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Jacobian discrepancies and rational singularities

Abstract. Inspired by several works on jet schemes and motivic integration, we consider an extension to singular varieties of the classical definition of discrepancy for morphisms of smooth varieties. The resulting invariant, which we call *Jacobian discrepancy*, is closely related to the jet schemes and the Nash blow-up of the variety. This notion leads to a framework in which adjunction and inversion of adjunction hold in full generality, and several consequences are drawn from these properties. The main result of the paper is a formula measuring the gap between the dualizing sheaf and the Grauert–Riemenschneider canonical sheaf of a normal variety. As an application, we give characterizations for rational and Du Bois singularities on normal Cohen–Macaulay varieties in terms of Jacobian discrepancies. In the case when the canonical class of the variety is $\mathbb{Q}$-Cartier, our result provides the necessary corrections for the converses to hold in theorems of Elkik, of Kovács, Schwede and Smith, and of Kollár and Kovács on rational and Du Bois singularities.

Keywords. Discrepancy, Jacobian, adjunction, Nash blow-up, jet scheme, multiplier ideal, rational singularity, Du Bois singularity

1. Introduction

The main result of the paper is a formula quantifying the difference between the dualizing sheaf and the Grauert–Riemenschneider canonical sheaf of a normal variety, and is stated below in Theorem C. As a motivation of this more technical result, we begin by first describing its implications to the study of rational and Du Bois singularities.

Rational singularities, first introduced and studied in dimension two as a generalization of Du Val singularities [Art66, Lip69b], can be thought of as those singularities that do not contribute to the cohomology of the structure sheaf of the variety. The connection with the singularities in the minimal model program was discovered by Elkik [Elk81]. Du Bois singularities [DB81, Ste83] form a wider, more mysterious class which arises naturally from the point of view of Hodge theory and satisfies good vanishing properties; see for instance [GKKP11]. The link between Du Bois singularities and log canonical singularities is a recent achievement first established in the Cohen–Macaulay case in [KSS10] and then, unconditionally, in [KK10]. For both classes of singularities, the connection with the singularities in the minimal model program appears to be unidirectional, as most

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rational and Du Bois singularities do not seem at first to satisfy any reasonable condition from the point of view of valuations and discrepancies.

As we shall see, there is in fact a deeper connection going the other way around which provides characterizations of these singularities when the variety is Cohen–Macaulay.

The precise connection depends on the tension between two closely related ideals attached to the singularities. The first one is the *lci-defect ideal* $\mathfrak{d}_X$ of X. This object is very natural from the point of view of liaison theory and is related to the Nash transformation of X with respect to the dualizing sheaf ω_X. In concrete terms, $\mathfrak{d}_X$ is the ideal generated by the equations of the residual intersections with all the reduced, locally complete intersection schemes $V \supset X$ of the same dimension. The second ideal, called the *lci-defect ideal of level r* of X and denoted by $\mathfrak{d}_{r,X}$, is defined when the canonical class K_X is $\mathbb{Q}$-Cartier and depends on the integer r such that $r K_X$ is Cartier. The two ideals agree when $r = 1$. In general, both ideals vanish precisely on the locus where X is not locally complete intersection.

In the following theorem we make the necessary correction for the converse to hold in the aforementioned results of [Elk81, KSS10, KK10]. Restricting to the Cohen–Macaulay case, the result yields a characterization of rational and Du Bois singularities in terms of discrepancies. In particular, since rational singularities are Cohen–Macaulay, the result provides a discrepancy characterization of all rational singularities.

Theorem A. *Let X be a normal variety, and assume that $r K_X$ is Cartier for some positive integer r.*

(a) *If X has rational singularities then the pair $(X, \mathfrak{d}_{r,X}^{1/r} \cdot \mathfrak{d}_X^{-1})$ is canonical.*

(b) *If X has Du Bois singularities then the pair $(X, \mathfrak{d}_{r,X}^{1/r} \cdot \mathfrak{d}_X^{-1})$ is log canonical.*

Moreover, the converse holds in both cases whenever X is Cohen–Macaulay.

In either case, the Cohen–Macaulay condition is essential for the converse to hold (see Example 7.6). If however the assumptions on the singularities of the pair is strengthened by removing the contribution of $\mathfrak{d}_X^{-1}$, then one obtains sufficient conditions for rational and Du Bois singularities holding in a much more general setting.

We arrive at the above result by considering a quite different set of questions that lead us to study an extension to arbitrary varieties of the notion of discrepancy of a divisorial valuation over a smooth variety.

The candidate is an integer that simply measures the difference between the Jacobian of the transformation (which gives the discrepancy in the smooth case) and the Jacobian of the singularity. Specifically, given a resolution of singularities $f : Y \to X$ of a complex variety X, we define the *Jacobian discrepancy* of a prime divisor $E \subset Y$ over X to be the integer

$$k_E^\diamond(X) := \operatorname{ord}_E(\mathfrak{j}_f) - \operatorname{ord}_E(\mathfrak{j}_X),$$

where $\mathfrak{j}_f \subset \mathcal{O}_Y$ is the Jacobian ideal of the morphism f and $\mathfrak{j}_X \subset \mathcal{O}_X$ is the Jacobian ideal of X.

Jacobian discrepancies are closely related to *Mather discrepancies*, where only the contribution coming from $\mathfrak{j}_f$ is taken into account. The latter are the main ingredient of

the change-of-variables formula in motivic integration [DL99] and are determined by the Nash blow-up of the variety. Geometric properties of Mather discrepancies were investigated in [dFEI08], and the recent work of Ishii [Ish11] is devoted to a study of singularities from the point of view of Mather discrepancies. When X is $\mathbb{Q}$-Gorenstein (that is, X is normal and its canonical class is $\mathbb{Q}$-Cartier), the relationship between Jacobian discrepancies, Mather discrepancies and usual discrepancies is implicit in the works [Kaw08, EM09, Eis10].

Like Mather discrepancies, Jacobian discrepancies can be read off from the jet schemes of the variety. In the following result, one should notice that not just the topology but also the scheme structure of the spaces of jets is relevant to this end.

Theorem B. *For every prime divisor $E \subset Y$ over a variety X we have*

$$k_E^{\diamond}(X) + 1 = 2n(m+1) - \dim_{\mathbb{C}}(TX_m|_{\eta_{E,m}}) \quad \text{for all } m \geq 2 \operatorname{ord}_E(\mathfrak{j}_X),$$

where X_m is the m-th jet scheme of X, TX_m is the tangent space to X_m, and $\eta_{E,m} \in X_m$ is the generic point of the image of the set of m-th jets in Y having order of contact one with E.

Using Jacobian discrepancies, one can formulate in a completely natural way a framework for singularities that runs parallel to the usual theory of singularities of $\mathbb{Q}$-Gorenstein varieties considered in the minimal model program. In particular, this leads to the notions of *J-canonical* and *log J-canonical singularities*, of *minimal log J-discrepancy*, and of *log J-canonical threshold*. This framework moves in a different direction with respect to the one proposed in [dFH09], where the asymptotic nature of the theory on $\mathbb{Q}$-Gorenstein varieties is instead taken into account. These invariants capture interesting new geometry of the singularities and we believe that they deserve investigation.

The theory is tailored to satisfy adjunction and inversion of adjunction (see Proposition 4.9 and Theorem 4.10). The latter, independently proven also by Ishii [Ish11], extends the main theorems of [EMY03, EM04, Kaw08, EM09] to the setting considered here, and has a number of consequences that were previously obtained in [Mus01, EM04, dFEM10] for normal, locally complete intersection varieties. These include the semicontinuity of minimal log J-discrepancies (see Corollaries 4.15 and 4.14, see also [Ish11]) and the fact that the set of all log J-canonical thresholds in any fixed dimension satisfies the ascending chain condition (see Corollary 4.13). Theorem B and inversion of adjunction also yield characterizations of the singularities of a variety that involve jet schemes and their tangent spaces (see Corollary 5.4).

The main application of the above viewpoint on singularities regards the Grauert–Riemenschneider canonical sheaf of a variety X, defined to be $f_*\omega_Y$ where $f : Y \to X$ is any resolution of singularities [GR70]. This is the natural generalization of the canonical line bundle of a manifold from the point of view of Kodaira vanishing and plays, for instance, a central role in Lipman's proof to resolution of singularities in dimension two [Lip78]. This sheaf agrees with the dualizing sheaf if X has rational singularities. In general there is an inclusion $f_*\omega_Y \subset \omega_X$ and our aim is to provide a measure of the gap.

To this end, we introduce the natural generalization of multiplier ideals to the above framework. Given a normal variety X, for any non-zero ideal $\mathfrak{a} \subset \mathcal{O}_X$ and every real

number c we use Jacobian discrepancies to define the *Jacobian multiplier ideal* $\mathcal{J}^\diamond(\mathfrak{a}^c)$. This is in general a fractional ideal sheaf; it is an ideal sheaf if either $c \geq 0$ or $\mathfrak{a}$ is trivial in codimension one. This agrees with the usual multiplier ideal if X is locally complete intersection.

The core result of the paper is the following formula which expresses the colon ideal of $f_*\omega_Y$ by ω_X as a Jacobian multiplier ideal.

Theorem C. *If $f \colon Y \to X$ is a resolution of singularities of a normal variety, then*

$$(f_*\omega_Y : \omega_X) = \mathcal{J}^\diamond(\mathfrak{d}_X^{-1}),$$

where the left-hand side is the colon of the sheaves viewed as $\mathcal{O}_X$-modules.

In light of this theorem and its 'twisted' version (stated below in Theorem 6.15), one can view the Grauert–Riemenschneider vanishing theorem and its generalizations as Nadel-type vanishings for Jacobian multiplier ideals.

More importantly, using Kempf's characterization of rational singularities on normal Cohen–Macaulay varieties and the analogous result on Du Bois singularities established in [KSS10], we deduce from Theorem C that, given any normal variety X,

(a) if X has rational singularities then the pair $(X, \mathfrak{d}_X^{-1})$ is J-canonical,

(b) if X has Du Bois singularities then the pair $(X, \mathfrak{d}_X^{-1})$ is log J-canonical,

and in both cases the converse holds if X is Cohen–Macaulay. This result appears in the main body of the paper as Corollary 7.2. When X is $\mathbb{Q}$-Gorenstein, this implies Theorem A.

As mentioned before, the Cohen–Macaulay condition is necessary for the converse to hold. If however one imposes stronger conditions on discrepancies, then a simple argument brought out to our attention by Mustață shows that the results of [Kaw98, KK10], in combination with inversion of adjunction, imply that any variety with J-canonical (resp., log J-canonical) singularities has rational (resp., Du Bois) singularities (see Theorem 7.7). This last result is in fact well-known to specialists if the hypotheses on the singularities are expressed in terms of the jet schemes of the variety.

We work over the field of complex numbers. Unless otherwise stated, we use the word *scheme* to refer to a separated scheme of finite type over $\mathbb{C}$. The word *variety* will refer to an irreducible reduced scheme.

2. Nash blow-up

The notions of discrepancy that will be defined in the paper are naturally related to the Nash blow-up. In this section we review the basic theory of this blow-up, and explore a modification of the construction, the blow-up of the dualizing sheaf.

2.1. Jacobian ideals

The *Jacobian ideal sheaf* of a reduced scheme X, denoted $\mathfrak{j}_X \subset \mathcal{O}_X$, is the smallest non-zero Fitting ideal of the cotangent sheaf Ω_X. If X is of pure dimension n, then $\mathfrak{j}_X = \mathrm{Fitt}^n(\Omega_X)$, the n-th Fitting ideal.

If Y is a smooth scheme and $f : Y \to X$ is a morphism to a scheme X, the *Jacobian ideal* of f is the 0-th Fitting ideal $\mathfrak{j}_f = \mathrm{Fitt}^0(\Omega_{Y/X})$ of $\Omega_{Y/X}$. If Y is equidimensional of dimension n, then Ω_Y^n is invertible and the image of the map induced at the level of top differentials $f^*\Omega_X^n \to \Omega_Y^n$ can be written as $\mathfrak{j}_f \otimes \Omega_Y^n$.

2.2. The classical Nash blow-up

The *Nash blow-up* of a reduced scheme X of pure dimension n is a surjective morphism

$$\nu : \widehat{X} \to X$$

satisfying the following universal property: a proper birational morphism of schemes $f : Y \to X$ factors through ν if and only if the sheaf $f^*\Omega_X$ has a locally free quotient of rank n. In general, if a resolution $f : Y \to X$ factors through the Nash blow-up of X, then the associated Jacobian ideal $\mathfrak{j}_f$ is locally principal.

The Nash blow-up of X is unique up to isomorphism, and can be constructed by taking the restriction of the projection $\mathrm{Gr}(\Omega_X, n) \to X$ to the closure $\widehat{X} \subset \mathrm{Gr}(\Omega_X, n)$ of the natural isomorphism $X_{\mathrm{reg}} \to \mathrm{Gr}(\Omega_{X_{\mathrm{reg}}}, n)$. Here, for a coherent sheaf $\mathcal{F}$ on X, we denote by $\mathrm{Gr}(\mathcal{F}, n)$ the Grassmannian of locally free quotients of $\mathcal{F}$ of rank n. If X is embedded in a smooth variety M, the natural quotient $\Omega_M|_X \to \Omega_X$ induces and inclusion $i : \mathrm{Gr}(\Omega_X, n) \hookrightarrow \mathrm{Gr}(\Omega_M, n)$, and $\widehat{X}$ is the closure of the natural embedding of X_{reg} in $\mathrm{Gr}(\Omega_M, n)$ given by i. Alternatively, using the Plücker embedding $\mathrm{Gr}(\Omega_X, n) \subset \mathbb{P}(\Omega_X^n)$, one can also view the Nash blow-up $\widehat{X}$ inside $\mathbb{P}(\Omega_X^n)$ as the closure of the natural isomorphism $X_{\mathrm{reg}} \cong \mathbb{P}(\Omega_{X_{\mathrm{reg}}}^n)$. Denoting for short

$$\widehat{\omega}_X := \Omega_X^n/\mathrm{torsion},$$

$\widehat{X}$ can also be viewed as the closure of X_{reg} in $\mathbb{P}(\widehat{\omega}_X)$ since the latter is closed in $\mathbb{P}(\Omega_X^n)$ and contains $\mathbb{P}(\Omega_{X_{\mathrm{reg}}}^n)$.

Remark 2.1. Since X is reduced, ω_X is torsion free (cf. Proposition (2.8) of [AK70]) and the canonical map $\Omega_X^n \to \omega_X$ (cf. Proposition 9.1 of [EM09]) factors through an inclusion $\widehat{\omega}_X \hookrightarrow \omega_X$.

