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Malte Witte

On a noncommutative Iwasawa main conjecture for varieties over finite fields

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Abstract. We formulate and prove an analogue of the noncommutative Iwasawa main conjecture for ℓ-adic Lie extensions of a separated scheme X of finite type over a finite field of characteristic prime to ℓ.

1. Introduction

In [CFK+05], Coates, Fukaya, Kato, Sujatha and Venjakob formulate a noncommutative Iwasawa main conjecture for ℓ-adic Lie extensions of number fields. Other, partly more general versions are formulated in [HK02], [RW04], and [FK06]. Following the approach of [FK06], we formulate and prove below an analogous statement for ℓ-adic Lie extensions of a separated scheme X of finite type over a finite field $\mathbb{F}_q$ with q elements, where ℓ does not divide q.

Assume for the moment that X is geometrically connected and let G be a factor group of the fundamental group of X such that $G \cong H \rtimes \Gamma$ where H is a compact ℓ-adic Lie group and $\Gamma = \mathrm{Gal}(\mathbb{F}_{q^{\ell^\infty}}/\mathbb{F}_q) \cong \mathbb{Z}_\ell$. We write

$$\mathbb{Z}_\ell[[G]] = \varprojlim \mathbb{Z}_\ell[G/U]$$

for the Iwasawa algebra of G. Let

$$S = \{f \in \mathbb{Z}_\ell[[G]] \colon \mathbb{Z}_\ell[[G]]/\mathbb{Z}_\ell[[G]]f \text{ is finitely generated as a } \mathbb{Z}_\ell[[H]]\text{-module}\}$$

denote Venjakob's canonical Ore set and write $\mathbb{Z}_\ell[[G]]_S$ for the localisation of $\mathbb{Z}_\ell[[G]]$ at S. We turn $\mathbb{Z}_\ell[[G]]$ into a smooth $\mathbb{Z}_\ell[[G]]$-sheaf $\mathcal{M}(G)$ on X by letting the fundamental group of X act contragrediently on $\mathbb{Z}_\ell[[G]]$.

Recall that there exists an exact localisation sequence of algebraic K-groups

$$\mathrm{K}_1(\mathbb{Z}_\ell[[G]]) \to \mathrm{K}_1(\mathbb{Z}_\ell[[G]]_S) \xrightarrow{d} \mathrm{K}_0(\mathbb{Z}_\ell[[G]], \mathbb{Z}_\ell[[G]]_S) \to 0.$$

Any endomorphism of perfect complexes of $\mathbb{Z}_\ell[[G]]_S$-modules which is a quasi-isomorphism gives rise to an element in the group $\mathrm{K}_1(\mathbb{Z}_\ell[[G]]_S)$. A system of generators for

M. Witte: Ruprecht-Karls-Universität Heidelberg, Mathematisches Institut, Im Neuenheimer Feld 288, D-69120 Heidelberg, Germany; e-mail: witte@mathi.uni-heidelberg.de

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the relative K-group $\mathrm{K}_0(\mathbb{Z}_\ell[[G]], \mathbb{Z}_\ell[[G]]_S)$ is given by perfect complexes of $\mathbb{Z}_\ell[[G]]$-modules whose cohomology groups are S-torsion.

For every continuous $\mathbb{Z}_\ell$-representation ρ of G, there exists a homomorphism

$$\rho\colon \mathrm{K}_1(\mathbb{Z}_\ell[[G]]_S) \to Q(\mathbb{Z}_\ell[[\Gamma]])^\times$$

into the units $Q(\mathbb{Z}_\ell[[\Gamma]])^\times$ of the field of fractions of $\mathbb{Z}_\ell[[\Gamma]]$. It is induced by sending $g \in G$ to $\det([g]\rho(g)^{-1})$, with $[g]$ denoting the image of g in Γ. On the other hand, ρ gives rise to a flat and smooth $\mathbb{Z}_\ell$-sheaf $\mathcal{M}(\rho)$ on X.

Let $\mathrm{R}\,\Gamma_c(X, \mathcal{F})$ and $\mathrm{R}\,\Gamma_c(\overline{X}, \mathcal{F})$ be the compact cohomology of a flat constructible $\mathbb{Z}_\ell$-sheaf $\mathcal{F}$ on X and on the base change $\overline{X}$ of X to the algebraic closure of $\mathbb{F}_q$, respectively. Furthermore, let $\mathfrak{F}_{\mathbb{F}_q} \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ denote the geometric Frobenius. The Grothendieck trace formula

$$L(\mathcal{F}, T) = \prod_{i\in\mathbb{Z}} \det(1 - \mathfrak{F}_{\mathbb{F}_q} T : \mathrm{H}^i_c(\overline{X}, \mathcal{F}))^{(-1)^{i+1}}$$

implies that the L-function $L(\mathcal{F}, T)$ of $\mathcal{F}$ is in fact a rational function.

The following theorem is our analogue of the noncommutative Iwasawa main conjecture in the special situation described above:

Theorem 1.1.

