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Ergodic properties of square-free numbers

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Abstract. We construct a natural invariant measure concentrated on the set of square-free numbers, and invariant under the shift. We prove that the corresponding dynamical system is isomorphic to a translation on a compact Abelian group. This implies that this system is not weakly mixing and has zero measure-theoretical entropy.

Keywords. Square-free numbers, correlation functions, dynamical systems with pure point spectrum, ergodicity

Introduction and notations

Let \mathcal{P} be the set of prime numbers. By p (with or without indices) we will always denote an element of \mathcal{P} . A positive integer n is *square-free* if $p^2 \nmid n$ for every p . Denote the set of all square-free numbers by \mathcal{Q} (for *quadratfrei*). The indicator of the set \mathcal{Q} is the function $n \mapsto \mu^2(n)$, where μ is the Möbius function:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ is not square-free,} \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes.} \end{cases}$$

The functions μ and μ^2 are of great importance in number theory because of their connection with the Riemann zeta function. For example,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}, \quad \sum_{n=1}^{\infty} \frac{\mu^2(n)}{n^s} = \frac{\zeta(s)}{\zeta(2s)}.$$

Furthermore, the estimate $|\sum_{n \leq N} \mu(n)| = \mathcal{O}_\varepsilon(N^{1/2+\varepsilon})$ as $N \rightarrow \infty$ is equivalent to the Riemann Hypothesis. P. Sarnak [12] has recently addressed a number of statistical and ergodic properties of the sequences $(\mu(n))_n$ and $(\mu^2(n))_n$.

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0.1. Notations

We shall use the standard notation $e(x) = e^{2\pi i x}$. For every integer n denote by $\omega(n)$ the number of its distinct prime factors. For example, $\omega(1) = 0$ and $\omega(2 \cdot 3) = \omega(2^{10} \cdot 3^7) = \omega(7 \cdot 23) = 2$. We shall also use the notations

$$\mathcal{P}(n) = \{p : p \mid n\}, \quad \mathcal{P}_2(n) = \{p : p^2 \mid n\}.$$

Notice that if $n \in \mathcal{Q}$, then $|\mathcal{P}(n)| = \omega(n)$, $\mathcal{P}_2(n) = \emptyset$, and $\mathcal{P}_2(n^2) = \mathcal{P}(n)$. For every finite set $\mathcal{A} \subset \mathcal{P}$, define

$$[\mathcal{A}] = \prod_{p \in \mathcal{A}} p.$$

In particular $[\emptyset] = 1$. Notice that if \mathcal{A}, \mathcal{B} are disjoint, then $[\mathcal{A} \cup \mathcal{B}] = [\mathcal{A}][\mathcal{B}]$ and $[\mathcal{A} \cap \mathcal{B}] = 1$.

1. Formulation of the results

The goal of this paper is to describe a dynamical system ‘naturally’ associated to \mathcal{Q} and study its statistical and ergodic properties.

1.1. Correlation functions

The first step is the construction of *correlation functions* for \mathcal{Q} . Choose r integers $0 \leq k_1 < \dots < k_r$ and consider the set

$$\mathcal{Q}_N(k_1, \dots, k_r) = \{n \leq N : \mu^2(n) = \mu^2(n + k_1) = \dots = \mu^2(n + k_r) = 1\}.$$

The ratio

$$\mathbb{E}_N(k_1, \dots, k_r) := |\mathcal{Q}_N(k_1, \dots, k_r)|/N \tag{1}$$

is the frequency of square-free integers $n \leq N$ for which $n + k_1, \dots, n + k_r$ are also square-free. It also gives the expectation (hence the notation \mathbb{E}) of the product $\mu^2(n)\mu^2(n + k_1) \dots \mu^2(n + k_r)$ with respect to the uniform measure on $\{1, \dots, N\}$. Notice, by taking $r = 1$ and $k_1 = 0$, that $\mathcal{Q}_N(0)$ is simply the set of all square-free numbers not greater than N . It is well known that

$$\lim_{N \rightarrow \infty} \mathbb{E}_N(0) = 6/\pi^2 \approx 0.6079271018. \tag{2}$$

We include the proof of (2) and some of its generalizations in Section 2 (see Theorems 2.1–2.3). The study of $\mathbb{E}_N(k_1, \dots, k_r)$ as $N \rightarrow \infty$ is also classical: see L. Mirsky [5], R. R. Hall [2], K. M. Tsang [14], D. R. Heath-Brown [4]. It is known that the limits

$$c_{r+1}(k_1, \dots, k_r) = \lim_{N \rightarrow \infty} \mathbb{E}_N(k_1, \dots, k_r) \tag{3}$$

exist. We shall refer to c_{r+1} as the $(r + 1)$ -st correlation function for \mathcal{Q} . Various formulæ for $c_{r+1}(k_1, \dots, k_r)$ are known (see Section 3). We shall rewrite the one by L. Mirsky [5] to express the correlation functions as a sum, namely

$$c_{r+1}(k_1, \dots, k_r) = \sum_{0 \leq l' < l'' \leq r} \sum_{\substack{\mu^2(d_{l',l''})=1 \\ d_{l',l''}^2 | k_{l'} - k_{l''}}} \sum_{m_0, m_1, \dots, m_r \geq 0} (-1)^{\sum_{i=0}^r m_i} \sum_{\substack{\mathcal{P}_l \subset \mathcal{P}, 0 \leq l \leq r \\ |\mathcal{P}_l|=m_l \\ [\mathcal{P}_{l'} \cap \mathcal{P}_{l''}] = d_{l',l''}}} \frac{1}{[\bigcup_{l=0}^r \mathcal{P}_l]^2}. \tag{4}$$

The above formula, although complicated, plays a role in the spectral analysis of the correlation functions. Let, for example, $r = 1$. For every $d \in \mathcal{Q}$ define

$$\sigma_d = \sum_{m_0, m_1 \geq 0} (-1)^{m_0+m_1} \sum_{\substack{\mathcal{P}_0, \mathcal{P}_1 \subset \mathcal{P} \\ |\mathcal{P}_0|=m_0, |\mathcal{P}_1|=m_1 \\ [\mathcal{P}_0 \cap \mathcal{P}_1]=d}} \frac{1}{[\mathcal{P}_0 \cup \mathcal{P}_1]^2}. \tag{5}$$

Explicit formulæ for σ_d are given in Section 3. Then

$$c_2(k) = \sum_{\substack{\mu^2(d)=1 \\ d^2 | k}} \sigma_d \tag{6}$$

and the corresponding spectral measure ν on \mathbb{S}^1 (i.e. satisfying $c_2(k) = \hat{\nu}(k)$ by Bochner's theorem) is pure point, given as a sum of δ -functions at the points $e(l/d^2)$, where $d \in \mathcal{Q}$. More precisely,

$$\nu = \sum_{\mu^2(d)=1} \sigma_d \sum_{l=0}^{d^2-1} \delta_{e(l/d^2)}, \tag{7}$$

where the convergence of the series is guaranteed by Lemma 3.1 below. The spectrum (i.e. the support of ν) is the group

$$\Lambda = \{e(l/d^2) : 0 \leq l \leq d^2 - 1, \mu^2(d) = \mu^2(\gcd(l, d^2)) = 1\} \subset \mathbb{S}^1.$$

Notice that every element of Λ is represented uniquely. Moreover, every rational number of the form l/d^2 such that d is square-free, $0 \leq l \leq d^2 - 1$, and $\gcd(l, d^2)$ is also square-free can be written as

$$\frac{l}{d^2} = \frac{l_1}{p_1^2} + \dots + \frac{l_r}{p_m^2} \tag{8}$$

for some l_1, \dots, l_m , where $\{p_1, \dots, p_m\} = \mathcal{P}(d)$. This representation (8) is unique if one imposes the restriction $0 \leq l_j \leq p_m^2 - 1, 1 \leq j \leq m$. In other words, the group Λ is isomorphic to the direct sum $\bigoplus_p \mathbb{Z}/p^2\mathbb{Z}$ (where only finitely many coordinates are non-zero). Therefore, Λ is the Pontryagin dual of the direct product group

$$\mathbb{G} = \prod_p \mathbb{Z}/p^2\mathbb{Z}, \tag{9}$$

which is an Abelian compact group (endowed with the product topology). In other words, $\hat{\mathbb{G}} \cong \Lambda$. Each element $\mathbf{g} \in \mathbb{G}$ is identified with a sequence $(g_{p^2})_{p \in \mathcal{P}}$ indexed by \mathcal{P} , where $g_{p^2} \in \mathbb{Z}/p^2\mathbb{Z}$:

$$\mathbf{g} \equiv (g_4, g_9, g_{25}, g_{49}, \dots).$$

Given $\mathbf{h} \in \mathbb{G}$, denote by $\mathbb{T}_{\mathbf{h}} : \mathbb{G} \rightarrow \mathbb{G}$ the translation $\mathbb{T}_{\mathbf{h}}(\mathbf{g}) = \mathbf{g} + \mathbf{h}$. Let \mathbb{B} be the natural σ -algebra on \mathbb{G} , and let us put the uniform measure on each $\mathbb{Z}/p^2\mathbb{Z}$. The corresponding product measure \mathbb{P} on \mathbb{B} is invariant under translations, and therefore it is the Haar measure.

The ergodic properties of translations on compact Abelian groups (also known as *Kronecker systems*) were studied for the first time by J. von Neumann [15]. He showed that two such ergodic translations with the same spectrum are isomorphic as measure-preserving dynamical systems. This is true in general for ergodic transformations with pure point spectrum and it plays an important role in our analysis. Later, P. R. Halmos and J. von Neumann [3] proved that every ergodic dynamical system with pure point spectrum is isomorphic to a translation on a compact Abelian group. This implies, for example, that every ergodic dynamical system with pure point spectrum is isomorphic to its inverse. For an historical survey on the isomorphism problem see [8].

1.2. A natural dynamical system

Consider the space X of all bi-infinite sequences $x = \{x(n), -\infty < n < \infty\}$ where each $x(n)$ takes value either 0 or 1. Denote by \mathcal{B} the natural σ -algebra generated by cylinder sets, and introduce the probability measure Π defined on \mathcal{B} as follows: For every $r \geq 0$ and every $-\infty < k_0 < k_1 < \dots < k_r < \infty$,

$$\Pi\{x \in X : x(k_0) = x(k_1) = \dots = x(k_r) = 1\} = c_{r+1}(k_1 - k_0, \dots, k_r - k_0), \quad (10)$$

where c_{r+1} is the $(r + 1)$ -st correlation function (4). It is clear that (10) determines the measure Π uniquely. We call Π *the natural measure corresponding to the set of square-free numbers*.

If T is the shift on X , i.e. $Tx = x'$, $x'(n) = x(n + 1)$, then it follows immediately from (10) that Π is invariant under T . We can now formulate the main result of this paper:

Main Theorem. (i) *The dynamical system (X, \mathcal{B}, Π, T) is ergodic and has pure point spectrum given by Λ .*

(ii) *(X, \mathcal{B}, Π, T) is isomorphic to $(\mathbb{G}, \mathbb{B}, \mathbb{P}, \mathbb{T}_{\mathbf{u}})$, where $\mathbf{u} = (1, 1, \dots)$.*

P. Sarnak [12] states that $(\mathbb{G}, \mathbb{B}, \mathbb{P}, \mathbb{T}_{\mathbf{u}})$ is a factor of (X, \mathcal{B}, Π, T) . His methods also allow one to show that the factor map is in fact an isomorphism. Our approach is rather different and is based on a spectral analysis of the dynamical system (X, \mathcal{B}, Π, T) . The statement in the following corollary can also be found in [12].

Corollary. *The dynamical system (X, \mathcal{B}, Π, T) is not weakly mixing, and its measure-theoretic entropy is zero.*

It is worth remarking that the main focus of [12] is on the topological dynamical systems $M = (O_{\mu(n)}, T)$, $S = (O_{\mu^2(n)}, T)$ given by the shifts on the orbit closures of $(\mu(n))_n$ and $(\mu^2(n))_n$, respectively. The topological entropy of S is positive, equal to $(6/\pi^2) \log 2$. R. Peckner [6] recently constructed a measure of maximal entropy for S ; he showed that this measure is unique, and the corresponding dynamical system is isomorphic to the direct product of $(\mathbb{G}, \mathbb{B}, \mathbb{P}, \mathbb{T}_{\mathbf{u}})$ and a Bernoulli shift with entropy $(6/\pi^2) \log 2$. In particular, the dynamical system (X, \mathcal{B}, Π, T) that we consider is its Pinsker factor.

Our paper is organized as follows. Section 2 includes the classical computation of the density of square-free numbers and its generalization to square-free numbers avoiding finite sets of prime factors (the proof is given in Appendix A). The latter will be used for the computation of certain relevant constants. Section 3 contains various formulæ for the correlation functions, including the derivation of (6) and (4) from the formula by L. Mirsky. Section 4 includes several useful lemmata (some of which are proven in Appendix B) concerning averages and exponential sums for the correlation functions. These results are crucial for the spectral analysis of the dynamical system (X, \mathcal{B}, Π, T) . The analysis is carried out in Section 5 and yields the first part of our Main Theorem. The analysis of the spectrum for $(\mathbb{G}, \mathbb{B}, \mathbb{P}, \mathbb{T}_{\mathbf{u}})$ is done in Section 6, and the second part of our Main Theorem follows from it, by means of a theorem by J. von Neumann [15].

