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Perron–Frobenius operators and the Klein–Gordon equation

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Abstract. For a smooth curve Γ and a set Λ in the plane $\mathbb{R}^2$, let $\mathrm{AC}(\Gamma;\Lambda)$ be the space of finite Borel measures in the plane supported on Γ, absolutely continuous with respect to arc length and whose Fourier transform vanishes on Λ. Following [12], we say that (Γ,Λ) is a Heisenberg uniqueness pair if $\mathrm{AC}(\Gamma;\Lambda)=\{0\}$. In the context of a hyperbola Γ, the study of Heisenberg uniqueness pairs is the same as looking for uniqueness sets Λ of a collection of solutions to the Klein–Gordon equation. In this work, we mainly address the issue of finding the dimension of $\mathrm{AC}(\Gamma;\Lambda)$ when it is nonzero. We will fix the curve Γ to be the hyperbola $x_1x_2=1$, and the set $\Lambda=\Lambda_{\alpha,\beta}$ to be the lattice-cross

$$\Lambda_{\alpha,\beta}=(\alpha\mathbb{Z}\times\{0\})\cup(\{0\}\times\beta\mathbb{Z}),$$

where α,β are positive reals. We will also consider Γ_+, the branch of $x_1x_2=1$ where $x_1>0$. In [12], it is shown that $\mathrm{AC}(\Gamma;\Lambda_{\alpha,\beta})=\{0\}$ if and only if $\alpha\beta\le 1$. Here, we show that for $\alpha\beta>1$, we get a rather drastic "phase transition": $\mathrm{AC}(\Gamma;\Lambda_{\alpha,\beta})$ is infinite-dimensional whenever $\alpha\beta>1$. It is shown in [13] that $\mathrm{AC}(\Gamma_+;\Lambda_{\alpha,\beta})=\{0\}$ if and only if $\alpha\beta<4$. Moreover, at the edge $\alpha\beta=4$, the behavior is more exotic: the space $\mathrm{AC}(\Gamma_+;\Lambda_{\alpha,\beta})$ is one-dimensional. Here, we show that the dimension of $\mathrm{AC}(\Gamma_+;\Lambda_{\alpha,\beta})$ is infinite whenever $\alpha\beta>4$. Dynamical systems, and more specifically Perron–Frobenius operators, play a prominent role in the presentation.

Keywords. Trigonometric system, inversion, Perron–Frobenius operator, Koopman operator, invariant measure, Klein–Gordon equation, ergodic theory

1. Introduction

1.1. Background: the Heisenberg uncertainty principle

The Heisenberg uncertainty principle asserts that it is not possible to have completely accurate information about the position and the momentum of a particle at the same time. If ψ is the spatial wave-function, which describes the position of the particle in question, and it is known that ψ is concentrated in a small region, then the deviation of the momentum wave-function of ψ from its mean must be large. The momentum wave-function

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is essentially the Fourier transform of the spatial wave-function. So, we may consider the Heisenberg uncertainty principle as the mathematical statement that a function and its Fourier transform cannot both be too concentrated simultaneously; cf. [2], [10], and [11].

1.2. Heisenberg uniqueness pairs

Let Γ be a finite disjoint union of smooth curves in the plane and Λ a subset of the plane. Let $\mathrm{AC}(\Gamma;\Lambda)$ be the space of bounded Borel measures μ in the plane supported on Γ, absolutely continuous with respect to arc length and whose Fourier transform

$$\widehat{\mu}(x_1,x_2)=\int_\Gamma e^{i\pi(x_1y_1+x_2y_2)}\,d\mu(y_1,y_2),\qquad (x_1,x_2)\in\mathbb{R}^2, \tag{1.1}$$

vanishes on Λ. Following [12], we say that (Γ,Λ) is a *Heisenberg uniqueness pair* if $\mathrm{AC}(\Gamma;\Lambda)=\{0\}$.

When Γ is an algebraic curve, that is, the zero locus of a polynomial p in two variables with real coefficients, the requirement that the support of μ be contained in Γ means that $\widehat{\mu}$ solves the partial differential equation

$$p\left(\frac{\partial_{x_1}}{\pi i},\frac{\partial_{x_2}}{\pi i}\right)\widehat{\mu}=0. \tag{1.2}$$

So, there is a natural interplay between the Heisenberg uniqueness pairs and the theory of partial differential equations (PDE) (cf. [12]). The most natural examples appear when we consider quadratic polynomials p corresponding to the standard conic sections: line, two parallel lines, two crossing lines, hyperbola, ellipse, and parabola. The natural invariance of Heisenberg uniqueness pairs under affine transformations of the plane allows us to reduce to the canonical models for these curves (cf. [12]). The case when Γ is either one line or the union of two parallel lines was solved completely for general $\Lambda\subset\mathbb{R}^2$ in [12]. In this direction, Blasi-Babot has solved particular cases when Γ is the union of three parallel lines (see [4]). The case when Γ is a circle (which also covers the ellipse case after an affine mapping) was recently studied independently by Lev and by Sjölin in [25], [20], where, e.g., circles and unions of straight lines are considered as sets Λ. Also subsets of unions of straight lines were considered, and a connection with the Beurling–Malliavin theory was made. More recently, Sjölin [26] has investigated the case of a parabola. However, very little seems to be known when Γ is the union of two intersecting lines.

1.3. Heisenberg uniqueness pairs for the hyperbola

The case of the hyperbola $\Gamma : x_1x_2=1$ and the lattice-cross

$$\Lambda_{\alpha,\beta}=(\alpha\mathbb{Z}\times\{0\})\cup(\{0\}\times\beta\mathbb{Z})$$

for given positive reals α,β was considered in [12], where the following result was obtained.

Theorem A (Hedenmalm, Montes-Rodríguez). *Let Γ be the hyperbola $x_1x_2=1$. Then (Γ,Λ) is a Heisenberg uniqueness pair if and only if $\alpha\beta\le 1$.*

When one of the branches of the hyperbola is considered, the critical density changes (see [13]).

Theorem B (Hedenmalm, Montes-Rodríguez). *Let* Γ_+ *be the branch of the hyperbola* $x_1x_2 = 1$ *where* $x_1 > 0$. *Then* $\mathrm{AC}(\Gamma_+; \Lambda_{\alpha,\beta}) = \{0\}$ *if and only if* $\alpha\beta < 4$. *Moreover, when* $\alpha\beta = 4$, $\mathrm{AC}(\Gamma_+; \Lambda_{\alpha,\beta})$ *is one-dimensional.*

For subcritical density of the lattice-cross, we have the following two theorems, corresponding to the hyperbola and a branch of the hyperbola.

Theorem 1.1. *Let* Γ *be the hyperbola* $x_1x_2 = 1$. *Then* $\mathrm{AC}(\Gamma; \Lambda_{\alpha,\beta})$ *is infinite-dimensional for* $\alpha\beta > 1$.

Theorem 1.2. *Let* Γ_+ *be the branch of the hyperbola* $x_1x_2 = 1$ *with* $x_1 > 0$. *Then the space* $\mathrm{AC}(\Gamma_+; \Lambda_{\alpha,\beta})$ *is infinite-dimensional for* $\alpha\beta > 4$.

Although the proofs of Theorem 1.1 and 1.2 exhibit a certain parallelism, the proof of Theorem 1.1 is more delicate. Mainly, the difference is that at the edge $\alpha\beta = 4$, the Perron–Frobenius operator which appears in the context of Theorem 1.2, induced by the classical Gauss map, has a spectral gap [acting on the space of functions of bounded variation], while the Perron–Frobenius operator associated to the edge case $\alpha\beta = 1$ in the context of Theorem 1.1 does not have such a gap; this is so because the Gauss-type transformation which defines the Perron–Frobenius operator has an indifferent fixed point.

The basic invariance properties of Heisenberg uniqueness pairs allow us to take $\alpha = 1$ and we may appeal to duality and reformulate Theorem A as follows (cf. [12]).

Theorem A′. *Let* $\mathcal{M}_\beta$ *be the linear subspace of* $L^\infty(\mathbb{R})$ *spanned by the functions* $x \mapsto e^{im\pi x}$ *and* $x \mapsto e^{in\pi\beta/x}$, *where* m, n *range over the integers and* β *is a fixed positive real. Then* $\mathcal{M}_\beta$ *is weak-star dense in* $L^\infty(\mathbb{R})$ *if and only if* $\beta \le 1$.

The analogous reformulation of Theorem 1.1 is as follows:

Theorem 1.3. *Let* $\mathcal{M}_\beta$ *be the linear subspace of* $L^\infty(\mathbb{R})$ *spanned by the functions* $x \mapsto e^{im\pi x}$ *and* $x \mapsto e^{in\pi\beta/x}$, *where* m, n *range over the integers. Then the weak-star closure of* $\mathcal{M}_\beta$ *in* $L^\infty(\mathbb{R})$ *has infinite codimension in* $L^\infty(\mathbb{R})$ *for* $\beta > 1$.

If we instead take $\alpha = 2$, we may reformulate Theorems B and 1.2 as follows.

Theorem B′. *Let* $\mathcal{N}_\beta$ *be the linear subspace of* $L^\infty(\mathbb{R}_+)$ *spanned by the functions* $x \mapsto e^{i2m\pi x}$ *and* $x \mapsto e^{in\pi\beta/x}$, *where* m, n *range over the integers and* β *is a fixed positive real. Then* $\mathcal{N}_\beta$ *is weak-star dense in* $L^\infty(\mathbb{R}_+)$ *if and only if* $\beta < 2$. *Moreover, the weak-star closure of* $\mathcal{N}_\beta$ *in* $L^\infty(\mathbb{R}_+)$ *has codimension* 1 *in* $L^\infty(\mathbb{R}_+)$ *for* $\beta = 2$.

Theorem 1.4. *Let* $\mathcal{N}_\beta$ *be the linear subspace of* $L^\infty(\mathbb{R}_+)$ *spanned by the functions* $x \mapsto e^{i2m\pi x}$ *and* $x \mapsto e^{in\pi\beta/x}$, *where* m, n *range over the integers and* β *is a fixed positive real. Then the weak-star closure of* $\mathcal{N}_\beta$ *has infinite codimension in* $L^\infty(\mathbb{R}_+)$ *for* $\beta > 2$.

By general functional analysis, the codimension of the weak-star closure of $\mathcal{M}_\beta$ equals the dimension of its pre-annihilator space

$$\mathcal{M}_\beta^\perp = \left\{ f \in L^1(\mathbb{R}) : \int_{\mathbb{R}} f(x) e^{in\pi x}\,dx = \int_{\mathbb{R}} f(x) e^{in\pi\beta/x}\,dx = 0 \text{ for all } n \in \mathbb{Z} \right\}. \tag{1.3}$$

Likewise, the codimension of the weak-star closure of $\mathcal{N}_\beta$ equals the dimension of its pre-annihilator space

$$\mathcal{N}_\beta^\perp = \left\{ f \in L^1(\mathbb{R}_+) : \int_{\mathbb{R}_+} f(x) e^{i2n\pi x}\,dx = \int_{\mathbb{R}_+} f(x) e^{in\pi\beta/x}\,dx = 0 \text{ for all } n \in \mathbb{Z} \right\}. \tag{1.4}$$

If $f \in \mathcal{N}_\beta^\perp$, then it is easy to see that the function $g(x) = f(\frac{1}{2}x)$, extended to vanish along the negative semi-axis $\mathbb{R}_-$, belongs to $\mathcal{M}_{2\beta}^\perp$. So, Theorems 1.3 and 1.4 show that there are elements in $\mathcal{M}_\beta^\perp$ with support on the positive semi-axis precisely when $\beta \geq 4$.

Corollary 1.5. *In the pre-annihilator $\mathcal{M}_\beta^\perp$ there exists a nontrivial element that vanishes on $\mathbb{R}_-$ if and only if $\beta \geq 4$. Moreover, if $\beta = 4$, there is only a one-dimensional subspace of such elements, while if $\beta > 4$, there is an infinite-dimensional subspace with this property.*

Remark 1.6. In the context of Theorems 1.3 and 1.4, we actually construct rather concrete infinite-dimensional subspaces of $\mathcal{M}_\beta^\perp$ and $\mathcal{N}_\beta^\perp$, respectively; cf. Theorems 8.2 and 8.6.

1.4. Discussion about harmonic extension and the codimension problem

If Γ is the hyperbola $x_1 x_2 = 1$, then for $\beta > 1$ the bounded harmonic extensions to the upper half-plane of the functions $x \mapsto e^{im\pi x}$ and $x \mapsto e^{in\pi\beta/x}$, where m, n range over the integers $\mathbb{Z}$, fails to separate all the points of the upper half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Indeed, if we consider

$$z_1 := m + i\sqrt{\frac{\beta}{mn} - 1}, \qquad z_2 := -m + i\sqrt{\frac{\beta}{mn} - 1}, \qquad \text{where } m, n \in \mathbb{Z}_+,\ mn < \beta,$$

then $f(z_1) = f(z_2)$ for every $f \in \mathcal{M}_\beta$, so that the differences of the Poisson kernels $P_{z_z} - P_{z_2}$ are in the pre-annihilator space $\mathcal{M}_\beta^\perp$. If we use the Cauchy kernel in place of the Poisson kernel here we also obtain elements of the pre-annihilator. But there are only finitely many combinations of $m, n \in \mathbb{Z}_+$ with $mn < \beta$, which corresponds to finitely many differences of Poisson or Cauchy kernels. This would lead us to suspect that that the pre-annihilator $\mathcal{M}_\beta^\perp$ might be finite-dimensional. Theorem 1.3 shows that this is far from being true.

1.5. Structure of the paper

In Section 2, we take a closer look at the link between Theorem 1.1 and the Klein–Gordon and Dirac equations. In order to make the paper accessible to a wider audience, we present in Section 3 the elementary aspects of the theory of dynamical systems and the standard notation for Perron–Frobenius operators needed here. In Section 4, we show how the theory of Perron–Frobenius operators is the natural tool to analyze Heisenberg uniqueness pairs for the hyperbola Γ and for one of its branches Γ_+. In particular, the famous Gauss–Kuzmin–Wirsing operator corresponds to the critical density case for Γ_+. In Section 5, we state some more involved results of the theory of Perron–Frobenius operators which are needed later on. In Section 6, we study the structure of the pre-annihilator space $\mathcal{M}_\beta^\perp$ associated with Γ. In Section 7, we show the existence and uniqueness of a absolutely continuous invariant measure for certain transformations acting on the interval $[-1, 1]$. This is the key point in the proof of Theorem 1.3, presented in Section 8. We end Section 8 by sketching the proof of Theorem 1.4, which turns out to be much simpler than that of Theorem 1.3. Finally, in Section 9, we apply our results to a problem involving the linear span of powers of two atomic singular inner functions in the Hardy space of the unit disk. In conclusion, we can say that the study of Heisenberg uniqueness pairs related to the Klein–Gordon equation leads to new and interesting problems involving Perron–Frobenius operators.

