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Null controllability of Grushin-type operators in dimension two

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Abstract. We study the null controllability of the parabolic equation associated with the Grushin-type operator $A = \partial_x^2 + |x|^{2\gamma}\partial_y^2$ ($\gamma > 0$) in the rectangle $\Omega = (-1, 1) \times (0, 1)$, under an additive control supported in an open subset ω of Ω. We prove that the equation is null controllable in any positive time for $\gamma < 1$ and that there is no time for which it is null controllable for $\gamma > 1$. In the transition regime $\gamma = 1$ and when ω is a strip $\omega = (a, b) \times (0, 1)$ ($0 < a, b \leq 1$), a positive minimal time is required for null controllability. Our approach is based on the fact that, thanks to the particular geometric configuration of Ω, null controllability is closely linked to the one-dimensional observability of the Fourier components of the solution of the adjoint system, uniformly with respect to the Fourier frequency.

Keywords. Null controllability, degenerate parabolic equations, Carleman estimates

1. Introduction

1.1. Main result

We consider the Grushin-type equation

$$\begin{cases} \partial_t f - \partial_x^2 f - |x|^{2\gamma}\partial_y^2 f = u(t, x, y)1_\omega(x, y), & (t, x, y) \in (0, \infty) \times \Omega, \\ f(t, x, y) = 0, & (t, x, y) \in (0, \infty) \times \partial\Omega, \end{cases} \tag{1}$$

where $\Omega := (-1, 1) \times (0, 1)$, $\omega \subset \Omega$, and $\gamma > 0$. Problem (1) is a linear control system in which

- the state is f,
- the control u is supported in the subset ω.

It is a degenerate parabolic equation, since the coefficient of $\partial_y^2 f$ vanishes on the line $\{x = 0\}$. We will investigate the null controllability of (1).

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Definition 1 (Null controllability). Let $T > 0$. System (1) is *null controllable in time* T if, for every $f_0 \in L^2(\Omega)$, there exists $u \in L^2((0,T)\times\Omega)$ such that the solution of

$$\begin{cases} \partial_t f - \partial_x^2 f - |x|^{2\gamma}\partial_y^2 f = u(t,x,y)1_\omega(x,y), & (t,x,y) \in (0,T)\times\Omega, \\ f(t,x,y) = 0, & (t,x,y) \in (0,T)\times\partial\Omega, \\ f(0,x,y) = f_0(x,y), & (x,y) \in \Omega, \end{cases} \tag{2}$$

satisfies $f(T,\cdot,\cdot) = 0$.

System (1) is *null controllable* if there exists $T > 0$ such that the system is null controllable in time T.

The main result of this paper is the following one.

Theorem 1. *Let ω be an open subset of $(0,1)\times(0,1)$.*

1. *If $\gamma \in (0,1)$, then system (1) is null controllable in any time $T > 0$.*
2. *If $\gamma = 1$ and $\omega = (a,b)\times(0,1)$ where $0 < a < b \le 1$, then there exists $T^* \ge a^2/2$ such that*
   - *for every $T > T^*$ system (1) is null controllable in time T,*
   - *for every $T < T^*$ system (1) is not null controllable in time T.*
3. *If $\gamma > 1$, then (1) is not null controllable.*

By duality, the null controllability of (1) is equivalent to an observability inequality for the adjoint system

$$\begin{cases} \partial_t g - \partial_x^2 g - |x|^{2\gamma}\partial_y^2 g = 0, & (t,x,y) \in (0,\infty)\times\Omega, \\ g(t,x,y) = 0, & (t,x,y) \in (0,\infty)\times\partial\Omega. \end{cases} \tag{3}$$

Definition 2 (Observability). Let $T > 0$. System (3) is *observable in ω in time T* if there exists $C > 0$ such that, for every $g_0 \in L^2(\Omega)$, the solution of

$$\begin{cases} \partial_t g - \partial_x^2 g - |x|^{2\gamma}\partial_y^2 g = 0, & (t,x,y) \in (0,T)\times\Omega, \\ g(t,x,y) = 0, & (t,x,y) \in (0,T)\times\partial\Omega, \\ g(0,x,y) = g_0(x,y), & (x,y) \in \Omega, \end{cases} \tag{4}$$

satisfies

$$\int_\Omega |g(T,x,y)|^2\,dx\,dy \le C\int_0^T\int_\omega |g(t,x,y)|^2\,dx\,dy\,dt.$$

System (3) is *observable in ω* if there exists $T > 0$ such that the system is observable in ω in time T.

Theorem 2. *Let ω be an open subset of $(0,1)\times(0,1)$.*

1. *If $\gamma \in (0,1)$, then system (4) is observable in ω in any time $T > 0$.*
2. *If $\gamma = 1$ and $\omega = (a,b)\times(0,1)$ where $0 < a < b \le 1$, then there exists $T^* \ge a^2/2$ such that*

- *for every $T > T^*$ system* (4) *is observable in ω in time T,*
- *for every $T < T^*$ system* (4) *is not observable in ω in time T.*

3. *If $\gamma > 1$, then system* (4) *is not observable in ω.*

Remark 1. When $\gamma = 1$, the geometric restriction on the control domain ω only affects our positive result. Indeed, Theorem 1 trivially implies that (1) fails to be null controllable (if $\gamma = 1$ and T is small) when ω is any connected open set at positive distance from the degeneracy region $\{x = 0\}$. It is also straightforward to observe that, if ω contains a strip containing $\{x = 0\}$, then null controllability holds for any $\gamma > 0$ thanks to standard localization arguments (see the Appendix).

1.2. Motivation and bibliographical comments

1.2.1. Null controllability of the heat equation. The null and approximate controllability of the heat equation are essentially well understood subjects for both linear and semilinear equations, and for bounded or unbounded domains (see, for instance, [15], [19], [21], [22], [23], [27], [31], [32], [36], [39], [40], [45], [46]). Let us summarize one of the existing main results. Consider the linear heat equation

$$\begin{cases} \partial_t f - \Delta f = u(t,x)1_\omega(x), & (t,x) \in (0,T) \times \Omega, \\ f(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ f(0,x) = f_0(x), & x \in \Omega, \end{cases} \tag{5}$$

where Ω is an open subset of $\mathbb{R}^d$, $d \in \mathbb{N}^*$, and ω is a subset of Ω. The following theorem is due, for the case $d = 1$, to H. Fattorini and D. Russell [20, Theorem 3.3], and, for $d \geq 2$, to O. Imanuvilov [29], [30] (see also the book [25] by A. Fursikov and O. Imanuvilov) and G. Lebeau and L. Robbiano [32] (see also [33]).

Theorem 3. *Let Ω be a bounded connected open set with boundary of class C^2 and ω be a nonempty open subset of Ω. Then the control system* (5) *is null controllable in any time $T > 0$.*

So, the heat equation on a smooth bounded domain is null controllable

- in arbitrarily small time;
- with an arbitrarily small control support ω.

Recently, null controllability results have also been obtained for uniformly parabolic operators with discontinuous (see, e.g., [16], [4], [5], [42]) or singular ([43] and [18]) coefficients.

It is then natural to wonder whether null controllability also holds for degenerate parabolic equations such as (1). Let us compare the known results for the heat equation with the results proved in this article. The first difference concerns the geometry of Ω: a more restrictive configuration is assumed in Theorem 1 than in Theorem 3. The second difference concerns the structure of the controllability results. Indeed, while the heat equation is null controllable in arbitrarily small time, the same result holds for the Grushin equation only when degeneracy is not too strong (i.e. $\gamma \in (0, 1)$). On the contrary, when degeneracy is too strong (i.e. $\gamma > 1$), null controllability does not hold any more. Of special interest is the transition regime ($\gamma = 1$), where the 'classical' Grushin operator appears: here, both behaviors live together, and a positive minimal time is required for null controllability.

1.2.2. Boundary-degenerate parabolic equations. The null controllability of parabolic equations degenerating on the boundary of the domain in one space dimension is well-understood, much less so in higher dimension. Given $0 < a < b < 1$ and $\gamma > 0$, let us consider the 1D equation

$$\partial_t w + \partial_x (x^{2\gamma} \partial_x w) = u(t, x) 1_{(a,b)}(x), \qquad (t, x) \in (0, \infty) \times (0, 1),$$

with suitable boundary conditions. Then it can be proved that null controllability holds if and only if $\gamma \in (0, 1)$ (see [12, 13]), while, for $\gamma \geq 1$, the best result one can show is "regional null controllability" (see [11]), which consists in controlling the solution within the domain of influence of the control. Several extensions of the above results are available in one space dimension; see [1, 37] for equations in divergence form, [10, 9] for nondivergence form operators, and [8, 24] for cascade systems. Fewer results are available for multidimensional problems, mainly in the case of two-dimensional parabolic operators which simply degenerate in the normal direction to the boundary of the space domain (see [14]). Note that, similarly to the above references, also for the Grushin equation null controllability holds if and only if the degeneracy is not too strong.

1.2.3. Parabolic equations degenerating inside the domain. In [38], the authors study linearized Crocco type equations

$$\begin{cases} \partial_t f + \partial_x f - \partial_{vv} f = u(t, x, v) 1_\omega(x, v), & (t, x, v) \in (0, T) \times (0, L) \times (0, 1), \\ f(t, x, 0) = f(t, x, 1) = 0, & (t, x) \in (0, T) \times (0, L), \\ f(t, 0, v) = f(t, L, v), & (t, v) \in (0, T) \times (0, 1). \end{cases}$$

For a given open subset ω of $(0, L) \times (0, 1)$, they prove regional null controllability. Notice that, in the above equation, diffusion (in v) and transport (in x) are decoupled.

In [3], the authors study the Kolmogorov equation

$$\partial_t f + v \partial_x f - \partial_{vv} f = u(t, x, v) 1_\omega(x, v), \quad (x, v) \in (0, 1)^2, \tag{6}$$

with periodic type boundary conditions. They prove null controllability in arbitrarily small time, when the control region ω is a strip parallel to the x-axis. We note that the above Kolmogorov equation degenerates on the whole space domain, unlike Grushin's equation. However, differently from the linearized Crocco equation, transport (in x at speed v) and diffusion (in v) are coupled. This is why the null controllability results are also different for these equations.

1.2.4. Unique continuation and approximate controllability. It is well-known that, for evolution equations, approximate controllability can be equivalently formulated as unique continuation (see [44]). The unique continuation problem for the elliptic Grushin-type operator

$$A = \partial_x^2 + |x|^{2\gamma} \partial_y^2$$

has been widely investigated. In particular, in [26] (see also the references therein) unique continuation is proved for every $\gamma > 0$ and every open set ω. For the parabolic Grushin-type operator studied in this paper, unique continuation holds for every $\gamma > 0$, $T > 0$, and any open set $\omega \subset \Omega$ (see Proposition 3).

1.2.5. Null controllability and hypoellipticity. It could be interesting to analyze the connections between null controllability and hypoellipticity. We recall that a linear differential operator P with C^∞ coefficients in an open set $\Omega \subset \mathbb{R}^n$ is called *hypoelliptic* if, for every distribution u in Ω, we have

$$\operatorname{sing\,supp} u = \operatorname{sing\,supp} Pu,$$

that is, u must be a C^∞ function in every open set where so is Pu. The following sufficient condition (which is also essentially necessary) for hypoellipticity is due to Hörmander (see [28]).

Theorem 4. *Let P be a second order differential operator of the form*

$$P = \sum_{j=1}^{r} X_j^2 + X_0 + c,$$

where $X_0, \dots, X_r$ denote first order homogeneous differential operators in an open set $\Omega \subset \mathbb{R}^n$ with C^∞ coefficients, and $c \in C^\infty(\Omega)$. Assume that there exist n operators among

$$X_{j_1},\ [X_{j_1}, X_{j_2}],\ [X_{j_1}, [X_{j_2}, X_{j_3}]], \dots,\ [X_{j_1}, [X_{j_2}, [X_{j_3}, [\dots, X_{j_k}] \dots]]],$$

where $j_i \in \{0, 1, \dots, r\}$, which are linearly independent at any given point in Ω. Then P is hypoelliptic.

Hörmander's condition is satisfied by the Grushin operator $A = \partial_x^2 + |x|^{2\gamma} \partial_y^2$ for every $\gamma \in \mathbb{N}^*$ (for other values of γ, the coefficients are not C^∞). Indeed, set

$$X_1(x, y) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2(x, y) := \begin{pmatrix} 0 \\ x^\gamma \end{pmatrix}.$$

Then

$$[X_1, X_2](x, y) = \begin{pmatrix} 0 \\ \gamma x^{\gamma-1} \end{pmatrix}, \quad [X_1, [X_1, X_2]](x, y) = \begin{pmatrix} 0 \\ \gamma(\gamma - 1) x^{\gamma-2} \end{pmatrix}, \quad \dots$$

Thus, if $\gamma = 1$, Hörmander's condition is satisfied with X_1 and $[X_1, X_2]$. In general, if $\gamma \geq 1$, then γ iterated Lie brackets are required.