Note that the tautological line bundle $\mathcal{O}_{\mathbb{P}(\Omega_X^n)}(1)$ of $\mathbb{P}(\Omega_X^n)$ restricts to the tautological line bundle $\mathcal{O}_{\mathbb{P}(\widehat{\omega}_X)}(1)$ of $\mathbb{P}(\widehat{\omega}_X)$. Following the terminology introduced in [dFEI08], we refer to the restriction $\mathcal{O}_{\widehat{X}}(1) := \mathcal{O}_{\mathbb{P}(\widehat{\omega}_X)}(1)|_{\widehat{X}}$ as the *Mather canonical line bundle* of X, and to any Cartier divisor $\widehat{K}_X$ on $\widehat{X}$ such that $\mathcal{O}_{\widehat{X}}(\widehat{K}_X) \cong \mathcal{O}_{\widehat{X}}(1)$ as a *Mather canonical divisor* of X.

Remark 2.2. The above terminology is motivated by the relationship with Mather–Chern classes (see Remark 1.5 of [dFEI08] for a discussion). Note that in [dFEI08] the symbol $\widehat{K}_X$ was used to denote the Mather canonical line bundle.

Remark 2.3. If $X \subset X'$ is the inclusion between two reduced equidimensional schemes of the same dimension and we denote by $\nu : \widehat{X} \to X$ and $\nu' : \widehat{X}' \to X'$ the respective Nash blow-ups, then $\widehat{X} \subset \widehat{X}'$, $\nu = \nu'|_{\widehat{X}}$ and $\mathcal{O}_{\widehat{X}}(1) = \mathcal{O}_{\widehat{X}'}(1)|_{\widehat{X}}$.

The Nash blow-up naturally relates to the blow-up of the Jacobian ideal of the scheme. Lemma 1 of [Lip69a] implies that for any reduced equidimensional scheme X, the blow-up of the Jacobian ideal of X factors through the Nash blow-up of X; see also [OZ91] for a detailed discussion of this property. A direct computation in local coordinates shows that when X is locally complete intersection the Nash blow-up of X is isomorphic to the blow-up of the Jacobian ideal of X (see [Nob75, OZ91]). One can use Remark 2.3 to see that in general, if X is a reduced equidimensional scheme and $V \supset X$ is a reduced, locally complete intersection scheme of the same dimension, then the Nash blow-up $\nu \colon \widehat{X} \to X$ is isomorphic to the blow-up of the ideal $j_V|_X$. Since the natural map $\Omega_X^n \to \omega_V|_X$ has image

$$\mathrm{Im}[\Omega_X^n \to \omega_V|_X] = j_V|_X \otimes \omega_V|_X$$

(see for instance [OZ91, EM09]), this fact also follows from the discussion given in the following subsection.

Proposition 2.4. *Let X be a reduced equidimensional scheme. Then, for every reduced, locally complete intersection scheme $V \supset X$ of the same dimension, there is a natural isomorphism*

$$j_V|_X \cdot \mathcal{O}_{\widehat{X}} \cong \mathcal{O}_{\widehat{X}}(1) \otimes \nu^*(\omega_V^{-1}|_X).$$

Proof. We view $\nu \colon \widehat{X} \to X$ as the blow-up of $j_V|_X$. Since by generic smoothness the kernel of the natural map $\Omega_X^n \to \omega_V|_X$ is the torsion of Ω_X^n, there is an isomorphism

$$\mathbb{P}(\widehat{\omega}_X) = \mathbb{P}(j_V|_X \otimes \omega_V|_X) \xrightarrow{\alpha} \mathbb{P}(j_V|_X).$$

This implies that

$$\mathcal{O}_{\mathbb{P}(\widehat{\omega}_X)}(1) \cong \alpha^* \mathcal{O}_{\mathbb{P}(j_V|_X)}(1) \otimes \pi^* \omega_V|_X$$

where $\pi \colon \mathbb{P}(\widehat{\omega}_X) \to X$ is the projection map. On the other hand, by the universal property of $\mathbb{P}(j_V|_X)$, the blow-up $\widehat{X} \to X$ of $j_V|_X$ factors through $\mathbb{P}(j_V|_X) \to X$, and $\mathcal{O}_{\mathbb{P}(j_V|_X)}(1)$ pulls back to $j_V|_X \cdot \mathcal{O}_{\widehat{X}}$. Since the map $\widehat{X} \to \mathbb{P}(j_V|_X)$ is an isomorphism onto its image, we obtain $\mathcal{O}_{\widehat{X}}(1) \cong (j_V|_X \cdot \mathcal{O}_{\widehat{X}}) \otimes \nu^* \omega_V|_X$. $\square$

2.3. The blow-up of the dualizing sheaf

Let X be a reduced equidimensional scheme. We denote by $\mathbb{C}(X)$ the sheaf of rational functions on X. Following the general definition given in [Kle79], this sheaf is given by the push-forward of the restriction of $\mathcal{O}_X$ to the associated primes, which in our case are just the generic points of the irreducible components of X.

In the construction of the Nash blow-up, one can replace the sheaf of differentials by any coherent sheaf $\mathcal{F}$ which is locally free of some rank r in an open dense subset of X. This idea was developed in detail in [OZ91], where it received the name of *Nash transformation of X relative to $\mathcal{F}$*.

The main result from [OZ91] that we will use is a description of the ideal whose blow-up gives the Nash transformation: it is the image of the composition

$$\bigwedge^r \mathcal{F} \to \bigwedge^r \mathcal{F} \otimes_{\mathcal{O}_X} \mathbb{C}(X) \xrightarrow{\cong} \mathbb{C}(X),$$

where the second arrow is induced by any trivialization of $\bigwedge^r \mathcal{F}$ at each of the generic points of X. Notice that different choices of trivialization give different ideals, but they differ by a Cartier divisor, so they give rise to the same blow-up. Also, for an arbitrary trivialization the ideal is possibly fractional, but there is always a choice that clears all the denominators.

For example, in the case of the classical Nash blow-up one is led to consider the natural sequence

$$\widehat{\omega}_X \to \omega_X \to \omega_V|_X \to \Omega^n_{\mathbb{C}(X)/\mathbb{C}},$$

where V is some reduced locally complete intersection n-dimensional scheme containing X. Since $\omega_V|_X$ is a line bundle, one can use it to trivialize $\Omega^n_{\mathbb{C}(X)/\mathbb{C}}$, and we see that the classical Nash blow-up is given by the ideal $\mathfrak{a}$ for which the image of $\widehat{\omega}_X$ in $\omega_V|_X$ is $\mathfrak{a} \otimes \omega_V|_X$, namely, the ideal $\mathfrak{j}_V|_X$.

For our purposes, it will be interesting to consider a second Nash transformation, this time relative to the dualizing sheaf. It is a proper birational morphism

$$\mu \colon \widetilde{X} \to X,$$

which is universal among those proper birational morphisms for which the pull-back of ω_X admits a line bundle quotient. Analogous to the case of the classical Nash blow-up, $\widetilde{X}$ can be constructed as the closure in $\mathbb{P}(\omega_X)$ of $X_{\text{reg}} \simeq \mathbb{P}(\omega_{X_{\text{reg}}})$.

Our interest in $\widetilde{X}$ arises from the ideals whose blow-up gives μ. After picking some locally complete intersection V as above, one sees that μ is the blow-up of X along the ideal $\mathfrak{d}_{X,V}$ for which the image of the inclusion of ω_X inside $\omega_V|_X$ is given by

$$\text{Im}[\omega_X \to \omega_V|_X] = \mathfrak{d}_{X,V} \otimes \omega_V|_X.$$

Note that μ is an isomorphism if and only if ω_X is invertible. One implication is obvious, and conversely, if μ is an isomorphism then $\mathfrak{d}_{X,V}$ is locally principal and hence $\omega_X = \mathfrak{d}_{X,V} \otimes \omega_V|_X$ is invertible.

Different choices of V give different ideals $\mathfrak{d}_{X,V}$, but their blow-up is always $\widetilde{X}$. These ideals are easy to describe (cf. [OZ91, EM09]). One embeds V in a smooth variety M, and considers the ideals I_X and I_V of X and V in M. Then, as $\mathcal{O}_V$-modules, one has

$$\omega_X \otimes \omega_V^{-1} = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{O}_X, \mathcal{O}_V) = (I_V : I_X)/I_V,$$

and therefore

$$\mathfrak{d}_{X,V} = ((I_V : I_X) + I_X)/I_X.$$

In other words, if we write $V = X \cup X'$, where X' is the residual part of V with respect to X (given by the ideal $(I_V : I_X)$), then $\mathfrak{d}_{X,V}$ is the ideal defining the intersection $X \cap X'$ in X. We may think of $\mathfrak{d}_{X,V}$ as giving the *residual intersection* of V with X.

As V varies, so does the residual intersection $\mathfrak{d}_{X,V}$. Thinking of this collection of intersections as a linear series in X, it is natural to consider its base locus, which is clearly supported on the points where X is not locally complete intersection. This motivates the next definition.

Definition 2.5. The *lci-defect ideal* of X is defined to be

$$\mathfrak{d}_X := \sum_V \mathfrak{d}_{X,V},$$

where the sum is taken over all reduced, locally complete intersection schemes $V \supset X$ of the same dimension.

In the case of the classical Nash blow-up, the analogous object is the Jacobian ideal $\mathfrak{j}_X$, which is spanned by the ideals $\mathfrak{j}_V|_X$ as V varies in any fixed embedding M of X. If we restrict the sum in the above definition to those schemes V varying in one fixed embedding M, the resulting ideal, which we denote by $\mathfrak{d}_{X/M}$, depends a priori on the embedding. Its integral closure however does not depend on M.

Proposition 2.6. *For any fixed embedding $X \subset M$, we have $\overline{\mathfrak{d}_{X/M}} = \overline{\mathfrak{d}_X}$.*

Proof. Since

$$\mathfrak{d}_X = \sum_{M \supset X} \mathfrak{d}_{X/M},$$

it suffices to prove that the integral closure of $\mathfrak{d}_{X/M}$ is independent of the embedding. Fix any embedding $X \subset M$. If $\mathcal{O}_{\widetilde{X}}(1)$ denotes the tautological quotient of $\mu^*\omega_X$ associated to the Nash transformation, then

$$\mathrm{Im}[\mu^*\widehat{\omega}_X \to \mu^*\omega_X \to \mathcal{O}_{\widetilde{X}}(1)] = \mathfrak{n} \otimes \mathcal{O}_{\widetilde{X}}(1)$$

for some ideal $\mathfrak{n} \subset \mathcal{O}_{\widetilde{X}}$. On the other hand, we have

$$\mathrm{Im}[\mu^*\widehat{\omega}_X \to \mu^*\omega_V|_X] = (\mathfrak{j}_V|_X \cdot \mathcal{O}_{\widetilde{X}}) \otimes \mu^*(\omega_V|_X).$$

By the same arguments of the proof of Proposition 2.4 there is a natural isomorphism

$$(\mathfrak{d}_{X,V} \cdot \mathcal{O}_{\widetilde{X}}) \otimes \mu^*(\omega_V|_X) \cong \mathcal{O}_{\widetilde{X}}(1),$$

and we see that $\mathfrak{n} \cdot (\mathfrak{d}_{X,V} \cdot \mathcal{O}_{\widetilde{X}}) = \mathfrak{j}_V|_X \cdot \mathcal{O}_{\widetilde{X}}$. By taking the sum as V varies, we obtain $\mathfrak{n} \cdot (\mathfrak{d}_{X/M} \cdot \mathcal{O}_{\widetilde{X}}) = \mathfrak{j}_X \cdot \mathcal{O}_{\widetilde{X}}$. This proves that the integral closure of $\mathfrak{d}_{X/M} \cdot \mathcal{O}_{\widetilde{X}}$ is independent of M. $\qquad\square$

Remark 2.7. For the purpose of this paper, the integral closure of $\mathfrak{d}_X$ is the only thing we need to control, and thus with slight abuse of notation one can always pretend that $\mathfrak{d}_X$ is determined from any embedding of X.

2.4. The $\mathbb{Q}$-Gorenstein case

When X is $\mathbb{Q}$-Gorenstein, one can exploit this property to give another scheme structure to the locus where X is not locally complete intersection, as explained in [EM09]. We review here this alternative theory, as it will be useful later to compare the new notions of discrepancy with the classical one.

Recall that a variety X is said to be $\mathbb{Q}$-*Gorenstein* if it is normal and its canonical class K_X is $\mathbb{Q}$-Cartier. Assume X is $\mathbb{Q}$-Gorenstein, and let r be a positive integer such that rK_X is Cartier. For any reduced, locally complete intersection scheme $V \supset X$ of the same dimension, we consider the ideal $\mathfrak{d}_{r,X,V} \subset \mathcal{O}_X$ such that the image of the natural map from $\mathcal{O}_X(rK_X)$ to $(\omega_V|_X)^{\otimes r}$ is given by

$$\mathrm{Im}[\mathcal{O}_X(rK_X) \to (\omega_V|_X)^{\otimes r}] = \mathfrak{d}_{r,X,V} \otimes (\omega_V|_X)^{\otimes r}.$$

Note that $\mathfrak{d}_{r,X,V}$ is a locally principal ideal since $\mathcal{O}_X(rK_X)$ is a line bundle.

Definition 2.8. With the above assumptions, we define the *lci-defect ideal of level r* of X to be

$$\mathfrak{d}_{r,X} := \sum_V \mathfrak{d}_{r,X,V},$$

where the sum is taken over all V as above.

Note also that, like $\mathfrak{d}_X$, the ideal $\mathfrak{d}_{r,X}$ vanishes exactly where X is not locally complete intersection. If we fix an embedding $X \subset M$ then the ideal $\mathfrak{d}_{r,X/M}$ obtained by restricting the sum in the above definition to schemes $V \subset M$ depends a priori on the embedding.

To better understand these ideals, for every V as above we consider the natural sequence

$$\widehat{\omega}_X^{\otimes r} \to \omega_X^{\otimes r} \to \mathcal{O}_X(rK_X) \to (\omega_V|_X)^{\otimes r}. \tag{2.1}$$

Using again the fact that $\mathcal{O}_X(rK_X)$ is a line bundle, we see that the image of the canonical map from $\widehat{\omega}_X^{\otimes r}$ to $\mathcal{O}_X(rK_X)$ is given by

$$\mathrm{Im}[\widehat{\omega}_X^{\otimes r} \to \mathcal{O}_X(rK_X)] = \mathfrak{n}_{r,X} \otimes \mathcal{O}_X(rK_X)$$

for some ideal sheaf $\mathfrak{n}_{r,X} \subset \mathcal{O}_X$.

Definition 2.9. In accordance with the terminology introduced in [EM09], we call $\mathfrak{n}_{r,X}$ the *Nash ideal of level r* of X.