2. Further motivation. The Klein–Gordon and Dirac equations

2.1. The Dirac equation in three spatial dimensions

In quantum mechanics the evolution of the position wave-function ψ associated to a physical system can be modelled by certain partial differential equations (PDE). According to the theory of spin, in the general setting, ψ has four components,

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4),$$

which should be thought of as written in column form, where each $\psi_j = \psi_j(t, x_1, x_2, x_3)$ is a mapping between an open set in $\mathbb{R}^4$ and a prescribed Hilbert space. Thus, these PDEs have to be understood as a system of equations for four separate wave-functions. The necessity of working with multiple-component wave-functions was pointed out by Pauli in order to understand the intrinsic angular momentum (spin) of atoms. There is no general equation whose solutions reflect faithfully the evolution of a given system from the relativistic point of view. Depending on the features of the system one must choose one or another type of equation. For instance, for a relativistic spin-0 particle with rest-mass m_0 we have the Klein–Gordon equation. Written in natural units it takes the form

$$(\partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2 - \partial_{x_3}^2 + m_0^2)\psi = 0.$$

Another example, perhaps the most important in this context, is the Dirac equation. It is used to describe the wave-function of the electron, although it remains valid when applied to a general relativistic spin-$\frac{1}{2}$ particle. In natural units it takes the form

$$(-i\gamma^0\partial_t - i\gamma^1\partial_{x_1} - i\gamma^2\partial_{x_2} - i\gamma^3\partial_{x_3} + m_0)\psi = 0,$$

commonly abbreviated as $(-i\,\partial\!\!\!/ + m_0)\psi = 0$, where $\gamma^0, \gamma^1, \gamma^2, \gamma^3$, the Dirac matrices, are the 4×4 matrices given by

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$
$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The algebraic properties of the matrices $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ allow us to obtain the following factorization of the Klein–Gordon equation:

$$(\partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2 - \partial_{x_3}^2 + m_0^2)\psi = (i\,\partial\!\!\!/ + m_0)(-i\,\partial\!\!\!/ + m_0)\psi = 0.$$

Hence a solution to the Dirac equation is always a solution to the Klein–Gordon equation. The converse statement is not true.

2.2. *The Dirac equation in one spatial dimension*

As before, let Γ be the hyperbola $x_1x_2 = 1$, and suppose $\mu \in \mathrm{AC}(\Gamma; \Lambda_{\alpha,\beta})$ for some positive reals α, β. Then, in view of (1.2), the Fourier transform $\widehat{\mu}$ given by (1.1) solves the partial differential equation

$$(\partial_{x_1}\partial_{x_2} + \pi^2)\widehat{\mu} = 0$$

in the sense of distribution theory. If we write $\psi(t, x) := \widehat{\mu}\big(\frac{1}{2}(t + x), \frac{1}{2}(t - x)\big)$, then ψ solves the one-dimensional Klein–Gordon equation for a particle of mass π,

$$(\partial_t^2 - \partial_x^2 + \pi^2)\psi = 0. \tag{2.1}$$

Theorem 1.1 asserts that if $\alpha\beta > 1$, then there exists an infinite-dimensional space of solutions ψ to (2.1) of the given form, subject to vanishing on the set

$$\Lambda'_{\alpha,\beta} = \{(m\alpha, m\alpha) \in \mathbb{R}^2 : m \in \mathbb{Z}\} \cup \{(n\beta, -n\beta) \in \mathbb{R}^2 : n \in \mathbb{Z}\}.$$

The corresponding Dirac equation in this context is

$$(-i\sigma^0\partial_t - i\sigma^1\partial_x + \pi)\psi = 0, \tag{2.2}$$

where σ^0, σ^1 are the 2×2 matrices given by

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here, $\psi = (\psi_1, \psi_2)$ in column form, and (2.2) may be written out more explicitly as the system

$$\begin{cases} -i\partial_t\psi_1 - i\partial_x\psi_2 + \pi\psi_1 = 0, \\ i\partial_t\psi_2 + i\partial_x\psi_1 + \pi\psi_2 = 0. \end{cases} \tag{2.3}$$

The question pops up whether the Dirac equation (2.2) has an infinite-dimensional space of solutions $\psi = (\psi_1, \psi_2)$ that vanish on $\Lambda'_{\alpha,\beta}$ for $\alpha\beta > 1$. As both ψ_1, ψ_2 automatically solve the Klein–Gordon equation (this is a consequence of the factorization we mentioned previously in the context of three spatial dimensions), the natural requirement is that both ψ_1, ψ_2 are Fourier transform of measures in $\mathrm{AC}(\Gamma', \Lambda'_{\alpha,\beta})$. Here, Γ' is the hyperbola $t^2 = x^2 + 1$, which corresponds to Γ after the change of variables. From the assumptions made on ψ_1, ψ_2, we have

$$\psi_j(t, x) = \int_{-\infty}^{\infty} f_j(v)\, e^{i\frac{1}{2}\pi[v(t+x)+v^{-1}(t-x)]}\, dv, \qquad j = 1, 2,$$

where f_1, f_2 belong to $\mathcal{M}_\beta^\perp$ (this subspace of $L^1(\mathbb{R})$ is defined by (1.3)). Note that in the last step, we tacitly imposed the normalizing assumption that $\alpha = 1$. As we implement this representation of ψ_1, ψ_2 into (2.3), we find that

$$\int_{-\infty}^{\infty} \{(v + v^{-1} + 2) f_1(v) + (v - v^{-1}) f_2(v)\} e^{i\frac{1}{2}\pi[v(t+x)+v^{-1}(t-x)]}\, dv = 0$$

and

$$\int_{-\infty}^{\infty} \{(v + v^{-1} - 2) f_2(v) + (v - v^{-1}) f_1(v)\} e^{i\frac{1}{2}\pi[v(t+x)+v^{-1}(t-x)]}\, dv = 0.$$

When we plug in $t = x$, we see from the uniqueness theorem for the Fourier transform that the above two equations are equivalent to having

$$(v + v^{-1} + 2) f_1(v) + (v - v^{-1}) f_2(v) = 0, \qquad v \in \mathbb{R},$$

and

$$(v - v^{-1}) f_1(v) + (v + v^{-1} - 2) f_2(v) = 0, \qquad v \in \mathbb{R},$$

in the almost-everywhere sense. These requirements are compatible, as each corresponds to having

$$f_2(v) = \frac{1 + v}{1 - v}\, f_1(v), \qquad v \in \mathbb{R}.$$

This means that we have reduced the study of the dimension of the space of solutions to the Dirac equation (2.2) subject to vanishing on $\Lambda'_{\alpha,\beta}$ (with $\alpha = 1$) plus the condition in terms of the Fourier transform to simply analyzing the dimension of the space

$$\{f \in \mathcal{M}_\beta^\perp : (1 + x)(1 - x)^{-1} f(x) \text{ is in } \mathcal{M}_\beta^\perp\}.$$

To answer this dimension question we would need to better understand the structure of the pre-annihilator space $\mathcal{M}_\beta^\perp$.

3. Perron–Frobenius operators

3.1. Dynamical systems

The theory of dynamical systems deals with the time evolution of a system of points under a fixed change rule. An important feature of a dynamical system is its attractors, sets of points towards which the points of the system converge. A dynamical system is a four-tuple $(I, \mathfrak{S}, \mu, \tau)$, where $(I, \mathfrak{S}, \mu)$ is a measure space and $\tau : I \to I$ is a measurable transformation. The measure μ is always positive and σ-finite; if it has finite total mass we renormalize and assume that the mass is 1, so that μ becomes a probability measure. We denote by τ^0 the identity map and write $\tau^n = \tau^{n-1} \circ \tau$ for $n = 1, 2, \ldots$. The evolution of a point $x \in I$ is described by its *orbit* under τ, i.e., the sequence

$$\{\tau^n(x)\}_{n=0}^{\infty}.$$

In a concrete situation, the actual expression for the iterates τ^n tends to explode already for rather modest values of n, which makes it extremely difficult to extract substantial information based on a direct approach. The most convenient approach is then the measure-theoretic one based on Perron–Frobenius operators. If we have a random variable $X : I \to \mathbb{R}$ distributed according to a density ρ, then the random variable $X \circ \tau$ will be distributed according to a new density, which is denoted by $\mathbf{P}_\tau \rho$. Instead of the orbits $\tau^n(x)$, we focus on the sequence of density functions

$$\{\mathbf{P}_\tau^n \rho\}_{n=0}^{\infty}.$$

The key point here is that, while τ may be nonlinear and discontinuous, the operator $\mathbf{P}_\tau$ is linear and bounded on the space $L^1(I, \mathfrak{S}, \mu)$ of integrable functions on I. The operator $\mathbf{P}_\tau$ is known as the *Perron–Frobenius operator* associated to the transformation τ. It turns out that in most situations the sequence of density functions $\mathbf{P}_\tau^n \rho$ converges to certain densities of measures on I that provide valuable information about the attractors of the system, which are known as *invariant measures*. More precisely, a σ-finite Borel measure ν on I is said to be *invariant* under τ if $\nu(\tau^{-1}(A)) = \nu(A)$ for every $A \in \mathfrak{S}$. The densities of the μ-absolutely continuous invariant measures can be recovered as eigenfunctions of the Perron–Frobenius operator corresponding to the eigenvalue $\lambda = 1$. Perron–Frobenius operators appear in many branches of pure and applied mathematics, such as stochastic processes, statistical mechanics, resonances, ordinary differential equations, thermodynamics, diffusion problems, positive matrices, and algorithms associated with continued fractions expansions. For background on Perron–Frobenius operators, we refer the reader to, e.g., [3], [5], [9]. In this work, we shall see how Perron–Frobenius operators are intimately related to the Heisenberg uniqueness pairs associated with the hyperbola $x_1 x_2 = 1$. This leads to new and interesting questions concerning this important class of operators.

3.2. Perron–Frobenius operators on bounded intervals

In our situation, the dynamical systems involved are of the form $(I, \mathfrak{B}_I, m, \tau)$, where I is a closed bounded interval of the real line, m is the Lebesgue measure defined on $\mathfrak{B}_I$, the

Borel σ-algebra of I, and τ denotes a measurable map from I into itself. For $1 \le p < \infty$, the Banach space $L^p(I)$ consists of those (equivalence classes of) measurable complex-valued functions f defined on I for which the norm

$$\|f\|_{L^p(I)}^p = \int_I |f|^p \, dm$$

is finite. The space $L^\infty(I)$ consists of the essentially bounded measurable complex-valued functions f supplied with the essential supremum norm. We shall use the following standard bilinear dual action:

$$\langle f, g\rangle_E := \int_E fg \, dm, \tag{3.1}$$

provided f, g are Borel measurable, and $fg \in L^1(E)$. Here, $E \subset \mathbb{R}$ is a Borel set with positive linear measure, $m(E) > 0$. For instance, if $f \in L^1(E)$ and $g \in L^\infty(E)$, the dual action is well-defined. When needed, we shall think of functions in $L^p(E)$ as extended to all of $\mathbb{R}$ by setting them equal to 0 off E.

We shall need the following concepts.

Definition 3.1. The map $\tau : I \to I$ is said to be a *filling C^2-smooth piecewise monotonic transformation* if there exists a countable collection $\{I_u\}_{u\in\mathcal{U}}$ of pairwise disjoint open intervals such that

(i) the set $I \setminus \bigcup\{I_u : u \in \mathcal{U}\}$ has linear Lebesgue measure 0,
(ii) for any $u \in \mathcal{U}$, the restriction of τ to I_u is strictly monotonic and extends to a C^2-smooth function on the closure of I_u, denoted τ_u, and $\tau_u' \neq 0$ in the interior of I_u,
(iii) for every $u \in \mathcal{U}$, τ_u maps the closure of I_u onto I.

Definition 3.2. If, in the setting of Definition 3.1, all conditions are fulfilled, except that (iii) is replaced by the weaker condition (iii′) below, we say that τ is a *partially filling C^2-smooth piecewise monotonic transformation*; the alternative condition is

(iii′) there exists a $\delta > 0$ such that for every $u \in \mathcal{U}$, the interval $\tau(I_u)$ has length $\ge \delta$.

In the context of the above two definitions, each interval I_u is known as a *fundamental interval*, and τ_u is said to be a *branch*. It is an important observation that each iterate τ^n, with $n = 1, 2, \ldots$, has the same basic structure as the transformation τ itself. The fundamental intervals associated with τ^n are given by

$$I^n_{(u_1,\ldots,u_n)} = \{x \in I : x \in I_{u_1}, \tau(x) \in I_{u_2}, \ldots, \tau^{n-1}(x) \in I_{u_n}\}, \qquad (u_1, \ldots, u_n) \in \mathcal{U}^n_\sharp,$$

where $\mathcal{U}^n_\sharp$ consists of those elements $(u_1, \ldots, u_n) \in \mathcal{U}^n$ such that the above interval $I^n_{(u_1,\ldots,u_n)}$ becomes nonempty. The corresponding branch on $I^n_{(u_1,\ldots,u_n)}$ is denoted $\tau^n_{(u_1,\ldots,u_n)} = \tau_{u_n} \circ \cdots \circ \tau_{u_1}$.

The *Koopman operator* associated with τ is the composition operator which acts on $L^\infty(I)$ by the formula $\mathbf{C}_\tau g = g \circ \tau$. Clearly, $\mathbf{C}_\tau$ is linear and norm-contractive on $L^\infty(I)$.

The Perron–Frobenius operator $\mathbf{P}_\tau : L^1(I) \to L^1(I)$ associated with τ is just the pre-adjoint of $\mathbf{C}_\tau$. Therefore, $\mathbf{P}_\tau$ is a norm contraction on $L^1(I)$ with

$$\langle \mathbf{P}_\tau f, g\rangle_I = \langle f, \mathbf{C}_\tau g\rangle_I, \qquad f \in L^1(I),\ g \in L^\infty(I). \tag{3.2}$$

It is immediate from (3.2) that an absolutely continuous measure $d\mu_f = f\,dm$ has the invariance property

$$\mu_f(\tau^{-1}(A)) = \mu(A) \qquad \text{for all } A \in \mathfrak{B}_I$$

if and only if

$$\mathbf{P}_\tau f = f. \tag{3.3}$$

It is clear that $\mathbf{C}_\tau^n = \mathbf{C}_{\tau^n}$, so by duality we have

$$\mathbf{P}_\tau^n = \mathbf{P}_{\tau^n}, \qquad n = 1, 2, \ldots. \tag{3.4}$$

Using (3.2), we find that

$$(\mathbf{P}_\tau f)(x) = \sum_{u\in\mathcal{U}} J_u(x) f(\tau_u^{-1}(x)), \qquad n = 1, 2, \ldots, \tag{3.5}$$

where $J_u \geq 0$ is the function on I that equals $|(\tau_u^{-1})'|$ on $\tau(I_u)$ and vanishes elsewhere. The map J_u is well defined, since τ is piecewise strictly monotonic. By (3.5) we see that $\mathbf{P}_\tau f \geq 0$ for $f \geq 0$ and $\|\mathbf{P}_\tau f\|_1 = \|f\|_1$. As $\mathbf{P}_\tau$ acts contractively on $L^1(I)$, its spectrum $\sigma(\mathbf{P}_\tau)$ is contained in the closed unit disk $\bar{\mathbb{D}}$.

4. Perron–Frobenius operators for Gauss-type maps and invariant measures

4.1. The Gauss-type maps and the corresponding Perron–Frobenius operators

The study of Heisenberg uniqueness pairs for a hyperbola (or a branch of it) involves the study of eigenfunctions of certain operators, which we introduce below.