Theorem 1 emphasizes that hypoellipticity is not sufficient for null controllability: Grushin's operator is hypoelliptic for all $\gamma \in \mathbb{N}^*$, but null controllability holds only when $\gamma = 1$.

A general result which relates null controllability to the number of iterated Lie brackets that are necessary to satisfy Hörmander's condition would be very interesting, but remains—for the time being—a challenging open problem.

1.2.6. Sensitivity to singular lower order terms. In [7], the authors study the Laplace–Beltrami operator on a 2D compact manifold endowed with a 2D almost Riemannian structure. Under very general assumptions, they prove that this operator is essentially selfadjoint. In the particular case of the Grushin metric, their result implies that any solution of

$$\partial_t f - \partial_x^2 f - x^2 \partial_y^2 f - \frac{1}{x}\partial_x f = 0, \qquad x \in \mathbb{R},\ y \in \mathbb{T},$$

such that $f(0, \cdot, \cdot)$ is supported in $\mathbb{R}_+^* \times \mathbb{T}$ stays supported in this set. As a consequence, with a distributed control as a source term in the right hand side, supported in $\mathbb{R}_+^* \times \mathbb{T}$, this system is not null controllable. This example shows that the control result studied in this article is sensitive to the addition of singular lower order terms.

1.3. Structure of the article

Section 2 is devoted to general results about Grushin's equation: well posedness in Section 2.1, Fourier decomposition of solutions and unique continuation in Section 2.2, dissipation rate of the Fourier components in Section 2.3.

Section 3 is devoted to the proof of the negative statements of Theorem 2, (and, equivalently, of Theorem 1), when $\gamma > 1$ or $\gamma = 1$ and T is small. In Section 3.1 we present the strategy for the proof, which relies on uniform observability estimates with respect to Fourier frequencies. Then we show the negative statements of Theorem 2, thanks to appropriate test functions to falsify uniform observability, in Section 3.2 for $\gamma > 1$ and in Section 3.3 for $\gamma = 1$.

Section 4 is devoted to the proof of the positive statements of Theorem 1, (and equivalently of Theorem 2) when $\gamma \in (0, 1)$ or $\gamma = 1$ and T is large. In Section 4.1 we prove a useful Carleman inequality for 1D heat equations with parameters. In Section 4.2, we obtain observability for such equations, uniformly with respect to the parameter. In Section 4.3, we prove Theorem 2 when $\gamma > 1$. Then, in Section 4.4, we conclude the proof of Theorem 2.

Finally, in Section 5, we briefly outline some open problems related to this paper. An appendix devoted to the case of $\{x = 0\} \subset \omega$ completes the analysis.

2. Well posedness and Fourier decomposition

2.1. Well posedness of the Cauchy problem

Let $H := L^2(\Omega)$, and denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_H$, respectively, the scalar product and norm in H. Define the product

$$(f, g) := \int_\Omega (f_x g_x + |x|^{2\gamma} f_y g_y)\, dx\, dy \tag{7}$$

for every f, g in $C_0^\infty(\Omega)$, and set $V = \overline{C_0^\infty(\Omega)}^{|\cdot|_V}$, where $|f|_V := (f, f)^{1/2}$.

Observe that $H_0^1(\Omega) \subset V \subset H$, thus V is dense in H. Consider the bilinear form a on V defined by

$$a(f, g) = -(f, g) \quad \forall f, g \in V. \tag{8}$$

Moreover, set

$$D(A) = \{f \in V : \exists c > 0 \text{ such that } |a(f, h)| \le c\|h\|_H \ \forall h \in V\}, \tag{9}$$
$$\langle Af, h\rangle = a(f, h) \quad \forall h \in V. \tag{10}$$

Then we can apply a result by Lions [35] (see also Theorem 1.18 in [44]) to conclude that $(A, D(A))$ generates an analytic semigroup $S(t)$ of contractions on H. Note that A is selfadjoint on H, and (10) implies that

$$Af = \partial_x^2 f + |x|^{2\gamma}\partial_y^2 f \quad \text{a.e. in } \Omega.$$

So, system (2) can be recast in the form

$$\begin{cases} f'(t) = Af(t) + u(t), & t \in [0, T], \\ f(0) = f_0, \end{cases} \tag{11}$$

where $T > 0$, $u \in L^2(0, T; H)$ and $f_0 \in H$.

Let us now recall the definition of weak solutions to (11).

Definition 3 (Weak solution). Let $T > 0$, $u \in L^2(0, T; H)$ and $f_0 \in H$. A function $f \in C([0, T]; H) \cap L^2(0, T; V)$ is a weak solution of (11) if for every $h \in D(A)$ the function $\langle f(t), h\rangle$ is absolutely continuous on $[0, T]$ and for a.e. $t \in [0, T]$,

$$\frac{d}{dt}\langle f(t), h\rangle = \langle f(t), Ah\rangle + \langle u(t), h\rangle. \tag{12}$$

Note that, as showed in [34], condition (12) is equivalent to the definition of solution by transposition, that is,

$$\int_\Omega [f(t^*, x, y)\varphi(t^*, x, y) - f_0(x, y)\varphi(0, x, y)]\,dx\,dy$$
$$= \int_0^{t^*}\int_\Omega \{f(\partial_t\varphi + \partial_x^2\varphi + |x|^{2\gamma}\partial_y^2\varphi) + u\varphi\}\,dx\,dy\,dt$$

for every $\varphi \in C^2([0, T] \times \Omega)$ and $t^* \in (0, T)$.

Let us recall that, for every $T > 0$ and $u \in L^2(0, T; H)$, the *mild solution* of (11) is defined as

$$f(t) = S(t)f_0 + \int_0^t S(t-s)u(s)\,ds, \qquad t \in [0, T]. \tag{13}$$

From [2], we know that the mild solution to (11) is also the unique weak solution in the sense of Definition 3. The following existence and uniqueness result follows.

Proposition 1. *For every $f_0 \in H$, $T > 0$ and $u \in L^2(0, T; H)$, there exists a unique weak solution of the Cauchy problem (11). This solution satisfies*

$$\|f(t)\|_H \le \|f_0\|_H + \sqrt{T}\,\|u\|_{L^2(0,T;H)} \quad \forall t \in [0, T]. \tag{14}$$

Moreover, $f(t) \in D(A)$ and $f'(t) \in H$ for a.e. $t \in (0, T)$.

Proof. (14) follows from (13). Moreover, since $S(\cdot)$ is analytic, $t \mapsto S(t)f_0$ belongs to $C^1((0, T]; H) \cap C^0((0, T]; D(A))$, and $t \mapsto \int_0^t S(t-s)u(s)\,ds$ belongs to $H^1(0, T; H) \cap L^2(0, T; D(A))$. In particular $f(t) \in D(A)$ and $f'(t) \in H$ for a.e. $t \in (0, T)$ (see, e.g., [6]). □

2.2. Fourier decomposition and unique continuation

Let us consider the solution of (4) in the sense of Definition 3, that is, the solution of system (11) with $u = 0$. Since g belongs to $C([0, T]; L^2(\Omega))$, the function $y \mapsto g(t, x, y)$ belongs to $L^2(0, 1)$ for a.e. $(t, x) \in (0, T) \times (-1, 1)$, thus it can be developed in Fourier series with respect to y as follows:

$$g(t, x, y) = \sum_{n \in \mathbb{N}^*} g_n(t, x)\varphi_n(y), \tag{15}$$

where

$$\varphi_n(y) := \sqrt{2}\sin(n\pi y) \quad \forall n \in \mathbb{N}^*$$

and

$$g_n(t, x) := \int_0^1 g(t, x, y)\varphi_n(y)\,dy \quad \forall n \in \mathbb{N}^*. \tag{16}$$

Proposition 2. *For every $n \ge 1$, g_n is the unique weak solution of*

$$\begin{cases} \partial_t g_n - \partial_x^2 g_n + (n\pi)^2|x|^{2\gamma} g_n = 0, & (t, x) \in (0, T) \times (-1, 1), \\ g_n(t, \pm 1) = 0, & t \in (0, T), \\ g_n(0, x) = g_{0,n}(x), & x \in (-1, 1), \end{cases} \tag{17}$$

where $g_{0,n} \in L^2(-1, 1)$ is given by $g_{0,n}(x) := \int_0^1 g_0(x, y)\varphi_n(y)\,dy$.

For the proof we need the following characterization of the elements of V. We denote by $L^2_\gamma(\Omega)$ the space of all square-integrable functions with respect to the measure $d\mu = |x|^{2\gamma}\,dx\,dy$.

Lemma 1. *For every $g \in V$ there exist $\partial_x g \in L^2(\Omega)$, $\partial_y g \in L^2_\gamma(\Omega)$ such that*

$$\int_\Omega (g(x,y)\partial_x\phi(x,y) + |x|^{2\gamma} g(x,y)\partial_y\phi(x,y))\,dx\,dy = -\int_\Omega (\partial_x g(x,y) + |x|^{2\gamma}\partial_y g(x,y))\phi(x,y)\,dx\,dy \qquad (18)$$

for every $\phi \in C_0^\infty(\Omega)$.

Proof. Let $g \in V$, and consider a sequence $(g^n)_{n\geq 1}$ in $C_0^\infty(\Omega)$ such that $g^n \to g$ in V, that is,

$$\int_\Omega [(g^n - g)_x^2 + |x|^{2\gamma}(g^n - g)_y^2]\,dx\,dy \to 0 \quad \text{as } n \to \infty.$$

Thus, $(\partial_x g^n)_{n\geq 1}$ is a Cauchy sequence in $L^2(\Omega)$ and $(\partial_y g^n)_{n\geq 1}$ is a Cauchy sequence in $L^2_\gamma(\Omega)$. So, there exist $h \in L^2(\Omega)$ and $k \in L^2_\gamma(\Omega)$ such that $\partial_x g^n \to h$ in $L^2(\Omega)$ and $\partial_y g^n \to k$ in $L^2_\gamma(\Omega)$. Hence,

$$\begin{array}{ccc}
\displaystyle\int_\Omega (g^n\partial_x\phi + |x|^{2\gamma} g^n\partial_y\phi)\,dx\,dy & = & \displaystyle -\int_\Omega (\partial_x g^n\phi + |x|^{2\gamma}\partial_y g^n\phi)\,dx\,dy \\
\downarrow & & \downarrow \\
\displaystyle\int_\Omega (g\partial_x\phi + |x|^{2\gamma} g\partial_y\phi)\,dx\,dy & = & \displaystyle -\int_\Omega (h\phi + |x|^{2\gamma} k\phi)\,dx\,dy
\end{array}$$

as $n \to \infty$. This yields the conclusion with $\partial_x g = h$ and $\partial_y g = k$. □

For any $n \geq 1$, system (17) is a first order Cauchy problem that admits the unique weak solution

$$\tilde{g}_n \in C^0([0,T]; L^2(-1,1)) \cap L^2(0,T; H_0^1(-1,1))$$

which satisfies

$$\frac{d}{dt}\left(\int_{-1}^1 \tilde{g}_n(t,x)\psi(x)\,dx\right) + \int_{-1}^1 \left[\tilde{g}_{n,x}(t,x)\psi_x(x) + (n\pi)^2|x|^{2\gamma}\tilde{g}_n(t,x)\psi(x)\right]dx = 0 \qquad (19)$$

for every $\psi \in H_0^1(-1,1)$.

