined on a sufficiently small normal neighborhood N_p of p, which maps each $q = \gamma(1) \in N_p$ into $s_p(q) = \gamma(-1)$, where γ is the uniquely determined geodesic in N_p from p to q. We collect in the following proposition some characterizations of locally symmetric semi-Riemannian manifolds, i.e., manifolds satisfying $\nabla R = 0$, for comparison with 2nd-symmetric spaces. See [35, pp. 219–223] for further details and results.

Proposition 3.1. *For a semi-Riemannian manifold* (M, g) *the following conditions are equivalent:*

(i) (M, g) *is locally symmetric.*
(ii) *If* $L : T_pM \to T_qM$ *is a linear isometry that preserves curvature (i.e., for any* $X_p, Y_p, Z_p \in T_pM$, $L(R(X_p, Y_p)Z_p) = R(L(X_p), L(Y_p))L(Z_p))$*, then there exist small normal neighborhoods* N_p *of* p *and* N_q *of* q *and a unique isometry* $\phi : N_p \to N_q$ *such that* $d\phi_{|p} = L$.
(iii) *The local geodesic symmetry* s_p *is an isometry at any* $p \in M$.
(iv) *If* X, Y *and* Z *are parallel vector fields along a curve* α*, then the vector field* $R(X, Y)Z$ *is also parallel along* α.
(v) *The sectional curvature is invariant under parallel translation: for any non-degenerate plane, its parallel transport along any curve has constant sectional curvature.*

Another important known result is [25, 35]:

Proposition 3.2. *Any semi-Riemannian symmetric space is an analytic, (geodesically) complete, homogeneous space* G/H*. Moreover, the universal covering of a complete connected locally symmetric space is symmetric.*

For the Riemannian case, we will need the following.

Proposition 3.3. *Let* (M, g) *be a locally symmetric Riemannian manifold. Then:*

(1) (M, g) *is locally isometric to the direct product of a finite number of irreducible locally symmetric spaces and a Euclidean space of dimension* $d \geq 0$.
(2) *If* (M, g) *is irreducible, then it is an Einstein manifold, i.e.,* $\mathrm{Ric} = cg$.
(3) *If* (M, g) *is Ricci-flat (that is, Einstein with* $c = 0$*), then it is flat.*

Proof. (1) This a consequence of the classical de Rham decomposition of M, as any irreducible part must be locally symmetric.

(2) As the Ricci tensor in a locally symmetric space is parallel, the result follows from a classical result by Eisenhart (Theorem 4.28 below).

(3) By hypothesis, (M, g) is locally isometric to a Ricci-flat symmetric space and, by a result in [2], this space must be flat.[4] □

The classification of Lorentzian simply-connected symmetric spaces by Cahen and Wallach [10] can be summarized as follows:

[4] In fact, Alekseevskii and Kimelfeld [2] proved that any Ricci-flat homogeneous Riemannian space is flat. It is worth pointing out that, in contrast to the locally symmetric case, locally homogeneus spaces may be non-regular, that is, not locally isometric to any homogeneous space [29]. However, Spiro [39] showed that all locally homogeneous spaces with non-positive Ricci curvature are regular, hence the result in [2] can be extended to the locally homogeneous case. Nevertheless, it cannot be extended to the Lorentzian case: it is easy to find a counterexample among the Cahen–Wallach spaces below.

Theorem 3.4. *Any simply-connected Lorentzian symmetric space* (M, g) *is isometric to the product of a simply-connected Riemannian symmetric space and one of the following Lorentzian manifolds:*

(a) $(\mathbb{R}, -dt^2)$,
(b) *the universal cover of d-dimensional de Sitter or anti-de Sitter spaces,* $d \geq 2$,
(c) *a Cahen–Wallach space* $CW^d(A) = (\mathbb{R}^d, g_A)$, $d \geq 2$, *where* $A = (A_{ij})$ *is a* $(d-2) \times (d-2)$ *symmetric constant matrix, and the metric is written*

$$g_A = -2du(dv + A_{ij}x^i x^j du) + \delta_{ij} dx^i dx^j, \tag{5}$$

where δ_{ij} *is the Kronecker delta (observe that* CW^2 *is just the Lorentz–Minkowski space* $\mathbb{L}^2$*, as A necessarily vanishes).*

Therefore, if a Lorentzian symmetric space admits a parallel lightlike vector field, then it is locally isometric to the product of a d-dimensional Cahen–Wallach space and an $(n-d)$*-dimensional Riemannian symmetric space with* $d \geq 2$.

3.2. Characterization of 2nd-symmetric spaces

There is no reason a priori to think that the characterization for 2nd-symmetric Lorentzian manifolds may include properties of local geodesic symmetries, as in Proposition 3.1(iii), or any other similar semi-local property. Of course, this does not mean that such a property might not be found a posteriori.

For any geodesic γ such that $p = \gamma(0)$ and tangent plane $\Pi \subset T_pM$, consider the parallel-transported plane $\tau \mapsto \Pi_\gamma(\tau) \subset T_{\gamma(\tau)}M$. Using that the scalar product between parallel-propagated vector fields is constant and that the sectional curvature $K(\Pi_p)$ on non-degenerate planes Π_p determines the full curvature tensor, one easily derives the next lemma.

Lemma 3.5. *The following three conditions are equivalent:*

(i) *For any non-degenerate tangent plane* Π, *its parallel transport* Π_γ *along any geodesic* γ *satisfies that* $\frac{d}{d\tau}(K(\Pi_\gamma))$ *remains constant along* γ.
(ii) *For any parallel-propagated vector fields* X, Y, Z *along any geodesic* γ, *the vector field* $(\nabla_{\gamma'} R)(X, Y)Z$ *is itself parallel-propagated along* γ.
(iii) $\nabla_X(\nabla_Y R) - \nabla_{\nabla_X Y} R$ *is skew-symmetric in* X, Y.

Moreover, if these conditions hold, then the following property follows:

(S) *if* Π *is a lightlike plane with radical spanned by* $\vec{v} \in T_pM$, *and* V *and* Π_γ *are the parallel transports of* $\vec{v}$ *and* Π *along any geodesic* γ, *respectively, then* $\frac{d}{d\tau}(K_V(\Pi_\gamma))$ *remains constant along* γ, *where* $K_V(\Pi_\gamma)$ *denotes the* null sectional curvature along γ *and is defined by (see* [23])

$$K_V(\Pi_\gamma) = \frac{R(V, X, V, X)}{g(X, X)}$$

with $\{V, X\}$ *spanning* Π_γ.

This result leads to the sought-for characterization of 2nd-symmetry.

Proposition 3.6. *The following statements are equivalent for a Lorentzian manifold* (M, g)*:*

(i) $\nabla\nabla R = 0$*, i.e.,* (M, g) *is* 2*nd-symmetric .*
(ii) *If* V, X, Y, Z *are parallel-propagated vector fields along any curve* α*, then* $(\nabla_V R)(X, Y)Z$ *is itself parallel-propagated along the curve.*
(iii) (M, g) *is semi-symmetric* (*i.e., the curvature tensor satisfies* (2)) *and satisfies the equivalent conditions in Lemma* 3.5 *for any geodesic* γ*.*

Proof. As (ii) characterizes when the tensor ∇R is parallel, this condition is equivalent to (i). Also, both conditions imply (iii) trivially. For the converse, notice that, by semi-symmetry, $\nabla\nabla R(\cdot\,; U_1, U_2) = \nabla_{U_2}(\nabla_{U_1} R) - \nabla_{\nabla_{U_2} U_1} R$ is symmetric in U_1, U_2, and, a fortiori, it vanishes by applying the condition (iii) of Lemma 3.5. □

Remark 3.7. For any Lorentzian manifold that satisfies the equivalent conditions in Lemma 3.5 the (null) sectional curvature of a plane parallel-propagated along a geodesic varies as an affine function on the affine parameter of the geodesics; in other words, the curvature along geodesics grows linearly. Recall that the constancy of the sectional curvature characterizes locally symmetric spaces (Proposition 3.1). On the other hand, there are obvious situations in which its non-constant linear growth can be excluded. For example, assume that $\gamma : [0, \infty) \to M$ is a complete lightlike half-geodesic such that its velocity is imprisoned in a compact subset $C \subset TM$. Then, for any lightlike plane Π with radical spanned by $\vec{v} = \gamma'(0)$, the derivative of $K_V(\Pi_\gamma)$ must vanish. (There exists a sequence $\tau_n \nearrow \infty$ such that $\gamma(\tau_n) \to q \in M$, $\gamma'(\tau_n) \to \vec{w}_q \in T_q M$ and $\Pi_\gamma(\tau_n) \to \Pi_q \subset T_q M$, hence $K_V(\Pi_\gamma(\tau_n)) \to K_{\vec{w}_q}(\Pi_q)$, and this is incompatible with linear growth of the curvature.) This sort of properties, together with Proposition 3.6, suggests global obstructions to the existence of compact proper 2nd-symmetric spaces.

Corollary 3.8. *Let* (M, g) *be a proper* (*connected*) 2*nd-symmetric space. Then* $\nabla R \neq 0$ *everywhere, and* (M, g) *admits a unique parallel lightlike direction.*

Proof. Obviously, if $(\nabla R)_p \neq 0$ at some $p \in M$, then (the parallel tensor) ∇R cannot vanish at any point. To prove the existence of a parallel lightlike direction recall that, as we have already mentioned, around each point p there exists a parallel lightlike vector K [37, Theorem 4.2]. Moreover, there cannot exist a second such K' that is independent of K at p, as otherwise a parallel timelike vector field T could be constructed as a linear combination of K and K'; hence, the metric would split around p as a product $-dt^2 \oplus g_R$, with $T = \nabla t$ and g_R a Riemannian metric, which should be locally symmetric, in contradiction with $(\nabla R)_p \neq 0$. Thus, the corresponding parallel lightlike directions locally generated by K and K' must agree, so that they match into a single global one. □

3.3. *Generalized Cahen–Wallach spaces of order r*

In this section we introduce the archetypes for rth-symmetric Lorentzian manifolds, which are generalizations of the d-dimensional Cahen–Wallach spaces $CW^d(A)$ introduced in Theorem 3.4.

Consider the larger family $CW_r^d(A)$ of all the *d-dimensional generalized Cahen–Wallach spaces of order r*, $CW_r^d(A) = (\mathbb{R}^d, g_A)$, defined by using the same expression (5) but letting now A depend on u as a matrix of polynomials of degree $\leq r-1$,

$$A \equiv A(u) = A^{(r-1)}u^{r-1} + \cdots + A^{(1)}u + A^{(0)}, \tag{6}$$

where $A^{(l)}$ is a constant symmetric $(d-2)\times(d-2)$ matrix for all $l \in \{0,\ldots,r-1\}$. As before, a generalized Cahen–Wallach space is *proper* if at least one of the polynomials has degree $r-1$, that is, if $A^{(r-1)} \neq 0$. Notice that all the spaces $CW_r^d(A)$ contain a parallel lightlike vector field (see Section 4.1), and $CW_1^d(A) = CW^d(A)$. In addition we have

Proposition 3.9. *Any proper generalized Cahen–Wallach space is analytic, (geodesically) complete and proper rth-symmetric.*

Proof. The analyticity is obvious and the completeness follows from [11, Proposition 3.5], where a more general type of plane waves was treated. The last property was already mentioned in [37] and follows by computing the derivatives of the curvature tensor in the basis $\{E_\alpha\} = \{\partial_u - H\partial_v, \partial_v, \partial_i - W_i\partial_v\}$ (see Section 4.3). The only non-vanishing components of $\nabla^l R$ for $l \in \{0,\ldots,r-1\}$ (for $l \geq r$, $\nabla^l R = 0$) are

$$\nabla_0 \overset{(l)}{\ldots} \nabla_0 R^1{}_{i0j} = \frac{d^l A_{ij}}{du^l} = \sum_{k=l}^{r-1} \frac{k!}{(k-l)!} A_{ij}^{(k)} u^{k-l}.$$

which leads to the result immediately. □

The following lemma will be used to reduce the global version of Theorem 1.1 to the local one.

Lemma 3.10. *Let (M,g) be a complete simply-connected Lorentzian manifold which is locally isometric to the product of some generalized Cahen–Wallach space with a simply connected Riemannian symmetric space. Then (M,g) is in fact globally isometric to such a product.*

Proof. By assumption, (M,g) is locally isometric to an analytical manifold due to Propositions 3.2 and 3.9, and thus it is analytical too. The result follows from the fact that, for any two complete simply-connected analytic semi-Riemannian manifolds (M,g) and (M',g'), every isometry defined between connected open subsets of M and M' can be uniquely extended to an isometry of the entire M and M' (see, for example [28, Cor. 6.4, Ch. VI] for the Riemannian case, and [28, Th. 6.1, Ch. VI], [35, Cor. 7.29] for its generalization to the semi-Riemannian case). □

4. Brinkmann spaces

A Lorentzian manifold is called a *Brinkmann space* if it admits a parallel lightlike vector field. Brinkmann spaces have attracted increasing attention in recent years: see e.g. [3], [4], [18]. In this section, we derive some of their properties relevant to our problem.

4.1. Basics on Brinkmann spaces

In what follows, the lightlike parallel vector field K of a Brinkmann space (M, g) will always be assumed to be fixed. It is well-known that any point $p \in M$ admits a coordinate chart $\{x^\alpha\} = \{u, v, x^i\}$, which we call a *Brinkmann chart*, such that the metric takes the form (3) and $K = -\partial_v$ ([8], see also [45]). Without loss of generality, we will assume that the range of the coordinates includes $|u|, |v|, |x^i| < \epsilon$ for some $\epsilon > 0$.

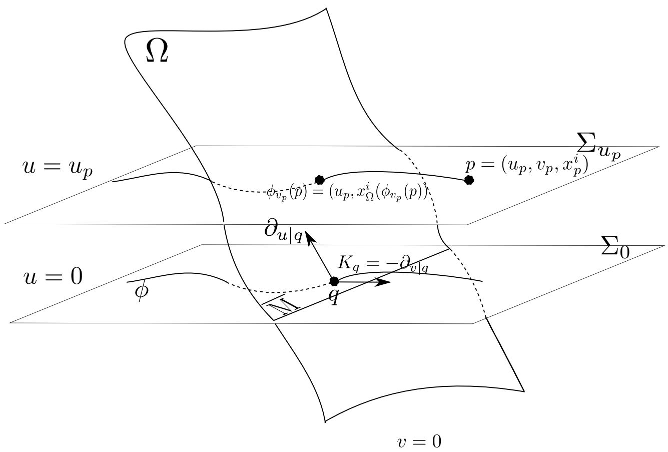

Fig. 1. Construction of a Brinkmann chart for a fixed lightlike parallel vector field $K = -\partial_v$.

Brinkmann charts can be obtained by means of the following process (see Figure 1). As K is parallel, choose a function u such that $K = \nabla u$ and with the value 0 in the image of u. Each level set $\Sigma_{u_0} = u^{-1}(u_0)$ is a lightlike integral manifold of the distribution $K^\perp$ orthogonal to K. Set $\Sigma = \Sigma_0$ and choose a hypersurface Ω which is transverse to both Σ and K. The function u will serve as a coordinate for M as well as for Ω. In Ω we choose a coordinate neighborhood $\{u, x^i_\Omega\}$ completing u. The coordinate v on M is defined by using the flow ϕ of K to move each point p to the point $\phi_{v_p}(p) \in \Sigma_{u_p} \cap \Omega$, and then we put $x^i_p = x^i_\Omega(\phi_{v_p}(p))$. Observe that we have chosen to move p by using the flow of $K = -\partial_v$ so that $1 \equiv du(\partial_u) = g(\nabla u, \partial_u) = g(K, \partial_u) = -g(\partial_v, \partial_u)$.

Conversely, the expression (3) selects K as $-\partial_v$, and Σ and Ω are the hypersurfaces $u = 0$ and $v = 0$ respectively. Locally, a pair (Σ, Ω) determines a *Brinkmann decomposition*, i.e., a pair of functions $\{u, v\}$ constructed as above, which may serve as the first two coordinates for different Brinkmann charts.

