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The critical nonlinear wave equation in two space dimensions

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Abstract. Extending our previous work [9], we show that the Cauchy problem for wave equations with critical exponential nonlinearities in two space dimensions is globally well-posed for arbitrary smooth initial data.

1. Introduction

Consider the equation

$$u_{tt} - \Delta u + ue^{u^2} = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^2. \tag{1}$$

In [2] Ibrahim, Majdoub, and Masmoudi demonstrated that the initial value problem for equation (1) is well-posed for smooth Cauchy data

$$(u, u_t)_{|t=0} = (u_0, u_1) \tag{2}$$

with initial energy

$$E(u(0)) = \int_{\mathbb{R}^2} e(u(0))\,dx \le 2\pi, \tag{3}$$

where

$$e(u) = \tfrac{1}{2}(|u_t|^2 + |\nabla u|^2 + e^{u^2} - 1). \tag{4}$$

Equation (1) is closely related to the critical Sobolev embedding in two space dimensions defined by the Moser–Trudinger inequality

$$\sup_{u\in H_0^1(\Omega),\ \|\nabla u\|^2_{L^2(\Omega)}\le 1} \int_\Omega e^{\alpha u^2}\,dx \le C(\alpha)|\Omega| \tag{5}$$

for any bounded domain $\Omega \subset \mathbb{R}^2$ having 2-dimensional Lebesgue measure $|\Omega|$ and any $\alpha \le 4\pi$, with a constant $C(\alpha) < \infty$ independent of Ω; see [6], [11]. For $\alpha > 4\pi$ the above supremum is infinite. In particular, when $E(u(0)) > 2\pi$, not even a locally uniform spatial L^1-bound is available for the term ue^{u^2}. In analogy with nonlinear wave equations

$$u_{tt} - \Delta u + u|u|^{p-2} = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^n \tag{6}$$

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with $p > 2n/(n-2)$ in $n \geq 3$ space dimensions, where the nonlinear term cannot be bounded in the dual space of H^1 in terms of the Dirichlet energy, the Cauchy problem for equation (1) was therefore termed "supercritical" for initial data with energy $E(u(0)) > 2\pi$. The recent results [1], [3] of Ibrahim, Jrad, Majdoub, and Masmoudi, showing that the local solution of the Cauchy problem (1), (2) does not depend on the initial data in a locally uniformly continuous fashion when $E(u(0)) > 2\pi$, seemed to further justify this classification.

However, in contrast with these results, in [9] we were able to show that the Cauchy problem (1), (2) is well-posed in the radially symmetric case, regardless of the size of the data. Here we show that the restriction (3) is not needed in the general case either.

Theorem 1.1. *For any $u_0, u_1 \in C^\infty(\mathbb{R}^2)$ there exists a unique, smooth solution u to the Cauchy problem* (1)*,* (2)*, defined for all time.*

The proof of Theorem 1.1 is strikingly different from the proof of the companion result in the spherically symmetric setting. In the latter case, locally uniform pointwise bounds for the solution away from $x = 0$ permit one to rule out blow-up by means of standard multipliers. In contrast, in the present setting the usual multiplier technique only seems to give decay of the energy in the interior of any light cone, and full control only of certain components. In particular, we cannot rule out outgoing waves concentrating energy near the lateral boundary of the light cone. However, in combination with a subtle improvement of the Moser–Trudinger inequality (5), stated as Lemma 4.3 below, the partial control of the energy that we achieve allows us to improve the bounds for the nonlinear term in equation (1) sufficiently for ruling out blow-up. Lemma 4.3 may also be of interest in itself.

Note that no weighted energy estimates are required in the proof, as would be expected in a truly "supercritical" context. It thus appears that problem (1), (2) still belongs to the realm of "critical" equations. More generally, it seems that this may be true for all problems where smallness of the energy implies regularity, as in the present case. See [3], [5], [8], [10] for recent results on supercritical wave equations, and [7] for background material on nonlinear wave equations in general.

2. Basic estimates

For the proof of Theorem 1.1 we argue indirectly, as in [9]; that is, we suppose that the local solution u to (1), (2) for certain Cauchy data $u_0, u_1 \in C^\infty(\mathbb{R}^2)$ cannot be smoothly extended to a neighborhood of some point (T_0, x_0) where $T_0 \geq 0$. As shown in [9], we may assume that u_0, u_1 are compactly supported, $T_0 > 0$, and that $u \in C^\infty([0, T_0[\times\mathbb{R}^2)$.

After translating the origin of our coordinate system to the point x_0, if necessary, we may assume that $x_0 = 0$. Also shifting time by T_0 and then reversing the arrow of time, in the following we may assume that we have a compactly supported solution $u \in C^\infty(]0, T_0] \times \mathbb{R}^2)$ of (1) blowing up at $(0, 0)$.

We now briefly recall some standard estimates from [9] that will also be needed for the present approach.

2.1. *Energy inequality and flux decay*

Upon multiplying (1) by u_t we obtain the conservation law

$$0 = \frac{d}{dt} e(u) - \mathrm{div}(\nabla u \cdot u_t) \tag{7}$$

for the energy density $e(u)$ and the density of momentum

$$m(u) = \nabla u \cdot u_t.$$

In the following, we only will make use of equation (7) on compact regions. In order to simplify later computations, we therefore now drop the term -1 in the definition of $e(u)$ above and henceforth let

$$e(u) = \tfrac{1}{2}(|u_t|^2 + |\nabla u|^2 + e^{u^2}).$$

The original definition (4) was made to ensure that compactly supported functions u have finite total energy.

Since clearly $|m(u)| \leq e(u)$, integration of (7) over a truncated light cone yields

$$E(u(t), B_R(x_0)) := \int_{B_R(x_0)} e(u(t))\,dx \leq E(u(s+t), B_{R+|s|}(x_0)) \tag{8}$$

for any $x_0 \in \mathbb{R}^2$, $R > 0$, and $0 < s + t, t \leq T_0$. In particular, energy will spread with speed at most 1.

Estimate (8) neglects the flux terms, which will be important later. Of particular interest will be the case when $x_0 = 0$. For $0 < S \leq T \leq T_0$ denote by $v(y) = u(|y|, y)$ the restriction of u to the lateral boundary

$$M_S^T = \{z = (t, x);\ S \leq t \leq T,\ |x| = t\}$$

of the truncated forward light cone

$$K_S^T = \{z = (t, x);\ S \leq t \leq T,\ |x| \leq t\}$$

with vertex at $z = (0, 0)$. Upon integrating (7) over K_S^T we then find the identity

$$E(u(S), B_S(0)) + \mathrm{Flux}(u, M_S^T) = E(u(T), B_T(0)) \tag{9}$$

for all $0 < S < T \leq T_0$, where

$$\mathrm{Flux}(u, M_S^T) := \frac{1}{2} \int_{B_T(0) \setminus B_S(0)} (|\nabla v|^2 + e^{v^2})\,dy$$

is the energy flux through M_S^T. Similar identities hold on any region with space-like or null boundary, for instance, in the intersection of a truncated forward light cone with a backward light cone, or with the complement of a backward light cone.