(1) $\mathrm{R}\,\Gamma_c(X, \mathcal{M}(G) \otimes_{\mathbb{Z}_\ell} \mathcal{F})$ *is a perfect complex of* $\mathbb{Z}_\ell[[G]]$*-modules whose cohomology groups are* S*-torsion. Moreover, the endomorphism* $\mathrm{id} - \mathfrak{F}_{\mathbb{F}_q}$ *of the complex* $\mathrm{R}\,\Gamma_c(\overline{X}, \mathcal{M}(G)_S \otimes_{\mathbb{Z}_\ell} \mathcal{F})$ *is a quasi-isomorphism, and hence it gives rise to an element*

$$\mathcal{L}_G(X/\mathbb{F}_q, \mathcal{F}) = [\mathrm{id} - \mathfrak{F}_{\mathbb{F}_q}]^{-1} \in \mathrm{K}_1(\mathbb{Z}_\ell[[G]]_S).$$

(2) $d\mathcal{L}_G(X/\mathbb{F}_q, \mathcal{F}) = [\mathrm{R}\,\Gamma_c(X, \mathcal{M}(G) \otimes_{\mathbb{Z}_\ell} \mathcal{F})]^{-1}$.

(3) *Assume that* ρ *is a continuous* $\mathbb{Z}_\ell$*-representation of* G*. Then*

$$\rho(\mathcal{L}_G(X/\mathbb{F}_q, \mathcal{F})) = L(\mathcal{M}(\rho) \otimes_{\mathbb{Z}_\ell} \mathcal{F}, [\mathfrak{F}_{\mathbb{F}_q}]^{-1})$$

in $Q(\mathbb{Z}_\ell[[\Gamma]])^\times$.

Theorem 1.1(3) implies the following interpolation property of $\mathcal{L}_G(X/\mathbb{F}_q, \mathcal{F})$ with respect to special values of L-functions: Let ϵ denote the cyclotomic character and decompose $\epsilon = \epsilon_f \times \epsilon_\infty$ according to the decomposition $\mathrm{Gal}(\mathbb{F}_q(\zeta_{\ell^\infty})/\mathbb{F}_q) = \Delta \times \Gamma$. Then for every $n \in \mathbb{Z}$, the leading term at $T = 0$ of the image of $\epsilon_\infty^n \rho(\mathcal{L}_G(X/\mathbb{F}_q, \mathcal{F}))$ under the isomorphism

$$Q(\mathbb{Z}_\ell[[\Gamma]]) \to Q(\mathbb{Z}_\ell[[T]]), \qquad [\mathfrak{F}_{\mathbb{F}_q}]^{-1} \mapsto T + 1,$$

agrees with the leading term of $L(\mathcal{M}(\epsilon_f^{-n}\rho) \otimes_{\mathbb{Z}_\ell} \mathcal{F}, q^{-n}T)$ at $T = 1$.

In Section 8 we will prove a version of the noncommutative Iwasawa main conjecture that is in several aspects more general than Theorem 1.1:

The above theorem is limited to geometrically connected schemes X. We overcome this limitation by allowing G to be the covering group of any suitable principal covering of X. Moreover, we will only require G to be a virtual pro-ℓ-group, which is a slightly weaker condition than being an ℓ-adic Lie group.

The ring $\mathbb{Z}_\ell$ will be replaced by more general rings of scalars. A good class of rings to work with is the class of adic $\mathbb{Z}_\ell$-algebras, which is also used in [FK06]. It contains all finite extensions of $\mathbb{Z}_\ell$ and is closed under forming profinite group rings with respect to virtual pro-ℓ-groups. Furthermore, it is more natural to state and prove the conjecture not only for flat and constructible sheaves, but to extend it to perfect complexes.

Beyond proving the main conjecture, we will study the transformation properties of $\mathcal{L}_G(X/\mathbb{F}_q, \mathcal{F})$ under scalar extensions and changes of the principal covering of X. Note that parts (1) and (2) of the above theorem and its generalisation remain true in the case $\ell \mid q$, but our element $\mathcal{L}_G(X/\mathbb{F}_q, \mathcal{F})$ does not satisfy (3).

The K-theoretical formulation of the conjecture will be based on Waldhausen's construction of higher K-groups [Wal85] and on the construction of the 1-type of the associated topological spectrum given by F. Muro and A. Tonks [MT07]. The localisation sequence may then be viewed as a consequence of Waldhausen's localisation theorem. In the appendix, we derive an explicit description of its rightmost connecting homomorphism. This allows us to obtain a particularly simple description of the elements of K-groups appearing in the conjecture.

In order to use Waldhausen's formalism, we need to introduce suitable Waldhausen categories replacing the usual triangulated categories of perfect complexes of adic sheaves. We also need to define Waldhausen exact functors calculating the classical derived functors such as higher derived images with proper support and derived tensor products. For this, we use the approach developed in [Wit08]. All necessary transformation properties for elements in K-groups can then be read off directly from the underlying transformation properties of the Waldhausen exact functors.