Proof of Proposition 2. In order to verify that the nth Fourier coefficient of g, defined by (16), satisfies system (17), observe that

$$g_n(0,\cdot) = g_{0,n}(\cdot), \qquad g_n(t,\pm 1) = 0 \quad \forall t \in (0,T)$$

and

$$g_n \in C^0([0,T]; L^2(-1,1)) \cap L^2(0,T; H_0^1(-1,1)).$$

Thus, it is sufficient to prove that g_n fulfills condition (19). Indeed, using the identity (16), for all $\psi \in H_0^1(-1,1)$ we obtain, for a.e. $t \in [0,T]$,

$$\frac{d}{dt}\left(\int_{-1}^{1} g_n \psi \, dx\right) + \int_{-1}^{1} (g_{n,x}\psi_x + (n\pi)^2 |x|^{2\gamma} g_n \psi)\, dx = \int_{-1}^{1}\int_{0}^{1} \{g_t \varphi_n \psi + g_x \varphi_n \psi_x + (n\pi)^2 |x|^{2\gamma} g \varphi_n \psi\}\, dy\, dx. \tag{20}$$

Observe that Proposition 1 ensures $g_t(t,\cdot) \in L^2(\Omega)$ and $g(t,\cdot) \in D(A)$ for a.e. $t \in (0,T)$. So, multiplying $g_t - Ag$ by $h(x,y) = \psi(x)\varphi_n(y) \in V$ and integrating over Ω we obtain, for a.e. $t \in (0,T)$,

$$\begin{aligned} 0 &= \int_0^1 \int_{-1}^{1} (g_t - Ag)\psi\varphi_n \, dx\, dy \\ &= \int_0^1 \int_{-1}^{1} g_t \psi \varphi_n \, dx\, dy + \int_0^1 \int_{-1}^{1} (g_x \psi_x \varphi_n + |x|^{2\gamma} g_y \psi \varphi_{n,y})\, dx\, dy \\ &= \int_0^1 \int_{-1}^{1} g_t \psi \varphi_n \, dx\, dy + \int_0^1 \int_{-1}^{1} (g_x \psi_x \varphi_n + (n\pi)^2 |x|^{2\gamma} g \psi \varphi_n)\, dx\, dy, \end{aligned} \tag{21}$$

where (in the last identity) we have used Lemma 1. Combining (20) and (21) completes the proof. □

Proposition 3. *Let $T > 0$, $\gamma > 0$, let ω be a bounded open subset of $(0,1) \times (0,1)$, and let $g \in C([0,T]; H) \cap L^2(0,T;V)$ be a weak solution of (3). If $g \equiv 0$ on $(0,T) \times \omega$, then $g \equiv 0$ on $(0,T) \times \Omega$.*

Proof. Let $\epsilon > 0$ be such that $\omega \subset (\epsilon, 1) \times (0,1)$. By unique continuation for uniformly parabolic 2D equations, we deduce that $g \equiv 0$ on $(0,T) \times (\epsilon,1) \times (0,1)$. Thus, $g_n \equiv 0$ on $(0,T) \times (\epsilon, 1)$ for every $n \in \mathbb{N}^*$. Then, by unique continuation for the uniformly parabolic 1D equation (17), we deduce that $g_n \equiv 0$ on $(0,T) \times (-1,1)$ for every $n \in \mathbb{N}^*$. □

2.3. Dissipation speed

Let us introduce, for every $n \in \mathbb{N}^*$, $\gamma > 0$, the operator $A_{n,\gamma}$ defined on $L^2(-1,1)$ by

$$D(A_{n,\gamma}) := H^2 \cap H_0^1(-1,1), \quad A_{n,\gamma}\varphi := -\varphi'' + (n\pi)^2 |x|^{2\gamma} \varphi. \tag{22}$$

The smallest eigenvalue of $A_{n,\gamma}$ is given by

$$\lambda_{n,\gamma} = \min\left\{ \frac{\int_{-1}^{1} [v'(x)^2 + (n\pi)^2 |x|^{2\gamma} v(x)^2]\, dx}{\int_{-1}^{1} v(x)^2\, dx} : v \in H_0^1(-1,1),\ v \neq 0 \right\}. \tag{23}$$

We are interested in the asymptotic behavior (as $n \to \infty$) of $\lambda_{n,\gamma}$, which quantifies the dissipation speed of the solution of (17).

Lemma 2. *Problem*

$$\begin{cases} -v''_{n,\gamma}(x) + (n\pi)^2|x|^{2\gamma} v_{n,\gamma}(x) = \lambda_{n,\gamma} v_{n,\gamma}(x), & x \in (-1,1), \\ v_{n,\gamma}(\pm 1) = 0, \end{cases} \tag{24}$$

admits a unique positive solution with $L^2(-1,1)$-norm one. Moreover, $v_{n,\gamma}$ is even.

Proof. Since (24) is a Sturm–Liouville problem, it is well-known that its first eigenvalue is simple, and the associated eigenfunction has no zeros. Thus, we can choose $v_{n,\gamma}$ to be strictly positive everywhere. Moreover, by normalization, we can find a unique positive solution satisfying the condition $\|v_{n,\gamma}\|_{L^2(-1,1)} = 1$. Finally, $v_{n,\gamma}$ is even. Indeed, if it is not, let us consider the function $w(x) = v_{n,\gamma}(|x|)$. Then w still belongs to $H_0^1(-1,1)$, it is a weak solution of (24) and it does not increase the functional in (23), i.e.

$$\frac{\int_{-1}^1 [w'(x)^2 + (n\pi)^2|x|^{2\gamma} w(x)^2]\,dx}{\int_{-1}^1 w(x)^2\,dx} \le \frac{\int_{-1}^1 [v'_{n,\gamma}(x)^2 + (n\pi)^2|x|^{2\gamma} v_{n,\gamma}(x)^2]\,dx}{\int_{-1}^1 v_{n,\gamma}(x)^2\,dx}.$$

The coefficients of the equation in (24) being regular, we deduce that w is a classical solution of (24). Since $\lambda_{n,\gamma}$ is simple, it follows $v_{n,\gamma}(x) = v_{n,\gamma}(|x|)$. □

The following result turns out to be a key point of the proof of Theorem 1.

Proposition 4. *For every $\gamma > 0$, there are constants $c_* = c_*(\gamma), c^* = c^*(\gamma) > 0$ such that*

$$c_* n^{2/(1+\gamma)} \le \lambda_{n,\gamma} \le c^* n^{2/(1+\gamma)} \quad \forall n \in \mathbb{N}^*.$$

Proof. First, we prove the lower bound. Let $\tau_n := n^{1/(1+\gamma)}$. With the change of variable $\phi(x) = \sqrt{\tau_n}\,\varphi(\tau_n x)$, we get

$$\begin{aligned} \lambda_{n,\gamma} &= \inf\left\{\int_{-1}^1 (\phi'(x)^2 + (n\pi)^2|x|^{2\gamma}\phi(x)^2)\,dx : \phi \in C_c^\infty(-1,1),\ \|\phi\|_{L^2(-1,1)} = 1\right\} \\ &= \tau_n^2 \inf\left\{\int_{-\tau_n}^{\tau_n} (\varphi'(y)^2 + \pi^2|y|^{2\gamma}\varphi(y)^2)\,dy : \varphi \in C_c^\infty(-\tau_n,\tau_n),\ \|\varphi\|_{L^2(-\tau_n,\tau_n)} = 1\right\} \\ &\ge c_*\tau_n^2 \end{aligned}$$

where

$$c_* := \inf\left\{\int_{\mathbb{R}} (\varphi'(y)^2 + \pi^2|y|^{2\gamma}\varphi(y)^2)\,dy : \varphi \in C_c^\infty(\mathbb{R}),\ \|\varphi\|_{L^2(\mathbb{R})} = 1\right\}$$

is positive (see [41] for the case of $\gamma = 1$).

Now, we prove the upper bound in Proposition 4. For every $k > 1$ let us consider the function $\varphi_k(x) := (1 - k|x|)^+$, which belongs to $H_0^1(-1,1)$. Easy computations show that

$$\int_{-1}^1 \varphi_k(x)^2\,dx = \frac{2}{3k}, \qquad \int_{-1}^1 \varphi'_k(x)^2\,dx = 2k, \qquad \int_{-1}^1 |x|^{2\gamma}\varphi_k(x)^2\,dx = 2c(\gamma)k^{-1-2\gamma},$$

where
$$c(\gamma) := \frac{1}{2\gamma+1} - \frac{1}{\gamma+1} + \frac{1}{2\gamma+3}.$$
Thus, $\lambda_{n,\gamma} \leq f_{n,\gamma}(k) := 3[k^2 + (\pi n)^2 c(\gamma) k^{-2\gamma}]$ for all $k > 1$. Since $f_{n,\gamma}$ attains its minimum at $\bar{k} = \tilde{c}(\gamma) n^{1/(\gamma+1)}$, we have $\lambda_{n,\gamma} \leq f_{n,\gamma}(\bar{k}) = C(\gamma) n^{2/(\gamma+1)}$. □

3. Proof of the negative statements of Theorem 2

The goal of this section is the proof of the following results:

- if $\gamma = 1$, $\omega \subset (a, 1) \times (0, 1)$ for some $a > 0$ and $T < a^2/2$, then system (4) is not observable in ω in time T,
- if $\gamma > 1$ and $T > 0$, then system (4) is not observable in ω in time T.

Without loss of generality, one may assume that $\omega = (a, b) \times (0, 1)$ with $0 < a < b < 1$.

3.1. *Strategy for the proof*

Let g be the solution of (4). Then g can be represented as in (15), and we emphasize that, for a.e. $t \in (0, T)$ and every $-1 \leq a_1 < b_1 \leq 1$,
$$\int_{(a_1,b_1)\times(0,1)} |g(t,x,y)|^2 \, dx \, dy = \sum_{n=1}^{\infty} \int_{a_1}^{b_1} |g_n(t,x)|^2 \, dx$$
(Bessel–Parseval equality). Thus, in order to prove Theorem 2, it is sufficient to study the observability of system (17) uniformly with respect to $n \in \mathbb{N}^*$.

Definition 4 (Uniform observability). Let $0 < a < b \leq 1$ and $T > 0$. System (17) is *observable in* (a, b) *in time* T *uniformly with respect to* $n \in \mathbb{N}^*$ if there exists $C > 0$ such that, for every $n \in \mathbb{N}^*$ and $g_{0,n} \in L^2(-1, 1)$, the solution of (17) satisfies
$$\int_{-1}^{1} |g_n(T,x)|^2 \, dx \leq C \int_0^T \int_a^b |g_n(t,x)|^2 \, dx.$$

System (17) is *observable in* (a, b) *uniformly with respect to* $n \in \mathbb{N}^*$ if there exists $T > 0$ such that the system is observable in (a, b) in time T uniformly with respect to $n \in \mathbb{N}^*$.

The negative parts of the conclusion of Theorem 2 follow from the result below.

Theorem 5. *Let* $0 < a < b \leq 1$.

1. *If* $\gamma = 1$ *and* $T < a^2/2$, *then system* (17) *is not observable in* (a, b) *in time* T *uniformly with respect to* $n \in \mathbb{N}^*$.
2. *If* $\gamma > 1$, *then system* (17) *is not observable in* (a, b) *uniformly with respect to* $n \in \mathbb{N}^*$.

The proof of Theorem 5 relies on the use of appropriate test functions that falsify uniform observability. This is proved thanks to a well adapted maximum principle (see Lemma 3) and explicit supersolutions (see (28)) for $\gamma > 1$, and thanks to direct computations for $\gamma = 1$.

3.2. Proof of Theorem 5 for $\gamma > 1$

Let $\gamma \in [1, \infty)$ be fixed and $T > 0$. For every $n \in \mathbb{N}^*$, we denote by λ_n (instead of $\lambda_{n,\gamma}$) the first eigenvalue of the operator $A_{n,\gamma}$ defined in Section 2.3, and by v_n the associated positive eigenvector of norm one, which satisfies

$$\begin{cases} -v_n''(x) + [(n\pi)^2|x|^{2\gamma} - \lambda_n]v_n(x) = 0, & x \in (-1, 1),\ n \in \mathbb{N}^*, \\ v_n(\pm 1) = 0, \quad v_n \geq 0, \\ \|v_n\|_{L^2(-1,1)} = 1. \end{cases}$$

Then, for every $n \geq 1$, the function

$$g_n(t, x) := v_n(x)e^{-\lambda_n t} \qquad \forall (t, x) \in \mathbb{R} \times (-1, 1)$$

solves the adjoint system (17). Let us note that

$$\int_{-1}^{1} g_n(T, x)^2\, dx = e^{-2\lambda_n T},$$

$$\int_0^T \int_a^b g_n(t, x)^2\, dx\, dt = \frac{1 - e^{-2\lambda_n T}}{2\lambda_n} \int_a^b v_n(x)^2\, dx.$$

So, in order to prove that uniform observability fails, it suffices to show that

$$\frac{e^{2\lambda_n T}}{\lambda_n} \int_a^b v_n(x)^2\, dx \to 0 \qquad \text{as } n \to \infty. \tag{25}$$

The above convergence will be obtained by comparing v_n with an explicit supersolution of the problem on a suitable subinterval of $[-1, 1]$.

