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On a question of Demailly–Peternell–Schneider

Dedicated to the memory of Eckart Viehweg

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Abstract. We give an affirmative answer to question posed by Demailly–Peternell–Schneider [DPS] in 2001 and recently by Peternell in [P]. Let $f : X \rightarrow Y$ be a surjective morphism from a log canonical pair (X, D) onto a \mathbb{Q} -Gorenstein variety Y . If $-(K_X + D)$ is nef, we show that $-K_Y$ is pseudo-effective.

1. Introduction

Projective varieties with numerically effective anticanonical bundles appear naturally in the minimal model program. In the surface case, these objects include del Pezzo surfaces, K3 surfaces, Abelian surfaces and so on. The higher dimensional case has been investigated intensively by several authors, for instance, Campana [C], Demailly–Peternell–Schneider [DPS], Peternell [P] and the second author [Z2]. A key step in understanding the geometry of these varieties is determining projective morphisms between them. Let $f : X \rightarrow Y$ be a surjective morphism with $-K_X$ nef. What can one say about $-K_Y$? In this direction, one of the main open questions was asked by Demailly–Peternell–Schneider [DPS] and recently by Peternell [P]:

Question A. *Let X be a complex projective manifold on which $-K_X$ is nef. Let $f : X \rightarrow Y$ be a projective morphism onto the projective manifold Y . Is $-K_Y$ pseudo-effective?*

Throughout, we work over the complex number field \mathbb{C} .

The main result in this paper is an affirmative answer to Question A. We prove the following stronger theorem:

Main Theorem. *Let X be a normal projective variety and D an effective \mathbb{Q} -divisor on X such that the pair (X, D) is log canonical. Let Y be a normal and \mathbb{Q} -Gorenstein projective variety. If $f : X \rightarrow Y$ is a surjective morphism and $-(K_X + D)$ is nef, then $-K_Y$ is pseudo-effective.*

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Remark 1.1. Naively, under the same condition as in the main theorem, one might expect that $-K_Y$ is nef. Unfortunately, this is false in general.

The following example was kindly provided to us by Y. Prokhorov:

Example 1.2. Embed $\mathbb{P}^1 \times \mathbb{P}^1$ into \mathbb{P}^5 through the linear system $|(1, 2)|$ to obtain a projective cone $W \subset \mathbb{P}^6$ over the surface $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ with the normal sheaf $\mathcal{N}_{S/W} \cong \mathcal{O}_S(-1, -2)$ and vertex $Q = [0, 0, 0, 0, 0, 1]$. Let $\pi : X \rightarrow W$ be the blow-up at the vertex $Q \in W \subset \mathbb{P}^6$. Then X is a nonsingular \mathbb{P}^1 -bundle over S with the exceptional divisor $E \cong S$. Since $K_X = \pi^*(K_W) + E$, $-K_X = -\pi^*(K_W) - E$ is nef, one of the curve families on S is K_X -trivial, while another curve family $\{l\}$ on S is K_X -negative. Mori's K -negative contraction $f = \text{Cont}_l : X \rightarrow Y$ maps S onto a curve $C = f(S) \cong \mathbb{P}^1$ on Y . By Mori's theorem [M], Y is nonsingular. According to Proposition 4.5 and Corollary 4.6 of Prokhorov [Pr], it follows that $K_Y \cdot C = 2 > 0$. Thus $-K_Y$ is not nef.