After these preparations, the proof of part (1) of the above theorem can be reduced to the case that $G \cong H \times \mathbb{Z}_\ell$ with H finite. In this situation the Hochschild–Serre spectral sequence implies that

$$H^i_c(X, \mathcal{M}(G) \otimes_{\mathbb{Z}_\ell} \mathcal{F}) \cong \varprojlim_n H^{i-1}_c(\overline{Y}, \mathcal{F})/(1 - \mathfrak{F}^{\ell^n}_{\mathbb{F}_q}) H^{i-1}_c(\overline{Y}, \mathcal{F}),$$

where Y is the Galois covering of X corresponding to H. Since $H^{i-1}_c(\overline{Y}, \mathcal{F})$ is finitely generated as $\mathbb{Z}_\ell$-module, it follows that $H^i_c(X, \mathcal{M}(G) \otimes_{\mathbb{Z}_\ell} \mathcal{F})$ is S-torsion. Part (2) is a formal consequence of part (1) and our description of the boundary homomorphism in the localisation sequence. Finally, part (3) may be reduced to the classical Grothendieck trace formula and the fact that the evaluation homomorphism

$$\mathbb{Z}_\ell[T] \to \mathbb{Z}_\ell[[\Gamma]], \qquad T \mapsto [\mathfrak{F}_{\mathbb{F}_q}]^{-1},$$

maps $1 + T\mathbb{Z}_\ell[T]$ to S.

The article is structured as follows. In Section 2 we recall the necessary terminology of principal coverings. Section 3 contains a brief account on adic rings. Furthermore, we give a convenient construction of a Waldhausen category calculating their K-theory. In Section 4 we introduce the Waldhausen categories used to calculate $\mathrm{K}_1(\mathbb{Z}_\ell[[G]]_S)$ and $\mathrm{K}_0(\mathbb{Z}_\ell[[G]], \mathbb{Z}_\ell[[G]]_S)$ in the localisation sequence. Our construction of the Waldhausen category of perfect complexes of adic sheaves is recalled in Section 5. In Section 6 we define an analogue of $\mathcal{M}(G)$ for arbitrary principal coverings and study its transformation properties. Section 7 treats the special case $G = \Gamma$. The precise formulations and the proofs of our main results are given in Section 8. In the appendix we derive an explicit description of the rightmost connecting homomorphism of Waldhausen's localisation sequence under the same assumptions under which the localisation sequence is known to exist.

Using the results of this article and of [EK01], D. Burns [Bur11] has recently constructed in the case $\ell \mid q$ a modification of $\mathcal{L}_G(X/\mathbb{F}_q, \mathcal{F})$ which has the right interpolation property. The results of this article together with the descent formalism developed by Venjakob and Burns in [BV11] can also be used to generalise the proof of an analogue of the equivariant Tamagawa number conjecture given in [Bur04]. An analogue of the noncommutative Iwasawa main conjecture for elliptic curves over function fields in the case that ℓ is equal to the characteristic p of the field in question has been considered in [OT09]. F. Trihan and D. Vauclair have announced proofs of more general main conjectures in this case. We also point out that tremendous progress towards a proof of the noncommutative Iwasawa main conjecture for totally real fields has been achieved in [Kat06], [Har10], [Kak11], [Kak13], and in [RW11].

2. Principal coverings

In this section, we recall the concept of principal coverings from [Gro03, Def. 2.8]. More precisely, we shall consider pro-objects over the category of finite principal coverings defined there.

If A is either a commutative ring or a scheme, we let $\mathbf{Sch}_A$ denote the category of schemes of finite type over A. If G is any profinite group, we write $\mathfrak{N}_G$ for the set of open normal subgroups of G, partially ordered by inclusion.

Definition 2.1. Let G be a profinite group and X a locally noetherian scheme. A *principal covering* $(f\colon Y \to X, G)$ of X with *Galois group* G is an inverse system of X-schemes

$$(f_U\colon Y_U \to X)_{U\in\mathfrak{N}_G},$$

together with a right operation of G on the system such that for any $U \in \mathfrak{N}_G$,

(1) f_U is finite, étale, and surjective,
(2) the operation of U on the scheme Y_U is trivial,
(3) the natural morphism

$$\coprod_{\sigma\in G/U} Y_U \xrightarrow{\coprod \mathrm{id}_{Y_U}\times\sigma} Y_U \times_X Y_U$$

is an isomorphism.

