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Sharp bounds for the intersection of nodal lines with certain curves

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Abstract. Let Y be a hyperbolic surface and let ϕ be a Laplacian eigenfunction having eigenvalue $-1/4 - \tau^2$ with $\tau > 0$. Let $N(\phi)$ be the set of nodal lines of ϕ . For a fixed analytic curve γ of finite length, we study the number of intersections between $N(\phi)$ and γ in terms of τ . When Y is compact and γ a geodesic circle, or when Y has finite volume and γ is a closed horocycle, we prove that γ is “good” in the sense of [TZ]. As a result, we show that the number of intersections between $N(\phi)$ and γ is $O(\tau)$. This bound is sharp.

1. Introduction

Let Y be a hyperbolic surface (a Riemannian surface with constant curvature -1) and let Δ be the Laplace–Beltrami operator on Y . Let $\phi \in L^2(Y)$ be a real valued eigenfunction for Δ with eigenvalue $-(1/4 + \tau^2)$, that is,

$$\Delta\phi = -(1/4 + \tau^2)\phi.$$

Let $N(\phi)$ denote the zero set of ϕ on Y . Since Y is an analytic manifold and ϕ is an analytic function, $N(\phi)$ is a union of analytic curves on Y . Therefore, for an analytic segment γ of finite length, either ϕ vanishes on γ , or $\gamma \cap N(\phi)$ consists of a finite number of points.

It has been shown in several ways that $N(\phi)$ shares common properties with the zero set of a polynomial of degree τ for large $\tau > 0$ (see [DF], [S3], and [Z]). Accordingly we expect that

$$|\gamma \cap N(\phi)| \ll_{Y,\gamma} \tau$$

will hold for all ϕ which do not vanish on γ . Here and elsewhere $A \ll_{\omega} B$ means $|A| < CB$ for some constant $C > 0$ depending only on ω . Unless otherwise stated, in what follows, we assume that $\tau > 0$ is sufficiently large (say $\tau > 100$).

The main result of the article is:

Theorem 1.1. *Assume either one of the following:*

- Y is a compact surface and γ is a geodesic circle,
- Y is a noncompact surface of finite volume and γ is a closed horocycle.

Then $N(\phi) \cap \gamma$ consists of a finite number of points and we have

$$|N(\phi) \cap \gamma| \ll_{Y,\gamma} \tau. \tag{1.1}$$

We remark that, in spite of its appearance, (1.1) is not a local feature of the eigenfunctions on hyperbolic surfaces. That is, without any assumptions on Y , the assertions in Theorem 1.1 are not true in general. We give two examples to emphasize the role of the global geometry of Y .

1. Let $Y = \Gamma_0 \backslash \mathbb{H}$ where $\Gamma_0 = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$. Let the horocycle γ be given by $\{x + i \mid x \in \mathbb{R}\}$. Consider a one-parameter family of eigenfunctions

$$f_\tau(x + iy) = \sqrt{y} K_{i\tau}(2\pi y) \cos 2\pi x.$$

Since there are infinitely many $\tau \in \mathbb{R}$ such that $K_{i\tau}(2\pi) = 0$, we can find a sequence of eigenfunctions f_{τ_n} with $\tau_n \rightarrow \infty$ such that they all vanish on γ .

2. Let Y be as above and let γ be an analytic curve which is not part of a vertical line. For $n > 0$, let

$$f_n(x + iy) = \sqrt{y} K_i(2\pi ny) \cos 2\pi nx$$

be a sequence of eigenfunctions having eigenvalue $-5/4$. Then for all sufficiently large n , $N(f_n)$ does not vanish identically on γ and we have

$$|N(f_n) \cap \gamma| \gg n.$$

In fact, if one assumes Y has finite volume, then a nonzero eigenfunction ϕ in $L^2(Y)$ cannot vanish either on a horocycle or on a geodesic circle. Let us sketch a proof for the horocycle case; the other case can be treated similarly.

Without loss of generality, assume that $Y = \Gamma \backslash \mathbb{H}$ where Γ is reduced at ∞ . We further assume that γ is given by $\{x + iy_0 \mid x \in \mathbb{R}\}$ for some $y_0 > 0$. Say the Fourier expansion of the eigenfunction $\phi \in L^2(Y)$ at ∞ is given by

$$\phi(x + iy) = \sum_{n \neq 0} a_\phi(n) \sqrt{y} K_{i\tau}(2\pi |n|y) e^{2\pi i n x}.$$

If ϕ vanishes on γ , then

$$\int_0^1 |\phi(x + iy_0)|^2 dx = \sum_{n \neq 0} |a_\phi(n)|^2 y_0 K_{i\tau}(2\pi |n|y_0)^2 = 0.$$

Since $K_{i\tau}(2\pi |n|y_0) = 0$ for only finitely many n , this implies that $a_\phi(n) = 0$ for all but finitely many n . Now recall that the horocycle becomes equidistributed on Y when one increases the length of the horocycle [S1]. (For the geodesic circle, the first equidistribution result is due to Delsarte [D]. We give a short proof of the result when Y is compact in the appendix.) Therefore, we have

$$\|\phi\|_{L^2(Y)}^2 = \lim_{y \rightarrow 0} \int_0^1 |\phi(x + iy)|^2 dx = \lim_{y \rightarrow 0} \sum_{n \neq 0} |a_\phi(n)|^2 y K_{i\tau}(2\pi |n|y)^2.$$

By interchanging the limit and the finite summation, we see that the above equals

$$\sum_{n \neq 0} |a_\phi(n)|^2 \lim_{y \rightarrow 0} y K_{i\tau}(2\pi |n|y)^2 = 0,$$

which contradicts the assumption $\phi \neq 0$.