If $\{u', v', x'^i\}$ denotes a second Brinkmann chart which overlaps $\{u, v, x^i\}$, the corresponding Σ' will also be a level set of u, and Ω' can be regarded as a graph on Ω. So, the change of coordinates can be written as

$$u' = u - u_0, \qquad v' = v + F(u, x^j), \qquad x'^i = x'^i(u, x^j). \tag{7}$$

Consequently, the relations between H, W_i, g_{ij} in the original coordinates $\{u, v, x^i\}$ and H', W'_i, g'_{ij} in the new ones $\{u', v', x'^i\}$ are (recall the conventions at the beginning of Section 2)

$$H = H' + \dot{F} + W'_i \dot{x}'^i - \tfrac{1}{2} g'_{ij} \dot{x}'^i \dot{x}'^j, \tag{8}$$

$$W_i = W'_j \frac{\partial x'^j}{\partial x^i} + F_{,i} - \tfrac{1}{2} g'_{jk} \left(\frac{\partial x'^j}{\partial x^i} \dot{x}'^k + \frac{\partial x'^k}{\partial x^i} \dot{x}'^j \right), \tag{9}$$

$$g_{ij} = g'_{kl} \frac{\partial x'^k}{\partial x^i} \frac{\partial x'^l}{\partial x^j}.$$

The freedom in the choice of Ω makes it possible to obtain a Brinkmann decomposition $\{u, v\}$ with a Brinkmann chart such that $H \equiv 0 \equiv W_i$, in particular ∂_u is lightlike and geodesic in the associated chart (see e.g. [38]). For the sake of completeness, let us construct such a coordinate chart. Choose some Σ as above and take any $(n-2)$-submanifold $\overline{M} \hookrightarrow \Sigma$ which is transverse to K. All such $\overline{M}$ are locally isometric, as K is a (parallel) lightlike direction in Σ. Now, consider for each $x \in \overline{M}$ the *unique lightlike direction* $\vec{l}_x$ orthogonal to $\overline{M}$ and linearly independent of K_x. In a small neighborhood, construct Ω by taking the geodesics with initial velocity $\vec{l}_x$ for all $x \in \overline{M}$. Complete the chart by choosing some local coordinates $\{x^i_{\overline{M}}\}$ in $\overline{M}$ and defining the coordinates x^i_Ω at each $y \in \Omega$ by $x^i_\Omega(y) := x^i_{\overline{M}}(x_y)$, where x_y is the unique point in $\overline{M}$ which lies on the same lightlike geodesic as y. Notice that Ω is a lightlike hypersurface, and the corresponding coordinate vector field ∂_u spans its radical, i.e., $H = 0 = W_i$, as required. We emphasize that the value of H and $W_i dx^i$ depend on the choice of the coordinates $\{x^i\}$. Moreover, given a Brinkmann decomposition $\{u, v\}$, there exists a Brinkmann chart $\{u, v, x^i\}$ with $H = 0 = W_i$ if and only if the hypersurface Ω obtained as $v = 0$ is lightlike (when Ω is lightlike, the integral curves of its radical must be lightlike pregeodesics of M, because of the local maximizing properties of these curves). In spite of its simplicity, such a Brinkmann decomposition will not be especially relevant for our study. It might simplify some intermediate computations, but they are not well adapted to the generalized Cahen–Wallach spaces of order 2 which will turn out to be, as already announced, the essential part of proper 2nd-symmetric Lorentzian manifolds.

Given a Brinkmann chart, each integral curve of ∂_v is labeled by $\{u = u_0, x^i = x^i_0\}$, and each hypersurface $\Omega_{v_0} \equiv \{v = v_0\}$ is a general pseudo-Riemannian hypersurface (possibly signature-changing, as studied systematically in [31]) and isometric to Ω. We will repeatedly use the two natural transverse foliations associated to each Brinkmann decomposition, namely:

(1) The $(n-2)$-dimensional foliation $\overline{\mathcal{M}}$ with leaves $\overline{M} = \Sigma \cap \Omega$ and the submanifolds obtained by moving $\overline{M}$ with the flows of ∂_u and ∂_v. Each leaf is defined by $\{u = u_0, v = v_0\}$ and represented by $\overline{M}_{(u_0,v_0)}$. The induced metric will be denoted by $\overline{g}$ ($\overline{g} : T\overline{\mathcal{M}} \times T\overline{\mathcal{M}} \to \mathbb{R}$), so that $\overline{g}_{ij} = g_{ij}$ in any Brinkmann chart. When necessary, $\overline{g}$ will be regarded as a metric on a single leaf. This foliation depends only on the Brinkmann decomposition (Σ, Ω), or equivalently on the chosen functions $\{u, v\}$ only.

(2) The 2-dimensional foliation $\mathcal{U}$ whose leaves are the surfaces obtained by moving each single point along the flows of ∂_u and ∂_v. Each leaf is given by $\{x^i = c_0^i\}$ for some constants c_0^i and the induced metric is $-2du(dv + Hdu)$. This foliation depends on the expression of ∂_u, and thereby on the coordinates chosen on Ω for the Brinkmann chart.

Any Brinkmann chart allows for the decomposition of the tensor bundles in different ways, some of them to be detailed here. By a *partly null frame* $\{E_\alpha\}$ for a Lorentzian manifold we mean a local basis of vector fields such that

$$g(E_0, E_0) = 0, \quad g(E_1, E_1) = 0, \quad g(E_0, E_1) = -1,$$
$$g(E_0, E_i) = 0, \quad g(E_1, E_i) = 0, \quad g(E_i, E_j) = a_{ij},$$

for some functions $a_{ij} = a_{ji}$, which must obviously define a positive definite metric. Its dual basis will be denoted by $\{\theta^\alpha\}$. Now, fix a Brinkmann chart and consider the associated foliations $\overline{\mathcal{M}}, \mathcal{U}$ and distributions $T\overline{\mathcal{M}}$, $T\mathcal{U}$ and $(T\mathcal{U})^\perp$. The (canonical) *partly null frame of the Brinkmann chart* is given by

$$\begin{aligned} \{E_\alpha\} &= \{\partial_u - H\partial_v,\ \partial_v,\ -W_i\partial_v + \partial_i\}, \\ \{\theta^\alpha\} &= \{du, dv + Hdu + W_j dx^j, dx^i\}, \end{aligned} \qquad i = 2, \ldots, n-1, \tag{10}$$

which has $a_{ij} = g_{ij}$. Notice that $(T\mathcal{U})^\perp$ is spanned by the vector fields $\{E_i\}$, and the brackets $[E_i, E_j] = (\partial_j W_i - \partial_i W_j)\partial_v \in \Gamma(T\mathcal{U})$ measure its lack of involutivity.

The decomposition $TM = T\mathcal{U} \oplus T\overline{\mathcal{M}}$ associated to any Brinkmann chart yields, on the one hand, the projection

$$\mathcal{P}_{\overline{\mathcal{M}}} : \Gamma(TM) \to \Gamma(T\overline{\mathcal{M}}), \qquad X \mapsto \overline{X},$$

so that $\mathcal{P}_{\overline{\mathcal{M}}}(\partial_u) = \mathcal{P}_{\overline{\mathcal{M}}}(\partial_v) = 0$ and $\mathcal{P}_{\overline{\mathcal{M}}}(E_i) = \partial_i$, and on the other hand the natural inclusion

$$\mathcal{I}_{\overline{\mathcal{M}}} : \Gamma(T\overline{\mathcal{M}}) \to \Gamma(TM)$$

with the corresponding dual map, which we denote

$$\mathcal{I}^*_{\overline{\mathcal{M}}} : \Gamma(T^*M) \to \Gamma(T^*\overline{\mathcal{M}}), \qquad \beta \mapsto \overline{\beta}.$$

Clearly, $\{\overline{E_i}\} = \{\partial_i\}$ is a basis in $T\overline{\mathcal{M}}$ whose cobasis is $\{\overline{\theta}^i = \overline{dx^i}\}$. If $X = X^\alpha E_\alpha \in \Gamma(TM)$ and $\beta = \beta_\alpha \theta^\alpha \in \Gamma(T^*M)$, then $\overline{X}^i = X^i$ and $\overline{\beta}_i = \beta_i$ in the given bases.

We also introduce two linear homomorphisms between spaces of sections defined as:

(1) $^-: \Gamma(T^r_s M) \to \Gamma(T^r_s \overline{\mathcal{M}})$, which maps each $T = T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} E_{\alpha_1} \otimes \cdots \otimes E_{\alpha_r} \otimes \theta^{\beta_1} \otimes \cdots \otimes \theta^{\beta_s}$ into $\overline{T} = T^{i_1\ldots i_r}_{j_1\ldots j_s} \overline{E}_{i_1} \otimes \cdots \otimes \overline{E}_{i_r} \otimes \overline{\theta}^{j_1} \otimes \cdots \otimes \overline{\theta}^{j_s}$,

(2) $^\circ: \Gamma(T^r_s \overline{\mathcal{M}}) \to \Gamma(T^r_s M)$, which maps each $T = T^{i_1\ldots i_r}_{j_1\ldots j_s} \overline{E}_{i_1} \otimes \cdots \otimes \overline{E}_{i_r} \otimes \overline{\theta}^{j_1} \otimes \cdots \otimes \overline{\theta}^{j_s}$ into $\mathring{T} = T^{i_1\ldots i_r}_{j_1\ldots j_s} E_{i_1} \otimes \cdots \otimes E_{i_r} \otimes \theta^{j_1} \otimes \cdots \otimes \theta^{j_s}$.

In general, $\overline{\mathring{T}} = T$ but $\mathring{\overline{T}} \neq T$, because in the first case $T \in \Gamma(T^r_s\overline{\mathcal{M}})$ necessarily, but not in the second.

Remark 4.1. The previous maps are $C^\infty(M)$-linear homomorphisms between spaces of sections, they commute with contractions, with tensor products and, when applicable, with wedge products in a natural way; moreover, they leave invariant the $C^\infty(M)$ functions considered as (0, 0)-tensor fields: $\mathring{f} = \overline{f} = f$ for all $f \in C^\infty(M)$. Both homomorphisms $^-$ and $^\circ$ have trivial expressions in the introduced bases. Essentially, all components of the tensor fields and their images remain equal except that, in the case of $T \mapsto \overline{T}$, the components along E_0, E_1, θ^0 or θ^1 must be dropped, while, in the case of $T \mapsto \mathring{T}$, these components must be restored with vanishing value. This simplifies our subsequent work in components substantially, as we will not need to distinguish notationally among different tensor fields derived from a single one: it will be enough to realize which space of sections is being considered.

As important illustrative examples, observe that the functions W_i in the Brikmann expression (3) can be regarded as the components in the basis $\{\overline{E_i}\} = \{\partial_i\}$ of a one-form on $\overline{\mathcal{M}}$, namely $W = W_i\overline{\theta}^i \in \Gamma(T^*\overline{\mathcal{M}})$, as well as the components in the basis $\{E_\alpha\}$ of the one-form $\mathring{W}$ on M. Analogously, g_{ij} can be regarded as the components of the projection $P_{\overline{\mathcal{M}}}(g) = \overline{g} \in \Gamma(T^0_2\overline{\mathcal{M}})$, which is the inherited metric on $\overline{\mathcal{M}}$. Therefore, the metric g can be rewritten as

$$g = -2du(dv + Hdu + \mathring{W}) + \mathring{\overline{g}}.$$

One should also keep in mind that, when a single leaf $\overline{M}_{(u,v)}$ is considered, g_{ij} will also denote the components of the induced metric $\overline{g}$ on the leaf, with no explicit mention of (u, v) or the underlying Brinkmann decomposition.

4.2. *Three differential operators adapted to $\overline{\mathcal{M}}$ and v-invariant sections*

By using the previously obtained vector bundle decomposition associated to each Brinkmann chart, three differential operators $\overline{d}$, $\overline{\nabla}$ and $^\cdot$ ("dot") are introduced next. The operator $\overline{\nabla}$ depends on the Brinkmann decomposition $\{u, v\}$ only, while the other two depend on the whole Brinkmann chart $\{u, v, x^i\}$.

4.2.1. v-invariant tensor fields. The variation on $\overline{\mathcal{M}}$ of $T \in \Gamma(T^r_s\overline{\mathcal{M}})$ under displacements along the direction $E_1 = \partial_v$ will be rather irrelevant for us. In fact, $E_1 = -K$ is parallel, and therefore all the interesting objects to be used here will also be invariant under its flow. More precisely:

Definition 4.2. A tensor field $T \in \Gamma(T^r_s\overline{\mathcal{M}})$ on $\overline{\mathcal{M}}$ is *v-invariant* if $\mathring{T}$ is Lie-parallel along the flow of E_1, that is, $\mathcal{L}_{E_1}\mathring{T} = 0$.

Remark 4.3. (1) As $E_1 = \partial_v$, a tensor field $T = T^{i_1\dots i_r}_{j_1\dots j_s}\partial_{i_1}\otimes\cdots\otimes\partial_{i_r}\otimes\overline{dx^{j_1}}\otimes\cdots\otimes\overline{dx^{j_s}}$ $\in \Gamma(T^r_s\overline{\mathcal{M}})$ is v-invariant if and only if $\partial_v(T^{i_1\dots i_r}_{j_1\dots j_s}) = 0$. For example, $\overline{g}$, H and W are v-invariant.

(2) Obviously, if T is v-invariant then it is determined by its value on any of the hypersurfaces $\Omega_{v_0} = \{v = v_0\}$, and any section $T_{\Omega_{v_0}} \in \Gamma(\Omega_{v_0}, T^r_s\overline{\mathcal{M}})$, defined only on Ω_{v_0}, can be extended to a unique v-invariant section $T_{\overline{\mathcal{M}}} \in \Gamma(T^r_s\overline{\mathcal{M}})$.

(3) As will become obvious later, the derivatives of a v-invariant section with respect to any of the three operators $\overline{d}$, $\overline{\nabla}$ and $\dot{}$ to be defined below are also v-invariant. Hence, these operators can be naturally defined for sections in $\Gamma(\Omega_{v_0}, T^r_s\overline{\mathcal{M}})$ (for instance, $\overline{\nabla}T_{\Omega_{v_0}}$ is the restriction of $\overline{\nabla}T_{\overline{\mathcal{M}}}$ to Ω_{v_0}).

There is an alternative definition of v-invariance.

Proposition 4.4. *$T \in \Gamma(T^r_s\overline{\mathcal{M}})$ is v-invariant if and only if $\mathring{T}$ is parallel in the direction E_1.*

Proof. As E_1 is parallel, $\nabla_{E_1}Q = \mathcal{L}_{E_1}Q$ for any $Q \in \Gamma(T^r_s M)$. □

4.2.2. The $\overline{\mathcal{M}}$ exterior derivative $\overline{d}$

Definition 4.5. The $\overline{\mathcal{M}}$ *exterior derivative* $\overline{d} : \Lambda^s\overline{\mathcal{M}} \to \Lambda^{s+1}\overline{\mathcal{M}}$ associated to a Brinkmann chart $\{u, v, x^i\}$ is defined by

$$\overline{d}\beta = \overline{d\mathring{\beta}} \qquad \forall \beta \in \Lambda^s\overline{\mathcal{M}},$$

where d is the usual exterior derivative on the manifold M.

It is straightforward to check that, if $\beta \in \Lambda^s M$, then $\overline{d}\,\overline{\beta} = \overline{d\beta}$. In particular, $\overline{d}f = \overline{df}$ for $f \in C^\infty(M)$ and thus $\overline{\theta^i} = \overline{dx^i} = \overline{d}x^i$. This allows us to use the expressions ∂_i and $\overline{d}x^i$ instead of $\overline{E_i}$ and $\overline{\theta}^i$, respectively. Moreover, using the notation introduced in (4),

$$df = \dot{f}du + \partial_v f dv + \overbrace{\overline{d}f}^{\circ}.$$

Proposition 4.6. *Let $\beta \in \Lambda^s\overline{\mathcal{M}}$. The differential $\overline{d}$ has the following properties:*

(1) *It is linear and $\overline{d}(\beta \wedge \tau) = \overline{d}\beta \wedge \tau + (-1)^s\beta \wedge \overline{d}\tau$ for all $\tau \in \Lambda^q\overline{\mathcal{M}}$.*
(2) $\overline{d}(\overline{d}\beta) = 0$.
(3) *If $\beta = \frac{1}{s!}\beta_{i_1\ldots i_s}\overline{d}x^{i_1} \wedge \cdots \wedge \overline{d}x^{i_s}$, then $\overline{d}\beta = \frac{1}{s!}\partial_k(\beta_{i_1\ldots i_s})\overline{d}x^k \wedge \overline{d}x^{i_1} \wedge \cdots \wedge \overline{d}x^{i_s}$.*
(4) (Poincaré Lemma) *If $\overline{d}\beta = 0$, then each $p \in M$ admits a neighborhood U and an $(s-1)$-form τ such that $\overline{d}\tau = \beta$ on U. Moreover, if the $\overline{d}$-closed s-form β is v-invariant, then so can be chosen the $(s-1)$-form τ.*

Proof. (1) Straightforward.

(2) $\overline{d}(\overline{d}\beta) := \overline{d}(\overline{d\mathring{\beta}}) = \overline{d(d\mathring{\beta})} = 0$.