2. The density of \mathcal{Q} and some of its subsets

Recall that $\mathbb{E}_N(0) = |\{n \leq N : n \in \mathcal{Q}\}|/N$. The following theorem is very classical.

Theorem 2.1.

$$\lim_{N \rightarrow \infty} \mathbb{E}_N(0) = 6/\pi^2. \quad (11)$$

Proof. We can write μ^2 as the indicator of the set of square-free numbers by imposing the condition that its argument avoids all arithmetic progressions modulo p^2 :

$$\mu^2(n) = \prod_p (1 - \chi_{p^2}(n)). \quad (12)$$

In the above expression $\chi_{p^2}(n)$ is the indicator of the arithmetic progression $\{p^2 l : l \in \mathbb{Z}\}$. Let us open the brackets in (12):

$$\mu^2(n) = 1 - \sum_p \chi_{p^2}(n) + \sum_{p_1 < p_2} \chi_{p_1^2}(n) \chi_{p_2^2}(n) - \sum_{p_1 < p_2 < p_3} \chi_{p_1^2}(n) \chi_{p_2^2}(n) \chi_{p_3^2}(n) + \dots$$

We can write

$$\begin{aligned} \mathbb{E}_N(0) &= \frac{1}{N} \sum_{n \leq N} \mu^2(n) \\ &= 1 - \sum_p \frac{1}{p^2} + \sum_{p_1 < p_2} \frac{1}{(p_1 p_2)^2} - \sum_{p_1 < p_2 < p_3} \frac{1}{(p_1 p_2 p_3)^2} + \dots + \varepsilon_N \\ &= \prod_p \left(1 - \frac{1}{p^2}\right) + \varepsilon_N = \frac{1}{\zeta(2)} + \varepsilon_N = \frac{6}{\pi^2} + \varepsilon_N. \end{aligned}$$

Here and below ε_N denotes a remainder that tends to zero as $N \rightarrow \infty$. \square

The statement of Theorem 1 can actually be refined as follows:

Theorem 2.2.

$$\mathbb{E}_N(0) = 6/\pi^2 + \mathcal{O}(N^{-1/2}) \quad \text{as } N \rightarrow \infty.$$

In other words, ε_N in the proof of Theorem 2.1 satisfies the estimate $|\varepsilon_N| = \mathcal{O}(N^{-1/2})$. This result is also very classical, and is a special case of Theorem 2.3 below. Let us fix a finite set $\mathcal{S} \subset \mathcal{P}$ and define the set

$$\mathcal{Q}_N^{\mathcal{S}}(0) = \{n \leq N : \mu^2(n) = 1, p \in \mathcal{S} \Rightarrow p \nmid n\} \tag{13}$$

of all square-free numbers not greater than N and not divisible by any of the primes $p \in \mathcal{S}$. For example, $\mathcal{Q}_N^{\{2\}}(0)$ is the set of odd square-free numbers not greater than N . Notice that when \mathcal{S} is empty we get the full set of square-free numbers, i.e. $\mathcal{Q}_N^{\emptyset}(0) = \mathcal{Q}_N(0)$. In analogy with (1), let us define

$$\mathbb{E}_N^{\mathcal{S}}(0) = |\mathcal{Q}_N^{\mathcal{S}}(0)|/N.$$

We have the following

Theorem 2.3. *For every finite $\mathcal{S} \subset \mathcal{P}$ we have*

$$\mathbb{E}_N^{\mathcal{S}}(0) = \frac{\alpha(\mathcal{S})}{\zeta(2)} + \mathcal{O}_{\mathcal{S}}(N^{-1/2}) \quad \text{as } N \rightarrow \infty, \quad \text{where } \alpha(\mathcal{S}) = \prod_{p \in \mathcal{S}} \frac{p}{p+1}$$

and the constant $C(\mathcal{S})$ implied by the $\mathcal{O}_{\mathcal{S}}$ -notation can be taken as

$$C(\mathcal{S}) = 4 \prod_{p \in \mathcal{S}} \frac{p-1}{p} + \left(\prod_{p \in \mathcal{S}} p - 1 \right) - \prod_{p \in \mathcal{S}} (p-1).$$

The proof of Theorem 2.3 is presented in Appendix A; it implies the existence of the asymptotic densities

$$\lim_{N \rightarrow \infty} \mathbb{E}_N^{\mathcal{S}}(0) = \alpha(\mathcal{S})/\zeta(2). \tag{14}$$

For example, the density of the set of odd square-free numbers is $4/\pi^2$ (i.e. odd and even square-free numbers are in 2 : 1 proportion). Analogously, by choosing $\mathcal{S} = \{p\}$, we see that the set of square-free numbers not divisible by p is “ p times as large” (in the sense of density) as the set of those divisible by p . If, for instance, we choose $\mathcal{S} = \{2, 3\}$ we obtain $\alpha(\{2, 3\}) = 1/2$, and we see that 50% of the square-free numbers are not divisible by either 2 or 3.

3. The formulæ for the correlation functions

L. Mirsky [5] proved that

$$c_{r+1}(k_1, \dots, k_r) = \prod_p \left(1 - \frac{A_p^{(r+1)}(k_1, \dots, k_r)}{p^2} \right), \tag{15}$$

where $A_p^{(r+1)}(k_1, \dots, k_r) = |\{0, k_1 \pmod{p^2}, \dots, k_r \pmod{p^2}\}|$. Notice that

$$1 \leq A_p^{(r+1)}(k_1, \dots, k_r) \leq r$$

for finitely many p and $A_p^{(r+1)}(k_1, \dots, k_r) = r + 1$ for infinitely many p . For $r = 1$, we have

$$A_p^{(2)}(k) = \begin{cases} 1, & p^2 \mid k, \\ 2, & \text{otherwise.} \end{cases}$$

This gives, for instance,

$$c_2(k) = \prod_{p^2 \mid k} \left(1 - \frac{1}{p^2} \right) \prod_{p^2 \nmid k} \left(1 - \frac{2}{p^2} \right). \tag{16}$$

It will be useful for us to write $c_2(k)$ (and in general $c_{r+1}(k_1, \dots, k_r)$) as a sum. Recall the definition of σ_d from Section 1.1. We prove the following formula for σ_d :

Lemma 3.1.

$$\sigma_d = \frac{1}{d^2} \prod_{p \nmid d} \left(1 - \frac{2}{p^2} \right). \tag{17}$$

Proof. Recall that, since d is square-free, $|\mathcal{P}(d)| = \omega(d)$. By setting $m = m_1 - \omega(d)$ and $M = m_1 + m_2 - 2\omega(d)$ in (5) we obtain

$$\begin{aligned} \sigma_d &= \sum_{0 \leq m \leq M} (-1)^{-2\omega(d)} (-1)^M \binom{M}{m} \sum_{\substack{\mathcal{P}' \subset \mathcal{P} \setminus \mathcal{P}(d) \\ |\mathcal{P}'|=M}} \frac{1}{d^2} \frac{1}{[\mathcal{P}']^2} \\ &= \frac{1}{d^2} \sum_{M \geq 0} \sum_{\substack{\mathcal{P}' \subset \mathcal{P} \setminus \mathcal{P}(d) \\ |\mathcal{P}'|=M}} \frac{(-2)^M}{[\mathcal{P}']^2} = \frac{1}{d^2} \prod_{p \nmid d} \left(1 - \frac{2}{p^2} \right). \quad \square \end{aligned}$$

In particular, Lemma 3.1 shows that σ_d is positive and bounded away from zero and infinity. More precisely

$$0 < \sigma_1 \leq \sigma_d < 6/\pi^2,$$

where $\sigma_1 = \prod_p (1 - 2/p^2) \approx 0.3226340989$. We can also rewrite $\sigma_d = \sigma_1 \cdot \prod_{p \mid d} \frac{1}{p^2 - 2}$.

Proposition 3.2. *Let k be an arbitrary integer. Then*

$$c_2(k) = \sum_{\substack{\mu^2(d)=1 \\ d^2|k}} \sigma_d. \tag{18}$$

Proof. Since $\mathcal{P}_2(k) = \{p : p^2 | k\}$ and $\mathcal{D}(k) = \{\prod_{p \in \mathcal{P}'} p : \mathcal{P}' \subset \mathcal{P}_2(k)\}$, Lemma 3.1 gives

$$\begin{aligned} \sum_{d \in \mathcal{D}(k)} \sigma_d &= \sum_{d \in \mathcal{D}(k)} \frac{1}{d^2} \prod_{p|d} \left(1 - \frac{2}{p^2}\right) = \prod_p \left(1 - \frac{2}{p^2}\right) \sum_{d \in \mathcal{D}(k)} \frac{1}{d^2} \prod_{p|d} \left(1 - \frac{2}{p^2}\right)^{-1} \\ &= \prod_p \left(1 - \frac{2}{p^2}\right) \sum_{d \in \mathcal{D}(k)} \prod_{p|d} \frac{1}{p^2 - 2} = \prod_p \left(1 - \frac{2}{p^2}\right) \prod_{p \in \mathcal{P}_2(k)} \left(1 + \frac{1}{p^2 - 2}\right) \\ &= \prod_{p^2|k} \left(1 - \frac{1}{p^2}\right) \prod_{p^2 \nmid k} \left(1 - \frac{2}{p^2}\right) = c_2(k) \end{aligned}$$

by (16). □

In particular, if $k = 0$, then $\mathcal{D}(0) = \mathcal{Q}$ and $\mathcal{P}_2(0) = \mathcal{P}$ and we retrieve the known fact

$$c_2(0) = \sum_{\mu^2(d)=1} \sigma_d = \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2}.$$

Remark 3.3. Proposition 3.2 shows that the value of $c_2(k)$ depends on the arithmetic properties of k . This fact is certainly very unusual from the point of view of probability theory and statistical mechanics. If k is square-free, then the function $c_2(k)$ takes the constant value σ_1 . Analogously, $c_2(k)$ is constant along any subsequence of numbers k sharing the same set of divisors that are the square of a square-free number. If we define $\mathcal{D}(k) := \{d : \mu^2(d) = 1, d^2 | k\}$, then $\mathcal{D}(k) = \mathcal{D}(k') \Rightarrow c_2(k) = c_2(k')$. The opposite implication follows from (17). Observe that every set $\mathcal{D}(k)$ is of the form

$$\mathcal{D}(k) = \left\{ \prod_{p \in \mathcal{P}'} p : \mathcal{P}' \subseteq \mathcal{P}_2(k) \right\}, \tag{19}$$

where $\mathcal{P}_2(k) = \{p : p^2 | k\}$. This means that $|\mathcal{D}(k)| = 2^{|\mathcal{P}_2(k)|}$ and $\mathcal{D}(k) = \mathcal{D}(k') \Leftrightarrow \mathcal{P}_2(k) = \mathcal{P}_2(k')$. The set of k such that $\mathcal{P}_2(k) = \emptyset$ is the set of square-free numbers, and we know that it has positive density (equal to $6/\pi^2$, given by (2)). In general, we have the following

Proposition 3.4 (Density of the level sets of c_2). *Fix a square-free number d . Then the density of those k 's such that $c_2(k) = c_2(d^2)$ exists and is given by*

$$\vartheta(d^2) := \lim_{N \rightarrow \infty} \frac{1}{N} |\{k \leq N : c_2(k) = c_2(d^2)\}| = \frac{6}{\pi^2} \prod_{p|d} \frac{1}{p^2 - 1}. \tag{20}$$

Proof. If $\mathcal{P}(d) = \{p_1, \dots, p_m\}$ then k satisfies $c_2(k) = c_2(d^2)$ if and only if it is of the form $k = p_1^{a_1} \cdots p_m^{a_m} q$, where $\mu^2(q) = 1$, $a_j \geq 2$, and $p_j \nmid q$ for every $j = 1, \dots, m$. Fix $a_1, \dots, a_m \geq 2$. Then

$$\begin{aligned} \frac{1}{N} |\{k \leq N : k = p_1^{a_1} \cdots p_m^{a_m} q, \mu^2(q) = 1, p_j \nmid q \text{ for } j = 1, \dots, m\}| \\ = \frac{1}{p_1^{a_1} \cdots p_m^{a_m}} \mathbb{E}_{(N/(p_1^{a_1} \cdots p_m^{a_m}))}^{\{p_1, \dots, p_m\}}(0) \end{aligned}$$

and, by Theorem 2.3, the limit as $N \rightarrow \infty$ is

$$\frac{6}{\pi^2} \frac{1}{p_1^{a_1} \cdots p_m^{a_m}} \prod_{j=1}^m \frac{p_j}{p_j + 1} = \prod_{j=1}^m \frac{1}{p_j^{a_j-1} (p_j + 1)}.$$

Now, by summing over all $a_j \geq 2$, we obtain

$$\frac{6}{\pi^2} \prod_{j=1}^m \frac{1}{p_j + 1} \sum_{a_j \geq 2} \frac{1}{p_j^{a_j-1}} = \frac{6}{\pi^2} \prod_{j=1}^m \frac{1}{(p_j + 1)(p_j - 1)} = \frac{6}{\pi^2} \prod_{j=1}^m \frac{1}{p_j^2 - 1}$$

and the proposition is proven. □

Remark 3.5. The argument in the proof of Proposition 3.4 will also be used in Appendix B. We can check that

$$\begin{aligned} \sum_{\mu^2(d)=1} \vartheta(d^2) &= \sum_{m \geq 0} \sum_{\substack{\mathcal{P}' \subset \mathcal{P} \\ |\mathcal{P}'|=m}} \frac{6}{\pi^2} \prod_{p \in \mathcal{P}'} \frac{1}{p^2 - 1} = \frac{6}{\pi^2} \prod_p \left(1 + \frac{1}{p^2 - 1}\right) \\ &= \frac{6}{\pi^2} \prod_p \left(1 - \frac{1}{p^2}\right)^{-1} = 1. \end{aligned}$$

Here we present the values of $\vartheta(d^2)$ for square-free numbers $d \leq 17$. The sum of the corresponding densities is $\approx 97.6\%$ and one can check that $\sum_{d \leq 42, \mu^2(d)=1} \vartheta(d^2) > 99\%$.

d	1	2	3	5	6	7	10	11	13	14	15	17	...
$\vartheta(d^2)$	$\frac{6}{\pi^2}$	$\frac{2}{\pi^2}$	$\frac{3}{4\pi^2}$	$\frac{1}{4\pi^2}$	$\frac{1}{4\pi^2}$	$\frac{1}{8\pi^2}$	$\frac{1}{12\pi^2}$	$\frac{1}{20\pi^2}$	$\frac{1}{28\pi^2}$	$\frac{1}{24\pi^2}$	$\frac{1}{32\pi^2}$	$\frac{1}{48\pi^2}$...