By (9), in particular, $\lim_{T\downarrow 0} E(u(T), B_T(0))$ exists and we deduce decay of the flux

$$\mathrm{Flux}(u, M_0^T) := \sup_{0<S<T} \mathrm{Flux}(u, M_S^T) \to 0 \quad \text{as } T \downarrow 0. \tag{10}$$

Finally, we also have

$$E(u(T), B_T(0)) \le E(u(T_0), B_{T_0}(0)) =: E_0 \tag{11}$$

for $0 < T < T_0$. Set $M^T = M_0^T$, $K^T = K_0^T$ for brevity.

2.2. *Blow-up criterion*

The work of Ibrahim, Majdoub, and Masmoudi [2] gives rise to the following characterization of blow-up through concentration of energy.

Lemma 2.1. *There exists* $\varepsilon_0 > 0$ *such that*

$$E(u(T), B_T(0)) \ge \varepsilon_0 \quad \textit{for all } 0 < T \le T_0. \tag{12}$$

The short proof of Lemma 2.1 given in [9] also works in the nonsymmetric case.

2.3. *Pointwise estimates*

Without any symmetry assumption we clearly cannot expect to obtain the same pointwise estimates away from $x = 0$ that we were able to employ in [9]. However, we can still obtain bounds for the spherical averages

$$\bar v = \bar v(t) = \frac{1}{2\pi}\int_0^{2\pi} v(te^{i\phi})\, d\phi$$

of v, the trace of u on M^{T_0}. Indeed, for $0 < t < T_1 \le T_0$ by Hölder's inequality we can bound

$$\begin{aligned}|\bar v(t)| &\le |\bar v(T_1)| + \int_t^{T_1} |\bar v'(s)|\, ds \le |\bar v(T_1)| + \left(\int_t^{T_1} |\nabla \bar v|^2 s\, ds \cdot \int_t^{T_1} \frac{ds}{s}\right)^{1/2} \\ &\le |\bar v(T_1)| + \pi^{-1/2}\,\mathrm{Flux}^{1/2}(u, M_t^{T_1}) \log^{1/2}(T_1/t).\end{aligned}$$

In view of (10) we may choose $0 < T_1 \le \min\{1, T_0\}$ to ensure that for all $0 < t \le T_1$,

$$\mathrm{Flux}^{1/2}(u, M_t^{T_1}) \le \mathrm{Flux}^{1/2}(u, M^{T_1}) \le 1/8.$$

We then fix $0 < T_2 \le T_1$ so that $8|\bar v(T_1)| \le \log^{1/2}(1/t)$ for $0 < t \le T_2$. Also observing that $\log(T_1/t) \le \log(1/t)$ for our choice of T_1, we thus obtain the bound

$$4|\bar v(t)| \le \log^{1/2}(1/t) \quad \text{for all } 0 < t \le T_2. \tag{13}$$

3. Partial energy decay

Introduce polar coordinates (r, ϕ). The conservation law (7) may then be written in the form

$$\partial_t(re) - \partial_r(rm) = r^{-1}\partial_\phi(u_t u_\phi), \tag{14}$$

where now

$$e = e(u) = \tfrac{1}{2}(u_t^2 + u_r^2 + r^{-2}u_\phi^2 + e^{u^2}), \qquad m = m(u) = u_t u_r.$$

Multiplying (1) by $x \cdot \nabla u$ we also obtain the identity

$$0 = \frac{d}{dt}(u_t x \cdot \nabla u) - \operatorname{div}\left(\nabla u x \cdot \nabla u - \frac{x}{2}(|\nabla u|^2 - |u_t|^2 + e^{u^2})\right) + |u_t|^2 - e^{u^2}.$$

In polar coordinates this reads

$$\partial_t(r^2 u_t u_r) - \tfrac{1}{2}\partial_r(r^2(u_t^2 + u_r^2 - e^{u^2} - r^{-2}u_\phi^2)) + r(u_t^2 - e^{u^2}) = \partial_\phi(u_r u_\phi),$$

that is, we have

$$\partial_t(r^2 m) - \partial_r(r^2(e - q)) + r(u_t^2 - e^{u^2}) = \partial_\phi(u_r u_\phi), \tag{15}$$

where

$$q = q(u) = r^{-2}u_\phi^2 + e^{u^2}.$$

Finally, we multiply (1) by $u - \bar{v}$ to obtain the equation

$$0 = \frac{d}{dt}(u_t(u - \bar{v})) - \operatorname{div}(\nabla u(u - \bar{v})) + |\nabla u|^2 - |u_t|^2 + u_t\bar{v}_t + u(u - \bar{v})e^{u^2},$$

that is,

$$\begin{aligned}\partial_t(ru_t(u - \bar{v})) - \partial_r(ru_r(u - \bar{v})) + r\big(|\nabla u|^2 - |u_t|^2 + u_t\bar{v}_t + u(u - \bar{v})e^{u^2}\big) \\ = r^{-1}\partial_\phi((u - \bar{v})u_\phi).\end{aligned} \tag{16}$$

Multiplying equation (14) by r/t, we obtain

$$\partial_t\left(\frac{r^2}{t}e\right) - \partial_r\left(\frac{r^2}{t}m\right) + \frac{r^2}{t^2}e + \frac{r}{t}m = t^{-1}\partial_\phi(u_t u_\phi). \tag{17}$$

Likewise, upon dividing (15) and (16) by t we find the expressions

$$\partial_t\left(\frac{r^2}{t}m\right) - \partial_r\left(\frac{r^2}{t}(e - q)\right) + \frac{r}{t}(u_t^2 - e^{u^2}) + \frac{r^2}{t^2}m = t^{-1}\partial_\phi(u_r u_\phi) \tag{18}$$

and

$$\partial_t\left(\frac{r}{t}u_t(u-\bar v)\right)-\partial_r\left(\frac{r}{t}u_r(u-\bar v)\right)+\frac{r}{t}\left(|\nabla u|^2-|u_t|^2+u_t\bar v_t+u_t\frac{u-\bar v}{t}+u(u-\bar v)e^{u^2}\right)$$
$$=\partial_t\left(\frac{r}{t}\left(u_t(u-\bar v)+\frac{|u-\bar v|^2}{2t}\right)\right)-\partial_r\left(\frac{r}{t}u_r(u-\bar v)\right)$$
$$+\frac{r}{t}\left(|\nabla u|^2-|u_t|^2+u_t\bar v_t+\bar v_t\frac{u-\bar v}{t}+\frac{|u-\bar v|^2}{t^2}+u(u-\bar v)e^{u^2}\right)$$
$$=\frac{1}{rt}\partial_\phi((u-\bar v)u_\phi), \tag{19}$$

respectively. Dividing both sides of (19) by 2, adding (17), and also adding (18), we then arrive at the equation

$$\partial_t\left(\frac{r^2}{t}\left(e+m+u_t\frac{u-\bar v}{2r}+\frac{|u-\bar v|^2}{4rt}\right)\right)-\partial_r\left(\frac{r^2}{t}\left(e-q+m+u_r\frac{u-\bar v}{2r}\right)\right)$$
$$+\frac{r}{t}\left(\left(1+\frac{r}{t}\right)(e+m)+\frac{1}{2}u_t\bar v_t+\bar v_t\frac{u-\bar v}{2t}+\frac{|u-\bar v|^2}{2t^2}+\frac{u(u-\bar v)-3}{2}e^{u^2}\right)$$
$$=t^{-1}\partial_\phi\left(\left(u_r+u_t+\frac{u-\bar v}{2r}\right)u_\phi\right). \tag{20}$$

Similarly, subtracting (17) from the sum of (18) and $1/2$ times (19), we obtain

$$\partial_t\left(\frac{r^2}{t}\left(m-e+u_t\frac{u-\bar v}{2r}+\frac{|u-\bar v|^2}{4rt}\right)\right)-\partial_r\left(\frac{r^2}{t}\left(e-q-m+u_r\frac{u-\bar v}{2r}\right)\right)$$
$$+\frac{r}{t}\left(\left(1-\frac{r}{t}\right)(e-m)+\frac{1}{2}u_t\bar v_t+\bar v_t\frac{u-\bar v}{2t}+\frac{|u-\bar v|^2}{2t^2}+\frac{u(u-\bar v)-3}{2}e^{u^2}\right)$$
$$=t^{-1}\partial_\phi\left(\left(u_r-u_t+\frac{u-\bar v}{2r}\right)u_\phi\right). \tag{21}$$

In the following we repeatedly make use of Young's inequality $2ab\le\delta a^2+\delta^{-1}b^2$ for any $a,b,\delta>0$. The letter C will denote a generic constant independent of u, T, etc., unless otherwise stated. Its value may change from line to line and even within the same line.