Proposition 2.10. *With the above notation, there are inclusions*

$$\mathfrak{n}_{r,X} \cdot \mathfrak{d}_{r,X/M} \subset \mathfrak{n}_{r,X} \cdot \mathfrak{d}_{r,X} \subset \mathfrak{j}_X^r$$

which induce identities on integral closures. In particular, for every embedding $X \subset M$ we have $\overline{\mathfrak{d}_{r,X/M}} = \overline{\mathfrak{d}_{r,X}}$.

Proof. Since the image of $\widehat{\omega}_X$ in $\omega_V|_X$ is given by $\mathfrak{j}_V|_X$, we have $\mathfrak{n}_{r,X} \cdot \mathfrak{d}_{r,X,V} = (\mathfrak{j}_V|_X)^r$, and hence

$$\sum_{V \subset M} (\mathfrak{j}_V|_X)^r = \mathfrak{n}_{r,X} \cdot \mathfrak{d}_{r,X/M} \subset \mathfrak{n}_{r,X} \cdot \mathfrak{d}_{r,X} = \sum_V (\mathfrak{j}_V|_X)^r \subset \mathfrak{j}_X^r,$$

and both inclusions become equalities once integral closures are taken. The last assertion is a direct consequence of the first part. $\qquad\square$

Remark 2.11. With the above notation, the ideal $\mathfrak{d}_{r,X}$ has the same integral closure as the colon ideals $J'_r := (\mathfrak{j}^r_X : \mathfrak{n}_{r,X})$ and $J_r := (\overline{\mathfrak{j}^r_X} : \mathfrak{n}_{r,X})$. First notice that there is a chain of inclusions $\mathfrak{d}_{r,X} \subset J'_r \subset J_r$, the first one following by Proposition 2.10 and the latter by the inclusion $\mathfrak{j}^r_X$ in its integral closure. Therefore it suffices to check that $\overline{\mathfrak{d}_{r,X}} = \overline{J_r}$. This follows by combining Proposition 2.10, which gives $\overline{\mathfrak{n}_{r,X} \cdot \mathfrak{d}_{r,X}} = \overline{\mathfrak{j}^r_X}$, with Proposition 9.4 of [EM09], which gives $\overline{\mathfrak{n}_{r,X} \cdot J_r} = \overline{\mathfrak{j}^r_X}$. In [EM09], the ideal J_r is chosen as a measure of the locus where X is not locally complete intersection, and the scheme it defines is called the *non-lci subscheme of level r* of X. For the purpose of this paper, there is no significant difference between the two ideals as we only need to take into consideration the valuative contributions of these ideals, which are equivalent.

Proposition 2.12. *With the above notation, we have $\mathfrak{d}^r_X \subset \overline{\mathfrak{d}_{r,X}}$.*

Proof. By the definitions and the sequence (2.1) we have an inclusion $\mathfrak{d}^r_{X,V} \subset \mathfrak{d}_{r,X,V}$, and hence $\sum_V \mathfrak{d}^r_{X,V} \subset \mathfrak{d}_{r,X}$. One concludes by observing that $\mathfrak{d}^r_X$ is contained in the integral closure of $\sum_V \mathfrak{d}^r_{X,V}$.   □

Remark 2.13. If ω_X is invertible then we can take $r = 1$, and it follows from the definitions that $\mathfrak{d}_X = \mathfrak{d}_{1,X}$. In general however the inclusion $\overline{\mathfrak{d}^r_X} \subset \overline{\mathfrak{d}_{r,X}}$ might be strict (cf. Remark 7.4 below).

3. Discrepancies

In this section we introduce the main invariants that we will study throughout the paper.

3.1. Mather and Jacobian discrepancies

We focus on the case of a reduced equidimensional scheme X. To fix terminology, by a *resolution* of a reduced equidimensional scheme X we intend a morphism $f : Y \to X$ from a smooth scheme Y that restricts to a proper birational map over each irreducible component of X and such that every irreducible component of Y dominates an irreducible component of X. In particular, if $f : Y \to X$ is a resolution and X has irreducible components X_i, then Y decomposes as a disjoint union of smooth varieties Y_i and f restricts to resolutions $f_i : Y_i \to X_i$.

We say that E is a prime divisor *over X* if E is a prime divisor on some resolution $f : Y \to X$, and that it is *exceptional* if f is not an isomorphism at the generic point of E. The *center* $c_X(E)$ of E in X is the generic point of the image of E in X. If Y_i is the component of Y containing E and $f_i : Y_i \to X_i$ is the induced resolution of the corresponding irreducible component of X, then associated to E one defines the divisorial valuation ord_E on the function field $\mathbb{C}(X_i)$: for any nonzero element $\phi \in \mathbb{C}(X_i)$, $\mathrm{ord}_E(\phi)$ is the order of vanishing (or pole) of $f_i^*(\phi)$ at the generic point of $E \subset Y_i$.

If $\mathfrak{a} \subset \mathcal{O}_X$ is an ideal sheaf on X, then we denote $\mathrm{ord}_E(\mathfrak{a}) := \mathrm{ord}_E(\mathfrak{a}|_{X_i})$. Equivalently, we have $\mathrm{ord}_E(\mathfrak{a}) = \mathrm{ord}_E(\mathfrak{a} \cdot \mathcal{O}_Y)$ where the right-hand side denotes the integer a for which the image of $\mathfrak{a} \cdot \mathcal{O}_Y$ in the discrete valuation ring $\mathcal{O}_{Y,E}$ is the a-th power of

the maximal ideal. Note that $\operatorname{ord}_E(\mathfrak{a}) = \infty$ if and only if $\mathfrak{a}$ vanishes identically along the irreducible component X_i of X dominated by the component of Y containing E.

Definition 3.1. Let X be a reduced equidimensional scheme. Given a resolution $f: Y \to X$, if E is a prime divisor over X, the *Mather discrepancy* and the *Jacobian discrepancy* of E over X are respectively defined by

$$\widehat{k}_E(X) := \operatorname{ord}_E(\mathfrak{j}_f) \quad \text{and} \quad k_E^\diamond(X) := \operatorname{ord}_E(\mathfrak{j}_f) - \operatorname{ord}_E(\mathfrak{j}_X).$$

The *relative Mather canonical divisor* and the *relative Jacobian canonical divisor* of f are, respectively,

$$\widehat{K}_{Y/X} := \sum_{E \subset Y} \widehat{k}_E(X) \cdot E \quad \text{and} \quad K_{Y/X}^\diamond := \sum_{E \subset Y} k_E^\diamond(X) \cdot E,$$

where the sum runs over all prime divisors E on Y.

Remark 3.2. If X is smooth, then $\widehat{k}_E(X) = k_E^\diamond(X) = k_E(X)$, the usual discrepancy of E over X, and $\widehat{K}_{Y/X} = K_{Y/X}^\diamond = K_{Y/X}$, the usual relative canonical divisor.

If X is a reduced equidimensional scheme and $f: Y \to X$ is a resolution factoring through the Nash blow-up of X, then

$$\mathfrak{j}_f \cdot \mathcal{O}_Y = \mathcal{O}_Y(-\widehat{K}_{Y/X}),$$

and $\widehat{K}_{Y/X}$ is the unique f-exceptional divisor linearly equivalent to $K_Y - \widehat{f}^* \widehat{K}_X$ for a choice of a canonical divisor K_Y on Y and of a Mather canonical divisor $\widehat{K}_X$ (see Proposition 1.7 of [dFEI08]). Furthermore, if $f: Y \to X$ factors through the blow-up of $\mathfrak{j}_X$, and we write $\mathfrak{j}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-B)$ for some effective divisor B on Y, then

$$K_{Y/X}^\diamond = \widehat{K}_{Y/X} - B.$$

One deduces the following property.

Proposition 3.3. *Let $f: Y \to X$ be a resolution of a reduced equidimensional scheme X that factors through the blow-up of the Jacobian ideal $\mathfrak{j}_X$. If $f': Y' \to X$ is another resolution factoring through f via a morphism $h: Y' \to Y$, then*

$$\widehat{K}_{Y'/X} = K_{Y'/Y} + h^* \widehat{K}_{Y/X} \quad \text{and} \quad K_{Y'/X}^\diamond = K_{Y'/Y} + h^* K_{Y/X}^\diamond.$$

3.2. Discrepancies over $\mathbb{Q}$-Gorenstein varieties

Suppose that X a $\mathbb{Q}$-Gorenstein variety. In this case one defines the relative canonical divisor of a resolution $f: Y \to X$ as the $\mathbb{Q}$-divisor $K_{Y/X} = K_Y - f^* K_X$ where K_Y is a canonical divisor on Y, $K_X = f_* K_Y$, and $f^* K_X$ is defined as the pull-back of a $\mathbb{Q}$-Cartier divisor. If E is a prime divisor on Y, we denote by $k_E(X) := \operatorname{ord}_E(K_{Y/X})$. This is the usual discrepancy of E over X.

The relation between Mather and Jacobian discrepancies and the usual discrepancies defined for this class of varieties is implicit in the works [Kaw08, EM09, Eis10], and is made explicit in the following statement.

Proposition 3.4. *Let X be a $\mathbb{Q}$-Gorenstein variety. Let r be any positive integer such that $r K_X$ is Cartier, let $\mathfrak{n}_{r,X}$ be the Nash ideal of level r of X, and let $\mathfrak{d}_{r,X}$ be the lci-defect ideal of level r of X. Then for every prime divisor E over X we have*

$$\widehat{k}_E(X) = k_E(X) + \tfrac{1}{r}\,\mathrm{ord}_E(\mathfrak{n}_{r,X}) \quad and \quad k_E^{\diamond}(X) = k_E(X) - \tfrac{1}{r}\,\mathrm{ord}_E(\mathfrak{d}_{r,X}).$$

Proof. Let $f \colon Y \to X$ be a log resolution of X such that E is a divisor on Y and both $\mathfrak{n}_{r,X} \cdot \mathcal{O}_Y$ and $\mathfrak{j}_f$ are locally principal. Fix a canonical divisor K_Y on Y such that $f_* K_Y = K_X$, and let D be an effective divisor on Y such that $r K_{Y/X} + D \geq 0$. Consider the composition γ of the maps

$$f^*(\Omega_X^n)^{\otimes r} \xrightarrow{\alpha} \mathcal{O}_Y(f^*(r K_X)) \xrightarrow{\beta} \mathcal{O}_Y(r K_Y + D) \xrightarrow{\cong} (\Omega_Y^n)^{\otimes r} \otimes \mathcal{O}_Y(D),$$

where β is induced by a global section of $\mathcal{O}_Y(r K_{Y/X} + D)$. The maps α, β and γ have images, respectively,

$$\begin{aligned}
\mathrm{Im}(\alpha) &= (\mathfrak{n}_{r,X} \cdot \mathcal{O}_Y) \otimes \mathcal{O}_Y(f^*(r K_X)), \\
\mathrm{Im}(\beta) &= \mathcal{O}_Y(-r K_{Y/X} - D) \otimes \big((\Omega_Y^n)^{\otimes r} \otimes \mathcal{O}_Y(D)\big), \\
\mathrm{Im}(\gamma) &= \big(\mathfrak{j}_f^r \otimes \mathcal{O}_Y(-D)\big) \otimes \big((\Omega_Y^n)^{\otimes r} \otimes \mathcal{O}_Y(D)\big).
\end{aligned}$$

By comparing images, we see that

$$\mathfrak{j}_f^r = \mathfrak{n}_{r,X} \cdot \mathcal{O}_Y(-r K_{Y/X}).$$

Since the ideals $\mathfrak{j}_X^r$ and $\mathfrak{n}_{r,X} \cdot \mathfrak{d}_{r,X}$ have the same integral closure (see Proposition 2.10), we conclude that

$$\begin{aligned}
\mathrm{ord}_E(\mathfrak{j}_f) &= \mathrm{ord}_E(K_{Y/X}) + \tfrac{1}{r}\,\mathrm{ord}_E(\mathfrak{n}_{r,X}) \\
&= \mathrm{ord}_E(K_{Y/X}) + \mathrm{ord}_E(\mathfrak{j}_X) - \tfrac{1}{r}\,\mathrm{ord}_E(\mathfrak{d}_{r,X}),
\end{aligned}$$

and both formulas follow. $\square$

Corollary 3.5. *If X is a locally complete intersection variety and $f \colon Y \to X$ is a resolution of singularities, then $K_{Y/X}^{\diamond} = K_{Y/X}$.*

Proof. By Proposition 3.4, since $\mathfrak{d}_{r,X}$ is trivial if X is locally complete intersection. $\square$

4. Singularities

This section is devoted to the study of singularities of pairs from the point of view of Jacobian discrepancies. We refer to [KM98] for an introduction to singularities of pairs in the usual setting.

4.1. Pairs and singularities

Throughout the paper, a *pair* $(X, \mathfrak{A})$ will always consist of a reduced equidimensional scheme X and a *proper $\mathbb{R}$-ideal* $\mathfrak{A} = \prod_k \mathfrak{a}^{c_k}$ of X, namely, a finite formal product, with

real exponents c_k, of ideal sheaves $\mathfrak{a}_k \subset \mathcal{O}_X$ such that $\mathfrak{a}_k|_{X_i} \neq (0)$ on every irreducible component X_i of X. The $\mathbb{R}$-ideal $\mathfrak{A}$, and the pair itself, are said to be *effective* if $c_k \geq 0$ for all k. They are said to be *effective in codimension one* if $c_k \geq 0$ for all k such that $\dim Z(\mathfrak{a}_k) = \dim X - 1$, where in general $Z(\mathfrak{a}) \subset X$ denotes the subscheme defined by an ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$. The *vanishing locus* of $\mathfrak{A}$ is the union of the supports of the $Z(\mathfrak{a}_k)$.

For any map $g \colon X' \to X$ we denote $\mathfrak{A} \cdot \mathcal{O}_{X'} := \prod_k (\mathfrak{a}_k \cdot \mathcal{O}_{X'})^{c_k}$. If g is an inclusion, then we also set $\mathfrak{A}|_{X'} := \mathfrak{A} \cdot \mathcal{O}_{X'}$. For any real number λ, we write $\mathfrak{A}^\lambda := \prod_k \mathfrak{a}_k^{\lambda c_k}$. If E is a prime divisor over X, then we denote $\mathrm{ord}_E(\mathfrak{A}) := \sum_k c_k \, \mathrm{ord}_E(\mathfrak{a}_k)$.