For any profinite group G and a locally noetherian scheme X, there is always the trivial principal covering $(X \times G \to X, G)$ given by

$$(X\times G)_U = \coprod_{\sigma\in G/U} X$$

for any open normal subgroup U of G. If X is connected and x is a geometric point of X, then there exists a distinguished principal covering $(f\colon \widetilde{X} \to X, \pi_1^{\text{ét}}(X,x))$ whose Galois group is the étale fundamental group $\pi_1^{\text{ét}}(X,x)$ of X. It is characterised by the property that any compatible system of geometric points $(x \to \widetilde{X}_U)_{U \in \mathfrak{N}_{\pi_1^{\text{ét}}(X,x)}}$ over the base point induces an isomorphism

$$\varinjlim_{U \in \mathfrak{N}_{\pi_1^{\text{ét}}(X,x)}} \operatorname{Hom}_X(\widetilde{X}_U, \cdot) \to \operatorname{Hom}_X(x, \cdot)$$

of functors from the category of finite étale X-scheme to sets. Moreover, the schemes $\widetilde{X}_U$ are connected.

If $(f\colon Y \to X, G)$ is a principal covering and $X' \to X$ is a locally noetherian X-scheme, then we write

$$(f \times_X X'\colon Y \times_X X' \to X', G)$$

for the principal covering of X' obtained by base change, i.e. $(Y \times_X X')_U = Y_U \times_X X'$.

If V is an open (not necessarily normal) subgroup of G and $U \subset V$ is an open normal subgroup of G then the quotient scheme

$$Y_V := Y_U/(V/U)$$

exists and is (up to canonical isomorphism) independent of the choice of U. Moreover, we obtain a principal covering

$$(f^V\colon Y \to Y_V, V)$$

of Y_V given by the inverse system $(f_U^V\colon Y_U \to Y_V)_{U \in \mathfrak{N}_V}$, where f_U^V denotes the canonical projection map.

If H is a closed normal subgroup of G and $\alpha\colon G \to G/H$ the natural projection, we define

$$(f_H\colon Y_H \to X, G/H)$$

to be the principal covering given by the inverse system

$$(f_{\alpha^{-1}(U)}\colon Y_{\alpha^{-1}(U)} \to X)_{U \in \mathfrak{N}_{G/H}}.$$

Definition 2.2. A *morphism*

$$a\colon (Y \to X, G) \to (Y' \to X, G')$$

of principal coverings of X is a continuous group homomorphism $\alpha\colon G \to G'$ together with a G-equivariant morphism of inverse systems

$$a\colon (Y_{\alpha^{-1}(U)} \to X)_{U \in \mathfrak{N}_{G'}} \to (Y'_U \to X)_{U \in \mathfrak{N}_{G'}}.$$

Lemma 2.3. *A morphism*

$$a\colon (Y \to X, G) \to (Y' \to X, G')$$

of principal coverings of X is an isomorphism if and only if the associated homomorphism of groups $\alpha\colon G \to G'$ is an isomorphism.

Proof. We may assume that $G = G'$ and that α is the identity. We may then reduce to the case that G is finite and that X is the spectrum of a local ring A. Then Y and Y' are the spectra of finite flat A-algebras B and B', respectively. The rank of both B and B' as free A-modules is equal to the cardinality of G. Since $a\colon Y \to Y'$ is finite étale, it follows that B is a finitely generated, projective B'-module of constant rank 1. Hence, $B \cong B'$. □

In the following, we will impose further restrictions on the group G.

Definition 2.4. Let ℓ be a prime. We call a profinite group G a *virtual pro-ℓ-group* if its ℓ-Sylow subgroups are of finite index. Without further comment, we require all virtual pro-ℓ-groups appearing in this article to be topologically finitely generated.

A principal covering is called a *virtual pro-ℓ-covering* if its Galois group is a virtual pro-ℓ-group.

Note that all compact ℓ-adic Lie groups are virtual pro-ℓ-groups in the above sense. The following example of a virtual pro-ℓ-covering will play an important role: Let $\mathbb{F}_q$ be a finite field with q elements. We fix an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. Let ℓ be any prime, let k be an integer prime to ℓ, and set

$$\mathbb{F}_{q^{k\ell^\infty}} = \bigcup_{n\geq 0} \mathbb{F}_{q^{k\ell^n}}$$

(as a subfield of $\overline{\mathbb{F}}_q$). This gives rise to a principal covering

$$(\operatorname{Spec} \mathbb{F}_{q^{k\ell^\infty}} \to \operatorname{Spec} \mathbb{F}_q, \Gamma_{k\ell^\infty})$$

with Galois group $\Gamma_{k\ell^\infty} \cong \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}_\ell$.

Definition 2.5. Let X be a scheme of finite type over the finite field $\mathbb{F}_q$ and set $X_{k\ell^\infty} = X \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \mathbb{F}_{q^{k\ell^\infty}}$. The principal covering

$$(X_{k\ell^\infty} \to X, \Gamma_{k\ell^\infty})$$

will be called the *cyclotomic $\Gamma_{k\ell^\infty}$-covering* of X.

We point out that with this definition, $X_{k\ell^\infty}$ is not necessarily connected, even if X itself is connected.

Definition 2.6. Let X be a scheme of finite type over a finite field $\mathbb{F}$, and ℓ an arbitrary prime number. We call a principal covering $(f\colon Y \to X, G)$ *admissible* if

(1) $G \cong H \rtimes \Gamma_{\ell^\infty}$ is the semidirect product of a closed normal virtual pro-ℓ-subgroup H and the group Γ_{ℓ^∞},
(2) $(f_H\colon Y_H \to X, \Gamma_{\ell^\infty})$ is isomorphic to the cyclotomic Γ_{ℓ^∞}-covering of X.

