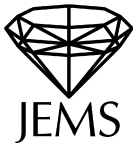
J. Eur. Math. Soc. 15, 731–754

DOI 10.4171/JEMS/374



Burt Totaro

Line bundles with partially vanishing cohomology

Received July 20, 2010 and in revised form October 22, 2010

Abstract. Define a line bundle L on a projective variety to be q-ample, for a natural number q, if tensoring with high powers of L kills coherent sheaf cohomology above dimension q. Thus 0-ampleness is the usual notion of ampleness. We show that q-ampleness of a line bundle on a projective variety in characteristic zero is equivalent to the vanishing of an explicit finite list of cohomology groups. It follows that q-ampleness is a Zariski open condition, which is not clear from the definition.

Ample line bundles are fundamental to algebraic geometry. The same notion of ampleness arises in many ways: geometric (some positive multiple gives a projective embedding), numerical (Nakai–Moishezon, Kleiman), or cohomological (Serre) [20, Chapter 1]. Over the complex numbers, ampleness of a line bundle is also equivalent to the existence of a metric with positive curvature (Kodaira).

The goal of this paper is to study weaker notions of ampleness, and to prove some of the corresponding equivalences. The subject began with Andreotti–Grauert's theorem that on a compact complex manifold X of dimension n, a hermitian line bundle L whose curvature form has at least $n-q$ positive eigenvalues at every point has $H^i(X, E \otimes L^{\otimes m}) = 0$ for every $i > q$, every coherent sheaf E on X, and every integer m at least equal to some m_0 depending on E [1]. Call the latter property *naive q-ampleness* of L, for a given natural number q. Thus naive 0-ampleness is the usual notion of ampleness, while every line bundle is naively n-ample. Demailly recently proved a form of the converse to Andreotti–Grauert's theorem for $(n-1)$-ample line bundles on a complex projective n-fold [9, Theorem 1.4]. We can try to understand naive q-ampleness on projective varieties over any field.

Sommese gave a clear geometric characterization of naive q-ampleness when in addition L is semi-ample (that is, some positive multiple of L is spanned by its global sections). In that case, naive q-ampleness is equivalent to the condition that the morphism to projective space given by some multiple of L has fibers of dimension at most q [28]. That has been useful, but the condition of semi-ampleness is restrictive, and in this paper we do not want to assume it. For example, the line bundle $O(a, b)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is naively 1-ample exactly when at least one of a and b is positive, whereas semi-ampleness would require both a and b to be nonnegative. Intuitively, q-ampleness means that a line bundle

B. Totaro: DPMMS, Cambridge University, Wilberforce Road, Cambridge CB3 0WB, England; e-mail: b.totaro@dpmms.cam.ac.uk

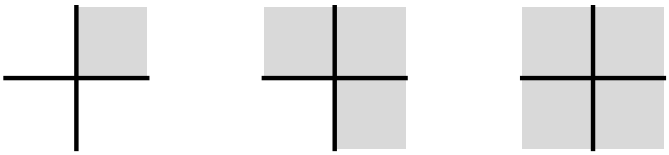


Fig. 1. The (open) naive q-ample cones in $N^1(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{R}^2$.

is positive "in all but at most q directions". One can hope to relate q-ampleness to the geometry of subvarieties of intermediate dimension.

The main results of this paper apply to projective varieties over a field of characteristic zero. In that situation, we show that naive q-ampleness (which is defined using the vanishing of infinitely many cohomology groups) is equivalent to the vanishing of finitely many cohomology groups, a condition we call q-T-ampleness (Theorem 6.3). A similar equivalence holds for all projective schemes over a field of characteristic zero (Theorem 7.1). These results are analogues of Serre's characterization of ampleness. Indeed, q-T-ampleness is an analogue of the geometric definition of ampleness by some power of L giving a projective embedding; the latter is also a "finite" condition, unlike the definition of naive q-ampleness. The equivalence implies in particular that naive q-ampleness is Zariski open on families of varieties and line bundles in characteristic zero, which is not at all clear from the definition.

Theorem 6.3 also shows that naive q-ampleness in characteristic zero is equivalent to uniform q-ampleness, a variant defined by Demailly–Peternell–Schneider [10]. It follows that naive q-ampleness defines an open cone (not necessarily convex) in the Néron–Severi vector space $N^1(X)$. (For example, the $(n-1)$-ample cone of an n-dimensional projective variety is the complement of the negative of the closed effective cone, by Theorem 9.1.) After these results, it makes sense to say simply "q-ample" to mean any of these equivalent notions for line bundles in characteristic zero.

The following tools are used for these equivalences. First, we use the relation found by Kawamata between Koszul algebras and resolutions of the diagonal (Theorem 2.1). Following Arapura [3, Corollary 1.12], we show that Castelnuovo–Mumford regularity behaves well under tensor products on any projective variety (Theorem 3.4). We prove the vanishing of certain Tor groups associated to the Frobenius homomorphism on any commutative $\mathbb{F}_p$-algebra (Theorem 4.1). Finally, Theorem 5.1 generalizes Arapura's positive characteristic vanishing theorem [3, Theorem 5.4] to singular varieties. The main equivalences of the paper, which hold in characteristic zero, are proved by the unusual method of reducing modulo many different prime numbers and combining the results.

Finally, we give a counterexample to a Kleiman-type characterization of q-ample line bundles. Namely, we define an $\mathbb{R}$-divisor D to be *q-nef* if $-D$ is not big on any $(q+1)$-dimensional subvariety of X. The q-nef cone is closed in $N^1(X)$ (not necessarily convex), and all q-ample line bundles are in the interior of the q-nef cone. The converse would be

a generalization of Kleiman's numerical criterion for ampleness (the case $q = 0$). This "Kleiman criterion" is true for $q = 0$ and $q = n - 1$, but Section 10 shows that it can fail for 1-ample line bundles on a complex projective 3-fold.

1. Notation

Define an $\mathbb{R}$-*divisor* on a projective variety X to be an $\mathbb{R}$-linear combination of Cartier divisors on X [20]. Two $\mathbb{R}$-divisors are called *numerically equivalent* if they have the same intersection number with all curves on X. We write $N^1(X)$ for the real vector space of $\mathbb{R}$-divisors modulo numerical equivalence, which has finite dimension.

The *closed effective cone* is the closed convex cone in $N^1(X)$ spanned by effective Cartier divisors. A line bundle is *pseudoeffective* if its class in $N^1(X)$ is in the closed effective cone. A line bundle is *big* if its class in $N^1(X)$ is in the interior of the closed effective cone. A line bundle L is big if and only if there are constants m_0 and $c > 0$ such that $h^0(X, L^{\otimes m}) \geq cm^n$ for all $m \geq m_0$, where n is the dimension of X [20, Vol. 1, Theorem 2.2.26].

Let X be a projective scheme of dimension n over a field. Assume that the ring $O(X)$ of regular functions on X is a field; this holds in particular if X is connected and reduced. Write $k = O(X)$. Define a line bundle $O_X(1)$ on X to be *N-Koszul,* for a natural number N, if the sections of $O_X(1)$ give a projective embedding of X and the homogeneous coordinate ring $A = \bigoplus_{j \geq 0} H^0(X, O(j))$ is N-Koszul. That is, the field k has a resolution as a graded A-module,

$$\cdots \to M_1 \to M_0 \to k \to 0,$$

with M_i a free module generated in degree i for $i \leq N$. (Note that Polishchuk–Positselski's book on Koszul algebras uses "N-Koszul" in a different sense [27, Section 2.4].) In particular, we say that a line bundle $O_X(1)$ is *Koszul-ample* if it is $2n$-Koszul. For example, the standard line bundle $O(1)$ on $\mathbb{P}^n$ is Koszul-ample. Backelin showed that a sufficiently large multiple of every ample line bundle on a projective variety is Koszul-ample (actually with M_i generated in degree i for all i, not just i at most $2n$, but we do not need that) [4]. Explicit bounds on what multiple is needed have been given [25, 11, 14].

One advantage of working with N-Koszulity for finite N rather than Koszulity in all degrees is that N-Koszulity is a Zariski open condition in families, as we will use in some arguments.

Given a Koszul-ample line bundle $O_X(1)$ on a projective scheme X, define the Castelnuovo–Mumford *regularity* of a coherent sheaf E on X to be the least integer m such that

$$H^i(X, E(m - i)) = 0$$

for all $i > 0$ [20, Vol. 1, Definition 1.8.4]. (Thus the regularity is $-\infty$ if E has zero-dimensional support.) We know that for any coherent sheaf E, $E(m)$ is globally generated and has vanishing higher cohomology for m sufficiently large, and one purpose of regularity is to estimate how large m has to be. Namely, if $\operatorname{reg}(E) \leq m$, then $E(m)$ is globally generated and has no higher cohomology [20, Vol. 1, Theorem 1.8.3]. We remark that if

"$H^i(X, E(m-i)) = 0$ for all $i > 0$" holds for one value of m, then it also holds for any higher value of m [20, Vol. 1, Theorem 1.8.5]. It is immediate from the definition that $\mathrm{reg}(E(j)) = \mathrm{reg}(E) - j$ for every integer j.

To avoid confusion, note that $\mathrm{reg}(O_X)$ can be greater than 0, in contrast to what happens in the classical case where X is $\mathbb{P}^n$ and $O_X(1)$ is the standard line bundle $O(1)$.

2. Resolution of the diagonal

In this section, we show that Koszul-ampleness of a line bundle $O_X(1)$ leads to an explicit resolution of the diagonal as a sheaf on $X \times X$. The resolution was constructed for $O_X(1)$ sufficiently ample by Orlov [24, Proposition A.1], and under the more convenient assumption that the coordinate ring of $(X, O_X(1))$ is a Koszul algebra by Kawamata [15, proof of Theorem 3.2]. Theorem 2.1 works out the analogous statement when $O_X(1)$ is only N-Koszul.

Let X be a projective scheme over a field. Assume that the ring $O(X)$ of regular functions on X is a field; this holds in particular if X is connected and reduced. Write $k = O(X)$. Let $O_X(1)$ be an N-Koszul line bundle, for a positive integer N. That is, $O_X(1)$ is very ample and the homogeneous coordinate ring $A = \bigoplus_{i \geq 0} H^0(X, O(i))$ is N-Koszul. (In later sections, we will work with a Koszul-ample line bundle $O_X(1)$, which means taking $N = 2\dim(X)$.)