Definition 4.1. Let $(X, \mathfrak{A})$ be a pair, and let E be a prime divisor over X. The *log Jacobian discrepancy* (or *log J-discrepancy*) of E over $(X, \mathfrak{A})$ is the number

$$a_E^\diamond(X, \mathfrak{A}) := k_E^\diamond(X) + 1 - \mathrm{ord}_E(\mathfrak{A}).$$

The pair $(X, \mathfrak{A})$ is said to be *log J-canonical* (resp., *log J-terminal*) if $a_E^\diamond(X, \mathfrak{A}) \geq 0$ (resp., $a_E^\diamond(X, \mathfrak{A}) > 0$) for all prime divisors E over X. The pair $(X, \mathfrak{A})$ is said to be *J-canonical* (resp., *J-terminal*) if $a_E^\diamond(X, \mathfrak{A}) \geq 1$ (resp., $a_E^\diamond(X, \mathfrak{A}) > 1$) for all exceptional prime divisors E over X. If $\mathfrak{A} = \mathcal{O}_X$, then we just drop it from the notation. In particular, we say that X is *J-canonical* or *log J-canonical* if so is the pair $(X, \mathcal{O}_X)$.

Remark 4.2. If one defines the *log Mather discrepancy* of E over (X, Z) to be $\widehat{a}_E(X, \mathfrak{A}) := \widehat{k}_E(X) + 1 - \mathrm{ord}_E(\mathfrak{A})$, then $a_E^\diamond(X, \mathfrak{A}) = \widehat{a}_E(X, \mathfrak{A} \cdot \mathfrak{j}_X)$. This invariant is studied in [Ish11].

Definition 4.3. Let $(X, \mathfrak{A})$ be a pair. If $(X, \mathfrak{A})$ is log J-canonical, then the *log J-canonical threshold* of $(X, \mathfrak{A})$ is

$$\mathrm{lct}^\diamond(\mathfrak{A}) := \sup\{\lambda \geq 0 \mid (X, \mathfrak{A}^\lambda) \text{ is log J-canonical}\} \in [0, \infty].$$

Here we set $\mathrm{lct}^\diamond(\mathfrak{A}) = 0$ if $\mathfrak{a}_k|_{X_i} = (0)$ for some i and some k such that $c_k > 0$. For any Grothendieck point $\eta \in X$, the *minimal log J-discrepancy* of (X, T) at η is

$$\mathrm{mld}_\eta^\diamond(X, \mathfrak{A}) := \inf_{c_X(E) = \eta} a_E^\diamond(X, \mathfrak{A}).$$

If η_{X_i} is the generic point of an irreducible component X_i of X, then we set by definition $\mathrm{mld}_{\eta_{X_i}}^\diamond(X, \mathfrak{A}) = 0$. If $T \subset X$ is a closed set, then we denote

$$\mathrm{mld}_T^\diamond(X, \mathfrak{A}) := \inf_{\eta \in T} \mathrm{mld}_\eta^\diamond(X, \mathfrak{A}).$$

Remark 4.4. If X is locally complete intersection, then the above invariants agree with the usual analogous invariants: $a_E^\diamond(X, \mathfrak{A}) = a_E(X, \mathfrak{A})$, $\mathrm{lct}^\diamond(X, \mathfrak{A}) = \mathrm{lct}(X, \mathfrak{A})$, and $\mathrm{mld}_T^\diamond(X, \mathfrak{A}) = \mathrm{mld}_T(X, \mathfrak{A})$, the usual log discrepancy, log canonical threshold, and minimal log discrepancy.

Remark 4.5. Let $(X, \mathfrak{A})$ be any pair as above. Let $X' \to X$ be the normalization, and let $\mathfrak{c}_X := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_X)$ be the conductor ideal; this is the largest ideal sheaf in $\mathcal{O}_X$ which is also an ideal sheaf in $\mathcal{O}_{X'}$. By Theorem 2.4 of [Yas07], there is an inclusion $\mathfrak{j}_X \subset \mathfrak{c}_X$, and hence for every prime divisor E over X one has

$$\widehat{a}_E(X, \mathfrak{A} \cdot \mathfrak{j}_X) \leq \widehat{a}_E(X', \mathfrak{A}' \cdot \mathfrak{c}_X)$$

where $\mathfrak{A}' := \mathfrak{A} \cdot \mathcal{O}_{X'}$. In the special case where X is a simple normal crossing divisor in a smooth variety M of dimension $n + 1$, we actually have an equality $\mathfrak{j}_X = \mathfrak{c}_X$, and thus if X_i is the irreducible component of X over which E lies then

$$a_E^{\diamond}(X, \mathfrak{A}) = a_E(X_i, \mathfrak{A}|_{X_i} \cdot \mathfrak{c}_X|_{X_i}).$$

Definition 4.6. A *log resolution* of a pair $(X, \mathfrak{A})$ consists of a log resolution $f : Y \to X$ of X such that $\mathfrak{a}_k \cdot \mathcal{O}_Y$ is locally principal for every k and the union of their vanishing loci, together with the vanishing locus of $\mathfrak{j}_X \cdot \mathcal{O}_Y$ and the exceptional locus $\mathrm{Ex}(f)$, form a simple normal crossing divisor.

The existence of log resolutions follows from Hironaka's resolution of singularities [Hir64]. Note that the above definition of log resolution differs slightly from the usual one (it is more restrictive) in that we also impose conditions on the pull-back of the Jacobian ideal of X. Note that, according to our definition, every log resolution factors through the blow-up of the Jacobian ideal of X, and thus through the Nash blow-up of X.

Minimal log J-discrepancies of a pair $(X, \mathfrak{A})$ are trivial at the generic point of X and are easy to compute at points of codimension one. Note also that since the normalization of X gives a log resolution of $(X, \mathfrak{A})$ in codimension one, minimal log J-discrepancies in codimension one are computed on any log resolution of the pair.

Proposition 4.7. *Let $(X, \mathfrak{A})$ be a pair as above, and let $T \subset X$ be a closed subset of codimension ≥ 1.*

(a) *If $(X, \mathfrak{A})$ is log J-canonical in a neighborhood of T, then $\mathrm{mld}_T^{\diamond}(X, \mathfrak{A})$ is realized on any given log resolution $f : Y \to X$ of $(X, \mathfrak{A} \cdot \mathcal{I}_T)$ as the log J-discrepancy $a_E^{\diamond}(X, \mathfrak{A})$ of some prime divisor E on Y.*
(b) *If $(X, \mathfrak{A})$ is not log J-canonical in any neighborhood of T and $\dim X \geq 2$, then $\mathrm{mld}_T^{\diamond}(X, \mathfrak{A}) = -\infty$.*

Proof. Using Proposition 3.3, the proof is the same as the one of the analogous properties for usual minimal log discrepancies on $\mathbb{Q}$-Gorenstein varieties. $\square$

4.2. Adjunction

Let X be a reduced equidimensional scheme, embedded in a smooth variety M. Let $\mathfrak{A}$ be a proper $\mathbb{R}$-ideal on M such that $\mathfrak{A}|_X$ is a proper $\mathbb{R}$-ideal on X. We denote by $\mathfrak{j}_{X,M} \subset \mathcal{O}_M$ the ideal defining the scheme $Z(\mathfrak{j}_X)$ viewed as a subscheme of M. We will use the following version of embedded resolution.

Definition 4.8. An *embedded log resolution* of $(X, \mathfrak{A}|_X)$ in $(M, \mathfrak{A})$ consists of a log resolution $g \colon N \to M$ of $(M, \mathfrak{A} \cdot j_{X,M})$ satisfying the following properties:

(i) g is an isomorphism at the generic point of each irreducible component X_i of X;

(ii) the restriction of g to the proper transform Y of X gives a log resolution $f \colon Y \to X$ of $(X, \mathfrak{A}|_X)$;

(iii) Y is transverse to the simple normal crossing divisor given by the union of $\mathrm{Ex}(g)$ and the vanishing locus of $(\mathfrak{A} \cdot j_{X,M}) \cdot \mathcal{O}_N$.

If $\mathfrak{A} = \mathcal{O}_X$, then we just say that f is an *embedded log resolution* of X in M. An embedded log resolution is said to be *factorizing* if, furthermore, we have $\mathcal{I}_X \cdot \mathcal{O}_N = \mathcal{I}_Y \cdot \mathcal{O}_N(-G)$ where G is a divisor on N.

The existence of factorizing resolutions is established in [BV03]. The following adjunction formula holds. For the proof we refer to Lemma 4.4 of [Eis10].

Proposition 4.9. *Let X be a reduced subscheme of pure codimension e of a smooth variety M. Let $g \colon N \to M$ be a factorizing embedded log resolution of X in M, and write $\mathcal{I}_X \cdot \mathcal{O}_N = \mathcal{I}_Y \cdot \mathcal{O}_N(-G)$, where Y is the proper transform of X and G is a divisor in N supported on the exceptional locus. Then*

$$K^{\diamond}_{Y/X} = (K_{N/M} - eG)|_Y.$$

4.3. Inversion of adjunction

The next theorem generalizes the inversion of adjunction formula for $\mathbb{Q}$-Gorenstein varieties proved in [EM09] and [Kaw08].

Theorem 4.10. *Let X be a reduced subscheme of pure dimension $n \geq 2$ and codimension e of a smooth variety M. Then for every proper, effective $\mathbb{R}$-ideal $\mathfrak{A}$ on M not containing any irreducible component of X is its vanishing locus, and every closed subset $T \subset X$, we have*

$$\mathrm{mld}^{\diamond}_T(X, \mathfrak{A}|_X) = \mathrm{mld}_T(M, \mathfrak{A} \cdot \mathcal{I}^e_X).$$

The theorem has been proven independently by Ishii [Ish11]. The line of arguments presented here is slightly different from (although equivalent at the core to) that given in [Ish11], which follows more closely [EM09].

The proof uses the jet schemes X_m and the space of arcs X_∞ of X. We refer the reader to [DL99, EM09] for the basic definitions and properties of the theory. We will use the description of divisorial valuations and discrepancies given in [ELM04] in the smooth case and then extended in [dFEI08] to the singular case. Since here we allow X to be reducible, the only further extension we need is that of the notion of quasi-cylinder and codimension. So, we say that a subset $C \subset X_\infty$ is a *quasi-cylinder* of *codimension k* if C is contained in the arc space $(X_i)_\infty$ of some irreducible component X_i of X and is a quasi-cylinder of codimension k there. Since X is equidimensional, this implies that if $\psi_m \colon X_\infty \to X_m$ is the truncation map, then $k = n(m+1) - \dim(\psi_m(C))$ for all $m \gg 1$, where $n = \dim X$.

Proof of Theorem 4.10. For every generic point η_{X_i} of an irreducible component X_i of X we have $\mathrm{mld}^{\diamond}_{\eta_{X_i}}(X, \mathfrak{A}|_X) = 0 = \mathrm{mld}_{\eta_{X_i}}(M, \mathfrak{A} \cdot \mathcal{I}_X^e)$. We can therefore reduce to the case in which T has codimension ≥ 1 in X.

To prove one inequality, we take a factorizing embedded log resolution $g \colon N \to M$ of $(X, \mathfrak{A}|_X)$ in $(M, \mathfrak{A})$, and let $f \colon Y \to X$ be the induced log resolution. Since every g-exceptional divisor that meets Y intersects it transversely, if F is any such divisor and E is the divisor cut out by F on Y, then $\mathrm{ord}_E(\mathfrak{A}|_X) = \mathrm{ord}_F(\mathfrak{A})$. Using Proposition 4.9, it follows by direct comparison along the g-exceptional divisors meeting Y and having center inside T that either $(X, \mathfrak{A}|_X)$ is not log J-canonical, in which case $(M, \mathfrak{A} \cdot \mathcal{I}_X^e)$ is not log canonical and both minimal log (J-)discrepancies are $-\infty$, or the first pair is log J-canonical and

$$\mathrm{mld}^{\diamond}_T(X, \mathfrak{A}|_X) \geq \mathrm{mld}_T(M, \mathfrak{A} \cdot \mathcal{I}_X^e).$$

Here we are using the fact that if $(X, \mathfrak{A}|_X)$ is J-canonical then its minimal log J-discrepancy is computed on any given log resolution of $(X, \mathfrak{A}|_X)$ (see Proposition 4.7).

The reverse inequality follows from the following claim.

Claim 4.11. *For every prime divisor F over M with center inside X, there is a prime divisor E over X with center contained in the center of F, and an integer $q \geq 1$, such that*

$$q \cdot a^{\diamond}_E(X, \mathfrak{A}|_X) \leq a_F(M, \mathfrak{A} \cdot \mathcal{I}_X^e). \tag{4.1}$$

To see that this implies the inequality $\mathrm{mld}^{\diamond}_T(X, \mathfrak{A}|_X) \leq \mathrm{mld}_T(M, \mathfrak{A} \cdot \mathcal{I}_X^e)$, note that if $\mathrm{mld}_T(M, \mathfrak{A} \cdot \mathcal{I}_X^e) \geq 0$ then we get the statement by dividing by q in (4.1), whereas if $\mathrm{mld}_T(M, \mathfrak{A} \cdot \mathcal{I}_X^e) < 0$ then the formula only implies that $\mathrm{mld}^{\diamond}_T(X, \mathfrak{A}|_X) < 0$, but then one deduces immediately that $\mathrm{mld}^{\diamond}_T(X, \mathfrak{A}|_X) = -\infty$ by Proposition 4.7.

It remains to prove Claim 4.11. We consider the *maximal divisorial set*

$$W = W^1(F) \subset M_\infty,$$

where M_∞ is the space of arcs of M. By definition, W is the closure in M_∞ of the set of arcs of N having order of contact one with F (cf. [ELM04]). The set W is an irreducible cylinder in M_∞. By the results of [ELM04], the valuation val_W determined by the vanishing order along the generic arc in W agrees with the divisorial valuation ord_F, and moreover

$$\mathrm{codim}(W, M_\infty) = k_F(M) + 1.$$

The intersection $W \cap X_\infty \subset X_\infty$ is a cylinder in X_∞ and is not contained in the arc space of the singular locus of X, by Lemma 8.3 of [EM09]. Let C be an irreducible component of $W \cap X_\infty$ that is not contained in the arc space of the singular locus of X. Then C is a quasi-cylinder in X_∞. By Propositions 3.10 and 2.12 of [dFEI08], we can find a prime divisor E over X, and an integer $q \geq 1$, such that the valuation val_C associated to the quasi-cylinder C is equal to the divisorial valuation $q \, \mathrm{ord}_E$, and moreover the maximal divisorial set $W^q(E) \subset X_\infty$ is a quasi-cylinder of codimension

$$\mathrm{codim}(W^q(E), X_\infty) = q \cdot (\widehat{k}_E(X) + 1),$$

where $\widehat{k}_E(X)$ is the Mather discrepancy of E over X.