Note that the semidirect product $H \rtimes \Gamma_{\ell^\infty}$ of a virtual pro-ℓ-group H and $\Gamma_{\ell^\infty} \cong \mathbb{Z}_\ell$ is itself a virtual pro-ℓ-group.

3. The K-theory of adic rings

In this section, we recall some facts about adic rings and their K-theory. Properties of these rings have previously been studied in [War93] and in [FK06]. We refer to [Wit08, Section 5.1–2] for a more complete treatment.

All rings will be associative with unity, but not necessarily commutative. For any ring R, we let

$$\mathrm{Jac}(R) = \{x \in R \colon 1 - rx \text{ is invertible for any } r \in R\}$$

denote the *Jacobson radical* of R. The ring R is called *semilocal* if $R/\mathrm{Jac}(R)$ is artinian.

Definition 3.1. A ring Λ is called an *adic ring* if for each integer $n \geq 1$, the ideal $\mathrm{Jac}(\Lambda)^n$ is of finite index in Λ and

$$\Lambda = \varprojlim_{n} \Lambda/\mathrm{Jac}(\Lambda)^n.$$

Note that Λ is adic precisely if it is compact, semilocal and the Jacobson radical is finitely generated [War93, Theorem 36.39].

For any adic ring Λ we denote by $\mathfrak{I}_\Lambda$ the set of open two-sided ideals of Λ, partially ordered by inclusion.

Proposition 3.2. *Let Λ be an adic $\mathbb{Z}_\ell$-algebra and let G be a virtual pro-ℓ-group. Then the profinite group ring*

$$\Lambda[[G]] = \varprojlim_{J \in \mathfrak{I}_\Lambda} \varprojlim_{U \in \mathfrak{N}_G} \Lambda/J[G/U]$$

is an adic $\mathbb{Z}_\ell$-algebra. Moreover, if U is any open normal pro-ℓ-subgroup of G, then the kernel of

$$\Lambda[[G]] \to \Lambda/\mathrm{Jac}(\Lambda)[G/U]$$

is contained in $\mathrm{Jac}(\Lambda[[G]])$.

Proof. We begin by proving the assertion about the Jacobson radical. Clearly, $\Lambda[[G]]$ is a compact ring. Hence,

$$\mathrm{Jac}(\Lambda[[G]]) = \varprojlim_{V \in \mathfrak{N}_G,\, n \geq 0} \mathrm{Jac}(\Lambda/\mathrm{Jac}(\Lambda)^n[G/V]).$$

We may thus assume that Λ and G are finite. A direct calculation shows that the two-sided ideal $\mathrm{Jac}(\Lambda)\Lambda[G]$ is nilpotent, and therefore it is contained in the Jacobson radical of $\Lambda[G]$. Consequently, we may assume that $n = 1$, i.e. Λ is a finite product of full matrix rings over finite fields of characteristic ℓ. Considering each factor of Λ separately and using that

$$\mathrm{Jac}(M_{k,k}(\Lambda)[G]) = \mathrm{Jac}(M_{k,k}(\Lambda[G])) = M_{k,k}(\mathrm{Jac}(\Lambda[G])),$$

we can restrict to Λ itself being a finite field of characteristic ℓ. We are thus reduced to the classical case treated in [CR90a, Prop. 5.26].

Hence, returning to the general situation, we find an open normal pro-ℓ subgroup U of G such that the kernel of

$$\Lambda[[G]] \to \Lambda/\mathrm{Jac}(\Lambda)[G/U]$$

is contained in $\mathrm{Jac}(\Lambda[[G]])$. This kernel is an open ideal of $\Lambda[[G]]$ generated by a system of generators of $\mathrm{Jac}(\Lambda)$ over Λ together with the elements $1 - u_i$ for a system of topological generators (u_i) of U. Thus, $\mathrm{Jac}(\Lambda[[G]])$ is also open and finitely generated. Therefore, we conclude that $\Lambda[[G]]$ is an adic ring. □

We will now examine the algebraic K-groups of Λ. For this, we will follow Waldhausen's approach [Wal85]. Recall that a *Waldhausen category* is a category $\mathbf{W}$ with zero object together with two classes of morphisms, called *cofibrations* and *weak equivalences*, that satisfy a certain set of axioms. Using Waldhausen's S-construction one can associate to each such category in a functorial manner a connected pointed topological space $X(\mathbf{W})$. By definition, the n-th K-group of $\mathbf{W}$ is the $(n+1)$-th homotopy group of this space:

$$\mathrm{K}_n(\mathbf{W}) = \pi_{n+1}(X(\mathbf{W})).$$

Waldhausen exact functors are functors that respect the additional structure of a Waldhausen category. Each such functor $F\colon \mathbf{W} \to \mathbf{W}'$ induces a continuous map between the associated topological spaces, and hence a homomorphism

$$\mathrm{K}_n(F)\colon \mathrm{K}_n(\mathbf{W}) \to \mathrm{K}_n(\mathbf{W}').$$

We refer to [TT90] for a more thorough introduction to the topic.

