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A support theorem for Hilbert schemes of planar curves

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Abstract. Consider a family of integral complex locally planar curves whose relative Hilbert scheme of points is smooth. The decomposition theorem of Beilinson, Bernstein, and Deligne asserts that the pushforward of the constant sheaf on the relative Hilbert scheme splits as a direct sum of shifted semisimple perverse sheaves. We will show that no summand is supported in positive codimension. It follows that the perverse filtration on the cohomology of the compactified Jacobian of an integral plane curve encodes the cohomology of *all* Hilbert schemes of points on the curve. Globally, it follows that a family of such curves with smooth relative compactified Jacobian ("moduli space of D-branes") in an irreducible curve class on a Calabi–Yau threefold will contribute equally to the BPS invariants in the formulation of Pandharipande and Thomas, and in the formulation of Hosono, Saito, and Takahashi.

1. Introduction

In this note a *curve* will always be *integral, complete, locally planar, and defined over* $\mathbb{C}$.[1]

Let C be a curve of arithmetic genus g. The Hilbert scheme of points $C^{[d]}$ parameterizes length d subschemes of C; it is complete, integral, d-dimensional, and l.c.i. [AIK, BGS]. If $\pi : \mathcal{C} \to B$ is a family of curves, there is a relative Hilbert scheme $\pi^{[d]} : \mathcal{C}^{[d]} \to B$ with fibres $(\mathcal{C}^{[d]})_b = (\mathcal{C}_b)^{[d]}$. Planarity of the curves ensures the existence of families in which the total space of $\mathcal{C}^{[d]}$ is smooth ([S]), see Theorem 8 below; ultimately this is a consequence of the smoothness of the Hilbert scheme of points on a surface. When $\mathcal{C}^{[d]}$ is smooth, the decomposition theorem of Beilinson, Bernstein and Deligne [BBD] applied to the proper map $\pi^{[d]} : \mathcal{C}^{[d]} \to B$ asserts that $\mathrm{R}\pi^{[d]}_*\mathbb{C}$ decomposes as a direct sum of shifted intersection complexes associated to local systems on constructible subsets of the base.

Let $\widetilde{\pi} : \widetilde{\mathcal{C}} \to \widetilde{B}$ denote the restriction of π to the smooth locus. The Hilbert schemes of a smooth curve are its symmetric products, and in particular the map $\widetilde{\pi}^{[d]}$ is smooth. Thus the summand of $\mathrm{R}\pi^{[d]}_*\mathbb{C}[d+\dim B]$ with support equal to B is $\bigoplus \mathrm{IC}(B, \mathrm{R}^{d+i}\widetilde{\pi}^{[d]}_*\mathbb{C})[-i]$.

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[1] This reflects a limitation of the authors rather than a certainty that the methods do not work in characteristic p.

As pointed out by Macdonald [M], the cohomology of the symmetric products is expressed in terms of the cohomology of the curves by the formula

$$\mathrm{R}^i\widetilde{\pi}_*^{[d]}\mathbb{C} = \bigoplus_{k=0}^{\lfloor i/2\rfloor}(\textstyle\bigwedge^{i-2k}\mathrm{R}^1\widetilde{\pi}_*\mathbb{C})(-k) = (\mathrm{R}^{2d-i}\widetilde{\pi}_*^{[d]}\mathbb{C})(d-i) \quad \text{for } i \leq d. \tag{1}$$

Even given this expression, computing $\mathrm{IC}(B, \mathrm{R}^i\widetilde{\pi}_*^{[d]}\mathbb{C})$ is a nontrivial matter, about which we say nothing here. But at least $\mathrm{R}\pi_*^{[d]}\mathbb{C}[d+\dim B]$ contains no other summands:

Theorem 1. *Let $\pi : \mathcal{C} \to B$ be a family of integral plane curves, and let $\widetilde{\pi} : \widetilde{\mathcal{C}} \to \widetilde{B}$ be its restriction to the smooth locus. If $\mathcal{C}^{[d]}$ is smooth, then*

$$\mathrm{R}\pi_*^{[d]}\mathbb{C}[d+\dim B] = \bigoplus_{i=-d}^{d}\mathrm{IC}(B, \mathrm{R}^{d+i}\widetilde{\pi}_*^{[d]}\mathbb{C})[-i].$$

From now on we will use the notation

$${}^p\mathrm{R}^i\pi_*^{[d]}\mathbb{C}[d+\dim B] := {}^p\mathcal{H}^i(\mathrm{R}\pi_*^{[d]}\mathbb{C}[d+\dim B])$$

for the perverse cohomology sheaves of $\mathrm{R}\pi_*^{[d]}\mathbb{C}[d+\dim B]$.

The central term of (1) can be reinterpreted in terms of the family of Jacobians of curves. Indeed, taking $\widetilde{\pi}^J : J(\widetilde{\mathcal{C}}) \to \widetilde{B}$ to be the family of Jacobians over the smooth locus, there is a (canonical) identification of local systems

$$\mathrm{R}^i\widetilde{\pi}_*^J\mathbb{C} = \textstyle\bigwedge^i(\mathrm{R}^1\widetilde{\pi}_*\mathbb{C}). \tag{2}$$

Consequently,

$$\mathrm{R}^i\widetilde{\pi}_*^{[d]}\mathbb{C} = \bigoplus_k(\mathrm{R}^{i-2k}\widetilde{\pi}_*^J\mathbb{C})(-k) = (\mathrm{R}^{2d-i}\widetilde{\pi}_*^{[d]}\mathbb{C})(d-i) \quad \text{for } i \leq d. \tag{3}$$

It can be convenient to express (1)–(3) in the following formula:

$$\sum_{d=0}^{\infty}\sum_{i=0}^{2d}q^d\mathrm{R}^i\widetilde{\pi}_*^{[d]}\mathbb{C} = \frac{\sum_{i=0}^{2g}q^i\bigwedge^i(\mathrm{R}^1\widetilde{\pi}_*\mathbb{C})}{(1-q\mathbb{C})(1-q\mathbb{C}(-1))} = \frac{\sum_{i=0}^{2g}q^i\mathrm{R}^i\widetilde{\pi}_*^J\mathbb{C}}{(1-q\mathbb{C})(1-q\mathbb{C}(-1))}. \tag{4}$$

The family of Jacobians can be extended over the singular locus of π to the *compactified Jacobian* [AK], $\pi^J : \bar{J}^d(\mathcal{C}) \to B$, whose fibre $\bar{J}^d(\mathcal{C})_b = \bar{J}^d(\mathcal{C}_b)$ parameterizes rank one, degree d torsion free sheaves on $\mathcal{C}$.[2] The map π^J is proper, and for Gorenstein curves there is an Abel–Jacobi map $AJ : \mathcal{C}^{[d]} \to \bar{J}^d(\mathcal{C})$ taking a subscheme to the dual of its ideal sheaf.[3] For $d > 2g-2$, the map AJ is a $\mathbb{P}^{d-g}$-bundle, and

[2] It also extends to the *generalized Jacobian* $J(\mathcal{C})$ whose fibre $J(\mathcal{C})_b$ parameterizes line bundles on $\mathcal{C}_b$; this is a commutative group scheme of dimension g of which the affine part is of dimension $\delta(\mathcal{C}_b)$. This is a subscheme of the compactified Jacobian, and acts on it. Such actions are central to Ngô's arguments, but play no role here.