One can also compute the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N c_2(n) = (6/\pi^2)^2 \approx 0.3695753612 \tag{21}$$

by considering the series $\sum_{\mu^2(d)=1} \vartheta(d^2) c_2(d^2)$ and using Proposition 3.4 and Lemma 3.1. We shall retrieve this fact from the more general result of Lemma 4.4. Figure 1 summarizes the structure of the second correlation function.

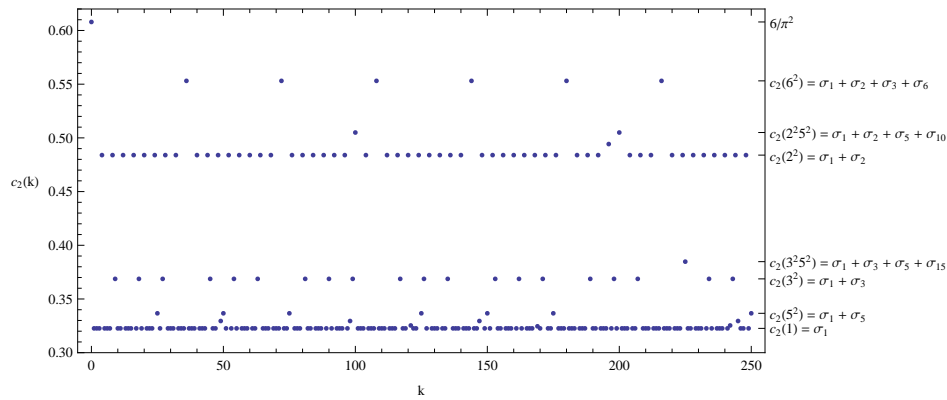


Fig. 1. The second correlation function $c_2(k)$ and its level sets.

Let us address the case of higher order correlation functions.

Proposition 3.6. *Let k_1, \dots, k_r be such that all the differences $k_{l'} - k_{l''}$, $0 \leq l' < l'' \leq r$, are square-free. Then*

$$c_{r+1}(k_1, \dots, k_r) = \sum_{m_0, m_1, \dots, m_r \geq 0} (-1)^{\sum_{l=0}^r m_l} \sum_{\substack{\mathcal{P}_l \subset \mathcal{P}, 0 \leq l \leq r \\ \mathcal{P}_{l'} \cap \mathcal{P}_{l''} = \emptyset \text{ for } l' \neq l''}} \frac{1}{[\bigcup_{l=0}^r \mathcal{P}_l]^2}. \quad (22)$$

For general k_1, \dots, k_r we have

$$c_{r+1}(k_1, \dots, k_r) = \sum_{0 \leq l' < l'' \leq r} \sum_{\substack{\mu^2(d_{l', l''})=1 \\ d_{l', l''}^2 | k_{l'} - k_{l''}}} \sum_{m_0, m_1, \dots, m_r \geq 0} (-1)^{\sum_{l=0}^r m_l} \sum_{\substack{\mathcal{P}_l, 0 \leq l \leq r \\ |\mathcal{P}_l|=m_l \\ [\mathcal{P}_{l'} \cap \mathcal{P}_{l''}]=d_{l', l''}}} \frac{1}{[\bigcup_{l=0}^r \mathcal{P}_l]^2}. \quad (23)$$

Proof. The case of $A_p^{(r+1)}(k_1, \dots, k_r) = r + 1$ corresponds to the case when $0, k_1, \dots, k_r$ are distinct modulo p^2 , for every prime p . This means that the differences $k_{l'} - k_{l''}$ are not divisible by p^2 , for every prime p . In other words, they are all square-free. In this case, by writing $\mathcal{P}' = \bigcup_{l=0}^r \mathcal{P}_l$ and $m = m_0 + \dots + m_r$, the rhs of (22) equals

$$\sum_{m \geq 0} (-1)^m \frac{m!}{m_0! m_1! \dots m_r!} \sum_{\substack{\mathcal{P}' \subset \mathcal{P} \\ |\mathcal{P}'|=m}} \frac{1}{[\mathcal{P}']^2} = \prod_p \left(1 - \frac{r+1}{p^2} \right).$$

Analogously, one can check that the quantity on the rhs of (23) equals (15) with no restrictions on k_1, \dots, k_r . □

Remark 3.7. Notice that the rhs of (22) depends neither on k_1, \dots, k_r nor on the values of their differences as long as they all are square-free. Moreover, it is not enough to check that the consecutive differences $k_1 - k_0, k_2 - k_1, \dots, k_r - k_{r-1}$ are square-free in order for all differences to be square-free. For example, if $(k_1, k_2, k_3, k_4) = (1, 6, 7, 10)$, all consecutive differences are square-free but $2^2 \mid k_4 - k_2$ and $3^2 \mid k_4 - k_1$.

Notice also that $c_{r+1}(k_1, \dots, k_r)$ might be zero if $r \geq 3$. For example, $c_4(1, 2, 3) = 0$ since there is no n such that $n, n + 1, n + 2, n + 3$ are all square-free. All cases when $c_{r+1}(k_1, \dots, k_r) = 0$ correspond to constraints modulo p^2 for some prime p . This fact is clearly reflected by the general formula (15) for $c_{r+1}(k_1, \dots, k_r)$.

Let us point out that the formulæ (18), (22), (23) could be derived directly by inclusion-exclusion using arithmetic progressions with step p^2 . That approach—as pointed out by the anonymous referee—generates an error term that cannot be estimated elementarily. We therefore prefer the derivation of the formulæ directly from Mirsky’s.

In Section 4 we shall use the following lemma by R. R. Hall [2].

Lemma 3.8. *For every $0 \leq k_1 < \dots < k_r$ we have*

$$c_{r+1}(k_1, \dots, k_r) = \sum_{s_0 \geq 1} \dots \sum_{s_r \geq 1} g(s_0) \dots g(s_r) \sum_{\substack{0 \leq t_j \leq s_j^2 - 1 \\ \mu^2(\gcd(t_j, s_j^2)) = 1 \\ 0 \leq j \leq r \\ t_0/s_0^2 + t_1/s_1^2 + \dots + t_r/s_r^2 \in \mathbb{Z}}} e\left(\frac{t_1}{s_1^2} k_1 + \dots + \frac{t_r}{s_r^2} k_r\right), \tag{24}$$

where

$$g(s) = \frac{6}{\pi^2} \mu(s) \prod_{p \mid s} \frac{1}{p^2 - 1}. \tag{25}$$

Moreover, the series in (24) converges absolutely.

3.1. Spectral analysis of c_2

Let us expand slightly the discussion given in Section 1.1. We can rewrite (18) as

$$c_2(k) = \sum_{\mu^2(d)=1} K_d(k), \quad \text{where } K_d(k) := \begin{cases} \sigma_d & \text{if } d^2 \mid k, \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

The function K_d is constant (equal to σ_d) along the arithmetic progression $\{ld^2 : l \in \mathbb{Z}\}$ and 0 elsewhere. This function is the Fourier transform of a measure on the circle \mathbb{S}^1 , given by a sum of δ -functions at the points $e(l/d^2), l = 0, 1, \dots, d^2 - 1$, with equal weights σ_d/d^2 . A corollary of this fact is the formula (7) for the spectral measure ν on \mathbb{S}^1 .

4. Averages of the correlation functions

This section is dedicated to the proof of some results generalizing (21). For instance, one can restrict the average to those integers belonging to a certain residue class modulo a square-free d (Lemmata 4.1–4.3). These averages are then used in the analysis of an exponential sums of the form $(1/N) \sum_{n=1}^N \lambda^n c_2(n)$, where λ is a complex number of modulus 1 (Lemma 4.4) in the case when $\lambda \in \Lambda$. The latter can be further extended to multiple averages of higher-order correlation functions (Lemma 4.6). These exponential sums play a crucial role in the spectral analysis of the Koopman operator for the “natural” dynamical systems (X, \mathcal{B}, Π, T) from Section 1.2, whose invariant measure is defined by means of the correlations $c_{r+1}(k_1, \dots, k_r)$ (see Section 5). For example, given an eigenfunction $\theta_\lambda : X \rightarrow \mathbb{C}$ with eigenvalue $\lambda \in \Lambda$ for the Koopman operator, we will see that its correlation with the projection onto the s -th coordinate $x \mapsto x(s) \in \{0, 1\}$, i.e. the inner product $\langle x(s), \theta_\lambda \rangle_{L^2(X, \mathcal{B}, \Pi)}$, is given by $\lambda^s \lim_{N \rightarrow \infty} (1/N) \sum_{n=1}^N \lambda^n c_2(n)$, and we will use the explicit form of this limit as function of λ to study the set $\{\theta_\lambda\}_{\lambda \in \Lambda}$ of all eigenfunctions. The proofs of the first three lemmata are given in Appendix B.

Lemma 4.1. *Let d be square-free. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{N} \sum_{l \leq N} c_2(d^2 l) = \left(\frac{6}{\pi^2}\right)^2 \prod_{p \in \mathcal{P}(d)} \frac{p^2}{p^2 - 1}.$$

Lemma 4.2. *Let d be square-free and let $1 \leq t \leq d^2 - 1$, $\gcd(d^2, t) = g \geq 1$, where g is square-free. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{N} \sum_{l \leq N} c_2(d^2 l + t) = \left(\frac{6}{\pi^2}\right)^2 \prod_{p \in \mathcal{P}(d)} \frac{p^2(p^2 - 2)}{(p^2 - 1)^2}.$$

Lemma 4.3. *Let d be square-free, and let $1 \leq t \leq d^2 - 1$, $\gcd(t, d^2) = g \geq 1$. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{N} \sum_{l \leq N} c_2(d^2 l + t) = \left(\frac{6}{\pi^2}\right)^2 \prod_{p \in \mathcal{P}(d)} \frac{p^2(p^2 - 2)}{(p^2 - 1)^2} \prod_{p \in \mathcal{P}_2(g)} \frac{p^2 - 1}{p^2 - 2}.$$

The following two lemmata deal with exponential sums involving the second and the third correlation functions. Recall the function g from (25).

Lemma 4.4. *Let $\lambda = e(l/d^2) \in \Lambda$. Then the limit*

$$\mathfrak{J}_2(\lambda) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \lambda^n c_2(n)$$

exists and equals $g(d)^2$.

Proof. We can write $n = d^2l + t$ for some $l \geq 0$ and $0 \leq t \leq d^2 - 1$ and set

$$I_N(\lambda) = \frac{1}{N} \sum_{n \leq N} \lambda^n c_2(n) = \sum_{t=0}^{d^2-1} I_N^{(t)}(l, d^2), \quad (27)$$

where

$$I_N^{(t)}(\lambda) = \frac{1}{N} \sum_{m \leq (N-t)/d^2} e\left(\frac{lt}{d^2}\right) c_2(d^2m + t).$$

Lemma 4.1 gives

$$\lim_{N \rightarrow \infty} I_N^{(0)}(\lambda) = \frac{1}{d^2} \left(\frac{6}{\pi^2}\right)^2 \prod_{p|d} \frac{p^2}{p^2 - 1}. \quad (28)$$

For $t \neq 0$, the value of $\lim_{N \rightarrow \infty} I_N^{(t)}(\lambda)$ is given by Lemmata 4.2–4.3. It depends only on $\mathcal{P}_2(g)$, where $g = \gcd(t, d^2)$. More explicitly,

$$\lim_{N \rightarrow \infty} I_N^{(t)}(\lambda) = e\left(\frac{lt}{d^2}\right) \frac{1}{d^2} \left(\frac{6}{\pi^2}\right)^2 \prod_{p|d} \frac{p^2(p^2 - 2)}{(p^2 - 1)^2} \prod_{p^2|g} \frac{p^2 - 1}{p^2 - 2}. \quad (29)$$