Let R be any ring. Recall that a complex $M^\bullet$ of left R-modules is called *strictly bounded* if there exists a number k such that $M^n = 0$ for $n < -k$ and for $n > k$. The complex $M^\bullet$ is called *strictly perfect* if it is strictly bounded and for every n, the module M^n is finitely generated and projective. The complex $M^\bullet$ is called *perfect* if it quasi-isomorphic to a strictly perfect complex in the category of all complexes of left R-modules.

Definition 3.3. We let $\mathbf{SP}(R)$ denote the Waldhausen category of strictly perfect complexes and $\mathbf{P}(R)$ the Waldhausen category of perfect complexes: The morphisms are morphisms of complexes $f\colon M^\bullet \to N^\bullet$ in the usual sense (not morphisms in the derived category). The weak equivalences are the quasi-isomorphisms and the cofibrations are morphisms of complexes $f\colon M^\bullet \to N^\bullet$ which are injective (i.e. for each n, $f\colon M^n \to N^n$ is injective). In the case of $\mathbf{SP}(R)$ we also require the cokernel of a cofibration f to be strictly perfect.

By the Gillet–Waldhausen Theorem [TT90, Theorem 1.11.7] we know that the Waldhausen K-theory of $\mathbf{SP}(R)$ and of $\mathbf{P}(R)$ coincide with the Quillen K-theory of R:

$$\mathrm{K}_n(\mathbf{P}(R)) = \mathrm{K}_n(\mathbf{SP}(R)) = \mathrm{K}_n(R).$$

Definition 3.4. Let R and S be rings. We denote by R^{op}-$\mathbf{SP}(S)$ the Waldhausen category of complexes of S-R-bimodules (with S acting from the left, R acting from the right) which are strictly perfect as complexes of S-modules. The weak equivalences and cofibrations are the same as in $\mathbf{SP}(S)$.

For complexes $M^\bullet$ and $N^\bullet$ of right and left R-modules, respectively, we let

$$(M \otimes_R N)^\bullet$$

denote the total complex of the bicomplex $M^\bullet \otimes_R N^\bullet$. Any complex $M^\bullet$ in R^{op}-$\mathbf{SP}(S)$ gives rise to a Waldhausen exact functor

$$(M \otimes_R (-))^\bullet \colon \mathbf{SP}(R) \to \mathbf{SP}(S)$$

and hence to homomorphisms $\mathrm{K}_n(R) \to \mathrm{K}_n(S)$.

Let now Λ be an adic ring. We introduce another Waldhausen category computing the K-theory of Λ which is more convenient for our purposes.

Definition 3.5. Let R be any ring. A complex $M^\bullet$ of left R-modules is called *DG-flat* if every module M^n is flat and for every acyclic complex $N^\bullet$ of right R-modules, the complex $(N \otimes_R M)^\bullet$ is acyclic.

Note that a strictly bounded above complex $M^\bullet$ is DG-flat precisely when every module M^n is flat. The notion of DG-flatness, which was introduced in [AF91], allows us to deal also with unbounded complexes. These are often the most natural objects to consider in our context. More precisely, we will have to deal with inverse systems of such objects:

Definition 3.6. Let Λ be an adic ring. We denote by $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ the following Waldhausen category. The objects of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ are inverse system $(P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ satisfying the following conditions:

(1) for each $I \in \mathfrak{I}_\Lambda$, $P_I^\bullet$ is a DG-flat perfect complex of left Λ/I-modules,
(2) for each $I \subset J \in \mathfrak{I}_\Lambda$, the transition morphism of the system

$$\varphi_{IJ} : P_I^\bullet \to P_J^\bullet$$

induces an isomorphism

$$\Lambda/J \otimes_{\Lambda/I} P_I^\bullet \cong P_J^\bullet.$$

A morphism of inverse systems $(f_I \colon P_I^\bullet \to Q_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ is a weak equivalence if every f_I is a quasi-isomorphism. It is a cofibration if every f_I is injective and the cokernel of f_I is a DG-flat complex of Λ/I-modules.