(3) Apply (1), (2) and $\overline{d}f = \overline{df}$ in $\overline{d}(\frac{1}{s!}\beta_{i_1\ldots i_s}\overline{d}x^{i_1} \wedge \cdots \wedge \overline{d}x^{i_s})$.

(4) We sketch the notationally simpler case $s = 1$ (to be used later). For $\beta = \beta_i\overline{d}x^i$, put

$$\tau(u, v, x^2, \ldots, x^{n-1}) = \sum_{i=2}^{n-1} x^i\left(\int_0^1 \beta_i(u, v, \sigma x^2, \ldots, \sigma x^{n-1})\, d\sigma\right) + h(u, v),$$

for arbitrary h. Then $\overline{d}\beta = 0$ (i.e., $\frac{\partial\beta_i}{\partial x^j} - \frac{\partial\beta_j}{\partial x^i} = 0$) yields $\partial_i\tau = \beta_i$, as required. □

4.2.3. The covariant derivative $\overline{\nabla}$ and its curvature tensor $\overline{R}$ on $\overline{\mathcal{M}}$

Definition 4.7. The *covariant derivative* $\overline{\nabla}$ on $\overline{\mathcal{M}}$ associated to a Brinkmann decomposition $\{u, v\}$ is the map

$$\overline{\nabla} : \Gamma(T\overline{\mathcal{M}}) \times \Gamma(T\overline{\mathcal{M}}) \to \Gamma(T\overline{\mathcal{M}}), \qquad (X, Y) \mapsto \overline{\nabla}_X Y,$$

defined at each point $p \in M$ by

$$(\overline{\nabla}_X Y)_p := (\overline{\nabla}^{\bar{g}}_{\check{X}} \check{Y})_p$$

where $\check{X}$ and $\check{Y}$ are the restrictions of X, Y to the leaf $M_{(u(p),v(p))}$ and $\overline{\nabla}^{\bar{g}}$ the Levi-Civita connection of the first fundamental form $\bar{g}$ on that leaf.

Remark 4.8. a) Clearly, $\overline{\nabla}$ has the formal properties of a symmetric covariant derivative, as well as $\overline{\nabla}\bar{g} = 0$.

b) The *Christoffel symbols* $\overline{\Gamma}^i_{jk}$ of $\overline{\nabla}$ in the basis $\{\partial_i\}$ are defined by the relation $\overline{\nabla}_{\partial_j} \partial_i = \overline{\Gamma}^k_{ij} \partial_k$. Trivially, they are smooth and invariant under the flow of ∂_v.

c) The covariant derivative on $\overline{\mathcal{M}}$ can be extended to sections of $T^r_s\overline{\mathcal{M}}$. For the coordinate basis $\{\partial_i\}$ and any $T \in \Gamma(T^r_s\overline{\mathcal{M}})$ one has

$$\overline{\nabla}_m T^{i_1\ldots i_r}_{j_1\ldots j_s} = \partial_m(T^{i_1\ldots i_r}_{j_1\ldots j_s}) + \sum_{a=1}^{r} \overline{\Gamma}^{i_a}_{km} T^{i_1\ldots i_{a-1} k i_{a+1}\ldots i_r}_{j_1\ldots j_s} - \sum_{b=1}^{s} \overline{\Gamma}^k_{j_b m} T^{i_1\ldots i_r}_{j_1\ldots j_{b-1} k j_{b+1}\ldots j_s}.$$

d) For $X, Y \in \Gamma(TM)$, $\overline{\nabla_X Y} \neq \overline{\nabla}_{\overline{X}}\overline{Y}$ in general (for example, if $X = E_0$ then $\overline{X} = 0$).

The covariant derivative $\overline{\nabla}$ yields a natural *curvature tensor* $\overline{\mathcal{R}}$ *of the foliation* $\overline{\mathcal{M}}$ defined formally as the usual curvature of $\overline{\nabla}$:

$$\overline{\mathcal{R}}(X, Y)Z = (\overline{\nabla}_X\overline{\nabla}_Y - \overline{\nabla}_Y\overline{\nabla}_X - \overline{\nabla}_{[X,Y]})Z \in \Gamma(T\overline{\mathcal{M}}), \quad \forall X, Y, Z \in \Gamma(T\overline{\mathcal{M}}), \quad (11)$$

and also its derived *Ricci tensor* $\overline{\mathcal{R}\mathrm{ic}}$ and *scalar curvature* $\overline{\mathcal{S}}$ *of* $\overline{\mathcal{M}}$. All of them satisfy the standard symmetries corresponding to a curvature tensor.

Definition 4.9. Fixed a Brinkmann decomposition $\{u, v\}$. The foliation $\overline{\mathcal{M}}$ is called *u-Einstein* if $\overline{\mathcal{R}\mathrm{ic}} = \mu\bar{g}$ for some function μ such that $d\mu \wedge du = 0$. In particular, when μ is constant, $\overline{\mathcal{M}}$ is called *Einstein*, and when $\mu \equiv 0$ we say that $\overline{\mathcal{M}}$ is *Ricci-flat*.

In the case $\overline{\mathcal{R}} = 0$ (respectively $\overline{\nabla}\,\overline{\mathcal{R}} = 0$), the foliation $\overline{\mathcal{M}}$ is said to be *flat* (respectively *locally symmetric*).

Some simple properties follow immediately.

Proposition 4.10. *Let (M, g) be a Brinkmann space with a fixed Brinkmann decomposition $\{u, v\}$.*

(1) *If $\overline{\nabla}^r\overline{\mathcal{R}} = 0$ for some $r > 1$, then $\overline{\mathcal{M}}$ is a locally symmetric foliation.*
(2) *If $\overline{\mathcal{M}}$ is locally symmetric and Ricci-flat, then it is flat.*

(3) *If $\overline{\mathcal{M}}$ is flat, the Brinkmann decomposition admits a chart $\{u, v, y^i\}$ such that the metric g becomes*

$$g = -2du(dv + Hdu + W_i dy^i) + \delta_{ij} dy^i dy^j.$$

Proof. (1) For each (u, v), the Riemannian result which states that $\overline{\nabla}^r\overline{\mathcal{R}} = 0$ implies local symmetry can be applied to each leaf $\overline{M}_{(u,v)}$.

(2) Apply Proposition 3.3(3) to each leaf of $\overline{\mathcal{M}}$.

(3) We sketch a procedure to obtain the required new coordinates $y^i = y^i(u, x^j)$, for the sake of completeness. Let γ be an integral curve of ∂_u along the hypersurface $\Omega = \{v = 0\}$, and put, for small $U \subset \mathbb{R}^{n-2}$ with $0 \in U$,

$$\varphi :]-\epsilon, \epsilon[\times U \to M, \qquad (u, w^i) \mapsto \overline{\exp}_{\gamma(u)}(w^i \partial_i|_{\gamma(u)}),$$

where $\overline{\exp}_{\gamma(u)}$ denotes the exponential at $\gamma(u)$ on the leaf $\overline{M}_{(\gamma(u),0)}$ for the (flat) metric $\overline{g}$. The classical Cartan theorem shows that each map $\varphi_u := \varphi(u, \cdot)$ is an affine transformation from U to the leaf. So, regarding $\{w^i\}$ as coordinates in $\mathbb{R}^{n-2}$,

$$\varphi_u^* \overline{g} = h_{ij}(u) dw^i dw^j$$

for some constants $h_{ij}(u)$ on each leaf (which define a Euclidean metric and depend smoothly on u). Now, φ allows us to consider $\{u, w^i\}$ as coordinates in Ω, with each coordinate vector field $\partial/\partial w^i$ parallel on each leaf of $\overline{\mathcal{M}}_{(u,0)}$. By using the Gram–Schmidt procedure, an orthonormal basis $\{V_j = B^i_j \frac{\partial}{\partial w^i}\}$ in $T\overline{\mathcal{M}}$ is obtained. Indeed, by construction the transition matrix (B^i_j) depends smoothly only on u, and thus, by Remark 4.3(3), $\overline{\nabla} V_j = \overline{d}(B^i_j)\frac{\partial}{\partial w^i} = 0$. Therefore, Proposition 4.6(4) implies that $(V_j)^\flat = dy^j_\Omega$ for some functions $\{y^j_\Omega(u, w^i)\}$ on some open subset of Ω. The required functions $\{y^j\}$ are obtained by extending $\{y^j_\Omega(u, w^i)\}$ to a neighborhood of M in a v-invariant way according to Remark 4.3(2). □

4.2.4. The $\dot{}$ (dot) derivative. We introduce the following simple derivative:

Definition 4.11. *The dot derivative $\dot{T} \in \Gamma(T^r_s\overline{\mathcal{M}})$ of a tensor field $T \in \Gamma(T^r_s\overline{\mathcal{M}})$ is* defined as $\dot{T} = \overline{\mathcal{L}_{\partial_u}\mathring{T}}$.

The components of $\dot{T}$ in the coordinate basis $\{\partial_i\}$ are $\dot{T}^{i_1\ldots i_r}_{j_1\ldots j_s} = \partial_u(T^{i_1\ldots i_r}_{j_1\ldots j_s})$. Indeed, the first usage of the dot for functions was the definition (4).

4.3. Some geometrical objects and the operator D_0

Our aim in this section is to obtain explicit expressions for the curvature and its derivatives adapted to a Brinkmann chart. We start by computing the curvature two-forms for a partly null frame. Then, we introduce a new operator D_0 which, together with those introduced in the last section, will allow us to simplify calculations and to provide manageable formulae for the derivatives of the curvature. This will be very helpful in order to solve the equations of 2nd-symmetry.

4.3.1. The connection 1*-forms and the curvature tensor.* We compute the connection and curvature forms by using Cartan techniques, according to the conventions in [32], [14]. In a standard manner, we define the connection one-forms $\overline{\omega}^i_j \in \Lambda^1(\overline{\mathcal{M}})$ and the corresponding curvature two-forms $\overline{\Omega}^i_j \in \Lambda^2(\overline{\mathcal{M}})$ of the foliation $\overline{\mathcal{M}}$ for the given coordinate basis $\{\partial_i\}$ as

$$\overline{\nabla}_X \partial_i = \overline{\omega}^j_i(X)\partial_j, \quad \overline{\mathcal{R}}(X,Y)\partial_j = \overline{\Omega}^i_j(X,Y)\partial_i, \quad \forall X, Y \in \Gamma(T\overline{\mathcal{M}}).$$

Observe that $\overline{\omega}^i_j = \overline{\Gamma}^i_{jk}\overline{d}x^k$, where $\overline{\Gamma}^i_{jk}$ are the Christoffel symbols of $\overline{\nabla}$ as introduced in Remark 4.8. Then, a simple computation shows that the first and second Cartan equations for $\overline{\mathcal{M}}$ still hold:

$$0 = \overline{\omega}^i_j \wedge \overline{d}x^j, \quad \overline{\Omega}^i_j = \overline{d}\,\overline{\omega}^i_j + \overline{\omega}^i_k \wedge \overline{\omega}^k_j.$$

In order to compute the connection and curvature of the Brinkmann spaces we introduce two tensor fields on $\overline{\mathcal{M}}$ in terms of the adapted differential $\overline{d}$ and the dot derivative, which will be especially relevant in the computations. We define $h \in \Lambda^1(\overline{\mathcal{M}})$ and $t \in T^0_2(\overline{\mathcal{M}})$ as

$$h = \overline{d}H - \dot{W}, \tag{12}$$
$$t = -\tfrac{1}{2}(\dot{g} + \overline{d}W). \tag{13}$$

Observe that the symmetric and skew-symmetric parts of t are precisely $-\dot{g}/2$ and $-\overline{d}W/2$, respectively. We emphasize that h and t depend on the Brinkmann chart. Equipped with these sections, one can check that the non-vanishing connection one-forms associated to a partly null frame $\{E_\alpha\}$ of any Brinkmann chart are

$$\omega^1_i = \omega^j_0 g_{ij} = h_i\theta^0 - t_{ij}\theta^j, \tag{14}$$
$$\omega^i_j = -g^{ik}t_{kj}\theta^0 + \mathring{\omega}^i_j, \tag{15}$$

where $h_i = H_{,i} - \dot{W}_i$ and $t_{ij} = \frac{1}{2}(-\dot{g}_{ij} + W_{i,j} - W_{j,i})$ are the components of the tensors defined in (12) and (13). It should be noted that $\overline{\omega^i_j} = \overline{\omega}^i_j$, but $\mathring{\omega}^i_j \neq \omega^i_j$. Then the non-vanishing curvature two-forms associated to the partly null frame $\{E_\alpha\}$ of any Brinkmann chart read

$$\Omega^1_i = -(\overline{\nabla}_j h_i + \dot{t}_{ij} + t^k{}_i t_{kj})\theta^0 \wedge \theta^j + \tfrac{1}{2}(\overline{\nabla}_k t_{ij} - \overline{\nabla}_j t_{ik})\theta^j \wedge \theta^k, \tag{16}$$
$$\Omega^i_j = (\overline{\nabla}_k t^i{}_j + \dot{\overline{\Gamma}}{}^i_{jk})\theta^0 \wedge \theta^k + \overline{\Omega}^i_j. \tag{17}$$

so that again $\overline{\Omega^i_j} = \overline{\Omega}^i_j$, but $\mathring{\Omega}^i_j \neq \Omega^i_j$.

From (14)–(15) it is immediate to obtain the components $\gamma^\alpha_{\beta\lambda}$ of the connection one-forms ω^α_β, defined by $\omega^\alpha_\beta = \gamma^\alpha_{\beta\lambda}\theta^\lambda$. These will be used later. Recall also the identity $\nabla_{E_\beta}E_\alpha = \gamma^\lambda_{\alpha\beta}E_\lambda$. This together with (14) provides the simple formula

$$h(X) = g(\mathring{X}, \nabla_{E_0}E_0), \quad \forall X \in \Gamma(T\overline{\mathcal{M}}). \tag{18}$$

Similarly, the non-vanishing components $R^{\alpha}{}_{\beta\gamma\delta}$ of the curvature tensor R, defined by $\Omega^{\alpha}_{\beta} = \frac{1}{2} R^{\alpha}{}_{\beta\gamma\delta}\theta^{\gamma} \wedge \theta^{\delta}$, can be read off from (16)–(17):

$$\begin{aligned} R^{1}{}_{i0j} &= -(\overline{\nabla}_j h_i + \dot{t}_{ij} + t^{k}{}_{i} t_{kj}), \qquad (19)\\ R^{1}{}_{ijk} &= \overline{\nabla}_k t_{ij} - \overline{\nabla}_j t_{ik},\\ R^{i}{}_{jkl} &= \overline{\mathcal{R}}^{i}{}_{jkl},\\ R^{i}{}_{j0k}\ (= -g^{ri} R^{1}{}_{krj}) &= \overline{\nabla}_k t^{i}{}_{j} + \dot{\overline{\Gamma}}^{i}{}_{jk}, \qquad (20) \end{aligned}$$

where, in the last expression, the dot acts on the functions $\overline{\Gamma}^{i}{}_{jk}$. From this, the non-vanishing components of the Ricci tensor can be easily computed (note that $R^{1}{}_{\alpha\beta\mu} = -R_{0\alpha\beta\mu}$):

$$\begin{aligned} R_{00} &= R^{i}{}_{0i0} = -g^{ij} R^{1}{}_{j0i} = \overline{\nabla}_i h^i + g^{ij}\dot{t}_{ji} + t^{ki} t_{ki},\\ R_{0i} &= \overline{\nabla}_i t^{j}{}_{j} - \overline{\nabla}_j t^{j}{}_{i},\\ R_{ij} &= \overline{\mathcal{R}}_{ij}, \end{aligned}$$

and the scalar curvature of M turns out to be equal to that of $\overline{\mathcal{M}}$,

$$S = \overline{\mathcal{S}}.$$

Therefore, we have proven the following result.

Proposition 4.12. *The curvature tensor $\overline{\mathcal{R}}$ of the foliation $\overline{\mathcal{M}}$ as defined in* (11)*, and its associated Ricci tensor $\overline{\mathcal{R}\mathrm{ic}}$ and scalar curvature $\overline{\mathcal{S}}$ satisfy*

$$\overline{\mathcal{R}} = \overline{R}, \qquad \overline{\mathcal{R}\mathrm{ic}} = \overline{\mathrm{Ric}}, \qquad \overline{\mathcal{S}} = \overline{S},$$

where $\overline{R}$, $\overline{\mathrm{Ric}}$ and $\overline{S}$ are the projections by the homomorphism $^{-}$ of the curvature tensor R, the Ricci tensor Ric *and the scalar curvature S of the Brinkmann space (M, g), respectively.*

Using this result, from now on there will be no need to distinguish between these objects and we will use only the notation $\overline{R}$, $\overline{\mathrm{Ric}}$ and $\overline{S}$ for them.