[3] In general, it is better to define the Abel–Jacobi map from the Quot scheme of the dualizing sheaf (see [AK]).

$\mathrm{R}\,AJ_*\mathbb{C} = \bigoplus_{i=0}^{d-g} \mathbb{C}[-2i]$; thus the statement in Theorem 1 is true in this range for the map π^J as well. Over a sufficiently small open set, π admits a section σ with image in the smooth locus of the curves; twisting by $\mathcal{O}(\sigma)$ identifies the $\bar{J}^d(\mathcal{C})$ for varying d and so $\pi^J_* \mathbb{C}$ does not depend on d. In particular, we recover the support theorem for the map π^J, a special case of Ngô's support theorem [N]. It can be shown [S, Prop. 14] that smoothness of the relative compactified Jacobian implies smoothness of all relative Hilbert schemes. Therefore taking IC sheaves in (1) and (2) yields the following corollary.

Corollary 2. *Let $\pi : \mathcal{C} \to B$ be a family of integral plane curves of arithmetic genus g. If the relative compactified Jacobian $\bar{J}(\mathcal{C})$ is smooth, then*

$$ {}^p\mathrm{R}^{i-d} \pi^{[d]}_* \mathbb{C}[d + \dim B] = \bigoplus_{k=0}^{\lfloor i/2 \rfloor} {}^p\mathrm{R}^{i-g-2k} \pi^J_* \mathbb{C}[g + \dim B](-k) \quad \text{for } 0 \le i \le d. $$

(The ${}^p\mathrm{R}^{i-d}$ for $i > d$ are determined similarly by duality.)

This corollary has a consequence for the enumerative geometry of Calabi–Yau threefolds, which we briefly sketch. Gopakumar and Vafa argued in [GV] that the cohomology of the moduli space of D-branes (roughly speaking, semistable sheaves supported on curves) on a Calabi–Yau Y should give rise to *integer* "BPS" invariants, one for each genus and homology class in $\mathrm{H}_2(Y, \mathbb{Z})$, which encode the Gromov–Witten invariants of Y. Hosono, Saito, and Takahashi [HST] use intersection cohomology and the tools of [BBD] to give a precise formulation; however, their proposal is known *not* to give the desired BPS numbers in general [BP]. A different definition of integer BPS invariants is given by Pandharipande and Thomas [PT] using the closely related spaces of "stable pairs", which for integral planar curves are just the Hilbert schemes of points. By the work of Behrend [B], the BPS invariants are extracted by a weighted Euler characteristic of these spaces, the weighting function depending only on the singularities of the moduli space. For BPS invariants associated to *irreducible* homology classes, it is sensible to discuss the contribution of an individual curve in both theories; if the moduli space of sheaves on Y is smooth along the locus of sheaves supported on a curve C, then the intersection cohomology considerations may be neglected in [HST], and likewise the weighting function of Behrend may be neglected in [PT]. In this case, taking Euler characteristics in the Corollary yields the equality of the contributions of the curve C to these two theories. Such curves will certainly appear if Y contains a Fano surface, and indeed the enumerative geometry of curves on surfaces is also illuminated by the stable pairs spaces [KST, KT].

The Hilbert schemes also appear in the conjecture of Oblomkov, Rasmussen, and the second author [OS, ORS]. This relates the cohomology of the Hilbert schemes of points on a locally planar curve to the Khovanov–Rozansky HOMFLY homology of the links of its singularities. The HOMFLY homology is a vector space carrying three gradings, and its Poincaré polynomial $\overline{\mathcal{P}}$ is written in the variables a, q, t. When the curve has a unique singularity, the conjecture implies that, up to normalization, the lowest coefficient of a in $\overline{\mathcal{P}}$ is $\sum q^{2n} \mathfrak{w}(C^{[n]})$, where $\mathfrak{w}$ is the weight polynomial $\mathfrak{w}(X) := \sum_{i,j} t^i (-1)^{i+j} \dim \mathrm{Gr}^i_W \mathrm{H}^j_c(X)$. The present work shows that the series on the RHS may be extracted instead from the perverse filtration on the cohomology of the compactified Jacobian of C. The perverse filtration is surely no easier to compute directly than the Hilbert scheme series, but at least in special cases the cohomology of the compactified Jacobian (under its alias of affine Springer fibre [L]) appears in the geometric construction of the spherical representations of the rational Cherednik algebra [VV]. The perverse filtration interacts well with these geometric constructions, and as a consequence the series above can sometimes be computed after passing to other incarnations of the Cherednik algebra and its representations which are more suitable for computations. Details appear in [ORS].

Theorem 1 is inspired by the support theorem of B. C. Ngô [N], and is a consequence of it when $d > 2g - 2$. Nonetheless our proofs—we give two—do not logically depend on his work, and rely on the deformation theory results in [S]; in particular, the analogue of the crucial "δ-regularity" assumption in [N] is automatically satisfied in our case once the total space $\mathcal{C}^{[d]}$ is smooth (see Corollary 9).

Conventions. We follow [BBD] in declaring $\mathcal{F} \in \mathrm{D}^b_c(X)$ perverse when $\dim \mathrm{Supp}\mathcal{H}^i(\mathcal{F}) \leq -i$, and the same holds for the Verdier dual. That is, if X is smooth and n-dimensional, $\mathbb{C}[n]$ is perverse. In arguments of a topological nature, we omit Tate twists. As mentioned at the outset, all curves are integral and have singularities of embedding dimension 2. All families of curves will enjoy a smooth base. For a curve C, we denote by $\overline{C}$ its normalization, and write $\delta(C)$ for the difference between its arithmetic and geometric genera, which we term the *cogenus*: $\delta(C) = p_a(C) - p_a(\overline{C})$.

2. Background on relative Hilbert schemes and versal deformations

The Hilbert schemes of points on integral planar curves are singular, but not hopelessly so:

Theorem 3 ([AIK, BGS]). *Let C be a complete integral planar curve. Then $C^{[d]}$ is integral, complete, d-dimensional, and locally a complete intersection.*

We systematically employ versal deformations of curve singularities. We will always mean this in the sense of analytic spaces (see [GLS] for a thorough treatment). The base of a versal deformation of a plane curve singularity is smooth. If $\pi : \mathcal{C} \to B$ is a family of curves, we say it is *locally versal* at b if it induces versal deformations of all the singularities of $\mathcal{C}_b$. This last condition may be rephrased as follows ([FGvS, Section A]): letting $\overline{\mathbb{V}}(\mathcal{C}_b)$ be the product of the versal deformations of the singularities of $\mathcal{C}_b$, there is a natural tangent map $T_b B \to T_b \overline{\mathbb{V}}(\mathcal{C}_b)$ at b coming from the local-to-global spectral sequence for first order deformations of $\mathcal{C}_b$. The family is locally versal if this tangent map is surjective. Such families have in particular the following properties:

Theorem 4 ([DH, T]). *Let $\pi : \mathcal{C} \to B$ be a family of curves. The cogenus is an upper semicontinuous function on B. Local versality is an open condition, and in a locally versal family the locus of curves of cogenus at least δ is equal to the closure of the locus of δ-nodal curves. In particular, the locus of curves of cogenus δ in a locally versal family has codimension δ.*

Any curve singularity can be found on a rational curve; for an explicit construction see e.g. [L]. Moreover, if $\mathcal{C} \to B$ is a family of curves, then locally near $b \in B$ one can find a different family $\mathcal{C}' \to B$ such that $\mathcal{C}'_b$ is rational with the same singularities as $\mathcal{C}_b$ and the two families induce the same deformations of the singularities of the central fibre.