For $t \in \mathbb{R}$, let $\{t\}_1$ be the number in $[0, 1[$ such that $t - \{t\}_1 \in \mathbb{Z}$. We also need the quantity $\{t\}_2$, which is in $]-1, 1]$ and is uniquely determined by the requirement $t - \{t\}_2 \in 2\mathbb{Z}$. For fixed $0 < \gamma < \infty$, let $\theta_\gamma : [0, 1[\to [0, 1[$ be the mapping defined by $\theta_\gamma(x) := \{\gamma/x\}_1$ for $x \neq 0$ and $\theta_\gamma(0) := 0$. For $1 \leq \gamma < \infty$, we let $\mathbf{C}_{\theta_\gamma} : L^\infty([0, 1[) \to L^\infty([0, 1[)$ be the Koopman operator

$$\mathbf{C}_{\theta_\gamma} g(x) := g \circ \theta_\gamma(x), \qquad x \in [0, 1[,$$

while for $0 < \gamma < 1$, we let $\mathbf{C}_{\theta_\gamma} : L^\infty([0, 1[) \to L^\infty([0, 1[)$ be the *weighted* Koopman operator

$$\mathbf{C}_{\theta_\gamma} g(x) := 1_{[0,\gamma[}(x)\, g \circ \theta_\gamma(x), \qquad x \in [0, 1[.$$

Here and below, we write 1_E for the characteristic function of a set $E \subset \mathbb{R}$, which equals 1 on E and vanishes elsewhere. The definition for $0 < \gamma < 1$ is motivated by the applications in [13] in the context of Heisenberg uniqueness pairs for a branch of the hyperbola. The pre-adjoint to $\mathbf{C}_{\theta_\gamma}$ is the operator $\mathbf{P}_{\theta_\gamma} : L^1([0,1[) \to L^1([0,1[)$ given by

$$\mathbf{P}_{\theta_\gamma} f(x) := \sum_{j=1}^{\infty} \frac{\gamma}{(j+x)^2} f\left(\frac{\gamma}{j+x}\right), \qquad x \in [0,1[, \tag{4.1}$$

with the understanding that f vanishes off $[0,1[$. The operator $\mathbf{P}_{\theta_\gamma}$ is a Perron–Frobenius operator when $1 \le \gamma < \infty$. Note that for $\gamma \ge 2$, it is possible to begin the summation in (4.1) from the integer part of γ, as the terms with lower summation index do not contribute to the sum. It is easy to see that the eigenfunction equation

$$\mathbf{P}_{\theta_\gamma} f = \lambda f, \qquad |\lambda| = 1,$$

fails to have a solution f in $L^1([0,1[)$ for $0 < \gamma < 1$, which implies that $(\Gamma_+, \Lambda_{\alpha,\beta})$ is a Heisenberg uniqueness pair for $\alpha\beta < 4$. In the case $\gamma = 1$, $\mathbf{P}_{\theta_1}$ is the famous *Gauss–Kuzmin–Wirsing* operator, which is connected with the continued fraction algorithm. It is known that $\mathbf{P}_{\theta_1} f = \lambda f$ with $|\lambda| = 1$ has a nontrivial solution only for $\lambda = 1$, in which case the solution f is unique (up to a scalar multiple). These observations are basic in the proof of Theorem B (or, which is the same, Theorem B′); the natural parameter choices are $\alpha = 2$ and $\beta = 2\gamma$.

For $0 < \beta < \infty$, let $\tau_\beta :]-1,1[\to]-1,1[$ be the mapping defined by $\tau_\beta(x) := \{-\beta/x\}_2$ for $x \neq 0$ and $\tau_\beta(0) := 0$. For $1 \le \beta < \infty$, we let $\mathbf{C}_{\tau_\beta} : L^\infty(]-1,1]) \to L^\infty(]-1,1])$ be the Koopman operator

$$\mathbf{C}_{\tau_\beta} g(x) := g \circ \tau_\beta(x), \qquad x \in]-1,1],$$

while for $0 < \beta < 1$, we let $\mathbf{C}_{\tau_\beta} : L^\infty(]-1,1]) \to L^\infty(]-1,1])$ be the *weighted* Koopman operator

$$\mathbf{C}_{\tau_\beta} g(x) := 1_{]-\beta,\beta]}(x)\, g \circ \tau_\beta(x), \qquad x \in]-1,1].$$

The definition for $0 < \beta < 1$ is motivated by the applications in [12] in the context of Heisenberg uniqueness pairs for the hyperbola. The pre-adjoint to $\mathbf{C}_{\tau_\beta}$ is the operator $\mathbf{P}_{\tau_\beta} : L^1(]-1,1[) \to L^1(]-1,1[)$ given by

$$\mathbf{P}_{\tau_\beta} f(x) := \sum_{j \in \mathbb{Z}^\times} \frac{\beta}{(2j-x)^2} f\left(\frac{\beta}{2j-x}\right), \qquad x \in]-1,1], \tag{4.2}$$

with the understanding that f vanishes off the interval $]-1,1]$. Here, we write $\mathbb{Z}^\times := \mathbb{Z} \setminus \{0\}$. The operator $\mathbf{P}_{\tau_\beta}$ is a Perron–Frobenius operator for $1 \le \beta < \infty$. A rather elementary argument shows that the eigenfunction equation

$$\mathbf{P}_{\tau_\beta} f = \lambda f, \qquad |\lambda| = 1,$$

fails to have a solution f in $L^1(]-1,1])$ for $0 < \beta < 1$, which implies that $(\Gamma, \Lambda_{\alpha,\beta})$ is a Heisenberg uniqueness pair for $\alpha\beta < 1$. For $\beta = 1$, the transformation $\tau_1(x) =$

$\{-1/x\}_2$ is related to the continued fraction algorithm with even partial quotients (cf. [18], [27]). The map $\tau_1(x) = \{-1/x\}_2$ has an indifferent fixed point at 1. This entails that the invariant absolutely continuous density has infinite total mass; in this case, the density is given explicitly by $(1-x^2)^{-1}$. Using some ergodicity properties, it is easy to show that the equation $\mathbf{P}_{\tau_1} f = \lambda f$ does not have a solution $f \in L^1(]-1,1])$ for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. These observations are basic to the proof of Theorem A (or, which is the same, Theorem A′).

4.2. Discrete and singular invariant measures for the Gauss map

It is well-known that the Gauss map $\theta_1(x) := \{1/x\}_1$ on the unit interval $[0,1[$ [with $\theta_1(0) := 0$] has infinitely many essentially different invariant measures. However, up to a constant multiple, there is only one that is absolutely continuous: $(1+x)^{-1}\,dx$. As we shall see, the discrete finite invariant measures of θ_1 have an explicit characterization. We write δ_a for the Dirac measure at point a. We shall need the set of fixed points of iterates of the Gauss map θ_1:

$$\Sigma_k := \{a \in [0,1[\,: \theta_1^k(a) = a\}, \qquad \Sigma_\infty := \bigcup_{k=1}^{\infty} \Sigma_k.$$

For $a \in \Sigma_\infty$, there exists a minimal $k \ge 1$ such that $\theta_1^k(a) = a$; we write $k(a)$ for this k. We put

$$\rho_a := \frac{1}{k(a)} \sum_{j=0}^{k(a)-1} \delta_{\theta_1^j(a)}, \qquad a \in \Sigma_\infty.$$

Theorem 4.1. *Let μ be a discrete finite invariant measure for the Gauss map θ_1. Then there is a function $\xi : \Sigma_\infty \to \mathbb{C}$ with $\sum_{a\in\Sigma_\infty} |\xi(a)| < \infty$, such that*

$$\mu = \sum_{a\in\Sigma_\infty} \xi(a)\rho_a. \tag{4.3}$$

Proof. That the measure μ is invariant means that

$$\int_{[0,1[} f\circ\theta_1(x)\,d\mu(x) = \int_{[0,1[} f(x)\,d\mu(x)$$

for every f integrable with respect to $|\mu|$. It is trivial to check that all measures of the given form are invariant. In the other direction, it is well-known that every discrete invariant measure μ may be decomposed into irreducible (ergodic) parts (see, e.g., [8, pp. 16–18]). We just need to show that up to a multiplicative constant, each irreducible part is of the form ρ_a. So, let ρ be an ergodic discrete invariant probability measure on $[0,1[$. Let $E \subset [0,1[$ be the minimal countable set which carries the mass of ρ. By the Birkhoff–Khinchin Ergodic Theorem, for each $x_0 \in E$ we have

$$\frac{1}{n}\sum_{j=0}^{n-1} f(\theta_1^j(x_0)) \to \int_E f\,d\rho \quad \text{as } n \to \infty.$$

In particular, if we let f equal the characteristic function of the one-point set $\{x_0\}$, and observe that from the minimality of E, we have $\rho(\{x_0\}) > 0$, we find that approximately a $\rho(\{x_0\})$ proportion of the time on the interval $0 \le j \le n-1$, we have $\theta_\gamma^j(x_0) = x_0$. This means that $x_0 \in \Sigma_\infty$. We may also conclude by picking other functions f that E equals the orbit of x_0:

$$E = \{\theta_\gamma^j(x_0) : 0 \le j \le k(x_0) - 1\}.$$

By measure invariance, each point of E must have equal mass, so that $\rho = \rho_a$ with $a = x_0$. This completes the proof. □

Remark 4.2. (a) The proof of Theorem 4.1 did not really use the fact that we are dealing with the Gauss map. In particular, the corresponding assertion holds for the transformation $\tau_1(x) = \{-1/x\}_2$ in place of the Gauss map.

(b) The set Σ_∞ consists of the reciprocals of the reduced quadratic irrationals plus the origin.

(c) The discrete measures provided by Theorem 4.1 above extend in a natural fashion to the positive half-axis $\mathbb{R}_+$. These extensions lift to the hyperbola branch Γ_+ where $x_1 x_2 = 1$ and $x_1 > 0$, and the lifted measures on Γ_+ have Fourier transforms that vanish on the lattice-cross $\Lambda_{\alpha,\beta}$ with $\alpha = \beta = 2$. Indeed, this is the only way to obtain such discrete measures on Γ_+.

4.3. *The Minkowski measure*

The most studied singular continuous measure for the Gauss–Kuzmin–Wirsing operator is the Minkowski measure. It belongs to a family of singular probability measures which we call Markovian measures (see below). The Minkowski question mark function was first introduced by Minkowski in 1904. Let $\{a_n(x)\}_{n=1}^\infty$ be the sequence of positive integers in the continued fraction expansion of x. Salem [23] proved that the question mark function can be defined by

$$?(x) = \sum_{j=1}^{\infty} \frac{2(-1)^{j+1}}{2^{a_1(x)+\cdots+a_j(x)}}, \qquad 0 \le x \le 1,$$

where the series is finite for rational x. Then ? is a strictly increasing continuous singular function. It takes rational numbers to dyadic numbers and quadratic surds to rationals. The Riemann–Stieltjes measure $d?$ on $[0, 1]$ is then a singular continuous probability measure, which can be shown to be invariant for the Gauss map $\theta_1(x) = \{1/x\}_1$. The numbers $a_j(x) \in \mathbb{Z}_+$ are the successive remainders which we throw away as we iterate the Gauss map at a point x. Let us say that a probability measure μ on $[0, 1]$ is *Markovian* with respect to the Gauss map if there are numbers $q(j)$ with $0 \le q(j) < 1$ and

$$\sum_{j=1}^{\infty} q(j) = 1$$

such that the μ-mass of the "cylinder set"

$$\{x \in [0, 1[\,:\, a_j(x) = b_j \text{ for } j = 1, \ldots, k\}$$

equals

$$\prod_{j=1}^{k} q(b_j).$$

Then the Minkowski measure is Markovian with $q(j) = 2^{-j}$. Problem (d) in [12] could be settled in the negative if we could find a singular continuous measure on Γ, the Fourier transform of which tends to zero at infinity, while it vanishes along $\Lambda_{\alpha,\beta}$ with $\alpha = \beta = 1$. A perhaps easier task is to find a singular invariant measure on [0, 1] with respect to the Gauss map such that the Fourier transform tends to zero at infinity (measures whose Fourier transforms decay to 0 at infinity are called *Rajchman measures*). The Markovian measures are all invariant and singular continuous. E.g., the Minkowski measure is of this type. But it is not known if it is Rajchman. Indeed, this question is a well-known problem raised by Salem [23].

5. Further properties of Perron–Frobenius operators

5.1. The spectral decomposition of Perron–Frobenius operators

Let I be a closed bounded interval. The total variation of a complex-valued function $h : I \to \mathbb{C}$ is

$$\mathrm{var}_I(h) = \sup\left\{\sum_{j=1}^{n-1} |h(t_{j+1}) - h(t_j)|\right\},$$

where the supremum is taken over all $t_1, \ldots, t_n \in I$ with $t_1 < \cdots < t_n$. The function h is said to be of *bounded variation* when $\mathrm{var}_I(h) < \infty$. We will denote by $\mathrm{BV}(I)$ the subspace of $L^1(I)$ functions which have representatives of bounded variation. The space $\mathrm{BV}(I)$ becomes a Banach space when supplied, e.g., with the norm

$$\|h\|_{\mathrm{BV}} = \|h\|_1 + \inf_{\tilde{h}\sim h} \mathrm{var}_I(\tilde{h}), \qquad h \in \mathrm{BV}(I), \tag{5.1}$$

where the infimum is taken over all elements in the equivalence class of h (so that $\tilde{h} = h$ except on a Lebesgue null set). It is well-known that for each $h \in \mathrm{BV}(I)$ there is a right-continuous function in the class of h where the infimum in the definition of $\|h\|_{\mathrm{BV}}$ in (5.1) is attained. In particular, $\mathrm{BV}(I)$ is a subspace of $L^\infty(I)$ (see [16]). A perhaps more precise description of $\mathrm{BV}(I)$ is that these are the primitive functions of finite complex-valued Borel measures on I.

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle in $\mathbb{C}$ and let $\sigma_{\mathrm{p}}(\mathbf{P}_\tau)$ denote the point spectrum of $\mathbf{P}_\tau$, where the Perron–Frobenius operator $\mathbf{P}_\tau$ is thought of as acting on $L^1(I)$. As a consequence of the Ionescu-Tulcea and Marinescu theorem (see [5] and [16, Section 5.3]), the following spectral decomposition holds for $\mathbf{P}_\tau$. We recall the notions of filling and partially filling C^2-smooth monotonic transformations in Definitions 3.1 and 3.2.

Theorem C. *Suppose $\tau : I \to I$ is a partially filling C^2-smooth piecewise monotonic transformation with the following properties:*

(i) [uniform expansiveness] *There exist an integer $m \geq 1$ and a positive real ϵ such that $|(\tau^m)'(x)| \geq 1 + \epsilon$ for all $x \in \bigcup\{I_u : u \in \mathcal{U}_\sharp^m\}$.*

(ii) [second derivative condition] *There exists a positive constant* M *such that* $|\tau''(x)| \le M|\tau'(x)|^2$ *for all* $x \in \bigcup\{I_u : u \in \mathcal{U}\}$.