Proposition 3.7. *The Waldhausen exact functor*

$$F \colon \mathbf{SP}(\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda), \qquad P^\bullet \mapsto (\Lambda/I \otimes_\Lambda P^\bullet)_{I \in \mathfrak{I}_\Lambda},$$

identifies $\mathbf{SP}(\Lambda)$ *with a full Waldhausen subcategory of* $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ *such that for every* $Q^\bullet$ *in* $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ *there exists a complex* $P^\bullet$ *in* $\mathbf{SP}(\Lambda)$ *and a quasi-isomorphism* $F(P^\bullet) \xrightarrow{\sim} Q^\bullet$. *Moreover, F induces isomorphisms*

$$\mathrm{K}_n(\mathbf{SP}(\Lambda)) \cong \mathrm{K}_n(\mathbf{PDG}^{\mathrm{cont}}(\Lambda)).$$

Proof. The main step is to show that for every object $(Q_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$, the complex

$$\varprojlim_{I\in\mathfrak{I}_\Lambda} Q_I^\bullet$$

is a perfect complex of Λ-modules. This is proved using the argument of [FK06, Proposition 1.6.5]. The assertion about the K-theory is then an easy consequence of the Waldhausen approximation theorem. We refer to [Wit08, Proposition 5.2.5] for the details. □

Remark 3.8. Definition 3.6 makes sense for any compact ring Λ. However, we do not expect Proposition 3.7 to be true in this generality. The argument of [FK06, Proposition 1.6.5] uses in an essential way that Λ is compact for its $\mathrm{Jac}(\Lambda)$-adic topology.

We can extend the definition of the tensor product to $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ as follows.

Definition 3.9. For $(P_I^\bullet)_{I\in\mathfrak{I}_\Lambda}\in\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ and $M^\bullet\in\Lambda^{\mathrm{op}}\text{-}\mathbf{SP}(\Lambda')$ we define a Waldhausen exact functor

$$\Psi_{M^\bullet}\colon \mathbf{PDG}^{\mathrm{cont}}(\Lambda)\to\mathbf{PDG}^{\mathrm{cont}}(\Lambda'),\qquad P^\bullet\mapsto\Big(\varprojlim_{J\in\mathfrak{I}_\Lambda}\Lambda'/I\otimes_{\Lambda'}(M\otimes_\Lambda P_J)^\bullet\Big)_{I\in\mathfrak{I}_{\Lambda'}}.$$

Note that for every $I\in\mathfrak{I}_{\Lambda'}$ there exists a $J_0\in\mathfrak{I}_\Lambda$ such that

$$\varprojlim_{J\in\mathfrak{I}_\Lambda}\Lambda'/I\otimes_{\Lambda'}(M\otimes_\Lambda P_J)^\bullet=(M/IM\otimes_{\Lambda/J_0}P_{J_0})^\bullet.$$

One checks easily that this definition is compatible with the usual tensor product with $M^\bullet$ on $\mathbf{SP}(\Lambda)$.

From [MT07] we deduce the following generators and relations for the group $\mathrm{K}_1(\Lambda)$.

Proposition 3.10. *The group* $\mathrm{K}_1(\Lambda)$ *is generated by quasi-isomorphisms*

$$(f_I\colon P_I^\bullet\xrightarrow{\sim}P_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$*. Moreover, the following relations are satisfied:*

(1) $[(f_I\colon P_I^\bullet\xrightarrow{\sim}P_I^\bullet)_{I\in\mathfrak{I}_\Lambda}]=[(g_I\colon P_I^\bullet\xrightarrow{\sim}P_I^\bullet)_{I\in\mathfrak{I}_\Lambda}][(h_I\colon P_I^\bullet\xrightarrow{\sim}P_I^\bullet)_{I\in\mathfrak{I}_\Lambda}]$ *if for each* $I\in\mathfrak{I}_\Lambda$*, one has* $f_I=g_I\circ h_I$*,*
(2) $[(f_I\colon P_I^\bullet\xrightarrow{\sim}P_I^\bullet)_{I\in\mathfrak{I}_\Lambda}]=[(g_I\colon Q_I^\bullet\xrightarrow{\sim}Q_I^\bullet)_{I\in\mathfrak{I}_\Lambda}]$ *if for each* $I\in\mathfrak{I}_\Lambda$*, there exists a quasi-isomorphism* $a_I\colon P_I^\bullet\xrightarrow{\sim}Q_I^\bullet$ *such that the square*

$$\begin{array}{ccc} P_I^\bullet & \xrightarrow{f_I} & P_I^\bullet \\ \downarrow a_I & & {}_I\downarrow a_I \\ Q_I^\bullet & \xrightarrow{g_I} & Q_I^\bullet \quad {}_I \end{array}$$

commutes up to homotopy,

(3) $[(g_I\colon P_I'^{\bullet} \xrightarrow{\sim} P_I'^{\bullet})_{I\in\mathfrak{I}_\Lambda}] = [(f_I\colon P_I^{\bullet} \xrightarrow{\sim} P_I^{\bullet})_{I\in\mathfrak{I}_\Lambda}][(h_I\colon P_I''^{\bullet} \xrightarrow{\sim} P_I''^{\bullet})_{I\in\mathfrak{I}_\Lambda}]$ *if for each* $I \in \mathfrak{I}_\Lambda$*, there exists an exact sequence* $P_I^{\bullet} \rightarrowtail P_I'^{\bullet} \twoheadrightarrow P_I''^{\bullet}$ *such that the diagram*

$$\begin{array}{ccccc} P_I^{\bullet} & \rightarrowtail & P_I'^{\bullet} & \twoheadrightarrow & P_I''^{\bullet} \\ \downarrow{\scriptstyle f_I} & & \downarrow{\scriptstyle g_I} & & \downarrow{\scriptstyle h_I} \\ P_I^{\bullet} & \rightarrowtail & P_I'^{\bullet} & \twoheadrightarrow & P_I''^{\bullet} \end{array}$$

commutes in the strict sense.