It is noteworthy that Theorem 1.1 does not require any arithmetic assumptions. With such an assumption, when γ is a closed horocycle for instance, the following strong result for the lower bound of the restricted L^2 norm is known [GRS]:

$$\|\phi\|_{L^2(\gamma)} \gg_{\epsilon} \tau^{-\epsilon}.$$

Therefore, by Lemma 4.2, we have the second part of Theorem 1.1. However, as shown in [TZ], a much weaker lower bound

$$\|\phi\|_{L^2(\gamma)} \gg e^{-B\tau}, \tag{1.2}$$

for some constant B , yields the same conclusion. In [TZ], they call a curve on a surface satisfying this lower bound “good.” The main result of this paper can be rephrased as:

- (1) any closed horocycle on a hyperbolic surface of finite volume is “good,” and
- (2) any geodesic circle on a compact hyperbolic surface is “good.”

We obtain such lower bounds by using purely analytic and geometric techniques. The main observation for our proof is Lemma 2.3 below. From this, to deduce Theorem 1.1, we note that it suffices to find a large Fourier coefficient ($b_{\phi}(n)$ as in (3.1) and $a_{\phi}(n) = b_{\phi,1}(n)$ in (3.2)) in the range $c_1\tau < |n| < c_2\tau$ for certain constants c_1 and c_2 .

When γ is a geodesic circle and Y is compact, one can study the partial sum $S(X) = \sum_{|n| < X} |b_{\phi}(n)|^2$ by integrating $|\phi|^2$ on a region bounded by two concentric geodesic circles that contains multiple copies of Y . By quantifying the lower bound and the upper bound of $S(X)$, we get

$$S(c_2\tau) - S(c_1\tau) \gg 1$$

for appropriate constants c_1 and c_2 (Lemma 5.1).

However, when Y is noncompact and γ is either a geodesic circle or a closed horocycle, the same method only yields $|N(\phi) \cap \gamma| \ll \tau \log^2 \tau$. Here the discrepancy between $\ll \tau \log^2 \tau$ and the desired bound $\ll \tau$ is closely related to the hypothetical phenomenon, known as “escape of mass” of the eigenfunctions. Hence we have to use a different method for noncompact surfaces.

When γ is a closed horocycle, we use the theory of Eisenstein series and their analytic continuation. By integrating $|\phi|^2$ against an Eisenstein series and using the functional equation for the Eisenstein series, we can study the Fourier coefficients $a_{\phi}(n)$ by using the approximate functional equation. After choosing an appropriate test function in the approximate functional equation, we deduce Lemma 6.3.

2. Special functions

We begin our discussion with the uniform approximation formulas for some special functions. Let us define

$$\tilde{K}_{i\tau}(x) = \frac{1}{|\Gamma(1/2 + i\tau)|} K_{i\tau}(x) \quad (x > 0),$$

and

$$C_\tau(m, x) = \frac{|\Gamma(1/2 + m + i\tau)|}{|\Gamma(1/2 + i\tau)|} P_{-1/2+i\tau}^{-m}(x) \quad (x > 1)$$

for a nonnegative integer m . Here $K_\tau(z)$ is the modified Bessel function of the second kind and $P_\tau^m(z)$ is the associated Legendre function of the first kind.

Lemma 2.1 ([E]). 1. When $x > \tau > 0$, put $\cos \theta = \tau/x$ where $\theta \in (0, \pi/2)$. Then

$$\tilde{K}_{i\tau}(x) = \frac{\sqrt{\pi/2} + O(1/x)}{(x^2 - \tau^2)^{1/4}} e^{-(\sin \theta - \theta \cos \theta)x}.$$

2. When $\tau > x > 0$, put $\psi(\tau, x) = \tau \cosh^{-1}(\tau/x) - \sqrt{\tau^2 - x^2} + \pi/4$. Then

$$\tilde{K}_{i\tau}(x) = \frac{\sqrt{2\pi}}{(\tau^2 - x^2)^{1/4}} \left(\sin(\psi(\tau, x)) + O\left(\min\left\{\frac{1}{x}, \frac{x^2}{\tau}\right\}\right) \right)$$

3. When $|\tau - x| < \tau^{1/3}$, put $C_0 = 2^{-2/3} 3^{-1/6} \pi \Gamma(1/3)$. Then

$$\tilde{K}_{i\tau}(x) \approx C_0 x^{-1/3} + O(x^{-2/3})$$

Lemma 2.2. 1. When $\beta := \tau/m \ll 1$, $m > 0$, and $x > 1$, we have ([Du, p. 325])

$$C_\tau(m, x) = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{\beta^2 - \eta}{x^2 \beta^2 - 1 - \beta^2}\right)^{1/4} \tilde{K}_{i\tau}(m\sqrt{\eta}) \left(1 + O\left(\frac{\sqrt{\eta}}{m}\right)\right).$$

Here the variable η is given by:

$$\begin{aligned} \xi &= \frac{1}{x^2 - 1}, \\ \int_{\beta^2}^{\eta} \frac{(s - \beta^2)^{1/2}}{2s} ds &= \int_{\beta^2}^{\xi} \frac{(s - \beta^2)^{1/2}}{2s(1+s)^{1/2}} ds \quad \text{when } \xi > \beta^2, \\ \int_{\beta^2}^{\eta} \frac{(\beta^2 - s)^{1/2}}{2s} ds &= \int_{\beta^2}^{\xi} \frac{(\beta^2 - s)^{1/2}}{2s(1+s)^{1/2}} ds \quad \text{when } \xi < \beta^2. \end{aligned}$$