Then $\Lambda_\tau := \sigma_{\mathrm{p}}(\mathbf{P}_\tau) \cap \mathbb{T}$ *is finite and nonempty, say* $\Lambda_\tau = \{\lambda_1, \dots, \lambda_p\}$. *Here, one of the eigenvalues is the point* 1, *say* $\lambda_1 = 1$. *If* E_i *denotes the eigenspace of* $\mathbf{P}_\tau$ *corresponding to* λ_i, *then* E_i *is finite-dimensional and* $E_i \subset \mathrm{BV}(I)$. *In addition,*

$$\mathbf{P}_\tau^n h = \sum_{i=1}^{p} \lambda_i^n \mathbf{P}_{\tau,i} h + \mathbf{Z}_\tau^n h, \qquad h \in L^1(I),\ n = 1, 2, \dots,$$

where the operators $\mathbf{P}_{\tau,i}$ *are* (*Banach space*) *projections onto* E_i, *and the operator* $\mathbf{Z}_\tau$ *acts boundedly on* $L^1(I)$ *as well as on* $\mathrm{BV}(I)$. *Moreover, the spectrum of* $\mathbf{Z}_\tau$ *as an operator acting on* $\mathrm{BV}(I)$ *is contained in the open unit disk* $\mathbb{D}$; *i.e.,* $\mathbf{Z}_\tau$ *acting on* $\mathrm{BV}(I)$ *has spectral radius* < 1.

Remark 5.1. (a) It is a by-product of Theorem C that $\mathbf{P}_\tau$ acts boundedly on $\mathrm{BV}(I)$. In fact, the way things work is that this rather elementary observation is the beginning of the analysis that leads up to Theorem C.

(b) Since 1 is an eigenvalue of $\mathbf{P}_\tau$, a corresponding eigenfunction (which is then in $\mathrm{BV}(I)$) is the density for an invariant measure. If there are several such eigenfunctions for $\lambda_1 = 1$, then one of them is ≥ 0, which we can normalize so that we get an absolutely continuous invariant probability measure with density in $\mathrm{BV}(I)$; compare with the proof of Theorem 7.2.

(c) The formulation of the Ionescu-Tulcea and Marinescu theorem in [16] initially assumes that τ is "filling", but it is later remarked that the theorem holds for "partially filling" transformations (cf. Definitions 3.1 and 3.2); see [16, p. 214], and also [6] and [8, p. 169].

(d) When considered as an operator on $L^1(I)$, the Perron–Frobenius operator $\mathbf{P}_\tau$ will usually have eigenvalues at all points of the open disk, with eigenfunctions in $L^\infty([-1, 1])$ (cf. [17]).

Remark 5.2. From the presentation in [16, Section 5.3], it is clear that if τ is "filling", we have a stronger assertion in Theorem C: $\lambda_1 = 1$ is the only eigenvalue of $\mathbf{P}_\tau$ on $\mathbb{T}$, and the τ-invariant absolutely continuous probability measure is unique, with a density that is bounded from above and below by two positive constants. Cf. also [8, p. 172], where it is shown that under the given assumptions, τ is *mixing*. We briefly outline the argument, following the presentation in [16, Section 5.3]. We write $f_u := \tau_u^{-1} : I \to \bar{I}_u$ for the inverse branches ($u \in \mathcal{U}$). The assumptions (i) and (ii) of Theorem C correspond to the conditions (E_m) and (A) of [16, pp. 191–192]. Next, by [16, Proposition 5.3.3], (E_m) forces $|f_u'(x)|$ to be uniformly bounded in $x \in I$ and $u \in \mathcal{U}$. In view of Condition (A) of [16, p. 192], we also know that $|f_u''(x)|$ is uniformly bounded in $x \in I$ and $u \in \mathcal{U}$. In particular, then, f_u' is absolutely continuous for all $u \in \mathcal{U}$. Next, by [16, Proposition 5.3.4], we use this absolute continuity together with condition (E_m) of [16, p. 191], to see that condition (C) [Rényi's distortion estimate] holds as well. Next, by [16, Theorem 5.3.5], we use condition (C) to find that there is a unique ergodic τ-invariant absolutely continuous probability measure, and that its density is bounded from above and below by positive constants. This means that there is only one eigenvalue, namely 1. We mention here that the condition (BV) of [16, p. 200]—which requires the sum of the variations of f_u' over $u \in \mathcal{U}$ to be bounded—is a trivial consequence of condition (A), because the sum of the variations of f_u' over $u \in \mathcal{U}$ amounts to summing the lengths of the intervals I_u, which add up to the length of I.

5.2. *The folklore theorem (Adler's theorem)*

We shall be interested in partially filling C^2-smooth monotonic transformations of a closed bounded interval I. In view of Remark 5.1(c), we can be sure that Theorem C also holds in this more general situation. Moreover, Remark 5.1(b) tells us that there exists a τ-invariant absolutely continuous probability measure, but it might not be unique. To get uniqueness, we need to make stronger assumptions on τ. In this direction we have Adler's theorem, also known as the folklore theorem (see [5]).

Theorem D (Adler's theorem). *Let $\tau : I \to I$ be a partially filling C^2-smooth piecewise monotonic transformation with the following properties:*

(i) [uniform expansiveness] *There exist an integer $m \geq 1$ and a positive real ϵ such that $|(\tau^m)'(x)| \geq 1 + \epsilon$ for all $x \in \bigcup\{I_u : u \in \mathcal{U}_\sharp^n\}$.*
(ii) [second derivative condition] *There exists a positive constant M such that $|\tau''(x)| \leq M|\tau'(x)|^2$ for all $x \in \bigcup\{I_u : u \in \mathcal{U}\}$.*
(iii) [Markov property 1] *For every $u \in \mathcal{U}$ there is $n = n(u) \geq 1$ such that* $\mathrm{clos}[\tau^n(I_u)] = I$.
(iv) [Markov property 2] *Whenever $\tau(I_u) \cap I_v \neq \emptyset$ for some two indices $u, v \in \mathcal{U}$, then $\tau(I_u) \supset I_v$.*

Then τ admits a unique absolutely continuous invariant probability measure $d\rho = \varrho\, dm$. Moreover, the density ϱ is bounded from above and below by positive constants.

Remark 5.3. (a) A transformation τ satisfying (iii)–(iv) above is said to be a *Markov map*.

(b) A well-known result which preceded Adler's theorem is the Lasota and Yorke theorem [19].

5.3. *Dynamical properties of Gauss-type maps*

We first consider the transformation τ_β of the interval $]-1, 1]$ defined by $\tau_\beta(0) := 0$ and

$$\tau_\beta(x) := \{-\beta/x\}_2, \qquad x \neq 0. \tag{5.2}$$

Here, we recall that for $t \in \mathbb{R}$, $\{t\}_2$ denotes the unique number in $]-1, 1]$ with $t - \{t\}_2$ in $2\mathbb{Z}$. We *restrict our attention to $\beta > 1$ only*. Let the index set $\mathcal{U} = \mathcal{U}_\beta$ be the subset of the nonzero integers u for which the corresponding branch interval is nonempty:

$$I_u := \left]\frac{\beta}{2u+1}, \frac{\beta}{2u-1}\right[\cap \,]-1, 1[\,\neq \emptyset. \tag{5.3}$$

We put $u_0 = u_0(\beta) := \frac{1}{2}(\beta - \{\beta\}_2)$, which is an integer ≥ 1. We note that if β is an *odd integer*, then

$$I_u = \left]\frac{\beta}{2u+1}, \frac{\beta}{2u-1}\right[, \qquad u \in \mathcal{U}, \tag{5.4}$$

and $\mathcal{U}$ consists of all $u \in \mathbb{Z}^\times$ with $|u| \ge \frac{1}{2}(\beta + 1)$. In this case, the "filling" requirement is fulfilled: $\tau_\beta(I_u) =]-1, 1[$ for all $u \in \mathcal{U}$. More generally, when β is not an odd integer, then $\mathcal{U}$ consists of all $u \in \mathbb{Z}^\times$ with $|u| \ge u_0$, and we have

$$I_u = \left]\frac{\beta}{2u+1}, \frac{\beta}{2u-1}\right[, \qquad u \in \mathcal{U} \setminus \{\pm u_0\}, \tag{5.5}$$

so that

$$\tau_\beta(I_u) =]-1, 1[, \qquad u \in \mathcal{U} \setminus \{\pm u_0\}.$$

We see that the deviation from the "filling" requirement is rather slight (just two branches fail).

Next, we quickly calculate the derivative of τ_β:

$$\tau_\beta'(x) = \beta/x^2 \ge \beta > 1, \qquad x \in \bigcup\{I_u : u \in \mathcal{U}\}, \tag{5.6}$$

so the uniform expansiveness condition is met already by τ_β (with $m = 1$). Moreover,

$$\frac{|\tau_\beta''(x)|}{|\tau_\beta'(x)|^2} \le \frac{2|x|}{\beta} \le 2, \qquad x \in \bigcup\{I_u : u \in \mathcal{U}\},$$

so we also have the second derivative control. Unfortunately, we cannot rely on Theorem C to give us the uniqueness of the absolutely continuous invariant measure for τ_β (although Remark 5.2 says that in the "filling" case we really do have uniqueness). We cannot rely on Adler's theorem either, as τ_β is not necessarily a Markov map. We will show that condition (iii) of Adler's theorem holds for all sufficiently large n, say $n \ge n(u)$; this has the interpretation of "strong mixing".

We are also interested in the Gauss-type map θ_γ of $[0, 1[$ defined by $\theta_\gamma(0) := 0$ and

$$\theta_\gamma(x) := \{\gamma/x\}_1, \qquad x \ne 0. \tag{5.7}$$

Here, we recall that $\{t\}_1$ is the fractional part of $t \in \mathbb{R}$, with values in $[0, 1[$ and $t - \{t\}_1 \in \mathbb{Z}$. We *restrict our attention to* $\gamma > 1$ *only*. Let the index set $\mathcal{V} = \mathcal{V}_\gamma$ be the subset of the positive integers v for which the corresponding branch interval is nonempty:

$$J_v := \left]\frac{\gamma}{v+1}, \frac{\gamma}{v}\right[\cap]0, 1[\ne \emptyset.$$

We note that if γ is an integer, then $\mathcal{V}$ consists of all positive integers v with $v \ge \gamma$, and

$$J_v = \left]\frac{\gamma}{v+1}, \frac{\gamma}{v}\right[, \qquad v \in \mathcal{V}.$$

More generally, if $v_0 = v_0(\gamma) := \gamma - \{\gamma\}_1 \in \mathbb{Z}_+$, then

$$J_v = \left]\frac{\gamma}{v+1}, \frac{\gamma}{v}\right[, \qquad v \in \mathcal{V} \setminus \{v_0\},$$

and

$$\theta_\gamma(J_v) =]0, 1[, \qquad u \in \mathcal{V} \setminus \{v_0\}.$$

So, the deviation from the "filling" requirement is rather slight (only one branch fails). Next, we calculate the derivative of θ_γ:

$$|\theta'_\beta(x)| = \gamma/x^2 \geq \beta > 1, \qquad x \in \bigcup\{J_v : v \in \mathcal{U}\},$$

so the uniform expansiveness condition is met already by θ_β (with $m = 1$). Moreover,

$$\frac{|\theta''_\gamma(x)|}{|\theta'_\gamma(x)|^2} \leq \frac{2x}{\gamma} \leq 2, \qquad x \in \bigcup\{J_v : v \in \mathcal{V}\},$$

so we also have the second derivative control. Again, we cannot unfortunately rely on Theorem C to give us the uniqueness of the absolutely continuous invariant probability measure for θ_γ (although Remark 5.2 says that in the "filling" case we have uniqueness). We cannot rely on Adler's theorem either, as θ_γ is not necessarily a Markov map. However, it appears that here, it is nevertheless known that the absolutely continuous θ_γ-invariant probability measure is unique and has strictly positive density almost everywhere. One way to see this is to show that condition (iii) of Adler's theorem is fulfilled for all $n \geq n(u)$, and proceed in an analogous fashion as we do for τ_β, with $\beta > 1$ (cf. Lemma 7.1 and Theorem 7.2). Actually, once the "strong mixing" property of condition (iii) of Adler's theorem has been verified, the proof of Theorem 7.2 applies more or less verbatim, and gives the asserted properties. Also compare with [8, pp. 168–177].

5.4. The explicit calculation of invariant measures

In general, the computation of the absolutely continuous invariant measures for τ_β (as well as for θ_γ) is intractable. Only in a few particular cases is it possible to supply explicit expressions for the corresponding densities. They all correspond to values of the parameters for which we are dealing with Markov maps. For instance, when $\beta > 1$ is an odd integer, τ_β is "filling" [i.e., we have complete branches], and the unique τ_β-invariant probability measure on $[-1, 1]$ is given by

$$\frac{c(\beta)}{1-(x/\beta)^2} 1_{[-1,1]}(x)\, dx, \quad \text{where}$$

$$\frac{1}{c(\beta)} = \int_{-1}^{1} \frac{dx}{1-(x/\beta)^2} = \beta \log \frac{\beta+1}{\beta-1} \qquad (\beta = 3, 5, 7, \ldots).$$

It is more interesting that it is possible to obtain the τ_β-invariant probability density in a more complicated situation, when $\beta = 3/2$. The uniqueness and ergodicity of that measure will be obtained in Section 7.

Proposition 5.4 ($\beta = 3/2$). *The density of the unique ergodic $\tau_{3/2}$-invariant absolutely continuous probability measure is given by*

$$\varrho_0(x) = c_0 \left\{ \frac{1}{1-(2x/3)^2} 1_{[-1/2,1/2]}(x) + \frac{3/4}{(1-|x|/3)(1+2|x|/3)} 1_{[-1,1]\setminus[-1/2,1/2]}(x) \right\},$$

where $c_0^{-1} = 3\log 5 - \frac{9}{2}\log 2$.