Lemma 3.1. *For $0<T<\min\{T_2,e^{-1}\}$ we have*

$$\int_{K^T}\left(\left(1\pm\frac{r}{t}\right)(e\pm m)+\frac{|u-\bar v|^2}{2t^2}+\frac{1}{4}|u-\bar v|^2e^{u^2}\right)\frac{dx\,dt}{t}\le C(1+E_0).$$

Proof. For fixed $0 < T < \min\{T_2, e^{-1}\}$ and $0 < S < T$ we integrate (20) over the region where $0 < S < t < T$, $0 < r < t$, $0 \le \phi < 2\pi$ corresponding to the truncated cone K_S^T to obtain

$$\begin{aligned} I_+ &:= \int_{K_S^T} \left(\left(1+\frac{r}{t}\right)(e+m) + \frac{|u-\bar{v}|^2}{2t^2} + \frac{1}{4}|u-\bar{v}|^2 e^{u^2} \right) \frac{dx\,dt}{t} \\ &= \int_S^T \int_0^t \int_0^{2\pi} \frac{r}{t} \left(\left(1+\frac{r}{t}\right)(e+m) + \frac{|u-\bar{v}|^2}{2t^2} + \frac{1}{4}|u-\bar{v}|^2 e^{u^2} \right) d\phi\, dr\, dt \\ &\le II + III + IV + V, \end{aligned}$$

with II, III, and IV corresponding to the boundary terms and with 'error' term

$$V = -\int_{K_S^T} \left(\frac{1}{2} u_t \bar{v}_t + \bar{v}_t \frac{u-\bar{v}}{2t} + \frac{u^2 - \bar{v}^2 - 6}{4} e^{u^2} \right) \frac{dx\,dt}{t}.$$

Recalling that $e + m \ge 0$ and using Young's inequality to estimate

$$\left| u_t \frac{u-\bar{v}}{2t} \right| \le \frac{1}{2}|u_t|^2 + \frac{|u-\bar{v}|^2}{4t^2} \le e + \frac{|u-\bar{v}|^2}{4t^2}$$

for any t, we can bound the top boundary term:

$$\begin{aligned} II &= -\int_{\{T\}\times B_T(0)} \frac{r}{t} \left(e + m + u_t \frac{u-\bar{v}}{2r} + \frac{|u-\bar{v}|^2}{4rt} \right) dx \\ &= -\int_{\{T\}\times B_T(0)} \left(\frac{r}{T}(e+m) + u_t \frac{u-\bar{v}}{2T} + \frac{|u-\bar{v}|^2}{4T^2} \right) dx \le \int_{\{T\}\times B_T(0)} e\, dx \le E_0. \end{aligned}$$

Also using Poincaré's inequality

$$\int_{\{t\}\times B_t(0)} \frac{|u-\bar{v}|^2}{t^2}\, dx \le C \int_{\{t\}\times B_t(0)} |\nabla u|^2\, dx \tag{22}$$

for any $0 < t < T$, in similar fashion we can bound the term corresponding to the lower boundary:

$$\begin{aligned} III &= \int_{\{S\}\times B_S(0)} \frac{r}{t} \left(e + m + u_t \frac{u-\bar{v}}{2r} + \frac{|u-\bar{v}|^2}{4rt} \right) dx \\ &\le \int_{\{S\}\times B_S(0)} \left(\frac{r}{S}(e+m) + e + \frac{|u-\bar{v}|^2}{2S^2} \right) dx \le C \int_{\{S\}\times B_S(0)} e\, dx \le C E_0. \end{aligned}$$

Moreover, for the lateral boundary component we have

$$\begin{aligned} IV &= \int_S^T \int_0^{2\pi} \frac{r^2}{t} \left(2(e+m) - q + (u_r + u_t)\frac{u-\bar{v}}{2r} + \frac{|u-\bar{v}|^2}{4rt} \right) d\phi\, dt \bigg|_{r=t} \\ &= \frac{1}{\sqrt{2}} \int_{M_S^T} \left((u_r+u_t)^2 + (u_r+u_t)\frac{v-\bar{v}}{2t} + \frac{|v-\bar{v}|^2}{4t^2} \right) do. \end{aligned}$$

But again by Poincaré's inequality, for any $0 < t < T$ we can estimate

$$\int_0^{2\pi} \frac{|v-\bar{v}|^2}{4t^2}\,d\phi \le C\int_0^{2\pi} t^{-2}|v_\phi|^2\,d\phi \le C\int_0^{2\pi}|\nabla v|^2\,d\phi, \tag{23}$$

and we conclude that

$$IV \le C\,\mathrm{Flux}(u, M^T) \le CE_0.$$

Finally, in order to bound V, for each t we write

$$\int_{B_t(0)} u_t\bar{v}_t\,dx = \int_{B_t(0)} ((u_t+u_r)\bar{v}_t - u_r\bar{v}_t)\,dx$$

and note that for any $0 < \delta < 1$ we can bound

$$\int_{B_t(0)} |(u_t+u_r)\bar{v}_t|\,dx \le \delta\int_{B_t(0)}(e+m)\,dx + \frac{1}{2\delta}\int_{B_t(0)}|\bar{v}_t|^2\,dx.$$

Next observe that

$$\begin{aligned}\int_{B_t(0)} u_r\bar{v}_t\,dx &= \int_0^{2\pi}\int_0^t (r(u-\bar{v})\bar{v}_t)_r\,dr\,d\phi - \int_{B_t(0)}\frac{u-\bar{v}}{r}\bar{v}_t\,dx\\ &= \int_{\partial B_t(0)}(v-\bar{v})\bar{v}_t\,do - \int_{B_t(0)}\frac{u-\bar{v}}{r}\bar{v}_t\,dx,\end{aligned}$$

where (23) allows us to bound

$$\begin{aligned}\int_{\partial B_t(0)}|(v-\bar{v})\bar{v}_t|\,do &\le \int_{\partial B_t(0)}\frac{|v-\bar{v}|^2}{2t}\,do + \frac{1}{2}\int_{\partial B_t(0)} t|\bar{v}_t|^2\,do\\ &\le Ct\int_{\partial B_t(0)}|\nabla v|^2\,do + Ct^2|\bar{v}_t|^2.\end{aligned}$$