Lemma 3. *Let $0 < a < b < 1$. For every $n \in \mathbb{N}^*$, set*

$$x_n := \left(\frac{\lambda_n}{(n\pi)^2}\right)^{1/(2\gamma)} \tag{26}$$

and let $W_n \in C^2([x_n, 1], \mathbb{R})$ be a solution of

$$\begin{cases} -W_n''(x) + [(n\pi)^2 x^{2\gamma} - \lambda_n]W_n(x) \geq 0, & x \in (x_n, 1), \\ W_n(1) \geq 0, \\ W_n'(x_n) < -\sqrt{x_n}\,\lambda_n. \end{cases} \tag{27}$$

Then there exists $n_* \in \mathbb{N}^*$ such that, for every $n \geq n_*$,

$$\int_a^b v_n(x)^2\, dx \leq \int_a^b W_n(x)^2\, dx.$$

Proof. First, observe that, thanks to Proposition 4, $x_n \to 0$ as $n \to \infty$. In particular, there exists $n_* \geq 1$ such that $x_n \leq a$ for every $n \geq n_*$. Now, let us prove that $|v_n'(x_n)| \leq \sqrt{x_n}\,\lambda_n$ for all $n \geq n_*$. Indeed, by Lemma 2, we have $v_n(x) = v_n(-x)$, thus $v_n'(0) = 0$. Hence, thanks to the Cauchy–Schwarz inequality and the relation $\|v_n\|_{L^2(-1,1)} = 1$,

$$\begin{aligned}
|v_n'(x_n)| &= \left| \int_0^{x_n} v_n''(s)\,ds \right| = \left| \int_0^{x_n} [(n\pi)^2|s|^{2\gamma} - \lambda_n] v_n(s)\,ds \right| \\
&\leq \left(\int_0^{x_n} [(n\pi)^2|s|^{2\gamma} - \lambda_n]^2\,ds \right)^{1/2} \left(\int_0^{x_n} v_n(s)^2\,ds \right)^{1/2} \leq \sqrt{x_n}\,\lambda_n.
\end{aligned}$$

Furthermore, we claim that $v_n(x) \leq W_n(x)$ for every $x \in [x_n, 1]$, $n \geq n_*$. Indeed, if not, there would exist $x_* \in [x_n, 1]$ such that

$$(W_n - v_n)(x_*) = \min\{(W_n - v_n)(x) : x \in [x_n, 1]\} < 0.$$

Since $(W_n - v_n)(1) \geq 0$ and $(W_n - v_n)'(x_n) < 0$, we have $x_* \in (x_n, 1)$. Moreover, the function $W_n - v_n$ has a minimum at x_*, thus $(W_n - v_n)'(x_*) = 0$ and $(W_n - v_n)''(x_*) \geq 0$. Therefore,

$$-(W_n - v_n)''(x_*) + [(n\pi)^2|x_*|^{2\gamma} - \lambda_n](W_n - v_n)(x_*) < 0,$$

which is a contradiction. Our claim follows and the proof is complete. □

In order to apply Lemma 3, we need an explicit supersolution W_n of (27) of the form

$$W_n(x) = C_n e^{-\mu_n x^{\gamma+1}}, \tag{28}$$

where $C_n, \mu_n > 0$. Notice that, in particular, $W_n(1) \geq 0$.

First step: *Let us prove that, for an appropriate choice of μ_n, the first inequality of (27) holds.* Since

$$\begin{aligned}
W_n'(x) &= -\mu_n(\gamma+1)x^\gamma W_n(x), \\
W_n''(x) &= [-\mu_n\gamma(\gamma+1)x^{\gamma-1} + \mu_n^2(\gamma+1)^2x^{2\gamma}]W_n(x),
\end{aligned}$$

the first inequality of (27) holds if and only if, for every $x \in (x_n, 1)$,

$$[(n\pi)^2 - \mu_n^2(\gamma+1)^2]x^{2\gamma} + \mu_n\gamma(\gamma+1)x^{\gamma-1} \geq \lambda_n. \tag{29}$$

In particular, it holds when

$$\mu_n \leq \frac{n\pi}{\gamma+1} \tag{30}$$

and

$$[(n\pi)^2 - \mu_n^2(\gamma+1)^2]x_n^{2\gamma} + \mu_n\gamma(\gamma+1)x_n^{\gamma-1} \geq \lambda_n. \tag{31}$$

Indeed, in this case, the left hand side of (29) is an increasing function of x. In view of (26), and after several simplifications, inequality (31) can be recast as

$$\mu_n \leq \frac{\gamma}{\gamma+1}\left(\frac{(n\pi)^2}{\lambda_n}\right)^{1/2+1/(2\gamma)}.$$

So, recalling (30), in order to satisfy the first inequality of (27) we can take

$$\mu_n := \min\left\{ \frac{n\pi}{\gamma+1}, \frac{\gamma}{\gamma+1}\left(\frac{(n\pi)^2}{\lambda_n}\right)^{1/2+1/(2\gamma)} \right\}. \tag{32}$$

For the following computations, it is important to notice that, thanks to (32) and Proposition 4, for n large enough μ_n is of the form

$$\mu_n = C_1(\gamma) n. \tag{33}$$

Second step: *let us prove that, for an appropriate choice of C_n, the third inequality of* (27) *holds.* Since

$$W_n'(x_n) = -C_n \mu_n (\gamma+1) x_n^{\gamma} e^{-\mu_n x_n^{\gamma+1}},$$

the third inequality of (27) is equivalent to

$$C_n > \frac{\lambda_n e^{\mu_n x_n^{\gamma+1}}}{(\gamma+1)\mu_n x_n^{\gamma-1/2}}.$$

Therefore, it is sufficient to choose

$$C_n := \frac{2\lambda_n e^{\mu_n x_n^{\gamma+1}}}{(\gamma+1)\mu_n x_n^{\gamma-1/2}}. \tag{34}$$

Third step: *Let us prove condition* (25). Thanks to Lemma 3, (28), (33) and (34), for every $n \geq n_*$,

$$\begin{aligned}\frac{e^{2\lambda_n T}}{\lambda_n}\int_a^b v_n(x)^2\,dx &\leq \frac{e^{2\lambda_n T}}{\lambda_n}\int_a^b W_n(x)^2\,dx \leq \frac{e^{2\lambda_n T}}{\lambda_n} W_n(a)^2 \\ &\leq \frac{e^{2\lambda_n T}}{\lambda_n} C_n^2 e^{-2\mu_n a^{1+\gamma}} \leq \frac{e^{2\lambda_n T}}{\lambda_n}\frac{4\lambda_n^2 e^{2\mu_n x_n^{\gamma+1}}}{(\gamma+1)^2\mu_n^2 x_n^{2\gamma-1}} e^{-2\mu_n a^{1+\gamma}}.\end{aligned}$$

By identities (26), (33) and Proposition 4, we have

$$\mu_n x_n^{\gamma+1} \leq C_2(\gamma) \quad \forall n \in \mathbb{N}^*,$$

thus

$$\frac{e^{2\lambda_n T}}{\lambda_n}\int_a^b v_n(x)^2\,dx \leq e^{2n(\frac{\lambda_n}{n}T - C_1(\gamma)a^{1+\gamma})}\frac{4\lambda_n e^{2C_2(\gamma)}}{(\gamma+1)^2\mu_n^2 x_n^{2\gamma-1}}. \tag{35}$$

Since $\gamma > 1$, we deduce from Proposition 4 that

$$\lambda_n/n \to 0 \quad \text{as } n \to \infty.$$

So, for every $T > 0$, there exists $n_\sharp \geq n_*$ such that, for every $n \geq n_\sharp$,

$$\frac{\lambda_n}{n}T - C_1(\gamma)a^{1+\gamma} < -\frac{1}{2}C_1(\gamma)a^{1+\gamma}. \tag{36}$$

Then inequality (35) yields condition (25) (since the term that multiplies the exponential behaves like a rational fraction of n).

3.3. Proof of Theorem 5 for $\gamma = 1$

In this section, we take $\gamma = 1$ and keep the abbreviated forms λ_n, v_n for $\lambda_{n,\gamma}, v_{n,\gamma}$ introduced in Section 2.3. Moreover, given two real sequences $\alpha_n \geq 0$ and $\beta_n > 0$, we write $\alpha_n \sim \beta_n$ to mean that $\lim_n \alpha_n/\beta_n = 1$.

With the above notation in mind, we have the following result.

Lemma 4. *Let a and b be real numbers such that $0 < a < b \leq 1$. Then*

$$\lambda_n \sim n\pi \tag{37}$$

and

$$\int_a^b v_n(x)^2\,dx \sim \frac{e^{-a^2 n\pi}}{2a\pi\sqrt{n}} \quad \text{as } n \to \infty. \tag{38}$$

When $T < a^2/2$, we can easily deduce from the above lemma that (25) holds; thus, system (17) is not observable in (a, b) uniformly with respect to $n \in \mathbb{N}^*$.

Proof of Lemma 4. The proof relies on the explicit expression

$$G(x) := \frac{e^{-x^2/2}}{\sqrt[4]{\pi}}$$

of the first eigenvector of the harmonic oscillator on the whole line, i.e.,

$$\begin{cases} -G''(x) + x^2 G(x) = G(x), & x \in \mathbb{R}, \\ \int_{\mathbb{R}} G(x)^2\,dx = 1. \end{cases}$$

First step: *Let us construct an explicit approximation, k_n, of v_n.* Fix $\epsilon > 0$ with

$$1 + (1-\epsilon)^2 > 2a^2, \tag{39}$$

and let $\theta \in C^\infty(\mathbb{R})$ be such that

$$\theta(\pm 1) = 1 \quad \text{and} \quad \operatorname{supp}(\theta) \subset (-1-\epsilon, -1+\epsilon) \cup (1-\epsilon, 1+\epsilon). \tag{40}$$

Define

$$k_n(x) = \frac{\sqrt[4]{n\pi}\, G(\sqrt{n\pi}\,x) - \sqrt[4]{n}\, e^{-n\pi/2}\theta(x)}{C_n}, \qquad x \in [-1, 1],$$

where $C_n > 0$ is such that $\|k_n\|_{L^2(-1,1)} = 1$. Note that $C_n^2 = C_{n,1} + C_{n,2} + C_{n,3}$ where

$$C_{n,1} = \sqrt{n} \int_{-1}^{1} e^{-n\pi x^2}\,dx = 1 + O\left(\frac{e^{-n\pi}}{\sqrt{n}}\right),$$

$$C_{n,2} = \sqrt{n}\, e^{-n\pi} \int_{-1}^{1} \theta(x)^2\,dx,$$

$$C_{n,3} = -2\sqrt{n}\, e^{-n\pi/2} \int_{-1}^{1} e^{-n\pi x^2/2}\theta(x)\,dx = O(\sqrt{n}\, e^{-\frac{n\pi}{2}(1+(1-\epsilon)^2)}).$$

Thus,

$$C_n = 1 + O(\sqrt{n}\, e^{-\frac{n\pi}{2}[1+(1-\epsilon)^2]}). \tag{41}$$

We have

$$\begin{cases} -k_n''(x) + (n\pi x)^2 k_n(x) = n\pi k_n(x) + E_n(x), & x \in (-1,1), \\ k_n(\pm 1) = 0, \end{cases}$$

where

$$E_n(x) := \frac{\sqrt[4]{n}\, e^{-n\pi/2}}{C_n}\big[\theta''(x) - (n\pi x)^2\theta(x) + n\pi\theta(x)\big].$$

Second step: *Let us prove* (37). As in the proof of Proposition 4, we have $\lambda_n \geq n\pi$. Moreover,

$$\begin{aligned} \lambda_n &\leq \int_{-1}^{1} [k_n'(x)^2 + (n\pi x)^2 k_n(x)^2]\, dx = n\pi + \int_{-1}^{1} k_n(x) E_n(x)\, dx \\ &\leq n\pi + O(n^{9/4} e^{-n\pi/2}), \end{aligned}$$

which proves (37).

Third step: *Let us prove that*

$$\int_a^b k_n(x)^2\, dx \sim \frac{e^{-a^2 n\pi}}{2a\pi\sqrt{n}}. \tag{42}$$

Indeed, the left hand side of (42) is the sum of three terms $(I_j)_{1\leq j\leq 3}$ that satisfy, thanks to (41),

$$\begin{aligned} I_1 &:= \frac{1}{\sqrt{\pi}\, C_n^2} \int_{a\sqrt{n\pi}}^{b\sqrt{n\pi}} e^{-y^2}\, dy = \frac{e^{-a^2 n\pi}}{2a\pi\sqrt{n}} + O\Big(\frac{e^{-a^2 n\pi}}{n^{3/2}}\Big), \\ I_2 &:= \frac{\sqrt{n}\, e^{-n\pi}}{C_n^2} \int_a^b \theta(x)^2\, dx = O(\sqrt{n}\, e^{-n\pi}), \\ I_3 &:= -\frac{2\sqrt{n}\, e^{-n\pi/2}}{C_n^2} \int_a^b e^{-n\pi x^2}\theta(x)\, dx = O(\sqrt{n}\, e^{-\frac{n\pi}{2}[1+(1-\epsilon)^2]}). \end{aligned}$$

So, (42) follows thanks to (39).

