J. Eur. Math. Soc. 15, 1457–1483

DOI 10.4171/JEMS/396

Bong H. Lian · Ruifang Song · Shing-Tung Yau

Period integrals and tautological systems

Received July 23, 2012 and in revised form December 17, 2012

Abstract. We study period integrals of CY hypersurfaces in a partial flag variety. We construct a regular holonomic system of differential equations which govern the period integrals. By means of representation theory, a set of generators of the system can be described explicitly. The results are also generalized to CY complete intersections. The construction of these new systems of differential equations has led us to the notion of a tautological system.

Keywords. Calabi–Yau, period integrals, Picard–Fuchs systems, partial flag varieties

Contents

1. Introduction

Let X be a nonsingular d-dimensional Fano variety, i.e. K_X^{-1} is ample. Assume that a general section $f_0 \in V^* = H^0(X, K_X^{-1})$ defines a nonsingular CY variety $Y_{f_0} = \{f_0 = 0\}$. The local Torelli theorem implies that the line $H^{d-1,0}(Y_f) \subset H^{d-1}(Y_{f_0})$ determines the isomorphism class of Y_f, for f close to a fixed f_0. Period integrals provide a way to parameterize the lines $H^{d-1,0}(Y_f)$, as f varies. By the Kodaira–Nakono–Akizuki vanishing

B. H. Lian: Department of Mathematics, Brandeis University,
Goldsmith MS 050, 415 South Street, Waltham, MA 02454, USA; e-mail: lian@brandeis.edu

R. Song: Department of Mathematics, University of Wisconsin-Madison,
480 Lincoln Drive, Madison, WI 53706, USA; e-mail: rsong@math.wisc.edu

S.-T. Yau: Department of Mathematics, Harvard University,
One Oxford Street, Cambridge, MA 02138, USA; e-mail: yau@math.harvard.edu

Mathematics Subject Classification (2010): Primary 14J32; Secondary 14M15

theorem and Serre duality, the Poincaré residue sequence collapses to give an isomorphism

$$\mathrm{Res} : H^0(X, \Omega^d(Y_f)) \to H^{d-1,0}(Y_f).$$

Let γ_i be a basis of the free part of $H_{d-1}(Y_{f_0}, \mathbb{Z})$, and $\tau(\gamma_i) \in H_d(X, \mathbb{Z})$ be a small tube over γ_i. We can choose a local family of meromorphic d-forms Ω_f with a single pole along Y_f, so that

$$\int_{\gamma_i} \mathrm{Res}\,\Omega_f = \int_{\tau(\gamma_i)} \Omega_f.$$

These period integrals determine the line $H^{d-1,0}(Y_f)$, and they fit together to form a sheaf of functions, which we call the period sheaf over $V^* - D$. Here D is the discriminant locus, consisting of f such that Y_f is singular. If Ω_f is globally defined on $V^* - D$, then its period integrals define a locally constant sheaf of finite rank over $V^* - D$.

Let $\pi : \mathcal{Y} \to V^* - D$ be the universal family of smooth CY hypersurfaces in X. For $f \in V^* - D$, let $j : Y_f \hookrightarrow X$ be the inclusion map. The vanishing cohomology of Y_f is defined as $H^{d-1}(Y_f)_{\mathrm{van}} := \mathrm{Ker}(H^{d-1}(Y_f) \xrightarrow{j_*} H^{d+1}(X))$. We have $H^{d-1}(Y_f) = j^* H^{d-1}(X) \oplus H^{d-1}(Y_f)_{\mathrm{van}}$ and the splitting preserves the Hodge structures. We also have the short exact sequence

$$0 \to H^d(X)_{\mathrm{prim}} \to H^d(X - Y_f) \xrightarrow{\mathrm{Res}} H^{d-1}(Y_f)_{\mathrm{van}} \to 0.$$

The groups $H^{d-1}(Y_f, \mathbb{C})$ form a flat vector bundle $\mathbb{H}^{d-1}_{\mathbb{C}}$ over $V^* - D$. Let $\mathcal{H}^{d-1} = \mathbb{H}^{d-1}_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{O}_{V^*-D}$ be the corresponding locally free sheaf over $V^* - D$. $\{H^{d-1}(Y_f)_{\mathrm{van}}\}_{f \in V^*-D}$ form a local system over $V^* - D$ which we denote by $\mathbb{H}^{d-1}_{\mathrm{van}}$. Let $\mathcal{H}^{d-1}_{\mathrm{van}} = \mathbb{H}^{d-1}_{\mathrm{van}} \otimes_{\mathbb{C}} \mathcal{O}_{V^*-D}$. The map $f \mapsto \mathrm{Res}\,\Omega_f$ gives a section of $\mathcal{H}^{d-1}_{\mathrm{van}}$. Thanks to [17], the vanishing cohomology of a sufficiently ample hypersurface $Y \subset X$ can be realized as residues of meromorphic forms on X with poles along Y. Moreover, this realization relates the Hodge level to the order of the pole.

Theorem 1.1 ([17]). *Suppose Y is a sufficiently ample hypersurface in X, i.e. for any $p \geq 0$, $q > 0$, and $s > 0$ we have $H^q(X, \Omega^p_X(sY)) = 0$. The residues* $\mathrm{Res}(\eta/f^k)$ *for $\eta \in H^0(X, K_X(kY))$ generate $F^{d-k-1}H^{d-1}(Y)_{\mathrm{van}}$. Here $F^\bullet H^{d-1}(Y)_{\mathrm{van}}$ is the induced Hodge filtration on $H^{d-1}(Y)_{\mathrm{van}}$.*

The result can be used to devise a reduction procedure for computing differential equations for period integrals. However, the procedure is difficult to implement, except in simple examples. Inspired by mirror symmetry [9], additional tools and alternative methods have been developed for hypersurfaces in a toric variety (see for e.g. [2], [20] and references therein). In this case, one can explicitly construct a global family of meromorphic top forms Ω_f, and a D-module that governs the period sheaf. The D-module turns out to be an extension of a GKZ hypergeometric system. General solutions to GKZ systems and their holonomic ranks have been found [15], [1], under certain nondegeneracy conditions. In one important degenerate case (for applications in mirror symmetry) a closed formula for the general solutions near a particular singular point, a point of maximal unipotent monodromy, has also been constructed [21], [26], giving explicit power series expansions for period integrals of CY hypersurfaces. We refer the reader to [27], [10] for surveys and the extensive bibliography therein for studies on the GKZ hypergeometric systems in other important contexts.

The main motivation of the present paper is to study period integrals and deformations of CY complete intersections in a homogeneous space. In this paper, we shall mostly restrict ourselves to partial flag varieties. We begin, in Section 2, by constructing period integrals for the universal family of these CY manifolds, by means of a global Poincaré residue formula. An explicit formula in the case of Grassmannians is given. We show that the family of CY is deformation complete. Next, we would like to explicitly construct, describe, and ultimately solve a D-module that governs the period integrals. One attempt would be to imitate the construction of GKZ systems for toric hypersurfaces. In this case, recall that the idea was to start with a "natural" basis of $H^0(X, K_X^{-1})$ (which is indexed by the integral points of some polytope); relations among the integral points then give rise to GKZ type binomial differential operators that govern the period integrals. The torus action on X yields additional first order operators. Together, the operators form a regular holonomic system. For homogeneous spaces, the standard monomial theory for representations of reductive groups provides a natural way to index bases of cohomology of line bundles over X. Thus one would expect that there should be a parallel approach to construct GKZ type systems in this case. Unfortunately, the D-modules one constructs this way are almost never holonomic—there would not be enough binomial differential operators to determine the period integrals—mainly because the variety defined by the binomial ideal typically has the wrong dimension in this case.

The present paper solves this problem by introducing a type of systems of differential equations, which we call *tautological systems*. For a fixed reductive algebraic group G, to every G-variety X equipped with a very ample equivariant line bundle L (or a list of such bundles), we attach a system of differential operators defined on $H^0(X, L)$, depending on a group character (Section 3). We show that the system is regular holonomic when X is a homogeneous space. A number of examples, including a toric variety, are discussed (Section 4). In this case, the construction recovers the GKZ hypergeometric system for CY hypersurfaces in a toric variety X, when G is the usual torus acting on X. Likewise the extended version of GKZ system is recovered when G is taken to be $\mathrm{Aut}(X)$. Finally, we show that the period integrals of the universal family of CY complete intersections in a partial flag variety are solutions to a tautological system. We also give an explicit description of this system (Section 5). In Section 6, we discuss some numerical examples and their solutions. Further generalizations and examples will appear in a future paper.

2. Poincaré residues for partial flag varieties

We follow the standard convention that the Lie algebra of a group H is denoted by the gothic letter $\mathfrak{h}$. Let X be a partial flag variety, i.e. $X = G/P$ where $G = \mathrm{SL}_n$ and P is a parabolic subgroup of G.

Our key step in explicitly describing Poincaré residues of complete intersections in X, is to first give a concrete construction of a nowhere vanishing G-invariant holomorphic form Ω of degree $k = \dim X$ on a certain principal torus bundle over X.

Construction of Ω. We shall fix a Borel subgroup B of G and assume that $P \supset B$. Let Φ, $\Delta = \{\alpha_1, \dots, \alpha_{n-1}\}$ be respectively the root system and the set of simple roots of G, relative to B. It is well-known that parabolic subgroups of G containing B are parameterized by subsets of Δ. The set

$$S = \Delta - \{\alpha_{d_1}, \dots, \alpha_{d_r}\} \quad (0 = d_0 < d_1 < \cdots < d_r < d_{r+1} := n)$$

corresponds to the parabolic subgroup P_S whose Lie algebra is

$$\mathfrak{p}_S = \mathfrak{b} + \sum_{\alpha \in [S]} \mathfrak{g}_{-\alpha}.$$

Here $[S]$ is the set of positive roots in the linear span of S. In this case, the homogeneous space $X = G/P_S$ can be identified with the partial flag variety $F(d_1, \dots, d_r, n)$, which consists of all r-step flags of the form

$$(0) \subset E_1 \subset \cdots \subset E_r \subset \mathbb{C}^n \quad (\dim E_i = d_i).$$

Put

$$\tilde{M} = M_{d_1,d_2} \times \cdots \times M_{d_r,n}$$

where $M_{a,b}$ $(a < b)$ denotes the space of $a \times b$ matrices of rank a. Put

$$\begin{aligned} K &= \mathrm{SL}_{d_1} \times \cdots \times \mathrm{SL}_{d_r}, \\ \hat{K} &= \mathrm{GL}_{d_1} \times \cdots \times \mathrm{GL}_{d_r}, \\ Z &= Z(\hat{K}) \cong (\mathbb{C}^\times)^r. \end{aligned} \tag{2.1}$$

Let $(g_1, \dots, g_r, g) \in \hat{K} \times G$ act on $\tilde{M}$ by

$$(g_1, \dots, g_r, g) \cdot (m_1, \dots, m_r) = (g_1 m_1 g_2^{-1}, \dots, g_r m_r g^{-1}).$$

The map $p : \tilde{M} \to X$, $(m_1, \dots, m_r) \mapsto (R(m_1 \cdots m_r) \subset \cdots \subset R(m_r))$, defines a G-equivariant principal $\hat{K}$-bundle over X. Here $R(m)$ denotes the span of the row vectors of the matrix m. The matrix entries of $\{m_i\}_{i=1}^r$ are called the *Stiefel coordinates* of $X = G/P_S$.

Let $z_1, \dots, z_N$ be the standard affine coordinates on $\mathbb{C}^N$. We call

$$\omega = \prod_{i=1}^N dz_i = dz_1 \wedge \cdots \wedge dz_N$$

the *coordinate top form* of $\mathbb{C}^N$. We view the matrix space $\tilde{M}$ as an open subset of the affine space $\mathbb{C}^{d_1 d_2} \times \cdots \times \mathbb{C}^{d_r d_{r+1}}$ and we let ω be the coordinate top form restricted to $\tilde{M}$. Observe that ω is G-invariant and nowhere vanishing.

Let $x_1, \dots, x_q$ $(q = \dim \hat{K})$ form a basis of holomorphic vector fields generated by the action of $\hat{K}$ on $\tilde{M}$. Let ι_{x_i} be the interior multiplication operator with respect to the vector field x_i. Since the holomorphic vector fields $x_1, \dots, x_q$ are everywhere linearly independent, and since ω is nowhere vanishing, the holomorphic form

$$\Omega = \iota_{x_1} \cdots \iota_{x_q} \omega$$

of degree $k = \dim X$ is nowhere vanishing. Since G and K commute, and since ω is G-invariant, Ω is also G-invariant. Since K acts trivially on $\bigwedge^q \hat{\mathfrak{k}}$, and since ω is K-invariant, Ω is also K-basic (i.e. K-invariant and K-horizontal). Note that Ω is Z-horizontal, but not Z-invariant because ω is not Z-invariant. It follows that Ω defines a form on the G-equivariant principal Z-bundle $\tilde{M}/K \to X$. This completes the construction of Ω.

