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The automorphism group of $\overline{M}_{0,n}$

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Abstract. The paper studies fiber type morphisms between moduli spaces of pointed rational curves. Via Kapranov's description we are able to prove that the only such morphisms are forgetful maps. This allows us to show that the automorphism group of $\overline{M}_{0,n}$ is the permutation group on n elements as soon as $n \geq 5$.

Keywords. Moduli space of curves, pointed rational curves, fibrations, automorphism

Introduction

The moduli space $M_{0,n}$ of smooth n-pointed stable rational curves, and its completion, the Knudsen-Mumford compactification $\overline{M}_{0,n}$, is a classical object of study that reflects many properties of families of rational pointed curves.

Already for small n, the moduli spaces $\overline{M}_{0,n}$ are quite intricate objects deeply rooted in classical algebraic geometry. Kapranov showed in [Ka] that $\overline{M}_{0,n}$ is identified with the closure of the subscheme of the Hilbert scheme parametrizing rational normal curves passing through n points in linearly general position in $\mathbb{P}^{n-2}$. Via this identification, given $n-1$ points in linearly general position in $\mathbb{P}^{n-3}$, $\overline{M}_{0,n}$ is isomorphic to an iterated blow-up of $\mathbb{P}^{n-3}$ at the strict transforms of all the linear spaces spanned by subsets of the points in order of increasing dimension (see Section 1).

The aim of this paper is to study automorphisms of $\overline{M}_{0,n}$ via Kapranov's construction. It is easy and well known that $\overline{M}_{0,4} \cong \mathbb{P}^1$ and that any permutation of the markings induces an automorphism of $\overline{M}_{0,n}$; we will refer to such automorphisms as *permutation automorphisms*. Moreover, if $n = 5$, $\overline{M}_{0,5}$ is a del Pezzo surface and again it is a classical result that its automorphism group is S_5. This suggests the following conjecture:

Conjecture 1. $\mathrm{Aut}(\overline{M}_{0,n}) \cong S_n$ *for* $n \geq 5$.

As already said, our idea to attack the conjecture is to study the elements in $\mathrm{Aut}(\overline{M}_{0,n})$ via fibrations of $\overline{M}_{0,n}$. To do this we identify base point free linear systems on $\overline{M}_{0,n}$ with

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linear systems on $\mathbb{P}^{n-3}$ having special base locus. From this point of view, for instance, the modular forgetful map $\phi_I : \overline{M}_{0,n} \to \overline{M}_{0,n-|I|}$, which forgets points indexed by $I \subset \{1,\dots,n\}$, corresponds, up to standard Cremona transformations, to a projection from a linear space.

A permutation automorphism has to permute the forgetful maps onto $\overline{M}_{0,n-1}$ as well. Hence to a permutation is associated a special birational map of $\mathbb{P}^{n-3}$ that switches lines through $n-1$ points in general position and rational normal curves. The idea of our approach is that if one can show that an automorphism of $\overline{M}_{0,n}$ preserves forgetful maps, then one obtains major restrictions on the corresponding birational map on the projective spaces.

Hence to achieve our goal we have to study fibrations on $\overline{M}_{0,n}$. In this direction a crucial result is the following.

Theorem 1. *Any dominant morphism $f : \overline{M}_{0,n} \to \overline{M}_{0,4} \cong \mathbb{P}^1$ factors as a composition of forgetful maps.*

Theorem 1 is proved by studying pencils of hypersurfaces on $\mathbb{P}^{n-3}$ induced by base point free pencils on $\overline{M}_{0,n}$. The possibility of classifying such pencils (see Definition 1.7) comes from the fact that their base locus cannot be too complicated (see Lemma 1.8). Sean Keel [Ke] proved that the intersection ring of $\overline{M}_{0,n}$ is the polynomial ring generated by the classes of the boundary divisors, modulo relations generated by forgetful maps to $\mathbb{P}^1 \cong \overline{M}_{0,4}$. Theorem 1 is closely related to Keel's result, and might be seen as a new perspective on it.

Pushing further these techniques via an inductive argument we are able to extend Theorem 1 to the following setting (see Definition 1.1).

Theorem 2. *Let $f : \overline{M}_{0,n} \to \overline{M}_{0,r_1} \times \cdots \times \overline{M}_{0,r_h}$ be a dominant morphism with connected fibers. Then f is a forgetful map.*

As expected, this together with some computation on certain birational endomorphisms of $\mathbb{P}^{n-3}$, is enough to describe the automorphism group of $\overline{M}_{0,n}$. So finally this yields the desired result.

Theorem 3. *Assume that $n \geq 5$. Then $\mathrm{Aut}(\overline{M}_{0,n}) = S_n$, the symmetric group on n elements.*

Theorem 3 has a natural counterpart in the Teichmüller-theoretic literature on automorphisms of moduli spaces $M_{g,n}$ developed in a series of papers by Royden, Earle–Kra, and others ([Ro], [EK], [Ko]), but we do not see a straightforward way to go from one to the other.

In this paper we study "modular" fibrations on $\overline{M}_{0,n}$ via the study of linear systems on $\mathbb{P}^{n-3}$. This program has been recently pursued also by Bolognesi [Bo] in his description of birational models of $\overline{M}_{0,n}$. In a forthcoming paper [BM], we plan to study fibrations of $\overline{M}_{0,n}$ onto either low dimensional varieties or with low n or with general linear fiber.

In the appendix we present an alternative proof, suggested by James M^{c}Kernan and Jenia Tevelev, of Theorem 1, as well as an example and suggestions of Rahul Pandharipande for $\overline{M}_{g,n}$.

1. Preliminaries

We work over the field of complex numbers. An *n-pointed stable curve of arithmetic genus* 0 is the datum $(C; q_1, \dots, q_n)$ of a tree of smooth rational curves and n ordered points on the nonsingular locus of C such that each component of C contains at least three points which are either marked or singular points of C. If $n \geq 3$, $\overline{M}_{0,n}$ is the smooth $(n-3)$-dimensional scheme constructed by Knudsen–Mumford, which is the fine moduli scheme of isomorphism classes $[(C; q_1, \dots, q_n)]$ of stable n-pointed curves of arithmetic genus 0.

For any $i \in \{1, \dots, n\}$ the forgetful map

$$\phi_i : \overline{M}_{0,n} \to \overline{M}_{0,n-1}$$

is the surjective morphism which associates to the isomorphism class $[(C; q_1, \dots, q_n)]$ of a stable n-pointed rational curve $(C; q_1, \dots, q_n)$ the isomorphism class of the $(n-1)$-pointed stable rational curve obtained by forgetting q_i and, if necessary, contracting to a point each component of C containing only q_i, one node of C and another marked point, say q_j. The locus of such curves forms a divisor, which we will denote by $E_{i,j}$. The morphism ϕ_i also plays the role of the universal curve morphism, so that its fibers are all rational curves transverse to $n-1$ divisors $E_{i,j}$. The divisors $E_{i,j}$ are the images of $n-1$ sections $s_{i,j} : \overline{M}_{0,n-1} \to \overline{M}_{0,n}$ of ϕ_i. The section $s_{i,j}$ associates to $[(C; q_1, \dots, q_i^{\vee}, \dots, q_n)]$ the isomorphism class of the n-pointed stable rational curve obtained by adding at q_j a smooth rational curve with marking of two points, labeled by q_i and q_j. Analogously, for every $I \subset \{1, \dots, n\}$, we have well defined forgetful maps $\phi_I : \overline{M}_{0,n} \to \overline{M}_{0,n-|I|}$. From our point of view the important part of a forgetful map is the set of forgotten indices, more than the actual marking of the remaining. For this we slightly abuse the language and introduce the following definition.

Definition 1.1. A *forgetful map* is the composition of $\phi_I : \overline{M}_{0,n} \to \overline{M}_{0,r}$ with an automorphism $g \in \mathrm{Aut}(\overline{M}_{0,r})$.

In order to avoid trivial cases we will always tacitly consider ϕ_I only if $n - |I| \geq 4$.