2. When $\alpha := m/\tau \ll 1$, $m \geq 0$, and $x > 1$, we have ([Du, p. 314])

$$C_\tau(m, x) = \left(\frac{\zeta - \alpha^2}{x^2 - \alpha^2 - 1}\right)^{\frac{1}{4}} J_m(\tau\sqrt{\zeta}) \left(1 + O\left(\frac{1}{\tau}\right)\right).$$

Here the variable ζ is given by:

$$\begin{aligned} f(\alpha, s) &= \frac{1 + \alpha^2 - s^2}{(s^2 - 1)^2}, \\ x_\alpha &= \sqrt{1 + \alpha^2}, \\ \int_{\alpha^2}^{\zeta} \frac{(s - \alpha^2)^{1/2}}{2s} ds &= \int_{x_\alpha}^x (-f(\alpha, s))^{1/2} ds \quad \text{when } x > x_\alpha, \\ \int_{\alpha^2}^{\zeta} \frac{(\alpha^2 - s)^{1/2}}{2s} ds &= \int_{x_\alpha}^x f(\alpha, s)^{1/2} ds \quad \text{when } x < x_\alpha. \end{aligned}$$

From these formulas we observe:

Lemma 2.3. *For any fixed $x > 0$, we can find a constant $c_1 > 0$ with the following property: For any constant $c_2 > c_1$, there exists a constant $A > 0$ depending on c_2 such that*

$$C_\tau(m, \cosh x) > e^{-Am}$$

for any $c_1\tau < m < c_2\tau$, for all sufficiently large $\tau > 0$. The same statement is true for $\tilde{K}_{i\tau}(mx)$.

3. Analysis on hyperbolic surfaces

3.1. Upper half-plane

In order to study eigenfunctions on the hyperbolic surface Y , we identify Y with a quotient space $\Gamma \backslash \mathbb{H}$ for some discrete subgroup $\Gamma \subset \text{SL}(2, \mathbb{R})$. Here $\mathbb{H} = \{z = x + iy \mid y > 0\}$ is the upper half-plane with the line element $ds^2 = y^{-2}(dx^2 + dy^2)$. Let $\Delta = \Delta_{\mathbb{H}}$ be the Laplace–Beltrami operator on \mathbb{H} and let $dA = y^{-2} dx dy$ be the volume element.

By a function on $\Gamma \backslash \mathbb{H}$, we mean a function on \mathbb{H} which is invariant under Γ . Let \mathcal{F} be a fundamental domain for the action of Γ on \mathbb{H} . For $1 \leq p < \infty$, we let $L^p(\Gamma \backslash \mathbb{H})$ be the space of measurable functions on $\Gamma \backslash \mathbb{H}$ such that

$$\|f\|_{L^p(\Gamma \backslash \mathbb{H})}^p := \int_{\mathcal{F}} |f(z)|^p dA < \infty.$$

We recall two theorems regarding eigenfunctions on the upper half-plane.

Lemma 3.1 ([T, p. 142]). *Fix a point $z_0 \in \mathbb{H}$ and consider geodesic polar coordinates (r, θ) centered at z_0 . For any bounded function ϕ on \mathbb{H} that satisfies*

$$-\Delta\phi(z) = (1/4 + \tau^2)\phi(z),$$

there exist constants $c_\phi(m)$ such that

$$\phi(r, \theta) = \sum_{m \in \mathbb{Z}} c_\phi(m) C_\tau(|m|, \cosh r) e^{im\theta}.$$

Lemma 3.2 ([T, p. 136]). *Assume ϕ is a function on \mathbb{H} satisfying the conditions*

$$-\Delta\phi(z) = (1/4 + \tau^2)\phi(z), \quad \phi(z) = \phi(z + a),$$

for some constant $a > 0$. Assume further that there exist constants $C > 0$ and $A > 0$ such that

$$|\phi(x + iy)| < Cy^A \quad \text{for } y > 1.$$

Then there exist constants α_ϕ, β_ϕ , and $c_\phi(m)$ such that

$$\phi(x + iy) = \alpha_\phi y^{1/2+i\tau} + \beta_\phi y^{1/2-i\tau} + \sum_{m \neq 0} c_\phi(m) \sqrt{y} \tilde{K}_{i\tau}(2\pi|m|y) e^{2\pi imx/a}.$$

Note that in both lemmas, it is the growth condition of the eigenfunction which determines the choice of the special functions appearing in the equations. In general, without the growth condition, the associated Legendre function of the second kind and the modified Bessel function of the first kind will appear.

3.2. Compact surface

Let $Y = \Gamma \backslash \mathbb{H}$ be a compact hyperbolic surface. Fix a point $z_0 \in \mathbb{H}$ and let (r, θ) be the geodesic polar coordinates centered at z_0 . Note that any geodesic circle centered at z_0 is given by $\gamma = \{(r, \theta) \mid r = r_0\}$ for some r_0 . Fix a fundamental domain \mathcal{F} containing z_0 which has diameter equal to the diameter of Y .

Lemma 3.3. *Let d be the diameter of Y and let $f \in L^2(Y)$ be an $L^2(Y)$ normalized function. For any $a, b \in \mathbb{R}^+$ with $2d < b - a$,*

$$\int_a^b \int_0^{2\pi} |f(r, \theta)|^2 \sinh r \, d\theta \, dr > \frac{2\pi}{\text{Area}(Y)} (\cosh(b - d) - \cosh(a + d)).$$

Proof. Let S be the set of $\gamma \in \Gamma$ such that $\gamma \mathcal{F} \subset \{(r, \theta) \mid a < r < b\}$. Observing that $\{(r, \theta) \mid a + d < r < b - d\} \subset S \mathcal{F}$,

$$\begin{aligned} \int_a^b \int_0^{2\pi} |f(r, \theta)|^2 \sinh r \, d\theta \, dr &\geq \int_{S \mathcal{F}} |f(r, \theta)|^2 \, dA = \frac{1}{\text{Area}(Y)} \int_{S \mathcal{F}} \, dA \\ &> \frac{1}{\text{Area}(Y)} \int_{a+d}^{b-d} \int_0^{2\pi} \sinh r \, d\theta \, dr = \frac{2\pi}{\text{Area}(Y)} (\cosh(b - d) - \cosh(a + d)). \quad \square \end{aligned}$$