Let us introduce the notation

$$\tau_t(d^2) = \prod_{p^2|\gcd(t, d^2)} \frac{p^2 - 1}{p^2 - 2}.$$

Notice that $\tau_t(d^2) = \tau_{d^2-t}(d^2)$ and therefore the limit $\lim_{N \rightarrow \infty} I_N(\lambda)$ is real. Using (28) and (29) we can write

$$\lim_{N \rightarrow \infty} I_N(\lambda) = \left(\frac{6}{\pi^2}\right)^2 \prod_{p|d} \frac{1}{p^2 - 1} \left(1 + \prod_{p|d} \frac{p^2 - 2}{p^2 - 1} \sum_{t=1}^{d^2-1} \cos\left(\frac{2\pi lt}{d^2}\right) \tau_t(d^2)\right) \quad (30)$$

and, if $\omega(d) = |\mathcal{P}(d)| = r$, then

$$\begin{aligned} \sum_{t=1}^{d^2-1} \cos\left(\frac{2\pi lt}{d^2}\right) \tau_t(d^2) &= \sum_{t \leq d^2-1} \cos\left(\frac{2\pi lt}{d^2}\right) - \sum_{\substack{p_1|d \\ p_1^2|t}} \sum_{t \leq d^2-1} \cos\left(\frac{2\pi lt}{d^2}\right) \left(1 - \frac{p_1^2 - 1}{p_1^2 - 2}\right) \\ &+ \sum_{\substack{p_1, p_2|d \\ p_1^2 p_2^2|t}} \sum_{t \leq d^2-1} \cos\left(\frac{2\pi lt}{d^2}\right) \left(1 - \frac{p_1^2 - 1}{p_1^2 - 2}\right) \left(1 - \frac{p_2^2 - 1}{p_2^2 - 2}\right) \\ &- \sum_{\substack{p_1, p_2, p_3|d \\ p_1^2 p_2^2 p_3^2|t}} \sum_{t \leq d^2-1} \cos\left(\frac{2\pi lt}{d^2}\right) \left(1 - \frac{p_1^2 - 1}{p_1^2 - 2}\right) \left(1 - \frac{p_2^2 - 1}{p_2^2 - 2}\right) \left(1 - \frac{p_3^2 - 1}{p_3^2 - 2}\right) + \dots \\ &+ (-1)^{r-1} \sum_{p_1, \dots, p_{r-1}|d} \sum_{\substack{t \leq d^2-1 \\ p_1^2 \dots p_{r-1}^2|t}} \cos\left(\frac{2\pi lt}{d^2}\right) \left(1 - \frac{p_1^2 - 1}{p_1^2 - 2}\right) \dots \left(1 - \frac{p_{r-1}^2 - 1}{p_{r-1}^2 - 2}\right). \quad (31) \end{aligned}$$

Recall that $\gcd(l, d^2)$ is square-free, and notice that for every $p_1, \dots, p_m \mid d, m < r,$

$$\sum_{\substack{t \leq d^{2-1} \\ p_1^2 \cdots p_m^2 \mid t}} \cos\left(\frac{2\pi lt}{d^2}\right) = -1.$$

Now (31) yields

$$\begin{aligned} \sum_{t=1}^{d^2-1} \cos\left(\frac{2\pi lt}{d^2}\right) \tau_t(d^2) &= (-1) \left(1 - \sum_{p_1 \mid d} \frac{-1}{p_1^2 - 2} + \sum_{p_1, p_2 \mid d} \frac{-1}{p_1^2 - 2} \frac{-1}{p_2^2 - 2} \right. \\ &\quad \left. - \sum_{p_1, p_2, p_3 \mid d} \frac{-1}{p_1^2 - 2} \frac{-1}{p_2^2 - 2} \frac{-1}{p_3^2 - 2} + \dots + (-1)^{r-1} \sum_{p_1, \dots, p_{r-1} \mid d} \frac{-1}{p_1^2 - 2} \cdots \frac{-1}{p_{r-1}^2 - 2} \right) \\ &= - \left(\prod_{p \mid d} \left(1 - \frac{-1}{p^2 - 2} \right) - (-1)^r \prod_{p \mid d} \frac{-1}{p^2 - 2} \right) = \prod_{p \mid d} \frac{1}{p^2 - 2} - \prod_{p \mid d} \frac{p^2 - 1}{p^2 - 2}, \end{aligned}$$

and (30) becomes

$$\begin{aligned} \lim_{N \rightarrow \infty} I_N(\lambda) &= \left(\frac{6}{\pi^2}\right)^2 \prod_{p \mid d} \frac{1}{p^2 - 1} \left(1 + \prod_{p \mid d} \frac{1}{p^2 - 1} - 1 \right) = \left(\frac{6}{\pi^2}\right)^2 \prod_{p \mid d} \frac{1}{(p^2 - 1)^2} \\ &= g(d)^2. \end{aligned} \quad \square$$

Remark 4.5. Since $\frac{3}{4}p^2 \leq p^2 - 1 \leq p^2$ for every $p,$ we have

$$\frac{1}{d^2} \leq \prod_{p \mid d} \frac{1}{p^2 - 1} \leq \left(\frac{4}{3}\right)^{\omega(d)} \frac{1}{d^2}. \tag{32}$$

Since d is square-free, if we want to give an upper bound for $\omega(d)$ in terms of d as $d \rightarrow \infty,$ it is enough to consider the case when d is the product of the first r prime numbers: $d = p_1 \cdots p_r.$ In this case $\omega(d) = r.$ It is known that $\log d = r \log r(1 + o(1))$ as $r \rightarrow \infty.$ This means that in general $\omega(d) \log \omega(d) \leq (1 + \varepsilon_1) \log d$ for every $\varepsilon_1 > 0,$ provided that $d \gg 1.$ This implies $\omega(d) \leq \frac{\log d^{1+\varepsilon}}{W(\log d^{1+\varepsilon_1})},$ where W denotes the Lambert function, i.e. the solution of the equation $x = W(x)e^{W(x)}.$ It is known that $W(x) \sim \log x$ as $x \rightarrow \infty.$ Therefore $\omega(d) \leq \frac{\log d^{1+\varepsilon_1}}{\log(\log d^{1+\varepsilon_1})^{1-\varepsilon_2}}$ for every $\varepsilon_2 > 0$ and thus

$$\left(\frac{4}{3}\right)^{\omega(d)} = \mathcal{O}_\varepsilon(d^\varepsilon) \tag{33}$$

for every $\varepsilon > 0$ as $d \rightarrow \infty.$ In other words, formulæ (28)–(29) give

$$\lim_{N \rightarrow \infty} I_N^{(t)}(l, d^2) = \mathcal{O}_\varepsilon\left(\frac{1}{d^{2-\varepsilon}}\right) \quad \text{as } d \rightarrow \infty,$$

for every $t = 0, 1, \dots, d^2 - 1$. However, the cancellations coming from the different exponential factors $e^{2\pi it/d^2}$ in $I_N(d)$ are responsible for the higher order of smallness shown in (30):

$$\lim_{N \rightarrow \infty} I_N(l, d^2) = \mathcal{O}\left(\frac{1}{d^{4-\varepsilon}}\right) \text{ as } d \rightarrow \infty.$$

Lemma 4.6. *Let $\lambda_1, \lambda_2 \in \Lambda$, $\lambda_1 = e(l_1/d_1^2)$, $\lambda_2 = e(l_2/d_2^2)$, and $\lambda = \lambda_1 \lambda_2 = e(l/d^2) \in \Lambda$. Then the 2-fold limit*

$$\mathfrak{J}_3(\lambda_1, \lambda_2) = \lim_{\substack{N_1 \rightarrow \infty \\ N_2 \rightarrow \infty}} \frac{1}{N_1 N_2} \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \lambda_1^{n_1} \lambda_2^{n_2} c_3(n_1, n_2) \tag{34}$$

exists and equals $g(d_1)g(d_2)g(d)$.

Proof. Using Lemma 3.8 we can write

$$\begin{aligned} \mathfrak{J}_3(\lambda_1, \lambda_2) &= \lim_{\substack{N_1 \rightarrow \infty \\ N_2 \rightarrow \infty}} \frac{1}{N_1 N_2} \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \lambda_1^{n_1} \lambda_2^{n_2} \sum_{s_0 \geq 1} \sum_{s_1 \geq 1} \sum_{s_2 \geq 1} g(s_0)g(s_1)g(s_2) \\ &\cdot \sum_{\substack{0 \leq t_j \leq s_j^2 - 1 \\ \mu^2(\gcd(t_j, s_j^2))=1 \\ j=0,1,2 \\ t_0/s_0^2 + t_1/s_1^2 + t_2/s_2^2 \in \mathbb{Z}}} e\left(\frac{t_1}{s_1^2}n_1 + \frac{t_2}{s_2^2}n_2\right). \end{aligned} \tag{35}$$

Let us bring the limit and the sums over n_1, n_2 in (35) inside the sum over t_0, t_1, t_2 . For fixed s_0, s_1, s_2 we have

$$\sum_{\substack{0 \leq t_j \leq s_j^2 - 1 \\ \mu^2(\gcd(t_j, s_j^2))=1 \\ j=0,1,2 \\ t_0/s_0^2 + t_1/s_1^2 + t_2/s_2^2 \in \mathbb{Z}}} \left(\lim_{N_1 \rightarrow \infty} \frac{1}{N_1} \sum_{n_1=1}^{N_1} e\left(\left(\frac{t_1}{s_1^2} + \frac{l_1}{d_1^2}\right)n_1\right) \right) \left(\lim_{N_2 \rightarrow \infty} \sum_{n_2=1}^{N_2} e\left(\left(\frac{t_2}{s_2^2}n_2 + \frac{l_2}{d_2^2}\right)n_2\right) \right)$$

The two sums over n_1, n_2 can be written as

$$\frac{1}{N_j} \sum_{n_j=1}^{N_j} e\left(\left(\frac{t_j}{s_j^2} + \frac{l_j}{d_j^2}\right)n_j\right) = \begin{cases} \frac{1}{N_j} \frac{\xi_j - \xi_j^{N+1}}{1 - \xi_j} & \text{if } \xi_j \neq 1, \\ 1 & \text{if } \xi_j = 1, \end{cases}$$

where $\xi_j = e(t_j/s_j^2 + l_j/d_j^2)$ and $j = 1, 2$. Thus, as $N \rightarrow \infty$, only the indices t_0, t_1, t_2 such that $\xi_1 = \xi_2 = 1$ give a non-zero contribution to (35). This condition means

$$\frac{t_j}{s_j^2} + \frac{l_j}{d_j^2} = \begin{cases} 1 & \text{if } \lambda_j \neq 1, \\ 0 & \text{if } \lambda_j = 1, \end{cases}$$

for $j = 1, 2$. However, because of the conditions $0 \leq t_j \leq s_j^2 - 1$ and $\mu^2(\gcd(t_j, s_j^2)) = 1$, the index

$$t_j = \begin{cases} (s_j^2/d_j^2)(d_j^2 - l_j) & \text{if } \lambda_j \neq 1, \\ 0 & \text{if } \lambda_j = 1, \end{cases}$$

will be considered only when $s_j = d_j$. The value of t_0 is given consequently by

$$\frac{t_0}{s_0^2} = \begin{cases} l/d^2 & \text{if } \lambda_1 \neq 1 \neq \lambda_2, \\ l_1/d_1^2 & \text{if } \lambda_1 \neq 1 = \lambda_2, \\ l_2/d_2^2 & \text{if } \lambda_1 = 1 \neq \lambda_2, \\ 0 & \text{if } \lambda_1 = 1 = \lambda_2. \end{cases}$$

In all cases this means $t_0/s_0^2 = l/d^2$, and the condition $\mu^2(\gcd(t_0, s_0^2)) = 1$ implies that $s_0 = d$ and $t_0 = l$. In other words, (35) becomes $\mathfrak{Y}_3(\lambda_1, \lambda_2) = g(d_1)g(d_2)g(d)$, and the lemma is proven. \square

Remark 4.7. Notice that if $\lambda_1 = e(l_1/d_1^2)$, $\lambda_2 = e(l_2/d_2^2) \in \Lambda$ satisfy $\gcd(d_1, d_2) = 1$, then

$$\mathfrak{Y}_3(\lambda_1, \lambda_2) = \left(\frac{6}{\pi^2}\right)^3 (-1)^{\omega(d_1 d_2)} \prod_{p|d_1 d_2} \frac{1}{(p^2 - 1)^2}.$$

The product $\prod_{p|d} (p^2 - 1)$ appears in several formulæ above. Concerning this product, we have the following basic

Lemma 4.8. *Let d be square-free. Then*

$$\prod_{p|d} (p^2 - 1) = |\{1 \leq l \leq d^2 - 1 : \mu^2(\gcd(l, d^2)) = 1\}|. \tag{36}$$

Proof. By standard inclusion-exclusion we can write the rhs of (36) as

$$\begin{aligned} d^2 - \sum_{p_1|d} \frac{d^2}{p_1^2} + \sum_{p_1, p_2|d} \frac{d^2}{(p_1 p_2)^2} - \sum_{p_1, p_2, p_3|d} \frac{d^2}{(p_1 p_2 p_3)^2} + \dots + (-1)^{\omega(d)} \\ = d^2 \prod_{p|d} \left(1 - \frac{1}{p^2}\right) = \prod_{p|d} (p^2 - 1). \quad \square \end{aligned}$$

5. The spectrum of the shift operator T

Recall the definition of the dynamical system (X, \mathcal{B}, Π, T) given in Section 1.2. Denote by U the operator on the Hilbert space $\mathcal{H} = L^2(X, \mathcal{B}, \Pi)$ given by

$$(Uf)(x) = f(Tx). \tag{37}$$

Since T is measure-preserving, the operator U is unitary. The goal of this section is to prove the following

Theorem 5.1. *The spectrum of the operator U is Λ .*

Let us show that Λ is contained in the spectrum of U . This fact is given by the following

Theorem 5.2. *Let $\lambda = e(l/d^2) \in \Lambda$. Then there exists a function $\theta_\lambda \in \mathcal{H}$ such that*

$$(U\theta_\lambda)(x) = \lambda \theta_\lambda(x) \quad (38)$$

for Π -almost every $x \in X$.