Besides the canonical class $K_{\overline{M}_{0,n}}$, on $\overline{M}_{0,n}$ are defined line bundles Ψ_i for each $i \in \{1, \dots, n\}$ as follows: the fiber of Ψ_i at a point $[(C; q_1, \dots, q_n)]$ is the tangent line T_{C,p_i}. Kapranov [Ka] proves the following:

Theorem 1.2. *Let $p_1, \dots, p_n \in \mathbb{P}^{n-2}$ be points in linear general position. Let H_n be the Hilbert scheme of rational curves of degree $n-2$ in $\mathbb{P}^{n-2}$. Then $\overline{M}_{0,n}$ is isomorphic to the subscheme $H \subset H_n$ parametrizing curves containing $p_1, \dots, p_n$. For each $i \in \{1, \dots, n\}$ the line bundle Ψ_i is big and globally generated and it induces a morphism $f_i : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ which is an iterated blow-up of the projections from p_i of the given points and of all strict transforms of the linear spaces they generate, in the order of increasing dimension.*

We will use the following notions:

Definition 1.3. A *Kapranov set* $\mathcal{K} \subset \mathbb{P}^{n-3}$ is an ordered set of $n-1$ points in linear general position, labeled by a subset of $\{1,\dots,n\}$. For any $J \subset \mathcal{K}$, the linear span of points in J is said to be a *vital linear subspace* of $\mathbb{P}^{n-3}$. A *vital cycle* is any union of vital linear subspaces.

To any Kapranov set, labeled by $\{1,\dots,i-1,i+1,\dots,n\}$, is uniquely associated a *Kapranov map* $f_i : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$, with $\Psi_i = f_i^*\mathcal{O}_{\mathbb{P}^{n-3}}(1)$, and to a Kapranov map is uniquely associated a Kapranov set up to projectivity.

Definition 1.4. Given a subset $I = \{i, i_1,\dots,i_s\} \subset \{1,\dots,n\}$ and the Kapranov map $f_i : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$, let $I^* = \{1,\dots,n\} \setminus I$. Then we indicate with

$$H^{i\vee}_{I^*} =: V^i_{I\setminus\{i\}} =: V^i_{i_1,\dots,i_s} := \langle p_{i_1},\dots,p_{i_s}\rangle \subset \mathbb{P}^{n-3}$$

the vital linear subspace generated by the p_{i_j}'s, and with

$$E_I := E_{i,i_1,\dots,i_s} := f_i^{-1}(V^i_I)$$

the divisor associated on $\overline{M}_{0,n}$.

Notice that $H^{h\vee}_{ij}$ is the hyperplane missing the points p_i and p_j and the set

$$\mathcal{K}' = (\mathcal{K} \setminus \{p_i, p_j\}) \cup (H^{\vee}_{ij} \cap \langle p_i, p_j\rangle)$$

is a Kapranov set in $H^{h\vee}_{ij}$.

In particular for any $i \in \{1,\dots,n\}$ and Kapranov set $\mathcal{K} = \{p_1,\dots,p_i^\vee,\dots,p_n\}$ the divisors $E_{i,j} = f_i^{-1}(p_j)$ are defined and this notation is compatible with the one adopted for the sections $E_{i,j}$ of ϕ_i. More generally, for any $i \in I \subset \{1,\dots,n\}$ the divisor E_I has the following property: its general point corresponds to the isomorphism class of a rational curve with two components, one with $|I|+1$ marked points, the other with $|I^*|+1$ marked points, glued together at the points not marked by elements of $\{1,\dots,n\}$. It follows from this picture that $E_I = E_{I^*}$ and that E_I is abstractly isomorphic to $\overline{M}_{0,|I|+1} \times \overline{M}_{0,|I^*|+1}$. The divisors E_I parametrise singular rational curves, and they are usually called *boundary divisors*. A further property of E_I is that for each choice of $i \in I$, $j \in I^*$, E_I is a section of the forgetful morphism

$$\phi_{I\setminus\{i\}} \times \phi_{I^*\setminus\{j\}} : \overline{M}_{0,n} \to \overline{M}_{0,|I^*|+1} \times \overline{M}_{0,|I|+1}.$$

This morphism is surjective and all fibers are rational curves. With our notations, $f_i(E_I)$ is a vital linear space of dimension $|I|-2$ if $i \in I$, and a vital linear space of dimension $|I^*|-2$ if $i \notin I$.

Definition 1.5. A dominant morphism $f : X \to Y$ is called a *fiber type morphism* if the dimension of the general fiber is positive, i.e. $\dim X > \dim Y$. A *fibration* is a fiber type morphism with connected fibers.

We are interested in describing linear systems on $\mathbb{P}^{n-3}$ that are associated to fibrations on $\overline{M}_{0,n}$. For this purpose we introduce some definitions.

Definition 1.6. A *linear system* on a smooth projective variety X is defined to be a pair (L, V), where $L \in \mathrm{Pic}(X)$ is a line bundle and $V \subseteq H^0(X, L)$ is a vector space. If no confusion is likely, we will let $\mathcal{L} = (L, V)$. In this case we will also abuse notation and write $A \in \mathcal{L}$ to mean $A \in \mathbb{P}(V)$, or in other words, to say that A is an effective divisor defined by some member of the linear system. Let $g : Y \to X$ be a birational morphism between smooth varieties. Let $A \in \mathcal{L}$ be a divisor and A_Y the strict transform on Y. Then $g^*L = A_Y + \Delta_A$ for some effective g-exceptional divisor Δ_A. For $A \in \mathcal{L}$ general, $\Delta_A = \Delta_{\mathcal{L}}$ does not depend on A. This allows one to define the strict transform of $\mathcal{L} = (L, V)$ via g as

$$\mathcal{L}_Y :=: g_*^{-1}\mathcal{L} := (g^*L - \Delta_{\mathcal{L}}, V_Y)$$

where V_Y is the vector space spanned by the strict transform of a base of general elements in V.

Definition 1.7. Let $\mathcal{K} \subset \mathbb{P}^{n-3}$ be a Kapranov set and $f_i : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ the associated map. An $M_{\mathcal{K}}$*-linear system* on $\mathbb{P}^{n-3}$ is a linear system $\mathcal{L} \subseteq |\mathcal{O}_{\mathbb{P}^{n-3}}(d)|$, for some d, such that $f_{i*}^{-1}\mathcal{L}$ is a base point free linear system.

Let $\mathcal{L}$ be an $M_{\mathcal{K}}$-linear system, and fix a Kapranov map $f_1 : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$. To better understand the properties of $M_{\mathcal{K}}$-linear systems let us look closer at f_1. Let $\epsilon : Y \to \mathbb{P}^{n-3}$ be the blow-up of $p_2 \in \mathcal{K}$ with exceptional divisor E. Then Kapranov's map f_1 can be factored as follows:

$$\begin{array}{ccc} \overline{M}_{0,n} & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle f_1} & \swarrow{\scriptstyle \epsilon} & \\ \mathbb{P}^{n-3} & & \end{array}$$

with a birational morphism $g : \overline{M}_{0,n} \to Y$. The map g is obtained by blowing up, in the prescribed order, the strict transform of every vital cycle of codimension at least 2 in Y. In particular it is an isomorphism on every codimension 1 point of $E \subset Y$. With this observation we are able to weakly control the base locus of $\mathcal{L}_Y$, the strict transform linear system.

Lemma 1.8. *Let $\mathcal{L}$ be an $M_{\mathcal{K}}$-linear system without fixed components, associated to the Kapranov map $f_i : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$. Let $\epsilon : Y \to \mathbb{P}^{n-3}$ be the blow-up of the Kapranov point $p_j \in \mathcal{K}$ with exceptional divisor E. Let $\mathcal{L}_Y$ be the strict transform. Then the linear system $\mathcal{L}_{Y|E}$ has no fixed components.*

Proof. Let

$$\begin{array}{ccc} \overline{M}_{0,n} & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle f_i} & \swarrow{\scriptstyle \epsilon} & \\ \mathbb{P}^{n-3} & & \end{array}$$

be the commutative diagram as above, with exceptional divisor $E \subset Y$. We noticed that g is an isomorphism on every codimension one point of E. That is, g is an isomorphism outside of a codimension two locus of E. By hypothesis $\mathcal{L}$ and hence $\mathcal{L}_Y$ have no fixed components. By construction g is a resolution of $\mathrm{Bs}\,\mathcal{L}_Y$. This yields

$$\mathrm{cod}_E(\mathrm{Bs}\,\mathcal{L}_Y \cap E) \geq 2,$$

and $\mathcal{L}_{Y|E}$ has no fixed components. □

A further property inherited from Kapranov's construction is the following.

Remark 1.9. Let $H_{ij}^{h\vee}$ be the hyperplane missing the points p_i and p_j and

$$\mathcal{K}' = (\mathcal{K} \setminus \{p_i, p_j\}) \cup (H_{ij}^{\vee} \cap \langle p_i, p_j \rangle)$$

the associated Kapranov set. Then $\mathcal{L}_{|H_{ij}^{\vee}}$ is an $M_{\mathcal{K}'}$-linear system.