Define vector spaces B_m inductively by $B_0 = k$, $B_1 = H^0(X, O(1))$, and

$$B_m = \ker\big(B_{m-1} \otimes H^0(X, O(1)) \to B_{m-2} \otimes H^0(X, O(2))\big).$$

(Products are over k unless otherwise specified.) By definition of N-Koszulity, the complex

$$B_N \otimes A(-N) \to \cdots \to B_1 \otimes A(-1) \to A \to k \to 0 \tag{1}$$

of graded A-modules is exact. (For an integer j and a graded module M, $M(j)$ means M with degrees lowered by j.) The vector space $\mathrm{Tor}_i^A(M, k)$ for a bounded-below A-module M can be viewed as the generators of the ith step of the minimal resolution of M. The Koszul resolution (1) of k as an A-module is clearly minimal, and so

$$B_m \cong \mathrm{Tor}_m^A(k, k)$$

for $0 \leq m \leq N$.

Let $\mathcal{R}_0 = O_X$, and

$$\mathcal{R}_m = \ker(B_m \otimes O_X \to B_{m-1} \otimes O_X(1))$$

for $m > 0$. The definition of B_m gives a complex of sheaves

$$0 \to \mathcal{R}_m \otimes_{O_X} O_X(-m+1) \to B_m \otimes_k O_X(-m+1) \to \cdots \to B_1 \otimes_k O_X \to O_X(1) \to 0. \tag{2}$$

This is exact for $0 < m \leq N$ by (1), using that (by Serre) a sequence of sheaves is exact if tensoring with $O(j)$ for j large and taking global sections gives an exact sequence. It follows that $\mathcal{R}_m$ is a vector bundle on X for $m \leq N$.

Theorem 2.1. *There is an exact sequence of sheaves on $X \times_k X$,*

$$\mathcal{R}_{N-1} \boxtimes O_X(-N+1) \to \cdots \to \mathcal{R}_1 \boxtimes O_X(-1) \to \mathcal{R}_0 \boxtimes O_X \to O_\Delta \to 0, \qquad (3)$$

where $\Delta \subset X \times_k X$ is the diagonal.

Here $E \boxtimes F$ denotes the external tensor product $\pi_1^*(E) \otimes \pi_2^*(F)$ on a product scheme $X_1 \times_k X_2$, for sheaves E on X_1 and F on X_2.

Proof. By Serre, this sequence of sheaves is exact if tensoring with $O(j, l)$ for all j and l sufficiently large and taking global sections gives an exact sequence. The definition of $\mathcal{R}_m$ implies that

$$\begin{aligned} H^0(X, \mathcal{R}_m(j)) &= \ker\bigl(B_m \otimes H^0(X, O(j)) \to B_{m-1} \otimes H^0(X, O(j+1))\bigr) \\ &= \ker(B_m \otimes A_j \to B_{m-1} \otimes A_{j+1}) \end{aligned}$$

for all $j \geq 0$. Thus we want to prove exactness of the complex of k-vector spaces

$$\begin{aligned} A_{l-N+1} \otimes \ker(B_{N-1} \otimes A_j \to B_{N-2} \otimes A_{j+1}) \to \cdots \\ \to A_{l-1} \otimes \ker(B_1 \otimes A_j \to A_{j+1}) \to A_l \otimes A_j \to A_{j+l} \to 0 \end{aligned} \qquad (4)$$

for all j and l sufficiently large. In fact, we will prove this for all $j, l \geq 0$.

Because A is an associative algebra with augmentation $A \to k$, the groups $\operatorname{Ext}_A^*(k, k)$ form an associative algebra (typically not graded-commutative, even when A is commutative). The product can be viewed as composition in the derived category of A. Since $\operatorname{Ext}_A^i(k, k) \cong \operatorname{Tor}_i^A(k, k)^*$, we can also say that $\operatorname{Tor}_*^A(k, k)$ is a coassociative coalgebra [27, Section 1.1]. Thus we have natural maps $B_{i+j} \to B_i \otimes B_j$ for i and j at most N. Coassociativity says in particular that the two compositions

$$B_i \to A_1 \otimes B_{i-1} \to A_1 \otimes B_{i-2} \otimes A_1 \quad \text{and} \quad B_i \to B_{i-1} \otimes A_1 \to A_1 \otimes B_{i-2} \otimes A_1$$

are equal, where we have identified B_1 with A_1. Also, there is a natural isomorphism $\operatorname{Ext}_A^*(k, k) \cong \operatorname{Ext}_{A^{\mathrm{op}}}^*(k, k)$ that reverses the order of multiplication [27, Section 1.1]. For a graded associative algebra A, it follows that A is N-Koszul if and only if A^{op} is N-Koszul. Therefore, for an N-Koszul algebra A, the following version of the Koszul complex (1) (using the maps $B_i \to A_1 \otimes B_{i-1}$ rather than $B_i \to B_{i-1} \otimes A_1$) is also exact:

$$A(-N) \otimes B_N \to \cdots \to A(-1) \otimes B_1 \to A \to k \to 0 \qquad (1')$$

Let us artificially define B_i to be zero for $i > N$. Consider the following triangular diagram, where the rows are obtained from the Koszul complex (1′) and the columns from the Koszul complex (1):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_0 \otimes B_0 \otimes A_{j+l} & & & & & \longrightarrow & 0 \\
 & & \uparrow & & & & & & \\
 & & \vdots & & & & 0 & & \\
 & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & A_0 \otimes B_{l-1} \otimes A_{j+1} & \longrightarrow \cdots \longrightarrow & A_{l-1} \otimes B_0 \otimes A_{j+1} & & & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \uparrow \\
0 & \longrightarrow & A_0 \otimes B_l \otimes A_j & \longrightarrow \cdots \longrightarrow & A_{l-1} \otimes B_1 \otimes A_j & \longrightarrow & A_l \otimes A_j & \longrightarrow & 0
\end{array}$$

The diagram commutes by the equality of the two maps $B_i \to A_1 \otimes B_{i-2} \otimes A_1$ mentioned above. View this diagram as having the group $A_0 \otimes B_0 \otimes A_{j+l}$ in position $(0, 0)$. Multiplying the vertical maps in odd columns by -1 makes this commutative diagram into a double complex C, meaning that the two composite maps in each square add up to zero. Compare the two spectral sequences converging to the cohomology of the total complex $\mathrm{Tot}(C)$ [22, Section 2.4]. In the first one, ${}_IE_0^{pq} = C^{pq}$ and the differential $d_0 : {}_IE_0^{pq} \to {}_IE_0^{p,q+1}$ is the vertical differential of C. Column p of C (for $0 \le p \le l$) is A_p tensored with the Koszul complex (1) in degree $j + l - p$, truncated at the step $\min\{l - p, N\}$. Therefore,

$${}_IE_1^{pq} = \begin{cases} A_p \otimes \ker(B_{l-p} \otimes A_j \to B_{l-p-1} \otimes A_{j+1}) & \text{if } q = -l \text{ and } p+q \ge -(N-1), \\ 0 & \text{if } q \ne -l \text{ and } p+q \ge -(N-1), \\ ? & \text{if } p+q \le -N. \end{cases}$$

So $H^p(\mathrm{Tot}(C))$ is isomorphic to the cohomology of the complex

$$\begin{aligned} A_{l-N+1} \otimes \ker(B_{N-1} \otimes A_j \to B_{N-2} \otimes A_{j+1}) \to \cdots \\ \to A_{l-1} \otimes \ker(B_1 \otimes A_j \to B_0 \otimes A_{j+1}) \to A_l \otimes A_j \to 0 \end{aligned}$$

for $-(N-2) \le p \le 0$, where $A_l \otimes A_j$ is placed in degree zero. (In this range, these groups in the E_1 term cannot be hit by any differential after d_1.)

The second spectral sequence converging to $H^*(\mathrm{Tot}(C))$ has ${}_{II}E_0^{pq} = C^{qp}$, and the differential $d_0 : {}_{II}E_0^{pq} \to {}_{II}E_0^{p,q+1}$ corresponds to the horizontal differential in C. Row $-r$ in C is the Koszul complex $(1')$ in degree r (with the group k omitted in the case $r = 0$), truncated at the Nth step, and tensored with A_{j+l-r}. So

$${}_{II}E_1^{pq} = \begin{cases} A_{j+l} & \text{if } p = q = 0, \\ 0 & \text{if } p+q \ge -(N-1) \text{ and } (p,q) \ne (0,0), \\ ? & \text{if } p+q \le -N. \end{cases}$$

Therefore $H^p(\mathrm{Tot}(C))$ is isomorphic to A_{j+l} if $p = 0$ and to zero if $-(N-1) \le p \le -1$ (although we only need this for $-(N-2) \le p \le -1$). Combining this with the previous description of $H^p(\mathrm{Tot}(C))$ gives the exact sequence (4), as we want. □

3. Castelnuovo–Mumford regularity of a tensor product

The properties of Castelnuovo–Mumford regularity discussed in Section 1 follow immediately from the classical case of sheaves on projective space. A deeper fact is that, *since $O_X(1)$ is Koszul-ample,* regularity behaves well under tensor products (Theorem 3.4). This will be used along with Lemma 3.3 in the proof of our main vanishing theorem, Theorem 5.1. The theorem that regularity behaves well under tensor products was proved by Arapura [3, Section 1], but the statements there have to be corrected slightly (we have to define Koszul-ampleness to be in degrees out to $2n$, not just n, for the proof of [3, Lemma 1.7] to work). Also, Theorem 2.1 simplifies the Koszulity assumption needed for these results, and [3, Section 1] works with a smooth variety, an assumption which can be dropped. So it seems reasonable to give the proofs here.

Throughout this section, let X be a projective scheme of dimension n over a field such that the ring $O(X)$ is a field (for example, any connected reduced projective scheme over a field). Write $k = O(X)$. Let $O_X(1)$ be a very ample line bundle, and define the vector bundles $\mathcal{R}_i$ as in Section 2. (We will only consider $\mathcal{R}_i$ when $O_X(1)$ is at least i-Koszul.)