Let ϕ be an eigenfunction for Δ on $\Gamma \backslash \mathbb{H}$ with eigenvalue $-(1/4 + \tau^2)$. We assume that ϕ is $L^2(Y)$ normalized. Since Y is compact, ϕ is a bounded function on \mathbb{H} . Hence we can find constants $b_\phi(m)$ such that

$$\phi(r, \theta) = \sum_{m \in \mathbb{Z}} b_\phi(m) C_\tau(|m|, \cosh r) e^{im\theta}. \tag{3.1}$$

Lemma 3.4. *We have*

$$\sum_{|m| < X} |b_\phi(m)|^2 \ll X + \tau.$$

Proof. From the asymptotics of $C_\tau(m, x)$, one proves

$$\int_1^R |C_\tau(|m|, x)|^2 \, dx \gg \frac{1}{\tau}$$

for $|m| < \tau \sqrt{R^2 - 1}$ and $1 \ll R$. Therefore,

$$\begin{aligned} B(R) &:= \int_0^{\cosh^{-1} R} \int_0^{2\pi} |\phi(r, \theta)|^2 \sinh r \, d\theta \, dr = \sum_{m \in \mathbb{Z}} |b_\phi(m)|^2 \int_1^R |C_\tau(|m|, x)|^2 \, dx \\ &\gg \frac{1}{\tau} \sum_{|m| < \tau \sqrt{R^2 - 1}} |b_\phi(m)|^2 \end{aligned}$$

for $R > \tau$. By using the same argument as in Lemma 3.3, we prove that $B(R) \ll R$. Therefore

$$\sum_{|m| \leq \tau \sqrt{R^2 - 1}} |b_\phi(m)|^2 \ll \tau R,$$

and substituting $\tau\sqrt{R^2 - 1} = X$ gives

$$\sum_{|m| \leq X} |b_\phi(m)|^2 \ll \sqrt{X^2 + \tau^2} < X + \tau. \quad \square$$

3.3. Noncompact surfaces and Eisenstein series

In this section we state the results regarding noncompact hyperbolic surfaces and Eisenstein series. For a detailed treatment of the subject, see [CS], [Ku], and [Se].

Assume that $Y = \Gamma \backslash \mathbb{H}$ is a noncompact hyperbolic surface which has finite volume. Let $\{\kappa_1, \dots, \kappa_h\}$ be a complete set of inequivalent cusps of Γ . Note that if κ is a cusp of Γ , then there exists a $\sigma \in \text{SL}(2, \mathbb{R})$ such that $\sigma\kappa = \infty$ and $\sigma^{-1}\Gamma_\kappa\sigma = \Gamma_0$. Here $\Gamma_\kappa \subset \Gamma$ is the stabilizer subgroup of κ and $\Gamma_0 = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$. This is equivalent to saying that $\sigma^{-1}\Gamma\sigma$ is reduced at ∞ . We fix our choice of such σ_j for each κ_j . Note that any closed horocycle on Y is given by

$$\{\sigma_j z \mid \text{Im}(z) = y_0\}$$

for some fixed $y_0 > 0$ and for some $1 \leq j \leq h$.

Let $\phi(z)$ be a $L^2(Y)$ normalized real eigenfunction corresponding to the eigenvalue $-(1/4 + \tau^2)$ with $\tau \geq 0$. Say the Fourier expansion of ϕ at κ_j is given by

$$\phi(\sigma_j z) = \sum_{n \neq 0} b_{\phi,j}(n) \sqrt{y} \tilde{K}_{i\tau}(2\pi|n|y) e^{2\pi i n x} \quad (3.2)$$

for some constants $b_{\phi,j}(n)$ for each $j = 1, \dots, h$. Note that the constant terms in Lemma 3.2 do not appear since $\phi \in L^2(Y)$ and $\tau \geq 0$. Also, because $\phi(z)$ is assumed to be real, $b_{\phi,j}(n) = \overline{b_{\phi,j}(-n)}$ for all $n \neq 0$.

Lemma 3.5. *For any $j \in \{1, \dots, h\}$,*

$$\sum_{|n| < X} |b_{\phi,j}(n)|^2 \ll X + \tau.$$

Proof. This is a slight generalization of [IS, p. 316], using the discreteness of Γ and the Bruhat decomposition of Γ . □

Now let $E_j(z, s)$ be the Eisenstein series corresponding to κ_j .

Theorem 3.6. *Let $\Phi(s) = (\varphi_{ij}(s))$ be the constant term matrix of $E_i(z, s)$. Then each $E_i(z, s)$ has a meromorphic continuation to the whole s -plane and the column vector $\mathcal{E}(z, s) = {}^t(E_1, \dots, E_h)$ satisfies the functional equation*

$$\mathcal{E}(z, s) = \Phi(s)\mathcal{E}(z, 1 - s).$$

We remark that each $\varphi_{ij}(s)$ can be represented as the ratio of two holomorphic functions of finite order ([K, p. 89] and [Se]), and that on $\text{Re}(s) \geq 1/2$, the functions $\varphi_{ij}(s)$ and E_i have only finite number of poles lying in $(1/2, 1]$. Therefore we may fix a set of

holomorphic functions $f(s), f_1(s), \dots, f_h(s)$ of finite order such that

$$f(s)E_1(z, s) = \sum_{j=1}^h f_j(1-s)E_j(z, 1-s) \tag{3.3}$$

with $f(s)E_1(z, s)$ having a simple pole only at $s = 1$ and $f_j(s)E_j(z, s)$ holomorphic in the region $\text{Re}(s) \geq 1/2$.