Proposition 5 ([FGvS]). *The map from the base of a versal deformation of an integral locally planar curve to the product of the versal deformations of its singularities is a smooth surjection.*

Corollary 6. *Let $\pi : \mathcal{C} \to B$ be a family of curves. Fix $b \in B$, and let $\overline{\mathcal{C}_b}$ be the normalization of $\mathcal{C}_b$. Then there exists a neighbourhood $b \in U \subset B$ and a family $\pi' : \mathcal{C}' \to U$ such that $\mathcal{C}'_b$ is rational with the same singularities as $\mathcal{C}_b$, and $\mathcal{C}$ and $\mathcal{C}'$ induce the same deformations of these singularities on U, and in particular have the same discriminant locus. Moreover, on U, we have an equality of local systems $\mathrm{R}^1\widetilde{\pi}_*\mathbb{C} = \mathrm{R}^1\widetilde{\pi}'_*\mathbb{C} \oplus \mathrm{H}^1(\overline{\mathcal{C}_b})$, the latter summand meaning the constant local system with the specified fibre.*

Proof. Let C' be a rational curve with the same singularities as $\mathcal{C}_b$; let $\mathcal{C}' \to \mathbb{V}(C')$ be a versal deformation of C', and, as above, let $\overline{\mathbb{V}}(\mathcal{C}_b)$ be the product of the versal deformations of the singularities of $\mathcal{C}_b$. We identify $\overline{\mathbb{V}}(\mathcal{C}_b) = \overline{\mathbb{V}}(C')$, and, by Proposition 5, the map $\mathbb{V}(C') \to \overline{\mathbb{V}}(\mathcal{C}_b)$ is a smooth surjection, so we may choose, possibly after shrinking, a local section σ. Pulling back, again possibly after shrinking B, the family $\mathcal{C}' \to \mathbb{V}(C')$ to B via the composition

$$B \to \mathbb{V}(\mathcal{C}_b) \to \overline{\mathbb{V}}(\mathcal{C}_b) \xrightarrow{\sigma} \mathbb{V}(C')$$

we obtain a family of rational curves $\pi' : \mathcal{C}'_B \to B$.

Shrink $U \subset B$ further so that the inclusion $\mathcal{C}_b \to \mathcal{C}|_U$ is a homotopy equivalence, and let $\tilde{b} \in U$ be a point with smooth fibre $\mathcal{C}_{\tilde{b}}$. Let $\mathcal{V}$ be the summand of $R^1\widetilde{\pi}_*\mathbb{C}$ whose fibre at $\tilde{b}$ is the kernel of the composition of the specialization map $\mathrm{H}^1(\mathcal{C}_{\tilde{b}}) \to \mathrm{H}^1(\mathcal{C}_b)$ with the pullback to the normalization $\mathrm{H}^1(\mathcal{C}_b) \to \mathrm{H}^1(\overline{\mathcal{C}_b})$. This is a symplectic summand; let $\mathcal{V}^\perp$ be its orthogonal complement. As $\mathcal{V}$ contains all vanishing cycles, the Picard–Lefschetz formula ensures $\mathcal{V}^\perp$ has trivial monodromy and thus extends to a trivial local system over B with fibre $\mathcal{V}^\perp_{\tilde{b}} = \mathrm{H}^1(\overline{\mathcal{C}}_b)$. On the other hand, $\mathcal{V}$ depends only on the deformation of the singularities, which is the same in $\mathcal{C}$ and $\mathcal{C}'$. ◻

To make use of such a replacement, it is necessary to know that the relative Hilbert scheme $\mathcal{C}'^{[d]}$ is smooth if $\mathcal{C}^{[d]}$ is. This follows from results of the second author on the smoothness of relative Hilbert schemes [S], which we now review. Recall that in view of our conventions, the base B of a family is always supposed to be smooth.

Proposition 7 ([S, Prop. 14]). *Let $\pi : \mathcal{C} \to B$ be a family of curves. If $\mathcal{C}^{[d]}$ is smooth, then $\mathcal{C}^{[n]}$ is smooth for any $n \leq d$.*

In particular, if $n = 1$, then $\mathcal{C}$ must be smooth.

Theorem 8. *Let $\mathcal{C} \to B$ be a family of curves. For $b \in B$, let I be the image of $T_b B$ in the product of the first-order deformations of the singularities of $\mathcal{C}_b$. Then:*

(1) *The smoothness of $\mathcal{C}^{[d]}$ along $\mathcal{C}_b^{[d]}$ depends only on I.*
(2) *If $\mathcal{C}^{[d]}$ is smooth along $\mathcal{C}_b^{[d]}$, then $\dim I \geq \min(d, \delta(\mathcal{C}_b))$.*
(3) *If $\dim I \geq d$ and I is general among such subspaces, then $\mathcal{C}^{[d]}$ is smooth along $\mathcal{C}_b^{[d]}$.*
(4) *$\mathcal{C}^{[d]}$ is smooth along $\mathcal{C}_b^{[d]}$ for all d if and only if I is transverse to the image of the "equigeneric ideal". It suffices for I to be generic of dimension at least $\delta(\mathcal{C}_b)$.*

Proof. For (1), take a subscheme z of $\mathcal{C}_b^{[d]}$ which decomposes as $z = \coprod z_i$ into subschemes of lengths d_i supported at points c_i. Let $(\overline{\mathcal{C}}_i, c_i) \to (\mathbb{V}_i, 0)$ be miniversal deformations of the curve singularities $(\mathcal{C}_b, c_i)$ and $(B, b) \to \prod(\mathbb{V}_i, 0)$ a map along which the multi-germ $\coprod(\mathcal{C}, c_i) \to (B, b)$ pulls back. Then analytically locally the germ $(\mathcal{C}^{[d]}, [z])$ pulls back from $\prod(\overline{\mathcal{C}}_i^{[d_i]}, [z_i])$ along the same map. As the fibres of $(\overline{\mathcal{C}}_i^{[d_i]}, [z_i]) \to (\mathbb{V}_i, 0)$ are reduced of dimension d_i by [AIK, BGS] and the total space is smooth by [S, Prop. 17], the smoothness of the pullback depends only on the image of $T_b B$ in $\prod T_0 \mathbb{V}_i$, which is well defined as the $\mathbb{V}_i$ were taken miniversal. The miniversal deformation of the germ of a curve at a smooth point being trivial, only the singularities contribute.