We have the following chain of inequalities:

$$q \cdot (\widehat{k}_E(X) + 1) = \mathrm{codim}(W^q(E), X_\infty) \leq \mathrm{codim}(C, X_\infty)$$
$$\leq \mathrm{codim}(W, M_\infty) + q \cdot \mathrm{ord}_E(\mathrm{j}_X) - e \, \mathrm{ord}_F(\mathcal{I}_X).$$

The first inequality follows from the fact that, as explained in the proof of Propositions 3.10 of [dFEI08], C is contained in $W^q(E)$, and the second by applying Lemma 8.4 in [EM09] as in the proof of Theorem 8.1 of [EM09].

Observe that, for any proper ideal $\mathfrak{b} \subset \mathcal{O}_M$ not vanishing on any component of X, we have $\mathrm{val}_C(\mathfrak{b}|_X) \geq \mathrm{val}_W(\mathfrak{b})$ by the inclusion $C \subset W$ and the fact that if $\gamma \in C$ then $\mathrm{ord}_\gamma(\mathfrak{b}|_X) = \mathrm{ord}_\gamma(\mathfrak{b})$. In particular, this implies that

$$q \cdot \mathrm{ord}_E(\mathfrak{A}|_X) \geq \mathrm{ord}_F(\mathfrak{A})$$

since $\mathfrak{A}$ is effective. Since $k_E^\diamond(X) = \widehat{k}_E(X) - \mathrm{ord}_E(\mathrm{j}_X)$, we deduce that

$$q \cdot (k_E^\diamond(X) + 1) \leq k_F(M) + 1 - e \, \mathrm{ord}_F(\mathcal{I}_X).$$

Claim 4.11 follows by combining these inequalities. □

Remark 4.12. With the same notation as in Theorem 4.10, suppose that X has dimension one. In this case most of the arguments of the proof go through, the only problem being that it is no longer true in general that $\mathrm{mld}_T^\diamond(X, \mathfrak{A}|_X)$ is $-\infty$ whenever it is negative, and one concludes in this case that either $\mathrm{mld}_T^\diamond(X, \mathfrak{A}|_X) = \mathrm{mld}_T(M, \mathfrak{A} \cdot \mathcal{I}_X^e) \geq 0$ or $0 > \mathrm{mld}_T^\diamond(X, \mathfrak{A}|_X) \geq \mathrm{mld}_T(M, \mathfrak{A} \cdot \mathcal{I}_X^e)$, and the latter is $-\infty$ if $e \geq 1$. These inequalities will suffice, however, in the applications of inversion of adjunction to the proofs of Corollary 5.4 and Theorem 7.7.

4.4. ACC and semicontinuity

Inversion of adjunction has several implications regarding the properties of the invariants of singularities related to J-discrepancies, which generalize analogous properties known for normal varieties with locally complete intersection singularities.

The first application gives the ACC property for the sets of log J-canonical thresholds in any fixed dimension. The problem arises from a conjecture of Shokurov for log canonical thresholds in the usual setting [Sho92], which has recently being solved for certain classes of singularities in [dFEM10, dFEM11]. In the framework considered in this paper we obtain the following unconditional result.

Corollary 4.13. *For every integer n, the set of log J-canonical thresholds in dimension n*

$$\{\mathrm{lct}^\diamond(\mathfrak{a}) \mid \mathfrak{a} \subset \mathcal{O}_X, \ X \ \textit{log J-canonical of pure dimension } n\}$$

satisfies the ascending chain condition. That is, every increasing sequence in the set is eventually constant.

The proof of this property is a straightforward extension of the corresponding proof given in [dFEM10] in the case of normal varieties with locally complete intersection singularities. In short, it goes as follows. The same argument of the proof of Proposition 6.3 of [dFEM10] shows that if X is a reduced equidimensional scheme with log J-canonical singularities then $\dim T_p X \leq 2 \dim X$ for every $p \in X$. Then, using this bound on the embedded dimension in combination with inversion of adjunction (Theorem 4.10) one deduces the above ACC property directly from the analogous property of *mixed log canonical thresholds* on smooth varieties, which is established in Theorem 6.1 of [dFEM10].

The second application of inversion of adjunction regards the semicontinuity of minimal log J-canonical discrepancies and the characterization of regular points in terms of these invariants. Once more, the question originates from a conjecture of Shokurov in the usual setting of minimal log discrepancies, later made more precise by Ambro; for a discussion of this we refer to [Amb99] and the references therein. Again, we have unconditional results in our setting. The first statement appears also in [Ish11].

Corollary 4.14. *For every effective pair* $(X, \mathfrak{A})$ *where* X *is a reduced equidimensional scheme, the function on closed points*

$$\mathrm{mld}^\diamond_-(X, \mathfrak{A}) \colon X \to \{-\infty\} \cup \mathbb{R}_{\geq 0}, \quad p \mapsto \mathrm{mld}^\diamond_p(X, \mathfrak{A}),$$

is lower semicontinuous in the Zariski topology.

Corollary 4.15. *Let* X *be a reduced equidimensional scheme. For every Grothendieck point* $\eta \in X$ *we have*

$$\mathrm{mld}^\diamond_\eta(X) \leq \mathrm{codim}(\eta, X),$$

and equality holds if and only if X *is smooth at* η.

Remark 4.16. Since $\mathrm{mld}^\diamond_\eta(X) \in \mathbb{Z} \cup \{-\infty\}$, it follows that X is regular at η if and only if $\mathrm{mld}^\diamond_\eta(X) > \mathrm{codim}(\eta, X) - 1$.

Once inversion of adjunction is in place, the proofs of these results are standard. The proof of Corollary 4.14 follows step by step the arguments of the corresponding result in [EM04]. Regarding Corollary 4.15 one uses induction on the embedded codimension of X at η, as explained for instance in Remark 4.2 of [dFE10] for the locally complete intersection case.

Remark 4.17. On a completely different topic, one also sees from inversion of adjunction that the bound on Castelnuovo–Mumford regularity proven in Corollary 1.4 of [dFE10] holds for every reduced equidimensional projective scheme $V \subset \mathbb{P}^n$ with log J-canonical singularities.

5. Jet schemes

It is known that Mather discrepancies can be computed using the codimension of certain sets in the arc space. We now give an analogous description for Jacobian discrepancies which involves the tangent space to the arc space. Throughout the section, let X be a reduced scheme of pure dimension n.

5.1. Mather discrepancies

Recall that to a divisor $E \subset Y$ over X one associates its *maximal divisorial set* $W^1(E)$, namely, the closure in X_∞ of the image of the set of arcs on Y having order of contact one with E. It is an irreducible quasi-cylinder in the arc space X_∞, and its codimension measures the Mather log discrepancy

$$\widehat{k}_E(X) + 1 = \operatorname{codim}(W^1(E), X_\infty)$$

(see [dFEI08] for details). More precisely, if $\psi_m \colon X_\infty \to X_m$ is the truncation map, then it follows by Lemma 3.4 of [DL99] (cf. Lemma 3.8 and the argument in the proof of Theorem 3.9 of [dFEI08]) that $W^1(E)$ is the closure of a quasi-cylinder over a constructible subset of X_{m_0} where $m_0 = 2 \operatorname{ord}_E(\mathfrak{j}_X)$, and thus we have

$$\widehat{k}_E(X) + 1 = n(m+1) - \dim(\psi_m(W^1(E))) \quad \text{for all } m \geq 2 \operatorname{ord}_E(\mathfrak{j}_X).$$

These formulas are the geometric manifestation of the change-of-variables formula in motivic integration [DL99].

5.2. Jacobian discrepancies

To obtain an analogous description for Jacobian discrepancies, we need to consider the total spaces of the tangent sheaves of the arc space and of the jet schemes. We denote them by TX_∞ and TX_m. These spaces are schemes, and their functors of points are easy to describe; note however that TX_∞ is not of finite type. For example, when X is affine the $\mathbb{C}$-valued points are given by

$$TX_\infty(\mathbb{C}) = \operatorname{Hom}\big(\operatorname{Spec} \mathbb{C}[\![t]\!][\epsilon]/(\epsilon^2), \, X\big),$$
$$TX_m(\mathbb{C}) = \operatorname{Hom}\big(\operatorname{Spec} \mathbb{C}[t, \epsilon]/(t^{m+1}, \epsilon^2), \, X\big).$$

We have natural projections

$$\pi_\infty \colon TX_\infty \to X_\infty, \qquad \pi_m \colon TX_m \to X_m$$

from the tangent spaces to their bases. Given an arc $\alpha \in X_\infty$ and a *liftable* jet $\beta \in \psi_m(X_\infty) \subset X_m$, we are interested in the restrictions of the tangent spaces over these points, which we denote by

$$TX_\infty|_\alpha := \pi_\infty^{-1}(\alpha), \qquad TX_m|_\beta := \pi_m^{-1}(\beta).$$

Proposition 5.1. *Let X be a reduced scheme of pure dimension n. Consider a liftable jet $\beta \in \psi_m(X_\infty)$, and let $L = \mathbb{C}(\beta)$ be the field of definition of β. Then*

$$\dim_L(TX_m|_\beta) = n(m+1) + \operatorname{ord}_\beta(\mathfrak{j}_X) \quad \text{for all } m \geq \operatorname{ord}_\beta(\mathfrak{j}_X).$$

Proof. Let $\alpha \in X_\infty$ be a lift of β. We can assume X is affine, and embedded in $M = \mathbb{A}^{n+e}$. We denote by $x_1, \ldots, x_{n+e}$ the coordinates in M, and let $I_X = (f_1, \ldots, f_r)$ be the equations of X. Then the arc α is given by a vector $(\alpha_1, \ldots, \alpha_{n+e})$ with entries $\alpha_i = \alpha_i(t)$ in the power series ring $L[\![t]\!]$. Since α is in X_∞, we know that $f_j(\alpha) = 0$ for all j.

The restriction $TM_\infty|_\alpha$ can be thought of as a free module over $L[\![t]\!]$ of rank $n + e$. Specifically, every element $\xi \in TM_\infty|_\alpha$ can be written in the form

$$\xi = \alpha + v\epsilon,$$

where $v = (v_1, \dots, v_{n+e})$ is a vector with coefficients in $L[\![t]\!]$ and ϵ is a fixed variable satisfying $\epsilon^2 = 0$. The tangent vector ξ belongs to $TX_\infty|_\alpha$ if $f_j(\xi) = 0$ for all j. Using the Taylor expansion, we get

$$f_j(\xi) = f_j(\alpha) + \sum_{i=1}^{n+e} \frac{\partial f_j}{\partial x_i}(\alpha)v_i\epsilon.$$

Let $J = \left(\frac{\partial f_j}{\partial x_i}\right)$ be the Jacobian matrix. Since $f_j(\alpha) = 0$, we see that

$$\xi \in TX_\infty|_\alpha \ \Leftrightarrow \ J(\alpha)v = 0.$$

In other words, $TX_\infty|_\alpha$ can be computed inside of $TM_\infty|_\alpha$ as the kernel of $J(\alpha)$.

Analogous statements hold for the jet β. In this case, $TM_m|_\beta$ is a free module over $L[t]/(t^{m+1})$, also of rank $n + e$, and $TX_m|_\beta$ is the kernel of $J(\beta)$:

$$\beta + w\epsilon \in TX_m|_\beta \ \Leftrightarrow \ J(\beta)w = 0 \ (\text{mod } t^{m+1}).$$

The goal is therefore to compute the dimension of the kernel of $J(\beta)$, and this can be done easily by diagonalizing the matrix. In our case, we can diagonalize both $J(\beta)$ and $J(\alpha)$ simultaneously. More precisely, the structure theorem for finitely generated modules over PIDs tells us that we can find invertible matrices A and B with coefficients in $L[\![t]\!]$ such that

$$A \cdot J(\alpha) \cdot B = D$$

where D is a diagonal matrix. Notice that D is not a square matrix: by *diagonal* we mean that all of its entries d_{ij} are zero except when $i = j$. We can also assume that D is of the form

$$D = \text{diag}(t^{a_1}, \dots, t^{a_s}, 0, \dots, 0)$$

with $0 \le a_1 \le \cdots \le a_s$.

Notice that $J(\beta)$ is the truncation of $J(\alpha)$ to order m, so if we denote by A_m, B_m, and D_m the truncations of A, B, and D, we have

$$A_m \cdot J(\beta) \cdot B_m = D_m.$$

The matrices A_m and B_m are also invertible, so in particular $J(\beta)$ and D_m have isomorphic kernels. The matrix D_m is given by

$$D_m = \text{diag}(t^{a_1}, \dots, t^{a_l}, 0, \dots, 0),$$

where $l \le s$ is picked so that $a_k > m$ for $k > l$. Its kernel is

$$(t^{m+1-a_1})/(t^{m+1}) \oplus \cdots \oplus (t^{m+1-a_l})/(t^{m+1}) \oplus (L[t]/(t^{m+1}))^{n+e-l},$$

which has dimension $a_1 + \cdots + a_l + (n + e - l)(m + 1)$ over L.

Recall that the matrix J gives a presentation for the module of differentials Ω_X. Therefore, the k-th Fitting ideal $\mathrm{Fitt}^k(\Omega_X)$ is generated by the minors of J of size $n + e - k$. In particular, the orders of vanishing

$$\mathrm{ord}_\alpha(\mathrm{Fitt}^k(\Omega_X)), \quad \mathrm{ord}_\beta(\mathrm{Fitt}^k(\Omega_X))$$

are the smallest order of vanishing of a minor of size $n + e - k$ of $J(\alpha)$ and $J(\beta)$. Since A, A_m, B, and B_m are invertible, these orders can be computed using D and D_m, and we see that

$$\mathrm{ord}_\alpha(\mathrm{Fitt}^k(\Omega_X)) = \begin{cases} a_1 + \cdots + a_{n+e-k} & \text{if } n + e - k \le s, \\ \infty & \text{if } n + e - k > s, \end{cases}$$

and

$$\mathrm{ord}_\beta(\mathrm{Fitt}^k(\Omega_X)) = \begin{cases} \min(a_1 + \cdots + a_{n+e-k}, m + 1) & \text{if } n + e - k \le l, \\ m + 1 & \text{if } n + e - k > l. \end{cases}$$

Since X has pure dimension n, we know that $\mathrm{Fitt}^{n-1}(\Omega_X) = 0$, so α vanishes along it with order ∞, and we find that $e + 1 > s$. Recall that $\mathrm{Fitt}^n(\Omega_X) = \mathrm{j}_X$. Using the hypothesis that $\mathrm{ord}_\beta(\mathrm{j}_X) \le m$, we get $e \le l$. Therefore $e = l = s$, and

$$\mathrm{ord}_\beta(\mathrm{j}_X) = a_1 + \cdots + a_e.$$

Finally, this implies that the kernel of D_m has dimension $\mathrm{ord}_\beta(\mathrm{j}_X) + n(m + 1)$ over L, as required. $\qquad\square$

Remark 5.2. It is essential in Proposition 5.1 that the jet β is liftable (that is, in the image of X_∞). In the proof, one cannot use the fact that $\mathrm{Fitt}^{n-1}(\Omega_X) = 0$ to show directly that $e + 1 > l$. This is due to the presence of the min in the formula for the order of β.