Proof. To simplify the notation, we write $\varrho_1(x) := c_0^{-1}\varrho_0(x)$, so that $\varrho_1(x)$ stands for the bracketed expression. We note that both ϱ_0, ϱ_1 are even functions. We need to check that

$$\sum_{j \in \mathbb{Z}^\times} \frac{3/2}{(2j-x)^2} \varrho_1\left(\frac{3/2}{2j-x}\right) = \varrho_1(x), \qquad x \in [-1,1], \tag{5.8}$$

where this equality should be understood in the almost-everywhere sense. Since

$$\frac{3/2}{2j-x} \in [-1/2, 1/2] \quad \text{for } |j| \geq 2,\ x \in [-1,1],$$

we may evaluate the sum of all but two terms on the left-hand side of (5.8), as most of the terms cancel:

$$\begin{aligned}
\sum_{j:\,|j|\geq 2} \frac{3/2}{(2j-x)^2} \varrho_1\left(\frac{3/2}{2j-x}\right) &= \sum_{j:\,|j|\geq 2} \frac{3/2}{(2j-x)^2} \times \frac{1}{1-\frac{1}{(2j-x)^2}} \\
&= \sum_{j:\,|j|\geq 2} \frac{3/2}{(2j-x)^2-1} = \frac{3}{4} \sum_{j:\,|j|\geq 2} \left[\frac{1}{2j-x-1} - \frac{1}{2j-x+1}\right] \\
&= \frac{3}{4}\left[\frac{1}{3-x} + \frac{1}{3+x}\right] = \frac{1/2}{1-(x/3)^2}, \qquad x \in [-1,1].
\end{aligned}$$

Next, we see that

$$\frac{3/2}{2-x} \in]1/2, 1], \qquad x \in]-1, 1/2],$$

and that this expression is in $]1, 3/2]$ for $x \in]1/2, 1]$. Here, we may of course replace x by $-x$ if we make the necessary adjustments. It follows that

$$\begin{aligned}
&\sum_{j:|j|=1} \frac{3/2}{(2j-x)^2} \varrho_1\left(\frac{3/2}{2j-x}\right) = \frac{3/2}{(2-x)^2} \varrho_1\left(\frac{3/2}{2-x}\right) + \frac{3/2}{(2+x)^2} \varrho_1\left(-\frac{3/2}{2+x}\right) \\
&= \frac{3/2}{(2-x)^2} \times \frac{3/4}{\left(1-\frac{1/2}{2-x}\right)\left(1+\frac{1}{2-x}\right)} 1_{[-1,1/2]}(x) \\
&\quad + \frac{3/2}{(2+x)^2} \times \frac{3/4}{\left(1-\frac{1/2}{2+x}\right)\left(1+\frac{1}{2+x}\right)} 1_{[-1/2,1]}(x) \\
&= \frac{1/4}{(1-2x/3)(1-x/3)} 1_{[-1,1/2]}(x) + \frac{1/4}{(1+2x/3)(1+x/3)} 1_{[-1/2,1]}(x) \qquad x \in]-1, 1[,
\end{aligned}$$

We may now express the whole sum in (5.8) as

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}^\times}\frac{3/2}{(2j-x)^2}\varrho_1\bigg(\frac{3/2}{2j-x}\bigg)\\
&=\frac{1/2}{1-(x/3)^2}+\frac{1/4}{(1-2x/3)(1-x/3)}1_{[-1,1/2]}(x)+\frac{1/4}{(1+2x/3)(1+x/3)}1_{[-1/2,1]}(x)\\
&=\frac{1}{4}\bigg[\frac{1}{1-x/3}+\frac{1}{1+x/3}\bigg]+\frac{1}{4}\bigg[\frac{2}{1-2x/3}-\frac{1}{1-x/3}\bigg]1_{[-1,1/2]}(x)\\
&\quad+\frac{1}{4}\bigg[\frac{2}{1+2x/3}-\frac{1}{1+x/3}\bigg]1_{[-1/2,1]}(x)\\
&=\frac{1}{4}\bigg[\frac{2}{1-2x/3}+\frac{2}{1+2x/3}\bigg]1_{[-1/2,1/2]}(x)+\frac{1}{4}\bigg[\frac{1}{1-x/3}+\frac{2}{1+2x/3}\bigg]1_{]1/2,1]}(x)\\
&\quad+\frac{1}{4}\bigg[\frac{1}{1+x/3}+\frac{2}{1-2x/3}\bigg]1_{]-1,-1/2[}(x)\\
&=\frac{1/2}{1-(2x/3)^2}1_{[-1/2,1/2]}(x)\\
&\quad+\frac{3/4}{(1-|x|/3)(1+2|x|/3)}1_{[-1,1]\setminus[-1/2,1/2]}(x)=\varrho_1(x),\qquad x\in\,]-1,1[.
\end{aligned}
$$

The constant c_0 is determined by the requirement that we should have a probability density, and is easily computed. The proof is complete. □

Remark 5.5. (a) It is possible to establish with similar means the τ_β-invariant absolutely continuous probability measure for $\beta = n(2n+1)/(n+1)$, where n is a positive integer.

(b) We mention here that the analogous θ_γ-invariant absolutely continuous probability measures are known explicitly for $\gamma \in \mathbb{Z}_+$ (see e.g. [7]):

$$
\frac{c(\gamma)}{1+x/\gamma}1_{[0,1]}(x)\,dx,\quad\text{where}\quad\frac{1}{c(\gamma)}=\int_0^1\frac{dx}{1+x/\gamma}=\gamma\log(1+1/\gamma)\quad(\gamma=1,2,\ldots).
$$

6. Characterization of the pre-annihilator space $\mathcal{M}_\beta^\perp$

6.1. Purpose of the section; some notation

In this section we provide a characterization of the subspace $\mathcal{M}_\beta^\perp$ in terms of certain operators. We proceed in a fashion somewhat similar to that used in the proof of Lemma 5.2 in [12]. We recall from Subsection 3.2 that the Koopman operator for τ_β is denoted by $\mathbf{C}_{\tau_\beta}$, and recall from Subsection 4.1 that the corresponding Perron–Frobenius operator is $\mathbf{P}_{\tau_\beta} : L^1([-1,1]) \to L^1([-1,1])$, given by

$$
\mathbf{P}_{\tau_\beta}h(x)=\sum_{j\in\mathbb{Z}^\times}\frac{\beta}{(2j-x)^2}h\bigg(\frac{\beta}{2j-x}\bigg),\qquad x\in[-1,1],
$$

with the understanding that $h \in L^1([-1,1])$ vanishes off $[-1,1]$. Following the notation of [12], we denote by $L^\infty_2(\mathbb{R})$ the subspace of $L^\infty(\mathbb{R})$ of 2-periodic functions, which is the same as the weak-star closure of

$$\mathrm{span}\{e^{in\pi x} : n \in \mathbb{Z}\}.$$

Likewise, we let $L^\infty_{\langle\beta\rangle}(\mathbb{R})$ be the weak-star closure of

$$\mathrm{span}\{e^{in\pi\beta/x} : n \in \mathbb{Z}\},$$

which also has a characterization in terms of periodicity [$f \in L^\infty_{\langle\beta\rangle}(\mathbb{R})$ if and only if the function $f(\beta/x)$ is in $L^\infty_2(\mathbb{R})$]. Let $\mathbf{S} : L^\infty([-1,1]) \to L^\infty(\mathbb{R} \setminus [-1,1])$ be the operator defined by

$$\mathbf{S}g(x) = g(\{x\}_2), \quad x \in \mathbb{R} \setminus [-1,1], \tag{6.1}$$

and let $\mathbf{T} : L^\infty(\mathbb{R} \setminus [-\beta,\beta]) \to L^\infty([-\beta,\beta])$ be the operator given by

$$\mathbf{T}g(x) = g\left(-\frac{\beta}{\{-\beta/x\}_2}\right), \quad x \in [-\beta,\beta] \setminus \{0\}. \tag{6.2}$$

It is clear that $\mathbf{S}$ and $\mathbf{T}$ are linear operators and that they both have norm 1 on the L^∞ spaces where they are defined. As a consequence, their pre-adjoints $\mathbf{S}^*$ and $\mathbf{T}^*$ are norm contractions on the corresponding L^1 spaces. The way things are set up, we have

$$L^\infty_2(\mathbb{R}) = \{g + \mathbf{S}g : g \in L^\infty(-1,1)\}, \tag{6.3}$$

$$L^\infty_{\langle\beta\rangle}(\mathbb{R}) = \{g + \mathbf{T}g : g \in L^\infty(\mathbb{R} \setminus [-\beta,\beta])\}. \tag{6.4}$$

We need the following restriction operators (recall that $\beta > 1$):

$$\begin{aligned}
&\mathbf{R}_1 : L^\infty(\mathbb{R} \setminus [-1,1]) \to L^\infty(\mathbb{R} \setminus [-\beta,\beta]),\\
&\mathbf{R}_2 : L^\infty([-\beta,\beta]) \to L^\infty([-1,1]),\\
&\mathbf{R}_3 : L^\infty([-\beta,\beta]) \to L^\infty([-\beta,\beta] \setminus [-1,1]),\\
&\mathbf{R}_4 : L^\infty(\mathbb{R} \setminus [-1,1]) \to L^\infty([-\beta,\beta] \setminus [-1,1]).
\end{aligned}$$

These operators just restrict the given function to a subset, which makes each a norm contraction. The corresponding pre-adjoints $\mathbf{R}_i^*$, for $i = 1,2,3,4$, act on the corresponding L^1 spaces, and just extend the given function to a larger set by setting it equal to zero where it was previously undefined. As $\beta > 1$, we easily check that $\mathbf{C}_\beta^2 = \mathbf{R}_2\mathbf{T}\mathbf{R}_1\mathbf{S}$. Taking the pre-adjoint of both sides, we get

$$\mathbf{P}^2_{\tau_\beta} = \mathbf{S}^*\mathbf{R}_1^*\mathbf{T}^*\mathbf{R}_2^*. \tag{6.5}$$

6.2. *The characterization of the pre-annihilator space $\mathcal{M}_\beta^\perp$*

We now supply the criterion which characterizes when a given $f \in L^1(\mathbb{R})$ belongs to $\mathcal{M}_\beta^\perp$.

Proposition 6.1 $(1 < \beta < \infty)$. *Let* $f \in L^1(\mathbb{R})$ *be written as*

$$f = f_1 + f_2 + f_3,$$

where $f_1 \in L^1([-1,1])$, $f_2 \in L^1([-\beta,\beta] \setminus [-1,1])$, *and* $f_3 \in L^1(\mathbb{R} \setminus [-\beta,\beta])$. *Then* $f \in \mathcal{M}_\beta^\perp$ *if and only if*

$$\text{(i)} \qquad (\mathbf{I} - \mathbf{P}_{\tau_\beta}^2) f_1 = \mathbf{S}^*(-\mathbf{R}_4^* + \mathbf{R}_1^* \mathbf{T}^* \mathbf{R}_3^*) f_2,$$

$$\text{(ii)} \qquad f_3 = -\mathbf{T}^* \mathbf{R}_2^* f_1 - \mathbf{T}^* \mathbf{R}_3^* f_2,$$

where $\mathbf{I}$ *is the identity on* $L^1([-1,1])$.

Proof. In view of the definition (1.3) of $\mathcal{M}_\beta^\perp$, and the representations (6.3) and (6.4) of $L_2^\infty(\mathbb{R})$ and $L_{\langle\beta\rangle}^\infty(\mathbb{R})$, we see that $f = f_1 + f_2 + f_3$ is in $\mathcal{M}_\beta^\perp$ if and only if

$$\langle f, g + \mathbf{S}g \rangle_\mathbb{R} = \langle f_1 + f_2 + f_3, g + \mathbf{S}g \rangle_\mathbb{R} = 0, \qquad g \in L^\infty([-1,1]),$$
$$\langle f, h + \mathbf{T}h \rangle_\mathbb{R} = \langle f_1 + f_2 + f_3, h + \mathbf{T}h \rangle_\mathbb{R} = 0, \qquad h \in L^\infty(\mathbb{R} \setminus [-\beta,\beta]).$$

Here, it is assumed that all functions f_1, f_2, f_3 are understood to vanish outside their domain of definition. We see that the above equations simplify to

$$\langle f_1, g \rangle_{[-1,1]} + \langle f_2 + f_3, \mathbf{S}g \rangle_{\mathbb{R}\setminus[-1,1]} = 0, \qquad g \in L^\infty([-1,1]),$$
$$\langle f_3, h \rangle_{\mathbb{R}\setminus[-\beta,\beta]} + \langle f_1 + f_2, \mathbf{T}h \rangle_{[-\beta,\beta]} = 0, \qquad h \in L^\infty(\mathbb{R} \setminus [-\beta,\beta]).$$

These equations are equivalent to having

$$f_1 = -\mathbf{S}^*(f_2 + f_3),$$
$$f_3 = -\mathbf{T}^*(f_1 + f_2).$$

A more precise formulation is

$$f_1 = -\mathbf{S}^* \mathbf{R}_4^* f_2 - \mathbf{S}^* \mathbf{R}_1^* f_3, \tag{6.6}$$
$$f_3 = -\mathbf{T}^* \mathbf{R}_2^* f_1 - \mathbf{T}^* \mathbf{R}_3^* f_2. \tag{6.7}$$

We note first that (6.7) is the same as (ii). Next, we substitute (6.7) into (6.6) and take into account (6.5); the result is (i). This completes the proof. □

7. Exterior spectrum of the Perron–Frobenius operator for a Gauss-type map on $[-1,1]$

7.1. Purpose of the section

In this section we will show that $\lambda_1 = 1$ is a simple eigenvalue of $\mathbf{P}_{\tau_\beta}$. This corresponds to having a unique absolutely continuous invariant probability measure for τ_β with $\beta > 1$. We will also prove that $\sigma_\mathrm{p}(\mathbf{P}_{\tau_\beta}) \cap \partial\mathbb{D} = \{1\}$. In view of Theorem C, these properties correspond to τ_β possessing *strong mixing*, with exponential decay of correlations (cf. [22, p. 122]; also, compare with weak mixing [8, p. 22, p. 29], and [16, p. 203]). Another useful reference is [1].

7.2. The iterates of an interval

We need the following lemma.

Lemma 7.1 $(1 < \beta < \infty)$. *Let J_0 be a nonempty open interval contained in* $[-1, 1]$. *Then, for large enough positive integers n, say $n \geq n_0$, we have $\tau_\beta^n(J_0) \supset]-1, 1[$.*

Proof. We begin with the observation that if $\tau_\beta^n(J_0) \supset]-1, 1[$ holds for $n = n_0$, then it also holds for all $n \geq n_0$, as most of the branches are complete (at most two may be incomplete).

Case I: *β is an odd integer*. Then τ_β is "filling", that is, all branches are complete; cf. Subsection 5.3. This case is well-understood, but it helps our presentation to review it. We recall from Subsection 5.3 that the fundamental intervals are given by (5.4) with $\mathcal{U}$ being the set of all nonzero integers u with $|u| \geq \frac{1}{2}(\beta + 1)$. We note that by (5.6), τ_β is expansive: as long as an interval J is contained in one of the fundamental intervals I_u, $u \in \mathcal{U}$, the image $\tau_\beta(J)$ is an interval of length at least β times the length of J. We observe that if our given interval J_0 contains one of the fundamental intervals I_u, $u \in \mathcal{U}$, then we are done, because $\tau_\beta(J_0) \subset]-1, 1[$ in this case. There are two other possibilities:

(a) *The interval J_0 is contained in I_u for some $u \in \mathcal{U}$:* In this case $\tau_\beta(J_0)$ is an interval of length at least $\beta m(J_0)$, by (5.6).

(b) *The interval J_0 has nonempty intersection with two neighboring fundamental intervals I_u, $I_{u'}$, and $J_0 \subset \operatorname{clos}[I_u \cup I_{u'}]$:* In this case the length of the intersection of J_0 with one of the two fundamental intervals, say I_u, is at least $\frac{1}{2}m(J_0)$. So, we have

$$m(\tau_\beta(J_0)) \geq m(\tau_\beta(J_0 \cap I_u)) \geq \frac{\beta}{2} m(J_0). \tag{7.1}$$

In particular, $\tau_\beta(J_0)$ contains an interval $\tau_\beta(J_0 \cap I_u)$ of length at least $\frac{1}{2}\beta\, m(J_0)$.

We see that in both cases (a)–(b), the image $\tau_\beta(J_0)$ contains an interval J_1 of length at least $\frac{1}{2}\beta\, m(J)$. We note that $\frac{1}{2}\beta \geq \frac{3}{2}$ as $\beta > 1$ is an odd integer. By running the same argument starting from J_1 in place of J_0, we see that unless J_1 contains a fundamental interval (in which case we are done), we obtain an interval J_2 contained in $\tau_\beta(J_1) \subset \tau_\beta^2(J_0)$ of length at least $(\frac{1}{2}\beta)^2 m(J_0)$. Continuing, we find intervals $J_1, J_2, \ldots$ of length $\geq (\frac{1}{2}\beta)^l m(J_0)$ with $J_l \subset \tau_\beta(J_{l-1}) \subset \tau_\beta^l(J_0)$, and we stop only when the interval J_l contains a fundamental interval. For a large enough l we must stop, at least because the length of J_l will eventually exceed twice the maximum length of a fundamental interval, and for that l we have $\tau_\beta^{l+1}(J_0) \supset]-1, 1[$.