Proof. Let $f_0(x) = x(0)$ and let U_λ be the unitary operator on \mathcal{H} defined by

$$(U_\lambda h)(x) = \lambda^{-1} h(Tx).$$

By the von Neumann Ergodic Theorem, the following limit exists in \mathcal{H} :

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N U_\lambda^n f_0(x) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \lambda^{-n} f_0(T^n x) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \lambda^{-n} x(n) =: \theta_\lambda(x). \end{aligned} \quad (39)$$

The function θ_λ is U_λ -invariant, i.e. $(U_\lambda \theta_\lambda)(x) = \theta_\lambda(x)$ for Π -almost every $x \in X$. This implies that $\lambda^{-1} \theta_\lambda(Tx) = \theta_\lambda(x)$, i.e. (38). \square

Denote by $x(s)$ the function $X \rightarrow \{0, 1\}$ given by the projection of $x \in X$ onto its s -th coordinate. Introduce the subspace $H \subseteq \mathcal{H}$,

$$H = \overline{\left\{ \sum_s a_s x(s) \right\}},$$

i.e. the closure of the set of all complex linear combinations of the $x(s)$'s. Then H is invariant under U , and by (39), all the eigenfunctions θ_λ belong to H . Let us remark that, since the operator U is unitary, the eigenfunctions θ_λ are orthogonal to one another for different λ . Let us write

$$x(s) = \sum_{\lambda \in \Lambda} \langle x(s), \theta_\lambda \rangle \theta_\lambda.$$

Recall (25). We have the following

Theorem 5.3. *Let $\lambda = e(l/d^2) \in \Lambda$. Then for every $s \in \mathbb{Z}$ we have*

$$\langle x(s), \theta_\lambda \rangle = \lambda^s g(d)^2. \quad (40)$$

Proof. Let us use (39) and write

$$\langle x(s), \theta_\lambda \rangle = \lim_{N \rightarrow \infty} \left\langle x(s), \frac{1}{N} \sum_{n=1}^N \lambda^{-n} x(n) \right\rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \lambda^n \langle x(s), x(n) \rangle. \quad (41)$$

Notice that $\langle x(s), x(n) \rangle = c_2(n - s)$, where c_2 is the second correlation function given by Proposition 3.2. Equation (41) becomes

$$\langle x(s), \theta_\lambda \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \lambda^n c_2(n - s) = \lambda^s \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \lambda^n c_2(n) = \lambda^s \mathfrak{J}_2(\lambda). \tag{42}$$

The needed statement follows now from Lemma 4.4. □

Theorem 5.3 immediately implies the following

Corollary 5.4. *All eigenfunctions θ_λ are non-zero.*

Remark 5.5. The formula (39) can be written for an arbitrary measure-preserving map, but in most cases (e.g. automorphisms with continuous spectrum) it gives zero. Theorem 5.3 shows that in our case it is non-zero.

We can also compute the L^2 -norm of each eigenfunction explicitly.

Theorem 5.6. *Let $\lambda = e(l/d^2) \in \Lambda$. Then*

$$\|\theta_\lambda\| = |g(d)|. \tag{43}$$

Proof. This is a straightforward application of Theorem 5.3:

$$\begin{aligned} \|\theta_\lambda\|^2 &= \langle \theta_\lambda, \theta_\lambda \rangle = \left\langle \theta_\lambda, \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \lambda^{-n} x(n) \right\rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \lambda^n \overline{\langle x(s), \theta_\lambda \rangle} \\ &= \left(\frac{6}{\pi^2}\right)^2 \prod_{p|d} \frac{1}{(p^2 - 1)^2}. \end{aligned} \tag{44}$$

Proposition 5.7. *The set $\{\theta_\lambda\}_{\lambda \in \Lambda}$ of eigenfunctions is a basis for H .*

Proof. Since the eigenfunctions are orthogonal it is enough to show that they span the space of all linear combinations of the $x(s)$'s. We know that each atom $\{\lambda\}$ of the spectral measure ν (associated to the second correlation function via Bochner's theorem) corresponds to θ_λ in the space H generated by linear forms, and these form a set of generators for H . □

Let us define the normalized eigenfunctions: for $\lambda \in \Lambda$ set

$$\tilde{\theta}_\lambda = \theta_\lambda / \|\theta_\lambda\|,$$

so that $\{\tilde{\theta}_\lambda\}_{\lambda \in \Lambda}$ is an orthonormal basis for H . Let us write

$$x(s) = \sum_{\lambda \in \Lambda} \langle x(s), \tilde{\theta}_\lambda \rangle \tilde{\theta}_\lambda.$$

Since $\{\tilde{\theta}_\lambda\}_\lambda$ is an orthonormal basis for H , by Lemma 4.8 and Theorem 5.6 we have

$$\begin{aligned} \|x(s)\|^2 &= \sum_{\lambda \in \Lambda} |\langle x(s), \tilde{\theta}_\lambda \rangle|^2 = \left(\frac{6}{\pi^2}\right)^2 \sum_{d \in \mathcal{Q}} \prod_{p|d} \frac{1}{p^2 - 1} \\ &= \left(\frac{6}{\pi^2}\right)^2 \sum_{\mathcal{P}' \subset \mathcal{P}, |\mathcal{P}'| < \infty} \prod_{p \in \mathcal{P}'} \frac{1}{p^2 - 1} = \left(\frac{6}{\pi^2}\right)^2 \prod_p \left(1 + \frac{1}{p^2 - 1}\right) = \frac{6}{\pi^2}. \end{aligned}$$

The same argument allows us to estimate the size of the error term in the following approximation of $x(s)$: for $D \geq 1$ let

$$x_D(s) = \sum_{\substack{\lambda = e(l/d^2) \in \Lambda \\ d \leq D}} \langle x(s), \tilde{\theta}_\lambda \rangle \tilde{\theta}_\lambda.$$

Arguing as in Remark 4.5, we have

$$\|x(s) - x_D(s)\|^2 = \sum_{\substack{\lambda = e(l/d^2) \in \Lambda \\ d > D}} |\langle x(s), \tilde{\theta}_\lambda \rangle|^2 = \frac{6}{\pi^2} \sum_{d > D} |g(d)| = \mathcal{O}_\varepsilon(D^{-1+\varepsilon}) \quad (44)$$

for every $\varepsilon > 0$.

Let us consider the product of two eigenfunctions $\tilde{\theta}_{\lambda_1}$ and $\tilde{\theta}_{\lambda_2}$. We have the following

Theorem 5.8. *Let $\lambda_1 = e(l_1/d_1^2)$, $\lambda_2 = e(l_2/d_2^2) \in \Lambda$. Then*

$$\tilde{\theta}_{\lambda_1} \tilde{\theta}_{\lambda_2} = \epsilon \tilde{\theta}_\lambda, \quad (45)$$

where $\lambda = e(l/d^2) = \lambda_1 \lambda_2$ and $\epsilon = \epsilon(\lambda_1, \lambda_2) = \mu(d_1)\mu(d_2)\mu(d) = \pm 1$.

Proof. It is enough to show that for every $s \in \mathbb{Z}$ we have

$$\langle \tilde{\theta}_{\lambda_1} \tilde{\theta}_{\lambda_2}, x(s) \rangle = \langle \tilde{\theta}_\lambda, x(s) \rangle.$$

Using the definition (39) we can write

$$\theta_{\lambda_1} \theta_{\lambda_2} = \lim_{\substack{N_1 \rightarrow \infty \\ N_2 \rightarrow \infty}} \frac{1}{N_1 N_2} \sum_{n=1}^{N_1} \sum_{n=2}^{N_2} \lambda_1^{-n_1} \lambda_2^{-n_2} x(n_1) x(n_2)$$

and thus

$$\langle \theta_{\lambda_1} \theta_{\lambda_2}, x(s) \rangle = \lim_{\substack{N_1 \rightarrow \infty \\ N_2 \rightarrow \infty}} \frac{1}{N_1 N_2} \sum_{n=1}^{N_1} \sum_{n=2}^{N_2} \lambda_1^{-n_1} \lambda_2^{-n_2} \langle x(n_1) x(n_2), x(s) \rangle.$$

Notice that $\langle x(n_1)x(n_2), x(s) \rangle = c_3(n_1 - s, n_2 - s)$. Therefore, by Lemma 4.6,

$$\begin{aligned} \langle \theta_{\lambda_1}\theta_{\lambda_2}, x(s) \rangle &= \lim_{\substack{N_1 \rightarrow \infty \\ N_2 \rightarrow \infty}} \frac{1}{N_1 N_2} \sum_{n=1}^{N_1} \sum_{n=2}^{N_2} \lambda_1^{-n_1} \lambda_2^{-n_2} c_3(n_1 - s, n_2 - s) \\ &= \lambda^{-s} \lim_{\substack{N_1 \rightarrow \infty \\ N_2 \rightarrow \infty}} \frac{1}{N_1 N_2} \sum_{n=1}^{N_1} \sum_{n=2}^{N_2} \lambda_1^{-n_1} \lambda_2^{-n_2} c_3(n_1, n_2) \\ &= \lambda^{-s} \mathfrak{Y}_3(\lambda_1^{-1}, \lambda_2^{-1}) = \lambda^{-s} g(d_1)g(d_2)g(d). \end{aligned}$$

On the other hand, by Theorem 5.3,

$$\langle \theta_\lambda, x(s) \rangle = e^{-\lambda} \mathfrak{Y}_2(\lambda) = \lambda^{-s} g(d)^2.$$

Therefore

$$\epsilon = \langle \tilde{\theta}_{\lambda_1}\tilde{\theta}_{\lambda_2}, x(s) \rangle \langle \tilde{\theta}_\lambda, x(s) \rangle^{-1} = \frac{g(d_1)g(d_2)g(d)}{|g(d_1)||g(d_2)|} \frac{|g(d)|}{g(d)^2} = \mu(d_1)\mu(d_2)\mu(d). \quad \square$$

By associativity of multiplication, $\epsilon(\lambda_1, \lambda_2)\epsilon(\lambda_1\lambda_2, \lambda_3) = \epsilon(\lambda_2, \lambda_3)\epsilon(\lambda_1, \lambda_2\lambda_3)$. Theorem 5.8 can be applied iteratively. It allows us to write all polynomial expressions in the eigenfunctions as linear expressions, and this is a very important fact.

We want to show that the set $\{\theta_\lambda\}_{\lambda \in \Lambda}$ of eigenfunctions is a basis for the whole space \mathcal{H} . We shall need the notion of *unitary rings* introduced by V. A. Rokhlin (see [10]).

Definition 5.9. A complex Hilbert space \mathfrak{H} is called a *unitary ring* if and only if, for certain pairs of elements, a product is defined satisfying:

- (I) If fg is defined, then $fg = gf$.
- (II) If $fg, (fg)h$ and gh are defined, then $(fg)h = f(gh)$.
- (III) If fh and gh are defined and $\alpha, \beta \in \mathbb{C}$, then $(\alpha f + \beta g)h = \alpha(fh) + \beta(gh)$.
- (IV) There exists $e \in \mathfrak{H}$ such that $ef = f$ for every $f \in \mathfrak{H}$.
- (V) If $f_n g$ are defined and $f_n \rightarrow f, f_n g \rightarrow h$, then $fg = h$.
- (VI) The set $\mathfrak{M} = \{f \in \mathfrak{H} : fg \text{ is defined for all } g \in \mathfrak{H}\}$ is dense in \mathfrak{H} ; moreover if fg is defined, then there exist $f_n \in \mathfrak{M}$ such that $f_n \rightarrow f$ and $f_n g \rightarrow fg$.
- (VII) For every $f \in \mathfrak{H}$, there exists $\tilde{f} \in \mathfrak{H}$ such that $\langle fg, h \rangle = \langle g, \tilde{f}h \rangle$ for all $f, g \in \mathfrak{M}$.

An important result by Rokhlin is that every unitary ring can be written as $\mathfrak{H} = L^2(M, \mathcal{M}, m)$, where (M, \mathcal{M}, m) is a Lebesgue space (see, e.g., V. A. Rokhlin¹ [9]). In our case we have the unitary ring $\mathcal{H} = L^2(X, \mathcal{B}, \Pi)$ and the subspace H which is a subring because of Theorem 5.8. In this representation a subring $\mathfrak{R} \subset \mathfrak{H}$ corresponds to a σ -subalgebra \mathcal{N} of \mathcal{M} , i.e. $\mathfrak{R} = L^2(M, \mathcal{N}, m|_{\mathcal{N}})$. Therefore H is a subspace of \mathcal{H} , which is a Hilbert space corresponding to some σ -subalgebra \mathcal{F} of \mathcal{B} . Let us show that

Proposition 5.10. *Up to sets of measure zero, $\mathcal{F} = \mathcal{B}$. In other words, $H = \mathcal{H}$.*

¹ The notion of *Lebesgue space* used here allows points with positive measure, unlike the classical case discussed by Ya. G. Sinai [13] in the context of K-systems.