In the above example, f is birational. Recently, we learned from Fujino and Gongyo's paper [FG] that H. Sato had constructed the following example where $f : X \rightarrow Y$ is flat:

Example 1.3. Let Σ be the fan in \mathbb{R}^3 whose rays are generated by $x_1 = (1 : 0 : 1)$, $x_2 = (0 : 1 : 0)$, $x_3 = (-1 : 3 : 0)$, $x_4 = (0 : -1 : 0)$, $y_1 = (0 : 0 : 1)$ and $y_2 = (0 : 0 : -1)$. Their maximal cones are $\langle x_1, x_2, y_1 \rangle$, $\langle x_1, x_2, y_2 \rangle$, $\langle x_2, x_3, y_1 \rangle$, $\langle x_2, x_3, y_2 \rangle$, $\langle x_3, x_4, y_1 \rangle$, $\langle x_3, x_4, y_2 \rangle$, $\langle x_4, x_1, y_1 \rangle$ and $\langle x_4, x_1, y_2 \rangle$. One can see that $X(\Sigma)$, the toric threefold corresponding to the fan Σ , has a \mathbb{P}^1 -fibration f onto $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(3))$ (see [FG] for more details). In fact, X is a weak Fano threefold, but $-K_Y$ is not nef.

Remark 1.4. If we replace the condition that $-(K_X + D)$ is nef by $-(K_X + D)$ pseudo-effective, then the conclusion is again false. This may explain the motivation behind Question A.

The following example is found in [Z1]:

Example 1.5. Let C be a smooth curve of genus g and A an ample line bundle on C such that $\deg A > 2 \deg K_C$. Let $S = \text{Proj}_C(\mathcal{O}_C(A) \oplus \mathcal{O}_C)$. Then $K_S = \pi^*(K_C + A) - 2L$, where L is the tautological bundle. Clearly $-K_S$ is big (since $h^0(S, -mK_S) \approx c \cdot m^2$ for $m \gg 0$ and some constant $c > 0$). Thus $-K_S$ is pseudo-effective. In this setting, S is a \mathbb{P}^1 -bundle over C , but $-K_C$ is never pseudo-effective if $g > 1$. *This example suggests the necessity of the condition that (X, D) is log canonical. Indeed, if we let $D = -K_S = 2(L - \pi^*A) + \pi^*(A - K_C)$, the pair (S, D) is not log canonical.*

2. Proof of the Main Theorem

Before we prove our Main Theorem, let us recall some related definitions and known results. For general references on the minimal model program, see (for instance) [K-M, KMM].

Definition 2.1 (Viehweg [V]). Let X be a smooth (quasi-)projective variety and \mathcal{F} a torsion-free coherent sheaf on X . We say that \mathcal{F} is *weakly positive on X* if there exists some Zariski open subvariety $U \subset X$ such that for every ample invertible sheaf \mathcal{H} and every positive integer m , there exists some positive integer n such that $\hat{S}^{mn}(\mathcal{F}) \otimes \mathcal{H}^n$ is generated by global sections over U , where \hat{S}^n denotes the reflexive hull of S^n .

In the proof of our theorem, we need the following weak positivity statement that was originally developed by Viehweg [V] and, later on, generalized by Campana [C] and Lu [L].

Lemma 2.2. *Let V be a smooth projective variety and M an effective \mathbb{Q} -divisor on V such that the pair (V, M) is log canonical. If $\varphi : V \rightarrow W$ is a surjective morphism onto the nonsingular variety W , then $\varphi_*m(K_{V/W} + M)$ is torsion free and weakly positive for every $m \in \mathbb{Z}_{>0}$ such that $m(K_{V/W} + M)$ is Cartier.*

Proof of the Main Theorem. Let $p : Y' \rightarrow Y$ be a resolution of Y . Let $\pi : X' \rightarrow X$ be a log resolution of (X, D) such that the induced rational map $X' \dashrightarrow Y'$ is in fact a morphism $g' : X' \rightarrow Y'$. We have the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ g' \downarrow & & \downarrow f \\ Y' & \xrightarrow{p} & Y \end{array}$$

Since the pair (X, D) is log canonical, we have

$$K_{X'} = \pi^*(K_X + D) + \sum d_i E_i,$$

where $d_i \geq -1$ for each i and $\sum E_i$ is a divisor with normal crossings only. We may also assume that the support of $f'^{-1}(\text{Sing}(Y))$ is a divisor with normal crossings only, where $f' = f \circ \pi$. Set $Y^0 := Y - \text{Sing}(Y)$.