Basic examples of fibrations are forgetful maps. Consider any set $I \subset \{1, \ldots, n\}$ and the associated forgetful map ϕ_I. If $j \notin I$ a typical diagram we will consider is

$$\begin{array}{ccc} \overline{M}_{0,n} & \xrightarrow{\phi_I} & \overline{M}_{0,n-|I|} \\ \downarrow{\scriptstyle f_j} & & \downarrow{\scriptstyle f_h} \\ \mathbb{P}^{n-3} & \overset{\pi_I}{\dashrightarrow} & \mathbb{P}^{n-3-|I|} \end{array}$$

where the f_j and f_h are Kapranov maps and π_I is the projection from V_I^j. In this case an $M_{\mathcal{K}}$-linear system associated to Φ_I is given by $|\mathcal{O}(1) \otimes \mathcal{I}_{V_I^j}|$ and if F_I is any fiber of ϕ_I, then $f_j(F_I)$ is a linear space of codimension $|I|$.

It is important for what follows to explicitly understand the rational map π_I when $j \in I$. For this we use Cremona transformations.

2. Cremona transformations and $M_{\mathcal{K}}$-linear systems

Definition 2.1. Let $\mathcal{K}$ be a Kapranov set with Kapranov map $f_i : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$. Then let

$$\omega_j^{\mathcal{K}} : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{n-3}$$

be the standard Cremona transformation centered on $\mathcal{K} \setminus \{p_j\}$. It is the map induced by forms of degree $n-3$ with points of multiplicity $n-4$ at $\mathcal{K} \setminus \{p_j\}$.

Remark 2.2. Via Kapranov's construction we may associate a Kapranov set labeled by $\{1, \ldots, n\} \setminus \{j\}$ to the right hand side $\mathbb{P}^{n-3}$, and in this notation Kapranov [Ka, Proposition 2.14] proved that

$$\omega_j^{\mathcal{K}} = f_j \circ f_i^{-1}$$

as birational maps. By a slight abuse of notation we may define $\omega_j^{\mathcal{K}}(V_I^i) := f_j(E_{I,i})$, even if ω_j is not defined on the general point of V_I^i.

Remark 2.3. Let $\omega_h^{\mathcal{K}}$ be the standard Cremona transformation centered on $\mathcal{K} \setminus \{p_h\}$, and $\mathcal{K}'$ the Kapranov set associated to the hyperplane $H_{jk}^{i\vee}$. Then for $h \neq j, k$ we have $\omega^{\mathcal{K}}_{h|H_{jk}^{i\vee}} = \omega_h^{\mathcal{K}'}$. This extends to arbitrary vital linear spaces. It follows from the definitions that $\omega_h^{\mathcal{K}}(V_I^i) = V^h_{I\setminus\{h\},i}$ if $h \in I$ and $\omega_h^{\mathcal{K}}(V_I^i) = V^h_{(I\cup\{i\})^*\setminus\{h\}}$ if $h \in I^*$.

Let us start with the special case of forgetful maps onto $\overline{M}_{0,4} \cong \mathbb{P}^1$. Let

$$\phi_I : \overline{M}_{0,n} \to \mathbb{P}^1 \cong \overline{M}_{0,4}$$

be a forgetful map and $\mathcal{M} = \phi_I^*\mathcal{O}(1)$. Choose a Kapranov map $f_i : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ with $\mathcal{L} := f_{i*}\mathcal{M} \subset |\mathcal{O}(1)|$. As already noticed, this is equivalent to choosing $i \notin I$. Then $\operatorname{Bs}\mathcal{L} = P$ is a codimension 2 linear space and we may assume, after reordering the indices, that

$$\mathcal{K} \setminus (\operatorname{Bs}\mathcal{L} \cap \mathcal{K}) = \{p_1, p_2, p_3\} \quad \text{and} \quad i = 4.$$

To understand what is the linear system $\mathcal{L}_5 := f_{5*}\mathcal{M}$ we use Kapranov's description of the map $\omega_5^{\mathcal{K}}$ (see Definition 2.1).

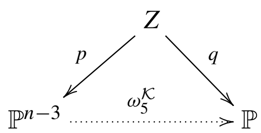
This is well known but we decided to write it down for readers less familiar with the classical subject of Cremona transformations. The map is the standard Cremona transformation centered on $\{p_1, p_2, p_3, p_6, \ldots, p_n\}$. Let

$$\omega_5^{\mathcal{K}} : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{n-3} =: \mathbb{P}$$

be the map given by the linear system

$$\left| \mathcal{O}_{\mathbb{P}^{n-3}}(n-3) \otimes \bigotimes_{i \in \{1,2,3,6,\ldots,n\}} \mathcal{I}_{p_i}^{n-4} \right|.$$

Let



be the usual resolution obtained by blowing up, in dimension increasing order, all linear spaces spanned by points in $\{p_1, p_2, p_3, p_6, \ldots, p_n\}$. Let $l \subset \mathbb{P}$ be a general line; then q^{-1} is well defined on l and $p(q^{-1}(l))$ is a rational normal curves passing through $\{p_1, p_2, p_3, p_6, \ldots, p_n\}$. Let $E_i \subset Z$ be the exceptional divisor corresponding to the blow-up of the points p_i. Then $q^{-1}(l) \cdot E_i = 1$, for $i \in \{1, 2, 3, 6, \ldots, n\}$, while $q^{-1}(l) \cdot F$ vanishes for any other p-exceptional divisor. This description allows us to easily compute the degree of $\mathcal{L}_5$:

$$\deg \mathcal{L}_5 = \deg \omega_5^{\mathcal{K}}(\mathcal{L}) = (n-3) - \sum_{h \neq 5} \operatorname{mult}_{p_h} \mathcal{L} = 2.$$

This yields $\omega_5^{\mathcal{K}}(\mathcal{L}) \subset |\mathcal{O}(2)|$.

To complete the analysis we have to understand the base locus of this system of quadrics. Let $\mathcal{K}_5$ be the Kapranov set labeled by $\{1, 2, 3, 4, 6, \ldots, n\}$.

In our convention (see Remark 2.2), for $\{i, j\} \subset \{1, 2, 3\}$ we have

$$\omega_5^{\mathcal{K}}(V_{i,j}^4) = V^5_{\{i,j,4\}^*\setminus\{5\}}.$$

That is, this line is sent to a codimension two linear space. Let $P_h = \omega_5^{\mathcal{K}}(V_{i,j}^4)$ for $\{i, j, h\} = \{1, 2, 3\}$. The general element in $\mathcal{L}$ intersects a general point of $V_{i,j}^4$ and therefore its transform via $\omega_5^{\mathcal{K}}$ has to contain P_h. The hypothesis $5 \in I$ tells us that the map $\omega_5^{\mathcal{K}}$ is well defined on the general point of $P = V_I^4$. Let $P_4 = \omega_5^{\mathcal{K}}(P)$ and $S := \omega_5^{\mathcal{K}}(\langle p_1, p_2, p_3\rangle)$. Then P_4 has to be contained in $\mathrm{Bs}\,\mathcal{L}_5$ and

$$S = \bigcap_{i=1}^{4} P_i.$$

In conclusion we have:

- $\mathrm{Bs}\,\mathcal{L}_4 = \bigcup_{i=1}^4 P_i \supset \mathcal{K}_5$,
- $\mathrm{Sing}(\mathcal{L}_5) = S$.

Hence the linear system $\mathcal{L}_5$ is a pencil of quadrics with four codimension two linear spaces in the base locus. It is the cone, in $\mathbb{P}^{n-3}$, over a pencil of conics through four general points and with vertex $V^5_{I\setminus\{5\}}$.

To study fibrations from $\overline{M}_{0,n}$ it is important to control the base locus of $M_{\mathcal{K}}$-linear systems. The easiest base loci are those of forgetful maps $\phi_I : \overline{M}_{0,n} \to \overline{M}_{0,r}$. For this we introduce the following definitions.

Definition 2.4. Let $\pi_i : \mathbb{P}^{r-2} \dashrightarrow \mathbb{P}^{r-3}$ be the projection from a Kapranov point p_i, and $\mathcal{L} = |\mathcal{O}_{\mathbb{P}^{r-2}}(1) \otimes \mathcal{I}_{p_i}|$. Define

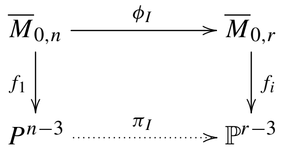
$$\mathcal{C}^i_{r-3} := \omega_i^{\mathcal{K}}(\mathcal{L}) \subset |\mathcal{O}_{\mathbb{P}^{r-2}}(r-2)|$$

to be the transform of hyperplanes through the point p_i. We say that an $M_{\mathcal{K}}$-linear system $\mathcal{M}$ on $\mathbb{P}^{n-3}$ *has base locus of type* Φ_r if $\mathrm{Supp}(\mathrm{Bs}\,\mathcal{M})$ is either a codimension $r-2$ linear space or the cone over $\mathrm{Bs}\,\mathcal{C}^i_{r-3}$ with vertex a linear space of codimension $r-1$. Equivalently $\mathcal{M}$ has base locus of type Φ_r if it is an $M_{\mathcal{K}}$-linear system with base locus supported on a linear space of codimension $r-2$ up to standard Cremona transformations.