To check (2), we may by (1) assume the map $T_b B \to I$ is an isomorphism and then identify locally B with its image in some representative $\overline{B}$ of $\prod(\mathbb{V}_i, 0)$. Shrink $\overline{B}$ until it can be written as $B \times \mathbb{D}$ for some polydisc $\mathbb{D}$; by openness of smoothness we may shrink $\mathbb{D}$ further until $\mathcal{C}^{[d]}|_{B\times\epsilon}$ is smooth for all $\epsilon \in \mathbb{D}$. It is known [DH, T] that the locus of nodal curves with $\delta(\mathcal{C}_b)$ nodes in $\prod \mathbb{V}_i$ is nonempty (and of codimension $\delta(\mathcal{C}_b)$); choose ϵ so that the slice $B \times \epsilon$ contains such a point p corresponding to such a curve. If $d \leq \delta$, there is a point $z \in \mathcal{C}_p^{[d]}$ naming a subscheme supported at d nodes. The Zariski tangent space $T_z \mathcal{C}_p^{[d]}$ is $2d$-dimensional, so $\mathcal{C}_p^{[d]}$ cannot be smoothed out over a base of dimension less than d.

For (3), again assume B is embedded in $\overline{B} = \prod \mathbb{V}_i$; now the situation is analytically locally smooth over that in [S, Thm. 19]. Finally, (4) is stated in [FGvS] for the compactified Jacobian; it follows for $\mathcal{C}^{[d]}$ for $d \gg 0$ because this space fibres smoothly over the Jacobian, and for lower d by Proposition 7. □

Corollary 9. *If $\mathcal{C} \to B$ is a family of curves with $\mathcal{C}^{[d]}$ smooth, then for $\delta \leq d$, the locus of curves with cogenus δ is of codimension at least δ in B.*

Proof. Suppose not; let B' be a generic $\delta - 1$-dimensional smooth subvariety of B. Then the restriction $\mathcal{C}^{[d]} \times_B B'$ is smooth and B' intersects the locus of curves of cogenus δ. This contradicts (2) of Theorem 8. □

Remark. Corollary 9 explains why we do not require a "δ-regularity" assumption as in [N]—in the case of Hilbert schemes and Jacobians, it follows from smoothness of the total space.

3. Estimates

The following is a variation on the "Goresky–MacPherson inequality" of [N, Section 7.3].

Lemma 10. *Let $\pi : X \to Y$ be a locally projective morphism of smooth varieties with irreducible fibres of dimension n. Then*

(1) ${}^p\mathrm{R}^j\pi_*\mathbb{C}[\dim X] = 0$ *if* $|j| > n$, *and* ${}^p\mathrm{R}^{\pm n}\pi_*\mathbb{C}[\dim X] = \mathbb{C}[\dim Y]$.
(2) $\mathcal{H}^i({}^p\mathrm{R}^j\pi_*\mathbb{C}[\dim X]) = 0$ *for* $|j| \neq n$ *and* $i \geq n - \dim Y - |j|$.

In particular, every summand of $\mathrm{R}\pi_*\mathbb{C}$ *is supported on a subvariety of codimension* $< n$.

Proof. The first statement follows immediately from the fact that the fibres are connected of dimension n. The estimate is symmetric in j and, by relative hard Lefschetz, ${}^p\mathrm{R}^j\pi_*\mathbb{C}[\dim X] \simeq {}^p\mathrm{R}^{-j}\pi_*\mathbb{C}[\dim X]$, thus we may assume $j \geq 0$. We check at a point $y \in Y$, where by [BBD], $\mathcal{H}^i({}^p\mathrm{R}^j\pi_*\mathbb{C}[\dim X])_y$ is a summand of $\mathrm{H}^{i+j+\dim X}(X_y, \mathbb{C})$. This vanishes for dimension reasons if $i+j+\dim X = i+j+\dim Y+n > 2\dim X_y = 2n$. Finally, as the fibres are irreducible, $R^{2n}\pi_*\mathbb{C} \simeq \mathbb{C}$. This top-dimensional cohomology is already accounted for by the summand ${}^p\mathrm{R}^n\pi_*\mathbb{C}[\dim X] = \mathbb{C}[\dim Y]$ and thus the vanishing for $j = i$ is ensured. The final statement follows because a summand supported on a subvariety Y' is the IC sheaf associated to some local system on an open subset of Y' and consequently the stalk of the cohomology sheaf in degree $-\dim Y'$ is nonzero on a general point of Y'; this is prohibited by the stated estimate when $\dim Y - \dim Y' \geq n$. □

Lemma 11. *Let $\pi : \mathcal{C} \to B$ be a family of curves such that $\mathcal{C}^{[d]}$ is smooth. Then for $i > 0$, and for every j, the support of the sheaf* $\mathcal{H}^i(\mathrm{IC}(B, \mathrm{R}^j\widetilde{\pi}^{[d]}_*\mathbb{C})[-\dim B])$ *is contained in the locus of curves of cogenus $> i$.*

Proof. We check at some point $b \in B$ and write δ for the cogenus of $\mathcal{C}_b$. By semicontinuity of cogenus, in some neighbourhood all curves have cogenus $\leq \delta$; we shrink B to this neighbourhood and show that $\mathcal{H}^i(\mathrm{IC}(B, \mathrm{R}^j\widetilde{\pi}^{[d]}_*\mathbb{C})[-\dim B]) = 0$ for all $i \geq \delta$. Shrinking B further if necessary, let $\pi' : \mathcal{C}' \to B$ be the family of curves constructed in Corollary 6, which we recall has the property that $\mathcal{C}'_b$ is rational, $\mathrm{R}^1\widetilde{\pi}_*\mathbb{C} = \mathrm{R}^1\widetilde{\pi}'_*\mathbb{C} \oplus \mathrm{H}^1(\overline{\mathcal{C}_b})$, and by item (1) of Theorem 8, $\mathcal{C}'^{[d]}$ is smooth. Taking exterior powers and comparing with equation (1), we see that $\mathrm{R}^j\widetilde{\pi}^{[d]}_*\mathbb{C}$ is a sum of $\mathrm{R}^{\leq j}\widetilde{\pi}'^{[d]}_*\mathbb{C}$; it will therefore suffice to check the assertion for the family $\mathcal{C}'$.

Note δ is the common arithmetic genus of the fibres of π'. From Macdonald's formula (1), all summands of $\mathrm{R}^i\widetilde{\pi}'^{[d]}_*\mathbb{C}$ appear already as summands of $\mathrm{R}^i\widetilde{\pi}'^{[\min(d,\delta)]}_*\mathbb{C}$. As $\mathcal{C}'^{[\min(d,\delta)]}$ is smooth by Proposition 7, we may as well assume $d \leq \delta$. By relative hard Lefschetz, it suffices to check the assertion for $j \leq d$. But now $j \leq d \leq \delta \leq i$, thus by the previous lemma, we are done. □

Remark. Being an IC sheaf ensures that the above mentioned cohomology is supported on *some* subspace of codimension $i + 1$. The force of the lemma is to show this subspace lies inside the codimension $i + 1$ locus of curves of cogenus $i + 1$. Experimental evidence suggests that the support is *much* smaller, and it would be interesting to have a precise characterization.