Pick an ample divisor L on Y and a rational number $\delta > 0$. Our goal is to show that $-K_Y + 2\delta L$ is equivalent to an effective \mathbb{Q} -divisor, which implies that $-K_Y$ is pseudo-effective.

Let A be a sufficiently very ample divisor on X . By Serre’s theorem, for every rational number $\varepsilon > 0$ and every divisible integer $m > 0$ such that ε/δ is sufficiently small and $m\delta$ and $m\varepsilon$ are integers, the sheaf

$$\mathcal{O}_Y(m\delta L) \otimes (f'_* \mathcal{O}_{X'}(m\varepsilon \pi^*(A)))^*$$

is generated by global sections. Thus, for all integers $n > 0$, it follows that

$$\mathcal{O}_Y(nm\delta L) \otimes \bigotimes^n (f'_* \mathcal{O}_{X'}(m\varepsilon \pi^*(A)))^* \quad \text{and} \quad \mathcal{O}_Y(nm\delta L) \otimes S^n((f'_* \mathcal{O}_{X'}(m\varepsilon \pi^*(A)))^*)$$

are both generated by global sections.

Since $-(K_X + D)$ is nef by assumption, $-(K_X + D) + \varepsilon A$ is \mathbb{Q} -ample. So we may assume that $-\pi^*(K_X + D) + \varepsilon\pi^*(A)$ is \mathbb{Q} -equivalent to an effective \mathbb{Q} -divisor. We may also assume that

$$\text{Supp}\left(-\pi^*(K_X + D) + \varepsilon\pi^*(A) + \sum E_i\right)$$

is a divisor with normal crossings only. Let $\Delta = -\pi^*(K_X + D) + \varepsilon\pi^*(A) + \sum a_i E_i$, where $a_i = -d_i$ whenever $d_i < 0$ and $a_i = 0$ otherwise.

We may write

$$K_{X'/Y} + \Delta \sim_{\mathbb{Q}} \varepsilon\pi^*(A) - f'^*K_Y + \hat{\Delta}$$

where $\hat{\Delta} := \sum_{j \in J} d_j E_j$, $d_j \geq 0$ and E_j is π -exceptional for each $j \in J$. Clearly the pair (X', Δ) is log canonical.

Choose a divisible sufficiently large integer $m > 0$ such that $m\varepsilon A$, $m\delta L$, $md_j E_j$ and $-mK_Y$ are all Cartier and define

$$\omega := m(\varepsilon\pi^*(A) - f'^*K_Y + \hat{\Delta}).$$

By Lemma 2.2, there is a Zariski open smooth subvariety $U \subset Y$ with $\text{codim}(Y - U) \geq 2$ such that

$$f'_*\omega = f'_*\mathcal{O}_{X'}(m\varepsilon\pi^*(A) + \hat{\Delta}) \otimes \mathcal{O}_Y(-mK_Y)$$

is locally free and weakly positive over U . Recall that δL is an ample \mathbb{Q} -divisor on Y . Since each E_j ($j \in J$) is exceptional with respect to π , the effect of $\hat{\Delta}$ can be neglected when $f'_*\omega$ is restricted to U . Thus

$$f'_*\mathcal{O}_{X'}(m\varepsilon\pi^*(A))|_U = f'_*\omega|_U \otimes \mathcal{O}_Y(mK_Y)|_U$$

is locally free over U . Furthermore, weak positivity implies that

$$\begin{aligned} & (\mathcal{S}^n f'_*\omega)|_U \otimes \mathcal{O}_Y(nm\delta L)|_U \\ &= \mathcal{S}^n(f'_*\mathcal{O}_{X'}(m\varepsilon\pi^*(A) + \sum md_j E_j))|_U \otimes \mathcal{O}_Y(-nmK_Y)|_U \otimes \mathcal{O}_Y(nm\delta L)|_U \\ &= \mathcal{S}^n(f'_*\mathcal{O}_{X'}(m\varepsilon\pi^*(A)))|_U \otimes \mathcal{O}_Y(-nmK_Y)|_U \otimes \mathcal{O}_Y(nm\delta L)|_U \end{aligned}$$