Remark 4.13. Concerning the consistency of the tensor equations on M:

(1) If, say, the first index of t and $\mathring{t}$ is raised, then the same expression $t^{i}{}_{j}$ is obtained, as the isomorphisms $\flat$ and $\sharp$ commute with the homomorphism $T \mapsto \mathring{T}$ ($t^{i}{}_{j} \equiv g^{i\alpha} t_{\alpha j} = g^{ir} t_{rj}$).

(2) Observe that $\overline{R} = \overline{\mathcal{R}}$ is related to R by means of the following formula:

$$\overline{R}(X, Y)Z = \overline{R(\mathring{X}, \mathring{Y})\mathring{Z}}, \qquad \forall X, Y, Z \in \Gamma(T\overline{\mathcal{M}}).$$

4.3.2. The D_0 derivation

Definition 4.14. The D_0 operator associated to any Brinkmann chart is defined by

$$D_0 : \Gamma(T^{r}_{s}\overline{\mathcal{M}}) \to \Gamma(T^{r}_{s}\overline{\mathcal{M}}), \qquad T \mapsto D_0 T = \overline{\nabla_{E_0}\mathring{T}}.$$

Consequently, $T \in \Gamma(T^{r}_{s}\overline{\mathcal{M}})$ is said to be D_0*-parallel* if $D_0 T = 0$.

D_0 measures the variation, projected to $T^r_s\overline{\mathcal{M}}$, of tensor fields $T \in \Gamma(T^r_s\overline{\mathcal{M}})$ under displacements along the direction E_0, which is transverse to the leaves of $\overline{\mathcal{M}}$.

Proposition 4.15. *D_0 has the formal properties of a tensor derivation on $\overline{\mathcal{M}}$:*

(i) *$\mathbb{R}$-linearity: $D_0(aA+bB) = aD_0A+bD_0B$ for all $a, b \in \mathbb{R}$ and $A, B \in \Gamma(T^r_s\overline{\mathcal{M}})$.*
(ii) *Leibniz rule: $D_0(A \otimes B) = (D_0A) \otimes B + A \otimes (D_0B)$ for all $A, B \in \Gamma(T^r_s\overline{\mathcal{M}})$.*
(iii) *Commuting with contractions: $D_0(C^i_j(A)) = C^i_j(D_0A)$ for all $A \in \Gamma(T^r_s\overline{\mathcal{M}})$, where C^i_j denotes the contraction of the ith contravariant slot with the jth covariant one.*

Proposition 4.16. *For $X \in \Gamma(T\overline{\mathcal{M}})$, the decomposition of $\nabla_{E_0}\mathring{X}$ in $T\mathcal{U} \oplus (T\mathcal{U})^\perp$ is*

$$\nabla_{E_0}\mathring{X} = h(X)E_1 + \overbrace{D_0X}^{\circ}.$$

Proof. Putting $\nabla_{E_0}\mathring{X} = X_1 + X_2$ with $X_1 \in \Gamma(T\mathcal{U})$ and $X_2 \in \Gamma((T\mathcal{U})^\perp)$, the definition of D_0 gives immediately $D_0X = \overline{X_2}$. For X_1, since $g(\mathring{X}, E_0) = g(\mathring{X}, E_1) = 0$ and E_1 is parallel using formula (18) we get

$$\begin{aligned} X_1 &= -g(\nabla_{E_0}\mathring{X}, E_1)E_0 - g(\nabla_{E_0}\mathring{X}, E_0)E_1 = g(\mathring{X}, \nabla_{E_0}E_1)E_0 + g(\mathring{X}, \nabla_{E_0}E_0)E_1 \\ &= h(X)E_1. \end{aligned}$$

□

Proposition 4.17. *$\overline{g}$ is D_0-parallel, that is,*

$$D_0\overline{g} = 0. \tag{21}$$

Proof. By definition, $D_0\overline{g} \equiv \overline{\nabla_{E_0}\mathring{g}}$ and $g = -\theta^0 \otimes \theta^1 - \theta^1 \otimes \theta^0 + \mathring{g}$. Since θ^0 is parallel and $\nabla_{E_0}g = 0$,

$$-\theta^0 \otimes \nabla_{E_0}\theta^1 - \nabla_{E_0}\theta^1 \otimes \theta^0 + \nabla_{E_0}\mathring{g} = 0.$$

Therefore, as $\overline{\theta^0 \otimes \nabla_{E_0}\theta^1 + \nabla_{E_0}\theta^1 \otimes \theta^0} = 0$, we have $\overline{\nabla_{E_0}\mathring{g}} = 0$. □

Remark 4.18. It is not difficult to show that, for a fixed partly null frame $\{E_\alpha\}$ and for arbitrary $\omega = \omega_i\overline{d}x^i \in \Gamma(T^*\overline{\mathcal{M}})$, the D_0-derivative acts such that

$$(D_0\omega)(X) = (E_0(\omega_i)\overline{d}x^i)(X) + t(\omega^\sharp, X), \qquad \forall X \in \Gamma(T\overline{\mathcal{M}}),$$

while, with the same notation and $X = X^i\partial_i$,

$$\omega(D_0X) = \omega(E_0(X^i)\partial_i) - t(\omega^\sharp, X).$$

These formulae can then be extended to arbitrary sections $T \in \Gamma(T^r_s\overline{\mathcal{M}})$, and can be expressed in the local basis $\{\partial_i\}$ of $\Gamma(T\overline{\mathcal{M}})$ by means of

$$\begin{aligned} (D_0X)^i &\equiv D_0X^i = (\partial_u - H\partial_v)(X^i) - t^i{}_jX^j, \\ (D_0\omega)_i &\equiv D_0\omega_i = (\partial_u - H\partial_v)(\omega_i) + t^j{}_i\omega_j. \end{aligned}$$

Its generalization to any section $T \in \Gamma(T^r_s\overline{\mathcal{M}})$ is

$$(D_0T)^{i_1\ldots i_r}_{j_1\ldots j_s} \equiv D_0T^{i_1\ldots i_r}_{j_1\ldots j_s} = (\partial_u - H\partial_v)(T^{i_1\ldots i_r}_{j_1\ldots j_s}) - \sum_{a=1}^{r} t^{i_a}{}_k T^{i_1\ldots i_{a-1}ki_{a+1}\ldots i_r}_{j_1\ldots j_s} + \sum_{b=1}^{s} t^k{}_{j_b} T^{i_1\ldots i_r}_{j_1\ldots j_{b-1}kj_{b+1}\ldots j_s}.$$

For v-invariant tensor fields the expression D_0T simplifies ($D_0T = \overline{\nabla_{\partial_u}\mathring{T}}$) so that

$$(D_0T)^{i_1\ldots i_r}_{j_1\ldots j_s} = \dot{T}^{i_1\ldots i_r}_{j_1\ldots j_s} - \sum_{a=1}^{r} t^{i_a}{}_k T^{i_1\ldots i_{a-1}ki_{a+1}\ldots i_r}_{j_1\ldots j_s} + \sum_{b=1}^{s} t^k{}_{j_b} T^{i_1\ldots i_r}_{j_1\ldots j_{b-1}kj_{b+1}\ldots j_s}. \tag{22}$$

Next, we collect some elementary properties of the D_0-parallel transport to be used later. First, let η be an integral curve of E_0 and take a vector field X_η along η which is everywhere tangent to $\overline{\mathcal{M}}$, i.e., $X_\eta \in \Gamma(T_\eta\overline{\mathcal{M}})$. The derivative $D_0(X_\eta)$ of X_η, as well as its D_0-parallelism, make an obvious sense.

Lemma 4.19. *Let $p \in M$, $\vec{v} \in T_p\overline{\mathcal{M}}$ and η the integral curve of E_0 with $\eta(0) = p = (u_p, v_p, x^i_p)$. Then there exists a unique vector field X_η obtained as the D_0-parallel transport along η such that $X_{\eta(0)} = \vec{v}$.*

A standard reasoning leads to:

Proposition 4.20. *The map which sends each $\vec{v} \in T_p\overline{\mathcal{M}}$ to its unique D_0-parallel transport $X_\eta(\tau) \in T_{\eta(\tau)}\overline{\mathcal{M}}$ along η is a linear isometry from $T_p\overline{\mathcal{M}}$ to $T_{\eta(\tau)}\overline{\mathcal{M}}$.*

Proof. The result follows from (21), since $(\overline{g}(X, Y))|_\eta$ depends only on the parameter u of η. ◻

Our last result yields an extension to all the leaves of any vector field on one leaf.

Proposition 4.21. *Let $X_{\overline{M}}$ be a vector field on a leaf $\overline{M}_{(u_0,v_0)}$ of $\overline{\mathcal{M}}$. Then there exists a unique v-invariant and D_0-parallel vector field $X_{\overline{\mathcal{M}}} \in \Gamma(T\overline{\mathcal{M}})$ which extends $X_{\overline{M}}$.*

Proof. Consider the hypersurface $\Omega_{v_0} = \{v = v_0\}$ and extend $X_{\overline{M}}$ to a vector field $X_{\Omega_{v_0}}$ on Ω_{v_0} by taking each integral curve η of ∂_u which starts at some $p \in \overline{M}_{(u_0,v_0)}$, and defining $X_{\Omega_{v_0}} \circ \eta$ as the D_0-parallel transport of X_p. Then, extend $X_{\Omega_{v_0}}$ to a v-invariant vector field according to Remark 4.3(2). ◻

4.3.3. Derivatives of the curvature tensor R. As E_1 is parallel, the curvature tensor will be determined on each Brinkmann chart by its value on quadruples of vectors tangent to $\overline{\mathcal{M}}$, plus some extra tensors which take care of the remaining components (partly) along the θ^1 or E_0 directions. We start by defining two such tensor fields on $\overline{\mathcal{M}}$.

Definition 4.22. For any Brinkmann chart and its associated partly null frame $\{E_\alpha\}$ we define $A \in \Gamma(T^0_2\overline{\mathcal{M}})$ and $B \in \Gamma(T^0_3\overline{\mathcal{M}})$ as $A := \theta^1(R(E_0, \cdot))$ and $B := \theta^1(R)$, that is,

$$A(X, Y) = \theta^1(R(E_0, \mathring{Y})\mathring{X}), \qquad B(X, Y, Z) = \theta^1(R(\mathring{Y}, \mathring{Z})\mathring{X}), \qquad \forall X, Y, Z \in \Gamma(T\overline{\mathcal{M}}).$$

From the symmetries of the curvature tensor it is obvious that A is symmetric:

$$A(X, Y) = A(Y, X), \quad \forall X, Y \in \Gamma(T\overline{\mathcal{M}}),$$

and that B is skew-symmetric in its last two slots and it satisfies a cyclic identity:

$$\begin{array}{l} B(X, Y, Z) = -B(X, Z, Y), \\ B(X, Y, Z) + B(Y, Z, X) + B(Z, X, Y) = 0, \end{array} \quad \forall X, Y, Z \in \Gamma(T\overline{\mathcal{M}}).$$

Remark 4.23. In the given basis $\{\partial_i\}$, $A_{ij} = A(\partial_i, \partial_j) = R^1{}_{i0j}$ and $B_{ijk} = B(\partial_i, \partial_j, \partial_k) = R^1{}_{ijk}$. The previous properties can be expressed using this notation as $A_{ij} = A_{(ij)}$, $B_{ijk} = B_{i[jk]}$, and $B_{[ijk]} = 0$. In what follows, and for the sake of brevity, we will resort to index notation in many cases, which is sufficient to illustrate these properties and reveals itself as very helpful in the required complicated calculations for 2nd-symmetry. As a starting example, note that we additionally have for instance

$$B_{ij}{}^k \; (= g^{kr} B_{ijr}) = R^k{}_{j0i}.$$

A direct computation of ∇R, for instance in the basis $\{E_\alpha\}$, provides the following formulae:

$$\begin{aligned} \overline{\theta^1(\nabla_{E_0} R(E_0, \cdot))} &= D_0 A + 2\mathcal{S}[C_3^1(h^\sharp \otimes B)], \\ \overline{\theta^1(\nabla R(E_0, \cdot))} &= \overline{\nabla} A - 2\mathcal{S}[C_{15}(t \otimes B)], \\ \overline{\theta^1(\nabla_{E_0} R)} &= D_0 B + h(\overline{R}), \\ \overline{\theta^1(\nabla R)} &= \overline{\nabla} B - C_1^1(t \otimes \overline{R}), \\ \overline{\nabla_{E_0} R} &= D_0 \overline{R}, \\ \overline{\nabla R} &= \overline{\nabla}\, \overline{R}, \end{aligned}$$

where $\mathcal{S}[T]$ gives the symmetric part of any covariant section $T \in \Gamma(T_s^0 \overline{\mathcal{M}})$ in its last two slots, and C_{ij} denotes the contraction of the ith and jth covariant indices (via the metric $\bar{g}^{-1}$).

We give names to the left-hand sides of these relations (except for the last one), thereby defining five tensor fields on $\overline{\mathcal{M}}$ which will allow for simpler expressions when computing $\nabla^2 R$.

Definition 4.24. For any Brinkmann chart and its associated partly null frame $\{E_\alpha\}$ we define $\widetilde{A} \in \Gamma(T_2^0 \overline{\mathcal{M}})$, $\widehat{A}, \widetilde{B} \in \Gamma(T_3^0 \overline{\mathcal{M}})$, $\widehat{B} \in \Gamma(T_4^0 \overline{\mathcal{M}})$ and $\widetilde{R} \in \Gamma(T_3^1 \overline{\mathcal{M}})$, for all $X, Y, Z, V \in \Gamma(T\overline{\mathcal{M}})$ by:

$$\begin{aligned} \widetilde{A}(X, Y) &= \theta^1((\nabla_{E_0} R)(E_0, \mathring{Y})\mathring{X}), \\ \widehat{A}(X, Y, Z) &= \theta^1((\nabla_{\mathring{X}} R)(E_0, \mathring{Z})\mathring{Y}), \end{aligned}$$

$$\widetilde{B}(X,Y,Z) = \theta^1((\nabla_{E_0}R)(\mathring{Y},\mathring{Z})\mathring{X}),$$
$$\widehat{B}(X,Y,Z,V) = \theta^1((\nabla_{\mathring{X}}R)(\mathring{Z},\mathring{V})\mathring{Y}),$$
$$\widetilde{R}(X,Y)Z = \overline{\nabla_{E_0}R(\mathring{X},\mathring{Y})\mathring{Z}}.$$

Observe that, therefore, one has $\tilde{R} = D_0\overline{R}$, $\tilde{B} = D_0B + h(\overline{R})$, etc. in agreement with the previous formulae.

Remark 4.25. In the given basis $\{\partial_i\}$, $\widetilde{A}_{ij} = \widetilde{A}(\partial_i,\partial_j) = \nabla_0 R^1{}_{i0j}$, $\widehat{A}_{sij} = \widehat{A}(\partial_s,\partial_i,\partial_j) = \nabla_s R^1{}_{i0j}$, $\widetilde{B}_{ijk} = \widetilde{B}(\partial_i,\partial_j,\partial_k) = \nabla_0 R^1{}_{ijk}$, $\widehat{B}_{sijk} = \widehat{B}(\partial_s,\partial_i,\partial_j,\partial_k) = \nabla_s R^1{}_{ijk}$ and $\widetilde{R}^i{}_{jkl} = \overline{d}x^i(\widetilde{R}(\partial_k,\partial_l)\partial_j) = \nabla_0 R^i{}_{jkl}$. Moreover, from the symmetries of the curvature tensor it is obvious that

(1) $\widetilde{A}$ and $\widehat{A}$ are symmetric in the last two indices: $\widetilde{A}_{ij} = \widetilde{A}_{(ij)}$; $\widehat{A}_{sij} = \widehat{A}_{s(ij)}$.
(2) $\widetilde{B}$, $\widehat{B}$, $\widetilde{R}$ are skew-symmetric in their last two indices: $\widetilde{B}_{ijk} = \widetilde{B}_{i[jk]}$; $\widehat{B}_{sijk} = \widehat{B}_{si[jk]}$; $\widetilde{R}_{sijk} = \widetilde{R}_{si[jk]}$.
(3) $\widetilde{B}$, $\widehat{B}$, $\widetilde{R}$ satisfy a cyclic identity: $\widetilde{R}_{i[jkl]} = 0$; $\widetilde{B}_{[ijk]} = 0$; $\widehat{B}_{s[ijk]} = 0$.
(4) $\widetilde{R}$ also satisfies $\widetilde{R}_{ijkl} = -\widetilde{R}_{jikl}$ and $\widetilde{R}_{ijkl} = \widetilde{R}_{klij}$, so that it has all the symmetries of a Riemann tensor.