Moreover, we have

$$\int_{B_t(0)}\left|\frac{u-\bar{v}}{r}\bar{v}_t\right|dx \le \delta\int_{B_t(0)}\frac{|u-\bar{v}|^2}{2rt}\,dx + \frac{1}{2\delta}\int_{B_t(0)}\frac{t}{r}|\bar{v}_t|^2\,dx,$$

with

$$\frac{1}{2\delta}\int_{B_t(0)}\frac{t}{r}|\bar{v}_t|^2\,dx \le C\delta^{-1}t^2|\bar{v}_t|^2.$$

We split the remaining term

$$\begin{aligned}\int_{B_t(0)}\frac{|u-\bar{v}|^2}{2rt}\,dx &\le \int_{B_{t/2}(0)}\frac{|u-\bar{v}|^2}{2rt}\,dx + \int_{B_t(0)}\frac{|u-\bar{v}|^2}{t^2}\,dx\\ &\le \int_{B_{t/2}(0)}\frac{|u-\tilde{u}|^2}{rt}\,dx + C|\tilde{u}-\bar{v}|^2 + \int_{B_t(0)}\frac{|u-\bar{v}|^2}{t^2}\,dx,\end{aligned} \tag{24}$$

where $\tilde{u} = \tilde{u}(t)$ is the average of $u(t)$ on $B_{t/2}(0)$. Note that we can bound

$$|\tilde{u} - \bar{v}|^2 \le C \int_{B_t(0)} \frac{|u - \bar{v}|^2}{t^2}\, dx,$$

while by Hölder's inequality and a variant of the Poincaré inequality we have

$$\int_{B_{t/2}(0)} \frac{|u - \tilde{u}|^2}{2rt}\, dx \le C \left(t^{-2} \int_{B_{t/2}(0)} |u - \tilde{u}|^6\, dx \right)^{1/3} \le C \int_{B_{t/2}(0)} |\nabla u|^2\, dx. \qquad (25)$$

Summarizing, we find

$$\left| \int_{B_t(0)} u_t \bar{v}_t\, dx \right| \le C\delta \int_{B_{t/2}(0)} |\nabla u|^2\, dx + C\delta \int_{B_t(0)} \left(e + m + \frac{|u - \bar{v}|^2}{t^2} \right) dx$$
$$+ Ct \int_{\partial B_t(0)} |\nabla v|^2\, do + C\delta^{-1} t^2 |\bar{v}_t|^2.$$

Similarly, with the help of Young's inequality we can bound

$$\int_{B_t(0)} \left| \bar{v}_t \frac{u - \bar{v}}{t} \right| dx \le \delta \int_{B_t(0)} \frac{|u - \bar{v}|^2}{t^2}\, dx + C \frac{1}{2\delta} \int_{B_t(0)} |\bar{v}_t|^2\, dx.$$

Thus we conclude that

$$\left| \int_{K_S^T} \left(u_t \bar{v}_t + \bar{v}_t \frac{u - \bar{v}}{t} \right) \frac{dx\, dt}{t} \right|$$
$$\le C\delta \int_S^T \int_{B_{t/2}(0)} |\nabla u|^2 \frac{dx\, dt}{t} + C\delta I_+ + C\delta^{-1}\, \mathrm{Flux}(u, M^T).$$

Finally, we observe that by (13),

$$(6 + \bar{v}^2 - u^2) e^{u^2} \le (6 + \bar{v}^2) e^{6 + \bar{v}^2} \le C(6 + \log(1/t)) t^{-1}$$

for all $0 < t < T_2$. Thus for $0 < T < \min\{T_2, e^{-1}\}$ we have

$$\int_{K_S^T} (6 + \bar{v}^2 - u^2) e^{u^2} \frac{dx\, dt}{t} \le C \int_{K^T} (6 + \log(1/t)) \frac{dx\, dt}{t^2} \le C,$$

and we conclude that

$$V \le C(1 + \delta I_+) + C\delta \int_S^T \int_{B_{t/2}(0)} |\nabla u|^2 \frac{dx\, dt}{t} + C\delta^{-1}\, \mathrm{Flux}(u, M^T).$$

Recalling that $\mathrm{Flux}(u, M^T) \le E_0$, together with our estimates for the boundary terms we find

$$I_+ \le C(1 + \delta I_+ + \delta^{-1} E_0) + C\delta \int_S^T \int_{B_{t/2}(0)} |\nabla u|^2 \frac{dx\, dt}{t}.$$

The analogous estimate

$$I_- := \int_{K_S^T} \left(\left(1 - \frac{r}{t}\right)(e-m) + \frac{|u-\bar{v}|^2}{2t^2} + \frac{1}{4}|u-\bar{v}|^2 e^{u^2} \right) \frac{dx\,dt}{t}$$
$$\le C(1 + \delta I_+ + \delta^{-1} E_0) + C\delta \int_S^T \int_{B_{t/2}(0)} |\nabla u|^2 \frac{dx\,dt}{t}$$

follows in the same fashion upon integrating (21) over K_S^T.

Finally, we note that we can bound $|\nabla u|^2 \le 2e = (e+m) + (e-m)$ and hence

$$\int_S^T \int_{B_{t/2}(0)} |\nabla u|^2 \frac{dx\,dt}{t} \le I_+ + 2I_-.$$

Thus for sufficiently small $\delta > 0$ with a constant C independent of $S > 0$ we have

$$I_+ + I_- \le C(1 + E_0).$$

Letting $S \downarrow 0$, we obtain the claim. □

4. Proof of Theorem 1.1

For given $0 < \varepsilon < 1$ in view of (10) and Lemma 3.1 we may fix $0 < T_\varepsilon < \min\{T_2, e^{-1}, \varepsilon^2\}$ so that

$$\mathrm{Flux}(u, M^{T_\varepsilon}) + \int_{K^{T_\varepsilon}} \left(\left(1 \pm \frac{r}{t}\right)(e \pm m) + \frac{|u-\bar{v}|^2}{t^2} + |u-\bar{v}|^2 e^{u^2} \right) \frac{dx\,dt}{t} < \varepsilon. \quad (26)$$

Introduce the characteristic coordinates

$$\xi = t + r, \qquad \eta = t - r.$$

Then we have

$$t = \frac{\xi + \eta}{2}, \qquad r = \frac{\xi - \eta}{2},$$

and

$$\partial_\xi = \tfrac{1}{2}(\partial_t + \partial_r), \quad \partial_\eta = \tfrac{1}{2}(\partial_t - \partial_r), \quad \partial_t = \partial_\xi + \partial_\eta, \quad \partial_r = \partial_\xi - \partial_\eta.$$

For any $0 < \xi_1 < T_\varepsilon$ let

$$\Gamma(\xi_1) = \{(t,x) \in K^{T_\varepsilon};\ \xi = t + |x| = \xi_1\}.$$

Integrating (7) over the region

$$\{(t,x) \in K^{T_\varepsilon};\ \xi = t + |x| \ge \xi_1\},$$

for any such ξ_1 we obtain

$$2\int_{\Gamma(\xi_1)} u_\eta^2\, do \le \int_{\Gamma(\xi_1)} (e-m)\, do \le E(u(T_\varepsilon), B_{T_\varepsilon}(0)) \le E_0 \tag{27}$$

as a useful variant of the energy inequality (9).