Lemma 12. *Let $\pi : \mathcal{C} \to B$ be a family of curves, $B' \subset B$ a smooth closed subvariety, and $\pi' : \mathcal{C}' \to B'$ the restricted family. Assume $\mathcal{C}^{[d]}$ and $\mathcal{C}'^{[d]}$ are smooth. Denote by $\widetilde{\pi}$ and $\widetilde{\pi}'$ the respective smooth loci of the maps. Then* $\mathrm{IC}(B, \mathrm{R}^i\widetilde{\pi}^{[d]}_*\mathbb{C})|_{B'}[\dim B' - \dim B] = \mathrm{IC}(B', \mathrm{R}^i\widetilde{\pi}'^{[d]}_*\mathbb{C})$.

Proof. By induction on the codimension of B' in B, we are reduced to proving the statement for B' a Cartier divisor in B. By [BBD, Cor. 4.1.12], the complex $K := \mathrm{IC}(B, \mathrm{R}^i\tilde{\pi}_*^{[d]}\mathbb{C})|_{B'}[-1]$ is a perverse sheaf. By proper base change, K is a summand of $\mathrm{R}\pi'^{[d]}_*\mathbb{C}[d + \dim B']$. As $\mathcal{C}'^{[d]}$ is smooth, K must be the sum of IC complexes, and by Corollary 9 the locus of curves of cogenus $\delta \leq d$ appears in codimension at least δ in B'. By Lemma 11 and the fact that the fibre is d-dimensional, $\dim \mathrm{Supp}\,\mathcal{H}^i(K) < -i$ for $i \neq -\dim B'$. Therefore no summand of K is an IC complex associated to a local system supported in positive codimension in B', and the claimed isomorphism follows from the obvious fact that, on the smooth locus, K coincides with the (shifted) local system $\mathrm{R}^i\tilde{\pi}'^{[d]}_*\mathbb{C}[\dim B']$. □

Corollary 13. *Let $\pi : \mathcal{C} \to B$ be a family of curves, and $\pi' : \mathcal{C}' \to B'$ its restriction to a smooth subvariety of the base; assume $\mathcal{C}^{[d]}$ and $\mathcal{C}'^{[d]}$ are smooth. Let $\mathcal{F}$ be the summand of $\mathrm{R}\pi_*^{[d]}\mathbb{C}[d + \dim B]$ not supported on all of B, i.e.*

$$\mathrm{R}\pi_*^{[d]}\mathbb{C}[d + \dim B] = \Big(\bigoplus_{i=-d}^{d} \mathrm{IC}(B, \mathrm{R}^{d+i}\tilde{\pi}_*^{[d]}\mathbb{C})[-i]\Big) \oplus \mathcal{F},$$

and similarly $\mathcal{F}'$ for B'. If $B' \not\subset \mathrm{Supp}\,\mathcal{F}$, then $\mathcal{F}' = \mathcal{F}|_{B'}[\dim B' - \dim B]$.

4. Proof via reduction to rational curves

Proposition 14. *Let $\pi : \mathcal{C} \to B$ be a family of curves. Then, in some neighbourhood of $b \in B$, Theorem 1 holds for $d \leq \delta(\mathcal{C}_b)$.*

Proof. Suppose not; let $\mathcal{C} \to B$ be a counterexample over a base of minimal dimension. Let $b \in B$ be any point in the support of a summand $\mathcal{F}$ of $\mathrm{R}\pi_*^{[d]}\mathbb{C}$ not supported on all of B. By Theorem 8 and Corollary 13, the restriction of the family to a general slice of dimension d passing through b remains a counterexample. Therefore $d \geq \dim B$. On the other hand, by (2) of Theorem 8, and the assumption that $d \leq \delta(\mathcal{C}_b)$, we must have $d \leq \dim B$. By Lemma 10, the support of $\mathcal{F}$ is of codimension $< d$, thus it intersects a general $d-1$-dimensional slice of B. Again by Corollary 13, the restricted family remains a counterexample, contradicting the assumption of minimal dimensionality. □

Now let $\pi : \mathcal{C} \to B$ be a family of curves; shrinking to a neighbourhood of some $b \in B$, let $\pi' : \mathcal{C}' \to B$ be the replacement family of Corollary 6. Then from equation (4), we see

$$\sum_{d=0}^{\infty}\sum_{i=0}^{2d} q^d \mathrm{R}^i\tilde{\pi}_*^{[d]}\mathbb{C} = \Big(\sum_{d=0}^{\infty}\sum_{i=0}^{2d} q^d \mathrm{H}^i(\overline{\mathcal{C}_b}^{[d]})\Big) \otimes \Big(\sum_{i=0}^{2\delta(\mathcal{C}_b)} q^i {\bigwedge}^i \mathrm{R}^1\tilde{\pi}'_*\mathbb{C}\Big). \tag{5}$$

As the final term is manifestly symmetric about q^δ, the series is determined by its first δ terms.

To finish the proof of Theorem 1, it would suffice to show that

$$\sum q^d \mathrm{H}^*(\mathcal{C}_b^{[d]}) = \Big(\sum q^d \mathrm{H}^*(\overline{\mathcal{C}_b}^{[d]})\Big) \mathcal{Z}_C(q) \tag{6}$$

for a generating polynomial of vector spaces $\mathcal{Z}_C(q)$ of degree 2δ with coefficients symmetric around q^δ. Indeed, then the fibre at b of both sides of the equality asserted in Theorem 1 would be determined in the same way by their values for $C^{[\leq\delta]}$, which by Proposition 14 are equal.

However, we know no direct way to establish equation (6), although of course it will follow as a consequence of Theorem 1. Instead, we prove the product formula and check the symmetry in the Grothendieck group of varieties, in which we denote by $\mathbb{L}$ the class of the affine line. This is still sufficient, because the *weight polynomial* both factors through the Grothendieck group of varieties and serves to witness the nonexistence of summands of $\mathrm{R}\pi_*^{[d]}\mathbb{C}[\dim B]$. For K a complex of vector spaces carrying a weight filtration, we write the weight polynomial $\mathfrak{w}(K) := \sum_{i,j} t^i(-1)^{i+j} \dim \mathrm{Gr}^i_W \mathrm{H}^j(K)$. For a variety Z, we abbreviate $\mathfrak{w}(Z)$ for $\mathfrak{w}(\mathrm{H}^*_c(Z))$.