Fourth step: *Let us prove that*

$$\|v_n - k_n\|^2_{L^2(-1,1)} = O(n^{9/2} e^{-n\pi}), \tag{43}$$

which ends the proof of (38). Let A_n be the operator defined by

$$D(A_n) = H^2 \cap H_0^1(-1,1), \qquad A_n\varphi(x) := -\varphi''(x) + (n\pi x)^2\varphi(x),$$

and let $(\lambda_n^j)_{j\in\mathbb{N}^*}$ be its eigenvalues, with associated eigenvectors $(v_n^j)_{j\in\mathbb{N}^*}$, that is, $A_n v_n^j = \lambda_n^j v_n^j$. We have $k_n = \sum_{j=1}^{\infty} z_j v_n^j$ where $z_j = \langle E_n, v_n^j\rangle/(\lambda_n^j - n\pi)$ for all $j \geq 2$. Thus,

$$\sum_{j=2}^{\infty} z_j^2 \leq C\|E_n\|^2_{L^2(-1,1)} = O(n^{9/2}e^{-n\pi})$$

and

$$z_1 = \Big(1 - \sum_{j=2}^{\infty} z_j^2\Big)^{1/2} = 1 + O(n^{9/2}e^{-n\pi}).$$

We can then recover (43) since $\|v_n - k_n\|^2_{L^2(-1,1)} = (1-z_1)^2 + \sum_{j=2}^{\infty} z_j^2$. □

4. Proof of the positive statements of Theorem 1

The goal of this section is the proof of the following results:

- if $\gamma \in (0,1)$, then system (1) is null controllable in any time $T > 0$,
- if $\gamma = 1$ and $\omega = (a,b)\times(0,1)$, with $0 < a < b \leq 1$, then there exists $T_1 > 0$ such that system (1) is null controllable in any time $T > T_1$ or, equivalently, system (3) is observable in ω in any time $T > T_1$.

The proof of these results relies on a new global Carleman estimate for solutions of (17), stated and proved in the next section.

4.1. A global Carleman estimate

For $n \in \mathbb{N}^*$, we introduce the operator

$$\mathcal{P}_n g := \frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} + (n\pi)^2|x|^{2\gamma} g.$$

Proposition 5. *Let $\gamma \in (0,1]$ and let $a, b \in \mathbb{R}$ be such that $0 < a < b \leq 1$. Then there exist a weight function $\beta \in C^1([-1,1];\mathbb{R}_+^*)$ and positive constants $\mathcal{C}_1, \mathcal{C}_2$ such that for every $n \in \mathbb{N}^*$, $T > 0$, and $g \in C^0([0,T];L^2(-1,1)) \cap L^2(0,T;H_0^1(-1,1))$ the following inequality holds:*

$$\begin{aligned}\mathcal{C}_1 \int_0^T\int_{-1}^1 \left(\frac{M}{t(T-t)}\left|\frac{\partial g}{\partial x}(t,x)\right|^2 + \frac{M^3}{(t(T-t))^3}|g(t,x)|^2\right) e^{-\frac{M\beta(x)}{t(T-t)}}\,dx\,dt \\ \leq \int_0^T\int_{-1}^1 |\mathcal{P}_n g|^2 e^{-\frac{M\beta(x)}{t(T-t)}}\,dx\,dt + \int_0^T\int_a^b \frac{M^3}{(t(T-t))^3}|g(t,x)|^2 e^{-\frac{M\beta(x)}{t(T-t)}}\,dx\,dt \qquad (44)\end{aligned}$$

where $M := \mathcal{C}_2 \max\{T + T^{3/2}, nT^2\}$.

Remark 2. In the case of $\gamma \in [1/2, 1]$, our weight β will be the classical one (see (46)–(49)). On the other hand, for $\gamma \in (0,1/2)$ we will follow the strategy of [1, 10, 37], adapting the weight β to the nonsmooth coefficient $|x|^{2\gamma}$ (see (78)–(79)).

Proof of Proposition 5. Without loss of generality, we may assume that $b < 1$. Let a', b' be such that $a < a' < b' < b$. All the computations of the proof will be made assuming, first, $g \in H^1(0, T; L^2(-1, 1)) \cap L^2(0, T; H^2 \cap H_0^1(-1, 1))$. Then the conclusion of Proposition 5 will follow by a density argument.

First case: $\boxed{\gamma \in [1/2, 1].}$ Consider the weight function

$$\alpha(t, x) := \frac{M\beta(x)}{t(T-t)}, \quad (t, x) \in (0, T) \times \mathbb{R}, \tag{45}$$

where $\beta \in C^2([-1, 1])$ satisfies

$$\beta \geq 1 \quad \text{on } (-1, 1), \tag{46}$$

$$|\beta'| > 0 \quad \text{on } [-1, a'] \cup [b', 1], \tag{47}$$

$$\beta'(1) > 0, \quad \beta'(-1) < 0, \tag{48}$$

$$\beta'' < 0 \quad \text{on } [-1, a'] \cup [b', 1], \tag{49}$$

and $M = M(T, n, \beta) > 0$ will be chosen later on. We also introduce the function

$$z(t, x) := g(t, x)e^{-\alpha(t,x)}, \tag{50}$$

which satisfies

$$e^{-\alpha}\mathcal{P}_n g = P_1 z + P_2 z, \tag{51}$$

where

$$\begin{aligned} P_1 z &:= -\frac{\partial^2 z}{\partial x^2} + (\alpha_t - \alpha_x^2)z + (n\pi)^2|x|^{2\gamma}z - \alpha_{xx}z, \\ P_2 z &:= \frac{\partial z}{\partial t} - 2\alpha_x\frac{\partial z}{\partial x}. \end{aligned} \tag{52}$$

We develop the classical proof (see [25]), taking the $L^2(Q)$-norm in the identity (51), then developing the double product, which leads to

$$\int_Q P_1 z P_2 z \, dx\, dt \leq \frac{1}{2}\int_Q |e^{-\alpha}\mathcal{P}_n g|^2 \, dx\, dt, \tag{53}$$

where $Q := (0, T) \times (-1, 1)$ and we compute precisely each term, paying attention to the behavior of the different constants with respect to n and T.

Terms involving $-\partial_x^2 z$: Integrating by parts, we get

$$-\int_Q \frac{\partial^2 z}{\partial x^2}\frac{\partial z}{\partial t} \, dx\, dt = \int_Q \frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial t \partial x} \, dx\, dt = \int_0^T \frac{1}{2}\frac{d}{dt}\int_{-1}^1 \left|\frac{\partial z}{\partial x}\right|^2 dx\, dt = 0, \tag{54}$$

because $\partial_t z(t, \pm 1) = 0$ and $z(0) \equiv z(T) \equiv 0$, which is a consequence of assumptions (50), (45) and (46). Moreover,

$$\int_Q \frac{\partial^2 z}{\partial x^2} 2\alpha_x \frac{\partial z}{\partial x}\, dx\, dt = -\int_Q \left|\frac{\partial z}{\partial x}\right|^2 \alpha_{xx}\, dx\, dt + \int_0^T \left(\alpha_x(t,1)\left|\frac{\partial z}{\partial x}(t,1)\right|^2 - \alpha_x(t,-1)\left|\frac{\partial z}{\partial x}(t,-1)\right|^2\right) dt. \tag{55}$$

Terms involving $(\alpha_t - \alpha_x^2)z$**:** Again integrating by parts, we have

$$\int_Q (\alpha_t - \alpha_x^2) z \frac{\partial z}{\partial t}\, dx\, dt = -\frac{1}{2}\int_Q (\alpha_t - \alpha_x^2)_t |z|^2\, dx\, dt. \tag{56}$$

Indeed, the boundary terms at $t = 0$ and $t = T$ vanish because, thanks to (50), (45), (46),

$$|(\alpha_t - \alpha_x^2)|z|^2| \leq \frac{1}{[t(T-t)]^2} e^{\frac{-M}{t(T-t)}} |M(T-2t)\beta + (M\beta')^2| \cdot |g|^2$$

tends to zero when $t \to 0$ and $t \to T$, for every $x \in [-1, 1]$. Moreover,

$$-2\int_Q (\alpha_t - \alpha_x^2) z \alpha_x \frac{\partial z}{\partial x}\, dx\, dt = \int_Q [(\alpha_t - \alpha_x^2)\alpha_x]_x |z|^2\, dx\, dt, \tag{57}$$

by integration by parts in the space variable.

Terms involving $(n\pi)^2|x|^{2\gamma}z$**:** First, since $z(0) \equiv z(T) \equiv 0$,

$$\int_Q (n\pi)^2 |x|^{2\gamma} z \frac{\partial z}{\partial t}\, dx\, dt = \frac{1}{2}\int_0^T \frac{d}{dt}\int_{-1}^1 (n\pi)^2 |x|^{2\gamma}|z|^2\, dx\, dt = 0. \tag{58}$$

Furthermore, by integration by parts in the space variable,

$$-2\int_Q (n\pi)^2 |x|^{2\gamma} z \alpha_x \frac{\partial z}{\partial x}\, dx\, dt = \int_Q [n^2\pi^2 |x|^{2\gamma}\alpha_x]_x z^2\, dx\, dt. \tag{59}$$

Terms involving $-\alpha_{xx}z$**:** Integrating by parts, we get

$$-\int_Q \alpha_{xx} z \frac{\partial z}{\partial t}\, dx\, dt = \int_Q \frac{1}{2}\alpha_{xxt}|z|^2\, dx\, dt. \tag{60}$$

The second term involving $-\alpha_{xx}z$ is

$$\int_Q 2z \frac{\partial z}{\partial x} \alpha_{xx}\alpha_x\, dx\, dt. \tag{61}$$

Combining (53)–(61) we conclude that

$$\int_Q |z|^2\big\{-\tfrac{1}{2}(\alpha_t-\alpha_x^2)_t+[(\alpha_t-\alpha_x^2)\alpha_x]_x+n^2\pi^2[|x|^{2\gamma}\alpha_x]_x+\tfrac{1}{2}\alpha_{xxt}\big\}\,dx\,dt$$
$$-\int_Q\left|\frac{\partial z}{\partial x}\right|^2\alpha_{xx}\,dx\,dt+\int_Q 2z\frac{\partial z}{\partial x}\alpha_{xx}\alpha_x\,dx\,dt$$
$$+\int_0^T\left(\alpha_x(t,1)\left|\frac{\partial z}{\partial x}(t,1)\right|^2-\alpha_x(t,-1)\left|\frac{\partial z}{\partial x}(t,-1)\right|^2\right)dt\le\frac{1}{2}\int_Q|e^{-\alpha}\mathcal{P}_n g|^2\,dx\,dt.$$

In view of (48), we have $\alpha_x(t,1)\ge 0$ and $\alpha_x(t,-1)\le 0$. Moreover,

$$\int_Q 2\alpha_{xx}\alpha_x z\frac{\partial z}{\partial x}\,dx\,dt\ge\int_Q\left(-\tfrac{1}{2}|\alpha_{xx}|\left|\frac{\partial z}{\partial x}\right|^2-2|\alpha_{xx}|\alpha_x^2|z|^2\right)dx\,dt.$$

Thus,

$$\int_Q|z|^2\big\{-\tfrac{1}{2}(\alpha_t-\alpha_x^2)_t+[(\alpha_t-\alpha_x^2)\alpha_x]_x+n^2\pi^2[|x|^{2\gamma}\alpha_x]_x+\tfrac{1}{2}\alpha_{xxt}-2|\alpha_{xx}|\alpha_x^2\big\}\,dx\,dt$$
$$+\int_Q\left|\frac{\partial z}{\partial x}\right|^2\big\{-\alpha_{xx}-\tfrac{1}{2}|\alpha_{xx}|\big\}\,dx\,dt\le\frac{1}{2}\int_Q|e^{-\alpha}\mathcal{P}_n g|^2\,dx\,dt.\qquad(62)$$

Now, in the left hand side of (62) we separate the terms on $(0,T)\times(a',b')$ and those on $(0,T)\times[(-1,a')\cup(b',1)]$. One has

$$\begin{aligned}-\alpha_{xx}(t,x)-\tfrac{1}{2}|\alpha_{xx}(t,x)|&\ge\frac{C_1M}{t(T-t)}\qquad\forall x\in[-1,a']\cup[b',1],\ t\in(0,T),\\ \big|-\alpha_{xx}(t,x)-\tfrac{1}{2}|\alpha_{xx}|\big|&\le\frac{C_2M}{t(T-t)}\qquad\forall x\in[a',b'],\ t\in(0,T),\end{aligned}\qquad(63)$$

where $C_1=C_1(\beta):=\frac{1}{2}\min\{-\beta''(x):x\in[-1,a']\cup[b',1]\}$ is positive thanks to the assumption (49) and $C_2=C_2(\beta):=\frac{3}{2}\sup\{|\beta''(x)|:x\in[a',b']\}$. Moreover,

$$\begin{aligned}&-\tfrac{1}{2}(\alpha_t-\alpha_x^2)_t+[(\alpha_t-\alpha_x^2)\alpha_x]_x+\tfrac{1}{2}\alpha_{xxt}-2|\alpha_{xx}|\alpha_x^2\\ &\quad=\frac{1}{(t(T-t))^3}\big\{M\beta(3Tt-T^2-3t^2)+M\beta''t(T-t)(t-T/2)\\ &\qquad+M^2(2t-T)(\beta''\beta+2\beta'^2)+M^3(-3\beta''-2|\beta''|)\beta'^2\big\}.\end{aligned}$$