In terms of ξ and η we can also write the first two terms in equation (20) in the form

$$\begin{aligned}\partial_t\Big(\frac{r^2}{t}\Big(e+m+u_t\frac{u-\bar v}{2r}+\frac{|u-\bar v|^2}{4rt}\Big)\Big)-\partial_r\Big(\frac{r^2}{t}\Big(e-q+m+u_r\frac{u-\bar v}{2r}\Big)\Big)\\ =\partial_\eta\Big(\frac{r^2}{t}\Big(2(e+m)-q+u_\xi\frac{u-\bar v}{r}+\frac{|u-\bar v|^2}{4rt}\Big)\Big)\\ +\partial_\xi\Big(\frac{r^2}{t}\Big(q+u_\eta\frac{u-\bar v}{r}+\frac{|u-\bar v|^2}{4rt}\Big)\Big).\end{aligned} \tag{28}$$

Observing that

$$2(e+m)-q=|u_t+u_r|^2=4u_\xi^2,$$

for $r/t \ge 3/4$ we have

$$\begin{aligned}&\frac{r}{t}\Big(2(e+m)-q+u_\xi\frac{u-\bar v}{r}+\frac{|u-\bar v|^2}{4rt}\Big)\\ &=\Big(4\frac{r}{t}-2\Big)u_\xi^2+2\Big(u_\xi+\frac{u-\bar v}{4t}\Big)^2+\frac{|u-\bar v|^2}{8t^2}\ge u_\xi^2+\frac{|u-\bar v|^2}{8t^2}.\end{aligned} \tag{29}$$

Fix $\lambda_0 = 3/4$. Given $0 < \xi_0 < 8^{-1}T_\varepsilon$, we set $\eta_0 = \frac{1-\lambda_0}{1+\lambda_0}\xi_0 = \xi_0/7$. For $\xi_1 \in [\xi_0, 8\xi_0]$ we then let

$$\Gamma_0(\xi_1) = \{(t,x) \in K^{T_\varepsilon};\ \xi = t+|x| = \xi_1,\ \eta = t-|x| < \eta_0\}$$

and we define

$$Q(\xi_1) := \int_{\Gamma_0(\xi_1)}\Big(q+\frac{|u-\bar v|^2}{t^2}\Big)\, do.$$

Note that $t < T_\varepsilon$ for any (t,x) with $\xi = t+|x| \le 8\xi_0 < T_\varepsilon$. Changing variables $(t,x) \mapsto (\xi = t+|x|, x)$, we see that for any $\xi_1 < T_\varepsilon/2$, with an absolute constant C,

$$\begin{aligned}\inf_{\xi_1<\xi<2\xi_1} Q(\xi) &\le \xi_1^{-1}\int_{\xi_1}^{2\xi_1} Q(\xi)\, d\xi\\ &\le C\int_{K^{T_\varepsilon}}\Big(\Big(1+\frac{r}{t}\Big)(e+m)+\frac{|u-\bar v|^2}{2t^2}\Big)\frac{dx\, dt}{t} < C\varepsilon.\end{aligned}$$

Thus, we can choose numbers $\xi_1 \in [\xi_0, 2\xi_0]$ and $\xi_2 \in [4\xi_0, 8\xi_0]$ such that

$$Q(\xi_1) \le 2\inf_{\xi_0<\xi<2\xi_0} Q(\xi) < C\varepsilon, \qquad Q(\xi_2) \le 2\inf_{4\xi_0<\xi<8\xi_0} Q(\xi) < C\varepsilon.$$

Lemma 4.1. *For any* $0 < \xi_0 < 8^{-1}T_\varepsilon$, *and any* $\xi_1 \in [\xi_0, 2\xi_0]$, $\xi_2 \in [4\xi_0, 8\xi_0]$ *as above,*

$$\sup_{2\xi_0 < \xi < 4\xi_0} Q(\xi) \leq \sup_{\xi_1 < \xi < \xi_2} Q(\xi) < C\sqrt{\varepsilon}.$$

Proof. Consider the set

$$R = R(\xi_1, \xi_2) = \{(t, x) \in K^{T_\varepsilon};\ \xi_1 < \xi < \xi_2,\ 0 < \eta < \eta_0\}$$

with boundary $\partial R = \bigcup_{i=1}^4 \Gamma_i$, where

$$\begin{aligned} \Gamma_1 &= \{(t, x);\ \xi_1 < \xi < \xi_2,\ \eta = 0\}, \qquad \Gamma_2 = \Gamma_0(\xi_2), \\ \Gamma_3 &= \{(t, x);\ \xi_1 < \xi < \xi_2,\ \eta = \eta_0\}, \qquad \Gamma_4 = \Gamma_0(\xi_1). \end{aligned}$$

Integrating (20) over R, we find the identity

$$A_0 + A_3 = A_1 - A_2 + A_4 + V, \tag{30}$$

where

$$A_0 = \int_R \left(\left(1 + \frac{r}{t}\right)(e + m) + \frac{|u - \bar{v}|^2}{2t^2} + \frac{1}{4}|u - \bar{v}|^2 e^{u^2} \right) \frac{dx\,dt}{t}$$

and where the terms A_i, $1 \leq i \leq 4$, correspond to integrals over the boundary components Γ_i, $1 \leq i \leq 4$. Finally, V again denotes the 'error' term

$$V = -\int_R \left(\frac{1}{2} u_t \bar{v}_t + \bar{v}_t \frac{u - \bar{v}}{2t} + \frac{u^2 - \bar{v}^2 - 6}{4} e^{u^2} \right) \frac{dx\,dt}{t}.$$

By (26),

$$0 \leq A_0 \leq \varepsilon.$$

Moreover, using (28), (29), and (23) we find

$$\begin{aligned} A_1 &= \int_{\Gamma_1} \left(\left(4\frac{r}{t} - 2\right) u_\xi^2 + 2\left(u_\xi + \frac{u - \bar{v}}{4t}\right)^2 + \frac{|u - \bar{v}|^2}{8t^2} \right) do \\ &\leq C \int_{\Gamma_1} \left(u_\xi^2 + \frac{|v - \bar{v}|^2}{t^2} \right) do \leq C\,\mathrm{Flux}(u, M^{8\xi_0}) \leq C\varepsilon. \end{aligned}$$

Using Young's inequality to bound

$$\left| u_\eta \frac{u - \bar{v}}{2t} \right| \leq \frac{\delta}{2} u_\eta^2 + \frac{|u - \bar{v}|^2}{8\delta t^2}, \tag{31}$$

and recalling that the energy inequality (27) allows us to bound

$$2 \int_{\Gamma_0(\xi)} u_\eta^2\, do \leq E_0 \tag{32}$$

for any $\xi < 8\xi_0$, we also find

$$|A_2| = \left|\int_{\Gamma_0(\xi_2)} \left(\frac{r}{t}q + u_\eta \frac{u-\bar{v}}{2t} + \frac{|u-\bar{v}|^2}{4t^2}\right) do\right| \leq (1+\delta^{-1})Q(\xi_2) + \delta E_0,$$

and similarly for A_4. Choosing $\delta = \sqrt{\varepsilon}$, by choice of ξ_1 and ξ_2 we obtain

$$|A_2| < C\sqrt{\varepsilon}, \quad |A_4| < C\sqrt{\varepsilon}.$$

In order to proceed, observe that by (29) we have

$$A_3 \geq \int_{\Gamma_3} \left(u_\xi^2 + \frac{|u-\bar{v}|^2}{8t^2}\right) do.$$