One can prove that, in addition, $\widetilde{B}_{ij}{}^k = \nabla_0 R^k{}_{j0i}$ and $\widehat{B}_{sij}{}^k = \nabla_s R^k{}_{j0i}$. We also point out the following basic relations:

$$\widetilde{R}_{ijkl} = -2\widehat{B}_{[ij]kl}, \tag{23}$$
$$\widetilde{B}_{kij} = 2\widehat{A}_{[ij]k}, \tag{24}$$
$$\widehat{B}_{[i|l|jk]} = 0 \quad (\text{so } \widehat{B}_{[i|l|j]k} = -\tfrac{1}{2}\widehat{B}_{klij}).$$

They follow by direct application of the second Bianchi identity $\nabla_{[\alpha}R_{\beta\sigma]\rho\mu} = 0$ in the partly null frame $\{E_\alpha\}$: the first two relations follow by taking $\{\alpha,\beta,\sigma\} = \{0,i,j\}$ and the last one for $\{\alpha,\beta,\sigma\} = \{i,j,k\}$.

Using all the above, another direct computation of $\nabla\nabla R$ leads to the following formulae:

$$\overline{\nabla\nabla R} = \overline{\nabla}\,\overline{\nabla}\,\overline{R}, \qquad \overline{\nabla_{E_0}\nabla R} = D_0\overline{\nabla}\,\overline{R}, \tag{25a}$$

$$\begin{aligned}\overline{\nabla\nabla_{E_0}R} &= \overline{\nabla}\,\widetilde{R} + C_{13}(t\otimes\overline{\nabla}\,\overline{R}),\\ \overline{\nabla_{E_0}\nabla_{E_0}R} &= D_0\widetilde{R} - C_1^1(h^\sharp\otimes\overline{\nabla}\,\overline{R}),\end{aligned} \tag{25b}$$

$$\begin{aligned}\overline{\theta^1(\nabla\nabla R)} &= \overline{\nabla}\widehat{B} - C_1^1(t\otimes\overline{\nabla}\,\overline{R}),\\ \overline{\theta^1(\nabla_{E_0}\nabla_s R)} &= D_0\widehat{B} + C_1^1(h\otimes\overline{\nabla}\,\overline{R}),\end{aligned} \tag{25c}$$

$$\begin{aligned}\overline{\theta^1(\nabla\nabla_{E_0}R)} &= \overline{\nabla}\widetilde{B} - C_1^1(t\otimes\widetilde{R}) + C_{13}(t\otimes\widehat{B}),\\ \overline{\theta^1(\nabla_{E_0}\nabla_{E_0}R)} &= D_0\widetilde{B} + C_1^1(h\otimes\widetilde{R}) - C_1^1(h^\sharp\otimes\widehat{B}),\end{aligned} \tag{25d}$$

$$\begin{aligned}\overline{\theta^1(\nabla\nabla R(E_0,\cdot))} &= \overline{\nabla}\widehat{A} - 2\mathcal{S}[C_{16}(t\otimes\widehat{B})],\\ \overline{\theta^1(\nabla_{E_0}\nabla R(E_0,\cdot))} &= D_0\widehat{A} + 2\mathcal{S}[C_4^1(h^\sharp\otimes\widehat{B})],\end{aligned} \tag{25e}$$

$$\overline{\theta^1(\nabla\nabla_{E_0}R(E_0,\cdot))} = \overline{\nabla}\widetilde{A} - 2\mathcal{S}[C_{15}(t\otimes\widetilde{B})] + C_{13}(t\otimes\widehat{A}), \tag{25f}$$

$$\overline{\theta^1(\nabla_{E_0}\nabla_{E_0}R(E_0,\cdot))} = D_0\widetilde{A} + 2\mathcal{S}[C_3^1(h^\sharp\otimes\widetilde{B})] - C_1^1(h^\sharp\otimes\widehat{A}). \tag{25g}$$

Recall that, for example, the equations in the first row can be written as

$$\nabla_m\nabla_s R^i{}_{jkl} = \overline{\nabla}_m\overline{\nabla}_s\overline{R}^i{}_{jkl}, \qquad \nabla_0\nabla_s R^i{}_{jkl} = D_0\overline{\nabla}_s\overline{R}^i{}_{jkl}. \tag{26}$$

4.4. Reducibility and a generalized Eisenhart theorem

The proof of our main Theorem 1.1 will be carried out by means of some local decompositions of the Brinkmann space. To that end, we need an appropriate notion of reducibility and a version of a classical theorem by Eisenhart adapted to Brinkmann spaces.

Definition 4.26. A Brinkmann decomposition $\{u, v\}$ of a Brinkmann space (M, g) is *spatially reducible* if there exists a Brinkmann chart $\{u, v, x^i\}$ around each point and a partition of the indices $I_1 = \{2, \dots, d+1\}, I_2 = \{d+2, \dots, n-1\}$ for some $d \in \{1, \dots, n-3\}$ such that $\overline{g}_{aa'} = 0$ and $\partial_{a'}\overline{g}_{ab} = 0$, where the unprimed indices a, b always belong to the same subset I_m ($m \in \{1, 2\}$) and the primed ones a', b' to the other one.

For such a Brinkmann chart, we say that $T \in \Gamma(T^r_s\overline{\mathcal{M}})$ is *reducible* whenever $T = T^{(1)} + T^{(2)}$ with $T^{(m)} = T^{(m)a_1\dots a_r}_{b_1\dots b_s}(u, x^c)\partial_{a_1}\otimes\cdots\otimes\partial_{a_r}\otimes\overline{d}x^{b_1}\otimes\cdots\otimes\overline{d}x^{b_s}$ and $a_1, \dots, a_r, b_1, \dots, b_s, c \in I_m$. We will denote this decomposition as $T = T^{(1)}\oplus T^{(2)}$.

Remark 4.27. (1) Observe that, if a Brinkmann decomposition is spatially reducible, then the metric $\overline{g}$ on $T\overline{\mathcal{M}}$ is reducible as a tensor field.

(2) Spatial reducibility implies the following property, which provides a more intrinsic expression for some of its consequences. For some Brinkmann decomposition $\{u, v\}$ there exist two foliations $\overline{\mathcal{M}}^{(1)}, \overline{\mathcal{M}}^{(2)}$ of M such that $T\overline{\mathcal{M}} = T\overline{\mathcal{M}}^{(1)}\oplus T\overline{\mathcal{M}}^{(2)}$, and this sum is orthogonal with respect to $\overline{g}$. In this case, we write $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{(1)}\times\overline{\mathcal{M}}^{(2)}$ and $\overline{g} = \overline{g}^{(1)}\oplus\overline{g}^{(2)}$, according to the notation in Definition 4.26. Then the metric g on M can be written as

$$g = -2du(dv + Hdu + \mathring{W}) + \mathring{\overline{g}}^{(1)}\oplus\mathring{\overline{g}}^{(2)}$$

(recall that H and W depend on the chosen coordinates $\{x^i\}$). In this context, a tensor field T is reducible if and only if $T = T^{(1)} + T^{(2)}$ where each $T^{(m)}$ is invariant under the flow of vectors in $\overline{\mathcal{M}}^{(m')}$ ($\mathcal{L}_X T^{(m)} = 0$ for all $X \in \Gamma(T\overline{\mathcal{M}}^{(m')})$) and it vanishes when applied to any element of $T\overline{\mathcal{M}}^{(m')}$ and $T^*\overline{\mathcal{M}}^{(m')}$. In particular, this happens for $\overline{g}$. The Riemannian manifold $(\overline{M}, \overline{g})$ can be written as the product of two manifolds which will also be denoted, abusing notation, by $(\overline{M}^{(1)}, \overline{g}^{(1)})$ and $(\overline{M}^{(2)}, \overline{g}^{(2)})$, each $\overline{M}^{(m)}$ generating $\overline{\mathcal{M}}^{(m)}$ as $\overline{M}$ generated $\overline{\mathcal{M}}$ (see Section 4). Thus, the equality $\overline{g} = \overline{g}^{(1)}\oplus\overline{g}^{(2)}$ may refer either to the metric decomposition in a leaf $\overline{M}_{(u_0,v_0)}$ or in $\overline{\mathcal{M}}$. The latter depends on u and is v-invariant. Even though this ambiguity is harmless, we will always refer to decompositions in $\overline{\mathcal{M}}$ unless otherwise specified.

(3) Sometimes, a Brinkmann chart may admit a partition $I_1, \dots, I_s$, $s \geq 2$, of the indices $\{2, \dots, n-1\}$ so that $\overline{g} \in \Gamma(T^0_2\overline{\mathcal{M}})$ has the properties given in Definition 4.26

for each $i, j \in I_m$, $k', l' \in I_{m'}$ and $m \neq m'$. In this case, the notation for orthogonal decomposition is naturally extended:

$$\overline{M} = \overline{M}^{(1)} \times \cdots \times \overline{M}^{(s)}, \quad \overline{\mathcal{M}} = \overline{\mathcal{M}}^{(1)} \times \cdots \times \overline{\mathcal{M}}^{(s)}, \quad \overline{g} = \overline{g}^{(1)} \oplus \cdots \oplus \overline{g}^{(s)},$$

and notions like being Einstein or flat are also extended to each $\overline{M}^{(m)}$, $\overline{\mathcal{M}}^{(m)}$ in a trivial way. However, the following caution must be kept in mind. The given decomposition of $\overline{\mathcal{M}}$ induces also an orthogonal decomposition at each leaf $\overline{M}_{(u,v)}$, in particular at $\overline{M}$. Nevertheless, as $\overline{g}$ is "u-dependent" such a decomposition may be irreducible in the sense of the traditional de Rham theorem for some leaves, but reducible for other ones. In principle, we will not care about this possible "spatial irreducibility" of the metrics $\overline{g}^{(m)}$ on the leaves. Eventually, though, we will arrive at a decomposition of $\overline{g}$ which will induce an irreducible decomposition of all the leaves, independent of u.

Let us turn to the Eisenhart theorem. Its classical version [17] states:

Theorem 4.28. *If a Riemannian manifold (N, g_R) admits a symmetric two-covariant tensor field $L \in \Gamma(T_2^0 N)$ not proportional to the metric g_R such that $\nabla^{g_R} L = 0$, then*

- *g_R is reducible: $g_R = g_R^{(1)} \oplus \cdots \oplus g_R^{(s)}$,*
- *$L = \sum_{m=1}^{s} \lambda_m \, g_R^{(m)}$ for some constants λ_m.*

Our aim now is to prove a version of this theorem adapted to the spatial reducibility of Definition 4.26 for Brinkmann decompositions. In our generalized version, the reduced metrics $\overline{g}^{(m)}$ will depend on u but the λ_m will still be constants, independent of u.

Theorem 4.29. *Let (M, g) be a Brinkmann space and fix a Brinkmann chart $\{u, v, x^i\}$. Assume that there exists a symmetric v-invariant, $\overline{\nabla}$-parallel and D_0-parallel section $\overline{L} \in \Gamma(T_2^0 \overline{\mathcal{M}})$ which is not proportional to $\overline{g}$. Then the Brinkmann decomposition $\{u, v\}$ is spatially reducible and $\overline{L}$ is reducible. Furthermore, the decomposition $\{u, v\}$ admits a Brinkmann chart $\{u, v, y^i\}$ such that:*

(1) *$\overline{g} = \overline{g}^{(1)} \oplus \cdots \oplus \overline{g}^{(s)}$ for some $s \geq 2$,*
(2) *$\overline{L} = \sum_{m=1}^{s} \lambda_m \, \overline{g}^{(m)}$ for some constants $\lambda_m \in \mathbb{R}$.*

Proof. Let p be any point of the chart. We construct an orthonormal basis of eigenvector fields $V_i \in T\overline{\mathcal{M}}$ defined on the hypersurface $\Omega_{v_p} = \{v = v_p\}$: consider the eigenvalue problem for $\overline{L}_p$ with respect to $\overline{g}_p$ on the vector space $T_p\overline{\mathcal{M}}$, i.e.,

$$\overline{L}_p(\cdot, \vec{v}) - \lambda \overline{g}_p(\cdot, \vec{v}) = 0, \tag{27}$$

and take an orthonormal basis $\{\vec{v}_i\}_{i=2}^{n-1}$ of eigenvectors of $\overline{L}_p$ in $T_p\overline{M}_{(u_p,v_p)}$. Extend this basis to a normal neighborhood $\overline{U}$ of p in the leaf $\overline{M}_{(u_p,v_p)}$ by defining $V_i|_q$ at each $q \in \overline{U}$ as the vector obtained by $\overline{\nabla}$-parallel transport of $\vec{v}_i$ along the unique geodesic $\gamma_q : [0, 1] \to \overline{U}$ from p to q. Clearly, if $\vec{v}_i$ is a λ-eigenvector, then $V_i|_q$ is an eigenvector of $\overline{L}_q$ with the same eigenvalue λ, because the one-forms on γ_q defined as $\tau \mapsto \overline{L}_{\gamma_q(\tau)}(\cdot, V_{\gamma_q(\tau)})$ and $\tau \mapsto \lambda \overline{g}_{\gamma_q(\tau)}(\cdot, V_{\gamma_q(\tau)})$ are parallel and coincide at p due to (27).

Moreover, $\{V_i\}$ is an orthonormal basis on $\overline{U}$. Now, obtain the sought-for basis $\{V_i\}$ on Ω_{v_p} by propagating each $V_i|_q$ in a D_0-parallel manner along the integral curve of ∂_u at $q \in \overline{U}$. Since $\overline{L}$ is D_0-parallel, $\{V_i\}$ is still an orthonormal basis of eigenvector fields of $\overline{L}$ and the eigenvalues of $\overline{L}$ are constant on Ω_{v_p}.

From v-invariance, the eigenvalues and the dimension of each eigenspace remain constant all over M. Therefore, if we denote any λ-eigenvector field as $V^{(\lambda)}$, and $\lambda_1, \ldots, \lambda_s$ are the eigenvalues of $\overline{L}$ with corresponding multiplicities $m_1, \ldots, m_s$, we can reorder the vector fields so that

$$\{V_1^{(\lambda_1)}, \ldots, V_{m_1}^{(\lambda_1)}, \ldots, V_1^{(\lambda_s)}, \ldots, V_{m_s}^{(\lambda_s)}\}$$

is an orthonormal basis of $T\overline{\mathcal{M}}$.

Let λ be one of the eigenvalues and let us prove that the distribution S_λ generated by its eigenvectors is involutive. Taking the $\overline{\nabla}_V$ covariant derivative of $\overline{L}(\cdot\,, V_i^{(\lambda)}) = \lambda\overline{g}(\cdot, V_i^{(\lambda)})$ for any $V \in \Gamma(T\overline{\mathcal{M}})$ and using the fact that $\overline{L}$ is $\overline{\nabla}$-parallel, we find that $\nabla_V V_i^{(\lambda)}$ lies in S_λ, and so does $[V_i^{(\lambda)}, V_j^{(\lambda)}]$. Analogously, the distribution $S_\lambda^\perp$ which assigns to each point p' the orthogonal complement of $(S_\lambda)_{p'}$ in $T_{p'}\overline{\mathcal{M}}$ is involutive. Regarding $S_\lambda^\perp$ as a distribution contained in $T\Omega$, there are $m_\lambda + 1$ functionally independent functions on Ω which are solutions of the equation $X(f) = 0$ for all $X \in S_\lambda^\perp$. The first of these functions can be chosen as $u|_\Omega$ for all λ. The other functions y_λ^i, $i = 1, \ldots, m_\lambda$, will complete a coordinate chart for Ω when the construction is repeated for all the eigenvalues $\lambda = \lambda_j$, $j = 1, \ldots, s$. From these coordinates $\{u|_\Omega, y_{\lambda_j}^{i_j} : i_j = 1, \ldots, m_{\lambda_j}, j = 1, \ldots, s\}$ on Ω we can construct a chart on M by extending the previous functions in a v-invariant way according to Remark 4.3, including the coordinate v. □

Remark 4.30. Clearly, under the hypotheses of Theorem 4.29, $\mathring{L} \in \Gamma(TM)$ is also diagonalizable (observe that ∂_u and ∂_v are associated to the eigenvalue 0). However, the hypotheses on $\overline{L}$ are not enough to ensure that $\mathring{L}$ is parallel for (M, g).