In this notation $\mathcal{C}^i_1$ is a pencil of plane conics through four fixed points. The above construction shows that to a forgetful map $\phi_I : \overline{M}_{0,n} \to \overline{M}_{0,4}$ are associated $M_{\mathcal{K}}$-linear systems with base locus of type Φ_4. This is actually the main motivation of our definition. We will use and improve this observation first in Proposition 2.5 and further in Lemma 3.5. The main point in our construction is that the base locus of $M_{\mathcal{K}}$-linear systems is enough to characterise linear systems inherited by forgetful maps.

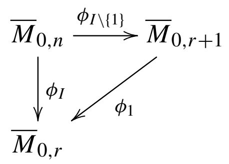
The special case of forgetful maps onto $\mathbb{P}^1$ is the one we use in this paper. Nonetheless we would like to stress that a similar behavior applies to an arbitrary forgetful map onto $\overline{M}_{0,r}$, for $r < n$.

Proposition 2.5. *Let $\phi_I : \overline{M}_{0,n} \to \overline{M}_{0,r}$ be a forgetful morphism. Assume that $1 \in I$ and let*

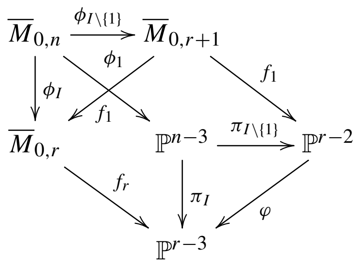


be the usual diagram. Then π_I is given by a sublinear system $\mathcal{L}_1 \subset |\mathcal{O}(r-2)|$, the general fiber of π_I is a cone, with vertex $V^1_{I\setminus\{1\}} \subset \mathbb{P}^{n-3}$, over a rational normal curve of degree $n-2-|I|$, and $\mathcal{L}_1$ has base locus of type Φ_r.

Proof. The morphism Φ_I can be factored as follows:



Hence we have the induced diagram



where $\pi_{I\setminus\{1\}}$ is a linear projection and φ is the map induced by $\mathcal{C}^2_r$. The claim follows. $\square$

3. Base point free pencils on $\overline{M}_{0,n}$

A *base point free pencil* $\mathcal{L}$ on $\overline{M}_{0,n}$ is the datum of a couple (L, V) on $\overline{M}_{0,n}$, where L is a line bundle on $\overline{M}_{0,n}$ and $V \subset H^0(\overline{M}_{0,n}, L)$ is a two-dimensional subspace. The natural map $V \otimes \mathcal{O}_{\overline{M}_{0,n}} \to L$ is surjective and this datum is equivalent to a surjective morphism $f : \overline{M}_{0,n} \to \mathbb{P}^1$ such that $L = f^*(\mathcal{O}(1))$.

Fix a Kapranov map $f_i : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ and let $\mathcal{L}_i := f_*\mathcal{L}$. Then the linear system $\mathcal{L}_i$ is an $M_{\mathcal{K}}$-linear system, inducing a birational map $\pi^i : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^1$. By definition of $M_{\mathcal{K}}$-linear system, $\mathcal{L} = f^{-1}_{i*}\mathcal{L}_i$ and f is a, not necessarily minimal, resolution of the indeterminacy of the map π^i. To the map f we therefore associate a diagram

$$\begin{array}{ccc} \overline{M}_{0,n} & \xrightarrow{f} & \mathbb{P}^1 \\ {\scriptstyle f_i}\downarrow & & \downarrow\cong \\ \mathbb{P}^{n-3} & \overset{\pi^i}{\dashrightarrow} & \mathbb{P}^1 \end{array}$$

where $\pi^i := f \circ f_i^{-1}$. The rational map π^i is uniquely associated to a pencil $\mathcal{L}_i \subset |\mathcal{O}_{\mathbb{P}^{n-3}}(d_i)|$ free of fixed divisors on $\mathbb{P}^{n-3}$. We have $\mathcal{L}_i = (\mathcal{O}_{\mathbb{P}^{n-3}}(d_i), W_i)$ where $W_i \subset H^0(\mathbb{P}^{n-3}, \mathcal{O}(d_i))$ has dimension two, and any element of V is the strict transform of an element of W_i, i.e. the strict transform map $f_{i*}^{-1} : W_i \to V$ is an isomorphism and the support of the cokernel of the evaluation map

$$\mathrm{ev}_i : W_i \otimes \mathcal{O}_{\mathbb{P}^{n-3}} \to \mathcal{L}_i$$

on $\mathbb{P}^{n-3}$ does not have divisorial components. In particular the support of the cokernel of ev_i is the base locus $\mathrm{Bs}\,\mathcal{L}_i$ of $\mathcal{L}_i$.

The most important example of dominant maps $f : \overline{M}_{0,n} \to \mathbb{P}^1$ is given by the forgetful maps already described in Section 2. The goal of this section will be to prove that in fact any surjective map with connected fibers $f : \overline{M}_{0,n} \to \mathbb{P}^1$ is in fact a forgetful map. The criterion we are going to use in order to understand whether a morphism $f : \overline{M}_{0,n} \to \mathbb{P}^1$ is a forgetful map is the following:

Proposition 3.1. *Let $f : \overline{M}_{0,n} \to X$ be a surjective morphism. Let $A \in \mathrm{Pic}(X)$ be a base point free linear system and $\mathcal{L}_i = f_{i*}(f^*(A))$. Assume that $\mathrm{mult}_{p_j} \mathcal{L}_i = \deg \mathcal{L}_i$ for some j. Then f factors through the forgetful map $\phi_j : \overline{M}_{0,n} \to \overline{M}_{0,n-1}$.*

Let $f : \overline{M}_{0,n} \to \overline{M}_{0,r}$ be a surjective morphism and $\pi : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{r-3}$ the induced map. Let $\mathcal{L}_i = f_{i*}(f^*(g_{j*}^{-1}(\mathcal{O}(1))))$ and assume that $\mathcal{L}_i \subset |\mathcal{O}_{\mathbb{P}^{n-3}}(1)|$. Then f is a forgetful map.

Proof. Let $\phi_j : \overline{M}_{0,n} \to \overline{M}_{0,n-1}$ be the forgetful morphism and F_j a general fiber. We already noticed that $f_i(F_j)$ is a line through the point p_j, for $i \neq j$. If $\mathrm{mult}_{p_j} \mathcal{L}_i = \deg \mathcal{L}_i$ then the restriction of $\mathcal{L}_i$ to each such line has a trivial moving part. This gives $f^*(A) \cdot F_j = 0$. The linear system $f^*(A)$ is base point free hence $f^*(A)$ is numerically trivial relatively to the morphism ϕ_j, i.e. $f^*(A) \cdot C = 0$ for any curve C contracted by ϕ_j. By construction, fibers of $\phi_j : \overline{M}_{0,n} \to \overline{M}_{0,n-1}$ are rational curves. This yields the vanishing $R^1\phi_{j*}\mathcal{O}_{\overline{M}_{0,n}} = 0$, and it is enough to conclude that any ϕ_j-numerically trivial linear system is the pull-back of a linear system in $\overline{M}_{0,n-1}$. Hence fibers of ϕ_j are contracted by f.

For any $h \neq i, j$ the map ϕ_j has a section $s_{j,h} : \overline{M}_{0,n-1} \to E_{j,h} \subset \overline{M}_{0,n}$ described in Section 1, and then a morphism $g := f \circ s_{j,h} : \overline{M}_{0,n-1} \to X$ is given such that $f = g \circ \phi_j$. Notice that g does not depend on the choice of $h \neq i, j$.