To summarize, we have

Theorem 2.1. *Let $X = G/P$, where $G = \mathrm{SL}_n$ and P is a parabolic subgroup of G. Then Ω constructed above is a nowhere vanishing G-invariant holomorphic form of degree $k = \dim X$, defined on a principal Z-bundle over X where Z is an algebraic torus.*

By construction, it is clear that Z acts on $\mathbb{C}\omega$, hence on Ω, by the character

$$Z \to \mathbb{C}^\times, \qquad t \mapsto \prod_{i=1}^{r} \det(t_i)^{d_{i+1}-d_{i-1}}. \tag{2.2}$$

Let $\lambda_1, \dots, \lambda_{n-1}$ be the fundamental dominant weights of $G = \mathrm{SL}_n$. Let $\mathcal{O}_d(1)$ be the hyperplane line bundle on $G(d,n) \hookrightarrow \mathbb{P}(\bigwedge^d \mathbb{C}^n)$. Note that $\bigwedge^d \mathbb{C}^n$ is the irreducible representation of G of highest weight λ_d. Let $\pi_i : X = F(d_1, \dots, d_r, n) \to G(d_i, n)$ be the ith natural projection. Then the line bundles

$$L_{\lambda_{d_i}} := \pi_i^* \mathcal{O}_{d_i}(1)$$

freely generate the group $\mathrm{Pic}(X) \overset{c_1}{\cong} H^2(X, \mathbb{Z})$ (cf. Proposition 5.1). Moreover we find that

$$c_1(TX) = c_1(K_X^{-1}) = \sum_{i=1}^{r} (d_{i+1} - d_{i-1}) c_1(L_{\lambda_{d_i}}) \tag{2.3}$$

whose coefficients agree with the exponents of the character (2.2).

Example 2.2 ($G(d,n)$, the Grassmannian of d-planes in $\mathbb{C}^n$). We will derive an explicit formula for Ω in this case. Let

$$\pi : M_{d,n} \to M_{d,n}/\mathrm{GL}_d = G(d,n)$$

be the natural projection. Then $\widehat{G}(d,n) := M_{d,n}/\mathrm{SL}_d$ is a principal $\mathbb{C}^*$-bundle over $G(d,n)$. Let $\{z_{il}\}_{1 \le i \le d; 1 \le l \le n}$ be the Stiefel coordinates of $G(d,n)$. The action of GL_d on $M_{d,n}$ generates d^2 linearly independent vector fields $u_{ij} = \sum_{l=1}^n z_{il} \frac{\partial}{\partial z_{jl}}$ $(1 \le i, j \le d)$. Let

$$\omega = \prod_{1 \le i \le d; 1 \le l \le n} dz_{il} = dz_{11} \wedge \cdots \wedge dz_{1n} \wedge dz_{21} \wedge \cdots \wedge dz_{2n} \wedge \cdots \wedge dz_{d1} \wedge \cdots \wedge dz_{dn}$$

be the coordinate top form on $M_{d,n}$. We will contract ω with $\{u_{ij}\}$ successively to obtain the desired SL_d-invariant $d(n-d)$-form Ω.

Let $\iota_{u_j} = \iota_{u_{dj}} \cdots \iota_{u_{1j}}$ and $\iota_u = \iota_{u_d} \cdots \iota_{u_1}$ for convenience.

Let $I_{d,n} = \{I = (1 \le i_1 < \cdots < i_d \le n)\}$ and let $p_I = \det Z_I$ be the minor of $(z_{ij})_{d\times n}$ indexed by I. Then $\{p_I\}_{I\in I_{d,n}}$ are the Plücker coordinates of $G(d,n)$. A simple computation shows that

$$\begin{aligned}\iota_{u_j} &= \iota_{u_{dj}} \cdots \iota_{u_{2j}} \iota_{u_{1j}} = \prod_{1\le i\le d} \iota_{z_{i1}\partial/\partial z_{j1}+\cdots+z_{in}\partial/\partial z_{jn}} \\ &= \sum_{I\in I_{d,n}} \det Z_I \prod_{l\in I} \iota_{\partial/\partial z_{jl}} = \sum_{I\in I_{d,n}} p_I \prod_{l\in I} \iota_{\partial/\partial z_{jl}}\end{aligned}$$

and

$$\begin{aligned}\iota_u &= \iota_{u_d}\cdots\iota_{u_1} = \Big(\sum_{I_d\in I_{d,n}} p_{I_d} \prod_{l\in I_d} \iota_{\partial/\partial z_{dl}}\Big)\cdots\Big(\sum_{I_1\in I_{d,n}} p_{I_1} \prod_{l\in I_1} \iota_{\partial/\partial z_{1l}}\Big) \\ &= \sum_{I_j\in I_{d,n}} p_{I_d}\cdots p_{I_1} \prod_{1\le j\le d;\, l\in I_j} \iota_{\partial/\partial z_{jl}} \\ &= \sum_{\sigma_i\in S_n/S_d\times S_{n-d}} (-1)^{\sigma_1}\cdots(-1)^{\sigma_d} p_{\sigma_1(I_0)}\cdots p_{\sigma_d(I_0)} \prod_{1\le j\le d;\, l\in\sigma_j(I_0)} \iota_{\partial/\partial z_{jl}}\end{aligned}$$

where $I_0 = (1 < \cdots < d)$, S_n is the symmetric group of n elements and $\sigma \in S_n$ acts on $\{p_I\}_{I\in I_{d,n}}$ by permuting the indices.

Applying the contraction operator ι_u to ω gives us (up to a sign $(-1)^{(d-1)d(n-d)/2}$)

$$\Omega = \sum_{\sigma_i\in S_n/S_d\times S_{n-d}} (-1)^{\sigma_1}\cdots(-1)^{\sigma_d} p_{\sigma_1(I_0)}\cdots p_{\sigma_d(I_0)} \prod_{1\le j\le d;\, d+1\le l\le n} dz_{j\sigma_j(l)}.$$

Let $L_1,\ldots,L_s$ be ample line bundles on the partial flag variety $X = F(d_1,\ldots,d_r,n)$ such that $\bigotimes_{i=1}^s L_i = K_X^{-1}$. By the Borel–Weil theorem, $H^0(X,L_i)$ are irreducible representations of G. Assume that $f_i \in H^0(X,L_i)$ are general sections.

Corollary 2.3. *The Poincaré residue* $\mathrm{Res}(\Omega/f_1\cdots f_s)$ *defines a nowhere vanishing holomorphic top form on the CY complete intersection* Y*:* $f_1 = \cdots = f_s = 0$.

Proof. Recall that every ample line bundle on X is of the form $L = \bigotimes_{i=1}^r (L_{\lambda_{d_i}})^{\otimes n_{d_i}}$, where $n_{d_i} \ge 1$ (cf. Proposition 5.1). By the Borel–Weil theorem, the sections of L can be represented as polynomials of the Plücker coordinates on the Grassmannians $G(d_i,n)$. It is easy to check that the multi-degree of such a polynomial is given by $(n_{d_1},\ldots,n_{d_r})$. Since the Plücker coordinates can be represented as polynomials of the Stiefel coordinates $(m_1,\ldots,m_r)$, we can view the section $f_1\cdots f_s$ of $\bigotimes_{i=1}^s L_i$ as a K-invariant polynomial of the Stiefel coordinates. Moreover, since $\bigotimes_{i=1}^s L_i = K_X^{-1}$, this polynomial transforms under $\hat{K}$ by the same character (2.3) as Ω does. It follows that $\Omega/f_1\cdots f_s$ is a $\hat{K}$-invariant meromorphic form on the matrix space $\tilde{M}$ of degree $k = \dim X$. Since $\Omega/f_1\cdots f_s$ is also $\hat{K}$-horizontal, it descends to a meromorphic top form on X. The pole of this form is clearly $\bigcup Y_i$, where $Y_i = \{f_i = 0\}$, hence the form is a global section of $\Omega_X^k(\bigcup Y_i)$. This implies that $\mathrm{Res}(\Omega/f_1 \cdots f_s)$ defines a holomorphic top form on Y. □

We will now briefly describe the cohomology of CY complete intersections in $X = F(d_1, \ldots, d_r, n)$, with focus on the case of hypersurfaces for simplicity. We begin with the cohomology of X [6]. The ring $H^*(X, \mathbb{Z})$ is isomorphic to the quotient of

$$S(x_1, \ldots, x_{d_1}) \otimes S(x_{d_1+1}, \ldots, x_{d_2}) \otimes \cdots \otimes S(x_{d_r+1}, \ldots, x_n)$$

by the ideal generated by $S^+(x_1, \ldots, x_n)$, where $S(x_1, \ldots, x_k)$ is the ring of symmetric functions in $x_1, \ldots, x_k$ with integer coefficients, and S^+ is the set of elements of degree > 0.

Let W_G, W_P be the Weyl groups of G and $P = P_S$ respectively. The B-orbits on G/P are in one-to-one correspondence with the elements in the coset W_G/W_P. The Bruhat decomposition $G/P = \bigcup_{w \in W_G/W_P} BwP$ is a cellular decompositon and the cells have the form $BwP \cong \mathbb{C}^{\mu(w)}$, where $\mu(w)$ is defined as follows. Let Φ^+ be the set of positive roots of G. Let $\Psi = \Phi^+ - [S]$ be the set of positive roots α such that $-\alpha$ is not a root for P. Then $\mu(w)$ is the number of elements in $w(\Phi^+) \cap (-\Psi)$. Thus the Poincaré polynomial of X is $P_t(X) = \frac{1}{|W_P|} \sum_{w \in W_G} t^{2\mu(w)}$.

Example 2.4 ([18], CY hypersurfaces in the Grassmannian $X = F(d, n) = G(d, n)$). Fix a full flag $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n$ in $\mathbb{C}^n$. For any sequence of integers $\underline{a} = (n - d \geq a_1 \geq \cdots \geq a_d \geq 0)$, the associated *Schubert cell* is defined to be

$$W_{a_1,\ldots,a_d} = \{\Lambda \in G(d, n) \mid \dim(\Lambda \cap V_{n-d+i-a_i}) = i\} \cong \mathbb{C}^{d(n-d)-\sum a_i}.$$

The closure $\overline{W}_{a_1,\ldots,a_d} = \{\Lambda \in G(d, n) \mid \dim(\Lambda \cap V_{n-d+i-a_i}) \geq i\}$ is called the *Schubert variety* associated to $\underline{a}$ and has dimension $d(n-d) - \sum a_i$. The associated *Schubert cycle* is the homology class $\sigma_{a_1,\ldots,a_d} = [\overline{W}_{a_1,\ldots,a_d}]$. The Schubert cells are all even-dimensional cells and therefore all boundary maps are 0. This implies that the homology ring $H_*(X, \mathbb{Z})$ has no torsion and is freely generated by all Schubert cycles $\sigma_{a_1,\ldots,a_d}$. In particular, the dimension of $H^{2p}(X, \mathbb{Z})$ is equal to the number of Schubert cycles of codimension p, i.e. the number of sequences of integers $\underline{a} = (n - d \geq a_1 \geq \cdots \geq a_d \geq 0)$ such that $\sum a_i = p$.

Let Y be a CY hypersurface in $X = G(d, n)$. The total Chern class of X is

$$c(TX) = \prod_{1 \leq i \leq d;\, d+1 \leq j \leq n} (1 - x_i + x_j)$$

and the total Chern class of the normal bundle of Y in X is

$$c(N_{Y/X}) = 1 + x_1 + \cdots + x_d.$$

Thus one gets

$$\begin{aligned} \chi(Y) &= \int_X \frac{c(TX)c_1(N_{Y/X})}{c(N_{Y/X})} \\ &= \int_X \prod_{1 \leq i \leq d;\, d+1 \leq j \leq n} (1 - x_i + x_j) \frac{x_1 + \cdots + x_d}{1 + x_1 + \cdots + x_d}. \end{aligned} \tag{2.4}$$

Since Y is ample, by the Lefschetz hyperplane theorem and the fact that $H^{p,q}(X)=0$ for $p\neq q$, we have

$$H^{p,q}(Y)\cong H^{p,q}(X)=0 \quad \text{if } p\neq q \text{ and } p+q\neq \dim Y. \tag{2.5}$$

Together with (2.4), these conditions determine the Betti numbers of the hypersurface Y. The Hodge numbers of Y can be computed as follows. Put $\bigwedge_y \Omega_X^1=\bigoplus_{p\geq 0} y^p\Omega_X^p$ and recall the χ_y-*genus*:

$$\chi_y(X)=\chi(X,\textstyle\bigwedge_y \Omega_X^1)=\displaystyle\sum_{p,q}(-1)^q h^{p,q}(X)y^p.$$

By the Hirzebruch–Riemann–Roch formula, we have

$$\begin{aligned}\chi_y(Y)&=\chi\big(X,\textstyle\bigwedge_y \Omega_X^1(1+y[-Y])^{-1}(1-[-Y])\big)\\&=\displaystyle\int_X \frac{1-e^{-\lambda}}{1+ye^{-\lambda}}\prod_{1\leq i\leq d;\, d+1\leq j\leq n}\frac{(x_j-x_i)(1+ye^{x_i-x_j})}{1-e^{x_i-x_j}}\end{aligned} \tag{2.6}$$

where $\lambda=c_1([Y])$. Together with (2.5), this equality determines the Hodge numbers of Y.