Hence, owing to (47) and (49), there exist $m_1=m_1(\beta)>0$, $C_3=C_3(\beta)>0$ and $C_4=C_4(\beta)>0$ such that, for every $M\ge M_1$ and $t\in(0,T)$,

$$\begin{aligned}&-\tfrac{1}{2}(\alpha_t-\alpha_x^2)_t+[(\alpha_t-\alpha_x^2)\alpha_x]_x+\tfrac{1}{2}\alpha_{xxt}-2|\alpha_{xx}|\alpha_x^2\\ &\qquad\ge\frac{C_3M^3}{[t(T-t)]^3}\qquad\forall x\in[-1,a']\cup[b',1],\\ &\big|-\tfrac{1}{2}(\alpha_t-\alpha_x^2)_t+[(\alpha_t-\alpha_x^2)\alpha_x]_x+\tfrac{1}{2}\alpha_{xxt}-2|\alpha_{xx}|\alpha_x^2\big|\le\frac{C_4M^3}{[t(T-t)]^3}\qquad\forall x\in[a',b']\end{aligned}\qquad(64)$$

where

$$M_1 = M_1(T,\beta) := m_1(\beta)(T + T^{3/2}). \tag{65}$$

Using (62)–(64), we deduce, for every $M \geq M_1$,

$$\begin{aligned}
&\int_0^T \int_{(-1,a')\cup(b',1)} \frac{C_1 M}{t(T-t)} \left|\frac{\partial z}{\partial x}\right|^2 dx\,dt \\
&\qquad + \int_0^T \int_{(-1,a')\cup(b',1)} \left[\frac{C_3 M^3}{(t(T-t))^3}|z|^2 + (n\pi)^2[|x|^{2\gamma}\alpha_x]_x |z|^2\right] dx\,dt \\
&\leq \int_0^T \int_{a'}^{b'} \left[\frac{C_2 M}{t(T-t)}\left|\frac{\partial z}{\partial x}\right|^2 + \frac{C_4 M^3}{(t(T-t))^3}|z|^2 - (n\pi)^2[|x|^{2\gamma}\alpha_x]_x|z|^2\right] dx\,dt \\
&\quad + \frac{1}{2}\int_Q |e^{-\alpha}\mathcal{P}_n g|^2\,dx\,dt.
\end{aligned} \tag{66}$$

Moreover, for every $x \in (-1,1)$, we have

$$|(n\pi)^2[|x|^{2\gamma}\alpha_x]_x| = \frac{M(n\pi)^2}{t(T-t)}\left|2\gamma\,\mathrm{sign}(x)|x|^{2\gamma-1}\beta'(x) + |x|^{2\gamma}\beta''(x)\right| \leq \frac{C_5 n^2 M}{t(T-t)}$$

where $C_5 = C_5(\beta) := \pi^2 \max\{2\gamma|x|^{2\gamma-1}|\beta'(x)| + |x|^{2\gamma}|\beta''(x)| : x \in [-1,1]\}$ is finite because $2\gamma - 1 \geq 0$. Let $M_2 = M_2(T,n,\beta)$ be defined by

$$M_2 = M_2(T,n,\beta) := \sqrt{2C_5/C_3}\, n(T/2)^2. \tag{67}$$

From now on, we take

$$M = M(T,n,\beta) := \mathcal{C}_2 \max\{T + T^2, nT^2\} \tag{68}$$

where

$$\mathcal{C}_2 = \mathcal{C}_2(\beta) := \max\{m_1, \sqrt{C_5/(8C_3)}\}$$

so that $M \geq M_1, M_2$ (see (65) and (67)). From $M \geq M_2$, we deduce that

$$|(n\pi)^2[|x|^{2\gamma}\alpha_x]_x| \leq \frac{C_3 M^3}{2[t(T-t)]^3} \qquad \forall (t,x) \in Q.$$

We have

$$\begin{aligned}
&\int_0^T \int_{(-1,a')\cup(b',1)} \left(\frac{C_1 M}{t(T-t)}\left|\frac{\partial z}{\partial x}\right|^2 + \frac{C_3 M^3}{2(t(T-t))^3}|z|^2\right) dx\,dt \\
&\qquad\qquad \leq \int_0^T \int_{a'}^{b'} \left(\frac{C_2 M}{t(T-t)}\left|\frac{\partial z}{\partial x}\right|^2 + \frac{C_6 M^3}{(t(T-t))^3}|z|^2\right) dx\,dt \\
&\qquad\qquad\quad + \frac{1}{2}\int_Q |e^{-\alpha}\mathcal{P}_n g|^2\,dx\,dt,
\end{aligned} \tag{69}$$

where $C_6 = C_6(\beta) := C_4 + C_3/2$. For every $\epsilon > 0$, we have

$$\frac{C_1 M}{t(T-t)}\left|\frac{\partial g}{\partial x} - \alpha_x g\right|^2 + \frac{C_3 M^3}{2(t(T-t))^3}|g|^2$$
$$\geq \left(1 - \frac{1}{1+\epsilon}\right)\frac{C_1 M}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2 + \frac{M^3}{(t(T-t))^3}\left(\frac{C_3}{2} - \epsilon C_1(\beta')^2\right)|g|^2. \tag{70}$$

Hence, choosing

$$\epsilon = \epsilon(\beta) := \frac{C_3}{4C_1\|\beta'\|_\infty^2},$$

from (69), (70) and (50) we deduce that

$$\int_0^T \int_{(-1,a')\cup(b',1)} \left(\frac{C_7 M}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2 + \frac{C_3 M^3 |g|^2}{4(t(T-t))^3}\right)e^{-2\alpha}\,dx\,dt$$
$$\leq \int_0^T \int_{a'}^{b'} \left(\frac{C_9 M^3 |g|^2}{(t(T-t))^3} + \frac{C_8 M}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2\right)e^{-2\alpha}\,dx\,dt$$
$$+ \frac{1}{2}\int_Q |e^{-\alpha}\mathcal{P}_n g|^2\,dx\,dt, \tag{71}$$

where $C_7 = C_7(\beta) := [1 - 1/(1+\epsilon)]C_1$, $C_8 = C_8(\beta) := 2C_2$ and $C_9 = C_9(\beta) := C_6 + 2C_2 \sup\{\beta'(x)^2 : x \in [a', b']\}$. So, adding the same quantity to both sides,

$$\int_Q \left(\frac{C_7 M}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2 + \frac{C_3 M^3 |g|^2}{4(t(T-t))^3}\right)e^{-2\alpha}\,dx\,dt \leq \frac{1}{2}\int_Q |e^{-\alpha}\mathcal{P}_n g|^2\,dx\,dt$$
$$+ \int_0^T \int_{a'}^{b'} \left(\frac{C_{11} M^3 |g|^2}{(t(T-t))^3} + \frac{C_{10} M}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2\right)e^{-2\alpha}\,dx\,dt, \tag{72}$$

where $C_{10} = C_{10}(\beta) := C_8 + C_7$ and $C_{11} = C_{11}(\beta) := C_9 + C_3/4$. Let us prove that the third term of the right hand side may be dominated by terms similar to the other two. We consider $\rho \in C^\infty(\mathbb{R}; \mathbb{R}_+)$ such that $0 \leq \rho \leq 1$ and

$$\rho \equiv 1 \quad \text{on } (a', b'), \tag{73}$$
$$\rho \equiv 0 \quad \text{on } (-1, a) \cup (b, 1). \tag{74}$$

We have

$$\int_Q (\mathcal{P}_n g)\frac{g\rho e^{-2\alpha}}{t(T-t)}\,dx\,dt = \int_0^T \int_{-1}^1 \left[\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} + (n\pi)^2|x|^{2\gamma} g\right]\frac{g\rho e^{-2\alpha}}{t(T-t)}\,dx\,dt.$$

Integrating by parts with respect to time and space, we obtain

$$\int_Q \frac{1}{2}\frac{\partial(g^2)}{\partial t}\frac{\rho e^{-2\alpha}}{t(T-t)}\,dx\,dt = \int_Q \tfrac{1}{2}|g|^2\rho\left(\frac{2\alpha_t}{t(T-t)} + \frac{T-2t}{(t(T-t))^2}\right)e^{-2\alpha}\,dx\,dt$$

and

$$-\int_Q \frac{\partial^2 g}{\partial x^2}\,\frac{g\rho e^{-2\alpha}}{t(T-t)}\,dx\,dt = \int_Q \frac{\rho e^{-2\alpha}}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2 dx\,dt \\ - \int_Q \frac{|g|^2 e^{-2\alpha}}{2t(T-t)}\big(\rho'' - 4\rho'\alpha_x + \rho(4\alpha_x^2 - 2\alpha_{xx})\big)\,dx\,dt. \tag{75}$$

Thus,

$$\int_Q \mathcal{P}_n g\,\frac{g\rho e^{-2\alpha}}{t(T-t)}\,dx\,dt \geq \int_Q \frac{\rho e^{-2\alpha}}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2 dx\,dt \\ - \int_Q \frac{|g|^2 e^{-2\alpha}}{2t(T-t)}\left(\rho'' - 4\rho'\alpha_x + \rho\left(4\alpha_x^2 - 2\alpha_{xx} - 2\alpha_t - \frac{T-2t}{t(T-t)}\right)\right)dx\,dt. \tag{76}$$

Therefore,

$$\begin{aligned}\int_0^T\int_{a'}^{b'} \frac{C_{10}M}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2 e^{-2\alpha}\,dx\,dt &\leq \int_Q \frac{C_{10}M\rho}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2 e^{-2\alpha}\,dx\,dt \\ &\leq \int_Q \mathcal{P}_n g\,\frac{C_{10}Mg\rho e^{-2\alpha}}{t(T-t)}\,dx\,dt \\ &\quad + \int_Q \frac{C_{10}M|g|^2 e^{-2\alpha}}{2t(T-t)}\left(\rho'' - 4\rho'\alpha_x + \rho\left(4\alpha_x^2 - 2\alpha_{xx} - 2\alpha_t - \frac{T-2t}{t(T-t)}\right)\right)dx\,dt \\ &\leq \int_Q |\mathcal{P}_n g|^2 e^{-2\alpha}\,dx\,dt + \int_0^T\int_a^b \frac{C_{12}M^3|g|^2 e^{-2\alpha}}{(t(T-t))^3}\,dx\,dt\end{aligned}$$

for some constant $C_{12} = C_{12}(\beta,\rho) > 0$. Combining (72) with the previous inequality, we get

$$\int_Q \left(\frac{C_7 M}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2 + \frac{C_3 M^3 |g|^2}{4(t(T-t))^3}\right)e^{-2\alpha}\,dx\,dt \\ \leq \frac{3}{2}\int_Q |e^{-\alpha}\mathcal{P}_n g|^2\,dx\,dt + \int_0^T\int_a^b \frac{C_{13}M^3|g|^2}{(t(T-t))^3}e^{-2\alpha}\,dx\,dt, \tag{77}$$

where $C_{13} = C_{13}(\beta,\rho) := C_{11} + C_{12}$. Then the global Carleman estimate (44) holds with

$$\mathcal{C}_1 = \mathcal{C}_1(\beta) := \frac{\min\{C_7, C_3/4\}}{\max\{3/2, C_{13}\}}.$$

Second case: $\boxed{\gamma \in (0, 1/2).}$ The previous strategy does not apply to $\gamma \in (0, 1/2)$ because the term $(n\pi)^2[|x|^{2\gamma}\alpha_x]_x$ (that diverges at $x = 0$) in (66) can no longer be bounded by $\frac{C_3 M^3}{(t(T-t))^3}$ (which is bounded at $x = 0$). Note that both terms are of the same order as M^3, because of the dependence of M on n in (68). In order to deal with this difficulty, we adapt the choice of the weight β and the dependence of M on n.