Proposition 15. *Suppose we are given a proper map $f : X \to Y$ between smooth varieties, and some summand $\mathcal{F}$ of $\mathrm{R}f_*\mathbb{C}[\dim X]$. If, for all $y \in Y$, we have $\mathfrak{w}(\mathcal{F}_y[-\dim X]) = \mathfrak{w}(X_y)$, then $\mathcal{F} = \mathrm{R}\pi_*\mathbb{C}[\dim X]$.*

Proof. Let $\mathrm{R}f_*\mathbb{C}[\dim X] = \mathcal{F} \bigoplus \mathcal{G}$; we must show that if $\mathfrak{w}(\mathcal{G}_y) = 0$ for all $y \in Y$, then $\mathcal{G} = 0$. Now $\mathcal{G}$ is a direct sum of shifted complexes of the form $\mathrm{IC}(L_i)$, with L_i local systems supported on locally closed subsets of B underlying pure variations of Hodge structures. Then for y a general point of the support, the purity of $\mathcal{G}$ and the vanishing of the weight polynomial force the vanishing of the local systems. □

Let C be a curve, C^{sm} its smooth locus, and $\overline{C}$ its normalization. For $p \in C$, we write $(C, p)^{[n]}$ for the subvariety of $C^{[n]}$ parameterizing subschemes set-theoretically supported at p; our notation is meant to recall that it depends only on the germ of C at p. Let $b(p)$ be the number of analytic local branches of C near p. Splitting subschemes according to their support gives the following equality in the Grothendieck group of varieties:

$$\begin{aligned}\sum q^n[C^{[n]}] &= \sum q^n[(C^{\mathrm{sm}})^{[n]}] \prod_{p\in C\setminus C^{\mathrm{sm}}} \sum q^n[(C,p)^{[n]}] \\ &= \Big(\sum q^n[\overline{C}^{[n]}]\Big)\Big(\prod_{p\in C\setminus C^{\mathrm{sm}}} (1-q)^{b(p)} \sum q^n[(C,p)^{[n]}]\Big). \qquad (7)\end{aligned}$$

This is the desired product formula. It remains to show that the final term of (7) is symmetric around q^δ. After passing to Euler characteristics, this is shown in [PT] using Serre duality; the argument below is similar.

Proposition 16. *Let C be a Gorenstein curve of cogenus δ, with smooth locus C^{sm} and $b(p)$ analytic local branches at a point $p \in C$. Define*

$$Z_C(q) := \prod_{p\in C\setminus C^{\mathrm{sm}}} (1-q)^{b(p)} \sum q^n[(C,p)^{[n]}].$$

Then $Z_C(q)$ is a polynomial in q of degree 2δ. Moreover, writing $\mathbb{L}$ for the class of the affine line, we have $Z_C(q) = (q^2\mathbb{L})^\delta Z_C(1/q\mathbb{L})$.

Proof. By (7), we may assume C is a rational curve of arithmetic genus g; note in this case $Z(C) = (1-q)(1-q\mathbb{L})\sum q^d[C^{[d]}]$. Fix a degree 1 line bundle $\mathcal{O}(1)$ on C. We map $C^{[d]} \to \bar{J}^0(C)$ by associating the ideal $I \subset \mathcal{O}_C$ to the sheaf $I^* = \mathcal{H}om(I, \mathcal{O}_C) \otimes \mathcal{O}(-d)$; the fibre is $\mathbb{P}(H^0(C, I^*))$. For $\mathcal{F}$ a rank one degree zero torsion free sheaf, we write the Hilbert function as $h_{\mathcal{F}}(d) = \dim \mathrm{H}^0(C, \mathcal{F} \otimes \mathcal{O}(d))$. Then since over the strata with constant Hilbert function, the map from the Hilbert schemes to the compactified Jacobian is the projectivization of a vector bundle, we have the equality $\sum q^d[C^{[d]}] = \sum_h [\{\mathcal{F} \mid h_{\mathcal{F}} = h\}] \sum q^d[\mathbb{P}^{h(d)-1}]$.

Fix $h = h_{\mathcal{F}}$ for some $\mathcal{F}$. Evidently h is supported in $[0, \infty)$, and by Riemann–Roch and Serre duality is equal to $d+1-g$ in $(2g-2, \infty)$. Inside $[0, 2g-2]$, it either increases by 0 or 1 at each step. Let $\phi_\pm(h) = \{d \mid 2h(d-1) - h(d-2) - h(d) = \pm 1\}$; evidently $\phi_- \subset [0, 2g]$ and $\phi_+ \subset [1, 2g-1]$, and

$$Z_h(q) := (1-q)(1-q\mathbb{L})\sum q^d[\mathbb{P}^{h(d)-1}] = \sum_{d\in\phi_-(\mathcal{F})} q^d \mathbb{L}^{h(d)-1} - \sum_{d\in\phi_+(\mathcal{F})} q^d \mathbb{L}^{h(d-1)}.$$

This is a polynomial in q of degree at most $2g$, hence so is $Z_C(q)$.

Now let $\mathcal{G} = \mathcal{F}^* \otimes \omega_C \otimes \mathcal{O}(2-2g)$, and $h^\vee = h_{\mathcal{G}}$. By Serre duality and Riemann–Roch, $h^\vee(d) = h(2g-2-d)+d+1-g$, so in particular $d \in \phi_\pm(h^\vee) \Leftrightarrow 2g-d \in \phi_\pm(h)$. It follows that $q^{2g}\mathbb{L}^g Z_h(1/q\mathbb{L}) = Z_{h^\vee}(q)$. As $Z_C(q) = \sum_h [\{\mathcal{F} \mid h_{\mathcal{F}} = h\}] Z_h(q)$, we obtain the final stated equality. □

This completes the (first) proof of Theorem 1.

5. Proof by reduction to nodal curves

Lemma 17. *If Theorem 1 holds for all versal families of curves, then it holds for all families.*

Proof. By Corollary 13, the hypothesis implies that Theorem 1 holds for any subfamily of a versal family. Now let $\pi : \mathcal{C} \to B$ be a family such that the theorem fails; let $\mathcal{F}$ be the summand of $\mathrm{R}\pi_*^{[d]}\mathbb{C}$ whose support is not all of B, and let $b \in B$ be a point such that $\mathcal{F}_b \neq 0$. Let $\phi : B \to \mathbb{V}(\mathcal{C}_b)$ be a map to the miniversal deformation, and let $B' \subset B$ be a smooth closed subvariety through b such that $\mathrm{d}\phi_b|_{B'}$ is an isomorphism onto the image of $\mathrm{d}\phi_b(T_b B)$. By Theorem 8(1), $\mathcal{C}^{[d]}|_{B'}$ is still smooth. According to Corollary 13, choosing $B' \not\subset \operatorname{Supp} \mathcal{F}$ ensures that the restricted family still provides a counterexample in any neighbourhood of b. Shrinking still further, the map $B' \to \mathbb{V}(\mathcal{C}_b)$ may be taken to be the embedding of a smooth subvariety, giving a contradiction. □

We now prove Theorem 1 for the versal family. The argument is induction on the cogenus, which depends crucially on the properties of the versal family identified in Theorems 4 and 8. For clarity, we separate topological generalities from the specific properties of the versal family.

Definition 18. Let X be a smooth complex analytic space with a constructible stratification $X = \coprod X_i$ such that X_i is everywhere of codimension $\geq i$. We write $\mathfrak{N}(\coprod X_i)$ for the full subcategory of $\mathrm{D}_c^b(X)$ whose objects $\mathcal{F}$ have the following property.

For $x \in X_i$ with $i < \dim X$ and for generic, sufficiently small, polydiscs $X \supset \mathbb{D}^i \times \mathbb{D} \supset \mathbb{D}^i \times 0 \ni x$, for sufficiently small $\epsilon \in \mathbb{D}$, the restriction

$$\mathcal{F}_x = \mathrm{R}\Gamma(\mathbb{D}^i \times 0, \mathcal{F}|_{\mathbb{D}^i \times 0}) = \mathrm{R}\Gamma(\mathbb{D}^i \times \mathbb{D}, \mathcal{F}|_{\mathbb{D}^i \times \mathbb{D}}) \to \mathrm{R}\Gamma(\mathbb{D}^i \times \epsilon, \mathcal{F})$$

is an isomorphism.