Following the standard Rankin–Selberg method, we obtain

Lemma 3.7. *Define $L_j(s, \phi) = \sum_{n=1}^\infty |b_{\phi,j}(n)|^2/n^s$ whenever the series is absolutely convergent. There exist constants $e_j \neq 0$ depending only on the choice of σ_j such that*

$$\int_Y E_j(z, s)|\phi(z)|^2 dA = e_j \cosh \pi \tau F(s, \tau)L_j(s, \phi)$$

where

$$F(s, \tau) = \pi^{-s} \frac{\Gamma^2(s/2)\Gamma(s/2 - i\tau)\Gamma(s/2 + i\tau)}{\Gamma(s)}.$$

From Lemma 3.5, the defining series of $L_j(s, \phi)$ is absolutely convergent and $L_j(s, \phi) \ll |\zeta(s)| + \tau$ for $\text{Re}(s) > 1$. Note that $L_j(s, \phi)$ has a meromorphic continuation to the whole complex plane by Theorem 3.6.

In the rest of the section, we study the growth of $F(\sigma + it, \tau)L_j(\sigma + it, \phi)$ at infinity for fixed $\sigma \geq 1/2$ and “fixed ϕ .” The only aim of this discussion is to take care of the technical convergence issues for the integrations in the proof of Lemma 6.1.

Using the Maass–Selberg relation, we have

Lemma 3.8. *Fix a fundamental domain Y_j for the action of $\sigma_j^{-1}\Gamma\sigma_j$ on \mathbb{H} so that it is contained in a strip of width 1 and contains a strip $Y_j(X)$ determined by $\text{Im}(z) > X$ for sufficiently large $X > 0$. Define E_j^X by*

$$E_j^X(\sigma_j z, s) := \begin{cases} E_j(\sigma_j z, s) - y^s & \text{if } \text{Im}(z) > X, \\ E_j(\sigma_j z, s) & \text{otherwise.} \end{cases}$$

for z in the fundamental domain. Then we have the inner product formula

$$(E_j^X(\sigma_j z, s), E_j^X(\sigma_j z, \bar{s}')) = \frac{X^{s+s'-1}}{s+s'-1} + \varphi_{jj}(s') \frac{X^{s-s'}}{s-s'} - \varphi_{jj}(s) \frac{X^{-s+s'}}{s-s'}$$

where $\text{Re}(s) > 1, \text{Re}(s') > 1$, and $s \neq s'$.

Because of the location of the poles of the Eisenstein series, this inner product formula is valid for all s and s' in the region $\{s \in \mathbb{C} \mid \text{Re}(s) > 1/2, s \notin (1/2, 1]\}$ [Ku, p. 32]. Since $E_j^X(\cdot, s) \in L^2(Y_j)$ [CS, p. 23] for any s in region, taking the limit $s' \rightarrow s$ on the right hand side yields

$$(E_j^X(\sigma_j z, s), E_j^X(\sigma_j z, \bar{s})) = \frac{X^{2s-1}}{2s-1} + 2 \log X \varphi_{jj}(s) + \varphi'_{jj}(s).$$

Let us denote the characteristic function of $Y_j(X)$ by $\chi_X(z)$. Then

$$\begin{aligned} \int_{Y_j} E_j(\sigma_j z, s) |\phi(\sigma_j z)|^2 dA &= \int_{Y_j} (E_j^X(\sigma_j z, s) + \chi_X(z) y^s) |\phi(\sigma_j z)|^2 dA \\ &\ll \|E_j^X(\cdot, s)\|_{L^2(Y_j)} \|\phi\|_{L^4(Y_j)}^2 + \int_{Y_j(X)} y^s |\phi(\sigma_j z)|^2 dA. \end{aligned}$$

Note that ϕ decays exponentially in y and $\varphi_{jj}(\sigma + it)$ has moderate growth at infinity for any fixed $\sigma > 1/2$ [Ku, p. 28]. Then by Lemma 3.8, we see that $F(\sigma + it, \tau) L_j(\sigma + it, \phi)$ grows no faster than $e^{|t|}$ at infinity for any fixed $\sigma > 1/2$. For $\sigma = 1/2$, by [S2] we have

$$\int_{Y_j} E_j(\sigma_j z, 1/2 + it) |\phi(\sigma_j z)|^2 dz \ll_{\phi} |t|^B e^{-\pi|t|/2}$$

for some $B \in \mathbb{R}$. To summarize:

Lemma 3.9. *For any fixed $\sigma \geq 1/2$ and for $|t| > 1$,*

$$F(\sigma + it, \tau) L_j(\sigma + it, \phi) \ll_{\phi} e^{|t|} \quad \text{for } j = 1, \dots, h.$$

4. Complexification

Let D_{ϵ} be an annulus on the complex plane given by $\{q \mid e^{-\epsilon} < |q| < e^{\epsilon}\}$. We first recall a lemma from [TZ].

Lemma 4.1. *Let h be a holomorphic function defined on D_{ϵ} . Normalize h so that*

$$\int_0^1 |h(e^{2\pi i x})|^2 dx = 1.$$

Then for any $\epsilon > 0$ there exists a constant $C(\epsilon) > 0$ such that the number of zeros of h in $D_{\epsilon/2}$, which we denote by $n_{\epsilon}(h)$, is bounded by

$$C(\epsilon) \max_{z \in D_{\epsilon}} \log |h(z)|.$$

Fix γ to be either a geodesic circle or a closed horocycle. Say we are given the Fourier expansion for an eigenfunction on γ so that

$$\phi|_{\gamma}(x) = \sum_{n \in \mathbb{Z}} a_{\phi}(n) e^{inx}.$$

From the asymptotics of special functions and by Lemmas 3.4 and 3.5, in either case we can find some constant $c > 1$ depending only on γ such that

$$|a_{\phi}(n)| \ll \begin{cases} \tau & \text{if } |n| < c\tau, \\ e^{-|n|} & \text{if } |n| \geq c\tau, \end{cases}$$

where the implied constants depend only on c and γ .