Let $\beta \in C^1([-1,1]) \cap C^2([-1,0) \cup (0,1])$ be such that

$$\beta'' < 0 \quad \text{on } [-1,0) \cup (0,a'] \cup [b',1] \tag{78}$$

and β has the following form on a neighborhood $(-\epsilon, \epsilon)$ of 0:

$$\beta(x) = \mathcal{C}_0 - \int_0^x \sqrt{\operatorname{sign}(s)|s|^{2\gamma} + \mathcal{C}_1}\, ds \quad \forall x \in (-\epsilon, \epsilon), \tag{79}$$

where $\mathcal{C}_0$, $\mathcal{C}_1$ are large enough to ensure that $\beta(x) \geq 1$, and $\operatorname{sign}(s)|s|^{2\gamma} + \mathcal{C}_1 \geq 0$ on $(-\epsilon, \epsilon)$. Notice that

$$\beta'(x) = -\sqrt{\operatorname{sign}(x)|x|^{2\gamma} + \mathcal{C}_1} \quad \forall x \in (-\epsilon, \epsilon), \tag{80}$$

thus β'' diverges at $x = 0$. Performing the same computations as in the previous case, we get inequality (62). Notice that one obtains (59) even if $\gamma \in (0, 1/2)$: the boundary terms vanish and $x \mapsto |x|^{2\gamma-1}$ is integrable at $x = 0$. Then, owing to (47) and (78), there exist $m_1 = m_1(\beta) > 0$, $C_3 = 1/2$ and $C_4 = C_4(\beta) > 0$ such that, for every $M \geq M_1$ and $t \in (0,T)$,

$$\begin{aligned} &-\tfrac{1}{2}(\alpha_t - \alpha_x^2)_t + [(\alpha_t - \alpha_x^2)\alpha_x]_x + \tfrac{1}{2}\alpha_{xxt} - 2|\alpha_{xx}|\alpha_x^2 \\ &\qquad \geq \frac{C_3 M^3}{[t(T-t)]^3} |\beta''(x)|\, \beta'(x)^2 \quad \forall x \in [-1,0) \cup (0,a'] \cup [b',1], \end{aligned}$$

and

$$\left| -\tfrac{1}{2}(\alpha_t - \alpha_x^2)_t + [(\alpha_t - \alpha_x^2)\alpha_x]_x + \tfrac{1}{2}\alpha_{xxt} - 2|\alpha_{xx}|\alpha_x^2 \right| \leq \frac{C_4 M^3}{[t(T-t)]^3} \quad \forall x \in [a', b'],$$

where $M_1 = M_1(T, \beta)$ is defined by (65). In view of (62) and (78), for every $M \geq M_1$,

$$\begin{aligned} &\int_0^T \int_{(-1,a')\cup(b',1)} \frac{M|\beta''|}{2t(T-t)} \left| \frac{\partial z}{\partial x} \right|^2 dx\, dt \\ &\qquad + \int_0^T \int_{(-1,a')\cup(b',1)} \left[\frac{C_3 M^3 |\beta''|\, \beta'^2}{(t(T-t))^3} + (n\pi)^2 (|x|^{2\gamma}\alpha_x)_x \right] |z|^2\, dx\, dt \\ &\quad \leq \frac{1}{2} \int_Q |e^{-\alpha} \mathcal{P}_n g|^2\, dx\, dt + \int_0^T \int_{a'}^{b'} \left[\frac{C_2 M}{t(T-t)} \left| \frac{\partial z}{\partial x} \right|^2 + \frac{C_4 M^3}{(t(T-t))^3} |z|^2 \right] dx\, dt \\ &\quad - (n\pi)^2 \int_0^T \int_{a'}^{b'} (|x|^{2\gamma}\alpha_x)_x |z|^2\, dx\, dt. \end{aligned} \tag{81}$$

Moreover,

$$\begin{aligned}|(n\pi)^2(|x|^{2\gamma}\alpha_x)_x| &= (n\pi)^2\frac{M}{t(T-t)}|2\gamma\,\mathrm{sign}(x)|x|^{2\gamma-1}\beta'(x)+|x|^{2\gamma}\beta''(x)| \\ &\le \frac{C_5 n^2 M}{t(T-t)}\big(|x|^{2\gamma-1}|\beta'(x)|+|x|^{2\gamma}|\beta''(x)|\big) \quad \forall x\in(-1,0)\cup(0,1),\end{aligned}$$

where $C_5 = \pi^2(2\gamma+1)$.

From now on, we take

$$M = M(T,n,\beta) := \mathcal{C}_2 \max\{T+T^{3/2}, nT^2\}, \tag{82}$$

where

$$\mathcal{C}_2 = \mathcal{C}_2(\beta) := \max\{m_1, 1/\lambda\}$$

and $\lambda = \lambda(\beta)$ is a (small enough) constant, to be chosen later on. From $M \ge nT^2/\lambda$, we deduce that, for every $x \in (-1,0)\cup(0,1)$,

$$|(n\pi)^2(|x|^{2\gamma}\alpha_x)_x| \le \frac{C_6\lambda^2 M^3}{(t(T-t))^3}\big(|x|^{2\gamma-1}|\beta'(x)| + |x|^{2\gamma}|\beta''(x)|\big),$$

where $C_6 = C_6(\gamma) > 0$. Let us verify that, for $\lambda = \lambda(\beta) > 0$ small enough and for every $x \in (-1,0)\cup(0,a')\cup(b',1)$, we have

$$\begin{aligned}\frac{C_6\lambda^2 M^3}{(t(T-t))^3}|x|^{2\gamma-1}|\beta'(x)| &\le \frac{C_3 M^3}{4(t(T-t))^3}|\beta''(x)|\,\beta'(x)^2, \\ \frac{C_6\lambda^2 M^3}{(t(T-t))^3}|x|^{2\gamma}|\beta''(x)| &\le \frac{C_3 M^3}{4(t(T-t))^3}|\beta''(x)|\,\beta'(x)^2,\end{aligned}$$

or, equivalently, for every $x \in (-1,0)\cup(0,a')\cup(b',1)$,

$$C_6\lambda^2|x|^{2\gamma-1} \le \frac{C_3}{4}|\beta''(x)|\cdot|\beta'(x)|, \qquad C_6\lambda^2|x|^{2\gamma} \le \frac{C_3}{4}\beta'(x)^2. \tag{83}$$

The second inequality is easy to satisfy for $\lambda = \lambda(\beta)$ small enough, because $|\beta'| > 0$ on $[-1,a']\cup[b',1]$. Thanks to (80), for every $x \in (-\epsilon,\epsilon)$,

$$\beta'(x)^2 = \mathrm{sign}(x)|x|^{2\gamma} + \mathcal{C}_1,$$

so

$$\beta''(x)\beta'(x) = \gamma|x|^{2\gamma-1}.$$

Therefore, for every $x \in (-\epsilon,\epsilon)\setminus\{0\}$, the first inequality in (83) is equivalent to

$$C_6\lambda^2 \le \frac{C_3}{4}\gamma,$$

which is trivially satisfied when $\lambda = \lambda(\beta)$ is small enough. Moreover, the first inequality of (83) holds for every $x \in [-1, -\epsilon] \cup [\epsilon, a'] \cup [b', 1]$ when $\lambda = \lambda(\beta)$ is small enough, since $|\beta''\beta'| > 0$ on this compact set. Finally, we have

$$\int_0^T \int_{(-1,a')\cup(b',1)} \left[\frac{M|\beta''|}{2t(T-t)} \left| \frac{\partial z}{\partial x} \right|^2 + \frac{C_3 M^3 |\beta''| \beta'^2}{2(t(T-t))^3} |z|^2 \right] dx\, dt$$
$$\leq \frac{1}{2} \int_Q |e^{-\alpha} \mathcal{P}_n g|^2 \, dx\, dt + \int_0^T \int_{a'}^{b'} \left[\frac{C_2 M}{t(T-t)} \left| \frac{\partial z}{\partial x} \right|^2 + \frac{C_5 M^3}{(t(T-t))^3} |z|^2 \right] dx\, dt, \tag{84}$$

where $C_5 = C_5(\beta) > 0$. The functions $|\beta''|\beta'^2$ and β'' are bounded from below by positive constants on $[-1, a'] \cup [b', 1]$, thus

$$\int_0^T \int_{(-1,a')\cup(b',1)} \left[\frac{C_6 M}{t(T-t)} \left| \frac{\partial z}{\partial x} \right|^2 + \frac{C_7 M^3}{(t(T-t))^3} |z|^2 \right] dx\, dt$$
$$\leq \frac{1}{2} \int_Q |e^{-\alpha} \mathcal{P}_n g|^2 \, dx\, dt + \int_0^T \int_{a'}^{b'} \left[\frac{C_2 M}{t(T-t)} \left| \frac{\partial z}{\partial x} \right|^2 + \frac{C_5 M^3}{(t(T-t))^3} |z|^2 \right] dx\, dt, \tag{85}$$

where $C_j = C_j(\beta) > 0$ for $j = 6, 7$. The rest of the proof goes as for $\gamma \in [1/2, 1]$. □

4.2. *Uniform observability*

The Carleman estimate of Proposition 5 allows us to prove the following uniform observability result.

Proposition 6. *Let $\gamma \in (0, 1)$ and let $a, b \in \mathbb{R}$ be such that $0 < a < b < 1$. Then there exists $C > 0$ such that for every $T > 0$, $n \in \mathbb{N}^*$, and $g_{0,n} \in L^2(-1, 1)$ the solution of* (17) *satisfies*

$$\int_{-1}^1 g_n(T, x)^2 \, dx \leq T^2 e^{C(1+T^{-(1+\gamma)/(1-\gamma)})} \int_0^T \int_a^b g_n(t, x)^2 \, dx\, dt.$$

Let us recall that explicit bounds on the observability constant of the heat equation with a potential are already known.

Theorem 6. *Let $-1 < a < b < 1$. There exists $c > 0$ such that, for every $T > 0$, $\alpha, \beta \in L^\infty((0, T) \times (-1, 1))$, $g_0 \in L^2(-1, 1)$, the solution of*

$$\begin{cases} \partial_t g - \partial_x^2 g + \beta \partial_x g + \alpha g = 0, & (t, x) \in [0, T] \times (-1, 1), \\ g(t, \pm 1) = 0, & t \in [0, T], \\ g(0, x) = g_0(x), & x \in (-1, 1), \end{cases}$$

satisfies

$$\int_{-1}^1 |g(T, x)|^2 \, dx \leq e^{cH(T, \|\alpha\|_\infty, \|\beta\|_\infty)} \int_0^T \int_a^b |g(t, x)|^2 \, dx\, dt,$$

where $H(T, A, B) := 1 + 1/T + TA + A^{2/3} + (1 + T)B^2$.

For the proof of the above result we refer the reader to [22, Theorem 1.3] in the case of $\beta \equiv 0$, and to [15, Theorem 2.3] for $\beta \not\equiv 0$. The optimality of the power $2/3$ of A in $H(T, A, B)$ has been proved in [17].

Proposition 6 may be seen as an improvement of the above estimate (relative to the asymptotic behavior as $n \to \infty$), in the special case of (17).

Proof of Proposition 6. We derive an explicit observability constant from the Carleman estimate of Proposition 5. For $t \in (T/3, 2T/3)$, we have

$$\frac{4}{T^2} \leq \frac{1}{t(T-t)} \leq \frac{9}{2T^2} \quad \text{and} \quad \int_{-1}^{1} g(T,x)^2\,dx \leq \int_{-1}^{1} g(t,x)^2\,dx\, e^{-\lambda_n T/3}.$$

Thus,

$$\mathcal{C}_1 \frac{64M^3}{T^6} e^{-\frac{9M\beta^*}{2T^2}} \frac{T}{3} e^{\lambda_n T/3} \int_{-1}^{1} g(T,x)^2\,dx \leq \mathcal{C}_3 \int_0^T \int_a^b g(t,x)^2\,dx\,dt$$

where $\beta^* := \max\{\beta(x) : x \in [-1,1]\}$, $\beta_* := \min\{\beta(x) : x \in [-1,1]\}$ and $\mathcal{C}_3 := \max\{x^3 e^{-\beta_* x}\}$. Using the inequality $M \geq \mathcal{C}_2[T + T^2]$ and Proposition 4, we get

$$\int_{-1}^{1} g(T,x)^2\,dx \leq \mathcal{C}_4 T^2 e^{c_1 M/T^2 - c_2 n^{2/(1+\gamma)} T} \int_0^T \int_a^b g(t,x)^2\,dx\,dt \tag{86}$$

for some constants $c_1, c_2, \mathcal{C}_4 > 0$ (independent of n, T and g).

First case: $n < 1 + 1/T$**.** Then $M = \mathcal{C}_2(T + T^2)$ and thus

$$\int_{-1}^{1} g(T,x)^2\,dx \leq \mathcal{C}_4 T^2 e^{c_1 \mathcal{C}_2 (1+1/T)} \int_0^T \int_a^b g(t,x)^2\,dx\,dt.$$

Second case: $n \geq 1 + 1/T$**.** Then $M = \mathcal{C}_2 n T^2$. The maximum value of the function $x \mapsto c_1 \mathcal{C}_2 x - c_2 x^{2/(1+\gamma)} T$ on $(0, \infty)$ is of the form $c_3 T^{-(1+\gamma)/(1-\gamma)}$ for some constant $c_3 > 0$ (independent of T). Thus,

$$\int_{-1}^{1} g(T,x)^2\,dx \leq \mathcal{C}_4 T^2 e^{c_3 T^{-(1+\gamma)/(1-\gamma)}} \int_0^T \int_a^b g(t,x)^2\,dx\,dt.$$

This gives the conclusion. □

In the case of $\gamma = 1$, we also have the following result.

