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Erratum to "Geometric rigidity of $\times m$ invariant measures"

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Regrettably, Lemma 2.3 of [3] is incorrect. This does not affect the main results of the paper, and all other statements remain correct, with the same proofs, if in the lemma in question the limit is replaced by a Cesàro limit, and the assertion is made for μ-a.e. x, as in Lemma 1 below. To state the modified lemma we adopt the notation of [3]. Fix a metric $d(\cdot,\cdot)$ on $\mathcal{P}([-1,1])$ compatible with the weak topology and say that 1-parameter families $(\theta_t)_{t\geq 0}, (\eta_t)_{t\geq 0} \subseteq \mathcal{P}([-1,1])$ are *mean-asymptotic* if $\frac{1}{T}\int_0^T d(\theta_t,\eta_t)\,dt = 0$ as $T\to\infty$.

Lemma 1. *Let $\mu \in \mathcal{P}(\mathbb{R})$ and $f \in \mathrm{diff}^1(\mathbb{R})$. Then for μ-a.e. x, writing $\nu = f\mu$, $y = f(x)$ and $s = \ln(f'(x))$, the sceneries $(\mu_{x,t})_{t\geq 0}$, $(\nu_{y,t-s})_{t\geq 0}$ are mean-asymptotic. In particular, μ generates P at x if and only if ν generates P at y.*

The last statement follows directly from the first, and is the content of [2, Proposition 1.9], which has yet to appear but predates [3]. Here we give a sketch of the proof of the first statement. A detailed treatment will appear in a forthcoming paper by Aspenberg, Ekström, Persson and Schmeling [1].

For $z,t\in\mathbb{R}$ define linear maps $U_{z,t}:\mathbb{R}\to[-1,1]$ by $U_{z,t}(w) = e^t(w-z)$ and write $I_t = U_{x,t}^{-1}([-1,1])$ and $J_t = (U_{y,t-s}f)^{-1}[-1,1]$, so that

$$\mu_{x,t} = \frac{1}{\mu(I_t)}\cdot U_{x,t}(\mu|_{I_t}) \quad\text{and}\quad \nu_{y,t-s} = \frac{1}{\mu(J_t)}\cdot U_{y,t-s}f(\mu|_{J_t}). \tag{1}$$

Evidently, to ensure that $d(\mu_{x,t},\nu_{y,t-s}) < \varepsilon$ it is sufficient, for an appropriate $\delta > 0$, to have (a) $|U_{x,t}(w) - U_{y,t-s}f(w)| < \delta$ for all $w \in I_t\cap J_t$, and (b) $\mu(I_t\cap J_t)/\mu(I_t)$, $\mu(I_t\cap J_t)/\mu(J_t)$ are within δ of 1. The linear approximation of f at x gives $f(w) = f(x) + e^s(w-x) + o(|w-x|)$, hence

$$U_{y,t-s}f(w) = U_{x,t}(w) + o(e^t|w-x|). \tag{2}$$

Since the diameter of I_t, J_t is $O(e^{-t})$, this implies (a) holds for all large t. (b) can fail for some large t, but only infrequently, as shown by the following lemma:

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Lemma 2. *Let θ be a probability measure on $\mathbb{R}$. For $0 < \alpha < \frac{1}{10}$ let $z \in \operatorname{supp}\theta$ and*

$$E_z = \left\{ t > 0 : \frac{\theta(B_{e^{-t-\alpha}}(z))}{\theta(B_{e^{-t+\alpha}}(z))} < 1 - \alpha^{1/3} \right\}.$$

Then, writing λ for Lebesgue measure,

$$\limsup_{T\to\infty} \frac{\lambda(E_z \cap [0,T])}{T} \le 2\alpha^{1/3} \quad \text{for } \theta\text{-a.e. } z.$$

Proof. Write $\beta = \alpha^{1/3}$, so $\beta < 1/2$. Fix z and suppose that $\lambda(E_z \cap [0,T]) > 2\beta T$ for some T. By the Besicovitch covering lemma, there is a disjoint family of intervals $[t_i - \alpha, t_i + \alpha]$ with $t_i \in E_z \cap [0,T]$ and having total length at least βT. Since each interval has length 2α, the number of intervals is at least $\beta T/2\alpha$. There is a $1-\beta$ drop in $\theta(B_{e^{-t}}(z))$ every time t "crosses" $[t_i - \alpha, t_i + \alpha]$. Thus

$$\theta(B_{e^{-T-\alpha}}(z)) \le \prod_i \frac{\theta(B_{e^{-t_i-\alpha}}(z))}{\theta(B_{e^{-t_i+\alpha}}(z))} < (1-\beta)^{\beta T/2\alpha} \le e^{-\beta^2 T/2\alpha} < (e^{-T})^{1/2\beta}.$$

If this holds for arbitrarily large T the (upper) pointwise dimension of θ at z is at least $1/2\beta$, which is greater than 1. This can happen only on a θ-null set of z. □

Returning to the proof of Lemma 1, by (2), $B_{e^{-t-\delta}}(x) \subseteq I_t \cap J_t \subseteq B_{e^{-t+\delta}}(x)$ for all large t, so by the previous lemma, $\mu(I_t \cap J_t)/\mu(I_t) \to 1$ and $\mu(I_t \cap J_t)/\mu(J_t) \to 1$ in the Cesàro sense. Hence the fraction of $t \in [0,T]$ for which conditions (a) and (b) hold tends to 1 as $T \to \infty$, and $(\mu_{x,t})$, $(\nu_{y,t-s})$ are mean-asymptotic.

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References

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