Define $\psi(q)$ by the relations $q = e^{ix}$ and $\psi(q) = \phi|_\gamma(x)$. If q is in $D_{1/2}$, then

$$|\psi(q)| \ll \sum_{|n| < c\tau} \tau e^{|n|/2} + \sum_{|n| > c\tau} e^{-|n|/2} \ll e^{c\tau}.$$

Therefore $\psi(q)$ is holomorphic on $D_{1/2}$, and applying Lemma 4.1 to $\psi(q)$, we obtain

Lemma 4.2. *Let Y be a hyperbolic surface which is compact (or has finite volume). Let γ be a fixed geodesic circle (or a fixed closed horocycle). Then*

$$|N(\phi) \cap \gamma| \ll_{Y,\gamma} \log \frac{e^{c\tau}}{\|\phi\|_{L^2(\gamma)}}$$

for some $c > 1$ depending only on Y and γ . Here $\|f\|_{L^2(\gamma)}$ is the restricted L^2 norm of f on γ .

Now, we are going to prove $\|\phi\|_{L^2(\gamma)}$ is not too small, from which Theorem 1.1 will follow.

5. Geodesic circle

Assume $Y = \Gamma \backslash \mathbb{H}$ is compact and γ is a fixed geodesic circle. Without loss of generality, we assume

$$\gamma = \{(r, \theta) \mid r = r_0\}$$

in the geodesic polar coordinates centered at some point $z_0 \in \mathbb{H}$. Let $b_\phi(m)$ be constants such that

$$\phi(r, \theta) = \sum_{m \in \mathbb{Z}} b_\phi(m) C_\tau(|m|, \cosh r) e^{2\pi i m \theta}$$

in the same coordinate system.

Lemma 5.1. *For any $c_1 > 0$, there exists c_2 greater than c_1 such that for all sufficiently large τ , there exists m such that $c_1\tau < |m| < c_2\tau$ and $1/\tau \ll |b_\phi(m)|^2$.*

Proof. If $\alpha = m/\tau \leq c_1$, then by Lemma 2.2,

$$|C_\tau(m, x)|^2 \ll \left(\frac{\zeta - \alpha^2}{x^2 - \alpha^2 - 1} \right)^{1/2} J_m(\tau\sqrt{\zeta})^2.$$

From the asymptotics of Bessel functions ([E]), for any x such that $\sqrt{\zeta} > 2\alpha$,

$$|C_\tau(m, x)|^2 \ll \left(\frac{\zeta - \alpha^2}{x^2 - \alpha^2 - 1} \right)^{1/2} \frac{1}{\tau\sqrt{\zeta}}.$$

Note that, because α is bounded by c_1 , $\sqrt{\zeta} > 2\alpha$ is ensured when x is greater than some sufficiently large constant C_1 depending only on c_1 . From the defining equation of ζ , we observe that $\zeta < x^2 - 1$ whenever $\zeta > \alpha^2$. Since $\frac{\zeta - \alpha^2}{\zeta}$ is an increasing function in ζ , we have

$$|C_\tau(m, x)|^2 \ll \frac{1}{\tau\sqrt{x^2 - 1}} \ll \frac{1}{x\tau}.$$

Now let $R > C_1$ and $a > 1$ be constants which will be determined later. In Lemma 2.2, note that if x is bounded from above by aR , then $\xi > \frac{1}{a^2R^2-1}$. Pick c_2 large enough so that

$$\frac{1}{a^2R^2-1} > \frac{1}{c_2^2} > 0$$

and

$$\int_{1/c_2^2}^{1/(a^2R^2-1)} \frac{(s-1/c_2^2)^{1/2}}{2s(1+s)^{1/2}} ds > \frac{1}{c_2} \int_1^4 \frac{(s-1)^{1/2}}{2s} ds.$$

Then it follows from the defining equation of η that $\eta > 4\beta^2$ for any $\beta \leq 1/c_2$. We also deduce from the same equation that there exists some constant $\epsilon > 0$ depending only on a and R such that $\eta > \epsilon$. Therefore, if $x < aR$ and $m \geq c_2\tau$, then the asymptotics of $\tilde{K}_{i\tau}(x)$ implies

$$|C_\tau(m, x)|^2 \ll e^{-Bm}$$

for some constant $B > 0$, depending only on a and R .

Now set $|d_\phi(m)|^2 = |b_\phi(m)|^2 + |b_\phi(-m)|^2$ for $m > 0$, and $d_\phi(0) = b_\phi(0)$. We have

$$\begin{aligned} B(a, R) &:= \int_{\cosh^{-1} R}^{\cosh^{-1} aR} \int_0^{2\pi} |\phi(r, \theta)|^2 \sinh r \, d\theta \, dr \\ &= \sum_{m=0}^{\infty} |d_\phi(m)|^2 \int_{\cosh^{-1} R}^{\cosh^{-1} aR} |C_\tau(m, \cosh r)|^2 \sinh r \, dr \\ &= \sum_{m=0}^{\infty} |d_\phi(m)|^2 \int_R^{aR} |C_\tau(m, x)|^2 \, dx. \end{aligned}$$