Lemma 3.1. *Let E be a vector bundle and F a coherent sheaf on X. Let $i \geq 0$, and assume that $O_X(1)$ is $(2n-i+1)$-Koszul. Suppose that for each pair of integers $0 \leq a \leq 2n-i$ and $b \geq 0$, either $H^b(X, E \otimes \mathcal{R}_a) = 0$ or $H^{i+a-b}(X, F(-a)) = 0$. Then*

$$H^i(X, E \otimes F) = 0.$$

Proof. This is essentially [3, Lemma 1.6]. Theorem 2.1 gives the first $2n-i$ steps of a resolution of the diagonal on $X \times_k X$. Tensoring with $E \boxtimes F$ gives a resolution of $E \otimes F$ on the diagonal in $X \times X$:

$$(E \otimes \mathcal{R}_{2n-i}) \boxtimes F(-2n+i) \to \cdots \to (E \otimes \mathcal{R}_0) \boxtimes F(0) \to E \otimes F \to 0.$$

To check that the latter complex really is exact, we have to show that the sheaves $\operatorname{Tor}_i^{O_X \otimes_k O_X}(E \otimes_k F, O_X)$ are zero for $i > 0$. Since E is a vector bundle, it suffices to show that $\operatorname{Tor}_i^{O_X \otimes_k O_X}(O_X \otimes_k F, O_X) = 0$ for $i > 0$. But this is isomorphic to $\operatorname{Tor}_i^{O_X}(F, O_X)$, which is indeed zero for $i > 0$.

It follows that $H^i(X, E \otimes F)$ is zero if $H^{i+a}(X \times X, (E \otimes \mathcal{R}_a) \boxtimes F(-a)) = 0$ for $0 \leq a \leq 2n-i$. (Because $X \times X$ has dimension $2n$, it does not matter how the resolution continues beyond degree $2n-i$.) This vanishing follows from our assumption by the Künneth formula. □

For $q = 0$, the following is a well-known property of Castelnuovo–Mumford regularity [20, Vol. 1, Theorem 1.8.5]. Keeler observed that the proof generalizes to give [16, Lemma 2.2]:

Lemma 3.2. *Let $O_X(1)$ be a basepoint-free ample line bundle on a projective scheme X over a field. Let F be a coherent sheaf and q a natural number such that*

$$0 = H^{q+1}(X, F(-1)) = H^{q+2}(X, F(-2)) = \cdots. \qquad (C_q)$$

Then $F(1)$ also satisfies (C_q). That is, $0 = H^{q+1}(X, F) = H^{q+2}(X, F(-1)) = \cdots$.

We return to our standing assumptions in this section: $O_X(1)$ is a very ample line bundle on a projective scheme X such that the ring $O(X)$ is a field k, and X has dimension n over k.

Lemma 3.3. *Let F be a coherent sheaf on X. Let q be a natural number such that*

$$H^{q+1}(X, F(-1)) = H^{q+2}(X, F(-2)) = \cdots .$$

Then $H^j(X, \mathcal{R}_i \otimes F) = 0$ for $j > q$ if $O_X(1)$ is $(n - j + 1 + i)$-Koszul.

Proof. This is [3, Corollary 1.9], with the Koszulity assumption corrected. The method is to show more generally that for any $a \geq 0$, $i \geq 0$, and $q+a < j$, $H^j(X, \mathcal{R}_i \otimes F(-a)) = 0$ if $O_X(1)$ is $(n - j + 1 + i)$-Koszul. We prove this by descending induction on a, starting with $a \geq n$ where the result is automatic (since $j > q + a \geq n$). We can assume that $j \leq n$; otherwise the cohomology group we consider is automatically zero. Therefore, $N := n - j + 1 + i$ is at least $i + 1$. The sequence (2) in Section 2 gives an exact sequence of vector bundles

$$0 \to \mathcal{R}_{i+1} \otimes O_X(-1) \to B_{i+1} \otimes O_X(-1) \to \mathcal{R}_i \to 0,$$

since $i + 1 \leq N$. This gives an exact sequence of cohomology

$$B_{i+1} \otimes H^j(X, F(-a-1)) \to H^j(X, \mathcal{R}_i \otimes F(-a)) \to H^{j+1}(X, \mathcal{R}_{i+1} \otimes F(-a-1)).$$

The first group is zero by our assumption on F and Lemma 3.2, and the last group is zero by our descending induction on a. Thus $H^j(X, \mathcal{R}_i \otimes F(-a)) = 0$ as we want. □

We now generalize a standard property of Castelnuovo–Mumford regularity from sheaves on projective space to sheaves on an arbitrary reduced projective scheme. We follow Arapura's proof [3, Corollary 1.12], with the Koszulity assumption corrected.

Theorem 3.4. *Let X be a projective scheme of dimension n over a field such that the ring of regular functions on X is a field (example: X connected and reduced). Let $O_X(1)$ be a $2n$-Koszul line bundle on X. Let E be a vector bundle and F a coherent sheaf on X. Then*

$$\operatorname{reg}(E \otimes F) \leq \operatorname{reg}(E) + \operatorname{reg}(F).$$

Proof. By definition, $E(\operatorname{reg}(E))$ and $F(\operatorname{reg}(F))$ have regularity zero. So we can assume that E and F have regularity at most zero, and we want to show that $E \otimes F$ has regularity at most zero. That is, we have to show that

$$H^i(X, E \otimes F(-i)) = 0$$

for $i > 0$. For any $0 \leq a \leq n + b - 1$ and $b > 0$, we have $H^b(X, E \otimes \mathcal{R}_a) = 0$ by Lemma 3.3 (in particular, we have arranged for the Koszulity assumption in Lemma 3.3 to hold). For any $b > 0$ and any $n + b \leq a \leq 2n - i$, we have

$$H^{i+a-b}(X, F(-i-a)) = 0$$

for dimension reasons. Finally, for any $0 \leq a \leq 2n - i$ and $b = 0$,

$$H^{i+a-b}(X, F(-i-a)) = 0$$

since F has regularity at most zero. Then Lemma 3.1 gives that $H^i(X, E \otimes F(-i)) = 0$ for $i > 0$. □

4. Hochschild homology and the Frobenius homomorphism

In this section we prove a flatness property of the Frobenius homomorphism $F(x) = x^p$ for all commutative $\mathbb{F}_p$-algebras, to be used in the proof of our main vanishing theorem, Theorem 5.1. This seems striking, since the Frobenius homomorphism on a noetherian $\mathbb{F}_p$-algebra A is flat only for A regular [18, Corollary 2.7]. The statement is that the Tor groups of certain $A \otimes A$-modules vanish in positive degrees, or equivalently that certain Hochschild homology groups vanish in positive degrees. The proof is inspired by Pirashvili's proof of vanishing for a related set of Tor groups in positive characteristic [26].

Let k be a field of characteristic $p > 0$, and let N be a natural number. For a commutative k-algebra A, write $\widetilde{A} = k \otimes_k A$, where k maps to k by the Nth iterate of Frobenius, $x \mapsto x^{p^N}$. We view $\widetilde{A}$ as a k-algebra using the left copy of k. The *relative Frobenius homomorphism* $\varphi : \widetilde{A} \to A$ is the k-algebra homomorphism given by $\varphi(x \otimes y) = xy^{p^N}$ for $x \in k$, $y \in A$. Note that $\widetilde{A}$ depends on the fixed number N throughout the following proof.

Theorem 4.1. *For any commutative k-algebra A, view $\widetilde{A} \otimes_k A$ as a module over $\widetilde{A} \otimes_k \widetilde{A}$ by $x \otimes y \mapsto x \otimes \varphi(y)$, and $\widetilde{A}$ as a module over $\widetilde{A} \otimes_k \widetilde{A}$ by $x \otimes y \mapsto xy$. Then*

$$\operatorname{Tor}_i^{\widetilde{A} \otimes_k \widetilde{A}}(\widetilde{A} \otimes_k A, \widetilde{A}) \cong \begin{cases} A & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

The Hochschild homology of a k-algebra R with coefficients in an R-bimodule M can be defined as $H_i(R, M) = \operatorname{Tor}_i^{R \otimes_k R^{\mathrm{op}}}(M, R)$ [21, Proposition 1.1.13]. So an equivalent formulation of Theorem 4.1 is that

$$H_i(\widetilde{A}, \widetilde{A} \otimes_k A) \cong \begin{cases} A & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

Proof of Theorem 4.1. In this proof, all tensor products are over k unless otherwise specified. The theorem is easy in degree zero: an isomorphism

$$(\widetilde{A} \otimes A) \otimes_{\widetilde{A} \otimes \widetilde{A}} \widetilde{A} \to A$$

is given by mapping $(x \otimes y) \otimes 1$ to $\varphi(x)y$.

The vanishing of Tor in positive degrees is clear for R a free commutative k-algebra (that is, a polynomial ring, possibly on infinitely many variables). Indeed, the relative Frobenius homomorphism $\varphi : \widetilde{A} \to A$ is flat in this case, so that $\widetilde{A} \otimes A$ is a flat $\widetilde{A} \otimes \widetilde{A}$-module.

For an arbitrary commutative k-algebra, let $P_* \to A$ be a *free resolution* of A, meaning an A-augmented simplicial commutative k-algebra $P_* \to A$ which is acyclic and such that each P_i is a free commutative k-algebra. This exists [21, Section 3.5.1].