Lemma 19. *$\mathfrak{N}(\coprod X_i)$ is a thick triangulated subcategory of $\mathrm{D}^b_c(X)$, i.e., it is closed under shifts, triangles, and taking summands.*

Lemma 20. *Let $X^+ \subset X$ be an open subset such that $X_i \setminus X^+$ is of codimension $> i$. Then the composition $\mathfrak{N}(\coprod X_i) \to \mathrm{D}^b_c(X) \to \mathrm{D}^b_c(X^+)$, where the second functor is given by restriction to the open set X^+, is faithful.*

Proof. Note that the condition on X^+ implies that $X_{\dim X} \subseteq X^+$. Consider $\mathcal{F} \in \mathfrak{N}(\coprod X_i)$ such that $\mathcal{F}|_{X^+} = 0$. We must show $\mathcal{F}_x = 0$ for all $x \in X$. Suppose by induction $\mathcal{F}_x = 0$ for $x \in X_{<i}$ and consider $x \in X_i \setminus X^+$. Evidently $(X_i \setminus X^+) \cup X_{>i}$ is of codimension $> i$, so the generic $\mathbb{D}^i \times \epsilon$ from the definition of $\mathfrak{N}(\coprod X_i)$ passing near x misses this locus completely. Thus by assumption and the induction hypothesis, $\mathcal{F}_x = \mathrm{R}\Gamma(\mathbb{D}^i \times \epsilon, \mathcal{F}) = 0$. □

We now apply the statements above to $X = B$, the base of a locally versal family, with the stratification $B = \coprod B_i$ given by the loci of curves of cogenus i, and $X^+ \subseteq B$ the open set parameterizing curves with at worst nodal singularities.

Proposition 21. *Let $\pi : \mathcal{C} \to B$ be a locally versal family of curves. Let B_i be the locus of curves of cogenus i. Then $\mathrm{R}\pi^{[d]}_* \mathbb{C}[\dim B] \in \mathfrak{N}(\coprod B_i)$.*

Proof. We check at some $b \in B_\delta$. The definition of $\mathfrak{N}$ is local on the base; as $\widetilde{\pi}$ is proper, after shrinking B the inclusion $\mathcal{C}^{[d]}_b \hookrightarrow \mathcal{C}^{[d]}$ becomes a homotopy equivalence. Any sufficiently small polydisc $b \in \mathbb{D}^\delta \times \mathbb{D} \subset B$ will induce homotopy equivalences $\mathcal{C}^{[d]}_b \to \mathcal{C}^{[d]}|_{\mathbb{D}^\delta \times 0} \to \mathcal{C}^{[d]}|_{\mathbb{D}^\delta \times \mathbb{D}}$. By Theorem 8(3), a generic choice ensures that the latter two spaces are smooth, possibly after further shrinking the discs; by openness of smoothness we can shrink $\mathbb{D}$ still further so that the projection $\mathcal{C}^{[d]}_{\mathbb{D}^\delta \times \mathbb{D}} \to \mathbb{D}$ is smooth. It follows that, possibly after shrinking $\mathbb{D}^\delta$ further, $\mathrm{H}^*(\mathcal{C}^{[d]}_{\mathbb{D}^\delta \times \mathbb{D}}) = \mathrm{R}\Gamma(\mathbb{D}^\delta \times \mathbb{D}, \mathrm{R}\pi^{[d]}_* \mathbb{C}) \to \mathrm{R}\Gamma(\mathbb{D}^\delta \times \epsilon, \mathrm{R}\pi^{[d]}_* \mathbb{C}) = \mathrm{H}^*(\mathcal{C}^{[d]}_{\mathbb{D}^\delta \times \epsilon})$ is an isomorphism (see [FGvS, proof of Thm. 1] for a similar argument). □

Proposition 22. *Theorem 1 holds for all locally versal families of curves.*

Proof. Let $\pi : \mathcal{C} \to B$ be a locally versal family of curves, and let B_i be the locus of curves of cogenus i. Let $\mathcal{F}$ be any summand of $\mathrm{R}\pi^{[d]}_* \mathbb{C}$ supported on a proper subvariety of B. Then by Lemma 19 and Proposition 21, $\mathcal{F} \in \mathfrak{N}(\coprod B_i)$. By Theorem 4 [DH, T], the locus of nodal curves is dense in each B_i; thus by Lemma 20 we need only check that the restriction of $\mathcal{F}$ to the locus of nodal curves is zero, i.e. Theorem 1 holds for families of nodal curves; this is done in the next lemma. □

Lemma 23. *Theorem 1 holds for locally versal families of nodal curves.*

Proof. Let $\pi : \mathcal{C} \to B$ be such a family. Let $b \in B$ be the base point, let $\{c_1, \dots, c_\delta\} \subseteq \mathcal{C}_b$ be the nodal set of the central curve $\mathcal{C}_b$, and denote by r its geometric genus. Shrinking B if necessary, we can assume:

(1) the discriminant locus is a normal crossing divisor $\Delta = \bigcup D_i$ with $i = 1, \dots, \delta$, where D_i is the locus in which the i-th node c_i is preserved.
(2) If b_0 is such that $\mathcal{C}_{b_0}$ is nonsingular, the vanishing cycles $\{\zeta_1, \dots, \zeta_\delta\}$ in $\mathcal{C}_{b_0}$ associated with the nodes of $\mathcal{C}_b$ are disjoint.

As the curve $\mathcal{C}_b$ is irreducible, the cohomology classes in $\mathrm{H}^1(\mathcal{C}_{b_0})$ of these vanishing cycles are linearly independent, and can then be completed to a symplectic basis.

Let T_i be the generators of the (abelian) local fundamental group $\pi_1(B \setminus \Delta, b_0)$ where T_i corresponds to "going around D_i". Then the monodromy defining the local system $\mathrm{R}^1\widetilde{\pi}_*\mathbb{C}$ on $B \setminus \Delta$ is given via the Picard–Lefschetz formula, and, in the symplectic basis above, has a Jordan form consisting of δ Jordan blocks of length 2. From this it is easy to compute the invariants of the local systems obtained by applying any linear algebra construction to $\mathrm{R}^1\widetilde{\pi}_*\mathbb{C}$, such as those that appear in $\mathrm{R}^i\widetilde{\pi}^{[d]}_*\mathbb{C}$. Let $\mathbb{S}^{i,[d]}$ be the linear algebra operation, described by formula (1), such that $\mathrm{R}^i\widetilde{\pi}^{[d]}_*\mathbb{C} = \mathbb{S}^{i,[d]}R^1\pi_*\mathbb{C}$. Denote by $j : B \setminus \Delta \to B$ the open inclusion.