We now turn to the question of deformation completeness. Let $L_1,\ldots,L_s$ be ample line bundles on X such that $\bigotimes_{i=1}^s L_i=K_X^{-1}$, as before. Put

$$V^*=H^0\Big(X,\bigoplus_{i=1}^s L_i\Big)=H^0(X,L_1)\times\cdots\times H^0(X,L_s)$$

and let $D\subset V^*$ be the locus of singular complete intersections. Consider the universal family $\mathcal{Y}$ of smooth CY complete intersections Y in $X=F(d_1,\ldots,d_r,n)$:

$$\mathcal{Y}=\{(p,f)\in X\times\mathbb{P}(V^*-D)\mid f(p)=0\}\to\mathbb{P}(V^*-D).$$

Suppose $\dim Y>2$. Adapting an argument of [4], we will show the deformation completeness and the discreteness of automorphisms of Y.

Theorem 2.5. *The family $\mathcal{Y}\to\mathbb{P}(V^*-D)$ is a complete deformation. Moreover, any fiber of this family has no nontrivial holomorphic vector fields.*

Proof. For any $f=(f_1,\ldots,f_s)\in V^*-D$, where $f_i\in H^0(X,L_i)$, let Y be the complete intersection defined by $f=0$. Let $\kappa:T_fV^*\to H^1(Y,T_Y)$ be the Kodaira–Spencer map.

The short exact sequence $0\to T_Y\to T_X|_Y\to\bigoplus_{i=1}^s L_i\to 0$ induces a long exact sequence

$$\begin{aligned}0&\to H^0(Y,T_Y)\to H^0(Y,T_X|_Y)\to H^0\Big(Y,\bigoplus_{i=1}^s L_i\Big)\xrightarrow{\kappa} H^1(Y,T_Y)\\&\to H^1(Y,T_X|_Y)\to\cdots.\end{aligned} \tag{2.7}$$

To show that every small deformation of Y is still a complete intersection, it suffices to show $H^1(Y, T_X|_Y) = 0$ and that the restriction map $H^0(X, \bigoplus_{i=1}^s L_i) \to H^0(Y, \bigoplus_{i=1}^s L_i)$ is surjective. We will use a spectral sequence argument and we will need the following vanishing result [7]:

$$H^q(X, \Omega_X^p) = 0 \quad \text{if } p \neq q, \quad \text{and} \quad H^p(X, T_X) = 0 \quad \text{if } p > 0. \tag{2.8}$$

Lemma 2.6. *Let $E = \bigoplus_{i=1}^s L_i$, $K^0 = \mathcal{O}_X$ and*

$$K^{-p} = \bigwedge^p \mathcal{O}_X(E^*) = \bigoplus_{1 \leq j_1 < \cdots < j_p \leq s} \mathcal{O}_X(L_{j_1}^{-1} \otimes \cdots \otimes L_{j_p}^{-1}).$$

The Koszul complex defined as follows is a resolution of $\mathcal{O}_Y$:

$$K^\bullet : 0 \to K^{-s} \xrightarrow{d^{-s}} \cdots \xrightarrow{d^{-2}} K^{-1} \xrightarrow{d^{-1}} K^0$$

where $d^{-p}(e_1 \wedge \cdots \wedge e_p) = \sum_{i=1}^p (-1)^{i-1} f_i e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_p$.

First, consider the two spectral sequences abutting to the hypercohomology of $K^\bullet \otimes \mathcal{O}_X(E)$. We have ${}'A_1^{p,q} = H^q(X, K^p \otimes \mathcal{O}_X(E))$ and ${}''A_2^{p,q} = H^q(X, \mathcal{H}^p(K^\bullet \otimes \mathcal{O}_X(E)))$. By the above lemma, the second one degenerates and

$$\mathbb{H}^0(X, K^\bullet \otimes \mathcal{O}_X(E)) = {}''A_2^{1,0} = H^0(Y, \mathcal{O}_Y(E)).$$

On the other hand, we will show that in the first spectral sequence, ${}'A_1^{-p,p} = 0$ for all $p \neq 0$, and thus ${}'A_1^{0,0} = H^0(X, \mathcal{O}_X(E))$ maps onto $H^0(Y, \mathcal{O}_Y(E))$. Now, ${}'A_1^{-p,p}$ can be different from 0 only for $0 \leq p \leq s$ and

$$\begin{aligned}
{}'A_1^{-p,p} &= H^p(X, K^{-p} \otimes \mathcal{O}_X(E)) \\
&\cong \bigoplus_{1 \leq j_1 < \cdots < j_p \leq s} H^p\Big(X, \mathcal{O}_X(L_{j_1}^{-1} \otimes \cdots \otimes L_{j_p}^{-1}) \otimes \bigoplus_{i=1}^s \mathcal{O}_X(L_i)\Big) \\
&\cong \bigoplus_{1 \leq j_1 < \cdots < j_p \leq s;\ 1 \leq i \leq s} H^p\big(X, \mathcal{O}_X(L_{j_1}^{-1} \otimes \cdots \otimes L_{j_p}^{-1} \otimes L_i)\big) \\
&\cong \bigoplus_{1 \leq j_1 < \cdots < j_p \leq s;\ 1 \leq i \leq s} H^{\dim X - p}\big(X, K_X \otimes \mathcal{O}_X(L_{j_1} \otimes \cdots \otimes L_{j_p} \otimes L_i^{-1})\big)^*.
\end{aligned} \tag{2.9}$$

Since $K_X = \bigotimes_{j=1}^s L_j^{-1}$, $K_X \otimes L_{j_1} \otimes \cdots \otimes L_{j_p} \otimes L_i^{-1}$ is a negative line bundle. By the Kodaira vanishing theorem, the above cohomology group vanishes unless $p = 0$.

Second, consider the two spectral sequences abutting to the hypercohomology of $K^\bullet \otimes T_X$. We have ${}'B_1^{p,q} = H^q(X, K^p \otimes T_X)$ and ${}''B_2^{p,q} = H^q(X, \mathcal{H}^p(K^\bullet \otimes T_X))$. The second one degenerates and

$$\mathbb{H}^1(X, K^\bullet \otimes T_X) = {}''B_2^{1,0} = H^1(Y, T_X|_Y).$$

On the other hand, we will show that in the first spectral sequence, ${}'B_1^{-p,p+1} = 0$ for all p, from which it follows that $H^1(Y, T_X|_Y) = 0$. We have

$$\begin{aligned} {}'B_1^{-p,p+1} &= H^{p+1}(X, K^{-p} \otimes T_X) \\ &= \bigoplus_{1 \le j_1 < \cdots < j_p \le s} H^{p+1}\big(X, \mathcal{O}_X(L_{j_1}^{-1} \otimes \cdots \otimes L_{j_p}^{-1}) \otimes T_X\big) \\ &= \bigoplus_{1 \le j_1 < \cdots < j_p \le s} H^{\dim X - p - 1}\big(X, K_X \otimes \mathcal{O}_X(L_{j_1} \otimes \cdots \otimes L_{j_p}) \otimes \Omega_X^1\big). \end{aligned} \tag{2.10}$$

When $p \neq s$ and $p > 0$, $K_X \otimes L_{j_1} \otimes \cdots \otimes L_{j_p}$ is a negative line bundle and thus

$$H^{\dim X - p - 1}\big(X, K_X \otimes \mathcal{O}_X(L_{j_1} \otimes \cdots \otimes L_{j_p}) \otimes \Omega_X^1\big) = 0$$

by the Kodaira vanishing theorem. When $p = 0$, $B_1^{0,1} = H^1(X, T_X) = 0$ by (2.8). When $p = s$, $B_1^{-s,s+1} = H^{\dim X - s - 1}(X, \Omega_X^1) = 0$ if $\dim X - s - 1 \neq 1$, i.e. if $\dim Y > 2$. (This excludes the K3 case.)

Finally, since K_Y is trivial, by Serre duality and by using the Lefschetz hyperplane theorem inductively we have

$$H^0(Y, T_Y) \cong H^{\dim Y}(Y, \Omega_Y^1)^* \cong H^{1,\dim Y}(Y)^* \cong H^{\dim Y - 1, 0}(Y) \cong H^{\dim Y - 1, 0}(X) = 0$$

and thus Y has no nontrivial holomorphic vector fields. This completes the proof. □

Remark 2.7. Therefore $\mathbb{P}(V^* - D)//\mathrm{SL}_n$ is a coarse moduli space for Y. It is unirational and $\dim \mathbb{P}(V^* - D)//\mathrm{SL}_n = \dim V^* - n^2$ ([11], [12]).

Remark 2.8. The result in the above theorem generalizes to CY complete intersections in any other homogeneous space G/P.

3. Tautological systems

In this section, let G be a connected linear reductive algebraic group and X be a projective G-manifold. For a finite-dimensional G-module V, we let V^* be the dual G-module. Let $\phi : X \hookrightarrow \mathbb{P}V$ be a G-equivariant embedding.

Let $\mathbb{C}[V] = \bigotimes_{k=1}^{\infty} \mathrm{Sym}^k V^*$ be the coordinate ring of V and $\mathbb{C}[V^*] = \bigotimes_{k=1}^{\infty} \mathrm{Sym}^k V$ be the coordinate ring of V^*. Denote by $I(X, \mathbb{P}V) \subset \mathbb{C}[V]$ the vanishing ideal of $\phi(X)$ in $\mathbb{P}V$. Since we have a canonical symplectic form $\langle\, , \rangle$ on $T^*V = V \times V^*$, each linear function $\zeta \in V^*$ uniquely defines a derivation $\partial_\zeta \in \mathrm{Der}\,\mathbb{C}[V^*]$, by the formula $\partial_\zeta a = \langle a, \zeta\rangle$, $a \in V$. The linear G-action $G \to \mathrm{Aut}\, V$ induces a Lie algebra action $\mathfrak{g} \to \mathrm{End}\, V \subset \mathrm{Der}\,\mathbb{C}[V^*]$, $x \mapsto Z_x$, with $Z_x a = x \cdot a$. We refer to the Z_x as the *G-operators*.

Let D_V be the sheaf of holomorphic differential operators on V, and D_{V^*} be the sheaf of holomorphic differential operators on V^*.

Definition 3.1. Fix an integer β, viewed as a character of $\mathbb{C}^\times$. The *tautological system* (or *D-module*) $\tau(X, \phi, G, \beta)$ is the D_{V^*}-module D_{V^*}/J where J is the left ideal of D_{V^*} generated by the following operators: $\{p(\partial_\zeta) \mid p(\zeta) \in I(X, \mathbb{P}V)\}$, $\{Z_x \mid x \in \mathfrak{g}\}$, together with the Euler operator $\sum_i a_i \partial_{\zeta_i} + \beta$, where the a_i and ζ_i are dual bases of V, V^*. We shall call $I(X, \mathbb{P}V)$ (and the ideal of differential operators corresponding to it) the *embedding ideal* of X relative to ϕ.

Remark 3.2. The correspondence $\zeta \mapsto \partial_\zeta$ may be thought of as part of the Fourier transform between D_V and D_{V^*}.

The definition can be made slightly more general and purely algebraic by starting with the initial data: a G-module V and a radical ideal $I \subset \mathbb{C}[V]$, possibly inhomogeneous. Since only the action of the Lie algebra of $G \times \mathbb{C}^\times$ on V^* enters the definition, it can be extended to allow an arbitrary (hence possibly nonintegrable) action of this Lie algebra, with the integer β being replaced by a character of the Lie algebra. Since the main examples of this paper do come equipped with group actions, we will defer this generality to a future study. However, for applications to complete intersections, we will later generalize the definition to allow multiple line bundles on X (Section 5).