5. Proper 2nd-symmetric Lorentzian manifolds

We are now equipped with all the elements that will allow us to solve the problem of 2nd-symmetry. The only remaining task is to solve the equations for 2nd-symmetry, given by setting all the expressions in (25) equal to zero. We are going to do this in several steps.

5.1. Reduction of the equations

We start by proving a fundamental simplification of 2nd-symmetric Brinkmann spaces (that is, of all proper 2nd-symmetric spaces), interesting in its own right.

Theorem 5.1. *Let (M, g) be a Brinkmann space with a fixed Brinkmann decomposition $\{u, v\}$. Then, if (M, g) is a 2nd-symmetric manifold, it follows that:*

(1) *the foliation $\overline{\mathcal{M}}$ is locally symmetric, i.e., $\overline{\nabla}\,\overline{R} = 0$,*
(2) *for any associated Brinkmann chart, the tensor fields $\widehat{B}$, $\widetilde{R}$, $\widehat{A}$, $\widetilde{B}$ in Definition 4.24 vanish,*
(3) *the scalar curvature S of the manifold is constant.*

The proof of this theorem will be carried out in two steps, one involving the intrinsic geometry of $\overline{\mathcal{M}}$ (beginning of Section 5.1.1), and the other involving the integrability equations (Propositions 5.6 and 5.7 in Section 5.1.3). An important consequence of this theorem, to be used later, is:

Corollary 5.2. *Let (M, g) be a 2nd-symmetric Brinkmann space with a fixed Brinkmann decomposition $\{u, v\}$. Then the section $\widetilde{A}$ is a tensor field on $\overline{\mathcal{M}}$, independent of the chosen Brinkmann chart. Moreover, (M, g) is proper 2nd-symmetric if and only if $\widetilde{A} \neq 0$.*

Proof. From Theorem 5.1, the only non-zero components of ∇R in any partly null frame associated to a Brinkmann chart $\{u, v, x^i\}$ are $\nabla_0 R^1{}_{i0j}$ $(= \widetilde{A}_{ij})$. Now, it is straightforward to check that

$$\widetilde{A}_{i'j'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \widetilde{A}_{ij}$$

under a change of the partly null frame associated to a transformation of the type (7). In particular, $\widetilde{A}$ behaves as a tensor field on $\overline{\mathcal{M}}$ for the given Brinkmann decomposition $\{u, v\}$. □

5.1.1. First step: local symmetry of $\overline{\mathcal{M}}$. To prove (1) of Theorem 5.1, observe that the first equation in (26) (when set to zero) together with Proposition 4.10 imply the result immediately. At this stage, we can also prove (3) *if we assume* (2): given that $S = \overline{S}$ and due to $\overline{\nabla}\,\overline{R} = 0$ the function $\overline{S}$ depends only on u. Thus, assuming $\widetilde{R} = 0$, which is part of (2), and recalling that $\widetilde{R} = D_0\overline{R}$, it follows that $0 = D_0\overline{S}$, so that $S = \overline{S}$ is constant, as required.

Using (25) and Theorem 5.1(1), the equations of 2nd-symmetry thus become

$$\overline{\nabla}_n \widehat{B}_{sijk} = 0, \qquad D_0 \widehat{B}_{sijk} = 0, \tag{28a}$$
$$\overline{\nabla}_s \widetilde{R}^i{}_{jkl} = 0, \qquad D_0 \widetilde{R}^i{}_{jkl} = 0, \tag{28b}$$
$$\overline{\nabla}_k \widehat{A}_{sij} = 2t^r{}_k \widehat{B}_{s(ij)r}, \qquad D_0 \widehat{A}_{sij} = -2h^r \widehat{B}_{s(ij)r}, \tag{28c}$$
$$\overline{\nabla}_s \widetilde{B}_{ijk} = t^r{}_s(\widetilde{R}_{rijk} - \widehat{B}_{rijk}), \qquad D_0 \widetilde{B}_{ijk} = -h^r(\widetilde{R}_{rijk} - \widehat{B}_{rijk}), \tag{28d}$$
$$\overline{\nabla}_k \widetilde{A}_{ij} = t^r{}_k(2\widetilde{B}_{(ij)r} - \widehat{A}_{rij}), \tag{28e}$$
$$D_0 \widetilde{A}_{ij} = -h^r(2\widetilde{B}_{(ij)r} - \widehat{A}_{rij}), \tag{28f}$$
$$\overline{\nabla}_s \overline{R}^i{}_{jkl} = 0, \tag{28g}$$

where, for completeness, we include the expressions for all these objects in this notation:

$$\widetilde{R}^i{}_{jkl} = D_0 \overline{R}^i{}_{jkl}, \tag{28h}$$
$$\widehat{B}_{sijk} = \overline{\nabla}_s B_{ijk} - t_{rs} \overline{R}^r{}_{ijk}, \tag{28i}$$

$$\widetilde{B}_{ijk} = D_0 B_{ijk} + h_r \overline{R}^r{}_{ijk}, \tag{28j}$$
$$\widehat{A}_{sij} = \overline{\nabla}_s A_{ij} - 2t^k{}_s B_{(ij)k}, \tag{28k}$$
$$\widetilde{A}_{ij} = D_0 A_{ij} + 2h^k B_{(ij)k}, \tag{28l}$$
$$B_{ijk} = \overline{\nabla}_k t_{ij} - \overline{\nabla}_j t_{ik}, \tag{28m}$$
$$A_{ij} = t_{ir} t^r{}_j - \overline{\nabla}_j h_i - D_0 t_{ij}, \tag{28n}$$
$$\overline{R}^i{}_{jkl} = R^i{}_{jkl}. \tag{28o}$$

5.1.2. Auxiliary algebraic results. As an interlude, before starting the second step we prove a couple of technical results about tensors with particular properties on a real vector space $\mathcal{V}$ of finite dimension. Tensors will be denoted in abstract index form.

Proposition 5.3. *Let $\mathcal{V}$ be an l-dimensional vector space with a positive definite inner product and T_{ijk} a three-covariant tensor such that:*

(a) *it is skew-symmetric in the last two indices: $T_{i[jk]} = T_{ijk}$,*
(b) *it satisfies a cyclic identity: $T_{ijk} + T_{jki} + T_{kij} = 0$.*

If

$$T_{(ij)}{}^r T_{rnm} = 0 \tag{29}$$

then $T_{ijk} = 0$.

Proof. Observe that, if we use (a), then (b) can be rewritten as $T_{[ijk]} = 0$. We use arguments inspired in [37, Lemma 4.1]. Suppose first that there does not exist a non-vanishing vector $v \in \mathcal{V}$ such that $v^r T_{rnm} = 0$. Define the set of vectors $\{Q^r(i,j)\}_{i,j=1}^l$ by $Q^r(i,j) = T_{(ij)}{}^r$ for each pair (i,j). Then equation (29) can be rewritten as $Q^r(i,j) T_{rnm} = 0$, that is, $Q^r(i,j) = 0$ for all i, j by our assumption. Consequently, $T_{(ij)k} = 0$, i.e., $T_{ijk} = T_{[ij]k}$. From this fact and (a), T_{ijk} is totally skew-symmetric, and so it vanishes by (b).

Suppose then that there does exist a non-vanishing vector $v \in \mathcal{V}$ such that $v^r T_{rnm} = 0$. Without loss of generality, we can assume that $\|v\| = 1$. The orthogonal splitting of T_{ijk} with respect to v is then

$$T_{ijk} = a_{ijk} + b_{ij} v_k - b_{ik} v_j \tag{30}$$

where

$$a_{[ijk]} = 0, \quad a_{i[jk]} = a_{ijk}, \quad a_{ijk} v^i = a_{ijk} v^j = 0,$$
$$b_{ij} = T_{ijk} v^k, \quad b_{ij} v^i (= T_{ijk} v^k v^i) = 0, \quad b_{ij} v^j = 0.$$

Since (b) implies $v^i (T_{ijk} + T_{jki} + T_{kij}) = 0$, we deduce from (30) that $b_{[jk]} = 0$, i.e., b_{ij} is a symmetric two-covariant tensor. But (29) implies $v^i v^m T_{(ij)}{}^r T_{rnm} = 0$, which in turn implies $b_j{}^r b_{rn} = 0$ by using (30) once more. Contracting the indices j, n and using that b_{ij} is symmetric, we find that $b_{ij} b^{ij} = 0$, and as the inner product is positive definite, $b_{ij} = 0$. Hence, $T_{ijk} = a_{ijk}$ so that T_{ijk} must actually be totally orthogonal to v. As the tensor a_{ijk} has the symmetries (a) and (b) of T_{ijk} but in the $(l-1)$-dimensional space $\langle v \rangle^\perp$, the result follows by induction. □

An immediate consequence, to be used later, is

Corollary 5.4. *Let $\mathcal{V}$ be an l-dimensional vector space with a positive definite inner product and T_{ijkl} a four-covariant tensor such that:*

(a) *it is skew-symmetric in the last two indices: $T_{si[jk]} = 0$,*
(b) *it satisfies the cyclic identity: $T_{ijkl} + T_{iklj} + T_{iljk} = 0$.*

If $T_{s(ij)}{}^r T_{lrnm} = 0$, then $T_{sijk} = 0$.

Proof. Let us consider for any $v \in \mathcal{V}$ the tensor $v^i T_{ijkl}$. It satisfies all the conditions in Proposition 5.3, so $v^i T_{ijkl} = 0$ for all $v \in \mathcal{V}$, that is, $T_{ijkl} = 0$. □

5.1.3. Second step: integrability equations. Obviously, 2nd-symmetry implies semi-symmetry. As a matter of fact, the equations of semi-symmetry happen to be some of the integrability conditions for the equations of 2nd-symmetry. To check this, and in order to exploit the integrability conditions in full, we introduce the expressions for the commutator of $\overline{\nabla}$ and D_0.

Proposition 5.5. *Given a fixed Brinkmann chart $\{u, v, x^i\}$:*

(1) *for any $T \in \Gamma(T^r_s\overline{\mathcal{M}})$, the Ricci identity reads*

$$(\overline{\nabla}_n\overline{\nabla}_s - \overline{\nabla}_s\overline{\nabla}_n)T^{i_1\dots i_r}_{j_1\dots j_s} = \sum_{b=1}^{s} \overline{R}^k{}_{j_b ns} T^{i_1\dots i_r}_{j_1\dots j_{b-1}kj_{b+1}\dots j_s} - \sum_{a=1}^{r} \overline{R}^{i_a}{}_{kns} T^{i_1\dots i_{a-1}ki_{a+1}\dots i_r}_{j_1\dots j_s}, \quad (31)$$

(2) *the commutator of D_0 with $\overline{\nabla}$ for any $F \in \Gamma(T^1_1\overline{\mathcal{M}})$ (trivially extendable to arbitrary $T \in \Gamma(T^r_s\overline{\mathcal{M}})$) is*

$$(\overline{\nabla}_k D_0 - D_0\overline{\nabla}_k)F^i{}_j = (H_{,k})(\partial_v F^i{}_j) + F^i{}_r B_{kj}{}^r - F^r{}_j B_{kr}{}^i - t^r{}_k\overline{\nabla}_r F^i{}_j,$$

and when F is v-invariant, this simplifies to

$$(\overline{\nabla}_k D_0 - D_0\overline{\nabla}_k)F^i{}_j = F^i{}_r B_{kj}{}^r - F^r{}_j B_{kr}{}^i - t^r{}_k\overline{\nabla}_r F^i{}_j. \quad (32)$$

Proof. (1) follows from the Ricci identities for $\overline{\nabla}$ on each leaf of $\overline{\mathcal{M}}$. To prove (2), using (22) and Remark 4.8 we get

$$\overline{\nabla}_k(D_0F^i{}_j) = \overline{\nabla}_k(\partial_u F^i{}_j - H\partial_v F^i{}_j - t^i{}_r F^r{}_j + t^r{}_j F^i{}_r),$$
$$D_0(\overline{\nabla}_k F^i{}_j) = D_0(\partial_k F^i{}_j + \overline{\Gamma}^i_{rk} F^r{}_j - \overline{\Gamma}^r_{jk} F^i{}_r).$$

The result follows by expanding these two expressions, and taking into account $\partial_v\overline{\Gamma}^i_{jk} = 0$ (Remark 4.8) plus formula (20). □

Now, we can easily prove the following statement:

Proposition 5.6. *Let (M, g) be a Brinkmann space with a fixed Brinkmann chart $\{u, v, x^i\}$. If (M, g) is a 2nd-symmetric manifold, the sections $\widehat{B}$ and $\widetilde{R}$ on $\overline{\mathcal{M}}$ vanish.*

Proof. By 2nd-symmetry, the integrability conditions associated to the first equations in (28a) and in (28c) read, respectively,

$$\overline{\nabla}_{[n}\overline{\nabla}_{m]}\widehat{B}_{ijkl} = 0, \quad \text{i.e.,}$$
$$\overline{R}^r{}_{snm}\widehat{B}_{rijk} + \overline{R}^r{}_{inm}\widehat{B}_{srjk} + \overline{R}^r{}_{jnm}\widehat{B}_{sirk} + \overline{R}^r{}_{knm}\widehat{B}_{sijr} = 0, \tag{33}$$
$$\overline{\nabla}_{[n}\overline{\nabla}_{m]}\widehat{A}_{sij} = B^r{}_{mn}\widehat{B}_{s(ij)r}, \quad \text{i.e.,}$$
$$\overline{R}^r{}_{snm}\widehat{A}_{rij} + \overline{R}^r{}_{inm}\widehat{A}_{srj} + \overline{R}^r{}_{jnm}\widehat{A}_{sir} = 2B^r{}_{nm}\widehat{B}_{s(ij)r}. \tag{34}$$

In both cases we used (31), and for the last case we also used the first equations in (28a) and in (28c), as well as (28m). If we differentiate (34) with respect to $\overline{\nabla}_k$, using the same information as before plus (28g), (28i) and (33), we obtain

$$\widehat{B}_{s(ij)}{}^r\widehat{B}_{lrnm} = 0.$$

Therefore, $\widehat{B}$ satisfies all the hypotheses in Corollary 5.4 at each leaf of $\overline{\mathcal{M}}$, so that $\widehat{B} = 0$ and, by the identity (23), $\widetilde{R} = 0$ too. □

This actually implies, due to (28h), (28g), (28d) and (28c), that the three sections $\overline{R}$, $\widetilde{B}$ and $\widehat{A}$ are $\overline{\nabla}$-parallel and D_0-parallel.

The integrability equations associated to (28e)–(28g), (28k), (28i) and the first equation in (28d) simplified via Proposition 5.6 are written, on using (28m) for the expressions containing $\overline{\nabla}t$, as

$$2\overline{\nabla}_{[n}\overline{\nabla}_{m]}\widetilde{B}_{ijk} = 0, \quad 2\overline{\nabla}_{[n}\overline{\nabla}_{m]}\widetilde{A}_{ij} = B^r{}_{mn}(2\widetilde{B}_{(ij)r} - \widehat{A}_{rij}), \quad \overline{\nabla}_{[n}\overline{\nabla}_{m]}\overline{R}_{ijkl} = 0,$$
$$2\overline{\nabla}_{[n}\overline{\nabla}_{m]}B_{ijk} = B^r{}_{mn}\overline{R}_{r(ij)k}, \quad \overline{\nabla}_{[n}\overline{\nabla}_{m]}A_{ij} = B^r{}_{mn}B_{(ij)r},$$

which via (31) provide

$$\overline{R}^r{}_{inm}\widetilde{B}_{rjk} + \overline{R}^r{}_{jnm}\widetilde{B}_{irk} + \overline{R}^r{}_{knm}\widetilde{B}_{ijr} = 0, \tag{35}$$
$$\overline{R}^r{}_{inm}\widetilde{A}_{rj} + \overline{R}^r{}_{jnm}\widetilde{A}_{ir} = B^r{}_{nm}(2\widetilde{B}_{(ij)r} - \widehat{A}_{rij}), \tag{36}$$
$$\overline{R}^r{}_{snm}\overline{R}_{rijk} + \overline{R}^r{}_{inm}\overline{R}_{srjk} + \overline{R}^r{}_{jnm}\overline{R}_{sirk} + \overline{R}^r{}_{knm}\overline{R}_{sijr} = 0, \tag{37}$$
$$\overline{R}^r{}_{inm}B_{rjk} + \overline{R}^r{}_{jnm}B_{irk} + \overline{R}^r{}_{knm}B_{ijr} = B^r{}_{nm}\overline{R}_{rijk}, \tag{38}$$
$$\overline{R}^r{}_{inm}A_{rj} + \overline{R}^r{}_{jnm}A_{ir} = 2B^r{}_{nm}B_{(ij)r}. \tag{39}$$

On the other hand, the integrability conditions derived from (32) applied to $\widehat{A}$, B and $\widetilde{A}$ become

$$B_{ms}{}^r\widehat{A}_{rij} + B_{mi}{}^r\widehat{A}_{irs} + B_{mj}{}^r\widehat{A}_{ijr} = 0, \tag{40}$$
$$B_{rjk}B_{mi}{}^r + B_{irk}B_{mj}{}^r + B_{ijr}B_{mk}{}^r = \overline{R}^r{}_{ijk}A_{rm}, \tag{41}$$
$$B_{mi}{}^r\widetilde{A}_{rj} + B_{mj}{}^r\widetilde{A}_{ir} = (2\widetilde{B}_{(ij)r} - \widehat{A}_{rij})A^r{}_m. \tag{42}$$

where we have used that $\widetilde{B}$ and $\widehat{A}$ are D_0-parallel and $\overline{\nabla}$-parallel together with (28n), (28f)–(28e) for $\widetilde{A}$ and (28i) reduced via Proposition 5.6 for B.