Proof. Let us use Rokhlin's technique of measurable partitions (see [11]). According to it \mathcal{F} corresponds to some measurable partition ξ of X . If $\mathcal{F} \subsetneq \mathcal{B}$, then there exists a bounded, non-negative function $h(x)$ and a subset $A \in \mathcal{F}$ such that $\mathbb{E}(h|C_\xi) \geq \alpha$ for almost all $C_\xi \in A$ and some positive α . Being a measurable function, h can be approximated arbitrarily well in $L^\infty(X, \mathcal{F}, \Pi|_{\mathcal{F}})$ by a function h' which is a polynomial in the $x(s)$'s. Using (44) we can approximate h' in measure by a finite polynomial in the eigenfunctions θ_λ . However, every such polynomial belongs to our Hilbert space $L^2(X, \mathcal{F}, \Pi|_{\mathcal{F}})$ and it is measurable with respect to \mathcal{F} . Therefore the conditional expectation of h' with respect to ξ is arbitrarily close to h' , but such a function cannot approximate h in measure. This shows that $H = \mathcal{H}$. \square

Propositions 5.10 and 5.7 immediately give the following

Corollary 5.11. *The set $\{\theta_\lambda\}_{\lambda \in \Lambda}$ of eigenfunctions is a basis in the space \mathcal{H} .*

This fact, together with Theorem 5.2 and Corollary 5.4, yields Theorem 5.1. It also implies the following

Theorem 5.12. *The dynamical system (X, \mathcal{B}, Π, T) is ergodic.*

Proof. By shift-invariance of Π we already know that the eigenspace spanned by the constants is at least one-dimensional. On the other hand, by Theorem 5.1, its dimension cannot be greater than one. This implies that the only invariant functions are constants Π -almost everywhere, and hence we have ergodicity. \square

Theorems 5.1 and 5.12 give part (i) of our Main Theorem.

Remark 5.13. One could also derive Corollary 5.11 in a different way and without using Rokhlin's theory of unitary rings and measurable partitions. The derivation, although explicit, is rather long. In fact, one can show that for every $-\infty < s_1 < \dots < s_r < \infty$ the product $x(s_1) \cdots x(s_r)$ belongs to the span of $\{\theta_\lambda\}_{\lambda \in \Lambda}$. For example, for $r = 2$, by Theorems 5.3 and 5.8,

$$\begin{aligned} x(s_1)x(s_2) &= \left(\sum_{\lambda_1 \in \Lambda} \langle x(s_1), \tilde{\theta}_{\lambda_1} \rangle \tilde{\theta}_{\lambda_1} \right) \left(\sum_{\lambda_2 \in \Lambda} \langle x(s_2), \tilde{\theta}_{\lambda_2} \rangle \tilde{\theta}_{\lambda_2} \right) \\ &= \sum_{\lambda_1, \lambda_2 \in \Lambda} \lambda_1^{s_1} \lambda_2^{s_2} g(d_1)g(d_2) \mu(d) \tilde{\theta}_{\lambda_1 \lambda_2} \\ &= \sum_{\lambda \in \Lambda} \left(\mu(d) \sum_{\substack{\lambda_1, \lambda_2 \in \Lambda \\ \lambda_1 \lambda_2 = \lambda}} \lambda_1^{s_1} \lambda_2^{s_2} g(d_1)g(d_2) \right) \tilde{\theta}_\lambda, \end{aligned}$$

and one can prove that

$$\sum_{\substack{\lambda_1, \lambda_2 \in \Lambda \\ \lambda_1 \lambda_2 = \lambda}} |g(d_1)g(d_2)| = \mathcal{O}_\varepsilon(d^{-2+\varepsilon})$$

for every $\varepsilon > 0$, where $\lambda = e(l/d^2)$. This implies that

$$\sum_{\lambda \in \Lambda} \left| \sum_{\substack{\lambda_1, \lambda_2 \in \Lambda \\ \lambda_1 \lambda_2 = \lambda}} \lambda_1^{s_1} \lambda_2^{s_2} g(d_1) g(d_2) \right|^2$$

is finite.

6. Spectral analysis for $(\mathbb{G}, \mathbb{B}, \mathbb{P}, \mathbb{T}_{\mathbf{u}})$

Recall the group \mathbb{G} defined in (9). Let us consider the space $\mathbb{H} = L^2(\mathbb{G}, \mathbb{B}, \mathbb{P})$, and the unitary operator \mathbb{U} on \mathbb{H} defined by

$$(\mathbb{U}f)(\mathbf{g}) = f(\mathbf{g} + (1, 1, \dots)).$$

Theorem 6.1. *The spectrum of \mathbb{U} is Λ .*

Proof. Consider the projection $\pi_{p^2} : \mathbb{G} \rightarrow \mathbb{Z}/p^2\mathbb{Z}$, $\pi_{p^2}(\mathbf{g}) = g_{p^2}$. It is immediate to see that the function $\xi_{e(1/p^2)}(\mathbf{g}) = e(\pi_{p^2}(\mathbf{g})/p^2)$ is an eigenfunction for \mathbb{U} with eigenvalue $e(1/p^2)$. By taking powers one can get any eigenfunction $\xi_{e(t/p^2)}$ with any eigenvalue $e(t/p^2)$ for $0 \leq t \leq p^2 - 1$. By multiplying different such eigenfunctions (with different p), one can obtain eigenfunctions ξ_λ with an arbitrary eigenvalue $\lambda \in \Lambda$. Since Λ is the character group of \mathbb{G} and $\mathbb{T}_{\mathbf{u}}$ is a translation in \mathbb{G} , there are no other eigenvalues. \square

To conclude the proof of part (ii) of our Main Theorem we need the following

Theorem 6.2 (J. von Neumann, [15]). *Two ergodic measure-preserving transformations with pure point spectrum are isomorphic if and only if they have the same spectrum.*

Theorems 6.1 and 6.2 imply that (X, \mathcal{B}, Π, T) and $(\mathbb{G}, \mathbb{B}, \mathbb{P}, \mathbb{T}_{\mathbf{u}})$ are isomorphic as measure-preserving dynamical systems. This concludes the proof of our Main Theorem.

Appendix A. The proof of Theorem 2.3

The proof is based on the identity

$$\left(\sum_{n=1}^{\infty} \frac{a(n)}{n^s} \right) \left(\sum_{n=1}^{\infty} \frac{b(n)}{n^s} \right) = \sum_{n=1}^{\infty} \frac{(a * b)(n)}{n^s}, \tag{46}$$

where $a * b$ is the Dirichlet convolution of a and b :

$$(a * b)(n) = \sum_{d|n} a(d)b(n/d). \tag{47}$$

We shall be considering only the case of $s = 2$ and bounded sequences $a(n)$ and $b(n)$, therefore there will be no question about convergence of the above series. We shall also use the classical identity

$$\sum_{d|n} \mu(d) = 0 \quad \text{for } n > 1. \tag{48}$$

First, let us consider the case of square-free numbers not divisible by a single prime p , i.e. $\mathcal{S} = \{p\}$. In this case, we shall prove Theorem 2.3 by means of three lemmata, and then we shall explain how to generalize this approach to general finite sets \mathcal{S} .

Let $w_p(n)$ be the indicator of the integers not divisible by p , i.e.

$$w_p(n) = \begin{cases} 0 & \text{if } p \mid n, \\ 1 & \text{otherwise.} \end{cases}$$

Lemma A.1.

$$\mu^2(n)w_p(n) = \sum_{d:d^2 \mid n} \mu(d)w_p(d)w_p(n/d). \quad (49)$$

Proof. If $p \mid n$, then $p \mid d$ or $p \mid n/d$ (possibly both) for every divisor d of n . Thus $w_p(n)w_p(n/d) = 0$ and the sum on the rhs of (49) is 0 (and obviously equals the lhs). If $p \nmid n$, then no divisor d of n will be divisible by p and $w_p(d) = w_p(n/d) = 1$. The sum in (49) then becomes $\sum_{d^2 \mid n} \mu(d)$. If n is square-free, then $d^2 = 1$ is the only perfect square that divides n , and the sum equals 1 (and clearly agrees with the lhs of (49)). If n is not square-free let us write it as $n_1 n_2^2$ where n_1 and n_2 are defined as follows. For every prime p let us define $l = l(p, n) = \max\{j : p^j \mid n\}$; then set $n_1 = \prod_{p: 2 \nmid l} p$ and $n_2 = \prod_{p: 2 \mid l} p^{l/2} \cdot \prod_{p: 2 \nmid l} p^{(l-1)/2}$. Since n_1 is square-free, if $d^2 \mid n$, then $d \mid n_2$. This means that the sum on the rhs of (49) becomes $\sum_{d \mid n_2} \mu(d)$ and equals 0 by (48) (thus matching the lhs). This concludes the proof. \square

Lemma A.2.

$$\sum_{n=1}^{\infty} \frac{w_p(n)}{n^2} = \frac{p^2 - 1}{p^2} \zeta(2).$$

Proof. The formula follows from the trivial computation

$$\sum_{n=1}^{\infty} \frac{w_p(n)}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(pn)^2} = \left(1 - \frac{1}{p^2}\right) \zeta(2). \quad \square$$

Let us denote by $\{\delta_1(n)\}_{n \geq 1}$ the sequence equal to 1 if $n = 1$ and 0 otherwise. Then we have

Lemma A.3.

$$(\mu w_p) * w_p = \delta_1.$$

Proof. For $n = 1$ the statement is obvious since $d = 1$ is the only divisor of n and we have $\mu(1)w_p(1)^2 = 1 = \delta_1(1)$. Let $n > 1$. Then $((\mu w_p) * w_p)(n) = \sum_{d \mid n} \mu(d)w_p(d)w_p(n/d)$. We can discuss the cases when $p \mid n$ and $p \nmid n$ separately, and argue as in the proof of Lemma A.1. In the first case we have $w_p(n)w_p(n/d) = 0$ and the sum is 0. In the second case $w_p(d) = w_p(n/d) = 1$ and the sum becomes $\sum_{d \mid n} \mu(d)$, that is, 0 by (48). In other words, we have shown that, for $n > 1$, we have $((\mu w_p) * w_p)(n) = 0 = \delta_1(n)$, and this concludes the proof. \square

Corollary A.4.

$$\sum_{n=1}^{\infty} \frac{\mu(n)w_p(n)}{n^2} = \frac{p^2}{p^2 - 1} \frac{1}{\zeta(2)}.$$

Proof. This is a straightforward application of Lemma A.2 and the formulæ (46)–(47) with $a = \mu w_p, b = w_p$, and (from Lemma A.3) $a * b = \delta_1$. \square

Proof of Theorem 2.3 when $\mathcal{S} = \{p\}$. Notice that $|\mathcal{Q}_N^{\{p\}}(0)| = \sum_{n \leq N} \mu^2(n)w_p(n)$. By Lemma A.1, we can write

$$|\mathcal{Q}_N^{\{p\}}(0)| = \sum_{n \leq N} \mu^2(n)w_p(n) = \sum_{n \leq N} \sum_{d^2 | n} \mu(d)w_p(d)w_p(n/d). \tag{50}$$

Now we want to exchange the two sums. Let us fix $d \leq \sqrt{N}$. For every n of the form $n = md^2$ we have $w_p(n/d) = w_p(m)w_p(d)$. Let $\eta_d^{\{p\}}(N)$ be the number of integers of the form md^2 where $m \leq N/d^2$ and $p \nmid m$. Then

$$|\mathcal{Q}_N^{\{p\}}(0)| = \sum_{d \leq \sqrt{N}} \eta_d^{\{p\}}(N)\mu(d)w_p(d).$$

We can estimate the number $\eta_d^{\{p\}}(N)$ as follows. Let $\lfloor N/d^2 \rfloor \equiv t \pmod{p}, t \in \{0, 1, \dots, p - 1\}$. Then

$$\eta_d^{\{p\}}(N) = \frac{\lfloor N/d^2 \rfloor - t}{p}(p - 1) + t = \frac{N}{d^2} \frac{p - 1}{p} + q_1^{\{p\}}(d, N),$$

where

$$|q_1^{\{p\}}(d, N)| \leq \frac{p - 1}{p} \left| \left\lfloor \frac{N}{d^2} \right\rfloor - \frac{N}{d^2} \right| + t \left(1 - \frac{p - 1}{p} \right) \leq 2 \frac{p - 1}{p} =: C'(\{p\}).$$

This gives

$$|\mathcal{Q}_N^{\{p\}}(0)| = N \frac{p - 1}{p} \sum_{d \leq \sqrt{N}} \frac{\mu(d)w_p(d)}{d^2} + q_2^{\{p\}}(N),$$

where $|q_2^{\{p\}}(N)| \leq C'(\{p\})\sqrt{N}$. Now, Corollary A.4 yields

$$|\mathcal{Q}_N^{\{p\}}(N)| = \frac{p}{p + 1} \frac{1}{\zeta(2)} N + q_2^{\{p\}}(N) + q_3^{\{p\}}(N),$$

where

$$\begin{aligned} |q_3^{\{p\}}(N)| &\leq N \frac{p - 1}{p} \sum_{d > \sqrt{N}} \frac{1}{d^2} \leq N \frac{p - 1}{p} \int_{\sqrt{N}}^{\infty} \frac{dx}{(x - 1)^2} \\ &= \frac{p - 1}{p} \frac{N}{\sqrt{N} - 1} \leq C''(\{p\})\sqrt{N} \end{aligned}$$

for every $N \geq 4$, where $C''(\{p\}) = 2(p - 1)/p$. This concludes the proof, with $\alpha(\{p\}) = p/(p + 1)$ and $C(\{p\}) = C'(\{p\}) + C''(\{p\})$. \square

Let us now discuss how to adapt the above proof to the case of a general finite set $\mathcal{S} \subset \mathcal{P}$. The sequence w_p has to be replaced by the indicator of the integers divisible by none of the primes in \mathcal{S} , i.e.