We are trying to show that Hochschild homology $H_i(\widetilde{A}, \widetilde{A} \otimes_k A)$ is zero for $i > 0$. The standard complex computing Hochschild homology $H_i(R, M)$ for a k-algebra R and an R-bimodule M consists of the k-vector spaces $C_n(R, M) = M \otimes R^{\otimes n}$. These form a

simplicial k-vector space, with boundary maps $d_i : M \otimes R^{\otimes n} \to M \otimes R^{\otimes n-1}$ given by [21, Section 1.1.1]:

$$\begin{aligned} d_0(m, a_1, \dots, a_n) &= (ma_1, a_2, \dots, a_n), \\ d_i(m, a_1, \dots, a_n) &= (m, a_1, \dots, a_i a_{i+1}, \dots, a_n) \quad \text{for } 1 \le i \le n-1, \\ d_n(m, a_1, \dots, a_n) &= (a_n m, a_1, \dots, a_{n-1}). \end{aligned}$$

So we can view Hochschild homology $H_i(R, M)$ as the homology groups of the chain complex $C_*(R, M)$ with boundary map

$$b = \sum_{i=0}^{n} (-1)^i d_i.$$

For any homomorphism $R \to S$ of k-algebras and homomorphism $M \to N|_R$ of R-bimodules, we have an obvious homomorphism $C_*(R, M) \to C_*(S, N)$ of the Hochschild chain complexes. Imitating the proof of [21, Theorem 3.5.8], we consider the homomorphisms between the chain complexes $C_*(P_n, \widetilde{P}_n \otimes P_n)$ given by the k-algebra homomorphisms $d_i : P_n \to P_{n-1}$ for $0 \le i \le n$. We get a commutative diagram, where the horizontal arrows are the alternating sum of the homomorphisms given by $d_0, \dots, d_n$:

$$\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \\
(\widetilde{P}_0 \otimes P_0) \otimes \widetilde{P}_0^{\otimes 2} & \longleftarrow & (\widetilde{P}_1 \otimes P_1) \otimes \widetilde{P}_1^{\otimes 2} & \longleftarrow & (\widetilde{P}_2 \otimes P_2) \otimes \widetilde{P}_2^{\otimes 2} & \longleftarrow & \\
{\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & \\
(\widetilde{P}_0 \otimes P_0) \otimes \widetilde{P}_0 & \longleftarrow & (\widetilde{P}_1 \otimes P_1) \otimes \widetilde{P}_1 & \longleftarrow & (\widetilde{P}_2 \otimes P_2) \otimes \widetilde{P}_2 & \longleftarrow & \\
{\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & \\
\widetilde{P}_0 \otimes P_0 & \longleftarrow & \widetilde{P}_1 \otimes P_1 & \longleftarrow & \widetilde{P}_2 \otimes P_2 & \longleftarrow &
\end{array}$$

Define a double complex M by changing b to $-b$ in the odd columns of this commutative diagram.

The homology of the ith column in the double complex M is the Hochschild homology of the polynomial ring P_i which we have already computed:

$$H_j(\widetilde{P}_i, \widetilde{P}_i \otimes P_i) = \begin{cases} P_i & \text{if } j = 0, \\ 0 & \text{if } j > 0. \end{cases}$$

Therefore the homology of the total complex, $H_i(\mathrm{Tot}(M))$, is isomorphic to

$$H_i(P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots) \cong \begin{cases} A & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

On the other hand, the homology of the jth row of the double complex M is

$$H_i(\widetilde{P}_* \otimes P_* \otimes \widetilde{P}_*^{\otimes j}) \cong \begin{cases} \widetilde{A} \otimes A \otimes \widetilde{A}^{\otimes j} & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

This calculation uses the Eilenberg–Zilber theorem: when we define the tensor product of two simplicial k-vector spaces by $(A \otimes B)_n = A_n \otimes_k B_n$, with boundary maps $d_i(x \otimes y) = d_i(x) \otimes d_i(y)$, the associated chain complex has homology groups $H_i(A \otimes B)$ isomorphic to $H_i(A) \otimes H_i(B)$ [21, Theorem 1.6.12]. By the second spectral sequence converging to the homology of $\mathrm{Tot}(M)$, we have an isomorphism

$$H_i(\widetilde{A} \otimes A \leftarrow (\widetilde{A} \otimes A) \otimes \widetilde{A} \leftarrow \cdots) \cong \begin{cases} A & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

The complex here (a quotient of the 0th column of M) is the one that computes Hochschild homology $H_i(\widetilde{A}, \widetilde{A} \otimes A)$. Thus we have shown that

$$H_i(\widetilde{A}, \widetilde{A} \otimes A) \cong \begin{cases} A & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$ □

5. Vanishing in positive characteristic

In this section, we prove a vanishing theorem for reduced projective schemes over a field of positive characteristic. For smooth projective varieties, the theorem is due to Arapura [3, Theorem 5.4]. The generalization to singular schemes follows the original proof, but with a new ingredient, a flatness property of the Frobenius morphism for arbitrary schemes over $\mathbb{F}_p$ (Theorem 4.1).

Theorem 5.1 is very different from the best-known vanishing theorem in positive characteristic, Deligne–Illusie–Raynaud's version of the Kodaira vanishing theorem. Their proof shows that (when a smooth projective variety X lifts from $\mathbb{Z}/p$ to $\mathbb{Z}/p^2$) vanishing of cohomology groups for the line bundle $K_X \otimes L^{p^b}$ with b large can imply vanishing for $K_X \otimes L$ [8, Lemme 2.9]. Theorem 5.1 goes the opposite way.

One result related to Theorem 5.1 is Siu's nonvanishing theorem. Siu's theorem says that if E is a pseudoeffective line bundle on a smooth projective variety X of dimension n over a field of characteristic zero, and $O_X(1)$ is an ample line bundle on X, then $H^0(X, K_X \otimes E(j)) \neq 0$ for some $1 \leq j \leq n+1$. (Ein pointed out this slight extension of Siu's theorem as stated in [20, Corollary 9.4.24].) By Serre duality, Siu's theorem gives that if L is a line bundle with $0 = H^n(X, L(-1)) = \cdots = H^n(X, L(-n-1))$, then L is naively $(n-1)$-ample (that is, L^* is not pseudoeffective, by Theorem 9.1).

We use the notion of regularity $\mathrm{reg}(M)$ from Section 1.

Theorem 5.1. *Let X be a projective scheme of dimension n over a field of characteristic $p > 0$ such that $O(X)$ is a field (example: X connected and reduced). Let $O_X(1)$ be a Koszul-ample line bundle on X. Let q be a natural number. Let L be a line bundle on X with*

$$0 = H^{q+1}(X, L(-n-1)) = H^{q+2}(X, L(-n-2)) = \cdots .$$

Then for any coherent sheaf M on X, we have

$$H^i(X, L^{\otimes p^b} \otimes M) = 0$$

whenever $i > q$ and $p^b \geq \mathrm{reg}(M)$.

Proof. We follow Arapura's proof in the smooth case as far as possible. Let k be the field $H^0(X, O_X)$. Let i be an integer greater than q, and b a natural number such that $p^b \geq \mathrm{reg}(M)$. Let $f' = F^b$, where $F : X \to X$ is the absolute Frobenius morphism (which acts as the identity on X as a set, and acts by pth powers on O_X). Let $\widetilde{X}$ be the base extension $X \times_k k$ where k maps to k by $x \mapsto x^{p^b}$. Then f' factors as

$$X \xrightarrow{f} \widetilde{X} \xrightarrow{g} X,$$

where f is a morphism of k-schemes and g is the natural morphism. The morphism f is called the *relative bth Frobenius morphism*. Let $\widetilde{L} = g^*L$ and $O_{\widetilde{X}}(1) = g^*O_X(1)$. Since g is given by a field extension, $O_{\widetilde{X}}(1)$ is Koszul-ample.

Let

$$C^i = \begin{cases} \widetilde{\mathcal{R}}_{-i} \boxtimes O_{\widetilde{X}}(i) & \text{if } -2n+1 \leq i \leq 0, \\ \ker(\widetilde{\mathcal{R}}_{2n-1} \boxtimes O_{\widetilde{X}}(-2n+1) \to \widetilde{\mathcal{R}}_{2n-2} \boxtimes O_{\widetilde{X}}(-2n+2)) & \text{if } i = -2n, \end{cases}$$

where $\mathcal{R}_i$ is defined as in Section 2 and $\widetilde{\mathcal{R}}_i = g^*\mathcal{R}_i$. The sheaves C^i form a resolution C^* of the diagonal Δ on $\widetilde{X} \times \widetilde{X}$, by Theorem 2.1. More generally, for vector bundles E_1 and E_2 on $\widetilde{X}$, $(E_1 \boxtimes E_2) \otimes C^*$ is quasi-isomorphic to $\delta_*(E_1 \otimes E_2)$, where $\delta : \widetilde{X} \to \widetilde{X} \times \widetilde{X}$ denotes the diagonal embedding. Therefore, $D^* = (O_{\widetilde{X}}(-n) \boxtimes O_{\widetilde{X}}(n)) \otimes C^*$ is another resolution of the diagonal.

Let $\Gamma \subset \widetilde{X} \times X$ be the graph of $f : X \to \widetilde{X}$, and write γ for the inclusion $X \cong \Gamma \subset \widetilde{X} \times X$. Then $\Gamma = (1 \times f)^{-1}(\Delta)$. Since D^* is a resolution of $O_\Delta \cong O_{\widetilde{X}}$, the cohomology sheaves of $(1 \times f)^*(D^*)$ are the groups $\mathrm{Tor}_*^{O_{\widetilde{X} \times \widetilde{X}}}(O_{\widetilde{X} \times X}, O_{\widetilde{X}})$. Since X may be singular, the relative Frobenius morphism need not be flat, and so $O_{\widetilde{X} \times X}$ need not be flat over $O_{\widetilde{X} \times \widetilde{X}}$. Nonetheless, Theorem 4.1 shows that these Tor groups are zero in positive degrees. Therefore, $(1 \times f)^*(D^*)$ is a resolution of O_Γ on $\widetilde{X} \times X$.

I claim that the complex of sheaves

$$G^* = (\widetilde{L} \boxtimes M) \otimes (1 \times f)^*(D^*)$$

is a resolution of the sheaf $\gamma_*(f^*\widetilde{L} \otimes M) \cong \gamma_*(L^{\otimes p^b} \otimes M)$. This follows if we can show that the sheaf $\mathrm{Tor}_i^{O_{\widetilde{X}} \otimes_k O_X}(\widetilde{L} \otimes_k M, O_\Gamma)$ is zero for $i > 0$, where $O_\Gamma \cong O_X$. Since $\widetilde{L}$ is a line bundle, it suffices to show that $\mathrm{Tor}_i^{O_{\widetilde{X}} \otimes_k O_X}(O_{\widetilde{X}} \otimes_k M, O_X)$ is zero for $i > 0$. But this is isomorphic to $\mathrm{Tor}_i^{O_X}(M, O_X)$, which is indeed zero for $i > 0$. Therefore, we can compute $H^i(X, L^{\otimes p^b} \otimes M) = H^i(\widetilde{X} \times X, \gamma_*(L^{\otimes p^b} \otimes M))$ using the resolution G^*.