In [23, Definition 1, p. 15], Hotta introduces the notion of an "L-twistedly G-equivariant" D-module over a G-variety X, where L is a connection on G. A result there can be applied to a tautological system. On p. 17 of [23], we can let V^* play the role of X there, $D_{V^*}I(X, \mathbb{P}V)$ the role of I there, and β the role of the G-character λ. Then Hotta's Theorem 2 implies that a tautological system has the λ-twisted G-equivariance property. In [23, Section 4, p. 22], he considers the special case when X is given by the closure of the G-orbit of a point in a linear representation of G; this can also be viewed as a special case of Definition 3.1.

Using the dual bases, we can write an element of $\mathbb{C}[V] = \mathbb{C}[\zeta_1, \zeta_2, \dots]$ as a polynomial $p(\zeta) = p(\zeta_1, \zeta_2, \dots)$, and $p(\partial_\zeta)$ as a partial differential operator $p(\partial/\partial a_1, \partial/\partial a_2, \dots)$ with constant coefficients, acting on functions of the variables $a_1, a_2, \dots$. If (x_{ji}) is the matrix representing $x \in \mathfrak{g}$ acting on V in the basis a_i, i.e. $x \cdot a_i = \sum_j x_{ji} a_j$, then

$$Z_x = \sum x_{ji} a_j \frac{\partial}{\partial a_i}.$$

Let $j : V \hookrightarrow W$ be a G-module homomorphism, and $\pi : W^* \twoheadrightarrow V^*$ the dual map. This induces a G-equivariant map on structure sheaves, $\pi^\# : \mathcal{O}_{V^*} \to \pi_* \mathcal{O}_{W^*}$, $f \mapsto f \circ \pi$ for $f \in \mathcal{O}_{V^*}(U)$, and the induced homomorphism on germs is also a $\mathfrak{g}$-module homomorphism:

$$(Z_x^V f) \circ \pi = Z_x^W (f \circ \pi) \tag{3.1}$$

where $f \in \mathcal{O}_{V^*}$. Here Z_x^V, Z_x^W are the G-operators on V^*, W^* respectively. Likewise for the Euler operators (with the same character β).

Now $\pi : W^* \twoheadrightarrow V^*$ induces the G-equivariant algebra homomorphisms $\pi : \mathbb{C}[W] \twoheadrightarrow \mathbb{C}[V]$ and

$$\pi : \mathbb{C}[\partial_{\zeta_W} \mid \zeta_W \in W^*] \twoheadrightarrow \mathbb{C}[\partial_{\zeta_V} \mid \zeta_V \in V^*], \qquad \partial_{\zeta^W} \mapsto \partial_{\pi\zeta^W}.$$

It is straightforward to check that for $f \in \mathcal{O}_{V^*}$ and $p(\partial_{\zeta w}) \in \mathbb{C}[\partial_{\zeta w}]$, we have

$$[(\pi p(\partial_{\zeta w}))f] \circ \pi = p(\partial_{\zeta w})(f \circ \pi). \tag{3.2}$$

Let $\mathcal{S}_{V^*}$ (likewise $\mathcal{S}_{W^*}$) be the subsheaf of $\mathcal{O}_{V^*}$ whose stalks consist of germs annihilated by the defining ideal J of the D-module $\mathcal{M} = \tau(X, \phi_V, G, \beta)$. Then we have a canonical isomorphism (of $\mathbb{C}_{V^*}$-modules) from $\mathcal{S}_{V^*}$ to the solution sheaf $\mathcal{H}om_{\mathcal{D}_{V^*}}(\mathcal{M}, \mathcal{O}_{V^*})$ of $\mathcal{M}$. Under this identification, we can therefore view $\mathcal{S}_{V^*}$ as the solution sheaf of $\mathcal{M}$.

Lemma 3.3 (Change of variables). *Let $\phi_V : X \hookrightarrow \mathbb{P}V$ be a G-equivariant embedding of a G-variety X, and $\phi_W = j \circ \phi_V$, where $j : V \hookrightarrow W$ is a G-module homomorphism, and let $\pi : W^* \twoheadrightarrow V^*$ be the dual of j. Let $\mathcal{S}_{V^*} \subset \mathcal{O}_{V^*}$ and $\mathcal{S}_{W^*} \subset \mathcal{O}_{W^*}$ be the solution sheaves of the D-modules $\tau(X, \phi_V, G, \beta)$ and $\tau(X, \phi_W, G, \beta)$. Then $\pi^\#$ maps $\mathcal{S}_{V^*}$ isomorphically onto $\mathcal{S}_{W^*}$.*

Proof. Since $\pi : \mathbb{C}[W]/I(X, \mathbb{P}W) \to \mathbb{C}[V]/I(X, \mathbb{P}V) \cong \mathbb{C}[X]$ is an isomorphism with $\pi I(X, \mathbb{P}W) = I(X, \mathbb{P}V)$, it follows from (3.2) and (3.1) that if $f \in \mathcal{S}_{V^*}$, then $f \circ \pi \in \mathcal{S}_{W^*}$. In other words, $\pi^\#$ sends a solution to $\tau(X, \phi_V, G, \beta)$ to a solution to $\tau(X, \phi_W, G, \beta)$, so $\pi^\# : \mathcal{S}_{V^*} \to \mathcal{S}_{W^*}$. It is injective, because it is so on the structure sheaves.

Fix bases $a_1, \dots, a_p$ of V and $b_1, \dots, b_q$ of W such that $j : a_i \mapsto b_i$ for $1 \le i \le p$. Then we can regard sections of $\mathcal{O}_{V^*}, \mathcal{O}_{W^*}$ as functions of the variables a and b respectively. Then $\pi^\#$ maps a function $f(a_1, \dots, a_p)$ on V^* to $f(b_1, \dots, b_p)$, and π maps $\partial/\partial b_i$ to $\partial/\partial a_i$ if $1 \le i \le p$, and to zero otherwise. So, the $\partial/\partial b_i$ $(p+1 \le i \le q)$ are generators in $\tau(X, \phi_W, G, \beta)$, hence they kill $\mathcal{S}_{W^*}$. Thus all solutions on W^* are independent of the variables $b_{p+1}, \dots, b_q$. Given any such solution $f(b_1, \dots, b_p)$, it is straightforward to check that the function $f(a_1, \dots, a_p)$ is a solution on V^*. This shows that $\pi^\#$ is surjective on the solution sheaves. □

The lemma will be used to give different descriptions to essentially the same D-module, by choosing different G-modules as targets for embedding X. As the proof shows, the net effect of changing the target from $\mathbb{P}V$ to $\mathbb{P}W$ in the initial data of our tautological system is that we introduce additional linear variables, and at the same time, additional first order operators corresponding to the linear forms in $V^\perp \subset W^*$.

Let $\phi : X \hookrightarrow \mathbb{P}V$ be a given G-equivariant embedding. Let $\mathcal{M}$ be the tautological D_{V^*}-module $\tau(X, \phi, G, \beta)$ for short. Let $\mathcal{H}$ be the solution sheaf $\mathcal{H} = \mathcal{H}om_{D_{V^*}}(\mathcal{M}, \mathcal{O}_{V^*})$. The following is an analogue of [24, Theorem 5.1.3].

Theorem 3.4. *Assume that the G-variety X has only a finite number of G-orbits. Then the following statements hold.*

(1) *The D_{V^*}-module $\mathcal{M}$ is regular holonomic. In particular, there is an open subset V^*_{gen} such that the restriction of the solution sheaf $\mathcal{H}$ to V^*_{gen} is locally constant of finite rank.*
(2) *More explicitly, let $X = \bigsqcup_{l=1}^r X_l$ be the decomposition into G-orbits and let $X_l^\vee \subset V^*$ be the conical variety whose projectivization $\mathbb{P}(X_l^\vee)$ is the projective dual to the Zariski closure of X_l in X. Then $V^*_{\text{gen}} = V^* - \bigcup_{l=1}^r X_l^\vee$.*

(3) *Suppose the coordinate ring* $\mathbb{C}[X]$ *is Cohen–Macaulay. Then the rank of the solution sheaf* $\mathcal{H}$ *over the generic stratum* V^*_{gen} *is less than or equal to the degree of* X *in* $\mathbb{P}V$.

Proof. We will adopt a mix of arguments of Kapranov [24] and Hotta [23].

The Fourier transform of the tautological D-module $\mathcal{M} = D_{V^*}/J$ is $\widehat{\mathcal{M}} = D_V/\widehat{J}$, where $\widehat{J}$ is the D_V-ideal generated by

$$I(X, \mathbb{P}V), \quad \left\{\sum x_{ji}\zeta_i \frac{\partial}{\partial \zeta_j} + \sum x_{ii} \;\middle|\; x \in \mathfrak{g}\right\}, \quad \sum_i \zeta_i \frac{\partial}{\partial \zeta_i} + \dim V - s.$$

It is a twisted $G\times\mathbb{C}^\times$-equivarant coherent D_V-module in the sense of [23] whose support $\operatorname{Supp}\widehat{\mathcal{M}}$ is the cone over X in V and thus consists of finitely many $G\times\mathbb{C}^\times$-orbits. Thus $\widehat{\mathcal{M}}$ is regular holonomic [5].

The tautological D-module $\mathcal{M} = D_{V^*}/J$ is homogeneous since the ideal J is generated by homogeneous elements under the gradation $\deg(\partial/\partial a_i) = -1$ and $\deg a_i = 1$. Thus $\mathcal{M}$ is regular holonomic since its Fourier transform $\widehat{\mathcal{M}}$ is regular holonomic [8].

The characteristic variety of $\mathcal{M}$ has the following explicit description. Let $\widehat{X}_l$ be the cone of the G-orbit X_l in $V - 0$. The group $\mathbb{C}^\times$ acts on the cone $\widehat{X}$ over $\phi(X)$ by scaling, and $G\times\mathbb{C}^\times$ acts on it with a finite number of orbits $\widehat{X}_l$, together with the fixed point $\widehat{X}_0 := 0$. Consider the characteristic variety $\operatorname{Ch}(\mathcal{M})$ of $\mathcal{M}$ as a subvariety of the symplectic variety $T^*V^* = V^*\times V$, equipped with the standard symplectic form $\langle\,,\rangle$. Then $\operatorname{Ch}(\mathcal{M})$ is contained in the following zero locus of principal symbols of the generators of J:

$$\begin{aligned} p(\zeta) &= 0, \qquad p \in I(X, V) \subset \mathbb{C}[\zeta], \\ \sum x_{ji}a_j\zeta_i &= 0, \qquad x \in \mathfrak{g}, \\ \sum a_i\zeta_i &= 0. \end{aligned}$$

The first set of equations says that if $(a,\zeta) \in \operatorname{Ch}(\mathcal{M})$, then ζ lies in $\widehat{X}$, hence in a unique orbit $\widehat{X}_l$. The second set of equations says that if $(a,\zeta) \in \operatorname{Ch}(\mathcal{M})$ then $\langle a, Z_x^*\zeta\rangle = 0$, where Z_x^* is viewed here as the tangent vector field corresponding to $x \in \mathfrak{g}$ generated by the dual G-action on V. The last equation says that a is normal to the Euler vector field generated by the $\mathbb{C}^\times$-action. In summary, if $(a,\zeta) \in \operatorname{Ch}(\mathcal{M})$, then ζ lies in a $G\times\mathbb{C}^\times$-orbit $\widehat{X}_l$ and $a \in V^*$ is normal to the orbit. In other words,

$$\operatorname{Ch}(\mathcal{M}) \subset \bigsqcup_{l=0}^{r} T^*_{\widehat{X}_l} V$$

where $T^*_{\widehat{X}_l} V$ is the conormal bundle of $\widehat{X}_l$ in V.

Lemma 3.5. *By the natural identification* $T^*V = T^*V^*$, *we have*

$$\bigsqcup_{l=0}^{r} T^*_{\widehat{X}_l} V = \bigcup_{l=0}^{r} T^*_{X_l^\vee} V^* \cup V\times\{0\} \tag{3.3}$$

where $X_l^\vee \subset V^*$ *the conical variety whose projectivization* $\mathbb{P}(X_l^\vee)$ *is the projective dual to the Zariski closure of* X_l *in* X.