Case II: *β is not an odd integer*. Then we have $-1 < \{\beta\}_2 < 1$, and with $u_0 := \frac{1}{2}(\beta - \{\beta\}_2) \in \mathbb{Z}_+$, the set $\mathcal{U}$ consists of all integers u with $|u| \geq u_0$. We see that $\beta = 2u_0 + \{\beta\}_2 \in]2u_0 - 1, 2u_0 + 1[$. The fundamental intervals I_u are given by (5.4) for $u \in \mathcal{U} \setminus \{\pm u_0\}$, while (cf. (5.3))

$$I_{u_0} = \left]\frac{\beta}{2u_0 + 1}, 1\right[, \qquad I_{-u_0} = \left]-1, -\frac{\beta}{2u_0 + 1}\right[.$$

On a fundamental interval I_u, the transformation τ_β is given by $x \mapsto 2u - \beta/x$.

Case II-A: *J_0 is an edge fundamental interval, i.e., $J_0 = I_{u_0}$ or $J_0 = I_{-u_0}$.* When $J_0 = I_{-u_0}$, we have

$$\tau_\beta(J_0) = \tau_\beta(I_{-u_0}) =]\beta - 2u_0, 1[\supset I^2_{u_0+1} := \left]\beta - 2u_0, \frac{\beta}{2u_0+1}\right[. \tag{7.2}$$

If $\beta - 2u_0 \le \beta/(2u_0+3)$, we have

$$\tau_\beta(I_{-u_0}) =]\beta - 2u_0, 1[\supset I^2_{u_0+1} \supset \left]\frac{\beta}{2u_0+3}, \frac{\beta}{2u_0+1}\right[= I_{u_0+1},$$

so that

$$\tau^2_\beta(I_{-u_0}) \supset \tau_\beta(I_{u_0+1}) =]-1, 1[.$$

It remains to treat the case when $\beta/(2u_0+3) < \beta - 2u_0$, so that $I^2_{u_0+1} \subset I_{u_0+1}$ and in particular $\beta - 2u_0 \in I_{u_0+1}$. We first claim that there exists a constant β' with $1 < \beta' \le \beta$, which only depends on β, such that

$$m\left(\tau_\beta\left(\left]y, \frac{\beta}{2u_0+1}\right[\right)\right) \ge \beta' m(]y, 1[), \qquad y \in I^1_{u_0+1} := \left[\frac{\beta}{2u_0+3}, \beta - 2u_0\right]. \tag{7.3}$$

Here, it is clear that $I^1_{u_0+1} \subset I_{u_0+1}$, and since τ_β is given by $x \mapsto 2u_0 + 2 - \beta/x$ on I_{u_0+1}, we have

$$\tau_\beta\left(\left]y, \frac{\beta}{2u_0+1}\right[\right) =]2u_0 + 2 - \beta/y, 1[.$$

We note that the estimate (7.3) is equivalent to having

$$\beta' y + \beta/y \ge 2u_0 + 1 + \beta', \qquad y \in I^1_{u_0+1}. \tag{7.4}$$

The function $f(y) = \beta' y + \beta/y$ is strictly decreasing in $]0, 1]$, so it suffices to check (7.4) at the right endpoint of $I^1_{u_0+1}$. It is a straightforward exercise to verify that (7.4) holds at $y = \beta - 2u_0 = \{\beta\}_2$ provided that β' is chosen sufficiently close to 1; it helps to observe that $\beta > 2$ because $\beta/(2u_0+3) < \beta - 2u_0$. So, (7.4) is valid for $\beta' > 1$ close enough to 1. If we put $J_1 := \tau_\beta(I_{-u_0}) =]\beta - 2u_0, 1[$ and $J_2 := \tau_\beta(I^2_{u_0+1}) =]\tau_\beta(\beta - 2u_0), 1[$ in accordance with (7.2), then, in view of (7.2) and (7.3),

$$J_2 \subset \tau_\beta(J_1) \quad \text{and} \quad m(J_2) \ge \beta' m(J_1).$$

If $\tau_\beta(\beta - 2u_0) \le \beta/(2u_0+3)$, then $J_2 \supset I_{u_0+1}$ and so $\tau_\beta(J_2) \supset]-1, 1[$ and hence $\tau^3_\beta(I_{-u_0}) \supset]-1, 1[$. If not, then we rerun the argument and get ever bigger intervals whose right endpoint is 1, so eventually the interval must contain I_{u_0+1}, and we are done. The case $J = I_{u_0}$ is analogous and therefore omitted.

Case II-B: *J_0 is a general nonempty open subinterval of* $[-1, 1]$. We put

$$x_0 = \frac{(2u_0+1)\beta}{2u_0(2u_0+1) + \beta}.$$

The point x_0 belongs to the fundamental interval I_{u_0} and has

$$\tau_\beta\left(\left]\frac{\beta}{2u_0+1}, x_0\right[\right) = I_{-u_0} \quad \text{and} \quad \tau_\beta\left(\left]-x_0, -\frac{\beta}{2u_0+1}\right[\right) = I_{u_0}. \tag{7.5}$$

A little calculation shows that

$$\tau'_\beta(x) \geq \beta/x_0^2 > 2, \qquad x \in [-x_0, x_0] \cap \bigcup\{I_u : u \in \mathcal{U}\}. \tag{7.6}$$

Next, put $\beta'' := \min(\beta, \beta/(2x_0^2))$, so that $\beta'' > 1$. We have a general nonempty subinterval J_0 of $[-1, 1]$, and want to show that $\tau_\beta^n(J_0)$ covers $]-1, 1[$ for some large enough n. We do this by showing that the length of $\tau_\beta^n(J_0)$ must otherwise continue growing geometrically. We observe that if J_0 contains one of the fundamental intervals I_u, $u \in \mathcal{U}$, then $\tau_\beta^n(J_0) \supset]-1, 1[$ with $n = 1$ for $|u| > u_0$, and with a possibly large n if $|u| = u_0$ by Case II-A above. If J_0 does not contain a fundamental interval, then we are left with the following two possibilities:

(a) *The interval J_0 is contained in a fundamental interval I_u for some $u \in \mathcal{U}$.* Then $J_1 := \tau_\beta(J_0)$ is an interval of length at least $\beta'' m(J_0)$, by (5.6).

(b) *The interval J_0 has nonempty intersection with two neighboring fundamental intervals I_u, $I_{u'}$, and $J_0 \subset$* clos $[I_u \cup I_{u'}]$. In this case we have two subcases.

(b1) *The interval J_0 is contained in* $[-x_0, x_0]$. Then one of the two intervals, say I_u, meets J_0 in a subinterval of length at least $\frac{1}{2}m(J_0)$, and $J_1 := \tau_\beta(J_0 \cap I_u)$ is an open interval contained in $\tau_\beta(J_0)$ of length (cf. (7.6))

$$m(J_1) = m(\tau_\beta(J_0 \cap I_u)) \geq \frac{\beta m(J_0)}{2x_0^2} \geq \beta'' m(J_0).$$

(b2) *The interval J_0 is not contained in* $[-x_0, x_0]$. Then

$$J_0 \cap I_{u_0} \supset \left[\frac{\beta}{2u_0 + 1}, x_0\right] \quad \text{or} \quad J_0 \cap I_{-u_0} \supset \left[-x_0, -\frac{\beta}{2u_0 + 1}\right],$$

so by (7.5), we have $\tau_\beta(J) \supset I_{-u_0}$ or $\tau_\beta(J) \supset I_{u_0}$.

If (b2) happens, we are done, because after one iteration of τ_β we cover one of the edge fundamental intervals. If (a) or (b1) takes place, then the set $\tau_\beta(J_0)$ contains an interval J_1 of length at least $\beta'' m(J_0)$. We may then consider J_1 in place of J_0, and we gain that J_1 is longer. Unless we stop, which occurs when the set contains a fundamental interval, we get a sequence of sets $J_0, J_1, J_2, \ldots,$ and their lengths grow geometrically. This is possible only finitely many times, which means that we eventually cover a fundamental interval. The proof is complete. □

7.3. *Exterior spectrum of the Perron–Frobenius operator*

For a real-valued function f, we use the standard convention to write $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$, so that $f = f^+ - f^-$.

Theorem 7.2 $(1 < \beta < \infty)$. *Let $\mathbf{P}_{\tau_\beta}$ be the Perron–Frobenius operator associated to τ_β acting on $L^1([-1, 1])$. Then $\lambda_1 = 1$ is a simple eigenvalue of $\mathbf{P}_{\tau_\beta}$ and is the only one with modulus one. Moreover, the eigenfunctions for eigenvalue* 1 *are nonzero scalar multiples of ϱ_0, where $\varrho_0\, dm$ is the unique ergodic τ_β-invariant absolutely continuous probability measure. Also, $\varrho_0 > 0$ almost everywhere.*

Proof. To simplify the notation, we write $\mathbf{P}$ in place of $\mathbf{P}_{\tau_\beta}$. The transformation τ_β : $]-1,1] \to]-1,1]$ is a partially filling C^2-smooth piecewise monotonic transformation, which meets the conditions (i) [with $m = 1$] and (ii) of Theorem C, so that by Remark 5.1(c), the assertion of Theorem C is valid also for τ_β. Consequently, $\lambda_1 = 1$ is an eigenvalue of $\mathbf{P}$ and so there is an eigenfunction $\varrho_0 \in \mathrm{BV}([-1,1])$ corresponding to it. From (3.5) together with the triangle inequality, we see that $|\mathbf{P}\varrho_0| \le \mathbf{P}|\varrho_0|$ pointwise. Since $\mathbf{P}\varrho_0 = \varrho_0$, we then have

$$\int_{[-1,1]} |\varrho_0|\, dm = \int_{[-1,1]} |\mathbf{P}\varrho_0|\, dm \le \int_{[-1,1]} \mathbf{P}|\varrho_0|\, dm = \int_{[-1,1]} |\varrho_0|\, dm.$$

This means that we must have $|\mathbf{P}\varrho_0| = \mathbf{P}|\varrho_0|$ almost everywhere on $[-1,1]$, and so in particular $\mathbf{P}|\varrho_0| = |\varrho_0|$. But then $|\varrho_0|$ is another eigenfunction for $\lambda_1 = 1$. We might as well replace ϱ_0 by $|\varrho_0|$, which amounts to assuming that $\varrho_0 \ge 0$, and after multiplication by a suitable positive constant we can assume that

$$\langle \varrho_0, 1\rangle_{[-1,1]} = \int_{[-1,1]} \varrho_0\, dm = 1.$$

We consider the set

$$A_+ := \{x \in [-1,1] : \varrho_0(x) > 0\}.$$

Using (3.2), we see that $\tau_\beta(A_+) \doteq A_+$ (the dot over the equality sign means that the sets are equal up to Lebesgue null sets). As an element of $\mathrm{BV}([-1,1])$, the function ϱ_0 can be assumed right-continuous. Then A_+ will contain some nontrivial open interval I_0. By iteration, we find that $\tau_\beta^n(A_+) \doteq A_+$ for $n = 1, 2, \dots$, so in particular, A_+ contains $\tau_\beta^n(I_0)$ (up to null sets). From Lemma 7.1 we know that for large enough n, $\tau^n(I_0)$ covers $]-1,1[$, and so $\varrho_0 > 0$ almost everywhere on $[-1,1]$.

Next, we show that the eigenspace for $\lambda_1 = 1$ is one-dimensional. We argue by contradiction, and suppose that there exists a nontrivial $\eta_1 \in L^1([-1,1])$ such that ϱ_0, η_1 are linearly independent and $\mathbf{P}\eta_1 = \eta_1$. From Theorem C we know that $\eta_1 \in \mathrm{BV}([-1,1])$. We consider the function

$$f := \{\langle \eta_1, 1\rangle_{[-1,1]}\}\, \varrho_0 - \eta_1 \in \mathrm{BV}([-1,1]),$$

which has $\langle f, 1\rangle_{[-1,1]} = 0$ and $\mathbf{P}f = f$. By replacing η_1 by its real or imaginary part (this is possible since $\mathbf{P}$ preserves real-valuedness), we may assume that η_1 is real-valued, so that f is real-valued. We can also assume that $f \in \mathrm{BV}([-1,1])$ is right-continuous. We now write $f = f^+ - f^-$, and observe that f^+, f^- are also right-continuous functions in $\mathrm{BV}([-1,1])$. Unless one of f^+, f^- vanishes almost everywhere, both must be positive on some open intervals I_+, I_-, respectively. In view of (3.5), for $n = 1, 2, \dots$, the functions $\mathbf{P}^n f^+, \mathbf{P}^n f^-$ are then positive almost everywhere on the sets $\tau^n(I_+), \tau^n(I_-)$, respectively. Lemma 7.1 then entails that there exists a positive integer n_0 such that $\mathbf{P}^{n_0} f^+, \mathbf{P}^{n_0} f^-$ are both positive almost everywhere on $[-1,1]$. Since $\mathbf{P}^n f = f$ for all $n = 1, 2, \dots$, we must then have

$$\|f\|_{L^1} = \|\mathbf{P}^{n_0} f\|_{L^1} = \|\mathbf{P}^{n_0} f^+ - \mathbf{P}^{n_0} f^-\|_{L^1} < \|\mathbf{P}^{n_0} f^+ + \mathbf{P}^{n_0} f^-\|_{L^1} = \|f\|_{L^1}, \tag{7.7}$$

where the L^1 norm is with respect to the interval $[-1, 1]$. This contradiction shows that at least one of f^+, f^- must be identically zero (almost everywhere). But as $\langle f, 1\rangle_{[-1,1]} = 0$, both f^+, f^- must then be the 0 function. This implies that η_1 is a scalar multiple of ϱ_0, a contradiction. We conclude that $\lambda_1 = 1$ is a simple eigenvalue (i.e., it has a one-dimensional eigenspace).

Next, we turn to the assertion that $\varrho_0\, dm$ is the unique ergodic τ_β-invariant absolutely continuous probability measure. This is a consequence of Proposition A.2.5 in [16], since $\varrho_0\, dm$ is an absolutely continuous measure with $\varrho_0 > 0$ almost everywhere and $\mathbf{P}\varrho_0 = \varrho_0$.