Given a prime divisor E over X, we denote by $\eta_{E,m}$ the generic point of the truncation $\psi_m(W^1(E))$ of the maximal divisorial set $W^1(E) \subset X_\infty$, and consider the restriction $TX_m|_{\eta_{E,m}}$ of the tangent space of X_m over this point.

Theorem 5.3. *With the above notation, for every E we have*

$$k_E^\diamond(X) + 1 = 2n(m + 1) - \dim_{\mathbb{C}}(TX_m|_{\eta_{E,m}}) \quad \text{for all } m \ge 2\,\mathrm{ord}_E(\mathrm{j}_X).$$

Proof. For $m \ge \mathrm{ord}_E(\mathrm{j}_X)$, the order of E along j_X is computed by the jet $\eta_{E,m}$, and the previous proposition gives

$$- \mathrm{ord}_E(\mathrm{j}_X) = n(m + 1) - \dim_{\mathbb{C}(\eta_{E,m})}(TX_m|_{\eta_{E,m}}).$$

On the other hand, the description of Mather discrepancies in terms of the codimension of $W(E)$ gives

$$\widehat{k}_E(X) + 1 = n(m + 1) - \dim_{\mathbb{C}}(\eta_{E,m})$$

for all $m \ge 2\,\mathrm{ord}_E(\mathrm{j}_X)$. The assertion then follows by combining the two formulas. $\quad\square$

Since $2n(m+1)$ is the dimension of the irreducible component of TX_m dominating X, one can consider the right-hand side in the formula in the theorem as a 'virtual codimension' of $TX_m|_{\eta_{E,m}}$, although not with respect to the full TX_m but rather in relation to this distinguished component.

It would be interesting to find a way to read the condition characterizing Jacobian discrepancies in Theorem 5.3 all the way up, on the tangent space of X_∞, similarly to Mather discrepancies which are detected by the codimension of certain quasi-cylinders. Unfortunately this seems difficult, as in general the map $TX_\infty|_{\eta_E} \to TX_m|_{\eta_{E,m}}$ is not dominant.

5.3. Jet interpretation of singularities

One more application of inversion of adjunction regards the characterization of J-canonical and log J-canonical singularities in terms of the dimensions of the jet schemes. In a similar fashion, Theorem 5.3 yields a characterization in terms of the dimension of the tangent spaces of the jet schemes. The next result extends and generalizes the analogous properties established for locally complete intersection varieties in [Mus01, EM04].

Corollary 5.4. *Let X be a reduced scheme of pure dimension n. For any prime divisor E over X and any m, we denote by $\eta_{E,m}$ the image in X_m of the generic point of the maximal divisorial set $W^1(E) \subset X_\infty$. Then the following are equivalent:*

(a) *X is log J-canonical.*
(b) *$\dim X_m = n(m+1)$ for every m.*
(c) *$\dim TX_m|_{\eta_{E,m}} \leq 2n(m+1)$ for all E and any $m \geq 2\operatorname{ord}_E(\mathfrak{j}_X)$.*

Similarly, the following are equivalent:

(a′) *X is J-canonical.*
(b′) *$\dim X_m = n(m+1)$ for every m, and every irreducible component of X_m of maximal dimension dominates an irreducible component of X.*
(c′) *$\dim TX_m|_{\eta_{E,m}} < 2n(m+1)$ for all E and any $m \geq 2\operatorname{ord}_E(\mathfrak{j}_X)$.*

Moreover, in (b) *and* (b′) *is enough to check the condition for all m such that $m+1$ is sufficiently divisible, and in* (c) *and* (c′) *it is enough to check the condition just for the prime divisors $E_1, \ldots, E_k$ appearing in a log resolution of X.*

Proof. The equivalences (a)⇔(b) and (a′)⇔ (b′) come from inversion of adjunction. The argument is quite standard. Let $S \subset X$ denote the singular locus of X. By definition X is log J-canonical if and only if $\operatorname{mld}_S^\diamond(X) \geq 0$ (resp., $\operatorname{mld}_S^\diamond(X) \geq 1$). If X is embedded in a smooth variety M, then by Theorem 4.10 (see Remark 4.12 if $\dim X = 1$) this is equivalent to $\operatorname{mld}_S(M, eX) \geq 0$ (resp., $\operatorname{mld}_S(M, eX) \geq 1$), where $e = \operatorname{codim}(X, M)$. Therefore the equivalences follow from the straightforward generalization of Theorems 3.1 and 3.2 of [Mus01] to reduced equidimensional schemes. Note that these theorems also imply that it is enough to check the conditions in (b) and (b′) when $m+1$ is sufficiently divisible.

The equivalences (a)⇔(c) and (a′)⇔(c′) and the last assertion are a straightforward consequence of the definitions of singularities, Theorem 5.3, and Proposition 4.7.   □

Remark 5.5. While the original proofs in [Mus01] make explicit use of motivic integration, it is now well understood by experts that the results only need the underlying geometric properties of the jet schemes, and one obtains quicker proofs using for instance the point of view of maximal divisorial sets developed in [ELM04].

6. Multiplier ideals

In this section we introduce multiplier ideals in our framework, and use them to measure the gap between the dualizing sheaf of a normal variety and its Grauert–Riemenschneider canonical sheaf. We refer to [Laz04] for an introduction to multiplier ideals in the usual setting.

6.1. Mather and Jacobian multiplier ideals

In the following, let X be a normal variety. Consider a proper $\mathbb{R}$-ideal $\mathfrak{A} = \prod_k \mathfrak{a}_k^{c_k}$ on X, and let $f : Y \to X$ be a log resolution of the pair $(X, \mathfrak{A})$. For short, we denote by $Z(\mathfrak{A} \cdot \mathcal{O}_Y) := \sum_k c_k \cdot Z(\mathfrak{a}_k \cdot \mathcal{O}_Y)$ the divisor determined by $\mathfrak{A}$ on Y.

Definition 6.1. The *Mather multiplier ideal* of $(X, \mathfrak{A})$ is the coherent sheaf of fractional ideals

$$\widehat{\mathcal{J}}(\mathfrak{A}) := f_* \mathcal{O}_Y(\widehat{K}_{Y/X} - \lfloor Z(\mathfrak{A} \cdot \mathcal{O}_Y) \rfloor).$$

and the *Jacobian multiplier ideal sheaf* of $(X, \mathfrak{A})$ is the coherent sheaf of fractional ideals

$$\mathcal{J}^\diamond(\mathfrak{A}) := f_* \mathcal{O}_Y(K_{Y/X}^\diamond - \lfloor Z(\mathfrak{A} \cdot \mathcal{O}_Y) \rfloor).$$

A standard argument using Proposition 3.3 shows that the definition of Mather and Jacobian multiplier ideals is independent of the particular log resolution. The proof follows the exact same lines of the proof of the analogous statement for multiplier ideals on smooth varieties (see Theorem 9.2.18 of [Laz04]). If X is locally complete intersection, then by Corollary 3.5 we have $\mathcal{J}^\diamond(\mathfrak{A}) = \mathcal{J}(\mathfrak{A})$, the usual multiplier ideal. If X is smooth, then we also have $\widehat{\mathcal{J}}(\mathfrak{A}) = \mathcal{J}(\mathfrak{A})$.

Remark 6.2. Since clearly $\mathcal{J}^\diamond(\mathfrak{A}) = \widehat{\mathcal{J}}(\mathfrak{A} \cdot \mathfrak{j}_X)$, the two theories of multiplier ideals are equivalent as long as one allows non-effective pairs. If one restricts the setting to effective pairs, then Jacobian multiplier ideals can be regarded as a special case of Mather multiplier ideals.

Proposition 6.3. *On a normal variety, both Mather and Jacobian multiplier ideals define an ideal sheaf (as opposed to a fractional ideal sheaf) if the pair is effective in codimension one.*

Proof. By the previous remark, it is enough to check this property for Mather multiplier ideals. Given a log resolution $f : Y \to X$ of a pair $(X, \mathfrak{A})$, write

$$\widehat{K}_{Y/X} - \lfloor Z(\mathfrak{A} \cdot \mathcal{O}_Y) \rfloor = P - N$$

where P and N are effective divisors with no common components, and consider the exact sequence

$$0 \to \mathcal{O}_Y(-N) \to \mathcal{O}_Y(P - N) \to \mathcal{O}_P(P - N) \to 0.$$

If the pair is effective in codimension one, then P is an exceptional divisor and $f_* \mathcal{O}_P(P - N) \subset f_* \mathcal{O}_P(P) = 0$ by a well-known lemma of Fujita (see Lemma 1-3-2 of [KMM87]). This implies that

$$\widehat{\mathcal{J}}(\mathfrak{A}) = f_* \mathcal{O}_Y(P - N) = f_* \mathcal{O}_Y(-N) \subset \mathcal{O}_X. \qquad \square$$

Mather and Jacobian multiplier ideals have similar properties to the usual multiplier ideals on smooth varieties. We list here a few, leaving to the reader the details of the proofs.

Proposition 6.4. *Let X be a normal variety.*

(a) *If $\overline{\mathfrak{A}} = \prod_k \overline{\mathfrak{a}}_k^{c_k}$, then $\widehat{\mathcal{J}}(\overline{\mathfrak{A}}) = \widehat{\mathcal{J}}(\mathfrak{A})$ and $\mathcal{J}^\diamond(\overline{\mathfrak{A}}) = \mathcal{J}^\diamond(\mathfrak{A})$.*
(b) *If $\mathfrak{B} = \prod_k \mathfrak{b}_k^{d_k}$ with $\mathfrak{b}_k \subset \mathfrak{a}_k$ and $d_k \geq c_k$ for all k, then $\widehat{\mathcal{J}}(\mathfrak{B}) \subset \widehat{\mathcal{J}}(\mathfrak{A})$ and $\mathcal{J}^\diamond(\mathfrak{B}) \subset \mathcal{J}^\diamond(\mathfrak{A})$.*
(c) *$\mathfrak{a} \subset \widehat{\mathcal{J}}(\mathfrak{a})$ for every ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$. In particular, $\widehat{\mathcal{J}}(\mathcal{O}_X) = \mathcal{O}_X$.*
(d) *$\mathfrak{a} \cdot \mathcal{J}^\diamond(\mathcal{O}_X) \subset \mathcal{J}^\diamond(\mathfrak{a})$ for every ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$.*

The proof of the next proposition is essentially the same as that of the transformation rule for the usual multiplier ideals (see Proposition 9.2.32 of [Laz04]). We outline it for the convenience of the reader. A similar property holds for Jacobian multiplier ideals: the same proof goes through, or equivalently one can deduce it from the case treated below using Remark 6.2.

Proposition 6.5. *Suppose that the $\mathbb{R}$-ideal $\mathfrak{A} = \prod_i \mathfrak{a}_i^{c_i}$ has integral exponents $c_i \in \mathbb{Z}$. Let $f: Y \to X$ be a resolution of X factoring through the blow-up of the Jacobian ideal $\mathfrak{j}_X$, and write $\mathfrak{a}_i \cdot \mathcal{O}_Y = \mathfrak{b}_i \cdot \mathcal{O}_Y(-D_i)$ where $\mathfrak{b}_i \subset \mathcal{O}_Y$ and D_i is a divisor. Let $\mathfrak{B} = \prod_i \mathfrak{b}_i^{c_i}$ and $D = \sum_i c_i D_i$. Then*

$$\widehat{\mathcal{J}}(\mathfrak{A}) = f_* \big(\mathcal{J}(\mathfrak{B}) \otimes \mathcal{O}_Y(\widehat{K}_{Y/X} - D) \big).$$

Proof. Let $f': Y' \to X$ be a log resolution of $(X, \mathfrak{A})$ factoring through f and a morphism $g: Y' \to Y$. Write $\mathfrak{a}_i \cdot \mathcal{O}_{Y'} = \mathcal{O}_{Y'}(-A_i)$ and $\mathfrak{b}_i \cdot \mathcal{O}_{Y'} = \mathcal{O}_{Y'}(-B_i)$, and let $A = \sum_i c_i A_i$ and $B = \sum_i c_i B_i$. Note that $A = B + g^* D$. By definition, we have $\mathcal{J}(\mathfrak{B}) = g_* \mathcal{O}_{Y'}(K_{Y'/Y} - B)$, and therefore

$$\widehat{\mathcal{J}}(\mathfrak{A}) = f'_* \mathcal{O}_{Y'}(\widehat{K}_{Y'/X} - A) = f'_* \big(\mathcal{O}_{Y'}(K_{Y'/Y} - B) \otimes g^* \mathcal{O}_Y(\widehat{K}_{Y/X} - D) \big)$$
$$= f_* \big(\mathcal{J}(\mathfrak{B}) \otimes \mathcal{O}_Y(\widehat{K}_{Y/X} - D) \big)$$

by the projection formula and Proposition 3.3. $\qquad \square$

The following characterizations of singularities come directly from the definitions.

Proposition 6.6. *Let X be a normal variety.*

(a) *X is J-canonical if and only if $\mathcal{J}^\diamond(\mathcal{O}_X) = \mathcal{O}_X$.*
(b) *X is log J-canonical if and only if $\mathcal{J}^\diamond(\mathfrak{j}_X^{-\lambda}) = \mathcal{O}_X$ for all $\lambda > 0$.*

Remark 6.7. The same notion of multiplier ideals has been independently introduced and studied in [EIM11].

6.2. *Grauert–Riemenschneider canonical sheaf of a variety*

Next we discuss how the canonical sheaf ω_X of a normal variety X relates to the *Grauert–Riemenschneider canonical sheaf* of X, which, we recall, is defined by

$$\omega_X^{\mathrm{GR}} := f_* \omega_Y$$

for any resolution $f : Y \to X$ (see [GR70]). There is a natural inclusion $\omega_X^{\mathrm{GR}} \subset \omega_X$ and our goal is to give a measure of the gap between the two sheaves.