Assume that $\mathcal{L}_i = |\mathcal{O}_{\mathbb{P}^{n-3}}(1) \otimes \mathcal{I}_{V_I^i}|$ for some vital cycle $V_I^i \subset \mathbb{P}^{n-3}$. Then for any vital point $p_k \in V_I^i$ we have $\mathrm{mult}_{p_k} \mathcal{L}_i = \deg \mathcal{L}_i$. Hence by the first statement we infer that f factorizes through ϕ_k for any $k \in I$. Then there is a map $g : \overline{M}_{0,r} \to \overline{M}_{0,r}$ such that $f = g \circ \phi_I$. Let $\gamma : \mathbb{P}^{r-3} \dashrightarrow \mathbb{P}^{r-3}$ be the induced map. Then γ is associated to a linear system of hyperplanes and it is therefore a projectivity that possibly permutes the Kapranov set on $\mathbb{P}^{r-3}$ (keep in mind our Definition 1.1). □

Definition 3.2. Let $\mathcal{L}_i := (L_i, W_i)$ be an $M_{\mathcal{K}}$-linear system on $\mathbb{P}^{n-3}$ and $A_i \in \mathcal{L}_i$ a general element. Let $H = H^{i\vee}_{h,k}$ be a vital hyperplane. We say that the restriction of $\mathcal{L}_i$ to H is *dominant* if the restriction map $\text{res}_H : W_i \to H^0(H, L_{i|H})$ is injective. Let $p_j \in \mathcal{K}$ be a Kapranov point. Let $\epsilon_j : Y_j \to \mathbb{P}^{n-3}$ be the blow-up of p_j with exceptional divisor E_j. Assume that $\epsilon_j^* A_i = A_{iY} + mE_j$. Then

$$\mathcal{L}_{i,Y_j} := (\epsilon_j^* L_i - mE_j, W_i^Y)$$

is the strict transform of $\mathcal{L}_i$. We say that $\mathcal{L}_i$ is *dominant* at the first order of p_j if the pull-back map $\epsilon_j^* : W_i \to H^0(E_j, (\epsilon_j^* L_i - mE_j)_{|E_j})$ is injective and $\mathcal{L}_{i,Y_j|E_j}$ is without fixed divisors.

We sketch here the ideas underlying our argument in order to characterise base point free pencils on $\overline{M}_{0,n}$.

We proceed by induction on n and assume that all base point free pencils on $\overline{M}_{0,n-1}$ inducing a surjective map $f : \overline{M}_{0,n-1} \to \mathbb{P}^1$ with connected fibers are forgetful maps. The result in Lemma 3.5 makes induction perfectly suitable for $n \geq 7$. Hence we prefer to leave as initial steps both the easy and well known case of the del Pezzo surface $\overline{M}_{0,5}$ (see also [BM]) and $\overline{M}_{0,6}$. The latter has been fully studied in [FG] and from our viewpoint is slightly more complicated than the others. Hence we decided to lean on [FG] instead of giving a direct proof; the interested reader can consult the arXiv version of our paper for this, [BMx].

According to Proposition 3.1, for the induction step it is enough to show that there exists a Kapranov map f_i and a Kapranov point p_j such that

$$\text{mult}_{p_j} \mathcal{L}_i = \deg \mathcal{L}_i.$$

To produce this point we find a vital hyperplane H such that the restriction of $\mathcal{L}_i$ to H is dominant. Then the hyperplane H has a Kapranov set and it is the image under a Kapranov map of $\overline{M}_{0,n-1}$ (see Remark 1.9). By induction we may find the required point for $\mathcal{L}_{i|H}$ and then lift it to the linear system $\mathcal{L}_i$. Here is a list of concerns in applying this idea:

- How can we find a vital hyperplane H such that the restriction of $\mathcal{L}_i$ to it is dominant?
- How to compare $\text{mult}_{p_j} \mathcal{L}_{i|H}$ with $\text{mult}_{p_j} \mathcal{L}_i$?
- What if the restricted morphism has either disconnected fibers or fixed components?

As a matter of fact, even if (L_i, W_i) is free of fixed divisors and if it induces a map with connected fibers, this may not be the case for the restricted linear system on any vital hyperplane $H \subset \mathbb{P}^{n-3}$. Keep in mind that there are only finitely many such hyperplanes.

The desired hyperplane H is produced in Lemmas 3.3 and 3.4. The basic idea is that the base locus of $\mathcal{L}_i$ cannot be empty because $\mathbb{P}^{n-3}$ does not carry base point free pencils so that there exists some point p_j contained in the base locus of $\mathcal{L}_i$. Notice that this does not mean that such a point is an isolated component of $\mathrm{Bs}\,\mathcal{L}_i$. Thanks to Lemma 1.8 we can prove that the $M_{\mathcal{K}}$-linear system (L_i, W_i) is dominant at the first order of p_j. To apply induction on the exceptional divisor over the point p_j we have to study pencils with possibly disconnected fibers. This is done in Lemma 3.5. With this and induction hypothesis we know that the pencil induced on the exceptional divisor has base locus of type Φ_4. Hence we may apply induction and find a hyperplane H such that $\mathcal{L}_i$ restricted to H is dominant.

Finally, we use Lemma 3.6 in order to show that we can in fact exclude the presence of fixed divisors on the restricted linear systems, so that we can really infer properties of (L_i, W_i) from properties of the restriction to some hyperplane H.

We now prove the above mentioned lemmas. Let us fix a pencil $\mathcal{L}_i$, without fixed components, together with the usual diagram

$$\begin{array}{ccc} \overline{M}_{0,n} & \xrightarrow{\ f\ } & \mathbb{P}^1 \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle \cong} \\ \mathbb{P}^{n-3} & \overset{\pi^i}{\dashrightarrow} & \mathbb{P}^1 \end{array}$$

Let $p_j \in \mathcal{K} \cap \mathrm{Bs}\,\mathcal{L}_i$ be a point and $\epsilon_j : Y_j \to \mathbb{P}^{n-3}$ the blow-up of p_j with exceptional divisor E_j. Let $\mathcal{L}_{i,Y_j} = \epsilon_j^* \mathcal{L}_i - m_j E_j$ be the strict transform of $\mathcal{L}_i$, for some positive m_j.

Lemma 3.3. *The linear system $\mathcal{L}_i = (L_i, W_i)$ is dominant at the first order of p_j and for any $A_1, A_2 \in \mathcal{L}_i$ we have*

$$\mathrm{mult}_{p_j} A_1 = \mathrm{mult}_{p_j} A_2.$$

Let $H = H^{i\vee}_{hk}$ be a vital hyperplane containing p_j, and $A \in \mathcal{L}_i$ a general element. Then

$$\mathrm{mult}_{p_j} A = \mathrm{mult}_{p_j} A_{|H}.$$

Proof. In the above notation we know, by Lemma 1.8, that $\mathcal{L}_{i,Y_j|E_j}$ has no fixed components. Hence the image of the pull-back map $\epsilon_j^* : W_i \to H^0(E_j, \mathcal{L}_{i,Y_j|E_j})$ is not one-dimensional. Since $\dim W_i = 2$ we conclude that ϵ_j^* is injective as required. The injectivity of ϵ_j^* forces every element in $\mathcal{L}_i$ to have the same multiplicity at p_j.

Let H_{Y_j} be the strict transform of H on Y_j. Then again by Lemma 1.8 we know that $\mathrm{Bs}\,\mathcal{L}_{i,Y_j} \not\supset E \cap H_{Y_j}$. Therefore the general element $A \in \mathcal{L}_i$ satisfies $\mathrm{mult}_{p_j} A = \mathrm{mult}_{p_j} A_{|H}$. □

Lemma 3.4. *Let $H = V^i_I \subset \mathbb{P}^{n-3}$ be a vital hyperplane such that $p_j \in H$. If $f(E_{i,I})$ is a point and if H_j is the strict transform of H under ϵ_j, then $L_{i,Y_j|E_j}$ is trivial along $H_j \cap E_j$.*

Proof. The morphism f_i is a resolution of indeterminacies of π^i and f_i factors through ϵ_j. Hence the result follows. □

In Section 2 we proved that every forgetful map onto $\mathbb{P}^1$ induces an $M_{\mathcal{K}}$-linear system of degree at most 2 with base locus of type Φ_4. One cannot expect that all morphisms to $\mathbb{P}^1$ have bounded degree. On the other hand, under suitable hypothesis, the base locus of $M_{\mathcal{K}}$-linear systems is unaffected by connectedness of fibers.

Lemma 3.5. *Assume that every dominant morphism $g : \overline{M}_{0,n} \to \mathbb{P}^1$ with connected fibers is a forgetful map. Then* $\mathrm{Bs}\,\mathcal{L}_i$ *is of type* Φ_4*. If moreover* $n \geq 6$ *there are vital points* p_j *satisfying* $\mathrm{mult}_{p_j}\mathcal{L}_i = \deg \mathcal{L}_i$.