By the spectral sequence $E_1^{i+c,-c} = H^{i+c}(\widetilde{X} \times X, G^{-c}) \Rightarrow H^i(\widetilde{X} \times X, G^*)$, the theorem holds if we can show that $H^{i+c}(\widetilde{X} \times X, G^{-c}) = 0$ for $i > q$ and $c \geq 0$. This is clear for the leftmost sheaf in the resolution, corresponding to $c = 2n$, because $\widetilde{X} \times X$ has dimension only $2n$ and q is nonnegative. For $0 \leq c \leq 2n-1$, the Künneth formula gives that

$$\begin{aligned} H^{i+c}(\widetilde{X} \times X, G^{-c}) &= H^{i+c}\big(\widetilde{X} \times X, (\widetilde{\mathcal{R}}_c \otimes \widetilde{L}(-n)) \boxtimes M(p^b(n-c))\big) \\ &\cong \bigoplus_{r+s=i+c} H^r(\widetilde{X}, \widetilde{\mathcal{R}}_c \otimes \widetilde{L}(-n)) \otimes H^s\big(X, M(p^b(n-c))\big). \end{aligned}$$

It remains to show that for all $i > q$, all $0 \leq c \leq 2n-1$, and all $r+s = i+c$, either the H^r group or the H^s group here is zero.

First suppose that $r > q$. We can assume that $s \leq n$; otherwise the H^s group is zero. So $r \geq i+c-n$, or equivalently $c \leq n+r-i$, and so $c \leq n+r-1$. By that fact together with $r > q$, Koszul-ampleness of $O_X(1)$, and our assumption on L, Lemma 3.3 gives that $H^r(X, \mathcal{R}_c \otimes L(-n)) = 0$. This implies the same vanishing on $\widetilde{X}$ for the same sheaf pulled back by the base extension $\widetilde{X} \to X$.

Otherwise, $r \leq q$. Since $r+s = i+c$ and $i > q$, we have $s > c$. In particular, s is greater than zero. We can assume that $s \leq n$; otherwise the H^s group is zero. So $c < s \leq n$. Then $p^b(n-c) \geq p^b \geq \operatorname{reg}(M)$, and so (since $s > 0$) $H^s(X, M(p^b(n-c))) = 0$. □

6. q-T-ampleness

Definition 6.1. Let $O_X(1)$ be a Koszul-ample line bundle (defined in Section 1) on a projective scheme X of dimension n over a field such that $O(X)$ is a field (example: X connected and reduced). Let q be a natural number. A line bundle L on X is called *q-T-ample* if, for some positive integer N, we have

$$\begin{aligned} 0 &= H^{q+1}(X, L^{\otimes N}(-n-1)) = H^{q+2}(X, L^{\otimes N}(-n-2)) = \cdots \\ &= H^n(X, L^{\otimes N}(-2n+q)). \end{aligned}$$

The details are not too important. Given that some multiple of L kills cohomology above dimension q for finitely many explicit line bundles on X, as in this definition, we will deduce that infinitely many multiples of L kill cohomology above dimension q for any given coherent sheaf. In particular, that will show that the definition of q-T-ampleness of a line bundle is independent of the choice of the Koszul-ample line bundle $O_X(1)$.

Theorem 5.1 makes it easy to characterize q-T-ampleness in characteristic $p > 0$ by a certain asymptotic vanishing of cohomology, as follows. In particular, property (2) shows that q-T-ampleness in characteristic $p > 0$ does not depend on the choice of the Koszul-ample line bundle $O_X(1)$.

Corollary 6.2. *Let X be a projective scheme over a field of characteristic $p > 0$ such that $O(X)$ is a field (example: X connected and reduced). Let q be a natural number. The following properties are equivalent, for a line bundle L on X:*

(1) *L is q-T-ample.*
(2) *Some positive multiple of L is q-F-ample. That is, there is a positive integer N such that for all coherent sheaves M on X, we have $H^i(X, M \otimes L^{\otimes Np^j}) = 0$ for all $i > q$ and all j sufficiently large depending on M.*

Proof. It is immediate that (2) implies (1). Here the number N in the definition of q-T-ampleness will be of the form Np^j for some j in the notation of (2). Theorem 5.1 shows that (1) implies (2), with the same value of N. □

In characteristic zero, q-T-ampleness has even stronger consequences, as we now show. Curiously, the proof involves reducing modulo many different prime numbers and using the prime number theorem. In particular, Theorem 6.3 shows that q-T-ampleness in any characteristic is independent of the choice of the Koszul-ample line bundle $O_X(1)$.

Theorem 6.3. *Let X be a projective scheme over a field of characteristic zero such that $O(X)$ is a field (example: X connected and reduced). Let q be a natural number. The following properties are equivalent, for a line bundle L on X:*

(1) *L is q-T-ample.*
(2) *L is naively q-ample. That is, for every coherent sheaf M on X, we have $H^i(X, M \otimes L^{\otimes m}) = 0$ for all $i > q$ and all m sufficiently large depending on M.*
(3) *L is uniformly q-ample. That is, there is a constant $\lambda > 0$ such that $H^i(X, L^{\otimes m}(-j)) = 0$ for all $i > q$, $j > 0$, and $m \geq \lambda j$.*

Proof. Demailly–Peternell–Schneider showed that (3) implies (2), by resolving any coherent sheaf by direct sums of the line bundles $O_X(-j)$ [10, Proposition 1.2]. Clearly (2) implies (1). We will show that (1) implies (3). Let L be a q-T-ample line bundle, and let N be the positive integer given in the definition.

To prove (3), we can work on a model of X over some finitely generated field extension of $\mathbb{Q}$. We can extend this to a projective model X_R of X over some domain R which is a finitely generated $\mathbb{Z}$-algebra. We can assume that $R = O(X_R)$, after replacing R by a finite extension if necessary; then all fibers X_t over closed points t of $\operatorname{Spec}(R)$ have $O(X_t)$ equal to a field (the residue field at t). After inverting a nonzero element of R, we can assume that X_R is also flat over R and $O_X(1)$ is Koszul-ample over R. (Recall from Section 1 that Koszul-ampleness is a Zariski open condition on a line bundle.) Choose an extension of the line bundle L to X over R.

By semicontinuity of cohomology [13, Theorem III.12.8], after inverting a nonzero element of R, we have

$$0 = H^{q+1}(X_t, L^{\otimes N}(-n-1)) = \cdots = H^n(X_t, L^{\otimes N}(-2n+q))$$

for all closed points $t \in \operatorname{Spec}(R)$, since that is true in characteristic zero. Let $r = \max\{1, \operatorname{reg}(L)\}$ (computed in characteristic zero). After inverting a nonzero element of R again, we can assume that $\operatorname{reg}(L|_{X_t}) \leq r$ for all closed points $t \in \operatorname{Spec}(R)$, again by semicontinuity. By Theorem 5.1, for each line bundle M on X_R and each closed point $t \in \operatorname{Spec}(R)$, we have

$$H^i(X_t, L^{\otimes Np^b} \otimes M|_{X_t}) = 0$$

whenever $i > q$ and $p^b \geq \operatorname{reg}(M|_{X_t})$, where p denotes the characteristic of the (finite) residue field of R at t. Again by semicontinuity of cohomology, it follows that for each closed point $t \in \operatorname{Spec}(R)$ and each line bundle M on X_R, we have

$$H^i(X, L^{\otimes Np^b} \otimes M) = 0 \tag{1}$$

in characteristic zero whenever $i > q$ and $p^b \geq \operatorname{reg}(M|_{X_t})$.

This already leads to an equivalent characterization of q-T-ampleness in characteristic zero; for example, it implies that q-T-ampleness is independent of the choice of the Koszul-ample line bundle $O_X(1)$. But we want to prove the stronger statements that L is naively q-ample and in fact uniformly q-ample.

The scheme $\mathrm{Spec}(R)$ has closed points of every characteristic p at least equal to some positive integer p_0. Therefore, for any positive integer a, we have

$$H^i(X, L^{\otimes Np+a}(-j)) = 0 \tag{2}$$

in characteristic zero if $i > q$, p is a prime $\geq p_0$, and $j+ar \leq p$. (Indeed, let t be a closed point of characteristic p in $\mathrm{Spec}(R)$. We have $j+a\ \mathrm{reg}(L|_{X_t}) \leq p$ since we arranged that $\mathrm{reg}(L|_{X_t}) \leq r$. By Theorem 3.4, we have $\mathrm{reg}(A \otimes B) \leq \mathrm{reg}(A) + \mathrm{reg}(B)$ for line bundles in any characteristic, and so $j + \mathrm{reg}(L^{\otimes a}|_{X_t}) \leq p$. Equivalently, $\mathrm{reg}(L^{\otimes a}(-j)|_{X_t}) \leq p$, and then equation (1) gives what we want.)

Let c be a real number in $(0, 1)$; we could take $c = 1/2$ for the current proof, but Theorem 6.4 will prove a stronger estimate by taking c close to zero. By the prime number theorem [23, Theorem 3.3.2], there is a positive integer m_1 (depending on c) such that for every integer $m \geq m_1$, there is a prime number p with

$$\frac{m}{N + c/r} \leq p < \frac{m}{N}.$$

Taking m_1 large enough, we can also assume that $p_0 \leq m_1/(N + c/r)$, so that the primes p produced above always have $p \geq p_0$.

Now, for each positive integer j, let $m_0 = m_0(j)$ be the maximum of m_1 and $\lceil (j/(1-c))(N + c/r) \rceil$. This will imply an inequality we want. First note that

$$\frac{j}{1-c}(N + c/r) \leq m_0,$$

and hence

$$\frac{j}{r}(N + c/r) \leq m_0 \frac{1-c}{r}.$$

Therefore

$$(m_0 + j/r)(N + c/r) \leq m_0(N + 1/r),$$

and hence

$$\frac{m_0 + j/r}{N + 1/r} \leq \frac{m_0}{N + c/r}.$$

It follows that the same inequality holds for all $m \geq m_0$ in place of m_0. Combining this with the previous paragraph's result, we find that for every $m \geq m_0$, there is a prime number $p \geq p_0$ with

$$\frac{m + j/r}{N + 1/r} \leq p < \frac{m}{N}.$$

Equivalently, if we define an integer a by $m = pN + a$, then $a > 0$ and $j + a\,r \leq p$. By (2), we have shown that for every $m \geq m_0(j)$, we have

$$H^i(X, L^{\otimes m}(-j)) = 0$$

for $i > q$.