Using these integrability equations we can prove the following further improvement of Proposition 5.6, which completes the proof of Theorem 5.1.

Proposition 5.7. *Under the hypotheses of Proposition 5.6 the sections $\widetilde{B}$ and $\widehat{A}$ vanish.*

Proof. We use the results of Proposition 5.6 throughout the proof to simplify the formulae to be combined in the calculations. Start by applying D_0 to (38), and use (28h), (28j), (37) and (35) in order to get

$$\widetilde{B}^r{}_{mk}\overline{R}_{sijr} = 0. \tag{43}$$

Apply $\overline{\nabla}_n$ to (41) and use (28g), (28i), (28k) and (38) to obtain

$$\widehat{A}^r{}_{mk}\overline{R}_{sijr} = 0. \tag{44}$$

Applying D_0 to (36) and using (34), the second equation in (28c) and (28d), (28f), (28h), (28j) and (35) we get

$$\widetilde{B}^r{}_{mk}(2\widetilde{B}_{(ij)r} - \widehat{A}_{rij}) = 0. \tag{45}$$

However, if we apply D_0 to (39) and use (28h), (28j), (28l), (36) and (38) we derive $2\widetilde{B}^r{}_{mk}B_{(ij)r} + B^r{}_{mk}\widehat{A}_{rij} = 0$. By applying D_0 once more to this last expression and using the second equation in (28c) and (28d), (28j), (43) and (44) we also get

$$\widetilde{B}^r{}_{mk}(2\widetilde{B}_{(ij)r} + \widehat{A}_{rij}) = 0. \tag{46}$$

In conclusion, comparing (45) and (46), we see that

$$\widetilde{B}^r{}_{mk}\widetilde{B}_{(ij)r} = 0,$$

so that $\widetilde{B}$ satisfies all the hypotheses in Proposition 5.3 for each leaf of $\overline{\mathcal{M}}$, and thus $\widetilde{B} = 0$. By the identity (24), we also get $\widehat{A}_{[ij]k} = 0$, i.e., $\widehat{A}_{ijk} = \widehat{A}_{(ij)k}$. D_0-differentiating (42), using (40) and putting $\widetilde{B} = 0$ in (28f), (28j) and (36), we get $\widehat{A}_{nrk}\widehat{A}^r{}_{ij} = 0$. Contracting all the indices and using the fact that $\widehat{A}$ is symmetric in the first two, we arrive at $0 = \widehat{A}_{nrk}\widehat{A}^{rnk} = \widehat{A}_{rnk}\widehat{A}^{rnk}$, i.e., $\widehat{A} = 0$. □

Therefore, we obtain

Corollary 5.8. *$\widetilde{A}$ is also a D_0-parallel and $\overline{\nabla}$-parallel section.*

Remark 5.9. We simply remark that all the equations written in this section, be they 2nd-symmetry ones or their integrability conditions, are satisfied for any Brinkmann chart: if we perform a new decomposition of type (7), the aforementioned equations must be satisfied in the new Brinkmann chart too.

5.2. Transformation into two independent Lorentzian problems

5.2.1. Reducibility of $\overline{g}$, $\overline{\mathrm{Ric}}$ and $\widetilde{A}$. For the following application of the generalized Eisenhart theorem recall Definitions 4.9 and 4.26.

Proposition 5.10. *Let (M, g) be a 2nd-symmetric Brinkmann space. Then any Brinkmann decomposition $\{u, v\}$ satisfies*

$$\overline{\mathcal{M}} = \overline{\mathcal{M}}^{(1)} \times \overline{\mathcal{M}}^{(2)}, \qquad \overline{g} = \overline{g}^{(1)} \oplus \overline{g}^{(2)},$$

where $\overline{\mathcal{M}}^{(1)}$ is a d-dimensional flat foliation and $\overline{\mathcal{M}}^{(2)}$ is a d'-dimensional locally symmetric foliation with $d, d' \geq 0$ and $d+d' = n-2$. Furthermore, $\overline{\mathcal{M}}^{(2)}$ is itself the product of s_2 locally symmetric (non-Ricci flat) Einstein foliations:

$$\overline{\mathcal{M}}^{(2)} = \overline{\mathcal{M}}^{(2,1)} \times \cdots \times \overline{\mathcal{M}}^{(2,s_2)}, \qquad \overline{g}^{(2)} = \overline{g}^{(2,1)} \oplus \cdots \oplus \overline{g}^{(2,s_2)}.$$

Proof. As $\overline{R}$ is $\overline{\nabla}$-parallel and D_0-parallel it follows that $D_0\overline{\mathrm{Ric}} = 0$ and $\overline{\nabla}\,\overline{\mathrm{Ric}} = 0$. Now, if the foliation is u-Einstein, the result is trivial (put $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{(1)}$, $\overline{g} = \overline{g}^{(1)}$ and use Proposition 4.10(2) for the Ricci-flat case; otherwise, from $D_0\overline{\mathrm{Ric}} = 0$ the foliation is Einstein and $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{(2)}$, $\overline{g} = \overline{g}^{(2)}$). If the foliation is not u-Einstein, apply Theorem 4.29 to $\overline{L} = \overline{\mathrm{Ric}}$ and, if $\overline{\mathrm{Ric}}$ happens to have a vanishing eigenvalue, choose $\overline{\mathcal{M}}^{(1)}$ as the corresponding Ricci-flat part, which, by Proposition 4.10(2), is actually flat. ◻

Therefore, under the conditions of Proposition 5.10 with $d, d' > 0$, the partition of the indices corresponds to the reducibility of $\overline{g}$ according to Definition 4.26, where I_1 yields the indices of the flat foliation and I_2 the indices of the locally symmetric one. Moreover, the associated curvature tensors satisfy $\overline{R}^{(1)} = 0$ and $\overline{R}^{(2)} \neq 0$, so that $\overline{R} = \overline{R}^{(2)}$ and $\overline{\nabla}\,\overline{R}^{(2)} = 0$. Moreover,

$$\overline{\mathrm{Ric}} = \sum_{m=1}^{s_2} \mu_m \overline{g}^{(2,m)}, \qquad \mu_m \in \mathbb{R} - \{0\},$$

and the flat coordinates $\{x^2, \ldots, x^{d+1}\}$ correspond to the zero eigenvalue $\mu_0 = 0$.

Convention 5.11. From now on:

(1) When dealing with a proper 2nd-symmetric Brinkmann manifold and using Proposition 5.10, we will restrict ourselves to spatially reducible Brinkmann decompositions $\{u, v\}$ such that $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{(1)} \times \overline{\mathcal{M}}^{(2)}$ as in the proposition. This includes the limit case when $\overline{\mathcal{M}}$ is Einstein, so that either $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{(1)}$ $(d' = 0)$ or $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{(2)}$ $(d = 0)$.
(2) The indices $a, b, c, \ldots$ will run from 2 to $d + 1$ and the indices $a', b', c', \ldots$ will run from $d + 2$ to $n - 1$.
(3) We will denote by μ^* any of the *non-zero* eigenvalues of $\overline{\mathrm{Ric}}$.

Next, our aim is to prove that $\widetilde{A}$ is reducible and it admits a similar decomposition to $\overline{\mathrm{Ric}}$. However, in contrast to $\overline{\mathrm{Ric}}$, the non-trivial part of $\widetilde{A}$ lies in $\overline{\mathcal{M}}^{(1)}$.

Proposition 5.12. *Choose any spatially reducible Brinkmann decomposition of a proper 2nd-symmetric Brinkmann space. Then $\widetilde{A}$ is reducible as $\widetilde{A}^{(1)} \oplus \widetilde{A}^{(2)}$ with $\widetilde{A}^{(2)} = 0$. In addition, there exists a Brinkmann chart $\{u, v, x^i\}$ such that $\overline{g}^{(1)} = \delta_{ab}dx^a dx^b$ and the matrix of components $(\widetilde{A}^{(1)}_{ij})$ is constant and diagonal.*

Proof. Applying D_0 to (41) and using (28l), (28j) and Proposition 5.7 together with the fact that $\overline{R}$ is D_0-parallel one derives

$$\overline{R}^r{}_{ijk}\widetilde{A}_{rm} = 0,$$

from which we get $\overline{R}^r{}_i\widetilde{A}_{rj} = 0$ for all i, j. However, $\widetilde{A} \neq 0$ by Corollary 5.2 so that, in the Brinkmann chart associated to the spatial reduction of Proposition 5.10, if we set $i \in I_2$, i.e., $i = b'$, we see that $\mu^*\widetilde{A}_{b'j} = 0$, that is, $\widetilde{A}_{b'j} = 0$ for all j and

$$\widetilde{A} = \widetilde{A}_{bc}(u, x^i)\overline{d}x^b\overline{d}x^c.$$

Now, from Corollary 5.8 we have $0 = \overline{\nabla}_{a'}\widetilde{A}_{bc} = \partial_{a'}\widetilde{A}_{bc}$ (the last equality because $\gamma^a_{ba'} = 0$), that is, $\widetilde{A}$ is reducible with $\widetilde{A}^{(2)} = 0$ and satisfies the hypotheses of Theorem 4.29. Moreover, the coordinate vector fields $\{\partial_{d+2}, \dots, \partial_{n-1}\}$ span a subspace S of the eigenspace S_0 associated to the zero eigenvalue of $\widetilde{A}$. Therefore, one can follow the steps of the proof of Theorem 4.29, but working just on $S^{\perp}$ ($X(x^a) = 0$ for all $X \in S$), to obtain

- $\overline{g}^{(1)} = \overline{g}^{(1,1)} \oplus \cdots \oplus \overline{g}^{(1,s_1)}$.
- $\widetilde{A} = \sum_{m=1}^{s_1} \lambda_m \overline{g}^{(1,m)}$.

As $\overline{g}^{(1)}$ is a flat metric, applying Proposition 4.10(3) to each of the mutually orthogonal $\overline{g}^{(1,m)}$, a further change of coordinates in the flat block yields $\overline{g}^{(1)} = \delta_{ab}dx'^a dx'^b$ (for example, use the proof of that proposition on the restriction to the block of coordinates associated to each $\overline{\mathcal{M}}^{(1,m)}$). In conclusion, in this Brinkmann chart the matrix of the tensor field $\widetilde{A}$ is a diagonal matrix of constants. □

Remark 5.13. Observe that the Brinkmann chart obtained in Proposition 5.12:

(1) maintains the Brinkmann decomposition $\{u, v\}$ of Proposition 5.10; as the tensor fields $\overline{\mathrm{Ric}}$ and $\widetilde{A}$ (Corollary 5.2) depend only on the decomposition, the conclusions obtained for them are independent of the remaining coordinates of the Brinkmann chart;
(2) is such that $\widetilde{A}$ and $\overline{\mathrm{Ric}}$ are orthogonal and occasionally they may vanish simultaneously for a subbundle on $\overline{\mathcal{M}}^{(1)}$;
(3) has a flat metric $\overline{g}^{(1)} = \delta_{ab}dx^a dx^b$ so that $\overline{\nabla}_a = \partial_a$; moreover, $\overline{\nabla}_{a'}T^a_b = \partial_{a'}(T^a_b)$.

5.2.2. The building blocks of the metric g. Until now, we have proven the reducibility of the metric $\overline{g}$ for any Brinkmann decomposition $\{u, v\}$. Our aim is to prove that, in fact, the 2nd-symmetry induces the existence of two simpler 2nd-symmetric Brinkmann spaces associated to the original Brinkmann space (M, g).

Lemma 5.14. *For a Brinkmann chart $\{u, v, x^i\}$ as in Proposition 5.12, the sections B and A in Definition 4.22 satisfy*

$$B_{aba'} = 0, \qquad B_{a'ab} = 0, \qquad B_{a'b'a} = 0, \qquad B_{aa'b'} = 0, \qquad A_{aa'} = 0.$$

Proof. The commutation property (32) applied to the $\overline{\nabla}$-parallel and D_0-parallel section $\overline{R}$ provides the following integrability condition:

$$B_{ms}{}^{r}\overline{R}_{rijk} + B_{mi}{}^{r}\overline{R}_{srjk} + B_{mj}{}^{r}\overline{R}_{sirk} + B_{mk}{}^{r}\overline{R}_{sijr} = 0.$$

(i) Contracting here i and k and
- taking $m = a$, $s = b$, $j = a'$, one gets $B_{ab}{}^{r}\overline{R}_{ra'} = B_{aba'}\mu^{*} = 0$, that is, $B_{aba'} = 0$,
- taking $m = a'$, $s = b'$, $j = a$, one gets $B_{a'a}{}^{r}\overline{R}_{b'r} = B_{a'ab'}\mu^{*} = 0$, that is, $B_{a'b'a} = 0$.

(ii) Using now $B_{[ijk]} = 0$, $B_{i[jk]} = 0$, and (i) above, and
- taking $i = a'$, $j = b'$ and $k = a$, one finds that $B_{aa'b'} = 0$,
- taking $i = a'$, $j = a$ and $k = b$, one finds that $B_{a'ab} = 0$.

(iii) Contracting i and s in (39) and taking $j = a, m = a'$, one derives $\overline{R}_{ra'}A^{r}{}_{a} = 2B_{(ia)}{}^{r}B_{ra'}{}^{i}$. Using (i) and (ii) above one arrives at $\mu^{*}A_{a'a} = 0$, so $A_{aa'} = 0$. □

Proposition 5.15. *For a Brinkmann chart $\{u, v, x^i\}$ as in Proposition 5.12, the sections t and h are reducible.*

Proof. We have to prove that

$$t_{aa'} = 0, \quad t_{a'a} = 0, \quad \partial_{a'}t_{ab} = 0, \quad \partial_{a}t_{a'b'} = 0, \quad \partial_{a}h_{a'} = 0, \quad \partial_{a'}h_{a} = 0.$$

The fact that $\overline{R}$ is D_0-parallel can be rewritten using (22) as

$$\dot{\overline{R}}_{ijkl} + t^{r}{}_{i}\overline{R}_{rjkl} + t^{r}{}_{j}\overline{R}_{irkl} + t^{r}{}_{k}\overline{R}_{ijrl} + t^{r}{}_{l}\overline{R}_{ijkr} = 0.$$

Taking $i = a$ one finds that $t^{r}{}_{a}\overline{R}_{rjkl} = 0$, so that contracting j and l and taking $k = a'$ yields

$$0 = t_{a'a} = -t_{aa'}. \tag{47}$$

By (28m) and Lemma 5.14 it follows that $0 = B_{aba'} = \overline{\nabla}_{a'}t_{ab} - \overline{\nabla}_{b}t_{aa'}$, which implies by Remark 5.13(2) and (47) that $\partial_{a'}t_{ab} = 0$. Analogously, $0 = B_{a'b'a} = \overline{\nabla}_{a}t_{a'b'} - \overline{\nabla}_{b'}t_{a'a}$ implies $\partial_{a}t_{a'b'} = 0$. Lemma 5.14 and (28n) yield $0 = A_{aa'} = t_{ar}t^{r}{}_{a'} - \overline{\nabla}_{a'}h_{a} - D_0 t_{aa'}$, so substituting (47) and using (22) and Remark 5.13(2) again gives $\partial_{a}h_{a'} = \partial_{a'}h_{a} = 0$. □

Theorem 5.16. *For a proper 2nd-symmetric Brinkmann space, there exists a spatially reducible Brinkmann decomposition $\{u', v'\}$ such that the function H and the one-form section W in (3) are reducible in the associated Brinkmann chart $\{u', v', x'^{i}\}$. This chart is related to that obtained in Proposition 5.12 by $\{u', v', x'^{i}\} = \{u, v + f(u, x^{j}), x^{i}\}$ for some function f.*

Proof. We must prove that there exists a Brinkmann chart $\{u', v', x'^{i}\}$ such that

$$H(u', x'^{i}) = H^{(1)}(u', x'^{a}) + H^{(2)}(u', x'^{a'}),$$

$$W_a = W_a(u', x'^{b}) \; (= W^{(1)}_a(u', x'^{b})), \quad W_{a'} = W_{a'}(u', x'^{b'}) \; (= W^{(2)}_{a'}(u', x'^{b'})).$$

Let $\{u, v, x^i\}$ be the Brinkmann chart obtained in Proposition 5.12, so that by Proposition 5.15, t and h are reducible.