The error term V can then be bounded as in the proof of Lemma 3.1, on noting that we can express

$$\int_R u_r \bar{v}_t \frac{dx\,dt}{t} = \int_R (r(u-\bar{v})\bar{v}_t)_r \frac{dr\,d\phi\,dt}{t} - \int_R \frac{u-\bar{v}}{r}\bar{v}_t \frac{dx\,dt}{t}$$

with

$$\begin{aligned}\left|\int_R (r(u-\bar{v})\bar{v}_t)_r \frac{dr\,d\phi\,dt}{t}\right| &\leq \int_{\partial R} |(u-\bar{v})\bar{v}_t| \frac{do}{t} \leq \delta \int_{\partial R} \frac{|u-\bar{v}|^2}{8t^2} do + \frac{2}{\delta}\int_{\partial R} |\bar{v}_t|^2 do \\ &\leq \delta(A_3 + Q(\xi_1) + Q(\xi_2)) + C\delta^{-1}\,\mathrm{Flux}(u, M^{4\xi_0}) \\ &\leq \delta A_3 + C\varepsilon + C\delta^{-1}\varepsilon \end{aligned} \tag{33}$$

for any $0 < \delta < 1$, in view of (23), (26), and our bounds for $Q(\xi_1)$ and $Q(\xi_2)$. Also note that in view of the fact that $r/t \geq \lambda_0 = 3/4$ on R we do not need to perform step (24); instead, we can easily estimate

$$\begin{aligned}\int_R \frac{|u-\bar{v}|}{r}|\bar{v}_t| \frac{dx\,dt}{t} &\leq 2\int_R \frac{|u-\bar{v}|}{t}|\bar{v}_t| \frac{dx\,dt}{t} \leq \int_R \frac{|u-\bar{v}|^2}{2t^2}\frac{dx\,dt}{t} + 2\int_R |\bar{v}_t|^2 \frac{dx\,dt}{t} \\ &\leq A_0 + C\,\mathrm{Flux}(u, M^{4\xi_0}) \leq C\varepsilon.\end{aligned}$$

Finally, recalling that $T_\varepsilon < 1$, in view of (13) we can estimate

$$(6+\bar{v}^2 - u^2)e^{u^2} \leq (6+\bar{v}^2)e^{6+\bar{v}^2} \leq C(6+\log(1/t))t^{-1/2}$$

for all $0 < t < T_\varepsilon$. Hence we have

$$\int_R (6+\bar{v}^2-u^2)e^{u^2}\frac{dx\,dt}{t} \leq C\int_{K^{8\xi_0}} (6+\log(1/t))\frac{dx\,dt}{t^{3/2}} \leq C\sqrt{\xi_0} \leq C\varepsilon.$$

Together with (33), when choosing $\delta = 1/2$, we thus obtain the bound

$$V \leq \tfrac{1}{2}A_3 + C\varepsilon.$$

From (30) we then conclude that

$$0 \le A_3 \le C\sqrt{\varepsilon}.$$

Thus for any $\xi_1 < \xi < \xi_2$, when integrating (20) over $R(\xi_1, \xi)$, from the analogue of (30) and choosing a sufficiently small number $\delta > 0$ in (33) we now find that

$$A_2(\xi) := \int_{\Gamma_0(\xi)} \left(\frac{r}{t} q + u_\eta \frac{u - \bar{v}}{t} + \frac{|u - \bar{v}|^2}{4t^2} \right) do \le C\sqrt{\varepsilon} + \tfrac{1}{2} Q(\xi). \tag{34}$$

This estimate implies the desired bound for $Q(\xi)$ once we control the middle term. But by Hölder's inequality, for any $\xi_1 < \xi < \xi_2$ we have

$$\begin{aligned} |(u - \bar{v})(\xi)|^2 &\le \left(|(u - \bar{v})(\xi_1)| + \int_{\xi_1}^{\xi_2} |u_\xi - \bar{v}_\xi|\, d\xi \right)^2 \\ &\le 2|(u - \bar{v})(\xi_1)|^2 + 2(\xi_2 - \xi_1) \int_{\xi_1}^{\xi_2} |u_\xi - \bar{v}_\xi|^2\, d\xi. \end{aligned}$$

Integrating over $\Gamma_0(\xi)$, observing that the surface measure do may be expressed as $r\, d\eta\, d\phi$, where $r = (\xi - \eta)/2$, and noting that throughout R we have $0 \le \eta \le \eta_0 = \xi_0/7$, $\xi_0 \le \xi \le 2t \le 16\xi_0$, we obtain

$$\begin{aligned} \int_{\Gamma_0(\xi)} \frac{|u - \bar{v}|^2}{t^2}\, do &\le C \int_{\Gamma_0(\xi_1)} \frac{|u - \bar{v}|^2}{t^2}\, do + C \int_R (|u_\xi|^2 + |\bar{v}_t|^2) \frac{dx\, dt}{t} \\ &\le C\varepsilon + CA_0 + C\,\mathrm{Flux}(u, M^{8\xi_0}) \le C\varepsilon \end{aligned} \tag{35}$$

for any $\xi_1 < \xi < \xi_2$. By (31) and (32), again choosing $\delta = \sqrt{\varepsilon}$, we can estimate

$$\begin{aligned} A_2(\xi) &= \int_{\Gamma_0(\xi)} \left(\frac{r}{t} q + (u_t - u_r) \frac{u - \bar{v}}{2t} + \frac{|u - \bar{v}|^2}{4t^2} \right) do \\ &\ge \lambda_0 Q(\xi) - \left(\lambda_0 + \frac{1}{8\delta} \right) \int_{\Gamma_0(\xi)} \frac{|u - \bar{v}|^2}{t^2}\, do - \frac{\delta}{2} \int_{\Gamma_0(\xi)} u_\eta^2\, do \ge \frac{3}{4} Q(\xi) - C\sqrt{\varepsilon}. \end{aligned}$$

Together with (34) it follows that

$$\sup_{\xi_1 < \xi < \xi_2} Q(\xi) < C\sqrt{\varepsilon},$$

as claimed. □

Combining Lemmas 3.1 and 4.1 we can bound the nonlinear term in equation (1) in any L^p-norm. A key role is played by the following improvement of the Moser–Trudinger inequality (5).

Lemma 4.2. *For any $E > 0$ and $p < \infty$ there exists a number $\varepsilon = 4\pi^2/(p^2 E) > 0$ and a constant $C > 0$ such that for any $\xi_0 > 0$ and $v \in H_0^1([0,1]^2)$ with*

$$\int_0^1\int_0^1 (\xi_0|v_y|^2 + \xi_0^{-1}|v_x|^2)\,dx\,dy \le E, \qquad \int_0^1\int_0^1 \xi_0^{-1}|v_x|^2\,dx\,dy \le \varepsilon$$

we have

$$\int_0^1\int_0^1 e^{pv^2}\,dx\,dy \le C.$$

Proof. Given $v \in H_0^1([0,1]^2)$ as above, set $\alpha = (\xi_0^2\varepsilon/E)^{1/4} > 0$ and let

$$v_\alpha(x,y) = v(x/\alpha, \alpha y) \in H_0^1([0,\alpha]\times[0,1/\alpha]),$$

satisfying

$$\begin{aligned}\int_0^\alpha\int_0^{1/\alpha} |\nabla v_\alpha|^2\,dx\,dy &= \int_0^1\int_0^1 (\alpha^{-2}|v_x|^2 + \alpha^2|v_y|^2)\,dx\,dy \\ &\le \varepsilon\xi_0\alpha^{-2} + \alpha^2 E/\xi_0 = 2(\varepsilon E)^{1/2} = 4\pi/p \end{aligned} \tag{36}$$

by our choice of ε. Note that the map $(x,y) \mapsto (x/\alpha, \alpha y)$ is measure-preserving; in particular, for any $s \ge 0$,

$$|\{(x,y);\ v_\alpha^2(x,y) \ge s\}| = |\{(x,y);\ v^2(x,y) \ge s\}|,$$

and

$$\int_0^1\int_0^1 e^{pv^2}\,dx\,dy = \int_0^\alpha\int_0^{1/\alpha} e^{pv_\alpha^2}\,dx\,dy.$$