$$w_{\mathcal{S}}(n) = \begin{cases} 0 & \text{if } p \mid n \text{ for some } p \in \mathcal{S}, \\ 1 & \text{otherwise.} \end{cases}$$

Lemma A.1 is still valid if we replace w_p by $w_{\mathcal{S}}$:

Lemma A.5.

$$\mu^2(n)w_{\mathcal{S}}(n) = \sum_{d:d^2 \mid n} \mu(d)w_{\mathcal{S}}(d)w_{\mathcal{S}}(n/d). \tag{51}$$

Lemma A.2 is replaced by an analogous statement given by inclusion-exclusion:

Lemma A.6.

$$\sum_{n=1}^{\infty} \frac{w_{\mathcal{S}}(n)}{n^2} = a(\mathcal{S})\zeta(2), \quad \text{where } a(\mathcal{S}) = \prod_{p \in \mathcal{S}} \frac{p^2 - 1}{p^2}.$$

Proof. If $\mathcal{S} = \{p_1, \dots, p_k\}$, then inclusion-exclusion gives

$$\begin{aligned} a(\mathcal{S}) &= \left(1 - \sum_{i=1}^k \frac{1}{p_i^2} + \sum_{1 \leq i_1 < i_2 \leq k} \frac{1}{(p_{i_1} p_{i_2})^2} - \sum_{1 \leq i_1 < i_2 < i_3 \leq k} \frac{1}{(p_{i_1} p_{i_2} p_{i_3})^2} + \dots + \frac{(-1)^k}{(p_1 \dots p_k)^2} \right) \\ &= \prod_{i=1}^k \left(1 - \frac{1}{p_i^2} \right). \quad \square \end{aligned}$$

Lemma A.3 also holds for $w_{\mathcal{S}}$:

Lemma A.7.

$$(\mu w_{\mathcal{S}}) * w_{\mathcal{S}} = \delta_1.$$

Finally, Corollary A.4 is replaced by

Corollary A.8.

$$\sum_{n=1}^{\infty} \frac{\mu(n)w_{\mathcal{S}}(n)}{n^2} = \frac{1}{a(\mathcal{S})\zeta(2)}.$$

Proof of Theorem 2.3 for general $\mathcal{S} = \{p_1, \dots, p_k\} \subset \mathcal{P}$. Lemma A.5 gives

$$|\mathcal{Q}_N^{\mathcal{S}}(0)| = \sum_{n \leq N} \sum_{d^2 \mid n} \mu(d)w_{\mathcal{S}}(d)w_{\mathcal{S}}(n/d). \tag{52}$$

Let us fix $d \leq \sqrt{N}$. For every n of the form $n = md^2$ we have $w_S(n/d) = w_S(m)w_S(d)$. Let $\eta_d^S(N)$ be the number of integers of the form md^2 where $m \leq N/d^2$ and $p \nmid m$ for every $p \in \mathcal{S}$. Then

$$|\mathcal{Q}_N^S(0)| = \sum_{d \leq \sqrt{N}} \eta_d^S(N) \mu(d) w_S(d).$$

The set of numbers not divisible by any $p \in \mathcal{S}$ has density given by

$$\begin{aligned} 1 - \sum_{i=1}^k \frac{1}{p_i} + \sum_{1 \leq i_1 < i_2 \leq k} \frac{1}{p_{i_1} p_{i_2}} - \sum_{1 \leq i_1 < i_2 < i_3 \leq k} \frac{1}{p_{i_1} p_{i_2} p_{i_3}} + \dots + \frac{(-1)^k}{p_1 \cdots p_k} \\ = \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right) = \prod_{p \in \mathcal{S}} \frac{p-1}{p}. \end{aligned}$$

The estimate of $\eta_d^S(N)$ comes from the following observation. If

$$\lfloor N/d^2 \rfloor \equiv t \pmod{[S]} \quad \text{for } t \in \{0, 1, \dots, [S] - 1\},$$

then

$$\eta_d^S(N) = \frac{\lfloor N/d^2 \rfloor - t}{[S]} \prod_{p \in \mathcal{S}} (p-1) + t = \frac{N}{d^2} \prod_{p \in \mathcal{S}} \frac{p-1}{p} + q_1^S(d, N),$$

where

$$\begin{aligned} |q_1^S(d, N)| &\leq \prod_{p \in \mathcal{S}} \frac{p-1}{p} \left(\left\lfloor \frac{N}{d^2} \right\rfloor - \frac{N}{d^2} \right) + t \left(1 - \prod_{p \in \mathcal{S}} \frac{p-1}{p} \right) \\ &\leq 2 \prod_{p \in \mathcal{S}} \frac{p-1}{p} + \left(\prod_{p \in \mathcal{S}} p - 1 \right) - \prod_{p \in \mathcal{S}} (p-1) =: C'(S). \end{aligned}$$

This gives

$$|\mathcal{Q}_N^S(0)| = N \prod_{p \in \mathcal{S}} \frac{p-1}{p} \sum_{d \leq \sqrt{N}} \frac{\mu(d) w_S(d)}{d^2} + q_2^S(N),$$

where $|q_2^S(N)| \leq C'(S)\sqrt{N}$. Now Corollary A.8 yields

$$|\mathcal{Q}_N^S(0)| = \prod_{p \in \mathcal{S}} \frac{p}{p+1} \frac{1}{\zeta(2)} N + q_2^S(N) + q_3^S(N),$$

where

$$|q_3^S(N)| \leq N \prod_{p \in \mathcal{S}} \frac{p-1}{p} \sum_{d > \sqrt{N}} \frac{1}{d^2} \leq C''(S)\sqrt{N}$$

and $C''(S) = 2 \prod_{p \in \mathcal{S}} (p-1)/p$ for $N \geq 4$. This concludes the proof of the general case of the theorem, with $\alpha(S) = \prod_{p \in \mathcal{S}} p/(p+1)$ and $C(S) = C'(S) + C''(S)$. \square

Appendix B. The proofs of Lemmata 4.1–4.3

Proof of Lemma 4.1. Let us write

$$l = \prod_{\bar{p} \in \bar{\mathcal{P}}} (\bar{p})^{a(\bar{p})} \cdot \prod_{p' \in \mathcal{P}'} (p')^{b(p')} \cdot q \tag{53}$$

where $\bar{\mathcal{P}} \subseteq \mathcal{P}(d)$, $a(\bar{p}) \geq 2$ for every $\bar{p} \in \bar{\mathcal{P}}$, $\mathcal{P}' \subset \mathcal{P} \setminus \mathcal{P}(d)$, $|\mathcal{P}'| < \infty$, $b(p') \geq 2$ for every $p' \in \mathcal{P}'$, q is square-free and $p \nmid q$ for every $p \in \bar{\mathcal{P}} \cup \mathcal{P}'$. It is clear that every $l \geq 1$ can be written uniquely as in (53). And the condition $l \leq N$ can be rewritten using the notation in (13) as

$$q \in \mathcal{Q}_{N / (\prod_{\bar{p} \in \bar{\mathcal{P}}} (\bar{p})^{a(\bar{p})} \cdot \prod_{p' \in \mathcal{P}'} (p')^{b(p')})}(\mathbb{0}).$$

Furthermore, notice that $c_2(d^2l)$ can depend only on d and \mathcal{P}' :

$$c_2(d^2l) = \prod_{p \in \mathcal{P}(d) \cup \mathcal{P}'} \left(1 - \frac{1}{p^2}\right) \prod_{p \notin \mathcal{P}(d) \cup \mathcal{P}'} \left(1 - \frac{2}{p^2}\right) = \sigma_1 \prod_{p \in \mathcal{P}(d) \cup \mathcal{P}'} \frac{p^2 - 1}{p^2 - 2}.$$

Now we can write

$$\begin{aligned} \frac{1}{N} \sum_{l \leq N} c_2(d^2l) &= \sigma_1 \prod_{p \in \mathcal{P}(d)} \frac{p^2 - 1}{p^2 - 2} \sum_{\bar{\mathcal{P}} \subseteq \mathcal{P}(d)} \sum_{\mathcal{P}' \subset \mathcal{P} \setminus \mathcal{P}(d)} \prod_{p' \in \mathcal{P}'} \frac{p'^2 - 1}{p'^2 - 2} \\ &\cdot \sum_{\substack{a(\bar{p}) \geq 2, \bar{p} \in \bar{\mathcal{P}} \\ b(p') \geq 2, p' \in \mathcal{P}'}} \frac{1}{\prod_{\bar{p} \in \bar{\mathcal{P}}} (\bar{p})^{a(\bar{p})} \cdot \prod_{p' \in \mathcal{P}'} (p')^{b(p')}} \mathbb{E}_{N / (\prod_{\bar{p} \in \bar{\mathcal{P}}} (\bar{p})^{a(\bar{p})} \cdot \prod_{p' \in \mathcal{P}'} (p')^{b(p')})}^{\bar{\mathcal{P}} \cup \mathcal{P}'}(\mathbb{0}). \end{aligned}$$

Now we can use (14) while taking the limit as $N \rightarrow \infty$, and sum over all $a(\bar{p}), b(p') \geq 2$ as in the proof of Proposition 3.4. Notice that the sets $\mathcal{P}(d)$ and \mathcal{P}' are disjoint. We obtain

$$\begin{aligned} &\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{l \leq N} c_2(d^2l) \\ &= \sigma_1 \prod_{p \in \mathcal{P}(d)} \frac{p^2 - 1}{p^2 - 2} \sum_{\bar{\mathcal{P}} \subseteq \mathcal{P}(d)} \sum_{\mathcal{P}' \subset \mathcal{P} \setminus \mathcal{P}(d)} \prod_{p' \in \mathcal{P}'} \frac{p'^2 - 1}{p'^2 - 2} \frac{6}{\pi^2} \prod_{p \in \bar{\mathcal{P}} \cup \mathcal{P}'} \frac{1}{p^2 - 1} \\ &= \sigma_1 \frac{6}{\pi^2} \prod_{p \in \mathcal{P}(d)} \frac{p^2 - 1}{p^2 - 2} \left(1 + \frac{1}{p^2 - 1}\right) \prod_{p \in \mathcal{P} \setminus \mathcal{P}(d)} \left(1 + \frac{1}{p^2 - 2}\right) \\ &= \frac{6}{\pi^2} \prod_{p \in \mathcal{P}(d)} \frac{p^2}{p^2 - 1} \prod_p \frac{p^2 - 1}{p^2} = \left(\frac{6}{\pi^2}\right)^2 \prod_{p \in \mathcal{P}(d)} \frac{p^2}{p^2 - 1}, \end{aligned}$$

and the lemma is proved. □

Proof of Lemma 4.2. Let us first consider the case $g = 1$. Numbers of the form $\prod_{p' \in \mathcal{P}'} p'^{b(p')} q$, where $\mathcal{P}' \subset \mathcal{P} \setminus \mathcal{P}(d)$, $|\mathcal{P}'| < \infty$, $b(p') \geq 2$ for $p' \in \mathcal{P}'$, q is square-free and $p \nmid q$ for every $p \in \mathcal{P}(d) \cup \mathcal{P}'$ can be represented as

$$\prod_{p' \in \mathcal{P}'} p'^{b(p')} \cdot q = d^2 l + h \tag{54}$$

for some $1 \leq h \leq d^2 - 1$, where $\gcd(h, d^2) = 1$. Since there are $\varphi(d^2)$ such h 's (here φ denotes Euler's totient function) and the various h 's appear with the same frequency, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{l \leq N} c_2(d^2 l + t) = \frac{1}{\varphi(d^2)} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{l \leq N} \sum_{\gcd(h, d^2)=1} c_2(d^2 l + h). \tag{55}$$

Notice that the condition $l \leq N$ becomes

$$q \in \mathcal{Q}_{(d^2 N + h) / \prod_{p' \in \mathcal{P}'} p'^{b(p')}}^{(\mathcal{P}(d) \cup \mathcal{P}')} (0)$$

and

$$c_2(d^2 l + t) = \prod_{p' \in \mathcal{P}'} \left(1 - \frac{1}{p'^2}\right) \prod_{p \notin \mathcal{P}'} \left(1 - \frac{2}{p^2}\right) = \sigma_1 \prod_{p' \in \mathcal{P}'} \frac{p'^2 - 1}{p'^2 - 2}.$$

Now

$$\begin{aligned} \frac{1}{N} \sum_{l \leq N} \sum_{\gcd(h, d^2)=1} c_2(d^2 l + h) &= \sigma_1 \sum_{\mathcal{P}' \subset \mathcal{P} \setminus \mathcal{P}(d)} \prod_{p' \in \mathcal{P}'} \frac{p'^2 - 1}{p'^2 - 2} \\ &\cdot \sum_{\substack{b(p') \geq 2, \\ p' \in \mathcal{P}'}} \frac{d^2}{\prod_{p' \in \mathcal{P}'} p'^{b(p')}} \frac{d^2 N + h}{d^2 N} \mathbb{E}_{(d^2 N + h) / \prod_{p' \in \mathcal{P}'} p'^{b(p')}}^{(\mathcal{P}(d) \cup \mathcal{P}')} (0), \end{aligned}$$

and by taking the limit as $N \rightarrow \infty$ we get

$$\begin{aligned} &\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{l \leq N} \sum_{\gcd(h, d^2)=1} c_2(d^2 l + h) \\ &= \sigma_1 \sum_{\mathcal{P}' \subset \mathcal{P} \setminus \mathcal{P}(d)} \prod_{p' \in \mathcal{P}'} \frac{p'^2 - 1}{p'^2 - 2} \frac{6}{\pi^2} \prod_{p \in \mathcal{P}(d)} \frac{p^3}{p + 1} \prod_{p' \in \mathcal{P}'} \frac{1}{p'^2 - 1} \\ &= \sigma_1 \frac{6}{\pi^2} \prod_{p \in \mathcal{P}(d)} \frac{p^3}{p + 1} \sum_{\mathcal{P}' \subset \mathcal{P} \setminus \mathcal{P}(d)} \prod_{p' \in \mathcal{P}'} \frac{1}{p'^2 - 2} \\ &= \sigma_1 \frac{6}{\pi^2} \prod_{p \in \mathcal{P}(d)} \frac{p^3}{p + 1} \prod_{p \in \mathcal{P} \setminus \mathcal{P}(d)} \left(1 + \frac{1}{p^2 - 2}\right) \\ &= \frac{6}{\pi^2} \prod_{p \in \mathcal{P}(d)} \frac{p^3(p^2 - 2)}{(p + 1)(p^2 - 1)} \prod_p \frac{p^2 - 1}{p^2} = \left(\frac{6}{\pi^2}\right)^2 \prod_{p \in \mathcal{P}(d)} \frac{p^3(p^2 - 2)}{(p + 1)(p^2 - 1)}. \tag{56} \end{aligned}$$

Let us apply the fact that φ is multiplicative and that $\varphi(p^2) = p(p - 1)$. We obtain

$$\frac{1}{\varphi(d^2)} = \prod_{p \in \mathcal{P}(d)} \frac{1}{p(p - 1)}. \tag{57}$$

Now (55)–(57) yield the desired result.