We split this summation into three pieces and apply the above estimations to obtain

$$B(a, R) \ll \frac{\log a}{\tau} \sum_{m \leq c_1\tau} |d_\phi(m)|^2 + R(a-1) \sum_{c_1\tau < m < c_2\tau} |d_\phi(m)|^2 + \sum_{c_2\tau \leq m} |d_\phi(m)|^2 e^{-Bm},$$

where we used the trivial bound $C_\tau(m, x) \ll 1$ for the second summation. Applying Lemma 3.4, we see that

$$B(a, R) \ll \log a + R(a-1) \sum_{c_1\tau < m < c_2\tau} |d_\phi(m)|^2 + \sum_{c_2\tau \leq m} m e^{-Bm}.$$

On the other hand, by Lemma 3.3, we have

$$B(a, R) > \frac{2\pi}{\text{Area}(\mathcal{F})} (\cosh(\cosh^{-1}(aR) - d) - \cosh(\cosh^{-1}(R) + d)) \gg aR.$$

Therefore by choosing a and R large enough and then τ large, we find $b_\phi(m)$ with the desired property. □

Note that Parseval’s theorem implies

$$\|\phi\|_{L^2(\gamma)}^2 = \sum |b_\phi(m)|^2 |C_\tau(m, \cosh r_0)|^2.$$

Hence by Lemmas 2.3 and 5.1, we obtain

$$\|\phi\|_{L^2(\gamma)} \gg_{Y,\gamma} e^{-B\tau}$$

for some constant $B > 0$. Therefore from Lemma 4.2, we deduce the first case of Theorem 1.1.

6. Closed horocycle

Let $Y = \Gamma \backslash \mathbb{H}$ and let γ be a fixed horocycle. With notation as in Section 3.3, we assume without loss of generality that Γ is reduced at ∞ (that is, $\kappa_1 = \infty$ and $\sigma_1 = \text{Id}$) and that γ is given by

$$\gamma = \{x + iy \mid y = y_0\}$$

for some $y_0 > 0$. Set $a_\phi(n) = b_{\phi,1}(n)$.

6.1. Approximate functional equation

First note that Theorem 3.6, Lemma 3.7, and equation (3.3) imply

$$f(s)F(s, \tau)L_1(s, \phi) = \sum_{j=1}^h e_j f_j(1-s)F(1-s, \tau)L_j(1-s, \phi). \tag{6.1}$$

By our choice of f, f_1, \dots, f_h , on $\text{Re}(s) \geq 1/2$, $f(s)L_1(s, \phi)$ has a simple pole at $s = 1$ and $f_j(s)L_j(s, \phi)$ is holomorphic. With this functional equation, we introduce a variation of the approximate functional equation ([IK, p. 98]).

Lemma 6.1. *For a sufficiently large integer m , define ψ by*

$$\psi(X) = \frac{1}{2\pi i} \int_{(2)} f(s)F(s, \tau)s \left(1 + \frac{s^2}{4\tau^2}\right) X^{-s} e^{-s^{4m}} ds.$$

Then for $X > 0$, we have

$$\sum_{n=1}^{\infty} |a_\phi(n)|^2 \psi\left(\frac{n}{X}\right) = \left(1 + \frac{1}{4\tau^2}\right) \frac{c}{\cosh \pi \tau} X + \frac{\tau^2}{\cosh \pi \tau} O\left(\frac{1}{X}\right),$$

for some constant c depending only on $f(s)$.

Proof. For simplicity, set

$$\begin{aligned} \Lambda(s, \phi) &= f(s)F(s, \tau)L_1(s, \phi), \\ \Lambda_j(s, \phi) &= e_j f_j(s)F(s, \tau)L_j(s, \phi), \\ H_m(s, \tau) &= s \left(1 + \frac{s^2}{4\tau^2}\right) e^{-s^{4m}}. \end{aligned}$$

By shifting the contour and using the functional equation (6.1), we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} |a_{\phi}(n)|^2 \psi\left(\frac{n}{X}\right) &= \frac{1}{2\pi i} \int_{(2)} \Lambda(s) H_m(s, \tau) X^s ds \\ &= \left(1 + \frac{1}{4\tau^2}\right) \frac{c}{\cosh \pi \tau} X + \frac{1}{2\pi i} \int_{(1/2)} \Lambda(s) H_m(s, \tau) X^s ds \\ &= \left(1 + \frac{1}{4\tau^2}\right) \frac{c}{\cosh \pi \tau} X + \sum_{j=1}^h \frac{1}{2\pi i} \int_{(1/2)} \Lambda_j(s) H_m(1-s, \tau) X^{1-s} ds \\ &= \left(1 + \frac{1}{4\tau^2}\right) \frac{c}{\cosh \pi \tau} X + \sum_{j=1}^h \frac{1}{2\pi i} \int_{(2)} \Lambda_j(s) H_m(1-s, \tau) X^{1-s} ds \\ &= \left(1 + \frac{1}{4\tau^2}\right) \frac{c}{\cosh \pi \tau} X + R(\phi) \end{aligned}$$

for some constant c depending only on $f(s)$. From

$$L_j(2 + it, \phi) \ll \tau, \quad F(2 + it, \tau) \ll \frac{\tau}{\cosh \pi \tau} e^t,$$

we get $R(\phi) = \frac{\tau^2}{\cosh \pi \tau} O\left(\frac{1}{X}\right)$.

We complete the proof by observing that we can choose m large enough to make all the summations and integrations converge absolutely, since every holomorphic function that appears in the computation is of finite order (by our choice of f, f_1, \dots, f_h and Lemma 3.9). \square

Lemma 6.2. For $X > 1$, we have

$$\psi(X) \ll \frac{1}{\cosh \pi \tau} \frac{1}{X \log^2 X}$$

and for $X > 0$, we have

$$\psi(X) \ll \frac{X}{\cosh \pi \tau}.$$

Proof. The first assertion follows after shifting the contour to $\text{Re}(s) = 1$ and using integration by parts. For the second assertion, note that $F(s, \tau)s(1 + \frac{s^2}{4\tau^2})$ has no pole for $\text{Re}(s) > -2$. Now shift the contour to $\text{Re}(s) = -1$.