Proposition 7. *Assume $\gamma = 1$. Let $a, b \in \mathbb{R}$ be such that $0 < a < b < 1$. Then there exists $T_1 > 0$ such that, for every $T > T_1$, system (17) is observable in (a, b) in time T uniformly with respect to $n \in \mathbb{N}^*$.*

Proof. One can follow the lines of the previous proof until (86). Then, for $n \geq 1 + 1/T$, we have $M = \mathcal{C}_2 n T^2$. Thus,

$$\int_{-1}^{1} g(T,x)^2\,dx \leq \mathcal{C}_4 T^2 e^{[c_1 \mathcal{C}_2 - c_2 T]n} \int_0^T \int_a^b g(t,x)^2\,dx\,dt.$$

This proves the proposition with $T_1 := c_1 \mathcal{C}_2 / c_2$. □

4.3. Construction of the control function for $\gamma \in (0,1)$

The goal of this section is the proof of null controllability in any time $T > 0$ for $\gamma \in (0,1)$. Our construction of the control steering the initial state to zero is the one of [5], which is in turn inspired by [32] (see also [33]).

For $n \in \mathbb{N}^*$, we define

$$\varphi_n(y) := \sqrt{2}\sin(n\pi y) \quad \text{and} \quad H_n := L^2(-1,1) \otimes \varphi_n,$$

which is a closed subspace of $L^2(\Omega)$. For $j \in \mathbb{N}$, we define $E_j := \bigoplus_{n \le 2^j} H_n$ and denote by Π_{E_j} the orthogonal projection onto E_j.

Proposition 8. *Let $\gamma \in (0,1)$, and let $a, b, c, d \in \mathbb{R}$ be such that $0 < a < b < 1$ and $0 < c < d < 1$. Then there exists a constant $C > 0$ such that for every $T > 0$, every $j \in \mathbb{N}^*$, and every $g_0 \in E_j$ the solution of (4) satisfies*

$$\int_\Omega g(T,x,y)^2\,dx\,dy \le T^2 e^{C(2^j + T^{-(1+\gamma)/(1-\gamma)})} \int_0^T \int_\omega g(t,x,y)^2\,dx\,dy\,dt$$

where $\omega := (a,b) \times (c,d)$.

For the proof of Proposition 8 we shall need the following inequality obtained in [32] (see also [33]).

Proposition 9. *Let $c, d \in \mathbb{R}$ be such that $c < d$. There exists $C > 0$ such that, for every $L \in \mathbb{N}^*$ and $(b_k)_{1 \le k \le L} \in \mathbb{R}^L$,*

$$\sum_{k=1}^{L} |b_k|^2 \le e^{CL} \int_c^d \Big|\sum_{k=1}^{L} b_k \varphi_k(y)\Big|^2\,dy.$$

Proof of Proposition 8. Let $(g_{0,n})_{1 \le n \le 2^j} \in L^2(-1,1)^{2^j}$ be such that

$$g_0(x,y) = \sum_{n=1}^{2^j} g_{0,n}(x)\varphi_n(y).$$

Then the solution of (4) is given by

$$g(t,x,y) = \sum_{n=1}^{2^j} g_n(t,x)\varphi_n(y)$$

where, for every $n \in \mathbb{N}^*$, g_n is the solution of (17). Applying Propositions 6 and 9, and recalling that $(\varphi_n)_{n \in \mathbb{N}^*}$ is an orthonormal sequence of $L^2(0,1)$, we deduce

$$\begin{aligned}\int_\Omega g(T,x,y)^2\,dx\,dy &= \sum_{n=1}^{2^j}\int_{-1}^{1} g_n(T,x)^2\,dx\\ &\le T^2 e^{C(1+T^{-(1+\gamma)/(1-\gamma)})}\sum_{n=1}^{2^j}\int_0^T\int_a^b g_n(t,x)^2\,dx\,dt\\ &\le T^2 e^{C(2^j+T^{-(1+\gamma)/(1-\gamma)})}\int_0^T\int_a^b\int_c^d\Big|\sum_{n=1}^{2^j} g_n(t,x)\varphi_k(y)\Big|^2\,dy\,dx\,dt\\ &= T^2 e^{C(2^j+T^{-(1+\gamma)/(1-\gamma)})}\int_0^T\int_\omega g(t,x,y)^2\,dx\,dy\,dt,\end{aligned}$$

where the constant C may change from line to line. □

Let $T>0$ and $f_0\in L^2(\Omega)$. We now proceed to construct a control $u\in L^2(0,T;L^2(\Omega))$ such that the solution of (2) satisfies $f(T,\cdot)\equiv 0$. Fix $\rho\in\mathbb{R}$ with

$$0<\rho<\frac{1-\gamma}{1+\gamma} \tag{87}$$

and let $K=K(\rho)>0$ be such that $K\sum_{j=1}^{\infty}2^{-j\rho}=T$. Let $(a_j)_{j\in\mathbb{N}}$ be defined by

$$\begin{cases} a_0=0,\\ a_{j+1}=a_j+2T_j, \quad j\ge 0,\end{cases}$$

where $T_j:=K2^{-j\rho}$ for every $j\in\mathbb{N}$. We now define the control u in the following way. On $[a_j,a_j+T_j]$, we apply a control u such that $\Pi_{E_j}f(a_j+T_j,\cdot)=0$ and

$$\|u\|_{L^2(a_j,a_j+T_j;L^2(\Omega))}\le \mathcal{C}_j\|f(a_j,\cdot)\|_{L^2(\Omega)}$$

where, in view of Proposition 8,

$$\mathcal{C}_j:=e^{C(2^j+T_j^{-(1+\gamma)/(1-\gamma)})}.$$

Observe that, in light of (14),

$$\|f(a_j+T_j,\cdot)\|_{L^2(\Omega)}\le(1+\sqrt{T_j}\,\mathcal{C}_j)\|f(a_j,\cdot)\|_{L^2(\Omega)}.$$

Then, on the interval $[a_j+T_j,a_{j+1}]$ we apply no control in order to take advantage of the natural exponential decay of the solution, thus obtaining

$$\|f(a_{j+1},\cdot)\|_{L^2(\Omega)}\le e^{-\lambda_{2^j}T_j}\|f(a_j+T_j,\cdot)\|_{L^2(\Omega)},$$

where λ_n is defined in (23). Combining the above inequalities, we conclude that

$$\|f(a_{j+1},\cdot)\|_{L^2(\Omega)}\le\exp\Big(\sum_{k=1}^{2^j}[\ln(1+\sqrt{T_k}\,\mathcal{C}_k)-C(2^k)^{2/(1+\gamma)}T_k]\Big)\|f_0\|_{L^2(\Omega)}.$$

The choice of ρ ensures that the sum in the exponential diverges to $-\infty$ as $j\to\infty$, forcing $f(T,\cdot)\equiv 0$. The fact that $u\in L^2(0,T;L^2(\Omega))$ can be checked by similar arguments. □

4.4. *End of the proof of Theorems 1 and 2*

Let ω be an open subset of $(0, 1) \times (0, 1)$. There exist $a, b, c, d \in \mathbb{R}$ with $0 < a < b < 1$, $0 < c < d < 1$ such that $(a, b) \times (c, d) \subset \omega$.

The first (resp. third) statement of Theorem 2 has been proved in Section 4.3 (resp. Section 3); let us prove the second one.

Let us consider $\gamma = 1$ and $\omega = (a, b) \times (0, 1)$. From Proposition 7, we deduce that (3) is observable in ω in any time $T > T_1$. From Theorem 5, we deduce that for any time $T < a^2/2$, (3) is not observable in ω in time T. Thus, the quantity

$$T^* := \inf\{T > 0 : \text{system (3) is observable in } \omega \text{ in time } T\}$$

is well defined and belongs to $[a^2/2, \infty)$. Clearly, observability in some time $T_\sharp$ implies observability in any time $T > T_\sharp$, so

- for every $T > T^*$, (4) is observable in ω in time T,
- for every $T < T^*$, (4) is not observable in ω in time T.

5. Conclusion and open problems

In this article we have studied the null controllability of the Grushin type equation (1), in the rectangle $\Omega = (-1, 1) \times (0, 1)$, with a distributed control localized on an open subset ω of $(0, 1) \times (0, 1)$. We have proved that null controllability:

- holds in any positive time when degeneracy is not too strong, i.e. $\gamma \in (0, 1)$,
- holds only in large time when $\gamma = 1$ and ω is a strip parallel to the y-axis,
- does not hold when degeneracy is too strong, i.e. $\gamma > 1$.

Null controllability when $\gamma = 1$, T is large enough, and the control region ω is more general is an open problem. When $\gamma = 1$, it would be interesting to characterize the minimal time T^* required for null controllability and possibly connect it with the associated diffusion process. We conjecture that $T^* = a^2/2$.

The technique of this paper should possibly extend to higher dimensional cylindrical domains of the form $(-1, 1) \times (0, 1)^m$. However, the generalization of this result to other muldimensional configurations (including $x \in (-1, 1)^n$, $y \in (0, 1)^m$ with $m, n \geq 1$) or boundary controls, is widely open.

Appendix. The case when $\{x = 0\} \subset \omega$

In this appendix we briefly explain why null controllability holds when degeneracy occurs inside the control region. Consider the control system

$$\begin{cases} \partial_t f - \partial_x^2 f - |x|^{2\gamma} \partial_y^2 f = u(t, x, y) 1_\omega(x, y), & (t, x, y) \in (0, T) \times \Omega, \\ f(t, x, y) = 0, & (t, x, y) \in (0, T) \times \partial\Omega, \\ f(0, x, y) = f_0(x, y), & (x, y) \in \Omega, \end{cases} \tag{88}$$

with $\omega = (-a,a)\times(0,1)$, $0<a\le 1$. Fix $b\in(0,a)$ and choose cut-off functions $\xi_i\in C^\infty(\mathbb{R})$, $i=0,1,2$, such that $0\le\xi_i\le 1$ and

$$\begin{cases} \xi_0+\xi_1+\xi_2\equiv 1, \\ \xi_0(x)=1 \quad \text{if } |x|\le b, \quad \xi_0(x)=0 \quad \text{if } |x|\ge a, \\ \xi_1(x)=0 \quad \text{if } x\le b, \quad \xi_0(x)=1 \quad \text{if } x\ge a, \\ \xi_2(x)=1 \quad \text{if } x\le -a, \quad \xi_2(x)=0 \quad \text{if } x\ge -b. \end{cases} \tag{89}$$

Let $\omega_1=(b,a)\times(0,1)$ and let $\Omega_1=(b,1)\times(0,1)$. There exists a control $u_1\in L^2((0,T)\times\Omega_1)$ such that the solution f_1 of

$$\begin{cases} \partial_t f-\partial_x^2 f-|x|^{2\gamma}\partial_y^2 f=u_1(t,x,y)1_{\omega_1}(x,y), & (t,x,y)\in(0,T)\times\Omega_1, \\ f(t,x,y)=0, & (t,x,y)\in(0,T)\times\partial\Omega_1, \\ f(0,x,y)=f_0(x,y), & (x,y)\in\Omega_1, \end{cases}$$

satisfies $f_1(T,\cdot)\equiv 0$ on Ω_1. Similarly, let $\omega_2=(-a,-b)\times(0,1)$ and let $u_2\in L^2((0,T)\times\Omega_2)$, where $\Omega_2=(-1,-b)\times(0,1)$, be such that the solution f_2 of

$$\begin{cases} \partial_t f-\partial_x^2 f-|x|^{2\gamma}\partial_y^2 f=u_2(t,x,y)1_{\omega_2}(x,y), & (t,x,y)\in(0,T)\times\Omega_2, \\ f(t,x,y)=0, & (t,x,y)\in(0,T)\times\partial\Omega_2, \\ f(0,x,y)=f_0(x,y), & (x,y)\in\Omega_2, \end{cases}$$

satisfies $f_2(T,\cdot)\equiv 0$ on Ω_2. Finally, let $\Omega_0=(-a,a)\times(0,1)$ and let f_0 be the solution of

$$\begin{cases} \partial_t f-\partial_x^2 f-|x|^{2\gamma}\partial_y^2 f=0, & (t,x,y)\in(0,T)\times\Omega_0, \\ f(t,x,y)=0, & (t,x,y)\in(0,T)\times\partial\Omega_0, \\ f(0,x,y)=\xi_0(x)f_0(x,y), & (x,y)\in\Omega_0. \end{cases}$$

Then

$$f(t,x,y):=\xi_1(x)f_1(t,x,y)+\xi_2(x)f_2(t,x,y)+\frac{T-t}{T}f_0(t,x,y)$$

satisfies (88) for a suitable control u, as well as $f(T,\cdot)\equiv 0$ on Ω.

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