Proof. The decomposition into $G \times \mathbb{C}^\times$-orbits, $\widehat{X} = \bigsqcup_{l=1}^r \widehat{X}_l \sqcup \{0\}$, is a Whitney stratification. In fact, for any pair $(\widehat{X}_s, \widehat{X}_l)$ such that $\widehat{X}_s \subset \overline{\widehat{X}_l}$, there exists an open dense set $U \subset \widehat{X}_l$ such that $(\widehat{X}_s, U)$ satisfies the Whitney condition; and since the $G \times \mathbb{C}^\times$-action on U generates $\widehat{X}_l$, $(\widehat{X}_s, \widehat{X}_l)$ also satisfies the Whitney condition. In particular, $\overline{T^*_{\widehat{X}_l} V} \cap \pi^{-1}(\widehat{X}_s) \subset T^*_{\widehat{X}_s} V$, where $\pi : T^*V \to V$ is the natural projection. Thus

$$\mathrm{Ch}(\mathcal{M}) \subset \bigsqcup_{l=1}^r T^*_{\widehat{X}_l} V \sqcup \{0\} \times V = \bigcup_{l=1}^r \overline{T^*_{\widehat{X}_l} V} \cup \{0\} \times V.$$

$\overline{T^*_{\widehat{X}_l} V} \subset T^*V$ is a closed conical Lagrangian submanifold. Under the natural identification $T^*V^* = T^*V = V \times V^*$, we have $\overline{T^*_{\widehat{X}_l} V} = \overline{T^*_{X_l^\vee} V^*}$, where $X_l^\vee$ is the conical variety dual to $\overline{\widehat{X}_l}$, i.e. $\mathbb{P}(X_l^\vee) \subset \mathbb{P}V^*$ is the projective dual to $\overline{X_l}$ (see [16]). $\square$

The singular locus of the D-module $\mathcal{M}$ is the Zariski closure of the image of $\mathrm{Ch}(\mathcal{M}) - V^* \times \{0\}$ under the projection $T^*V^* \to V^*$ (see [27]) and it is contained in the union $\bigcup_{l=1}^r X_l^\vee$. Thus the restriction of the solution sheaf $\mathcal{H}$ to $V^*_{\mathrm{gen}} = V^* - \bigcup_{l=1}^r X_l^\vee$ is a locally constant sheaf of finite rank. This proves parts (1)–(2) of Theorem 3.4.

To prove (3), we apply the following lemma of Kapranov [24].

Lemma 3.6 ([24]). *Let $\widehat{X}$ be the cone over X in V, and let $E \subset V$ be a linear subspace such that* $\dim E + \dim \widehat{X} = \dim V$ *and* $\widehat{X} \cap E = \{0\}$. *If $\mathbb{C}[X]$ is Cohen–Macaulay, then*

$$\dim_{\mathbb{C}}(\mathbb{C}[E] \otimes_{\mathbb{C}[V]} \mathbb{C}[X]) = \deg X.$$

The characteristic ideal $\widetilde{J}$ of $\mathcal{M} = D_{V^*}/J$ is the ideal in $\mathbb{C}[a, \zeta]$ generated by the principal symbols of differential operators in J. The holonomic rank of $\mathcal{M}$ is (see [27])

$$\mathrm{rank}(\mathcal{M}) = \dim_{\mathbb{C}(a)}(\mathbb{C}(a)[\zeta]/\mathbb{C}(a)[\zeta] \cdot \tilde{J}).$$

This gives the rank of the solution sheaf $\mathcal{H}$ over the generic stratum.

For any point $a \in V^*_{\mathrm{gen}}$, let E_a be the linear subspace of V defined by the linear equations

$$\sum x_{ji} a_j \zeta_i = 0, \qquad x \in \mathfrak{g}; \qquad \sum a_i \zeta_i = 0.$$

Then $E_a \cap \widehat{X} = 0$ by the construction of V^*_{gen} and $\dim E_a + \dim \widehat{X} = \dim V$. Thus the quotient of $\mathbb{C}[X]$ by the ideal generated by those linear equations has dimension equal to $\deg X$. This is the quotient obtained by moding out the principal symbols of generators of J and its dimension is greater than or equal to the dimension of the quotient by the full characteristic ideal $\widetilde{J}$. Therefore $\mathrm{rank}(\mathcal{M}) \leq \deg X$, i.e. the rank of the solution sheaf $\mathcal{H}$ is less than or equal to $\deg X$. $\square$

The homogeneous coordinate ring of the Grassmannian $G(d, n)$ is Cohen–Macaulay (see [13]) and the degree of $G(d, n) \hookrightarrow \mathbb{P}(\bigwedge^d \mathbb{C}^n)$, under the Plücker embedding, is given by the following formula (see [19]):

$$\deg G(d, n) = (d(n-d))! \prod_{i=0}^{d-1} \frac{i!}{(n-d+i)!}.$$

It follows that the degree of $\phi : X = G(d, n) \hookrightarrow \mathbb{P}V$, $V = H^0(X, K_X^{-1})^*$, is the above number times $n^{d(n-d)}$.

Thus we have the following

Corollary 3.7. *The tautological system $\tau(G(d, n), \phi, \mathrm{SL}_n, 1)$ is regular holonomic, and the solution sheaf $\mathcal{H}$ is locally constant of finite rank over $V^* - X^\vee$, where $X^\vee \subset V^*$ is the discriminant locus parameterizing singular CY hypersurfaces in $G(d, n)$. Moreover, the rank of $\mathcal{H}$ is less than or equal to $n^{d(n-d)}(d(n-d))! \prod_{i=0}^{d-1} i!/(n-d+i)!$.*

4. Examples

We keep the same notations as in preceding sections.

Example 4.1 (Very ample equivariant line bundle). Let X be a G-manifold, L be a very ample G-equivariant line bundle on X, and $\varphi_L : X \hookrightarrow \mathbb{P}V$, $V = H^0(X, L)^*$, be the G-equivariant embedding provided by L. Note that the embedding ideal $I(X, \mathbb{P}V)$ contains no nontrivial linear forms in this case. In particular, the Euler operator and the G-operators are the only first order generators of our tautological system in this case. The change-of-variables lemma shows that if we can realize the same G-module V inside another module W, then we obtain an alternative description for the solution sheaf of $\tau(X, \varphi_L, G, \beta)$.

Example 4.2 (Projective toric variety). Let $\mathcal{A} = \{\bar{\mu}_0, \dots, \bar{\mu}_p\} \subset 1 \times \mathbb{Z}^n \subset \mathbb{Z}^{n+1}$ be a finite list of distinct vectors generating $\mathbb{Z}^{n+1}$, and $\mathcal{L} \subset \mathbb{Z}^{p+1}$ be the lattice consisting of all vectors l such that $\sum_i l_i \bar{\mu}_i = 0$. Note that $\sum_i l_i = 0$. For each $l \in \mathcal{L}$, pick $l^\pm \in \mathbb{Z}_{\geq}^{p+1}$ such that $l = l^+ - l^-$. Let $X_\mathcal{A}$ be the projective variety defined by the homogeneous (because $\sum_i l_i = 0$) polynomials

$$\zeta^{l^+} - \zeta^{l^-} \qquad (l \in \mathcal{L})$$

in the variables $\zeta_0, \dots, \zeta_p$. This is an n-dimensional irreducible projective toric variety. The algebraic torus $T = (\mathbb{C}^\times)^n$ acts on $X_\mathcal{A}$ by

$$t \cdot [\zeta_0, \dots, \zeta_p] = [t^{\mu_0}\zeta_0, \dots, t^{\mu_p}\zeta_p]$$

where $\bar{\mu}_i = (1, \mu_i)$. Let $\phi : X_\mathcal{A} \hookrightarrow \mathbb{P}^p$ be the inclusion map, and β any integer. The tautological D-module $\tau(X_\mathcal{A}, \phi, T, \beta)$ coincides with a GKZ $\mathcal{A}$-hypergeometric system, as introduced in [15], and has important applications in mirror symmetry [2], [20]. If we replace T by $\mathrm{Aut}\, X_\mathcal{A}$, the resulting tautological D-module becomes an extended GKZ $\mathcal{A}$-hypergeometric system, as introduced in [21].

An important case that often arises in mirror symmetry is that one starts with a smooth projective toric variety X such that K_X^{-1} is semi-ample, i.e. $c_1(X)$ lies in the closure of the ample cone of X. The sections of the bundle define a rational map $\varphi : X \dashrightarrow \mathbb{P}V$ with $V = H^0(X, K_X^{-1})^*$, away from the base locus of the linear system. The closure of the image in $\mathbb{P}V$ can then be identified with $X_\mathcal{A}$ above, where $\mathcal{A}$ can be explicitly determined.

Example 4.3 (Kapranov's A-hypergeometric systems). In [24] Kapranov introduced a generalization of the GKZ $\mathcal{A}$-hypergeometric systems by, roughly speaking, replacing the algebraic torus T by a general reductive group H, a finite set of Laurent monomials by a finite set $A = \{V_\alpha\}$ of irreducible representations of H, and finally the "index" $\beta \in \mathfrak{t}^*$ by a character χ of the Lie algebra $\mathfrak{h}$. Put $M_A^* = \bigoplus_\alpha \operatorname{End} V_\alpha$ and let $\rho_A : H \to \prod_\alpha \mathrm{GL}(V_\alpha) \subset M_A^* - 0$ be the direct sum module. Then the closure of $\rho_A(H)$ in $M_A^* - 0$ is a spherical variety Y_A on which $H \times H$ acts naturally. Kapranov's A-hypergeometric system associated with the data A is the differential system defined on the affine space M_A given by [24, Eqn. 5.1.1]:

$$\begin{cases} L_h\Phi = R_h\Phi = \chi(h)\Phi, & h \in \mathfrak{h}, \\ P_f\Phi = 0, & f \in I_A. \end{cases} \tag{4.1}$$

Here $S^\bullet(M_A) \to \mathbb{C}[\partial]$, $f \mapsto P_f$, is the standard isomorphism between the polynomial ring on M_A and the ring of differential operators on M_A with constant coefficients; L_h and R_h are the infinitesimal generators of the left and right actions of H on M_A; and I_A is the ideal of $X_A := \mathbb{P}(Y_A)$. According to Definition 3.1, we see that Kapranov's system coincides with $\tau(X_A, \phi_A, H \times H, (-\chi, -\chi))$, where $\phi_A : X_A \hookrightarrow \mathbb{P}M_A^*$ is the inclusion map.

Example 4.4 (D-modules with "residual" symmetry). Let X be a G-variety, V a G-module, and $\phi : X \hookrightarrow \mathbb{P}V$ a G-equivariant embedding. Let K be a closed subgroup of G. Then the D-module $\tau(X, \phi, K, \beta)$ admits a "residual" group action by the centralizer H of K in G. In particular, H acts on the solution sheaf of this D-module. One interesting example is $X = \mathbb{P}M$ where M is the space of $n \times m$ ($n \geq m$) matrices, and $G = \mathrm{SL}_n \times \mathrm{SL}_m$ act by the usual left-right multiplications. Put $K = \mathrm{SL}_n \times 1$. Then there are exactly m K-orbits in X—the matrices in M of a given rank (≥ 1) form a single orbit. In particular, our D-module is regular holonomic. Since $X = \mathbb{P}M$ is also a toric variety (under the action of a maximal torus of $\mathrm{SL}(M)$), we can construct general solutions to $\tau(X, \mathcal{O}(\dim M), \mathrm{SL}(M), \beta)$ by using the method of [21]. Since $K \subset G = K \times H \subset \mathrm{SL}(M)$, our general solutions are a priori solutions to the D-module $\tau(X, \phi, K, \beta)$ which are also H-invariant.

Example 4.5 (Partial flag variety). Let $X = F(d_1, \ldots, d_r, n)$, and let $G = \mathrm{SL}_n$ act on $\mathbb{C}^n$ as usual. This induces a transitive G-action on X, and so there is exactly one G-orbit in X. Let L be an ample line bundle on X, and put $V = H^0(X, L)^*$. Then by the Borel–Weil theorem, V is an irreducible representation of G. Moreover, given a highest vector $v \in V$, we have a unique G-equivariant embedding $\varphi_L : X \hookrightarrow \mathbb{P}V$ which maps the standard flag in X to $[v]$. By Theorem 3.4, it follows that the tautological system $\tau(X, \varphi_L, G, \beta)$ is regular holonomic.

In the next section, we show that for $L = K_X^{-1}$ and $\beta = 1$, this system governs the period integrals of the universal family of CY hypersurfaces in X. Moreover, we will give an explicit description of this system by enumerating its generators.

5. Tautological systems for partial flag varieties

In this section, we shall study a tautological system associated to the partial flag variety

$$X = F(d_1, \dots, d_r, n).$$

As in Section 2, we fix a Borel subgroup B of $G = \mathrm{SL}_n$, and let

$$\Delta = \{\alpha_1, \dots, \alpha_{n-1}\}$$

be the simple roots of G. Put

$$S = \Delta - \{\alpha_{d_1}, \dots, \alpha_{d_r}\}$$

and let P_S be the parabolic subgroup of $G = \mathrm{SL}_n$ corresponding to S. As before, denote by $\lambda_1, \dots, \lambda_{n-1}$ the fundamental dominant weights of SL_n, so that $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$. Let L_{λ_i} be as in Section 2. Thanks to results of [7], the Picard group (G-equivariant or otherwise) of X has the following description.