Proof. Let $h : \overline{M}_{0,n} \to C$ be the Stein factorization of f. Let $\nu : \tilde{C} \to C$ be the normalization. Then there is a unique map $f' : \overline{M}_{0,n} \to \tilde{C}$ such that $h = \nu \circ f'$. The variety $\overline{M}_{0,n}$ is rational, therefore $\tilde{C} \cong \mathbb{P}^1$ and $|f^*\mathcal{O}(1)| \subset |f'^*\mathcal{O}(\gamma)|$ for some integer γ. By hypothesis f' is a forgetful map, so we may choose i in such a way that $|f_{i*}f'^*\mathcal{O}(1)| \subset |\mathcal{O}_{\mathbb{P}^{n-3}}(1)|$ satisfies

$$\mathrm{Bs}\,|f_{i*}f'^*\mathcal{O}(1)| = V_I^j,$$

where V_I^j is a codimension two irreducible vital space. Then elements in the linear system $\mathcal{L}_i$ are unions of γ hyperplanes containing V_I^j. Hence

$$\mathrm{Supp}(\mathrm{Bs}\,\mathcal{L}_i) := \mathrm{Supp}(\mathrm{Bs}\,|f_{i*}f^*\mathcal{O}(1)|) = \mathrm{Supp}(\mathrm{Bs}\,|f_{i*}f'^*\mathcal{O}(1)|) = V_I^j$$

and $\mathrm{mult}_{V_I^j}\mathcal{L}_i = \gamma$. To conclude it is enough to apply standard Cremona transformations to this configuration as described in Section 2. In particular if $n \geq 6$ all linear systems of type Φ_4 are cones with nonempty vertex and therefore there is at least one point p_j with $\mathrm{mult}_{p_j}\mathcal{L}_i = \deg \mathcal{L}_i$. □

We conclude this technical part by taking into account possible fixed divisors.

Lemma 3.6. *Let $H = H^{i\vee}_{h,k}$ be a vital hyperplane in $\mathbb{P}^{n-3}$, with $j \neq h, k$. Assume that $\mathcal{L}_i = (L_i, W_i)$ has a dominant restriction to H. Assume that F is a fixed divisor of the restricted system $\mathcal{L}_{i|H}$ and that $p_j \notin F$. Then*

$$F = \langle p_l \mid l \neq h, k, j\rangle = V^i_{\{i,h,k,j\}^*}.$$

Proof. The fixed divisor $F \subset H$ is in the base locus of (L_i, W_i). By hypothesis, $\mathcal{L}_i$ has no fixed divisors. Hence the support of F must be an irreducible component of $\mathrm{Bs}\,\mathcal{L}_i$. In particular F does not contain p_h, p_k, and therefore cannot intersect the line $\langle p_h, p_k\rangle = V^i_{h,k}$. By hypothesis, $p_j \notin F$. Hence the only possibility left is $F = \langle p_l \mid l \neq h, k, j\rangle$. □

We are ready for the proof of the following:

Theorem 3.7. *Let $f : \overline{M}_{0,n} \to \mathbb{P}^1 \cong \overline{M}_{0,4}$ be a fibration. Then f is a forgetful map.*

Proof. We use induction on n. The result is known for $n \leq 6$ [FG]. Let $f_n : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ be a Kapranov map with $\mathcal{K} = \{p_1, \ldots, p_{n-1}\}$. Let $\mathcal{L}_n = (L_n, V_n) = f_n(f^*(\mathcal{O}(1)))$ be the associated linear system. Then the linear system $\mathcal{L}_n$ is a pencil of hypersurfaces, without fixed components, say $\mathcal{L}_n = \{A_1, A_2\}$, and it is an $M_{\mathcal{K}}$-linear system.

From Lemma 3.3 we know that there exists $p_1 \in \mathcal{K} \cap \mathrm{Bs}\,\mathcal{L}_n$ such that $\mathcal{L}_n$ is dominant at the first order of p_1. From Lemma 3.3 we get

$$m = \mathrm{mult}_{p_1} A_1 = \mathrm{mult}_{p_1} A_2 \quad \text{and} \quad \mathrm{mult}_{p_1}\mathcal{L} = \mathrm{mult}_{p_1}\mathcal{L}_{|H^{n\vee}_{h,k}} \text{ for } h, k \neq 1. \tag{1}$$

Let $\epsilon_j : Y_j \to \mathbb{P}^{n-3}$ be the blow-up of p_j with exceptional divisor E_j, for $j = 1, \ldots, n-1$. By induction and Lemma 3.5, the base locus $\mathrm{Bs}\,\mathcal{L}_{n,Y_1}|_{E_1}$ is of type Φ_4.

Claim 1. *We may assume that* $\mathrm{Supp}(\mathrm{Bs}\,\mathcal{L}_{n,Y_1}|_{E_1})$ *is a codimension two linear space, obtained by intersecting the* ϵ_1*-strict transform of* $V^n_{1,\dots,n-4}$ *with* E_1.

Proof of Claim 1. Since $\mathrm{Bs}\,\mathcal{L}_{n,Y_1}|_{E_1}$ is of type Φ_4, if its support is not a codimension two linear space, it is the intersection with E_1 of the strict transform of the union of four codimension two spaces P_{n-i} for $i = 1, \dots, 4$, containing a codimension three linear space S. Let us assume that $S = V^n_{1,\dots,n-5}$, and that $P_{n-i} = V^n_{1,\dots,n-5,n-i}$ for $i = 1, \dots, 4$.

In particular $\mathcal{K} \subset \mathrm{Bs}\,\mathcal{L}_n$ and we apply Lemma 3.3 to all points of $\mathcal{K}$. Let us focus on the point p_{n-4}. If $\mathrm{Supp}(\mathrm{Bs}\,\mathcal{L}_{n,Y_{n-4}}|_{E_{n-4}})$ is not a codimension two linear space, it consists of the strict transform of four codimension two linear spaces intersecting at a codimension three space, obtained by removing a point from $V^n_{1,\dots,n-5,n-4}$. By hypothesis, $n \geq 7$, hence we may assume that this point is not p_1. This produces new components in $\mathrm{Supp}(\mathrm{Bs}\,\mathcal{L}_{n,Y_1}|_{E_1})$, and contradicts the type Φ_4. □

Let H be a vital hyperplane containing p_1 but not containing $V^n_{1,\dots,n-4}$. Then Claim 1 and Lemma 3.4 imply that the restriction of $\mathcal{L}_n$ to H is dominant.

Claim 2. *We may choose* H *in such a way that* $\mathcal{L}_{n|H}$ *is free of fixed divisors.*

Proof of Claim 2. Let $H_1 = H^{n\vee}_{n-4,n-3}$ and $H_2 = H^{n\vee}_{n-5,n-2}$ be two vital hyperplanes. Assume that the restriction of $\mathcal{L}_n$ to H_i has a nonempty fixed divisor F_i for $i = 1, 2$. Then from Lemma 3.6 the supports of F_1 and of F_2 are respectively $V^n_{2,\dots,n-5,n-2,n-1}$ and $V^n_{2,\dots,n-6,n-4,n-3,n-1}$.

Lemma 3.3 applied to the point p_{n-1} shows that $\mathcal{L}_n$ dominates p_{n-1} at first order. Let $\mathcal{K}_{n-1}$ be the induced Kapranov set on E_{n-1}. Then $\mathrm{Supp}(\mathrm{Bs}\,\mathcal{L}_{n,E_{n-1}}|_{E_{n-1}})$ has two irreducible components meeting in codimension 4. This is not possible for a linear system with base locus of type Φ_4. □

Let H be a vital hyperplane such that the restriction of $\mathcal{L}_n$ to H is dominant and $\mathcal{L}_{n|H}$ is free of fixed divisors. Then by Lemmas 3.5 and 3.3 there is a Kapranov point $p_h \in H$ such that

$$\mathrm{mult}_{p_h}\,\mathcal{L}_n = \mathrm{mult}_{p_h}\,\mathcal{L}_{n|H} = \deg \mathcal{L}_{n|H} = \deg \mathcal{L}_n$$

so that

$$\mathrm{mult}_{p_h}\,\mathcal{L}_n = \deg \mathcal{L}_n.$$

We conclude by Proposition 3.1 that f factors via the forgetful map ϕ_h. That is, $f = g \circ \phi_h$ for some morphism $g : \overline{M}_{0,n-1} \to \mathbb{P}^1$ with connected fibers. By induction hypothesis, g is a forgetful map. Hence f is forgetful. □

Corollary 3.8. *Let* $f : \overline{M}_{0,n} \to \mathbb{P}^1 \cong \overline{M}_{0,4}$ *be a nonconstant morphism. Then* f *factors through a forgetful map and a finite morphism.*

Proof. Taking the Stein factorization and the normalization, as in Lemma 3.5, we reduce the claim to Theorem 3.7. □

4. Morphisms to $\overline{M}_{0,r}$

In this final section we apply Theorem 3.7 to deduce that in fact any fibration $f : \overline{M}_{0,n} \to \overline{M}_{0,r}$ is a forgetful map. Note that f is dominant, hence $r < n$. As a corollary, we compute the automorphism group of $\overline{M}_{0,n}$.