But by (36), with the constant $C(4\pi)$ in (5), we have

$$\int_0^\alpha\int_0^{1/\alpha} e^{pv_\alpha^2}\,dx\,dy \le C(4\pi),$$

and our claim follows. □

Lemma 4.3. *There exists $\varepsilon > 0$ and a constant $C < \infty$ such that for any $0 < T < 4^{-1}T_\varepsilon$,*

$$\int_{K^T} e^{4u^2}\,dx\,dt \le CT.$$

Proof. Given $0 < \xi_0 < 8^{-1}T_\varepsilon$, let $\eta_0 = \frac{1-\lambda_0}{1+\lambda_0}\xi_0$, where $\lambda_0 = 3/4$ as before. For $\xi_0 < \xi_4 \le 8\xi_0$ recall the definitions

$$\Gamma(\xi_4) = \{(t,x) \in K^{T_\varepsilon};\ \xi = t + |x| = \xi_4\}$$
$$\Gamma_0(\xi_4) = \{(t,x) \in K^{T_\varepsilon};\ \xi = t + |x| = \xi_4,\ \eta = t - |x| < \eta_0\}$$

from the beginning of this section. Also let

$$\Gamma_1(\xi_4) = \{(t,x) \in K^{T_\varepsilon};\ \xi = t + |x| = \xi_4,\ \eta = t - |x| \geq 3\eta_0/4\},$$
$$\Gamma_2(\xi_4) = \{(t,x) \in K^{T_\varepsilon};\ \xi = t + |x| = \xi_4,\ \eta = t - |x| \geq \eta_0/2\}.$$

With the help of Lemma 4.2 and (38) below, respectively, for any fixed $0 < \xi_0 < 8^{-1}T_\varepsilon$ as above and any $2\xi_0 \leq \xi \leq 4\xi_0$ we now bound the integral of e^{4u^2} over $\Gamma(\xi)$, uniformly in ξ. Note that for each such ξ we have $\Gamma(\xi) \subset \Gamma_0(\xi) \cup \Gamma_1(\xi)$.

First consider $\Gamma_1(\xi) \subset \Gamma_2(\xi)$. Note that we have $r/t \leq 1 - \nu_0 < 1$ throughout $\Gamma_2(\xi)$ for any $\xi_0 \leq \xi \leq 8\xi_0$, with a uniform constant $\nu_0 > 0$ determined by our choice of λ_0. By Fubini's theorem and in view of (26) for each $0 < \xi_0 < 8^{-1}T_\varepsilon$ there is $\xi_4 \in [4\xi_0, 8\xi_0]$ such that

$$\begin{aligned}
\int_{\Gamma_2(\xi_4)} &\left(\nu_0(e-m) + \frac{|u-\bar{v}|^2}{2t^2}\right) do \\
&\leq \int_{\Gamma_2(\xi_4)} \left(\left(1 - \frac{r}{t}\right)(e-m) + \frac{|u-\bar{v}|^2}{2t^2}\right) do \\
&\leq 2 \inf_{4\xi_0<\xi<8\xi_0} \int_{\Gamma_2(\xi)} \left(\left(1 - \frac{r}{t}\right)(e-m) + \frac{|u-\bar{v}|^2}{2t^2}\right) do \\
&\leq C \int_{K^{T_\varepsilon}} \left(\left(1 - \frac{r}{t}\right)(e-m) + \frac{|u-\bar{v}|^2}{2t^2}\right) \frac{dx\,dt}{t} \leq C\varepsilon. \qquad (37)
\end{aligned}$$

In particular, we have

$$\int_{\Gamma_2(\xi_4)} (e-m)\,do \leq C\varepsilon.$$

Upon integrating the conservation law (7) over the region

$$\{(t,x) \in K^{T_\varepsilon};\ \xi_3 \leq \xi = t + |x| \leq \xi_4,\ \eta = t - |x| \geq \eta_0/2\}$$

for any $\xi_3 \in [2\xi_0, 4\xi_0]$, we then also obtain

$$\sup_{2\xi_0<\xi<4\xi_0} \int_{\Gamma_2(\xi)} (e-m)\,do \leq C\varepsilon.$$

Estimating as in (35), from (26) and (37) for any $2\xi_0 < \xi < 4\xi_0$ we likewise find the estimate

$$\begin{aligned}
\int_{\Gamma_2(\xi)} \frac{|u-\bar{v}|^2}{t^2}\,do &\leq C \int_{\Gamma_2(\xi_4)} \frac{|u-\bar{v}|^2}{t^2}\,do + C \int_{K^{T_\varepsilon}} (|u_\xi|^2 + |\bar{v}_t|^2)\,\frac{dx\,dt}{t} \\
&\leq C\varepsilon + C\,\mathrm{Flux}(u, M^{\xi_0}) \leq C\varepsilon.
\end{aligned}$$

Hence we obtain the uniform bound

$$\sup_{2\xi_0<\xi<4\xi_0} \int_{\Gamma_2(\xi)} \left((e-m) + \frac{|u-\bar{v}|^2}{t^2}\right) do \leq C\varepsilon. \qquad (38)$$

Fix a smooth cut-off function $0 \le \varphi_1 \le 1$ on $\mathbb{R}$ such that $\varphi_1(\eta) = 1$ for $\eta \ge 3\eta_0/4$ and $\varphi_1(\eta) = 0$ for $\eta \le \eta_0/2$, with $|\varphi_1'| \le 8/\eta_0 \le C/\xi_0$, and set $u_1 = \varphi_1(\eta)(u - \bar{v})$. For a point $z \in \Gamma(\xi)$ write $z = (t, y) = (\xi - |y|, y)$ with $y = re^{i\phi} \in B_{\xi/2}(0)$. Note that $\eta = t - r = \xi - 2|y|$. Letting $v_1(y) = u_1(\xi - |y|, y) \in H_0^1(B_{\xi/2}(0))$, for sufficiently small $\varepsilon > 0$ by (38) we have

$$\int_{B_{\xi/2}(0)} |\nabla v_1|^2 \, dy = \int_{\Gamma_2(\xi)} (4|\partial_\eta u_1|^2 + r^{-2}|\partial_\phi u_1|^2) \, do$$
$$\le C \int_{\Gamma_2(\xi)} \left((e - m) + \frac{|u - \bar{v}|^2}{t^2} \right) do \le C\varepsilon \le \pi/4,$$

uniformly in $2\xi_0 < \xi < 4\xi_0$. In view of (5) it follows that

$$\int_{\Gamma_1(\xi)} e^{16|u-\bar{v}|^2} \, dx \le \int_{\Gamma_2(\xi)} e^{16u_1^2} \, do \le C \int_{B_{\xi/2}(0)} e^{16v_1^2} \, dy \le C,$$

uniformly in $2\xi_0 < \xi < 4\xi_0$, with absolute constants $C > 0$.