is generated by global sections at the generic point of U for some sufficiently large n . Since there is a functorial isomorphism between $\mathcal{S}^n(\mathcal{F}^*)$ and $(\mathcal{S}^n \mathcal{F})^*$ for locally free sheaves, we have a nontrivial trace map

$$\mathcal{S}^n((f'_*\mathcal{O}_{X'}(m\varepsilon\pi^*(A)))^*)|_U \otimes \mathcal{S}^n(f'_*\mathcal{O}_{X'}(m\varepsilon\pi^*(A)))|_U \rightarrow \mathcal{O}_U.$$

As we have seen,

$$\mathcal{O}_Y(nm\delta L)|_U \otimes \mathcal{S}^n((f'_*\mathcal{O}_{X'}(m\varepsilon\pi^*(A)))^*)|_U$$

is generated by global sections over U . Thus

$$\begin{aligned} \mathcal{E}_U &:= \mathcal{O}_Y(nm\delta L)|_U \otimes \mathcal{S}^n((f'_*\mathcal{O}_{X'}(m\varepsilon\pi^*(A)))^*)|_U \otimes (\mathcal{S}^n f'_*\omega)|_U \otimes \mathcal{O}_Y(nm\delta L)|_U \\ &= \mathcal{O}_Y(-nmK_Y + 2nm\delta L)|_U \\ &\quad \otimes (\mathcal{S}^n((f'_*\mathcal{O}_{X'}(m\varepsilon\pi^*(A)))^*)|_U \otimes \mathcal{S}^n(f'_*\mathcal{O}_{X'}(m\varepsilon\pi^*(A)))|_U) \end{aligned}$$

is generated by global sections at the generic point of U . There are induced maps

$$\bigoplus \mathcal{O}_U \xrightarrow{\alpha} \mathcal{E}_U \xrightarrow{\beta} \mathcal{O}_Y(-nmK_Y + 2nm\delta L)|_U$$

such that α is surjective and β is nonzero. This proves that

$$H^0(U, \mathcal{O}_Y(-nmK_Y + 2nm\delta L)|_U) \neq 0.$$

Because $\text{codim}(Y - U) \geq 2$, the divisor $n(-mK_Y + 2m\delta L)$ is effective on Y . \square

As instant applications, we can improve some earlier results in [Z1, Z2].

Corollary 2.3. *Let X be a normal projective variety and D an effective \mathbb{Q} -divisor on X such that the pair (X, D) is log canonical and that $-(K_X + D)$ is nef. Let $f : X \rightarrow Y$ be a surjective morphism onto Y which is a \mathbb{Q} -Gorenstein projective variety. Then either*

- (1) Y is uniruled, or
- (2) $K_Y \sim_{\mathbb{Q}} 0$.

Proof. If Y is not uniruled, then K_Y is pseudo-effective by [BDPP]. However our Main Theorem says $-K_Y$ is pseudo-effective, which means $K_Y \sim_{\mathbb{Q}} 0$. \square

Remark 2.4. The weaker statement corresponding to (2) in [Z2] is that the Kodaira dimension $\kappa(Y)$ is zero.

Corollary 2.5 (cf. [Z1]). *Let X be a smooth projective variety with $-K_X$ nef. Then the Albanese map $\text{Alb}_X : X \rightarrow \text{Alb}(X)$ is surjective and has connected fibers.*

Proof. Let $a : X \rightarrow V$ be the induced morphism with connected fibers after the Stein factorization of Alb_X . We may even assume that V is smooth. Then $\kappa(V) = 0$ by the Main Theorem, and Kawamata's theorem [K] implies that V is an abelian variety. Thus it follows that $V = \text{Alb}(X)$ due to the universal property of Alb_X . \square

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