Proposition 5.1. *Put $X = F(d_1, \dots, d_r, n)$. Then*

$$H^2(X, \mathbb{Z}) \overset{c_1}{\cong} \mathrm{Pic}(X) = \mathrm{Pic}_G(X) = \mathrm{Hom}(P_S, \mathbb{C}^\times).$$

Moreover, $L = \bigotimes_{i=1}^r L_{\lambda_{d_i}}^{\otimes n_{d_i}}$ is ample iff it is very ample iff $n_{d_i} \geq 1$ for all i.

Using (2.3), we deduce

Proposition 5.2. *For $X = F(d_1, \dots, d_r, n)$, $K_X^{-1} = \bigotimes_{i=1}^r L_{\lambda_{d_i}}^{\otimes(d_{i+1} - d_{i-1})}$ is very ample.*

Fix an ample line bundle $L = \bigotimes_{i=1}^r L_{\lambda_{d_i}}^{\otimes n_{d_i}}$, put $\lambda := \sum_i n_{d_i} \lambda_{d_i}$, and let

$$V_\lambda := H^0(X, L)^*.$$

We can factor the embedding $\varphi_L : X \hookrightarrow \mathbb{P}V_\lambda$ in a canonical way, as follows. We define the *incidence map*

$$\iota : X = F(d_1, \dots, d_r, n) \hookrightarrow G(d_1, n) \times \cdots \times G(d_r, n), \qquad x \mapsto (\iota_1 x, \dots, \iota_r x),$$

where the $\iota_i : X \to G(d_i, n)$ are the natural projection maps. For each i, we have the standard Plücker embedding of the Grassmannian,

$$\pi_i : G(d_i, n) \hookrightarrow \mathbb{P}V_{\lambda_{d_i}},$$

where $V_{\lambda_{d_i}} := \bigwedge^{d_i} \mathbb{C}^n$ is the fundamental representation of highest weight λ_{d_i}. We also have the Veronese maps

$$\nu_i : \mathbb{P}V_{\lambda_{d_i}} \hookrightarrow \mathbb{P}\,\mathrm{Sym}^{n_{d_i}} V_{\lambda_{d_i}}, \qquad [v] \mapsto [v \otimes \cdots \otimes v],$$

and the Segre map

$$\psi : \mathbb{P}\,\mathrm{Sym}^{n_{d_1}} V_{\lambda_{d_1}} \times \cdots \times \mathbb{P}\,\mathrm{Sym}^{n_{d_r}} V_{\lambda_{d_r}} \hookrightarrow \mathbb{P}W, \qquad ([u_1], \ldots, [u_r]) \mapsto [u_1 \otimes \cdots \otimes u_r],$$

where

$$W = \mathrm{Sym}^{n_{d_1}} V_{\lambda_{d_1}} \otimes \cdots \otimes \mathrm{Sym}^{n_{d_r}} V_{\lambda_{d_r}}.$$

Put

$$\nu = \nu_1 \times \cdots \times \nu_r, \qquad \pi = \pi_1 \times \cdots \times \pi_r.$$

Then we get a G-equivariant embedding

$$\phi = \psi \circ \nu \circ \pi \circ \iota : X \hookrightarrow \mathbb{P}W$$

such that $\phi^* \mathcal{O}_{\mathbb{P}W}(1) = L$. By the Borel–Weil theorem, $H^0(X, L)^* = V_\lambda$ is an irreducible module of highest weight λ. Clearly, λ is the highest weight in W of multiplicity 1, implying that W contains a unique copy of V_λ. It follows that the image $\phi(X)$ in $\mathbb{P}W$ lies in the linear subspace defined by $V_\lambda^\perp \subset W^*$, consisting of the linear forms on W annihilating $V_\lambda \subset W$. Moreover, $V_\lambda^\perp$ contains every linear form vanishing on $\phi(X)$.

We now specialize to $L = K_X^{-1}$ and proceed to describe the image $\phi(X)$ in $\mathbb{P}W$. Put

$$U = \mathbb{P}V_{\lambda_{d_1}} \times \cdots \times \mathbb{P}V_{\lambda_{d_r}}.$$

We first enumerate generators of the vanishing ideal $I(U, \mathbb{P}W)$ of U in $\mathbb{P}W$, under the embedding $\psi \circ \nu : U \hookrightarrow \mathbb{P}W$. Fix a basis z_{ij} $(1 \le j \le m_i := \dim V_{\lambda_{d_i}})$ of $V_{\lambda_{d_i}}^* = H^0(\mathbb{P}V_{\lambda_{d_i}}, \mathcal{O}(1))$, and introduce the notation

$$z^v = z_1^{v_1} \cdots z_r^{v_r} = \prod_{i,j} z_{ij}^{v_{ij}}, \qquad v = (v_1, \ldots, v_r) \in \mathbb{Z}_{\ge 0}^{m_1} \times \cdots \times \mathbb{Z}_{\ge 0}^{m_r}.$$

Let $\mathcal{E}$ be the set of exponents $v = (v_1, \ldots, v_r)$ such that $|v_i| := \sum_j v_{ij} = n_{d_i}$ for each component vector v_i. For $v \in \mathcal{E}$, we can view z^v as a monomial function on $V_{\lambda_{d_1}} \times \cdots \times V_{\lambda_{d_r}}$. Let ξ_{i,v_i} $(|v_i| = n_{d_i})$ be the basis of $H^0(\mathbb{P}\,\mathrm{Sym}^{n_{d_i}} V_{\lambda_{d_i}}, \mathcal{O}(1))$ such that $\nu_i^* : \xi_{i,v_i} \mapsto z_i^{v_i}$ (the restriction map), and let ζ_v be the basis of $H^0(\mathbb{P}W, \mathcal{O}(1))$ such that $\psi^* : \zeta_v \mapsto \xi_{1,v_1} \cdots \xi_{r,v_r}$. It is easy to show that the degree 2 binomials on $\mathbb{P}W$

$$\zeta_u \zeta_v - \zeta_w \zeta_t, \qquad u + v = w + t \quad (u, v, w, t \in \mathcal{E}), \tag{5.1}$$

vanish on U. On the other hand it is also known that $I(U, \mathbb{P}W)$ is generated by quadratic forms. Then by termwise elimination, we find that any quadratic form vanishing on U is a linear combination of the binomials above. Finally, by Proposition 5.2, the defining ideal $I(X, \mathbb{P}W)$ is generated by the linear forms $V_\lambda^\perp \subset W^*$ together with $I(U, \mathbb{P}W)$ (see [14]).

Theorem 5.3. *Let $X = F(d_1, \ldots, d_r, n) = G/P_S$. Put $n_{d_i} = d_{i+1} - d_{i-1}$, and let $\phi : X \hookrightarrow \mathbb{P}W$ with $W = \mathrm{Sym}^{n_{d_1}} V_{\lambda_{d_1}} \otimes \cdots \otimes \mathrm{Sym}^{n_{d_r}} V_{\lambda_{d_r}}$ be as defined above. Let $f \in H^0(\mathbb{P}W, \mathcal{O}(1))$ be a general section, and put $Y_f = \{f = 0\} \cap \phi(X)$. For $\gamma \in H_{d-1}(Y_f, \mathbb{Z})$, let $\tau(\gamma)$ be a tube over the cycle γ in X. Then the period integral $\int_{\tau(\gamma)} \Omega/f$ is a solution to the tautological system $\tau(X, \phi, G, 1)$. The system is generated by the G-operators Z_x ($x \in \mathfrak{g}$), the Euler operator $\sum a_v \partial/\partial a_v + 1$, the first order operators $\sum \langle a_v, \zeta\rangle \partial/\partial a_v$ ($\zeta \in V_\lambda^\perp \subset W^*$), together with the second order binomial operators* (cf. (5.1))

$$\frac{\partial}{\partial a_u}\frac{\partial}{\partial a_v} - \frac{\partial}{\partial a_w}\frac{\partial}{\partial a_t}, \qquad u + v = w + t \quad (u, v, w, t \in \mathcal{E}).$$

Proof. A general section has the form $f = f(a, \zeta) = \sum_v a_v \zeta_v$. By a direct calculation, for $p(\zeta) \in I(X, \mathbb{P}W)$ of degree s, we find that

$$p(\partial_\zeta)\frac{1}{f} = (-1)^s s! \frac{p(\zeta)}{f^{s+1}},$$

which is zero on X. This implies that the period integral is killed by $p(\partial_\zeta)$. Let g be any automorphism of X. Since the period integral is the Poincaré pairing $\langle \tau(\gamma), \Omega/f\rangle$ on $X - Y_f$, it is invariant under g:

$$\langle \tau(\gamma), \Omega/f\rangle = \langle (g_*)^{-1}\tau(\gamma), g^*(\Omega/f)\rangle.$$

Now let $g \in G$ be close to identity. Then $(g_*)^{-1}\tau(\gamma) = \tau(\gamma)$. By Theorem 2.1, Ω is G-invariant, so

$$\langle \tau(\gamma), \Omega/f\rangle = \langle \tau(\gamma), g^*(1/f)\Omega\rangle.$$

For $x \in \mathfrak{g}$, consider the action of the 1-parameter subgroup $g = g_t = \exp(tx)$ of G. We have

$$\left.\frac{d}{dt}\right|_{t=0} g_t^*(1/f) = -\frac{x \cdot f}{f^2} = -\frac{Z_x f}{f^2} = -Z_x(1/f).$$

It follows that

$$0 = \langle \tau(\gamma), Z_x(1/f)\Omega\rangle = Z_x\langle \tau(\gamma), \Omega/f\rangle.$$

Finally, the period is killed by the Euler operator $\sum a_v \partial/\partial a_v + 1$ because $1/f$ is homogeneous of degree -1 in the variables a_v. Thus we have shown that the period is killed by all generators of the tautological system.

The last assertion of the theorem follows from the argument preceding the statement. □

Remark 5.4. For a general ample line bundle L, when the condition $n_{d_i} \geq 2$ does not necessarily hold, the generators of $I(X, \mathbb{P}W)$ can be much more complicated, involving quadratic forms which are not necessarily binomials (cf. (5.1)). For example, if $X = G(d, n)$, $L = \mathcal{O}_X(1)$ and $W = H^0(X, L)^*$, then $I(X, \mathbb{P}W)$ is generated by the Plücker relations, which are of course not binomials.

Remark 5.5. The generators of the tautological system associated to $X = G(d, n)$ and $L = K_X^{-1}$ are explicitly written down in 6.5.

By the change-of-variables lemma, we can use the linear forms in $V_\lambda^\perp$ to eliminate variables by expressing the coordinate functions a_v of W^* in terms of a basis of $V_\lambda = H^0(X, L)^*$. Then the D-module $\tau(X, \phi, G, 1)$ reduces to the D-module $\tau(X, \varphi_L, G, 1)$, corresponding to the canonical embedding $\varphi_L : X \hookrightarrow \mathbb{P}V_\lambda$. Thus, we can view the preceding theorem as giving an alternative description of $\tau(X, \varphi_L, G, 1)$, by introducing more variables to the differential equation system, by factoring φ_L in terms of the four classical maps, ι, π, ν, ψ. The reward is that the factorization gives a system whose quadratic operators are all *binomials* with simple (and universal) description, while the price is the introduction of a collection of first order operators with constant coefficients corresponding to the linear forms in $V_\lambda^\perp \subset W^* \subset I(X, \mathbb{P}W)$.

Our results on hypersurfaces can be generalized to complete intersections as follows. In Section 2, we have already seen the Poincaré residue formula for CY complete intersections in X. Let L_i be G-equivariant ample line bundles, and $\beta_i \in \mathbb{Z}$, $i = 1, \ldots, s$. Let $V_i = H^0(X, L_i)^*$ and $V = V_1 \times \cdots \times V_s$. Let a_l^i be a basis of V_i and ζ_l^i be the dual basis of V_i^*. The line bundles L_i define an equivariant embedding $\phi : X \hookrightarrow \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_s$. Denote by $I(X, \mathbb{P}V)$ the ideal of polynomial functions in $\mathbb{C}[V]$ which vanish on $\phi(X) \subset \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_s$. Note that $I(X, \mathbb{P}V_i) \subset I(X, \mathbb{P}V)$. Let Z_x^i, $x \in \mathfrak{g}$, be the infinitesimal form of the G-action on V_i and let $Z_x = \sum_{i=1}^s Z_x^i$.