By our construction of the number $m_0(j)$, there is a positive constant $\lambda > 0$ such that $m_0(j) \leq \lambda j$ for all $j > 0$. (Here λ depends on the chosen constant $c \in (0, 1)$; we can take $c = 1/2$, for example.) Thus we have shown that for all $i > q$, $j > 0$, and all $m \geq \lambda j$ we have

$$H^i(X, L^{\otimes m}(-j)) = 0.$$

That is, L is uniformly q-ample. □

The proof shows something more precise.

Theorem 6.4. *Let X be a projective scheme over a field of characteristic zero such that $O(X)$ is a field (example: X connected and reduced). Let L be a q-T-ample (or naively q-ample, or uniformly q-ample) line bundle on X. Then there are positive numbers m_1 and λ such that*

$$H^i(X, E \otimes L^{\otimes m}) = 0$$

for all $i > q$, all coherent sheaves E on X, and all $m \geq \max\{m_1, \lambda \operatorname{reg}(E)\}$. Moreover, for N the number in the definition of q-T-ampleness, we can take λ to be any real number greater than N (and some m_1 depending on λ).

Demailly–Peternell–Schneider proved this for some constant λ when L is uniformly q-ample [10, Proposition 1.2]. The point here is that the same holds for the a priori weaker notions of q-T-ampleness and naive q-ampleness in characteristic zero, and that we have an explicit estimate for the constant λ.

Proof of Theorem 6.4. The proof of Theorem 6.3 shows that there are positive constants λ and m_1 such that for all $i > q$, $j > 0$, and $m \geq \max\{m_1, \lambda j\}$, we have

$$H^i(X, L^{\otimes m}(-j)) = 0.$$

In terms of a constant $c \in (0, 1)$ which we were free to choose, the proof shows that we can take $\lambda = (1/(1-c))(N + c/r)$, where $r = \max\{1, \operatorname{reg}(L)\}$. Thus, by taking c close to zero, we can make λ arbitrarily close to N.

Let E be any coherent sheaf on X, and let $s = \operatorname{reg}(E)$, $R = \max\{1, \operatorname{reg}(O_X)\}$. Then E has a resolution by vector bundles of the form

$$\cdots \to O(-s-2R)^{\oplus a_2} \to O(-s-R)^{\oplus a_1} \to O(-s)^{\oplus a_0} \to E \to 0$$

[2, Corollary 3.2]. For integers $i > q$ and $m \geq \max\{m_1, \lambda s\}$, we want to show that $H^i(X, E \otimes L^{\otimes m}) = 0$. By the given resolution, this holds if $H^{i+j}(X, L^{\otimes m}(-s-jR)) = 0$ for all $j \geq 0$. By the previous paragraph, this holds if $m \geq \max\{m_1, \lambda(s + (n-q-1)R)\}$, using that cohomology vanishes in dimensions greater than n. This is enough to deduce the statement of the theorem, after slightly increasing λ and increasing m_1 as needed. □

7. Projective schemes

In this section we generalize the equivalence between the three notions of q-ampleness to arbitrary projective schemes over a field of characteristic zero.

Theorem 7.1. *Let X be a projective scheme over a field of characteristic zero, with an ample line bundle $O_X(1)$. Let q be a natural number. Then there is a positive integer C such that, for all line bundles L on X, the following properties are equivalent:*

(1) *There is a positive integer N such that $H^i(X, L^{\otimes N}(-j)) = 0$ for all $i > q$ and all $1 \leq j \leq C$.*
(2) *L is naively q-ample. That is, for every coherent sheaf M on X, we have $H^i(X, M \otimes L^{\otimes m}) = 0$ for all $i > q$ and all m sufficiently large depending on M.*
(3) *L is uniformly q-ample. That is, there is a constant $\lambda > 0$ such that $H^i(X, L^{\otimes m}(-j)) = 0$ for all $i > q$, $j > 0$, and $m \geq \lambda j$.*

In contrast to the case of reduced schemes (Theorem 6.3), we choose not to specify the value of C in condition (1). The proof gives an explicit value for C, probably far from optimal.

Proof of Theorem 7.1. By the same arguments as in Theorem 6.3, (3) implies (2) and (2) implies (1), for any fixed choice of the positive integer C. It remains to show that there is a positive integer C such that (1) implies (3), for all line bundles L on X. The idea is to reduce to the case of a reduced scheme.

We can assume that X is connected. Let X_0 be the underlying reduced scheme of X. Then $k := O(X_0)$ is a field. After replacing the given ample line bundle $O_X(1)$ by a positive multiple, we can assume that $O_X(1)$ restricts to a Koszul-ample line bundle on X_0 (by Backelin's theorem, as in Section 1).

I claim that there is a positive integer C (depending on X and $O_X(1)$) such that any line bundle on X satisfying (1) restricts to a q-T-ample line bundle on X_0. To see this, look at a resolution of O_{X_0} as a sheaf of O_X-modules. Explicitly, if we define $s = \mathrm{reg}_X(O_{X_0})$ and $R = \max\{1, \mathrm{reg}_X(O_X)\}$, then O_{X_0} has a resolution on X of the form

$$\cdots \to O_X(-s-2R)^{\oplus a_2} \to O_X(-s-R)^{\oplus a_1} \to O_X(-s)^{\oplus a_0} \to O_{X_0} \to 0$$

[2, Corollary 3.2]. Then, for any line bundle L on X, and any positive integer c, tensoring this resolution with $L(-c)$ gives a resolution of $L(-c)|_{X_0}$ as a sheaf of O_X-modules:

$$\cdots \to L(-c-s-2R)^{\oplus a_2} \to L(-c-s-R)^{\oplus a_1} \to L(-c-s)^{\oplus a_0} \to L(-c)|_{X_0} \to 0.$$

This gives a spectral sequence

$$E_1^{-u,v} = H^v(X, L(-c-s-uR)^{\oplus a_u}) \Rightarrow H^{v-u}(X_0, L(-c)).$$

Let n be the dimension of X, and let $C = n + q + 1 + s + (n-q-1)R$ (which does not depend on the line bundle L). Then the spectral sequence shows that for any line bundle L on X such that $H^i(X, L(-j)) = 0$ for all $i > q$ and all $1 \leq j \leq C$, we have

$$0 = H^{q+1}(X_0, L(-n-1)) = H^{q+2}(X_0, L(-n-2)) = \cdots.$$

Thus, for the given value of C, any line bundle L on X which satisfies condition (1) restricts to a q-T-ample line bundle on X_0. By Theorem 6.3, L is uniformly q-ample on X_0. To complete the proof, we have to show that L is uniformly q-ample on X.

Let I be the ideal sheaf $\ker(O_X \to O_{X_0})$. There is a natural number r such that $I^{r+1} = 0$. Thus the sheaf O_X has a filtration with quotients $O_X/I, I/I^2, \dots, I^r/I^{r+1}$, and these quotients are all modules over $O_X/I = O_{X_0}$. By considering a resolution of the sheaves I^l/I^{l+1} (for $l = 0, \dots, r$) by line bundles $O(-j)$ on X_0, the uniform q-ampleness of $L|_{X_0}$ implies that there is a constant $\lambda > 0$ such that $H^i(X_0, L^{\otimes m}(-j) \otimes I^l/I^{l+1}) = 0$ for all $i > q$, all $0 \leq l \leq r$, all $j > 0$, and all $m > \lambda j$. By our filtration of O_X, it follows that $H^i(X, L^{\otimes m}(-j)) = 0$ for all $i > q$, all $j > 0$, and all $m > \lambda j$. That is, L is uniformly q-ample on X. □

Therefore, on any projective scheme over a field of characteristic zero, we can say "q-ample" to mean any of the equivalent conditions on a line bundle in Theorem 7.1. We mention a consequence of the proof:

Corollary 7.2. *Let X be a projective scheme over a field of characteristic zero. Then a line bundle L is q-ample on X if and only if the restriction of L to the underlying reduced scheme X_0 is q-ample.*

8. Openness properties of q-ampleness

In this section we check that q-T-ampleness is Zariski open on families of varieties and line bundles over $\mathbb{Z}$. It follows that q-ampleness is Zariski open in characteristic zero, where we can use any of the three equivalent definitions in Theorem 6.3. Using Demailly–Peternell–Schneider's results, we also find that q-ampleness defines an open cone (typically not convex) in the Néron–Severi vector space $N^1(X)$ for X of characteristic zero. Neither property was known for naive q-ampleness. We discuss counterexamples to these good properties in positive characteristic.

Theorem 8.1. *Let $\pi : X \to B$ be a flat projective morphism of schemes over $\mathbb{Z}$. Suppose that π has connected fibers in the sense that $\pi_*(O_X) = O_B$. Let L be a line bundle on X, and let q be a natural number. Then the set of points b of the scheme B such that L is q-T-ample on the fiber over b is Zariski open.*

Proof. This is straightforward. Suppose that L is q-T-ample on the fiber X_b over a point $b \in B$. There is an affine open neighborhood U of b and a Koszul-ample line bundle $O_X(1)$ on the inverse image of U. (Koszul-ampleness is a Zariski open condition on a line bundle, as discussed in Section 1.) We can use this line bundle $O_X(1)$ in the definition of q-T-ampleness. (We have shown that the definition is independent of this choice.) By definition of q-T-ampleness, there is a positive integer N with

$$0 = H^{q+1}(X_b, L^{\otimes N}(-n-1)) = \cdots = H^n(X_b, L^{\otimes N}(-2n+q)).$$

By semicontinuity of cohomology, the same is true over some neighborhood of b. □

By Theorem 6.3, it follows that naive q-ampleness and uniform q-ampleness are Zariski open conditions in characteristic zero.

Demailly–Peternell–Schneider showed that uniform q-ampleness defines an open cone in the vector space $N^1(X)$ of $\mathbb{R}$-divisors modulo numerical equivalence. For completeness, we give the relevant definitions.

Definition 8.2. Let X be a projective variety over a field of characteristic zero. An $\mathbb{R}$-divisor D on X is *q-ample* if D is numerically equivalent to a sum $cL + A$ with L a q-ample divisor, c a positive real number, and A an ample $\mathbb{R}$-divisor. (By definition, an $\mathbb{R}$-divisor is ample if it is a positive linear combination of ample Cartier divisors [20, Definition 3.11].)