When X is locally complete intersection (or, more generally, when ω_X is invertible) we have

$$\omega_X^{\mathrm{GR}} \cong \omega_X \otimes \mathcal{J}(\mathcal{O}_X).$$

Indeed, by definition $\mathcal{J}(\mathcal{O}_X) = f_* \mathcal{O}_Y(K_{Y/X}) \cong f_* \omega_Y \otimes \omega_X^{-1}$ where $f : Y \to X$ is any log resolution of X. We can rewrite this formula using the Mather multiplier ideal of the Jacobian. The observation is that, if X is locally complete intersection and $Y \to X$ is a log resolution, then $K_{Y/X} = K_{Y/X}^{\diamond}$ and thus $\mathcal{J}(\mathcal{O}_X) = \mathcal{J}^{\diamond}(\mathcal{O}_X) = \widehat{\mathcal{J}}(\mathfrak{j}_X)$ (see Corollary 3.5 and Remark 6.2). Therefore the formula can be written as

$$\omega_X^{\mathrm{GR}} \cong \omega_X \otimes \widehat{\mathcal{J}}(\mathfrak{j}_X).$$

Remark 6.8. Since $\widehat{\omega}_X \cong \omega_X \otimes \mathfrak{j}_X$ when X is locally complete intersection, there is in this case a correspondence between the chain of inclusions $\widehat{\omega}_X \subset \omega_X^{\mathrm{GR}} \subset \omega_X$ and the inclusions $\mathfrak{j}_X \subset \widehat{\mathcal{J}}(\mathfrak{j}_X) \subset \mathcal{O}_X$, determined by tensoring by ω_X^{-1}.

In general, when X is not locally complete intersection, we pick a reduced, locally complete intersection scheme V containing X, of the same dimension. If $f : Y \to X$ is a log resolution of $(X, \mathfrak{j}_V|_X)$ and $\widehat{f} : Y \to \widehat{X}$ is the induced map on the Nash blow-up of X, then we have $\mathfrak{j}_V|_X \cdot \mathcal{O}_Y \cong \mathcal{O}_Y(\widehat{f}^* \widehat{K}_X) \otimes f^*(\omega_V^{-1}|_X)$ by Proposition 2.4. This gives $\widehat{\mathcal{J}}(\mathfrak{j}_V|_X) \cong f_* \omega_Y \otimes \omega_V^{-1}|_X$, and therefore

$$\omega_X^{\mathrm{GR}} \cong \omega_V|_X \otimes \widehat{\mathcal{J}}(\mathfrak{j}_V|_X). \tag{6.1}$$

Remark 6.9. For any V as above, there is a correspondence between the inclusions $\widehat{\omega}_X \subset \omega_X^{\mathrm{GR}} \subset \omega_V|_X$ and the inclusions $\mathfrak{j}_V|_X \subset \widehat{\mathcal{J}}(\mathfrak{j}_V|_X) \subset \mathcal{O}_X$, given by tensoring by $\omega_V^{-1}|_X$.

The idea now is to assemble the isomorphisms given in (6.1) by letting V vary. The next theorem is the main result of this section. The key ingredient is the *lci-defect ideal* $\mathfrak{d}_X$ of X, which was defined in Section 2 by

$$\mathfrak{d}_X := \sum_V \mathfrak{d}_{X,V}$$

where $\mathfrak{d}_{X,V}$ is the ideal determined by the image of $\omega_X \to \omega_V|_X$. Note that since X is normal, $\mathfrak{d}_X$ is trivial in codimension one and thus $\mathcal{J}^{\diamond}(\mathfrak{d}_X^{-1})$ is an ideal sheaf by Proposition 6.3.

Theorem 6.10. *For every normal variety X, we have*

$$(\omega_X^{\mathrm{GR}} : \omega_X) = \mathcal{J}^\diamond(\mathfrak{d}_X^{-1}).$$

Remark 6.11. Using Lemma 6.13 below and Remark 6.2, the formula in the theorem can be rewritten in the following equivalent ways:

$$(\omega_X^{\mathrm{GR}} : \omega_X) = (\mathcal{J}^\diamond(\mathcal{O}_X) : \mathfrak{d}_X) = (\widehat{\mathcal{J}}(\mathfrak{j}_X) : \mathfrak{d}_X).$$

Proof of Theorem 6.10. Since the question is local, we can assume that X is affine. We fix an embedding of X in a smooth affine variety M, and denote $e = \mathrm{codim}(X, M)$.

Let T be an irreducible algebraic family parametrizing reduced, complete intersections $V \subset M$ of codimension e containing X. The family is constructed as an open set of the Grassmannian of e-tuples of linear combinations among a fixed set of generators of the ideal $\mathcal{I}_X$ of X in M. We have $\omega_X = \mathfrak{d}_{X,V} \otimes \omega_V|_X$ by definition and $\omega_X^{\mathrm{GR}} = \widehat{\mathcal{J}}(\mathfrak{j}_V|_X) \otimes \omega_V|_X$ by (6.1), and hence

$$(\omega_X^{\mathrm{GR}} : \omega_X) = \big(\widehat{\mathcal{J}}(\mathfrak{j}_V|_X) \otimes \omega_V|_X : \mathfrak{d}_{X,V} \otimes \omega_V|_X\big) = (\widehat{\mathcal{J}}(\mathfrak{j}_V|_X) : \mathfrak{d}_{X,V})$$

since $\omega_V|_X$ is invertible. Using Lemma 6.13 below, we get

$$(\omega_X^{\mathrm{GR}} : \omega_X) = \widehat{\mathcal{J}}(\mathfrak{j}_V|_X \cdot \mathfrak{d}_{X,V}^{-1}).$$

Therefore, in order to prove the theorem we are reduced to proving the following identity:

$$\widehat{\mathcal{J}}(\mathfrak{j}_V|_X \cdot \mathfrak{d}_{X,V}^{-1}) = \widehat{\mathcal{J}}(\mathfrak{j}_X \cdot \mathfrak{d}_X^{-1}). \tag{6.2}$$

Since the left-hand side does not depend on V, we can assume that V is general in T.

Let $f \colon Y \to X$ be a log resolution of $(X, \mathfrak{j}_X \cdot \mathfrak{d}_X^{-1})$, and write

$$\mathfrak{j}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-A), \quad \mathfrak{d}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-B).$$

We have $\mathfrak{j}_X = \sum_{V \in T} \mathfrak{j}_V|_X$, and $\mathfrak{d}_X$ has the same integral closure of $\mathfrak{d}_{X/M} = \sum_{V \in T} \mathfrak{d}_{X,V}$ (see Proposition 2.6). Therefore, if V is sufficiently general then by Lemma 6.14 below we have

$$\mathfrak{j}_V|_X \cdot \mathcal{O}_Y = \mathfrak{a}_V \cdot \mathcal{O}_Y(-A), \quad \mathfrak{d}_{X,V} \cdot \mathcal{O}_Y = \mathfrak{b}_V \cdot \mathcal{O}_Y(-B),$$

where $\mathfrak{a}_V, \mathfrak{b}_V \subset \mathcal{O}_Y$ do not vanish along any exceptional divisor. Moreover, we have

$$\sum_{V \in T} \mathfrak{a}_V = \mathcal{O}_Y.$$

This follows again from the fact that, since $\sum_{V \in T} \mathfrak{j}_V|_X = \mathfrak{j}_X$, for every divisorial valuation ν we can find a V in T such that $\nu(\mathfrak{j}_V|_X) = \nu(\mathfrak{j}_X)$.

By taking $V = X \cup X'$ general, we can assume by Bertini that X' intersects X transversely in codimension one (recall that X is normal, hence smooth in codimension one). In particular, it follows that $\mathfrak{a}_V$ and $\mathfrak{b}_V$ agree in codimension one. Indeed, since these sheaves do not vanish on exceptional divisors, this becomes a computation on X, and it

is easy to see that if X' intersects X transversely in codimension one, then at the generic point of each irreducible component of $X \cap X'$ both ideals are reduced.

Note on the other hand that $j_V|_X \cdot \mathcal{O}_Y$ is locally principal, since f factors through the blow-up of j_X and hence through the Nash blow-up of X, which is isomorphic to the blow-up of $j_V|_X$. Therefore $\mathfrak{a}_V$ is locally principal, and hence there is an inclusion $\mathfrak{b}_V \subset \mathfrak{a}_V$ that is an identity in codimension one. This implies that

$$\mathcal{J}(\mathfrak{a}_V \cdot \mathfrak{b}_V^{-1}) = \mathcal{O}_Y.$$

Using Proposition 6.5, we get

$$\widehat{\mathcal{J}}(j_V|_X \cdot \mathfrak{d}_{V,X}^{-1}) = f_*\big(\mathcal{J}(\mathfrak{a}_V \cdot \mathfrak{b}_V^{-1}) \cdot \mathcal{O}_Y(\widehat{K}_{Y/X} - A + B)\big)$$
$$= f_*\mathcal{O}_Y(\widehat{K}_{Y/X} - A + B) = \widehat{\mathcal{J}}(j_X \cdot \mathfrak{d}_X^{-1}).$$

This proves the identity (6.2). $\qquad\square$

Remark 6.12. If in the proof one takes f so that it also factors through the Nash transformation of X relative to ω_X, then $\mathfrak{d}_{X,V} \cdot \mathcal{O}_Y$ is locally principal too and hence $\mathfrak{a}_V = \mathfrak{b}_V$. This step is however not necessary in the proof.

Lemma 6.13. *On a normal variety X, for any two ideals $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_X$ we have*

$$(\widehat{\mathcal{J}}(\mathfrak{a}) : \mathfrak{b}) = \widehat{\mathcal{J}}(\mathfrak{a} \cdot \mathfrak{b}^{-1}).$$

Proof. Let $f : Y \to X$ be a log resolution of $(X, \mathfrak{a} \cdot \mathfrak{b})$, and write $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-A)$ and $\mathfrak{b} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-B)$. Then

$$x \in (\widehat{\mathcal{J}}(\mathfrak{a}) : \mathfrak{b}) \;\Leftrightarrow\; x \cdot \mathfrak{b} \subset \widehat{\mathcal{J}}(\mathfrak{a}) \;\Leftrightarrow\; f^*x \cdot \mathcal{O}_Y(-B) \subset \mathcal{O}_Y(\widehat{K}_{Y/X} - A)$$
$$\Leftrightarrow\; f^*x \in \mathcal{O}_Y(\widehat{K}_{Y/X} - A + B) \;\Leftrightarrow\; x \in \widehat{\mathcal{J}}(\mathfrak{a} \cdot \mathfrak{b}^{-1}). \qquad\square$$

Lemma 6.14. *On a variety M, let $\mathfrak{a}_t \subset \mathcal{O}_M$, $t \in T$, be an algebraic family of ideal sheaves, and let $\mathfrak{a} = \sum_{t \in T} \mathfrak{a}_t$. Then for every divisorial valuation v of $\mathcal{O}_M$ there is a non-empty open set $T_v \subset T$ such that $v(\mathfrak{a}_v) = v(\mathfrak{a})$ for every $v \in T_v$.*

Proof. For every v we have $v(\mathfrak{a}) = \min_{t \in T} v(\mathfrak{a}_t)$, and hence the assertion follows from the semicontinuity of the function $\mathfrak{a}_t \mapsto v(\mathfrak{a}_t)$. $\qquad\square$

6.3. Grauert–Riemenschneider canonical sheaf of a pair

Let $\mathfrak{A} = \prod_k \mathfrak{a}_k^{c_k}$ be a proper $\mathbb{R}$-ideal on a normal variety X. Associated to the pair $(X, \mathfrak{A})$, we consider the sheaf

$$\omega_{(X,\mathfrak{A})}^{\mathrm{GR}} := f_*\omega_Y(-\lfloor Z(\mathfrak{A} \cdot \mathcal{O}_Y) \rfloor)$$

where $f : Y \to X$ is any log resolution of $(X, \mathfrak{A})$. We call $\omega_{(X,\mathfrak{A})}^{\mathrm{GR}}$ the *Grauert–Riemenschneider canonical sheaf of the pair*.

It is well-known that the definition of $\omega_{(X,\mathfrak{A})}^{\mathrm{GR}}$ is independent of the choice of log resolution. If $(X, \mathfrak{A})$ is effective in codimension one, then $\omega_{(X,\mathfrak{A})}^{\mathrm{GR}}$ is a subsheaf of ω_X^{GR} and

thus of ω_X. Motivated by an analogous definition in positive characteristic due to Smith [Smi95], this sheaf has been considered before with the name of *multiplier submodule* by several authors (see in particular [HS03, Bli04, ST08]). We however prefer to view $\omega_{(X,\mathfrak{A})}^{\mathrm{GR}}$ as a 'perturbation' of the Grauert–Riemenschneider canonical sheaf of X, hence the terminology and notation adopted here.

Theorem 6.10 generalizes as follows.

Theorem 6.15. *For every proper $\mathbb{R}$-ideal $\mathfrak{A}$ on a normal variety X, we have*

$$(\omega_{(X,\mathfrak{A})}^{\mathrm{GR}} : \omega_X) = \mathcal{J}^{\diamond}(\mathfrak{A} \cdot \mathfrak{d}_X^{-1}).$$

Proof. The proof proceeds along the lines of that of Theorem 6.10. Using

$$\omega_{(X,\mathfrak{A})}^{\mathrm{GR}} = \widehat{\mathcal{J}}(\mathfrak{A} \cdot \mathrm{j}_V|_X) \otimes \omega_V|_X,$$

which is a straightforward generalization of (6.1), we get this time

$$(\omega_{(X,\mathfrak{A})}^{\mathrm{GR}} : \omega_X) = \widehat{\mathcal{J}}(\mathfrak{A} \cdot \mathrm{j}_V|_X \cdot \mathfrak{d}_{V,X}^{-1}).$$

It is therefore enough to prove that

$$\widehat{\mathcal{J}}(\mathfrak{A} \cdot \mathrm{j}_V|_X \cdot \mathfrak{d}_{V,X}^{-1}) = \widehat{\mathcal{J}}(\mathfrak{A} \cdot \mathrm{j}_X \cdot \mathfrak{d}_X^{-1}).$$

This follows by the same arguments leading to (6.2) in the proof of Theorem 6.10. □

Remark 6.16. When ω_X is invertible, the theorem implies that $\mathcal{J}^{\diamond}(\mathfrak{A} \cdot \mathfrak{d}_X^{-1}) = \mathcal{J}(X, \mathfrak{A})$ for every proper $\mathbb{R}$-ideal $\mathfrak{A}$ on X, since in this case $\omega_{(X,\mathfrak{A})}^{\mathrm{GR}} \cong \omega_X \otimes \mathcal{J}(X, \mathfrak{A})$. In particular, X is canonical (resp., log canonical) if and only if the pair $(X, \mathfrak{d}_X^{-1})$ is J-canonical (resp., log J-canonical). Both properties can also be deduced directly from Proposition 3.4 as in this case $\mathfrak{d}_X = \mathfrak{d}_{1,X}$.