Finally, we show that $\mathbf{P}$ has no other eigenvalues than 1 on the unit circle $\mathbb{T}$. Suppose λ, with $|\lambda| = 1$ and $\lambda \neq 1$, is an eigenvalue of $\mathbf{P}$, and $\eta_2 \in L^1([-1, 1])$ is the nontrivial eigenfunction corresponding to λ, which we may normalize: $\|\eta_2\|_{L^1} = 1$. From (3.5) together with the triangle inequality, we have $|\mathbf{P}\eta_2| \leq \mathbf{P}|\eta_2|$ pointwise, and so, since $\mathbf{P}\eta_2 = \lambda\eta_2$,

$$\int_{[-1,1]} |\eta_2|\, dm = \int_{[-1,1]} |\lambda\, \eta_2|\, dm = \int_{[-1,1]} |\mathbf{P}\eta_2|\, dm \leq \int_{[-1,1]} \mathbf{P}|\eta_2|\, dm$$
$$= \int_{[-1,1]} |\eta_2|\, dm.$$

This means that we must have $|\mathbf{P}\eta_2| = \mathbf{P}|\eta_2|$ almost everywhere on $[-1, 1]$, and consequently $\mathbf{P}|\eta_2| = |\eta_2|$. But then $|\eta_2| = \varrho_0$, as the eigenspace for $\lambda_1 = 1$ was one-dimensional and spanned by ϱ_0. We write $\eta_2 = \chi\varrho_0$, where the function $\chi \in L^\infty([-1, 1])$ has $|\chi| = 1$ almost everywhere. When we take another look at the argument we just used involving equality in the triangle inequality, we realize that χ must have the property

$$\chi(\tau_\beta(x)) = \bar{\lambda}\chi(x), \qquad x \in [-1, 1],$$

in the almost-everywhere sense. By iteration, we get

$$\chi(\tau_\beta^n(x)) = \bar{\lambda}^n \chi(x), \qquad x \in [-1, 1],\ n = 1, 2, \ldots. \tag{7.8}$$

We pick a point x_0 where $\rho_0(x_0) > 0$, and by right-continuity there exists a nonempty (short) interval $]x_0, x_1[$ where ρ_0, η_2 are both very close to the value at x_0, so that χ is close to its value at x_0 as well: say, for some small $\epsilon > 0$,

$$|\chi(x) - \chi(x_0)| < \epsilon, \qquad x \in]x_0, x_1[.$$

Next, let n be such that $\tau_\beta^n(]x_0, x_1[) \supset]-1, 1[$. Then, by (7.8),

$$|\chi(\tau_\beta^n(x)) - \lambda^n \chi(x_0)| < \epsilon, \qquad x \in]x_0, x_1[,$$

so that

$$|\chi(y) - \lambda^n \chi(x_0)| < \epsilon, \qquad y \in]-1, 1[.$$

This means that χ is within distance ϵ of a constant function. As ϵ can be made as small as we like, the only possibility is that χ is equal to a constant. But then (7.8) is impossible unless $\lambda = 1$, contrary to assumption. The proof is complete. □

Remark 7.3 $(1 < \beta < \infty)$. A dynamical system $(I, \mathfrak{S}, \mu, \tau)$, where μ is finite and invariant under τ, is said to be *exact* when

$$\lim_{n\to\infty} \mu(\tau^n(A)) = \mu(I), \quad A \in \mathfrak{S}.$$

It is known that τ is strong mixing whenever $(I, \mathfrak{S}, \mu, \tau)$ is exact [22, p. 125]. If ϱ_0 stands for the density of the unique absolutely continuous τ_β-invariant probability measure, then

$$([-1, 1], \mathfrak{B}_{[-1,1]}, \varrho_0\, dm, \tau_\beta)$$

is exact. In particular,

$$\lim_{n\to\infty} m(\tau_\beta^n(A)) = m([-1, 1]), \quad A \in \mathfrak{B}_{[-1,1]}.$$

This can be obtained from Lemma 7.1 directly, by localizing around a point of A with density 1 and using the distortion control available from the control on the second derivative; cf. [14]. This suggests another (shorter) way to obtain the assertion of Theorem 7.2. We deduce that τ_β possesses strong mixing from exactness, and then only the eigenvalue 1 can occur and it must be simple.

8. Proofs of the main results

8.1. Proof of Theorem 1.3

To prove Theorem 1.3, we will need to consider the space $\mathrm{BV}([-\beta, \beta] \setminus [-1, 1])$ of complex-valued integrable functions defined on $[-\beta, -1] \cup [1, \beta]$ whose restrictions to $[-\beta, -1]$ belong to $\mathrm{BV}([-\beta, -1])$ and whose restrictions to $[1, \beta]$ belong to $\mathrm{BV}([1, \beta])$.

Lemma 8.1 $(1 < \beta < \infty)$. *The operator* $-\mathbf{S}^*\mathbf{R}_4^* + \mathbf{S}^*\mathbf{R}_1^*\mathbf{T}^*\mathbf{R}_3^*$ *maps* $\mathrm{BV}([-\beta, \beta] \setminus [-1, 1])$ *into* $\mathrm{BV}([-1, 1])$.

Proof. Let $f_2 \in \mathrm{BV}([-\beta, \beta]\setminus[-1, 1])$. The operator $\mathbf{S}^* : L^1(\mathbb{R}\setminus[-1, 1]) \to L^1([-1, 1])$ is given by

$$\mathbf{S}^*h(x) = \sum_{k\in\mathbb{Z}^\times} h(x + 2k), \quad x \in [-1, 1].$$

As

$$\mathbf{R}_4^* : L^1([-\beta, \beta] \setminus [-1, 1]) \to L^1(\mathbb{R} \setminus [-1, 1])$$

extends the function by letting it vanish where it was previously undefined, the function $\mathbf{S}^*\mathbf{R}_4^* f_2$ is just a finite sum of functions of bounded variation, so that we have

$$\mathbf{S}^*\mathbf{R}_4^* f_2 \in \mathrm{BV}([-1, 1]). \tag{8.1}$$

On the other hand, we can easily check that $\tilde{\mathbf{C}}^2 = \mathbf{T}\mathbf{R}_1\mathbf{S}\mathbf{R}_2$, where $\tilde{\mathbf{C}} = \mathbf{C}_{\tilde{\tau}_\beta}$ is the Koopman operator associated to the transformation $\tilde{\tau}_\beta : [-\beta, \beta] \to [-\beta, \beta]$ given by $\tilde{\tau}_\beta(0) = 0$ and

$$\tilde{\tau}_\beta(x) = \{-\beta/x\}_2, \quad x \in [-\beta, \beta] \setminus \{0\}.$$

If $\tilde{\mathbf{P}} = \mathbf{P}_{\tilde{\tau}_\beta} : L^1([-\beta, \beta]) \to L^1([-\beta, \beta])$ is the corresponding Perron–Frobenius operator, whose adjoint is $\tilde{\mathbf{C}}$, we find that

$$\tilde{\mathbf{P}}^2 = \mathbf{R}_2^* \mathbf{S}^* \mathbf{R}_1^* \mathbf{T}^*. \tag{8.2}$$

We easily check that $\tilde{\tau}_\beta$ satisfies conditions (i) (with $m = 2$) and (ii) of Theorem C. Although $\tilde{\tau}_\beta$ is only "partially filling", Theorem C remains nevertheless valid in view of Remark 5.1. In particular, $\tilde{\mathbf{P}}$ transforms $\mathrm{BV}([-\beta, \beta])$ into itself. Since $\mathbf{R}_3^*$ and $\mathbf{R}_2^*$ act as extension by zero, it follows from (8.2) that

$$\mathbf{S}^* \mathbf{R}_1^* \mathbf{T}^* \mathbf{R}_3^* f_2 \in \mathrm{BV}([-1, 1]). \tag{8.3}$$

Adding up, we see from (8.1) and (8.3) that

$$-\mathbf{S}^* \mathbf{R}_4^* f_2 + \mathbf{S}^* \mathbf{R}_1^* \mathbf{T}^* \mathbf{R}_3^* f_2 \in \mathrm{BV}([-1, 1])$$

for every $f_2 \in \mathrm{BV}([-\beta, \beta] \setminus [-1, 1])$. The proof is complete. □

We have now developed the tools needed to obtain Theorem 1.3. Actually, we formulate a more precise result.

Theorem 8.2 ($1 < \beta < \infty$). *There exists a bounded operator* $\mathbf{E} : \mathrm{BV}([-\beta, \beta] \setminus [-1, 1]) \to L^1(\mathbb{R})$ *with the following properties:*

(i) $\mathbf{E}$ *is an extension operator, in the sense that* $\mathbf{E} f(x) = f(x)$ *almost everywhere on* $[-\beta, \beta] \setminus [-1, 1]$ *for all* $f \in \mathrm{BV}([-\beta, \beta] \setminus [-1, 1])$.
(ii) *The range of* $\mathbf{E}$ *is infinite-dimensional, and contained in* $\mathcal{M}_\beta^\perp$.

Proof. To simplify the notation, we write $\mathbf{P}, \mathbf{C}$ in place of $\mathbf{P}_{\tau_\beta}, \mathbf{C}_{\tau_\beta}$, respectively. By Theorem C (valid by Remark 5.1(c)) together with Theorem 7.2, we have the following representation for the iterates of $\mathbf{P}$:

$$\mathbf{P}^n h = \{\langle h, \phi_0 \rangle_{[-1,1]}\} \varrho_0 + \mathbf{Z}^n h, \quad h \in L^1([-1, 1]),\ n = 1, 2, \ldots, \tag{8.4}$$

where we write $\mathbf{Z}$ in place of $\mathbf{Z}_{\tau_\beta}$, and $\phi_0 \in L^\infty([-1, 1])$ has $\langle \varrho_0, \phi_0 \rangle_{]-1,1]} = 1$. Here, $\varrho_0 \geq 0$ is the density of the absolutely continuous τ_β-invariant probability measure on $[-1, 1]$; we have $\varrho_0 \in \mathrm{BV}([-1, 1])$. The operator $\mathbf{Z}$ acts on $\mathrm{BV}([-1, 1])$ and its spectral radius is < 1. Since ϱ_0 is invariant under $\mathbf{P}$, we must have $\mathbf{Z}\varrho_0 = 0$. We now argue that $\phi_0 = 1$ (the constant function). Indeed, for each $h \in \mathrm{BV}([-1, 1])$, (8.4) gives

$$\begin{aligned} \langle h, 1 \rangle_{[-1,1]} &= \langle h, \mathbf{C}^n 1 \rangle_{[-1,1]} = \langle \mathbf{P}^n h, 1 \rangle_{[-1,1]} \\ &= \langle h, \phi_0 \rangle_{[-1,1]} \langle \varrho_0, 1 \rangle_{[-1,1]} + \langle \mathbf{Z}^n h, 1 \rangle_{[-1,1]} \\ &= \langle h, \phi_0 \rangle_{[-1,1]} + \langle \mathbf{Z}^n h, 1 \rangle_{[-1,1]} \to \langle h, \phi_0 \rangle_{[-1,1]} \end{aligned} \tag{8.5}$$

as $n \to \infty$, since $\mathbf{Z}^n h \to 0$ exponentially. Here, we used that $\langle \varrho_0, 1 \rangle_{[-1,1]} = 1$, which follows since ϱ_0 is the density of a probability measure. Finally, as $\mathrm{BV}([-1, 1])$ is dense in $L^1([-1,1])$, we conclude from (8.5) that $\phi_0 = 1$ a.e. on $[-1,1]$. The formula (8.4) is now even simpler:

$$\mathbf{P}^n h = \{\langle h, 1\rangle_{[-1,1]}\}\,\varrho_0 + \mathbf{Z}^n h, \qquad h \in L^1([-1,1]),\ n = 1, 2, \ldots. \tag{8.6}$$

We may now read off from (8.5) that

$$\langle \mathbf{Z}^n h, 1\rangle_{[-1,1]} = 0, \qquad h \in L^1([-1,1]),\ n = 1, 2, \ldots, \tag{8.7}$$

if we observe that the calculation preceding the limit in the last step of (8.5) works for general $h \in L^1([-1,1])$. Let us take an arbitrary element $f \in \mathrm{BV}([-\beta,\beta]\setminus[-1,1])$. From Lemma 8.1 above, we know that

$$-\mathbf{S}^*\mathbf{R}_4^* f + \mathbf{S}^*\mathbf{R}_1^*\mathbf{T}^*\mathbf{R}_3^* f \in \mathrm{BV}([-1,1]).$$

Since $\mathbf{Z}$ has spectral radius < 1 on $\mathrm{BV}([-1,1])$, the Neumann series

$$(\mathbf{I} - \mathbf{Z}^2)^{-1} = \mathbf{I} + \mathbf{Z}^2 + \mathbf{Z}^4 + \cdots$$

converges to a bounded operator on $\mathrm{BV}([-1,1])$, and we may put

$$\mathbf{E}_1 f := (\mathbf{I} - \mathbf{Z}^2)^{-1}\mathbf{S}^*\{-\mathbf{R}_4^* f + \mathbf{R}_1^*\mathbf{T}^*\mathbf{R}_3^* f\} \in \mathrm{BV}([-1,1]). \tag{8.8}$$

We observe that

$$\langle -\mathbf{S}^*\mathbf{R}_4^* f + \mathbf{S}^*\mathbf{R}_1^*\mathbf{T}^*\mathbf{R}_3^* f, 1\rangle_{[-1,1]} = \langle f, 1\rangle_{[-\beta,\beta]\setminus[-1,1]} - \langle f, 1\rangle_{[-\beta,\beta]\setminus[-1,1]} = 0,$$

and so, by (8.7) and (8.8),

$$\langle \mathbf{E}_1 f, 1\rangle_{[-1,1]} = \langle (\mathbf{I} - \mathbf{Z}^2)\mathbf{E}_1 f, 1\rangle_{[-1,1]} = 0. \tag{8.9}$$

Finally, we put

$$\mathbf{E}_3 f := -\mathbf{T}^*(\mathbf{R}_2^*\mathbf{E}_1 + \mathbf{R}_3^*) f \in L^1(\mathbb{R}\setminus[-\beta,\beta]). \tag{8.10}$$

We define the operator $\mathbf{E}$ to be the mapping

$$\mathbf{E} : \mathrm{BV}([-\beta,\beta]\setminus[-1,1]) \to L^1(\mathbb{R}), \qquad f \mapsto f + \mathbf{E}_1 f + \mathbf{E}_3 f,$$

with the understanding that each of the functions $f, \mathbf{E}_1 f, \mathbf{E}_2 f$ is extended to $\mathbb{R}$ by putting it equal to 0 where it was previously undefined. Then $\mathbf{E}$ is clearly bounded and linear, and has the property (i). In view of (8.9) and (8.6), we have $\mathbf{P}^n\mathbf{E}_1 f = \mathbf{Z}^n\mathbf{E}_1 f$ for $n = 1, 2, \ldots$. This means that in condition (i) of Proposition 6.1, we may replace $\mathbf{P}^2$ by $\mathbf{Z}^2$, and we just obtain the condition (i) of Proposition 6.1 rather immediately from (8.8). The condition (ii) of Proposition 6.1 is immediate from (8.10). By Proposition 6.1, we have

$$\mathrm{im}\,\mathbf{E} \subset \mathcal{M}_\beta^\perp,$$

which proves (ii), since the range of $\mathbf{E}$ must be infinite-dimensional (the restriction to $[-\beta,\beta]\setminus[-1,1]$ of the range is infinite-dimensional, being the space of all functions of bounded variation). This completes the proof of Theorem 8.2. □

Proof of Theorem 1.3. This is immediate from Theorem 8.2(ii). □

The next theorem shows that the range of $\mathbf{E}$ constructed in in the proof of Theorem 1.3 is a subspace of the weighted space $L^2(\mathbb{R}, \omega)$, where $\omega(x) := 1 + x^2$, with norm

$$\|f\|^2_{L^2(\mathbb{R},\omega)} := \int_{\mathbb{R}} |f|^2 \omega \, dm.$$

Proposition 8.3. *The range of* $\mathbf{E}$ *is contained in* $L^2(\mathbb{R}, \omega)$.