We have a natural isomorphism

$$(\mathbb{S}^{i,[d]}\mathrm{H}^1(\mathcal{C}_{b_0}))^{\pi_1(B\setminus\Delta,b_0)} = \mathcal{H}^{-\dim B}(\mathrm{IC}(B, \mathrm{R}^i\widetilde{\pi}^{[d]}_*\mathbb{C}))_b$$

between the monodromy invariants on $\mathbb{S}^{i,[d]}\mathrm{H}^1(\mathcal{C}_{b_0})$ and the stalk at b of the first non-vanishing cohomology sheaf of the intersection cohomology complex of $\mathrm{R}^i\widetilde{\pi}^{[d]}_*\mathbb{C}$. The decomposition theorem in [BBD] then implies that $\mathrm{H}^*(\mathcal{C}_b^{[d]})$ contains the Hodge structure

$$\mathbb{H}^{[d]} := \bigoplus_i (\mathbb{S}^{i,[d]}\mathrm{H}^1(\mathcal{C}_{b_0}))^{\pi_1(B\setminus\Delta,b_0)}$$

as a direct summand, with the weight filtration defined in the standard way by the logarithms of the monodromy operators (see [CK]).

It is easy to compute $\mathbb{H}^{[d]}$ explicitly; presumably $\mathrm{H}^*(\mathcal{C}_b^{[d]})$ can be computed by elementary methods and shown to match; this would complete the proof. In the absence of such a calculation, we use Proposition 15 and instead compare weight polynomials. On the one hand, we compute $\sum q^d \mathfrak{w}(\mathbb{H}^{[d]}) = (1 - q + t^2q^2)^\delta(1 + tq)^{2r}/(1 - q)(1 - t^2q)$.

On the other hand, when $C = \mathbb{P}^1_+$ is a rational curve with a single node, Riemann–Roch ensures that the Abel map is a projective bundle for any $d \geq 1$; when $d = 1$ we have $[\bar{J}^0(\mathbb{P}^1_+)] = [\mathbb{P}^1_+] = \mathbb{L}$. Thus we get the formula

$$\sum q^d[(\mathbb{P}^1_+)^{[d]}] = (1 - q + q^2\mathbb{L})/((1 - q)(1 - q\mathbb{L})).$$

Comparison with equation (7) gives $\sum q^d[\mathcal{C}_b^{[d]}] = (\sum q^d[\overline{\mathcal{C}_b}^{[d]}])(1 - q + q^2\mathbb{L})^\delta$; taking weight polynomials gives the desired result. □

This completes the (second) proof of Theorem 1.

6. An application

Given a projective map $F : X \to Y$ with X nonsingular, the decomposition theorem ([BBD]) $\mathrm{R}F_*\mathbb{C}[\dim X] = \bigoplus_{i=-r}^{r} {}^p\mathrm{R}^i F_*\mathbb{C}[\dim X][-i]$ gives, by proper base change, an isomorphism

$$\mathrm{H}^k(F^{-1}(y)) \xleftarrow{\Phi_F} \bigoplus_{i=-r}^{r} \mathcal{H}^{k-\dim X-i}({}^p\mathrm{R}^i F_*\mathbb{C}[\dim X])_y,$$

for every $y \in Y$ and $k \in \mathbb{N}$. The filtration

$$\begin{aligned}\mathrm{H}^k_{\leq 0}(F^{-1}(y)) \subseteq \mathrm{H}^k_{\leq 1}(F^{-1}(y)) \subseteq \cdots \subseteq \mathrm{H}^k_{\leq 2r-1}(F^{-1}(y)) \\ \subseteq \mathrm{H}^k_{\leq 2r}(F^{-1}(y)) = \mathrm{H}^k(F^{-1}(y)),\end{aligned}$$

where

$$\mathrm{H}^k_{\leq l}(F^{-1}(y)) := \Phi_F\Big(\bigoplus_{i=-r}^{-r+l} \mathcal{H}^{k-\dim X-i}({}^p\mathrm{R}^i F_*\mathbb{C}[\dim X])_y\Big),$$

is called the *perverse filtration* on $\mathrm{H}^k(F^{-1}(y))$ associated with F.

Let C be a complete integral locally planar curve. A deformation $C \subset \mathcal{C} \xrightarrow{\pi} B \ni b$, with $C = \pi^{-1}(b)$, such that $\mathcal{C}^{[d]}$ is nonsingular, defines, by the above, a perverse filtration $\mathrm{H}^k_{\leq 0}(C^{[d]}) \subseteq \mathrm{H}^k_{\leq 1}(C^{[d]}) \subseteq \cdots \subseteq \mathrm{H}^k_{\leq 2d-1}(C^{[d]}) \subseteq \mathrm{H}^k_{\leq 2d}(C^{[d]}) = \mathrm{H}^k(C^{[d]})$ on the cohomoloy groups of the $d-$th Hilbert scheme of C. The following proposition shows that this filtration is in fact intrinsic, i.e. it does not depend on π:

Proposition 24. *Let C be a complete integral curve, Let $C \subset \mathcal{C} \xrightarrow{\pi} B \ni b$, and $C \subset \mathcal{C}' \xrightarrow{\pi'} B \ni b'$ be two deformations of C with the property that $\mathcal{C}^{[d]}$ and $\mathcal{C}'^{[d]}$ are nonsingular. Then the perverse filtrations on $\mathrm{H}^k(C^{[d]})$ associated with π and π' coincide.*

Proof. By Theorem 1 we have

$$\mathrm{H}^k_{\leq l}(C^{[d]}) := \Phi_\pi\Big(\bigoplus_{i=-d}^{-d+l} \mathcal{H}^{k-d-\dim B-i}(\mathrm{IC}(B, \mathrm{R}^{d+i}\widetilde{\pi}^{[d]}_*\mathbb{C}))_b\Big), \tag{8}$$

and similarly for π'. By appropriately shrinking B and B' around b and b' respectively, we may assume that the families are the pullback from a versal deformation $C \subset \mathcal{C}_{\mathrm{v}} \xrightarrow{\pi_{\mathrm{v}}} B_{\mathrm{v}} \ni b_{\mathrm{v}}$ of C. By arguments analogous to those in Section 2 we are reduced to proving the statement in the case when $\pi : \mathcal{C} = \mathcal{C}_{\mathrm{v}} \times_{B_{\mathrm{v}}} B \to B$ and $\pi' : \mathcal{C}' = \mathcal{C}_{\mathrm{v}} \times_{B_{\mathrm{v}}} B' \to B'$ are the restriction of $\mathcal{C}_{\mathrm{v}} \to B_{\mathrm{v}}$ to two smooth subvarieties $B \overset{i}{\hookrightarrow} B_{\mathrm{v}} \overset{i'}{\hookleftarrow} B'$ with $b_{\mathrm{v}} = b = b' \in B \cap B'$. The natural restriction maps

$$i_*\mathrm{R}\pi^{[d]}_*\mathbb{C} = i_*i^*\mathrm{R}\pi^{[d]}_{\mathrm{v}\,*}\mathbb{C} \leftarrow \mathrm{R}\pi^{[d]}_{\mathrm{v}\,*}\mathbb{C} \to i'_*i'^*\mathrm{R}\pi^{[d]}_{\mathrm{v}\,*}\mathbb{C} = i'_*\mathrm{R}\pi'^{[d]}_*\mathbb{C}$$

induce isomorphisms on the cohomology sheaves at the point b_{v}, and the statement now follows immediately from the expression (8) applied to the maps π, π' and π_{v}, and Lemma 12. $\square$

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