Theorem 4.1. *Let $f : \overline{M}_{0,n} \to \overline{M}_{0,r}$ be a fibration. Then f is a forgetful map.*

Proof. We use induction on r. The first step, for $r = 4$, is the content of Theorem 3.7.

Let us fix a Kapranov map $f_r : \overline{M}_{0,r} \to \mathbb{P}^{r-3}$ and consider the forgetful map

$$\phi_{r-1} : \overline{M}_{0,r} \to \overline{M}_{0,r-1},$$

the Kapranov map $f_{r-1} : \overline{M}_{0,r-1} \to \mathbb{P}^{r-4}$ and the projection $\pi : \mathbb{P}^{r-3} \dashrightarrow \mathbb{P}^{r-4}$ given by the linear system $\Lambda_{r-1} = |\mathcal{O}_{\mathbb{P}^{r-3}}(1) \otimes \mathcal{I}_{p_{r-1}}|$. Then by induction hypothesis $\phi_{r-1} \circ f : \overline{M}_{0,n} \to \overline{M}_{0,r-1}$ is dominant and with connected fibers, hence a forgetful map. This means that we may choose a Kapranov map $f_n : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ such that

$$f_{n*}((f_r \circ f)_*^{-1}(\Lambda_{r-1})) \subset |\mathcal{O}_{\mathbb{P}^{n-3}}(1)|.$$

Recall that, from Proposition 3.1, we obtain the conclusion if we show that

$$f_{n*}((f_r \circ f)^*(\mathcal{O}_{\mathbb{P}^{r-3}}(1))) \subset |\mathcal{O}_{\mathbb{P}^{n-3}}(1)|.$$

By construction we have $\Lambda_{r-1} = |\mathcal{O}_{\mathbb{P}^{r-3}}(1) \otimes \mathcal{I}_{p_{r-1}}|$ and $f_r^{-1}(p_{r-1}) = E_{r,r-1}$. To conclude it is enough to show that $f^*(E_{r,r-1})$ is f_n-exceptional.

The following diagram will help us along the proof:

$$\begin{array}{ccccc}
\overline{M}_{0,n} & \xrightarrow{f} & \overline{M}_{0,r} & \xrightarrow{\phi_{r-1}} & \overline{M}_{0,r-1} \\
\downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_r} & & \downarrow{\scriptstyle f_{r-1}} \\
\mathbb{P}^{n-3} & \overset{\varphi}{\dashrightarrow} & \mathbb{P}^{r-3} & \overset{\pi}{\dashrightarrow} & \mathbb{P}^{r-4}
\end{array}$$

Let $\mathcal{L}$ be the linear system associated to the map

$$\pi \circ \varphi = f_{r-1} \circ \phi_{r-1} \circ f \circ f_n^{-1}.$$

By induction hypothesis we may assume that f_n is such that $\mathcal{L} = |\mathcal{O}(1) \otimes \mathcal{I}_P|$, where $P = \langle p_{r-1}, \ldots, p_{n-1} \rangle$. We fix notations in such a way that $(\pi \circ \varphi)(p_j) = p_j$ for $j < r-1$, and $\pi(p_j) = p_j$ for $j < r - 1$.

For any $E_{j,r} \neq E_{r-1,r}$ the map $\phi_{r-1|E_{j,r}} : \overline{M}_{0,r-1} \to \overline{M}_{0,r-1}$ is a forgetful map onto $\overline{M}_{0,r-2}$. Then for any $E_{i,r} \subset \overline{M}_{0,r}$ with $i < r - 1$, we have

$$f^*(E_{i,r}) = (\phi_{r-1} \circ f)^*(E_{i,r-1}) = E_{i,n}.$$

This shows that $f^*(E_{i,r})$ is f_n-exceptional for $i < r - 1$.

Notice that, once we fix f_r and choose the forgetful map ϕ_{r-1}, we have f_n such that $f^*(E_{i,r}) = E_{i,n}$ for $i < r - 1$.

We are assuming $r \geq 5$, hence, once we fix the Kapranov map $f_r : \overline{M}_{0,r} \to \mathbb{P}^{r-3}$ there are at least four possible forgetful maps $\phi_i : \overline{M}_{0,r} \to \overline{M}_{0,r-1}$ with $i < r$. To any such ϕ_i we may associate a Kapranov map $f_{n_i} : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ in such a way that $f^*(E_{j,r}) = E_{j,n_i}$ for $j \neq i$. On the other hand, the divisor $E_{i,j} \subset \overline{M}_{0,n}$ is sent to a point only by f_i and f_j. We may assume that $n_1 = n_2 = n$. The image of the divisor $E_{i,r}$ via $f_{n*}f^*$ does not depend on the map ϕ_i. Therefore $f_{n*}f^*(E_{i,r})$ is a point for any $i = 1, \ldots, r-1$.

By definition

$$f_r^*(\mathcal{O}(1)) = f_{r*}^{-1}\Lambda_{r-1} + E_{r-1,r},$$

hence

$$\mathcal{L} = f_{n*}((f_r \circ f)^*(\mathcal{O}(1))) = f_{n*}((f_r \circ f)^*(\Lambda_{r-1})) \subset |\mathcal{O}_{\mathbb{P}^{n-3}}(1)|,$$

and φ is given by a linear system of hyperplanes. This is equivalent to our statement by Proposition 3.1. □

This easily extends to morphisms onto products of $\overline{M}_{0,r_i}$.

Corollary 4.2. *Let $f : \overline{M}_{0,n} \to \overline{M}_{0,r_1} \times \cdots \times \overline{M}_{0,r_h}$ be a dominant fibration. Then f is a forgetful map.*

Proof. It is enough to compose f with projections onto factors. □

From Theorem 4.1 an automorphism of $\overline{M}_{0,n}$ must preserve all forgetful maps. This gives a very strong condition on the induced linear system of $\mathbb{P}^{n-3}$. We are ready to prove the main result on $\mathrm{Aut}(\overline{M}_{0,n})$. This is classical for $n = 5$.

Theorem 4.3. *Assume that $n \geq 5$. Then $\mathrm{Aut}(\overline{M}_{0,n}) = S_n$, the symmetric group on n elements.*

Proof. Let $g \in \mathrm{Aut}(\overline{M}_{0,n})$. Let $\phi_i : \overline{M}_{0,n} \to \overline{M}_{0,n-1}$ be the i-th forgetful map. Then by Theorem 4.1, $g \circ \phi_i$ is a map forgetting an index $j_i \in \{1, \ldots, n\}$.

This means that we can associate to g the permutation $\{j_1, \ldots, j_n\} \in S_n$. Let $\chi_n : \mathrm{Aut}(\overline{M}_{0,n}) \to S_n$ be the associated map. It can be easily checked that χ_n is a surjective morphism. A simple transposition is realized, for instance, by the standard Cremona transformations we recalled in Definition 2.1.

The main point is to determine the kernel. Assume that $\chi_n(g) = 1$. That is, $g \circ \phi_i$ forgets the i-th index for any $i \in \{1, \ldots, n\}$. Fix a Kapranov map $f_n : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$. The automorphism g induces a Cremona transformation γ_n on $\mathbb{P}^{n-3}$ that stabilizes the lines through the Kapranov points and also the rational normal curves through $\mathcal{K}$. Let $\mathcal{H}_n \subset |\mathcal{O}(d)|$ be the linear system associated to γ_n. Let $l_i \subset \mathbb{P}^{n-3}$ be a general line through p_i and Γ_n a general rational normal curve through $\mathcal{K}$. Then

$$\deg(\gamma_n(l_i)) = d - \mathrm{mult}_{p_i} \mathcal{H}_n = 1$$

for any $i \in \{1, \ldots, n-1\}$, and

$$\deg(\gamma_n(\Gamma_n)) = (n-3)d - \sum_{i=1}^{n-1} \mathrm{mult}_{p_i} \mathcal{H}_n = n - 3.$$

These yield

$$n-3=(n-3)d-(n-1)(d-1)$$

and finally $d=1$. That is, γ_n is a projectivity that fixes $n-1$ points. Therefore γ_n and hence g are the identity. □

Appendix

In this appendix we collect results which came out of conversations and help to improve and clarify the content of this paper. We start by reporting a nice proof of Theorem 3.7 kindly suggested by James McKernan and Jenia Tevelev [McT]. We thank James and Jenia for their help in translating our projective arguments into a better known terminology and also for producing a proof that best shows the “almost toric” nature of $\overline{M}_{0,n}$.