Simplification of W. $\partial_{a'} t_{ab} = 0$ implies that $W_{a,b} - W_{b,a}$ depends only on the coordinates $\{u, x^a\}$, hence there exists a function $f_1(u, x^i)$ such that $W_a(u, x^j) = f_{1,a}(u, x^j) + w_a(u, x^b)$. Analogously, $\partial_a t_{a'b'} = 0$ implies the existence of a function $f_2(u, x^i)$ such that $W_{a'}(u, x^i) = f_{2,a'}(u, x^i) + w_{a'}(u, x^{b'})$. Then (47) tells us that $f_{1,a,a'} = f_{2,a',a}$ so that $f_1(u, x^i) - f_2(u, x^i) = F(u, x^a) + G(u, x^{a'})$ for some functions F and G which can be absorbed into $w_a(u, x^b)$ and $w_{a'}(u, x^{b'})$, respectively. Consequently, there exists a function $f(u, x^j)$ such that

$$W_a(u, x^j) = f_{,a}(u, x^j) + w_a(u, x^b), \qquad W_{a'}(u, x^j) = f_{,a'}(u, x^j) + w_{a'}(u, x^{b'}). \tag{48}$$

Simplification of H. By using (12), $\partial_a h_{a'} = \partial_{a'} h_a = 0$ provides $H_{,a}(u, x^i) = h_a(u, x^b) + \dot{W}_a(u, x^i)$ and $H_{,a'}(u, x^i) = h_{a'}(u, x^{b'}) + \dot{W}_{a'}(u, x^i)$, which after use of (48) become $H_{,a}(u, x^i) = h_a(u, x^b) + \dot{f}_{,a}(u, x^i) + \dot{w}_a(u, x^b)$ and $H_{,a'}(u, x^i) = h_{a'}(u, x^{b'}) + \dot{f}_{,a'}(u, x^i) + \dot{w}_{a'}(u, x^{b'})$. Hence it is easy to deduce that

$$H(u, x^i) = \dot{f}(u, x^i) + H^{(1)}(u, x^a) + H^{(2)}(u, x^{a'})$$

for some functions $H^{(1)}, H^{(2)}$.

By choosing now the new Brinkmann decomposition defined by $v' = v + f(u, x^i)$, the conclusion follows on using (8) and (9). □

Remark 5.17. (1) What we have proven is that there exists a Brinkmann chart $\{u, v, x^i\}$ and a partition of the indices $I_1 = \{2, \ldots, d+1\}$, $I_2 = \{d+2, \ldots, n-1\}$ for some $d \in \{0, \ldots, n-2\}$ (recall Convention 5.11) such that $\overline{g}$, H and W are simultaneously reducible in the sense of Definition 4.26.

(2) Recall that if $\{u, v\}$ is spatially reducible, there exist two foliations $\overline{\mathcal{M}}^{(1)}, \overline{\mathcal{M}}^{(2)}$ with associated leaves $(\overline{M}^{(1)}, \overline{g}^{(1)})$, $(\overline{M}^{(2)}, \overline{g}^{(2)})$ such that $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{(1)} \times \overline{\mathcal{M}}^{(2)}$, $\overline{M} = \overline{M}^{(1)} \times \overline{M}^{(2)}$ and $\overline{g} = \overline{g}^{(1)} \oplus \overline{g}^{(2)}$ (Remark 4.27). Observe that the metric g can be written as

$$g = -2du(dv + (H^{(1)} + H^{(2)})du + \mathring{W}^{(1)} + \mathring{W}^{(2)}) + \mathring{\overline{g}}^{(1)} \oplus \mathring{\overline{g}}^{(2)},$$

so that to any 2nd-symmetric Brinkmann space we can associate a pair of lower-dimensional Brinkmann spaces $(M^{[m]}, g^{[m]})$, $m \in \{1, 2\}$, by $M^{[m]} = \mathbb{R}^2 \times \overline{M}^{(m)}$ and

$$g^{[m]} = -2du(dv + H^{(m)}du + W^{(m)}) + \mathring{\overline{g}}^{(m)}.$$

Of course, such a pair of spaces may be non-unique. $(M^{[m]}, g^{[m]})$ are the building blocks of any proper 2nd-symmetric Lorentzian manifold, and they are extremely helpful in the resolution of the equations for 2nd-symmetry because they actually simplify to the equations corresponding to each of the two simpler Brinkmann spaces $(M^{[m]}, g^{[m]})$, $m \in \{1, 2\}$, as the next proposition proves.

Proposition 5.18. *The pair* $(M^{[m]}, g^{[m]})$ *of Brinkmann spaces associated to any 2nd-symmetric Brinkmann space* (M, g) *according to the previous remark are themselves 2nd-symmetric.*

Proof. Note first of all that $\overline{R}$, h and t are reducible, so that A, B and their $\overline{\nabla}$-derivations and D_0-derivations are reducible too, hence $\widetilde{A}$ is also reducible. Therefore, for $m \in \{1, 2\}$, the reduced sections $\overline{R}^{(m)}$, $h^{(m)}$, $t^{(m)}$, $A^{(m)}$, $B^{(m)}$ correspond to the geometrical objects for the Brinkmann spaces $(M^{[m]}, g^{[m]})$ as defined in (11), (12), (13) and Definition 4.22 respectively. It is then straightforward to check that, in the Brinkmann chart of Theorem 5.16, the equations of 2nd-symmetry for (M, g) are exactly identical with the combination of the equations of 2nd-symmetry for the two building blocks $(M^{[m]}, g^{[m]})$. □

Remark 5.19. In conclusion, applying Proposition 5.18 to the Brinkmann decompositions $\{u', v'\}$ of Theorem 5.16, we can reorganize the equations of 2nd-symmetry in two simpler sets:

- The equations associated to the Brinkmann space $(M^{[1]}, g^{[1]})$ with coordinates $\{u, v, x^a\}$, such that $\overline{g}^{(1)} = \delta_{ab} dx^a dx^b$, $\overline{R}^{(1)} = 0$ (ergo $\overline{\mathrm{Ric}}^{(1)} = 0$) and $\widetilde{A}^{(1)} = \sum_{l=1}^{s_1} \lambda_l \overline{g}^{(1,l)}$ with some $\lambda_l \neq 0$ (in the proper case).
- Those associated to $(M^{[2]}, g^{[2]})$ with coordinates $\{u, v, x^{a'}\}$, such that $\overline{g}^{(2)} = \overline{g}^{(2,1)} \oplus \cdots \oplus \overline{g}^{(2,s_2)}$, $\overline{\nabla}\, \overline{R}^{(2,l)} = 0$ but $\overline{R}^{(2,l)} \neq 0$, $\overline{\mathrm{Ric}}^{(2)} = \sum_{l=1}^{s_2} \mu_l \overline{g}^{(2,l)}$ with each $\mu_l \neq 0$ and $\widetilde{A}^{(2)} = 0$.

5.3. *Proof of the main Theorem 1.1*

As a consequence of Corollary 3.8, we can apply Lemma 3.10, and consequently only the local version of the result must be proven. We will start the computations in the Brinkmann chart $\{u', v', x'^i\}$ of Theorem 5.16 but dropping the primes for clarity of notation.

By Remark 5.19, we first consider the equations for $(M^{[2]}, g^{[2]})$. Since $\widetilde{A}^{(2)} = 0$, Corollary 5.2 informs us that this is a locally symmetric Lorentzian manifold with a parallel lightlike vector field. Therefore, Theorem 3.4 implies that it is locally isometric to the product of a symmetric (non-flat) Riemannian space and $\mathbb{L}^2 = (\mathbb{R}^2, -2du dv)$. In particular, up to a change of coordinates of type

$$u' = u, \qquad v' = v + F(u, x^{a'}), \qquad y^{a'} = y^{a'}(u, x^{b'}), \tag{49}$$

$(M^{[2]}, g^{[2]})$ has $H^{(2)} = 0$, $W^{(2)} = 0$ and $\dot{\overline{g}}^{(2)} = 0$.

Consider now the 2nd-symmetry equations associated to $(M^{[1]}, g^{[1]})$ according to Remark 5.19. Using $\overline{R}^{(1)} = 0$ in (39) we have $B_{(ab)c} B^c{}_{de} = 0$, hence Proposition 5.3 leads us to

$$B^{(1)} = 0. \tag{50}$$

From (28m) it follows that $\overline{\nabla}_c t_{ab} - \overline{\nabla}_b t_{ac} = 0$, which together with (13) and Remark 5.13(2) provides $\partial_c t_{ab} - \partial_b t_{ac} = \partial_a t_{cb} = 0$. We conclude that $t^{(1)}$ is $\overline{\nabla}$-parallel. We have two consequences of this fact:

- $W_{a,b} - W_{b,a} = 2t_{ab}(u)$ depend only on u. Consequently, $W^{(1)}$ can be written as

  $$W_a(u, x^b) = f_{,a}(u, x^b) + t_{ac}(u) x^c$$

  for some function $f(u, x^b)$.

- Setting $\widehat{A} = 0$ in (28k) and by Lemma 5.14 and (50) we see that $\partial_c A_{ab} = 0$, which from (28n) and Proposition 5.15 implies that $\partial_c \partial_b h_a = 0$, that is, $h^{(1)}$ takes the form

$$h_a(u, x^b) = \Lambda_{ac}(u) x^c + B_a(u)$$

for some functions Λ_{ac} and B_a. From (12) we then have

$$\begin{aligned} H_{,a}(u, x^b) &= \Lambda_{ac}(u) x^c + B_a(u) + \dot{W}_a(u, x^b) \\ &= \Lambda_{ac}(u) x^c + B_a(u) + \dot{f}_{,a}(u, x^b) + \dot{t}_{ac}(u) x^c. \end{aligned}$$

Observe that the symmetry of $H_{,a,b}$ implies that $\Lambda_{ab} + \dot{t}_{ab} = \Lambda_{ba} + \dot{t}_{ba} = \Lambda_{(ab)}$ where the antisymmetric character of $\dot{t}$ has been used in the last equality.

In conclusion, the metric for $M^{[1]}$ becomes

$$g^{[1]} = -2du(dv + H^{(1)}(u, x^a)du + \mathring{W}^{(1)}) + \delta_{ab} dx^a dx^b \tag{51}$$

with

$$\begin{aligned} W^{(1)} &= W_a(u, x^b) \overline{d} x^a = [f_{,a}(u, x^b) + t_{ac}(u) x^c] \overline{d} x^a, \\ H^{(1)}(u, x^a) &= \dot{f}(u, x^a) + \tfrac{1}{2} \Lambda_{(bc)}(u) x^b x^c + B_c(u) x^c + C(u). \end{aligned}$$

Next, we use the following claim, to be proven later.

Claim 5.20. *For $(M^{[1]}, g^{[1]})$, there exists a change of Brinkmann chart of type*

$$u' = u, \qquad v' = v + \chi(u, x^a), \qquad y^a = R^a_b(u) x^b + D^a(u), \tag{52}$$

(where $R^a_b(u)$ is the matrix for a Euclidean rotation at each u) such that the metric becomes

$$g^{[1]} = -2du'(dv' + H'(u', y^a)du') + \delta_{ab} dy^a dy^b \tag{53}$$

where $H'(u', y^a) = -A_{ab}(u') y^a y^b$, A_{ab} being the components of the section A in Definition 4.22 associated to the Brinkmann chart $\{u', v', y^i\}$.

Combining the change of coordinates in this claim and the one given in (49), there exists a change of coordinates in the entire Brinkmann space (M, g) of type

$$\begin{aligned} u' &= u, & v' &= v + F(u, x^{a'}) + \chi(u, x^a), \\ y^{a'} &= y^{a'}(u, x^{b'}), & y^a &= R^a_b(u) x^b + D^a(u) \end{aligned}$$

such that the metric of (M, g) becomes

$$g = -2du'(dv' + H'(u', y^a)du') + \delta_{ab} dy^a dy^b + \overline{g}_{a'b'}(y^{c'}) dy^{a'} dy^{b'}$$

where $H'(u', y^a) = -A_{ab}(u') y^a y^b$. To end the proof, note that $\widetilde{A} = \dot{A}$ and therefore Corollary 5.8 gives, via $D_0 \widetilde{A} = 0$, $\ddot{A}_{ab}(u) = 0$. Of course, we need $\dot{A}_{a_0 b_0}(u) \neq 0$ for some a_0, b_0 in order for the manifold not to be locally symmetric, due to Corollary 5.2.

Proof of the claim. In order to find the required χ, D^a and R^a_b, put $H'(u', y^a) = -A_{ab}(u')y^a y^b$, substitute (52) in (53) and require that the resulting expression equals (51). Then the following equations arise:

$$\begin{aligned} &\dot{\chi} = H^{(1)} - H' + \tfrac{1}{2}\delta_{ab}(\dot{R}^a_c x^c + \dot{D}^a)(\dot{R}^b_d x^d + \dot{D}^b), \\ &\chi_{,a} = W_a + \tfrac{1}{2}\delta_{bc}\big((\dot{R}^b_d x^d + \dot{D}^b)R^c_a + (\dot{R}^c_d x^d + \dot{D}^c)R^b_a\big), \\ &\delta_{ab}R^a_c R^b_d = \delta_{cd}. \end{aligned} \tag{54}$$

Here, the known data are $t_{ab}(u)$, $B_c(u)$ and $\Lambda_{(ab)}(u)$, while the unknowns are $R^a_b(u)$, $D^b(u)$ and $\chi(u, x^a)$. The integrability conditions (given by the cross derivatives) of the first two expressions yield

$$t_{cd} = \tfrac{1}{2}\delta_{ab}(\dot{R}^a_c R^b_d - R^b_c \dot{R}^a_d), \tag{55}$$

$$\Lambda_{(cd)} = -2A_{be}R^b_c R^e_d - \tfrac{1}{2}\delta_{ab}(R^a_c \ddot{R}^b_d + R^b_c \ddot{R}^a_d), \tag{56}$$

$$B_c = -2A_{be}R^b_c D^e + \delta_{ab}\ddot{D}^a R^b_c. \tag{57}$$

To prove that these equations have solutions, proceed as follows. Define $R^c_b(u)$ as a solution of the ODE

$$\dot{R}^c_b = -\delta^{dc}(R^{-1})^a_d\, t_{ab}$$

for the given $t_{ab}(u)$. Then equation (55) is automatically satisfied. Concerning (54), note that, for such a solution, the derivative of $\delta_{cd}R^c_a R^d_b$ vanishes due to the antisymmetry of $t_{ab}(u)$. Therefore, by imposing as initial condition that $R^b_a(0)$ is any orthogonal matrix, the necessary condition (54) holds for all u (that is, $R^a_b(u)$ is a curve of rotation matrices, which can be chosen by using $d(d-1)/2$ free parameters codified in $R^b_a(0)$). Once $R^b_c(u)$ is determined, equation (56) fixes $A_{ab}(u)$ and, using this, the existence of $D^b(u)$ (which will depend on two new constant vectors, that is, on $2d$ new parameters) and, a posteriori, of χ, is ensured by (57) plus, again, standard results in differential equations. □

Remark 5.21. As (M_1, g_1) in Theorem 1.1 is a proper Cahen–Wallach space of order 2, let O^i_j be the orthogonal matrix that diagonalizes the symmetric matrix $A^{(1)}$ in (6). Then, performing a change of coordinates of type $u' = u - u_0$ and $y^i = O^i_j x^j$ we can write $A^{(1)}$ as a diagonal matrix and cancel one of the elements of the symmetric matrix $A^{(0)}$. In conclusion, the number of essential parameters of a proper 2nd-symmetric Lorentzian manifold with fixed $K = -\partial_v$ is given by $d - 1 + d(d+1)/2$ (observe that $A^{(1)}$ is a $d \times d$ square matrix). If K is not fixed, by a change of coordinates of type $u' = \varepsilon u - u_0$ with $v' = v/\varepsilon$ for some constant $\varepsilon \neq 0$ and $y^i = O^i_j x^j$, the same simplifications can be achieved and furthermore one of the non-zero eigenvalues of $A^{(1)}$ can be set to ± 1, in which case the number of essential parameters is one less.

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