Let us now consider the case when $\gcd(t, d^2) = g = \bar{p}$. In this case $d^2 = \bar{p}^2 d_1^2$ and $t = \bar{p} t_1$, where d_1 is square-free, $\bar{p} \nmid d_1$, and $\bar{p} \nmid t_1$. We can write

$$\frac{d^2}{\bar{p}} l + \frac{t}{\bar{p}} = \bar{p} d_1^2 l + t_1 = \prod_{p' \in \mathcal{P}'} p'^{b(p')} \cdot q_1,$$

where $\mathcal{P}' \subset \mathcal{P} \setminus \mathcal{P}(d)$, $|\mathcal{P}'| < \infty$, q_1 is square-free and $p \nmid q$ for every $p \in \mathcal{P}(d) \cup \mathcal{P}'$. The condition $l \leq N$ now reads

$$q_1 \in \mathcal{Q}_{(\bar{p}d_1^2 N + t_1) / \prod_{p' \in \mathcal{P}'} p'^{b(p')}}^{\mathcal{P}(d) \cup \mathcal{P}'}(0).$$

Since, by assumption, $\bar{p}^2 \nmid d^2 l + t$, we have

$$c_2(d^2 l + t) = c_2(\bar{p} d_1^2 l + t_1) = \prod_{p' \in \mathcal{P}'} \left(1 - \frac{1}{p'^2}\right) \prod_{p \notin \mathcal{P}'} \left(1 - \frac{2}{p^2}\right) = \sigma_1 \cdot \prod_{p' \in \mathcal{P}'} \frac{p'^2 - 1}{p'^2 - 2}.$$

Now, since $\gcd(t_1, \bar{p} d_1^2) = 1$, we can use (55):

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{l \leq N} c_2(d^2 l + t) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{l \leq N} c_2(\bar{p} d_1^2 l + t_1) \\ &= \frac{1}{\varphi(\bar{p} d_1^2)} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{l \leq N} \sum_{\gcd(h_1, \bar{p} d_1^2) = 1} c_2(\bar{p} d_1^2 + h_1), \end{aligned} \tag{58}$$

and we can write

$$\begin{aligned} \frac{1}{N} \sum_{l \leq N} \sum_{\gcd(h_1, \bar{p} d_1^2) = 1} c_2(\bar{p} d_1^2 + h_1) &= \sigma_1 \sum_{\mathcal{P}' \subset \mathcal{P} \setminus \mathcal{P}(d)} \prod_{p' \in \mathcal{P}'} \frac{p'^2 - 1}{p'^2 - 2} \\ &\cdot \sum_{\substack{b(p') \geq 2, \\ p' \in \mathcal{P}'}} \frac{\bar{p} d_1^2}{\prod_{p' \in \mathcal{P}'} p'^{b(p')}} \frac{\bar{p} d_1^2 N + h_1}{\bar{p} d_1^2 N} \mathbb{E}_{(\bar{p} d_1^2 N + h_1) / \prod_{p' \in \mathcal{P}'} p'^{b(p')}}^{\mathcal{P}(d) \cup \mathcal{P}'}(0). \end{aligned}$$

Notice that $\mathcal{P}(d) = \mathcal{P}(d_1) \cup \{\bar{p}\}$. By taking the limit as $N \rightarrow \infty$ we obtain

$$\begin{aligned}
 & \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{l \leq N} \sum_{\gcd(h_1, \bar{p}d_1^2) = 1} c_2(\bar{p}d_1^2 l + h_1) \\
 &= \sigma_1 \sum_{\mathcal{P}' \subset \mathcal{P} \setminus \mathcal{P}(d)} \prod_{p' \in \mathcal{P}'} \frac{p'^2 - 1}{p'^2 - 2} \bar{p}d_1^2 \frac{6}{\pi^2} \prod_{p \in \mathcal{P}(d)} \frac{p}{p+1} \prod_{p' \in \mathcal{P}'} \frac{1}{p'^2 - 1} \\
 &= \sigma_1 \frac{6}{\pi^2} \frac{\bar{p}^2}{\bar{p} + 1} \prod_{p \in \mathcal{P}(d_1)} \frac{p^3}{p+1} \sum_{\mathcal{P}' \subset \mathcal{P} \setminus \mathcal{P}(d)} \prod_{p' \in \mathcal{P}'} \frac{1}{p'^2 - 2} \\
 &= \sigma_1 \frac{6}{\pi^2} \frac{\bar{p}^2}{\bar{p} + 1} \prod_{p \in \mathcal{P}(d_1)} \frac{p^3}{p+1} \prod_{p \in \mathcal{P} \setminus \mathcal{P}(d)} \left(1 + \frac{1}{p^2 - 2}\right) \\
 &= \frac{6}{\pi^2} \frac{\bar{p}^2}{\bar{p} + 1} \frac{\bar{p}^2 - 2}{\bar{p}^2 - 1} \prod_{p \in \mathcal{P}(d_1)} \frac{p^3(p^2 - 2)}{(p+1)(p^2 - 1)} \prod_p \frac{p^2 - 1}{p^2} \\
 &= \left(\frac{6}{\pi^2}\right)^2 \frac{\bar{p}^2(\bar{p}^2 - 2)}{(\bar{p} + 1)(\bar{p}^2 - 1)} \prod_{p \in \mathcal{P}(d_1)} \frac{p^3(p^2 - 2)}{(p+1)(p^2 - 1)}. \tag{59}
 \end{aligned}$$

Let us use the fact that $\varphi(\bar{p}d_1^2) = \varphi(\bar{p})\varphi(d_1^2) = (\bar{p} - 1)\varphi(d_1^2)$ to obtain the formula

$$\frac{1}{\varphi(\bar{p}d_1^2)} = \frac{1}{\bar{p} - 1} \prod_{p \in \mathcal{P}(d_1)} \frac{1}{p(p-1)}. \tag{60}$$

Now we can combine (58)–(60) to conclude the proof of the lemma. The case of a general square-free g is treated analogously. \square

Proof of Lemma 4.3. The case when g is square-free (i.e. $\mathcal{P}_2(g) = \emptyset$) is already included in Lemma 4.2. Thus, assume $\mathcal{P}_2(g) \neq \emptyset$. Let, for simplicity, $\gcd(t, d^2) = g = \bar{p}^2$ (i.e. $\omega(g) = 1$), the case of $\omega(g) > 1$ being analogous. We have $d^2 = \bar{p}^2 d_1^2$ and $t = \bar{p}^2 t_1$, where d_1 is square-free and $\bar{p} \nmid d_1$. In particular $\gcd(t_1, d_1^2) = 1$. We can write

$$\frac{d^2}{\bar{p}^2} l + \frac{t}{\bar{p}^2} = d_1^2 l + t_1 = \bar{p}^a \prod_{p \in \mathcal{P}'} p^{b(p')} q_1,$$

where $a \geq 0$, $\mathcal{P}' \subset \mathcal{P} \setminus \mathcal{P}(d)$, $|\mathcal{P}'| < \infty$, q_1 is square-free and $p \nmid q_1$ for every $p \in \mathcal{P}(d) \cup \mathcal{P}'$. The condition $l \leq N$ can be written as

$$q_1 \in \mathcal{Q}_{(d_1^2 N + t_1) / \prod_{p' \in \mathcal{P}'} p^{b(p')}}(0),$$

and

$$\begin{aligned}
 c_2(d^2 l + t) &= c_2(\bar{p}^2(d_1^2 l + t_1)) = \prod_{p \in \mathcal{P}' \cup \{\bar{p}\}} \left(1 - \frac{1}{p^2}\right) \prod_{p \notin \mathcal{P}' \cup \{\bar{p}\}} \left(1 - \frac{2}{p^2}\right) \\
 &= \sigma_1 \prod_{p \in \mathcal{P}' \cup \{\bar{p}\}} \frac{p^2 - 1}{p^2 - 2}.
 \end{aligned}$$

Notice that \mathcal{P}' and $\{\bar{p}\}$ are disjoint by construction. Using (55) we see that

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{l \leq N} c_2(d^2l + t) &= \lim_{N \rightarrow \infty} \sum_{l \leq N} c_2(\bar{p}^2(d_1^2l + t_1)) \\ &= \frac{1}{\varphi(d_1^2)} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{l \leq N} \sum_{\gcd(h_1, d_1^2)=1} c_2(\bar{p}^2(d_1^2l + h_1)). \end{aligned} \tag{61}$$

We have

$$\begin{aligned} \frac{1}{N} \sum_{l \leq N} \sum_{\gcd(h_1, d_1^2)=1} c_2(\bar{p}^2(d_1^2l + h_1)) &= \sigma_1 \sum_{\mathcal{P}' \subset \mathcal{P} \setminus \mathcal{P}(d)} \frac{\bar{p}^2 - 1}{\bar{p}^2 - 2} \prod_{p' \in \mathcal{P}'} \frac{p'^2 - 1}{p'^2 - 2} \\ &\cdot \sum_{a \geq 0} \sum_{b(p') \geq 2, p' \in \mathcal{P}'} \frac{d_1^2}{\bar{p}^a \prod_{p' \in \mathcal{P}'} p'^{b(p')}} \frac{d_1^2 N + h_1}{d_1^2 N} \mathbb{E}_{(d_1^2 N + h_1) / (\bar{p}^a \prod_{p' \in \mathcal{P}'} p'^{b(p')})}^{\mathcal{P}(d) \cup \mathcal{P}'}(0), \end{aligned}$$

and by taking the limit as $N \rightarrow \infty$ we get

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{l \leq N} \sum_{\gcd(h_1, d_1^2)=1} c_2(\bar{p}^2(d_1^2l + h_1)) &= \sigma_1 \frac{\bar{p}^2 - 1}{\bar{p}^2 - 2} \sum_{\mathcal{P}' \subset \mathcal{P} \setminus \mathcal{P}(d)} \prod_{p' \in \mathcal{P}'} \frac{p'^2 - 1}{p'^2 - 2} \frac{d_1^2 \bar{p}}{\bar{p} - 1} \frac{6}{\pi^2} \prod_{p \in \mathcal{P}(d)} \frac{p}{p + 1} \prod_{p' \in \mathcal{P}'} \frac{1}{p'^2 - 1} \\ &= \sigma_1 \frac{6}{\pi^2} \frac{\bar{p} + 1}{\bar{p}(\bar{p}^2 - 2)} \prod_{p \in \mathcal{P}(d)} \frac{p^3}{p + 1} \sum_{\mathcal{P}' \subset \mathcal{P} \setminus \mathcal{P}(d)} \prod_{p' \in \mathcal{P}'} \frac{1}{p'^2 - 2} \\ &= \sigma_1 \frac{6}{\pi^2} \frac{\bar{p} + 1}{\bar{p}(\bar{p}^2 - 2)} \prod_{p \in \mathcal{P}(d)} \frac{p^3}{p + 1} \prod_{p \in \mathcal{P} \setminus \mathcal{P}(d)} \left(1 + \frac{1}{p^2 - 2}\right) \\ &= \frac{6}{\pi^2} \frac{\bar{p}^2}{\bar{p}^2 - 1} \prod_{p \in \mathcal{P}(d_1)} \frac{p^3}{p + 1} \frac{p^2 - 2}{p^2 - 1} \prod_p \frac{p^2 - 1}{p^2} \\ &= \left(\frac{6}{\pi^2}\right)^2 \frac{\bar{p}^2}{\bar{p}^2 - 1} \prod_{p \in \mathcal{P}(d_1)} \frac{p^3(p^2 - 2)}{(p + 1)(p^2 - 1)}. \end{aligned} \tag{62}$$

We use again the fact that

$$\frac{1}{\varphi(d_1^2)} = \prod_{p \in \mathcal{P}(d_1)} \frac{1}{p(p - 1)}, \tag{63}$$

and combining (61)–(63), we obtain the lemma. □

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