Proof of Theorem 3.7 ([McT]). We proceed by induction on n. We may assume that $n \geq 7$. Let f_{ij} be the restriction of f to $E_{i,j} \simeq \overline{M}_{0,n-1}$. There are two cases:

(1) f_{ij} is never constant.
(2) f_{ij} is constant for at least one pair $\{i, j\}$.

Suppose we have (1). We will derive a contradiction. By induction, we know that each f_{ij} is a composition of a forgetful map $f'_{ij}\colon E_{ij} \to \mathbb{P}^1$ and a finite morphism $g_{ij}\colon \mathbb{P}^1 \to \mathbb{P}^1$. Notice that the forgetful maps f'_{ij} and f'_{kl} agree on the intersections $E_{ij} \cap E_{kl}$ each time these divisors have a nonempty intersection, i.e. when $\{i, j\}$ and $\{k, l\}$ do not contain common elements. Indeed, both f'_{ij} and f'_{kl} restrict to some forgetful maps $E_{ij} \cap E_{kl} \simeq \overline{M}_{0,n-2} \to \overline{M}_{0,4} \simeq \mathbb{P}^1$. But

$$(f'_{ij})^*\mathcal{O}_{\mathbb{P}^1}(a) \simeq (f_{ij})^*\mathcal{O}_{\mathbb{P}^1}(1) \simeq (f_{kl})^*\mathcal{O}_{\mathbb{P}^1}(1) \simeq (f'_{kl})^*\mathcal{O}_{\mathbb{P}^1}(b)$$

for some positive integers a and b, and a forgetful map $\overline{M}_{0,n-2} \to \overline{M}_{0,4} \simeq \mathbb{P}^1$ is uniquely determined by the pull-back of $\mathcal{O}_{\mathbb{P}^1}(1)$ (up to a multiple).

There are two cases, up to the obvious symmetries:

$$f'_{12} = \begin{cases} \pi_{3,4,5,6}, \\ \pi_{\{1,2\},3,4,5}. \end{cases}$$

Consider f_{67}. Up to even more symmetries, we must have

$$f'_{67} = \begin{cases} \pi_{3,4,5,\{6,7\}} & \text{if } f'_{12} = \pi_{3,4,5,6}, \\ \pi_{2,3,4,5} & \text{if } f'_{12} = \pi_{\{1,2\},3,4,5}. \end{cases}$$

Possibly switching $\{1, 2\}$ and $\{6, 7\}$ we might as well assume that

$$f'_{12} = \pi_{\{1,2\},3,4,5} \quad \text{and} \quad f'_{67} = \pi_{2,3,4,5}.$$

It follows that f restricted to both $\delta_{12} \cap \delta_{34}$ and $\delta_{34} \cap \delta_{67}$ is constant. But then $f'_{34} = \pi_{1267}$ and so $f'_{15} = \pi_{\{1,5\},2,6,7}$. On the other hand $f'_{15} = \pi_{\{1,5\},2,3,4}$, a contradiction.

So we must have (2). Assume that f contracts, say, E_{1n}. Let $f_n\colon \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ be the Kapranov map associated to the Kapranov set $\{p_1, \dots, p_{n-1}\}$, that is, the map that blows up $n-1$ points $\{p_1, \dots, p_{n-1}\}$ in linear general position, and every linear space spanned by these points. Then f_n contracts $E_{1,n}$ to the point p_1. Let $\psi\colon L_n \to \mathbb{P}^{n-3}$ be the birational morphism which blows up every linear space blown up by π, except those which contain p_1. Notice that L_n is a toric variety, and there is a birational morphism $\varphi\colon \overline{M}_{0,n} \to L_n$ which factors $f_n = \psi \circ \varphi$. This yields an induced rational map $g = f \circ \varphi^{-1}\colon L_n \dashrightarrow \mathbb{P}^1$. Then the rational map $g\colon L_n \dashrightarrow \mathbb{P}^1$

(a) is regular at p_1;
(b) has a base locus of codimension 2 (as for any rational pencil);
(c) has a base locus contained in the indeterminacy locus of the birational map $\varphi^{-1}\colon L_n \dashrightarrow \overline{M}_{0,n}$. The latter is the union of linear subspaces passing through p_1 (in the Kapranov model).

It follows that the map $g\colon L_n \dashrightarrow \mathbb{P}^1$ is actually regular. As for any morphism to $\mathbb{P}^1$ with connected fibers, it is given by a complete linear series. Therefore it is a toric morphism. To conclude, we prove that it is one of the forgetful maps by studying the induced map of fans.

The fan F_n of L_n is obtained by taking the standard fan for $\mathbb{P}^{n-3}$ (with rays $R_1, \dots, R_{n-2}$, of which the first $n-2$ correspond to coordinate hyperplanes) followed by its barycentric subdivision. A toric morphism $L_n \to \mathbb{P}^1$ corresponds to a linear map $g\colon \mathbb{R}^{n-3} \to \mathbb{R}$ that sends each cone of F_n to either $\{0\}$, the positive ray $\mathbb{R}_+$, or the negative ray $\mathbb{R}_-$. We may assume without loss of generality that $g(R_1) = \mathbb{R}_+$ and $g(R_2) = \mathbb{R}_-$. The fan of L_n contains the ray $C = R_1 + R_2$ and we should have $g(C) = 0$. Therefore, g sends primitive generators of R_1 and R_2 to opposite vectors v_1 and v_2 in $\mathbb{R}$.

We claim that $g(R_i) = 0$ for $i > 2$. Assuming the claim, we then have $v_1, v_2 = \pm 1$ and the toric morphism is a resolution of the linear projection from the intersection of the first two coordinate hyperplanes, which corresponds to one of the forgetful maps.

Back to the claim, suppose for contradiction that $g(R_3) = \mathbb{R}_+$. Then $g(-R_3) = \mathbb{R}_-$. But $-R_3$ is the barycenter of the top-dimensional cone of F_n spanned by all R_i for $i \neq 3$. Hence F_n contains a cone with rays R_1 and $-R_3$, which does not map to any cone. □

Remark A.1. It is interesting to note that the space L_n is a moduli space in its own right, introduced by Losev and Manin [LM], and the morphism of $\overline{M}_{0,n}$ to L_n has a natural modular interpretation.

Recall that [GKM, Corollary 0.10] proves that for $g \geq 2$ and $n \geq 1$ any fibration of $\overline{M}_{g,n}$ factors via a forgetful map. Pandharipande [Pa2] observed that this could be the starting point to compute the automorphism group of this moduli space (see [Ma]). We do not dwell on this here but simply report a clever example of Pandharipande in genus 1 (see also [Pa1] for similar constructions). Quite surprisingly, this is the only genus in which there are fiber type morphisms that do not factor via forgetful maps.

Example A.2 (Pandharipande). If $[(E, p)]$ is any stable 1-pointed elliptic curve, the complete series $\mathcal{O}_E(2p)$ is a pencil and induces a map $i_p : E \to P^1$. The map which associates to $[E, p]$ the fourtuple of its ramification points gives an isomorphism

$$f : \overline{M}_{1,1} \to \overline{M}_{0,4}/S_3 = \mathbb{P}^1,$$

where the symmetric group S_3 acts by permuting three of the marked points, in such a way that the distinguished point in $f([E, p])$ is $i_p(p)$.

Take a nonsingular elliptic curve $[E, p, q]$ with $p \neq q$. Let x, y, z be the three points in $\mathbb{P}^1$ over which i_p ramifies and distinct from $i_p(p)$. We can associate to $[(E; p, q)]$ the fourtuple $[i_p(q); x, y, z] \in \overline{M}_{0,4}/S_3 = \mathbb{P}^1$. This defines a rational map

$$\phi : \overline{M}_{1,2} \dashrightarrow \overline{M}_{0,4}/S_3 = \mathbb{P}^1.$$

Claim. *The rational map ϕ extends to a dominant morphism which does not factor through any forgetful map.*

Proof of the Claim. The map ϕ surely is surjective when restricted to fibers of the forgetful maps ϕ_q and ϕ_p; in particular this shows that ϕ does not factor through any forgetful map. Moreover, this implies that ϕ extends to the section $F \cong \overline{M}_{1,1}$ of both ϕ_p and ϕ_q as the map f defined above. □

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