Proof. Let $f \in \mathrm{BV}([-\beta, \beta] \setminus [-1, 1])$. Following the proof of Theorem 8.2, we see that $\mathbf{E}_1 f \in \mathrm{BV}([-1, 1])$, so that the restriction of $\mathbf{E}f$ to the interval $[-\beta, \beta]$ has bounded variation. In particular, $\mathbf{E}f$ is bounded on $[-\beta, \beta]$, so we just need to estimate the weighted L^2-norm integral on $\mathbb{R} \setminus [-\beta, \beta]$. The restriction of $\mathbf{E}f$ to $\mathbb{R} \setminus [-\beta, \beta]$ equals $\mathbf{E}_3 f$, given by (8.8), which we understand as $\mathbf{E}_3 f = -\mathbf{T}^* h$. The operator $\mathbf{T}^* : L^1([-\beta, \beta]) \to L^1(\mathbb{R} \setminus [-\beta, \beta])$ is given explicitly by

$$\mathbf{T}^* h(x) = \sum_{j \in \mathbb{Z}^\times} \frac{\beta^2}{(\beta + 2jx)^2} h\left(\frac{\beta x}{\beta + 2jx}\right), \tag{8.11}$$

with the understanding that h vanishes off $[-\beta, \beta]$. We write

$$h_j(x) = h\left(\frac{\beta x}{\beta + 2jx}\right) \frac{\beta^2}{(\beta + 2jx)^2}, \qquad x \in \mathbb{R} \setminus [-\beta, \beta].$$

As h is bounded, it is clear from (8.11) that $\mathbf{E}_3 f$ is bounded on $\mathbb{R} \setminus [-\beta, \beta]$. Since $\mathbf{E}_3 f$ is also summable, we must have

$$\int_{\mathbb{R} \setminus [-\beta, \beta]} |\mathbf{E}_3 f(x)|^2 \, dx < \infty.$$

Hence, it is enough to show that

$$\int_{\mathbb{R} \setminus [-\beta, \beta]} |\mathbf{E}_3 f(x)|^2 x^2 \, dx = \int_{\mathbb{R} \setminus [-\beta, \beta]} \Bigg| \sum_{j \in \mathbb{Z}^\times} h_j(x) \Bigg|^2 x^2 \, dx < \infty. \tag{8.12}$$

A straightforward computation shows that

$$\begin{aligned} \int_{\mathbb{R} \setminus [-\beta, \beta]} |h_j(x)|^2 x^2 \, dx &= \int_{\frac{\beta}{2j+1}}^{\frac{\beta}{2j-1}} |h(x)|^2 x^2 \, dx \le \|h\|^2_{L^\infty([-\beta, \beta])} \int_{\frac{\beta}{2j+1}}^{\frac{\beta}{2j-1}} x^2 \, dx \\ &\le \frac{\beta^3}{j^4} \|h\|^2_{L^\infty([-\beta, \beta])}. \end{aligned}$$

As a consequence, we obtain

$$\sum_{j \in \mathbb{Z}^\times} \left\{ \int_{\mathbb{R} \setminus [-\beta, \beta]} |h_j(x)|^2 x^2 \, dx \right\}^{1/2} < \infty,$$

which entails (8.12). The proof is complete. □

Remark 8.4. The range of the operator $\mathbf{E}$ is a proper subspace of $\mathcal{M}_\beta^\perp$, even if we consider the closure of the range. Actually, if in the context of Proposition 6.1 we plug in $f_1 := \varrho_0$ (notation as in Theorem 7.2) and $f_2 := 0$, and put $f_3 := -\mathbf{T}^*\mathbf{R}_2\varrho_0$, then we obtain a function $\psi_0 := f_1 + f_2 + f_3 = \varrho_0 - \mathbf{T}^*\mathbf{R}_2\varrho_0$ which is in the annihilator $\mathcal{M}_\beta^\perp$, but ψ_0 is not in the closure of the range of $\mathbf{E}$. So a natural question is whether

$$\operatorname{span}\{\psi_0\} \oplus \operatorname{clos}[\operatorname{im}\mathbf{E}] = \mathcal{M}_\beta^\perp.$$

This would be quite reasonable from the point of view of the proof of Proposition 6.1.

8.2. *Proof of Theorem 1.4*

We turn to the proof of Theorem 1.4. We know from [13] that $\mathcal{N}_\beta^\perp$ is one-dimensional for $\beta = 2$. We shall write $\beta = 2\gamma$, and suppose that $\gamma > 1$. We need the operators

$$\mathbf{S}_+ : L^\infty([0,1]) \to L^\infty([1,\infty[), \quad \mathbf{S}_+ g(x) := g(\{x\}_1),$$

and

$$\mathbf{T}_+ : L^\infty([\gamma,\infty[) \to L^\infty([0,\gamma]), \quad \mathbf{T}_+ g(x) := g\left(\frac{\gamma}{\{\gamma/x\}_1}\right).$$

Their pre-adjoints map contractively

$$\mathbf{S}_+^* : L^1([1,\infty[) \to L^1([0,1]), \quad \mathbf{T}_+^* : L^1([0,\gamma]) \to L^1([\gamma,\infty[).$$

We need the following restriction operators:

$$\begin{aligned}
\mathbf{R}_5 &: L^\infty([1,\infty[) \to L^\infty([\gamma,\infty[),\\
\mathbf{R}_6 &: L^\infty([0,\gamma]) \to L^\infty([0,1]),\\
\mathbf{R}_7 &: L^\infty([0,\gamma]) \to L^\infty([1,\gamma]),\\
\mathbf{R}_8 &: L^\infty([1,\infty[) \to L^\infty([1,\gamma])
\end{aligned}$$

and their pre-adjoints $\mathbf{R}_5^*, \mathbf{R}_6^*, \mathbf{R}_7^*, \mathbf{R}_8^*$, which act on the corresponding L^1-spaces. We let $\mathbf{P}_+ := \mathbf{P}_{\theta_\gamma}$ denote the Perron–Frobenius operator associated to the Gauss-type transformation $\theta_\gamma : [0,1[\to [0,1[$, where $\theta_\gamma(0) := 0$ and $\theta_\gamma(x) := \{\gamma/x\}_1$ for $x \in]0,1[$; cf. Section 4. The analogue of (6.5) in this context is

$$\mathbf{P}_+^2 = \mathbf{S}_+^*\mathbf{R}_5^*\mathbf{T}_+^*\mathbf{R}_6^*. \tag{8.13}$$

We also have an analogue of Proposition 6.1.

Proposition 8.5 $(1 < \gamma < \infty)$. *Let $f \in L^1([0,\infty[)$ be written as*

$$f = f_1 + f_2 + f_3,$$

where $f_1 \in L^1([0,1[)$, $f_2 \in L^1([1,\gamma[)$, and $f_3 \in L^1([\gamma,\infty[)$. Then $f \in \mathcal{N}_{2\gamma}^\perp$ if and only if

(i) $$(\mathbf{I} - \mathbf{P}_+^2)f_1 = \mathbf{S}_+^*(-\mathbf{R}_8^* + \mathbf{R}_5^*\mathbf{T}_+^*\mathbf{R}_7^*)f_2,$$

(ii) $$f_3 = -\mathbf{T}_+^*\mathbf{R}_6^* f_1 - \mathbf{T}_+^*\mathbf{R}_7^* f_2,$$

where $\mathbf{I}$ is the identity on $L^1([0,1])$.

The proof is completely analogous to that of Proposition 6.1, and we omit it. Since $\gamma > 1$, the transformation θ_γ is uniformly expansive, and if we analyze its spectral properties on $\mathrm{BV}([0,1])$, we see that $\mathbf{P}_+$ has a spectral gap. More precisely, in the context of Theorem C [valid by Remark 5.1(c)], we can show that $\sigma(\mathbf{P}_+) \cap \mathbb{T} = \{1\}$ and that the eigenvalue 1 is simple. This leads to the following assertion, analogous to Theorem 8.2. Again, we omit the proof.

Theorem 8.6 $(1 < \gamma < \infty)$. *There exists a bounded operator* $\mathbf{E}_+ : \mathrm{BV}([1,\gamma]) \to L^1([0,\infty[)$ *with the following properties:*

(i) $\mathbf{E}_+$ *is an extension operator, in the sense that* $\mathbf{E}_+ f(x) = f(x)$ *almost everywhere on* $[1,\gamma]$ *for all* $f \in \mathrm{BV}([1,\gamma])$.

(ii) *The range of* $\mathbf{E}_+$ *is infinite-dimensional, and contained in* $\mathcal{N}_{2\gamma}^{\perp}$.

Proof of Theorem 1.4. This is immediate from Theorem 8.6(ii). □

9. Final remarks

9.1. A related problem in the Hardy space of the unit disk

An algebra of inner functions in the Hardy spaces H^p of the unit disk was considered by Matheson and Stessin [21]. This algebra depends on a parameter $\beta > 0$, and can be assumed to be

$$\mathcal{A}_\beta = \mathrm{span}\{e^{-\pi m \frac{1-z}{1+z}}\, e^{-\pi n \beta \frac{1+z}{1-z}} : m, n = 0, 1, 2, \ldots\}, \tag{9.1}$$

where the span is in the sense of finite linear combinations. The main result of [21] asserts that for any finite p, $\mathcal{A}_\beta$ is dense in H^p for $\beta < 1$, while it fails to be dense for $\beta > 1$. It is natural to also consider the (smaller) space

$$\mathcal{S}_\beta = \mathrm{span}\{e^{-\pi m \frac{1-z}{1+z}},\, e^{-\pi n \beta \frac{1+z}{1-z}} : m, n = 0, 1, 2, \ldots\}. \tag{9.2}$$

For $n = 0, 1, 2, \ldots$, let φ^n denote the function

$$\varphi^n(z) := e^{-\pi n \frac{1-z}{1+z}}\, e^{-\pi n \beta \frac{1+z}{1-z}}, \qquad z \in \mathbb{D}.$$

Then we clearly have the decomposition

$$\mathcal{A}_\beta = \mathcal{S}_\beta + \varphi \mathcal{S}_\beta + \varphi^2 \mathcal{S}_\beta + \cdots,$$

in the sense of finite sums. As a consequence of the main result in [12], Theorem A, we know that $\mathcal{S}_1$ is dense in the weak-star topology of BMOA, and hence in the norm topology in H^p for all finite p. It would appear rather plausible that the codimension of the H^p-closure of $\mathcal{A}_\beta$ might be finite for $\beta > 1$, as there are only finitely many points in the unit disk which are not separated by the generating inner functions. However, so far, we cannot provide an answer to this question. With the aid of Theorem 1.3 we can however prove that the H^2-closure of the space $\mathcal{S}_\beta$ has infinite codimension for $\beta > 1$. An outline of the argument is provided below.

We assume $\beta > 1$, and pick an arbitrary natural number $N \geq 1$. By Theorem 8.2, we can pick linearly independent elements $f_1, \dots, f_N \in \mathcal{M}_\beta^\perp$ in the range of $\mathbf{E}$, which all vanish on some proper subinterval of $[1, \beta]$, say on $[1, \beta']$, where $1 < \beta' < \beta$. We define linearly independent functions $\tilde{f}_j$, $j = 1, \dots, N$, on the unit circle $\mathbb{T}$ as follows:

$$\frac{1}{1-ix}\,\tilde{f}_j\!\left(\frac{1+ix}{1-ix}\right) = (1+ix)f_j(x), \qquad j = 1, \dots, N.$$

Next, by Proposition 8.3, the functions $\tilde{f}_j$ belong to $L^2(\mathbb{T})$ and they all vanish on a certain arc of $\mathbb{T}$. Let $\mathbf{Q} : L^2(\mathbb{T}) \to H^2$ denote the orthogonal (Szegö) projection.

We claim that the projected functions $\mathbf{Q}\tilde{f}_j$, $j = 1, \dots, N$, are linearly independent. Indeed, a relation of the form

$$\sum_{j=1}^{N} c_j \mathbf{Q}\tilde{f}_j = 0, \qquad c_j \in \mathbb{C},$$

implies that the function $\tilde{f}_{\mathrm{sum}} := \sum_{j=1}^N c_j \tilde{f}_j$ belongs to $L^2(\mathbb{T}) \ominus H^2 = \operatorname{conj} H_0^2$, where "conj" means complex conjugation, and H_0^2 is the subspace of H^2 of functions that vanish at the origin. Now the function $\tilde{f}_{\mathrm{sum}}$ is in $\operatorname{conj} H_0^2$ and vanishes along an arc of the circle $\mathbb{T}$, so by e.g. Privalov's theorem, $\tilde{f}_{\mathrm{sum}} = 0$ on all of $\mathbb{T}$. From this and the linear independence of the functions $\tilde{f}_1, \dots, \tilde{f}_N$, we obtain $c_j = 0$ for all $j = 1, \dots, N$. So, the projected functions $\mathbf{Q}\tilde{f}_j$, $j = 1, \dots, N$, are linearly independent, as claimed. Finally, we claim that $\mathbf{Q}\tilde{f}_j$, $j = 1, \dots, N$, belong to the ortho-complement of $\mathcal{S}_\beta$ in H^2. If $(\cdot, \cdot)_{H^2}$ denotes the sesquilinear inner product of H^2, we calculate that for $m = 0, 1, 2, \dots$,

$$\begin{aligned}\big(\mathbf{Q}\tilde{f}_j, e^{-\pi m \frac{1-z}{1+z}}\big)_{H^2} &= \frac{1}{2\pi}\int_{\mathbb{T}} \mathbf{Q}\tilde{f}_j(\zeta)\operatorname{conj}\big(e^{-\pi m \frac{1-\zeta}{1+\zeta}}\big)|d\zeta| \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \tilde{f}_j(\zeta)\operatorname{conj}\big(e^{-\pi m \frac{1-\zeta}{1+\zeta}}\big)|d\zeta| = \frac{1}{\pi}\int_{\mathbb{R}} f_j(x)e^{i\pi m x}\,dx = 0,\end{aligned}$$

where we use $f_j \in \mathcal{M}_\beta^\perp$. In an analogous fashion, we find that for $n = 0, 1, 2, \dots$,

$$\begin{aligned}\big(\mathbf{Q}\tilde{f}_j, e^{-\pi\beta n \frac{1+z}{1-z}}\big)_{H^2} &= \frac{1}{2\pi}\int_{\mathbb{T}} \mathbf{Q}\tilde{f}_j(\zeta)\operatorname{conj}\big(e^{-\pi\beta n \frac{1+\zeta}{1-\zeta}}\big)\,|d\zeta| \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \tilde{f}_j(\zeta)\operatorname{conj}\big(e^{-\pi\beta n \frac{1+\zeta}{1-\zeta}}\big)\,|\partial\zeta| = \frac{1}{\pi}\int_{\mathbb{R}} f_j(x)e^{-i\pi\beta n/x}\,dx = 0,\end{aligned}$$

where we we again use $f_j \in \mathcal{M}_\beta^\perp$. It follows that the codimension of the H^2-closure of $\mathcal{S}_\beta$ must be $\geq N$. As N was arbitrary, this means that the H^2-closure of $\mathcal{S}_\beta$ must have infinite codimension in H^2.

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