Theorem 8.3. *For any projective variety X over a field of characteristic zero, q-ampleness for $\mathbb{R}$-divisors agrees with the earlier definitions in the case of line bundles. Also, q-ampleness defines an open cone in the real vector space $N^1(X)$.*

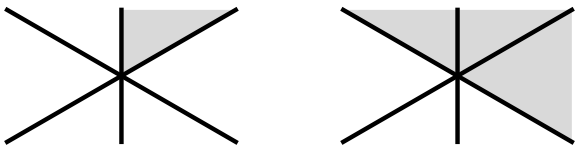
Also, the sum of a q-ample $\mathbb{R}$-divisor and an r-ample $\mathbb{R}$-divisor is $(q + r)$-ample.

Demailly–Peternell–Schneider proved Theorem 8.3 for uniform q-ampleness [10, Propositions 1.4 and 1.5]. This is equivalent to the other notions of q-ampleness for X projective over a field of characteristic zero, by Theorem 7.1.

Theorem 8.3 gives a simple insight into why the q-ample cone need not be convex for $q > 0$: the sum of two q-ample divisors is typically $2q$-ample, not q-ample. An example is the $(n-1)$-ample cone of a projective variety of dimension n, which is equal to the complement of the negative of the closed effective cone by Theorem 9.1. (Thus the $(n-1)$-ample cone is the *complement* of a closed convex cone.)

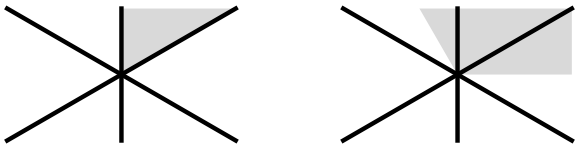
In positive characteristic, the right notion of q-ampleness remains to be found. In particular, naive q-ampleness and uniform q-ampleness are not Zariski open conditions in mixed characteristic, as one can check in the example of the three-dimensional flag manifold $\mathrm{SL}(3)/B$ over $\mathbb{Z}$ with $q = 1$. The figures show the (naive or uniform) q-ample cone in characteristic zero and in any characteristic $p > 0$, where the 1-ample cone is different from the one in characteristic zero (and independent of p).

On the other hand, q-T-ampleness is also not a good notion in positive characteristic, in the sense that the sum of a q-T-ample divisor and an r-T-ample divisor need not be $(q + r)$-T-ample. This happens with $q = r = 1$ on $(\mathrm{SL}(3)/B)^2$ in any characteristic $p > 0$.



0-T-ample cone 1-T-ample cone

Fig. 2. The q-ample cones in $N^1(\mathrm{SL}(3)/B) \cong \mathbb{R}^2$ in characteristic 0.



naive 0-ample cone naive 1-ample cone

Fig. 3. The naive q-ample cones in $N^1(\mathrm{SL}(3)/B) \cong \mathbb{R}^2$ in characteristic $p > 0$.

9. The $(n-1)$-ample cone

The 0-ample and $(n-1)$-ample cones of an n-dimensional variety are better understood than the intermediate cases. In this section we show that the $(n-1)$-ample cone of a projective variety of dimension n is the complement of the negative of the closed effective cone. This is well known for smooth varieties, but with care the proof works for arbitrary varieties.

Theorem 9.1. *Let X be a projective variety of dimension n over a field k of characteristic zero. Let L be a line bundle on X. Then L is $(n-1)$-ample if and only if $[L]$ in $N^1(X)$ is not in the negative of the closed effective cone.*

Proof. Let ω_X be what Hartshorne calls the dualizing sheaf of $\pi : X \to Y = \mathrm{Spec}(k)$, that is, the cohomology sheaf in dimension $-n$ of the dualizing complex $\pi^! O_Y$. Then, for every coherent sheaf F on X, there is a canonical isomorphism

$$\mathrm{Hom}_X(F, \omega_X) \cong H^n(X, F)^*$$

[13, Proposition III.7.5]. It follows that the sheaf ω_X is torsion-free. Indeed, if F is any coherent sheaf on X whose support has dimension less than n, then $H^n(X, F) = 0$ and so $\mathrm{Hom}_X(F, \omega_X) = 0$.

Let $O_X(1)$ be an ample line bundle on X. For a coherent sheaf F, write F^* for the sheaf $\mathcal{H}om_{O_X}(F, O_X)$. Then the sheaf ω_X^* is generically a line bundle on X (in particular, it is not zero), and so $H^0(X, \omega_X^*(j)) \neq 0$ for some $j > 0$. Equivalently, there is a nonzero map $f : \omega_X \to O(j)$ for some $j > 0$, which we fix. Since ω_X is torsion-free, f must be an injection of sheaves.

Suppose that $[L]$ in $N^1(X)$ is not in the negative of the closed effective cone. That is, L^* is not pseudoeffective. Then for any line bundle F on X, we have $H^0(X, F^* \otimes (L^*)^{\otimes m} \otimes O(j)) = 0$ for all m at least equal to some $m_0 = m_0(F)$. Using the injection f, it follows that $H^0(X, F^* \otimes (L^*)^{\otimes m} \otimes \omega_X) = 0$ for all $m \geq m_0(F)$. That is, $\mathrm{Hom}_X(F \otimes L^{\otimes m}, \omega_X) = 0$, and hence $H^n(X, F \otimes L^{\otimes m}) = 0$ for all $m \geq m_0(F)$. It follows that L is $(n-1)$-T-ample, which we call simply $(n-1)$-ample after Theorem 6.3.

Conversely, let L be an $(n-1)$-ample line bundle on X. Then for any line bundle F on X, we have $H^n(X, F \otimes L^{\otimes m}) = 0$ for all m at least equal to some $m_0 = m_0(F)$. Therefore $\mathrm{Hom}_X(F \otimes L^{\otimes m}, \omega_X) = 0$, which we write as $H^0(X, F^* \otimes (L^*)^{\otimes m} \otimes \omega_X) = 0$. Since the sheaf ω_X is not zero, there is a nonzero map $g : O_X(-j) \to \omega_X$ for some

$j > 0$, which we fix. Here g is an injection of sheaves because the line bundle $O_X(-j)$ is torsion-free. Therefore, $H^0(X, F^*(-j) \otimes (L^*)^{\otimes m}) = 0$ for all $m \geq m_0(F)$. In particular, for every $(n-1)$-ample line bundle L, L^* is not big. But the $(n-1)$-ample cone is open in $N^1(X)$ (Theorem 8.3). So for every $(n-1)$-ample line bundle L, L^* is not pseudoeffective. □

10. The q-nef cone

It is an open problem to give a numerical characterization of q-ampleness, analogous to the Kleiman or Nakai–Moishezon criteria for $q = 0$. We know from Theorem 8.3 that q-ampleness on a smooth projective variety of characteristic zero only depends on the numerical equivalence class of a divisor, but that leaves the problem of describing the q-ample cone by explicit inequalities. In this section, we disprove the most obvious attempt at a Kleiman criterion for q-ampleness.

Let X be a projective variety of dimension n over a field. For a natural number q, define the *q-nef cone* as the set of $D \in N^1(X)$ such that for every $(q+1)$-dimensional subvariety $Z \subset X$, $-D$ restricted to Z is not big. The q-nef cone is clearly a closed cone in $N^1(X)$, not necessarily convex. For example, the 0-nef cone is the usual nef cone, because $-D$ is not big on a curve Z exactly when $D \cdot Z \geq 0$. Another simple case is the $(n-1)$-nef cone, which is the complement of the negative of the big cone in $N^1(X)$.

Another description of q-nef divisors comes from the theorem of Boucksom–Demailly–Păun–Peternell. On any complex projective variety X, BDPP characterized the dual of the closed effective cone as the closed convex cone spanned by curves that move on X [6], [20, Vol. 2, Theorem 11.4.19]. For this statement, we say that a curve *moves* on a projective variety X if it is the image under some resolution $X' \to X$ of a complete intersection of ample divisors in X', $D_1 \cap \cdots \cap D_{n-1}$. By BDPP's theorem, an $\mathbb{R}$-divisor D on X is q-nef if and only if for every $q+1$-dimensional subvariety Z of X, D has nonnegative degree on some curve that moves on Z.

Let X be a projective variety over a field of characteristic zero. It is straightforward to check that the q-ample cone in $N^1(X)$ is contained in the interior of the q-nef cone, essentially because the restriction of a q-ample divisor to each $(q+1)$-dimensional subvariety is q-ample. In the extreme cases $q = 0$ and $q = n-1$, the q-ample cone is equal to the interior of the q-nef cone. But this can fail for the 1-ample cone of a smooth projective 3-fold, as we now show. The problem remains to give a Kleiman-type description of the q-ample cone.

Lemma 10.1. *Let X be the $\mathbb{P}^1$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ given by $X = P(O \oplus O(1,-1))$, over the complex numbers. (This is a smooth projective toric Fano 3-fold. It is the blow-up of $\mathbb{P}^3$ along two disjoint lines.) Then the 1-ample cone of X is strictly smaller than the interior of the 1-nef cone of X.*

Proof. Let E be the vector bundle $O \oplus O(1,-1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Write $X = P(E)$ for the variety of codimension-1 subspaces of E, with projection $\pi : X \to \mathbb{P}^1 \times \mathbb{P}^1$. Every

line bundle on X has the form $\pi^*O(a,b) \otimes O_{P(E)}(c)$ for some integers a, b, c. The cohomology of a line bundle on X is given by

$$H^i(X, \pi^*O(a,b) \otimes O_{P(E)}(c)) = H^i(\mathbb{P}^1 \times \mathbb{P}^1, O(a,b) \otimes R\pi_* O_{P(E)}(c)).$$

If $c > 0$, then $\pi_* O_{P(E)}(c) = S^c(E) = \bigoplus_{j=0}^{c} O(j, -j)$ and the higher direct images are zero. Thus, for $c > 0$,

$$H^i(X, \pi^*O(a,b) \otimes O_{P(E)}(c)) = H^i\Big(\mathbb{P}^1 \times \mathbb{P}^1, \bigoplus_{j=0}^{c} O(a+j, b-j)\Big).$$

We will compute the 1-ample cone of X intersected with the open half-space $c > 0$, which is enough for our purpose. If $c > 0$, then $H^3(X, \pi^*O(a,b) \otimes O_{P(E)}(c)) = 0$ by the previous paragraph. It follows that $L = \pi^*O(a,b) \otimes O_{P(E)}(c)$ is 2-ample whenever $c > 0$. Next,

$$H^2(X, \pi^*O(a,b) \otimes O_{P(E)}(c)) = H^2\Big(\mathbb{P}^1 \times \mathbb{P}^1, \bigoplus_{j=0}^{c} O(a+j, b-j)\Big).$$

Here $H^2(\mathbb{P}^1 \times \mathbb{P}^1, O(p,q))$ is zero if and only if $p \geq -1$ or $q \geq -1$. It follows that a line bundle L with $c > 0$ is 1-ample if and only if $a > 0$ or $b > c$ or $a + b > 0$.