Remark 6.17. Grauert and Riemenschneider proved that the Kodaira vanishing theorem holds on any normal projective variety X if one considers the sheaf ω_X^{GR} in place of the canonical sheaf ω_X [GR70]. More generally, a standard application of the Kawamata–Viehweg vanishing theorem implies the following general vanishing property. Let $\mathfrak{A} = \prod_i \mathfrak{a}_i^{c_i}$ be an effective proper $\mathbb{R}$-ideal on X, and suppose that, for every i, D_i is a Cartier divisor on X such that $\mathcal{O}_X(D_i) \otimes \mathfrak{a}_i$ is globally generated. Then for every Cartier divisor L such that $L - \sum_i c_i D_i$ is a nef and big $\mathbb{R}$-divisor, we have

$$H^j(\omega_{(X,\mathfrak{A})}^{\mathrm{GR}} \otimes \mathcal{O}_X(L)) = 0 \quad \text{for all } j > 0.$$

In view of Theorem 6.15, such vanishing can be interpreted as a Nadel-type vanishing theorem for the Jacobian multiplier ideal $\mathcal{J}^{\diamond}(\mathfrak{A} \cdot \mathfrak{d}_X^{-1})$.

7. Rational and Du Bois singularities

As an application of the main results of the paper, we give in this section necessary and sufficient conditions for rational and Du Bois singularities on normal varieties and provide a characterization for these classes of singularities in the Cohen–Macaulay case.

We recall that a variety X has *rational singularities* if given a resolution of singularities $f: Y \to X$ such that $f_*\mathcal{O}_Y = \mathcal{O}_X$ and $R^i f_*\mathcal{O}_Y = 0$ for $i > 0$, or in other words, such that the natural map $\mathcal{O}_X \to Rf_*\mathcal{O}_Y$ is a quasi-isomorphism. The original definition of Du Bois singularities is more complicated, and we will not recall it here. Several alternative definitions were given by many authors throughout the years, and we will adopt here the one given in [Sch07] for which a reduced scheme X embedded in a smooth variety M has *Du Bois singularities* if and only if, given a log resolution $g: N \to M$ of $(M, \mathcal{I}_X)$ (note: not an embedded log resolution of X) that is an isomorphism away from X, and denoting by $F = (g^{-1}(X))_{\mathrm{red}}$ the reduced pre-image of X, the natural map $\mathcal{O}_X \to Rg_*\mathcal{O}_F$ is a quasi-isomorphism. For more generalities on these classes of singularities, we refer to [KM98, Sch07, KSS10, KK10] and the references therein.

7.1. Necessary condition and characterization on Cohen–Macaulay varieties

It is a well-known result of Kempf [KKMS73] that a normal variety X has rational singularities if and only if it is Cohen–Macaulay and $f_*\omega_Y = \omega_X$. An analogous property proven more recently by Kovács, Schwede and Smith says that a normal Cohen–Macaulay variety X has Du Bois singularities if and only if $f_*\omega_Y(E) = \omega_X$ where $f: Y \to X$ is a log resolution and E is the reduced exceptional divisor (see Theorem 1.1 of [KSS10]). Furthermore, it follows by Theorem 3.8 of [KSS10] that the identity $f_*\omega_Y(E) = \omega_X$ holds for all normal varieties with Du Bois singularities, regardless of whether or not they are Cohen–Macaulay.

These facts motivate the following result.

Theorem 7.1. *Let X be a normal variety, and let $\mathfrak{d}_X \subset \mathcal{O}_X$ be the lci-defect ideal of X. Let $f: Y \to X$ be a log resolution of X, and denote by E the reduced exceptional divisor. Then the following properties hold:*

(a) *The pair $(X, \mathfrak{d}_X^{-1})$ is J-canonical if and only if $f_*\omega_Y = \omega_X$.*
(b) *The pair $(X, \mathfrak{d}_X^{-1})$ is log J-canonical if and only if $f_*\omega_Y(E) = \omega_X$.*

Proof. Recall that $f_*\omega_Y = \omega_X^{\mathrm{GR}}$. By Theorem 6.10, $\omega_X^{\mathrm{GR}} = \omega_X$ if and only if $\mathcal{J}^\diamond(\mathfrak{d}_X^{-1}) = \mathcal{O}_X$, which is equivalent to $(X, \mathfrak{d}_X^{-1})$ being J-canonical. This proves (a).

To prove (b), first note that if f is an isomorphism over the regular locus of X and λ is a sufficiently small positive number, then $\lfloor Z(\mathrm{j}_X^{-\lambda} \cdot \mathcal{O}_Y) \rfloor = -E$, and thus

$$f_*\omega_Y(E) = \omega_{(X, \mathrm{j}_X^{-\lambda})}^{\mathrm{GR}} \quad \text{for } 0 < \lambda \ll 1.$$

Therefore $(X, \mathfrak{d}_X^{-1})$ is log J-canonical if and only if $\omega_{(X, \mathrm{j}_X^{-\lambda})}^{\mathrm{GR}} = \omega_X$ for every sufficiently small $\lambda > 0$. On the other hand, by definition of multiplier ideal and the fact that the radical of $\mathfrak{d}_X$ contains j_X, we have that $(X, \mathfrak{d}_X^{-1})$ is log J-canonical if and only if $\mathcal{J}^\diamond(\mathrm{j}_X^{-\lambda} \cdot \mathfrak{d}_X^{-1}) = \mathcal{O}_X$ for every $\lambda > 0$. Therefore (b) follows from Theorem 6.15. $\square$

Corollary 7.2. *Let X be a normal variety, and let $\mathfrak{d}_X \subset \mathcal{O}_X$ be the lci-defect ideal of X.*

(a) *If X has rational singularities then $(X, \mathfrak{d}_X^{-1})$ is J-canonical.*
(b) *If X has Du Bois singularities then $(X, \mathfrak{d}_X^{-1})$ is log J-canonical.*

Moreover, the converse holds in both cases whenever X is Cohen–Macaulay.

In the special case when X is $\mathbb{Q}$-Gorenstein, we obtain the result stated in the Introduction as Theorem A.

Corollary 7.3. *With the same assumptions of Corollary 7.2, suppose that rK_X is Cartier for some positive integer r, and let $\mathfrak{d}_{r,X}$ be the lci-defect ideal of level r of X.*

(a) *If X has rational singularities then $(X, \mathfrak{d}_{r,X}^{1/r} \cdot \mathfrak{d}_X^{-1})$ is canonical.*
(b) *If X has Du Bois singularities then $(X, \mathfrak{d}_{r,X}^{1/r} \cdot \mathfrak{d}_X^{-1})$ is log canonical.*

Moreover, the converse holds in both cases whenever X is Cohen–Macaulay.

If ω_X is invertible then we can take $r = 1$, and since in this case $\mathfrak{d}_{1,X} = \mathfrak{d}_X$ the corollary recovers the well-known characterization of rational singularities and Du Bois singularities on Gorenstein varieties.

In general, assuming a priori that the variety is Cohen–Macaulay, Corollary 7.3 gives new proofs to the facts that a variety with log terminal (resp., log canonical) singularities has rational (resp., Du Bois) singularities, which we know from the results of [Elk81, KSS10, KK10]. To see this, first notice that $(X, \mathfrak{d}_{r,X}^{1/r} \cdot \mathfrak{d}_X^{-1})$ is canonical if and only if it is log terminal, since for every prime divisor E over X,

$$a_E(X, \mathfrak{d}_{r,X}^{1/r} \cdot \mathfrak{d}_X^{-1}) = a_E^{\diamond}(X, \mathfrak{d}_X^{-1}) \in \mathbb{Z}$$

by Proposition 3.4. Since $\mathfrak{d}_X^r \subset \overline{\mathfrak{d}_{r,X}}$ by Proposition 2.12, we have $a_E(X, \mathfrak{d}_{r,X}^{1/r} \cdot \mathfrak{d}_X^{-1}) \geq a_E(X)$, and thus the aforementioned properties follow, under the Cohen–Macaulay hypothesis, from the corollary.

More interestingly, the corollary provides the necessary correction on discrepancies for the converses of such results to hold.

Remark 7.4. One should think of the difference between $\mathfrak{d}_X$ and $\mathfrak{d}_{r,X}^{1/r}$, from a valuation-theoretic point of view, as the cause of failure of the converses in the theorems in [Elk81, KSS10, KK10]. Any $\mathbb{Q}$-Gorenstein variety with rational singularities that is not log terminal (for instance, the cone over an Enriques surface embedded by a sufficiently positive line bundle) gives an instance where the inclusion $\overline{\mathfrak{d}_X^r} \subset \overline{\mathfrak{d}_{r,X}}$ is strict.

7.2. On the Cohen–Macaulay condition

We discuss here an example showing that the Cohen–Macaulay hypothesis cannot be dropped in the closing assertions of the above corollaries. The example is known to experts. For the convenience of the reader, we first review some facts about cone singularities.

To fix notation, let S be a smooth projective variety of dimension $n-1 \geq 2$, embedded in a projective space by a projectively normal ample line bundle $\mathcal{O}_S(1)$. Let then $X = \mathrm{Spec} \bigoplus_{m \geq 0} H^0(\mathcal{O}_S(m))$ be the cone over S, and let $f : Y \to X$ be the resolution given by the total space of $\mathcal{O}_S(-1)$. The zero section of that line bundle is the exceptional divisor E of f. Note that X is normal. Let $x \in X$ be the vertex of the cone.

We have

$$\omega_X \cong \bigoplus_{m \in \mathbb{Z}} H^0(\omega_S(m)) \quad \text{and} \quad f_*\omega_Y \cong \bigoplus_{m > 0} H^0(\omega_S(m))$$

by Theorem (2.8) of [Wat81] and Proposition (1.6) of [Wat83], and therefore $f_*\omega_Y = \omega_X$ if and only if $H^0(\omega_S(m)) = 0$ for $m \leq 0$. One can similarly see that $f_*\omega_Y(E) = \omega_X$ if and only if $H^0(\omega_S(m)) = 0$ for $m < 0$, but we will not use this fact.

It is well-known that X is Cohen–Macaulay if and only if $H^i(\mathcal{O}_S(m)) = 0$ for $n-1 > i > 0$ and $m \geq 0$, and the singularity is rational if and only if the same vanishing holds for $i > 0$ and $m \geq 0$. It was proven by Du Bois [DB81] (see also [Ste83]) that $x \in X$ is a Du Bois singularity if and only if the natural map $R^i f_* \mathcal{O}_Y \to R^i f_* \mathcal{O}_E$ is an isomorphism for all $i > 0$, or equivalently, if and only if $R^i f_* \mathcal{O}_Y(-E) = 0$ for $i > 0$. We will use the following consequence of this property, which we learned from Karl Schwede.

Lemma 7.5. *With the above notation, if X has Du Bois singularities then $H^i(\mathcal{O}_S(m)) = 0$ for all $n > i > 0$ and $m > 0$.*

Proof. By Lemma (2.3) of [Wat83],

$$H_x^{i+1}(\mathcal{O}_X) \cong \bigoplus_{m \in \mathbb{Z}} H^i(\mathcal{O}_S(m)).$$

Note that $H_x^{i+1}(\mathcal{O}_X) \cong H_x^{i+1}(\mathfrak{m})$, where $\mathfrak{m} = f_* \mathcal{O}_Y(-E) \subset \mathcal{O}_X$ is the maximal ideal of x. The vanishing of the higher direct images of $\mathcal{O}_Y(-E)$ gives the degeneration of the appropriate Leray spectral sequence, and thus using duality (see Proposition (11.6) of [Kol97]) in combination with the relative version of the Grauert–Riemenschneider vanishing theorem, we have

$$H_x^{i+1}(\mathcal{O}_X) \cong H_E^{i+1}(\mathcal{O}_Y(-E)) \overset{\text{dual}}{\sim} R^{n-i-1} f_* \omega_Y(E) \cong R^{n-i-1} f_* \omega_Y \overset{\text{dual}}{\sim} H_E^i(\mathcal{O}_Y)$$
$$\cong H^i(\mathcal{O}_S).$$

The assertion follows by comparing the two formulas. □

We are now ready to discuss the example.

Example 7.6. Let C be a non-hyperelliptic curve of genus g. Fix a non-special divisor B on C such that $\deg B \geq 2g + 1$ (note that the linear system $|2(B - K_C)|$ is very ample and thus contains smooth elements), and let $\mathcal{E} = \mathcal{O}_C \oplus \omega_C(-B)$. Following [FP05], the ruled surface $\pi : S = \mathbb{P}_C(\mathcal{E}) \to C$ is a *canonical geometrically ruled surface*. Moreover, if $H = C_0 + \pi^*B$ where C_0 is the section determined by the quotient $\mathcal{E} \to \mathcal{O}_C$, then the linear system $|H|$ determines a projectively normal embedding of S as a *canonical scroll* in some $\mathbb{P}^N$ (see Theorem 6.16 of [FP05]). Let X be the cone over $S \subset \mathbb{P}^N$. Since

$H^1(\mathcal{O}_S(1)) \cong H^1(C, \mathcal{E}) \neq 0$, the singularity is not Du Bois (and thus not rational) by Lemma 7.5. On the other hand we have $H^0(\omega_S(m)) = 0$ for all $m \leq 0$, and so $f_*\omega_Y = \omega_X$. We conclude that the pair $(X, \mathfrak{d}_X^{-1})$ is J-canonical (and thus log J-canonical) by Theorem 7.1.

7.3. A general sufficient condition

Using inversion of adjunction and the results of [Kaw98, KK10] in place of Theorem 6.10, one obtains the following general sufficient condition. The argument was brought to our attention by Mircea Mustaţă.

Theorem 7.7. *Let X be a reduced equidimensional scheme.*

(a) *If X is J-canonical, then X is the disjoint union of its irreducible components, each of which is log terminal in the sense of* [dFH09] *and has rational singularities. In particular, X is normal and Cohen–Macaulay.*

(b) *If X is log J-canonical, then X has Du Bois singularities. In particular, X is seminormal.*

Proof. We can assume that X is embedded in a smooth variety M, with codimension e. If X is J-canonical then by Theorem 4.10 (see Remark 4.12 if $\dim X = 1$) the pair (M, eX) is canonical, and thus each irreducible component of X is an isolated log canonical center of the pair. Since intersections of log canonical centers are log canonical centers, it follows in particular that X is the disjoint union of its irreducible components. Moreover, it follows by the main result of [Kaw98] that each irreducible component of X has rational singularities and is log terminal in the sense of [dFH09]. If X is only log J-canonical, then (M, eX) is log canonical (again by Theorem 4.10 and Remark 4.12) and X is a log canonical center of (M, eX), so the result follows in this case by Theorem 1.4 of [KK10]. Regarding the last assertion in (b), see Remark 1.11 of [KK10].   □

The above theorem is well-known to specialists once the assumptions on the singularities of X are expressed in terms of the jet schemes of X_m. Indeed, by Corollary 5.4, the result can be rephrased by saying that if $\dim X_m = n(m+1)$ for every m, then X has Du Bois singularities, and if moreover every irreducible component of maximal dimension $n(m+1)$ of X_m dominates an irreducible component of X, then X has rational singularities.

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