Definition 5.6. We define the *tautological system* $\tau(X, G, L_1, \ldots, L_s, \beta_1, \ldots, \beta_s)$ as the D_{V^*}-module D_{V^*}/J, where J is the left D_{V^*}-ideal generated by the following operators: $\{p(\partial_\zeta) \mid p(\zeta) \in I(X, \mathbb{P}V)\}$, $\{Z_x \mid x \in \mathfrak{g}\}$, and the Euler operators $\sum_l a_l^i \partial_{\zeta_l^i} + \beta_i$, $i = 1, \ldots, s$.

The argument for Theorem 3.4 can be generalized to show that the system

$$\tau(X, G, L_1, \ldots, L_s, \beta_1, \ldots, \beta_s)$$

is a regular holonomic D-module. One considers the $G \times (\mathbb{C}^\times)^s$-action on V, where $(\mathbb{C}^\times)^s$ acts by scaling on the s factors of the representation $V = V_1 \times \cdots \times V_s$ of G. The characteristic variety of the D-module is then shown to be a subvariety of the disjoint union of the conormal bundles of the finitely many $G \times (\mathbb{C}^\times)^s$-orbits in the cone $\widehat{X}$ over $\phi(X)$, where each of the conormal bundles is Lagrangian in T^*V^*.

Now assuming $\bigotimes_{i=1}^s L_i = K_X^{-1}$, generic sections f_i in $H^0(X, L_i)$ define a smooth CY complete intersection $Y_f := \bigcap_{i=1}^s \{f_i = 0\}$ in X. For $\gamma \in H_{\dim Y_f}(Y_f, \mathbb{Z})$, let $\tau(\gamma)$ be a tube over the cycle γ in X. Let $\beta_i = 1$, $i = 1, \ldots, s$. Then the period integral

$$\int_\gamma \operatorname{Res} \frac{\Omega}{f_1 \cdots f_s} = \int_{\tau(\gamma)} \frac{\Omega}{f_1 \cdots f_s}$$

is a solution to the tautological system $\tau(X, G, L_1, \ldots, L_s, \beta_1, \ldots, \beta_s)$. This follows by the same argument as in the case of hypersurfaces (Theorem 5.3).

6. Explicit examples

Example 6.1 ($X = \mathbb{P}^{n-1}$). Let $z_1, \ldots, z_n$ be homogeneous coordinates on X and let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Let

$$\Delta_1 = \mathrm{conv}\{ne_i\}_{i=1}^n \subset \mathbb{R}^n,$$

i.e. the convex hull of the set of exponents of degree n monomials in the z_i. Then Δ_1 lies in an affine hyperplane in $\mathbb{R}^n$. Shifting it by $(1, \ldots, 1) \in \mathbb{Z}^n$ and then projecting it to a coordinate plane, we get the following polytope in $\mathbb{R}^{n-1}$:

$$\Delta = \mathrm{conv}\{(-1, \ldots, -1), (n-1, -1, \ldots, -1), \ldots, (-1, -1, \ldots, n-1)\}.$$

Let $A = \Delta \cap \mathbb{Z}^n = \{v_0, v_1, \ldots, v_{N-1}\}$ and suppose $v_0 = (0, \ldots, 0)$. Let $t_j = z_j/z_n$, $1 \le j \le n-1$, be affine coordinates on X. Then

$$V^* = H^0(X, \mathcal{O}(n)) = \bigoplus_{i=1}^{N} \mathbb{C} z_1 \cdots z_n t^{v_i}.$$

Let $\phi : X \hookrightarrow \mathbb{P}V$ be the degree n Veronese embedding. Let

$$\mathcal{L} = \left\{ l \in \mathbb{Z}^N \;\middle|\; \sum_{i=0}^{N-1} l_i v_i = 0 \right\}$$

be the lattice of integral relations among the v_i.

In this case, our construction in 2.1 recovers the well-known form

$$\Omega = \sum_{i=1}^{n} (-1)^{i-1} z_i dz_1 \cdots \widehat{dz_i} \cdots dz_n = z_1 \cdots z_n \prod_{i=1}^{n-1} \frac{dt_i}{t_i}.$$

Let $f(a,t) = z_1 \cdots z_n \sum a_i t^{v_i} \in V^*$ be a generic section such that $Y = \{f = 0\}$ is smooth. Then $\mathrm{Res}(\Omega/f)$ is a nowhere vanishing holomorphic top form on Y. We have an isomorphism $H^{n-1}(X - Y) \underset{\cong}{\xrightarrow{\mathrm{Res}}} H^{n-2}(Y)_{\mathrm{van}}$. When n is odd, $H^{n-2}(Y)_{\mathrm{van}} = H^{n-2}(Y)$.

The period integrals $\Pi_\gamma(a) = \int_{\tau(\gamma)} (\Omega/f(a,t))$, where $\gamma \in H_{n-1}(X - Y)$, are solutions to the system $\tau(X, \phi, \mathrm{SL}_n, 1)$, which coincides with the following extended GKZ system [21]:

$$\begin{cases} \dfrac{\partial}{\partial a_i} \dfrac{\partial}{\partial a_j} \Pi_\gamma(a) = \dfrac{\partial}{\partial a_{i'}} \dfrac{\partial}{\partial a_{j'}} \Pi_\gamma(a) & \text{if } v_i + v_j = v_{i'} + v_{j'} \in \mathbb{Z}^n, \\ Z_x \Pi_\gamma(a) = 0 \quad \forall x \in \mathfrak{sl}_n, \\ \displaystyle\sum_i a_i \frac{\partial}{\partial a_i} \Pi_\gamma(a) = -\Pi_\gamma(a). \end{cases} \tag{6.1}$$

Integrating $\mathrm{Res}(\Omega/f)$ along a particular cycle

$$\gamma_0 = \{|t_1| = \cdots = |t_{n-1}| = 1\},$$

we get

$$\begin{aligned}\Pi_{\gamma_0}(a) &= \int_{|t_j|=1,\,\forall j} \frac{\sum_{j=1}^{n}(-1)^{j-1} z_j dz_1 \cdots \widehat{dz_j} \cdots dz_n}{\sum a_i z^{v_i}} \\ &= \int_{|t_j|=1,\,\forall j} \frac{1}{\sum a_i t^{v_i}} \prod_{i=1}^{n-1} \frac{dt_j}{t_j} \\ &= \frac{1}{a_0} \sum_{l\in\mathcal{L},\, l_0<0,\, l_i\geq 0 \text{ if } i\neq 0} (-1)^{l_0} \frac{(-l_0)!}{\prod_{i\neq 0} l_i!} a^l. \end{aligned} \tag{6.2}$$

Remark 6.2. Similar computation can be carried out for $G(d,n)$. See 6.3 where a detailed computation is carried out for $G(2,4)$.

Remark 6.3. Note that the period integral above gives a power series solution to the system (6.1) near $a_0 = \infty$, where a_0 is the coefficient of the monomial $z_1 \cdots z_n$ in $f(a,z)$. By a result in [22], this is the only regular solution near this infinity. All other solutions have log singularities. An explicit formula for them can be given in terms of the Gamma function. The formula also generalizes to an arbitrary Fano toric manifold. See [21], [26] for details.

Example 6.4 ($X = G(2,4)$). Under the Plücker embedding, $X = G(2,4) \hookrightarrow \mathbb{P}^5$ is a quadratic hypersurface. The moduli space of CY hypersurfaces in X is given by $\mathbb{P}H^0(X, \mathcal{O}(4))//\mathrm{SL}_4$, whose dimension is 89.

A CY hypersurface in X can be considered as a complete intersection of type $(2,4)$ in $\mathbb{P}^5$ and the procedure to find periods on CY complete intersections in toric varieties applies. Let $z_0 = p_{12}, z_1 = p_{13}, z_2 = p_{23}, z_3 = p_{14}, z_4 = p_{24}, z_5 = p_{34}$ be homogeneous coordinates on $\mathbb{P}^5$. Then $X = \{z_0 z_5 + z_2 z_3 - z_1 z_4 = 0\}$. The particular period

$$\Pi(a) = \int_{|z_0|=\cdots=|z_5|} \frac{\sum_i (-1)^i z_i dz_0 \cdots \widehat{dz_i} \cdots dz_5}{(z_0 z_5 + z_2 z_3 - z_1 z_4)(\sum a_i z^{v_i})},$$

where $\sum a_i z^{v_i} \in H^0(\mathbb{P}^5, \mathcal{O}(4))$, can be computed as a double residue.

On the other hand, we can evaluate the period along a cycle in an affine chart in $G(2,4)$ explicitly as follows. The weight polytope of the SL_4-action on $H^0(X, \mathcal{O}(4))$ is $\Delta_w = \mathrm{conv}\{4(e_i + e_j) \mid 1 \leq i < j \leq 4\} \subset \mathbb{R}^4$.

On the affine chart $U = \{(\mathrm{id}_{2\times 2}, *_{2\times 2})\} \cong \mathbb{C}^4$, we have $p_{12} = 1$; moreover, $p_{13} = z_1$, $p_{23} = z_2$, $p_{14} = z_3$, $p_{24} = z_4$ are affine coordinates on this patch and $p_{34} = z_1 z_4 - z_2 z_3$. We have a linear map $H^0(X, \mathcal{O}(4)) \to \bigoplus_{|v|\leq 8} \mathbb{C} z^v$ expanding degree 4 polynomials in p_{ij} to polynomials in z of degree ≤ 8. Let v_0 be the exponent such that $p^{v_0} = p_{13} p_{23} p_{14} p_{24} = z_1 z_2 z_3 z_4$.

The period integral $\Pi_{\gamma_0}(a) = \int_{\gamma_0} (\Omega / \sum a_i p^{v_i})$, where $\sum a_i p^{v_i} \in H^0(X, \mathcal{O}(4))$, along the cycle

$$\gamma_0 = \{|z_j| = 1 \mid 1 \leq j \leq 4\}$$

can be computed as follows:

$$\begin{aligned}
\Pi_{\gamma_0}(a) &= \int_{|z_j|=1} \frac{z_1z_2z_3z_4}{\sum a_i p^{v_i}} \frac{dz_1dz_2dz_3dz_4}{z_1z_2z_3z_4} \\
&= \int_{|z_j|=1} \frac{1}{a_0 + \sum_{i\neq 0} a_i p^{v_i - v_0}} \prod \frac{dz_j}{z_j} \\
&= \frac{1}{a_0} \sum_{n=0}^{\infty} \text{constant term in } \left(\frac{\sum a_i p^{v_i}}{z_1z_2z_3z_4} \right)^n \frac{1}{a_0^n} \\
&= \frac{1}{a_0} \sum_{n=0}^{\infty} \frac{n!}{\prod l_i!} a^l \cdot \text{constant term in } \frac{p^{\sum l_i v_i}}{(z_1z_2z_3z_4)^n} \\
&= \frac{1}{a_0} \sum_{n=0}^{\infty} \frac{n!}{\prod l_i!} a^l \cdot \text{constant term in } \frac{p^{\sum l_i v_i}}{(z_1z_2z_3z_4)^n} \\
&= \frac{1}{a_0} \sum_{l\in\mathcal{L},\, l_0\le 0,\, l_i\ge 0 \text{ if } i\neq 0} \frac{(-l_0)!}{\prod_{i\neq 0} l_i!} \binom{n_5(l)}{n_5(l)+n_2(l)-n} a^l
\end{aligned} \tag{6.3}$$

where $n_5(l)$ is the exponent of p_{34}, $n_2(l)$ is the exponent of p_{23}, and $n_1(l)$ is the exponent of p_{13} in $p^{\sum l_i v_i}$; and $\mathcal{L}$ is the integral lattice defined by $\{\sum l_i w_i = 0\}$, where w_i is the weight of the $(\mathbb{C}^\times)^4$-action on p^{v_i}.

Example 6.5 ($X = G(d,n)$). We have $\mathrm{Pic}(X) = \mathbb{Z}\mathcal{O}_X(1)$ and $K_X^{-1} = \mathcal{O}_X(n)$, where $\mathcal{O}_X(1)$ is the pullback of the hyperplane bundle via the Plücker embedding $X \hookrightarrow \mathbb{P}(\bigwedge^d \mathbb{C}^n)$. Let $V = H^0(X, \mathcal{O}_X(n))^*$. A generic element in V^* defines a smooth CY hypersurface in X. Let

$$W^* = \mathrm{Sym}^n(\textstyle\bigwedge^d \mathbb{C}^n) = \displaystyle\bigoplus_{i=1}^{N} \mathbb{C}p^{v_i}$$

be the vector space spanned by all degree n monomials in Plücker coordinates. Then $V^* = W^*/Q_n$, where Q_n is the degree n part of the ideal genereated by Plücker relations. Let $a_1, \ldots, a_N$ be coordinates on W^*.