We now compute the 1-nef cone of X (intersected with $c > 0$) using toric geometry [12]. Every line bundle L on X can be made T-equivariant, where $T \cong (G_m)^3$ acts on X. By definition, for L to be 1-nef means that L^* is not big on any surface Y in X. Because L is T-equivariant, if L^* is big on some surface Y, then L^* is big on any translate tY for $t \in T$. Every action of T on a projective variety has a fixed point, and so there is a fixed point in the closure of the T-orbit of Y in the Hilbert scheme. Using upper semicontinuity of h^0, it follows that L^* is big on some T-invariant subscheme Z of dimension 2, and hence on some irreducible T-invariant surface. Thus L is 1-nef if and only if L^* is not big on the finitely many T-surfaces in X. (This argument shows that the q-nef cone of a toric variety is rational polyhedral, for any $q \geq 0$.)

The toric surfaces in X are the two sections $S_1 = P(O)$ and $S_2 = P(O(1,-1))$ of $X \to \mathbb{P}^1 \times \mathbb{P}^1$ and the inverse images $Y_1, \dots, Y_4$ of the four curves $\mathbb{P}^1 \times 0$, $\mathbb{P}^1 \times \infty$, $0 \times \mathbb{P}^1$, and $\infty \times \mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $L = \pi^*O(a,b) \otimes O_{P(E)}(c)$ be a line bundle with $c > 0$. Then L has positive degree on each fiber $\cong \mathbb{P}^1$, and these curves cover the surfaces $Y_1, \dots, Y_4$. It follows that L^* cannot be big on $Y_1, \dots, Y_4$. Thus L is 1-nef if and only if L^* is not big on S_1 and not big on S_2. Here $O_{P(E)}(1)$ restricts to the trivial bundle on S_1 and to $O(1,-1)$ on S_2, so L restricts to $O(a,b)$ on $S_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and to $O(a+c, b-c)$ on $S_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$. The big cone of $\mathbb{P}^1 \times \mathbb{P}^1$ consists of the line bundles $O(a,b)$ with $a > 0$ and $b > 0$. We conclude that a line bundle L with $c > 0$ is 1-nef if and only if ($a \geq 0$ or $b \geq 0$) and ($a + c \geq 0$ or $b - c \geq 0$).

For example, the line bundle $\pi^*O(-2,1) \otimes O_{P(E)}(3)$ is in the interior of the 1-nef cone but is not 1-ample. □

11. Questions

We raise two questions. The first is about the relation between q-ampleness and Küronya's asymptotic cohomological functions [19]. Let X be a projective variety of dimension n over a field of characteristic zero. For a line bundle L on X, define

$$\widehat{h}^i(L) = \limsup_{m\to\infty} \frac{h^i(X, L^{\otimes m})}{m^n/n!}.$$

Küronya showed that $\widehat{h}^i$ extends uniquely to a continuous function on $N^1(X)$ which is homogeneous of degree n. For example, for a nef $\mathbb{R}$-divisor D, we have $\widehat{h}^i(D) = 0$ for $i > 0$, and $\widehat{h}^0(D)$ is the intersection number D^n.

Question 11.1. Let D be an $\mathbb{R}$-divisor on a projective variety X over a field of characteristic zero. Let q be a natural number. Suppose that $\widehat{h}^i(E)$ is zero for all $i > q$ and all $\mathbb{R}$-divisors E in some neighborhood of D in $N^1(X)$. Is D q-ample?

The converse is clear, since the q-ample cone is open in $N^1(X)$ (Theorem 8.3). The question has a positive answer for $q = 0$, by de Fernex, Küronya, and Lazarsfeld [7]. It is also true for $q = n - 1$, using that the $(n-1)$-ample cone is the complement of the negative of the closed effective cone (Theorem 9.1).

Another question was raised by Dawei Chen and Rob Lazarsfeld.

Question 11.2. Let X be a Fano variety, say over a field of characteristic zero. Let q be a natural number. Is the q-ample cone in $N^1(X)$ the interior of a finite union of rational polyhedral convex cones?

The answer is positive for $q = 0$: the nef cone of a Fano variety is rational polyhedral, by the Cone theorem [17, Theorem 3.7]. It is also true for $q = n - 1$, since the closed effective cone of a Fano variety is rational polyhedral by Birkar, Cascini, Hacon, and M$^{\text{c}}$Kernan [5, Corollary 1.3.1]. So the first open case is the 1-ample cone of a Fano 3-fold.

Acknowledgments. Thanks to Anders Buch, Jean-Pierre Demailly, Lawrence Ein, and Marcus Zibrowius for useful discussions.

References

[1] Andreotti, A., Grauert, H.: Théorèmes de finitude pour la cohomologie des espaces complexes. Bull. Soc. Math. France **90**, 193–259 (1962) Zbl 0106.05501 MR 0150342

[2] Arapura, D.: Frobenius amplitude and strong vanishing theorems for vector bundles. Duke Math. J. **121**, 231–267 (2004) Zbl 1067.14018 MR 2034642

[3] Arapura, D.: Partial regularity and amplitude. Amer. J. Math. **128**, 1025–1056 (2006) Zbl 1105.14018 MR 2251592

[4] Backelin, J.: On the rates of growth of the homologies of Veronese subrings. In: Algebra, Algebraic Topology and Their Interactions (Stockholm, 1983), Lecture Notes in Math. 1183, Springer, 79–100 (1986) Zbl 0588.13011 MR 0846440

[5] Birkar, C., Cascini, P., Hacon, C., M$^{\mathrm{c}}$Kernan, J.: Existence of minimal models for varieties of log general type. J. Amer. Math. Soc. **23**, 405–468 (2010) Zbl 1210.14019 MR 2601039

[6] Boucksom, S., Demailly, J.-P., Păun, M., Peternell, T.: The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension. J. Algebraic Geom., to appear

[7] de Fernex, T., Küronya, A., Lazarsfeld, R.: Higher cohomology of divisors on a projective variety. Math. Ann. **337**, 443–455 (2007) Zbl 1127.14012 MR 2262793

[8] Deligne, P., Illusie, L.: Relèvements modulo p^2 et décomposition du complexe de de Rham. Invent. Math. **89**, 247–270 (1987) Zbl 0632.14017 MR 0894379

[9] Demailly, J.-P.: A converse to the Andreotti–Grauert theorem. Ann. Fac. Sci. Toulouse Math. **20**, fasc. spécial, 123–135 (2011) Zbl 1228.32020 MR 2858170

[10] Demailly, J.-P., Peternell, T., Schneider, M.: Holomorphic line bundles with partially vanishing cohomology. In: Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), Bar-Ilan Univ., 165–198 (1996) Zbl 0859.14005 MR 1360502

[11] Eisenbud, D., Reeves, A., Totaro, B.: Initial ideals, Veronese subrings, and rates of algebras. Adv. Math. **109**, 168–187 (1994) Zbl 0839.13013 MR 1304751

[12] Fulton, W.: Introduction to Toric Varieties. Princeton Univ. Press (1993) Zbl 0813.14039 MR 1234037

[13] Hartshorne, R.: Algebraic Geometry. Springer (1977). Zbl 0367.14001 MR 0463157

[14] Hering, M.: Multigraded regularity and the Koszul property. J. Algebra **323**, 1012–1017 (2010) Zbl 1184.14012 MR 2578590

[15] Kawamata, Y.: Equivalences of derived categories of sheaves on smooth stacks. Amer. J. Math. **126**, 1057–1083 (2004) Zbl 1076.14023 MR 2089082

[16] Keeler, D.: Ample filters and Frobenius amplitude. J. Algebra **323**, 3039–3053 (2010) Zbl 1231.13005 MR 2609191

[17] Kollár, J., Mori, S.: Birational Geometry of Algebraic Varieties. Cambridge Univ. Press (1998) Zbl 0926.14003 MR 1658959

[18] Kunz, E.: Characterizations of regular local rings for characteristic p. Amer. J. Math. **91**, 772–784 (1969) Zbl 0188.33702 MR 0252389

[19] Küronya, A.: Asymptotic cohomological functions on projective varieties. Amer. J. Math. **207**, 1475–1519 (2006) Zbl 1114.14005 MR 2275909

[20] Lazarsfeld, R.: Positivity in Algebraic Geometry, 2 vols. Springer (2004) Zbl 1093.14500 MR 2095472

[21] Loday, J.-L.: Cyclic Homology. Springer (1992) Zbl 0780.18009 MR 1217970

[22] McCleary, J.: A User's Guide to Spectral Sequences. Cambridge Univ. Press (2001) Zbl 0959.55001 MR 1793722

[23] Murty, M. R.: Problems in Analytic Number Theory. Springer (2008) Zbl 1190.11001 MR 2376618

[24] Orlov, D.: Equivalences of derived categories and K3 surfaces. J. Math. Sci. **84**, 1361–1381 (1997) Zbl 0938.14019 MR 1465519

[25] Pareschi, G.: Koszul algebras associated to adjunction bundles. J. Algebra **157**, 161–169 (1993) Zbl 0796.13012 MR 1219663

[26] Pirashvili, T.: Hochschild homology, Frobenius homomorphism and Mac Lane homology. Algebr. Geom. Topol. **7**, 1071–1079 (2007) Zbl 1137.19001 MR 2336249

[27] Polishchuk, A., Positselski, L.: Quadratic Algebras. Amer. Math. Soc. (2005). Zbl 1145.16009 MR 2177131

[28] Sommese, A.: Submanifolds of abelian varieties. Math. Ann. **233**, 229–256 (1978) Zbl 0381.14007 MR 0466647