In both cases we have used

$$\frac{\partial^k}{\partial t^k} (F(\sigma + it, \tau)) \ll_{k,\sigma} \frac{\tau^{\max\{\sigma-1,0\}}}{\cosh \pi \tau} e^t \quad (\sigma > -2, k \geq 0),$$

which follows from Stirling's formula. \square

6.2. Lower bound for $\|\phi\|_{L^2(\gamma)}$

Lemma 6.3. For any $c_1 > 0$, there exists c_2 greater than c_1 such that for all sufficiently large τ , we can find some n with $c_1\tau < n < c_2\tau$ which satisfies $1 \ll |a_{\phi}(n)|^2$.

Proof. From Lemma 6.1, we have

$$\sum_{n=1}^{\infty} |a_{\phi}(n)|^2 \psi\left(\frac{n}{X}\right) = \left(1 + \frac{1}{4\tau^2}\right) \frac{c}{2 \cosh \pi \tau} X + \frac{\tau^2}{\cosh \pi \tau} O\left(\frac{1}{X}\right).$$

Let $c_2 > 1$ and β be the constants to be determined such that $0 < c_1 < \beta < c_2$. Put $X = \beta\tau$. Split the summation into three ranges:

$$\sum_{n=1}^{\infty} |a_{\phi}(n)|^2 \psi\left(\frac{n}{X}\right) = \sum_{n < c_1 \tau} + \sum_{c_1 \tau \leq n \leq c_2 \tau} + \sum_{c_2 \tau < n}.$$

Call these sums I_1 , I_2 , and I_3 respectively. For I_1 , we use Lemmas 3.5 and 6.2 to obtain

$$|I_1| \ll \frac{c_1(c_1 + 1)}{\beta} \frac{\tau}{\cosh \pi \tau}.$$

For I_2 , we use Lemma 6.2 to obtain

$$|I_2| \ll \frac{1}{\cosh \pi \tau} \frac{c_2}{c_1} \sum_{c_1 \tau \leq n \leq c_2 \tau} |a_{\phi}(n)|^2.$$

For I_3 , we use Lemmas 3.5 and 6.2 and summation by parts to obtain

$$|I_3| \ll \sum_{c_2 \tau < n} \frac{|a_{\phi}(n)|^2}{\cosh \pi \tau} \left(\frac{n}{X} \log^2 \frac{n}{X}\right)^{-1} \ll \frac{\tau}{\cosh \pi \tau} \frac{\beta}{\log c_2 - \log \beta}.$$

These estimates imply that there exists a fixed constant $C > 0$ such that

$$\sum_{c_1 \tau \leq n \leq c_2 \tau} |a_{\phi}(n)|^2 > \frac{c_1}{c_2} \left(\frac{\beta}{C} - \frac{1}{\beta} - \frac{c_1(c_1 + 1)}{\beta} - \frac{\beta}{\log c_2 - \log \beta}\right) \tau$$

uniformly in c_1, c_2, β , and τ . Therefore for any given $c_1 > 0$, we can choose β and then c_2 so that

$$\sum_{c_1 \tau \leq n \leq c_2 \tau} |a_{\phi}(n)|^2 \gg_{c_1} \tau$$

uniformly in τ . □

By Parseval’s theorem, we have

$$\|\phi\|_{L^2(\gamma)}^2 = \sum_{n \neq 0} |a_{\phi}(n)|^2 y_0 \tilde{K}_{i\tau}(2\pi|n|y_0)^2.$$

Hence by Lemmas 2.3 and 6.3, we find a constant $B > 0$ such that

$$\|\phi\|_{L^2(\gamma)} \gg_{Y, \gamma} e^{-B\tau}.$$

Therefore, from Lemma 4.2, we deduce the second case of Theorem 1.1.

7. Appendix

7.1. Equidistribution of the geodesic circles

Let Y be a compact hyperbolic surface and fix a point $z_0 \in Y$. In this section we prove that the geodesic circle centered at z_0 becomes equidistributed on Y when we increase the radius of the circle. Accordingly, for any fixed continuous function f on Y , we prove

$$\lim_{r \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} f(\theta, r) d\theta = \int_Y f dA \quad (7.1)$$

where (r, θ) are the geodesic polar coordinates centered at z_0 . Since f is continuous and Y is compact, f is a bounded function, hence in $L^2(Y)$. Therefore by the spectral expansion of f in terms of the Laplacian eigenfunctions on Y , it is sufficient to prove (7.1) for the eigenfunctions.

Assume $f \in L^2(Y)$ satisfies

$$\Delta f = s(1 + s)f$$

with $s(1 + s) < 0$. Then by Lemma 3.1,

$$\frac{1}{2\pi} \int_0^{2\pi} f(\theta, r) d\theta = c_f P_s^0(\cosh r) = c_f P_s(\cosh r)$$

for some constant c_f , and by the orthogonality of the eigenfunctions,

$$\int_Y f dA = 0.$$

Therefore (7.1) is equivalent to

$$\lim_{r \rightarrow \infty} P_s(\cosh r) = 0.$$

When $s = -1/2 + i\tau$ for some $\tau > 0$, this follows from Lemma 2.2, and when $-1 < s < 0$, we use the integral representation ([L, p. 172])

$$P_s(\cosh r) = \frac{1}{\pi} \int_0^\pi (\cosh r + \sinh r \cos \theta)^s d\theta.$$

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