Consider the composition of the Plücker embedding and the Veronese embedding $\phi : X \hookrightarrow \mathbb{P}(\bigwedge^d \mathbb{C}^n) \hookrightarrow \mathbb{P}W$, which commutes with the anticanonical embedding $X \hookrightarrow \mathbb{P}V \hookrightarrow \mathbb{P}W$. The ideal defining X in $\mathbb{P}W$ is the sum of the ideals defining $\mathbb{P}(\bigwedge^d \mathbb{C}^n)$ and $\mathbb{P}V$ in $\mathbb{P}W$, and hence is generated by quadratic Veronese relations and linear relations in Q_n. Corresponding to these generators, we define the following differential operators:

- Veronese operators:

$$\left\{ \frac{\partial}{\partial a_i}\frac{\partial}{\partial a_j} - \frac{\partial}{\partial a_{i'}}\frac{\partial}{\partial a_{j'}} \;\middle|\; v_i + v_j = v_{i'} + v_{j'} \in \mathbb{Z}^{\binom{n}{d}} \right\}.$$

- First order operators:

$$\left\{ \sum_i c_i \frac{\partial}{\partial a_i} \;\middle|\; \sum_i c_i p^{v_i} \in Q_n \right\}.$$

Let $\rho : \mathfrak{sl}_n \to \mathrm{End}(W^*)$ be the infinitesimal form of the SL_n-action on W^*. The vector field generated by any $x \in \mathfrak{sl}_n$ on W^* can be written as

$$Z_x = \sum_{i,j} \rho_{ij}(x) a_j \frac{\partial}{\partial a_i}.$$

The period integrals on CY hypersurfaces in $G(d,n)$ are solutions to the following system of differential equations, which is an equivalent form of the tautological system $\tau(X, \phi, \mathrm{SL}_n, 1)$, where $\phi : X \hookrightarrow \mathbb{P}W$ is as above

$$\begin{cases} \dfrac{\partial}{\partial a_i}\dfrac{\partial}{\partial a_j}\Pi_\gamma(a) = \dfrac{\partial}{\partial a_{i'}}\dfrac{\partial}{\partial a_{j'}}\Pi_\gamma(a) \quad \text{if } v_i + v_j = v_{i'} + v_{j'} \in \mathbb{Z}^{\binom{n}{d}}, \\ \displaystyle\sum_i c_i \frac{\partial}{\partial a_i}\Pi_\gamma(a) = 0 \quad \text{if } \sum_i c_i p^{v_i} \in Q_n, \\ Z_x \Pi_\gamma(a) = 0 \quad \forall x \in \mathfrak{sl}_n, \\ \displaystyle\sum_i a_i \frac{\partial}{\partial a_i}\Pi_\gamma(a) = -\Pi_\gamma(a). \end{cases} \tag{6.4}$$

7. Note added: on holonomic rank

Consider the universal family of CY hypersurfaces in a given partial flag variety X (Example 4.5). The well-known applications of variation of Hodge structures in mirror symmetry show that it is important to decide which solutions of a differential system come from period integrals. The central object of our study is the period sheaf, i.e. the sheaf generated by the period integrals of the CY hypersurfaces. By Lemma 3.3 and Theorem 5.3, the period sheaf is a subsheaf of the solution sheaf of the module

$$\mathcal{M} = \tau(X, \varphi_{K_X^{-1}}, \mathrm{SL}_n, 1).$$

Thus an important open problem is to decide when the two sheaves coincide. If they do not coincide, how much larger is the solution sheaf? From Hodge theory, it is well-known that the rank of the period sheaf is given by the dimension of the varying middle cohomology of the smooth hypersurfaces Y_f:

$$\dim H^N(Y_f, \mathbb{C}) - \dim i^* H^N(X, \mathbb{C})$$

where $N = \dim Y_f$, and $i : Y_f \hookrightarrow X$ denotes the inclusion map. Therefore, to answer those questions, it is clearly desirable to know precisely the holonomic rank of $\mathcal{M}$.

Let us recall what is known about those general questions. In the case of CY hypersurfaces in, say, a Fano toric manifold X, it is known [15], [1] that the rank of the GKZ hypergeometric system (cf. Example 4.2) in this case is the normalized volume of the polytope generated by the exponents of the monomial sections in $H^0(X, K_X^{-1})$. This number is also the same as the degree of the anticanonical embedding $X \hookrightarrow \mathbb{P}H^0(X, K_X^{-1})^*$. It is also known [21] that this number always exceeds (and is usually much larger than) the rank of the period sheaf. If one considers the extended GKZ hypergeometric system, where the torus T acting on X is replaced by the full automorphism group $\mathrm{Aut}\, X$, one would expect that the rank of this system should be closer to that of the period sheaf. In fact, based on numerical evidence, it was conjectured [21] that for $X = \mathbb{P}^n$ (which lives in both the toric world and the homogeneous world), the holonomic rank of $\mathcal{M}$ coincides with the rank of the period sheaf. In the case when $X = X_A$ is a spherical variety of a reductive group G corresponding to a given set of irreducible G-modules A, Kapranov [24] showed that the holonomic rank of his A-hypergeometric system (see Example 4.3) is bounded above by the degree of embedding $X_A \subset \mathbb{P}M_A^*$, if $\mathbb{C}[Y_A]$ is assumed to be Cohen–Macaulay. Theorem 3.4 generalizes this to an arbitrary tautological system $\tau(X, \phi, G, \beta)$ where X has only a finite number of G-orbits. Note, however, that the rank upper bound in each case cited above can be obtained without using (therefore does not take advantage of) assumptions about whether the underlying D-module arises from the variation of Hodge structures of CY hypersurfaces.

Since the release of the current paper in May 2011, progress has been made toward the problem of holonomic rank for CY hypersurfaces. We mention the following recent result.

Theorem 7.1 ([3]). *Let $X = G(d, n)$. Then the holonomic rank of the D-module $\mathcal{M}$ at $f \in H^0(X, K_X^{-1})$ is precisely* $\dim H^{d(n-d)}(X - Y_f)$.

Note that the theorem holds not just at generic sections, but at every hyperplane section f. The theorem has also been generalized to an arbitrary flag variety. The proof is beyond the scope of this paper, and is expected to appear shortly [3]. The theorem also implies the above mentioned conjecture of [21]:

Corollary 7.2. *For $X = \mathbb{P}^n$, the solution sheaf of $\mathcal{M}$ coincides with the period sheaf of CY hypersurfaces in X.*

Let us compare this with the upper bound given by Corollary 3.7. In this case, $X = G(1, n+1) = \mathbb{P}^n$ and the latter bound is $(n+1)^n$. We claim that this always exceeds the rank of the period sheaf, which is given by the dimension of the vanishing cohomology of a smooth CY hyperplane section in X. In fact, by using the Lefschetz hyperplane theorem, we find that the rank of the period sheaf is

$$\frac{n}{n+1}(n^n - (-1)^n) < n^n + 1 \leq (n+1)^n.$$

Acknowledgments. The first author would like to thank Peter Littelmann and Gerry Schwarz for helpful communications. We thank B. Song who first worked out the computation in Example 6.4. Finally, we thank the referee for helpful comments. B.H.L. is partially supported by NSF FRG (grant DMS-0854965), and S.T.Y. by NSF FRG (grant DMS-0804454).

References

[1] Adolphson, A.: Hypergeometric functions and rings generated by monomials. Duke Math. J. **73**, 269–290 (1994) Zbl 0804.33013 MR 1262208

[2] Batyrev, V.: Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori. Duke Math. J. **69**, 349–409 (1993) Zbl 0812.14035 MR 1203231

[3] Bloch, S., Huang, A., Lian, B. H., Srinivas, V., Yau, S.-T.: On the holonomic rank problem. To appear

[4] Borcea, C.: Smooth global complete intersections in certain compact homogeneous complex manifolds. J. Reine Angew. Math. **344**, 65–70 (1983) Zbl 0511.14027 MR 0716247

[5] Borel, A. et al.: Algebraic D-modules. Academic Press (1987) Zbl 0642.32001 MR 0882000

[6] Borel, A.: Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. Ann. of Math. (2) **57**, 115–207 (1953) Zbl 0052.40001 MR 0051508

[7] Bott, R.: Homogeneous vector bundles. Ann. of Math. (2) **66**, 203–248 (1957) Zbl 0094.35701 MR 0089473

[8] Brylinski, J.-L.: Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques. Astérisque **140–141**, 3–134, 251 (1986) Zbl 0624.32009 MR 0864073

[9] Candelas, P., de la Ossa, X., Green, P., Parkes, L.: A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory. Nuclear Phys. B **359**, 21–74 (1991) Zbl 1098.32506 MR 1115626

[10] Cattani, E.: Three Lectures on Hypergeometric Functions. www.math.umass.edu/˜cattani/hypergeom_lectures.pdf (2006)

[11] Dolgachev, I.: Rationality of fields of invariants. In: Algebraic Geometry, Bowdoin, 1985 (Brunswick, MN, 1985), Proc. Sympos. Pure Math. 46, Part 2, Amer. Math. Soc., Providence, RI, 3–16 (1987) Zbl 0659.14009 MR 0927970

[12] Dolgachev, I.: Lectures on Invariant Theory. London Math. Soc. Lecture Note Ser. 296, Cambridge Univ. Press, Cambridge (2003) Zbl 1023.13006 MR 2004511

[13] Eisenbud, D.: Commutative Algebra with a View Toward Algebraic Geometry. Grad. Texts in Math. 150, Springer, New York (1995) Zbl 0819.13001 MR 1322960

[14] Fulton, W.: Young Tableaux. With Applications to Representation Theory and Geometry. London Math. Soc. Student Texts 35, Cambridge Univ. Press, Cambridge (1997) Zbl 0878.14034 MR 1464693

[15] Gel'fand, I., Kapranov, M., Zelevinsky, A.: Hypergeometric functions and toral manifolds. Functional Anal. Appl. **23**, 94–106 (1989) Zbl 0721.33006 MR 1011353

[16] Gel'fand, I., Kapranov, M., Zelevinsky, A.: Discriminants, Resultants and Multidimensional Determinants. Reprint of the 1994 ed., Modern Birkhäuser Classics, Birkhäuser Boston, Boston, MA (2008) MR 2394437

[17] Griffiths, P.: On the periods of certain rational integrals, I, II. Ann. of Math. (2) **90**, 460–495, 496–541 (1969) Zbl 0215.08103 MR 0260733

[18] Griffiths P., Harris, J.: Principles of Algebraic Geometry. Wiley (1978) Zbl 0408.14001 MR 0507725

[19] Harris, J.: Algebraic Geometry: A First Course. Springer, New York (1992) Zbl 0779.14001 MR 1416564

[20] Hosono, S., Klemm, A., Theisen, S., Yau, S.-T.: Mirror symmetry, mirror map and applications to Calabi–Yau hypersurfaces. Comm. Math. Phys. **167**, 301–350 (1995) Zbl 0814.53056 MR 1316509

[21] Hosono, S., Lian, B. H., Yau, S.-T.: GKZ-generalized hypergeometric systems in mirror symmetry of Calabi–Yau hypersurfaces. Comm. Math. Phys. **182**, 535–577 (1996) Zbl 0870.14028 MR 1461942

[22] Hosono, S., Lian, B. H., Yau, S.-T.: Maximal degeneracy points of GKZ systems. J. Amer. Math. Soc. **10**, 427–443 (1997) Zbl 0874.32007 MR 1423031

[23] Hotta, R.: Equivariant D-modules. arXiv:math/9805021v1

[24] Kapranov, M.: Hypergeometric functions on reductive groups. In: Integrable Systems and Algebraic Geometry (Kobe/Kyoto, 1997), World Sci., River Edge, NJ, 236–281 (1998) Zbl 0987.33008 MR 1672049

[25] Kashiwara, M.: Systems of Microdifferential Equations, Birkhäuser, Boston (1983) Zbl 0521.58057 MR 0725502

[26] Li, S., Lian, B. H., Yau, S.-T.: Picard–Fuchs equations for relative periods and Abel–Jacobi map for Calabi–Yau hypersurfaces. Amer. J. Math. **134**, 1345–1384 (2012) Zbl 1253.14036 MR 2975239

[27] Saito, M., Sturmfels, B., Takayama, N.: Gröbner Deformations of Hypergeometric Differential Equations. Springer, Berlin (2000) Zbl 0946.13021 MR 1734566