Also let $0 \le \varphi_2 = \varphi_2(\eta) \le 1$ be a smooth cut-off function such that $\varphi_2(\eta) = 1$ for $\eta \le 3\eta_0/4$ and $\varphi_2(\eta) = 0$ for $\eta \ge \eta_0$, with $|\varphi_2'| \le 8/\eta_0 \le C/\xi_0$. Finally, fix a smooth cut-off function $0 \le \chi = \chi(\phi) \le 1$ satisfying $\chi(\phi) = 1$ for $|\phi| \le \pi/8$ and $\chi(\phi) = 0$ for $|\phi| \ge \pi/4$. Set $u_2 = \varphi_2(\eta)(u - \bar{v})$. After extending $u_2(\xi, \eta, \phi) = u_2(\xi, -\eta, \phi)$ for $\eta < 0$ for fixed ξ, also let $u_{2k} = u_{2k}(\xi, \eta, \phi) = \chi(\phi - k\pi/4)u_2$, $1 \le k \le 8$. Note that $u_{2k}(\xi, \cdot, \cdot) \in H_0^1([-\eta_0, \eta_0] \times [(k-1)\pi/4, (k+1)\pi/4])$, $1 \le k \le 8$, and we have

$$\int_{-\eta_0}^{\eta_0} \int_{(k-1)\pi/4}^{(k+1)\pi/4} (|\partial_\eta u_{2k}|^2 + r^{-2}|\partial_\phi u_{2k}|^2) r \, d\phi \, d\eta$$
$$\le C \int_{\Gamma_0(\xi)} \left((e - m) + \frac{|u - \bar{v}|^2}{t^2} \right) do \le C E_0,$$

whereas Lemma 4.1 yields the bound

$$\int_{-\eta_0}^{\eta_0} \int_{(k-1)\pi/4}^{(k+1)\pi/4} (r^{-2}|\partial_\phi u_{2k}|^2) r \, d\phi \, d\eta$$
$$\le C \int_{\Gamma_0(\xi)} \left(q + \frac{|u - \bar{v}|^2}{t^2} \right) do = C Q(\xi) \le C\sqrt{\varepsilon},$$

uniformly in $1 \le k \le 8$ and $2\xi_0 < \xi < 4\xi_0$. Also observe that

$$\frac{\xi_0 \lambda_0}{1 + \lambda_0} = \frac{\xi_0 - \eta_0}{2} \le r = \frac{\xi - \eta}{2} \le 2\xi_0 + \eta_0/2 \le 3\xi_0$$

for $2\xi_0 < \xi < 4\xi_0$ and $|\eta| \le \eta_0$. Thus by Lemma 4.2 for sufficiently small $\varepsilon > 0$ with an absolute constant $C > 0$ we find that

$$\sup_{2\xi_0 < \xi < 4\xi_0} \int_{\Gamma_0(\xi)} e^{16u_{2k}^2} \, dx \le C$$

uniformly in $1 \le k \le 8$.

Now observe that $|u| \le |u-\bar v| + |\bar v| \le 2\max\{|u-\bar v|, |\bar v|\}$. Thus, by choice of φ_1, φ_2, and χ, and in view of (11), we can bound

$$\begin{aligned}\int_{\Gamma(\xi)} e^{4u^2}\,do &\le \int_{\Gamma(\xi)} e^{16|u-\bar v|^2}\,do + \int_{\Gamma(\xi)} e^{16|\bar v|^2}\,do\\ &\le \int_{\Gamma_1(\xi)} e^{16u_1^2}\,do + \sum_{1\le k\le 8}\int_{\Gamma_0(\xi)} e^{16u_{2k}^2}\,do + \int_{\Gamma(\xi)} t^{-1}\,do \le C,\end{aligned}$$

uniformly in $2\xi_0 < \xi < 4\xi_0$. Hence for any $0 < \xi_0 < 8^{-1}T_\varepsilon$ with a constant C independent of ξ_0 we find

$$\int_{2\xi_0}^{4\xi_0}\int_{\Gamma(\xi)} e^{4u^2}\,do\,d\xi \le C\xi_0.$$

Note that the collection $(\Gamma(\xi))_{0<\xi<4\xi_0}$ covers the cone $K^{2\xi_0}$. Replacing ξ_0 by $2^{-k}\xi_0$ and adding the resulting estimates, after the change of variables $(t,x)\mapsto(\xi = t+|x|, x)$ we then obtain

$$\int_{K^{2\xi_0}} e^{4u^2}\,dx\,dt \le \sum_{k\in\mathbb{N}_0}\int_{2^{1-k}\xi_0}^{2^{2-k}\xi_0}\int_{\Gamma(\xi)} e^{4u^2}\,do\,d\xi \le C\sum_{k\in\mathbb{N}_0} 2^{-k}\xi_0 \le C\xi_0,$$

as desired. □

Proof of Theorem 1.1. Fix $\varepsilon > 0$, $0 < T \le 4^{-1}T_\varepsilon$ as in Lemma 4.3 and let $u^{(0)}$ be the solution to the homogeneous wave equation $u^{(0)}_{tt} - \Delta u^{(0)} = 0$ in K^T with initial data $u^{(0)}(T) = u(T)$, $u^{(0)}_t(T) = u_t(T)$. Multiplying the equation

$$(u-u^{(0)})_{tt} - \Delta(u-u^{(0)}) + ue^{u^2} = 0$$

with $(u-u^{(0)})_t$ and integrating over K^T_S, we obtain the estimate

$$\begin{aligned}\frac12\int_{B_S(0)} |D(u-u^{(0)})(S)|^2\,dx &\le \int_{K^T_S(0)} |(u-u^{(0)})_t|\,|u|e^{u^2}\,dx\,dt\\ &\le \left(\sup_{S\le t\le T}\int_{B_t(0)} |D(u-u^{(0)})(t)|^2\,dx\right)^{1/2}\left(T\int_{K^T_S(0)} u^2e^{2u^2}\,dx\,dt\right)^{1/2},\end{aligned}$$

where $D = (\partial_t, \nabla)$. Replacing S by a suitable $t\in[S,T]$, we arrive at

$$\sup_{S\le t\le T}\int_{B_t(0)} |D(u-u^{(0)})(t)|^2\,dx \le 4T\int_{K^T_S(0)} u^2e^{2u^2}\,dx\,dt \le 4T\int_{K^T(0)} e^{4u^2}\,dx\,dt.$$

But choosing $T > 0$ sufficiently small, in view of Lemma 4.3 we can achieve that

$$4T\int_{K^T(z_0)} e^{4u^2}\,dx\,dt < \varepsilon_0,$$

where $\varepsilon_0 > 0$ is the constant defined in Lemma 2.1. Since

$$\lim_{t\downarrow 0}\int_{B_t(0)} |Du^{(0)}(t)|^2\,dx = 0$$

and since by Lemma 3.1 we also have

$$\liminf_{t\downarrow 0}\int_{B_t(0)} e^{u(t)^2}\,dx = 0,$$

we then find that

$$\liminf_{t\downarrow 0} E(u(t), B_t(0)) < \varepsilon_0,$$

contradicting Lemma 2.1. The proof is complete. □

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