Proof. The description of $\mathrm{K}_1(\mathbf{PDG}^{\mathrm{cont}}(\Lambda))$ as the kernel of

$$\mathcal{D}_1\mathbf{PDG}^{\mathrm{cont}}(\Lambda) \xrightarrow{\partial} \mathcal{D}_0\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$$

given in [MT07, Def. 1.4] (see also A.4) shows that all endomorphisms which are quasi-isomorphisms do indeed give rise to elements of $\mathrm{K}_1(\mathbf{PDG}^{\mathrm{cont}}(\Lambda))$. Together with the isomorphism

$$\mathrm{K}_1(\Lambda) \cong \varprojlim_{I\in\mathfrak{I}_\Lambda} \mathrm{K}_1(\Lambda/I)$$

[FK06, Prop. 1.5.3] this description also implies that relations (1) and (3) are satisfied. For (2), one can use [Wit08, Lemma 3.1.6]. Finally, the classical description of $\mathrm{K}_1(\Lambda)$ implies that $\mathrm{K}_1(\mathbf{PDG}^{\mathrm{cont}}(\Lambda))$ is already generated by isomorphisms of finitely generated, projective modules viewed as strictly perfect complexes concentrated in degree 0. □

4. Localisation

Localisation is considered a difficult topic in noncommutative ring theory. To be able to localise at a set of elements S in a noncommutative ring R one needs to show that this set is a denominator set. In particular, one must verify the Ore condition which is often a tedious task. We can avoid this topic by localising the associated Waldhausen category of perfect complexes instead of the ring itself.

Let Λ be an adic $\mathbb{Z}_\ell$-algebra, H a closed subgroup of a profinite group G and assume that both G and H are virtual pro-ℓ-groups. We define the following Waldhausen categories.

Definition 4.1. We write $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$ for the full Waldhausen subcategory of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ of objects $(P_J^{\bullet})_{J\in\mathfrak{I}_{\Lambda[[G]]}}$ such that

$$\varprojlim_{J\in\mathfrak{I}_{\Lambda[[G]]}} P_J^{\bullet}$$

is a perfect complex of $\Lambda[[H]]$-modules.

We write $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ for the Waldhausen category with the same objects, morphisms and cofibrations as $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$, but with a new set of weak equivalences given by those morphisms whose cones are objects of the category $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$.

Note that $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$ is a full additive subcategory of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ and that it is closed under weak equivalences, shifts, and extensions. This implies immediately that both $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$ and $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ are indeed Waldhausen categories (see e. g. [HM08, Section 3]) and that the natural functors

$$\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]]) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]]) \to w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$$

induce a cofibre sequence of the associated K-theory spaces, and hence a long exact localisation sequence

$$\begin{aligned}\cdots \to \mathrm{K}_i(\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])) \to \mathrm{K}_i(\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]]))\\ \to \mathrm{K}_i(w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])) \to \mathrm{K}_{i-1}(\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])) \to \cdots\end{aligned}$$

[TT90, Theorem 1.8.2].

Assume for the moment that $\Lambda = \mathbb{Z}_\ell$ and that $G \cong H \rtimes \Gamma_{\ell^\infty}$ with H a compact ℓ-adic Lie group and $\Gamma_{\ell^\infty} \cong \mathbb{Z}_\ell$. Then Venjakob constructed a left and right Ore set S of nonzerodivisors in $\mathbb{Z}_\ell[[G]]$. In particular, the quotient ring $\mathbb{Z}_\ell[[G]]_S$ exists and is flat as a right $\mathbb{Z}_\ell[[G]]$-module. Moreover, a finitely generated $\mathbb{Z}_\ell[[G]]$-module is S-torsion if and only if it is finitely generated as a $\mathbb{Z}_\ell[[H]]$-module [CFK+05, Section 2].

Since any choice of a generator of Γ_{ℓ^∞} identifies $\mathbb{Z}_\ell[[G]]$ with a skew power series ring over the noetherian ring $\mathbb{Z}_\ell[[H]]$, we see that

$$\mathbb{Z}_\ell[[G]] \cong \prod_{n\geq 0} \mathbb{Z}_\ell[[H]]$$

is flat as a $\mathbb{Z}_\ell[[H]]$-module. In particular, a complex $(P^\bullet_J)_{J\in\mathfrak{I}_{\mathbb{Z}_\ell[[G]]}}$ in $\mathbf{PDG}^{\mathrm{cont}}(\mathbb{Z}_\ell[[G]])$ is in $\mathbf{PDG}^{\mathrm{cont},w_H}(\mathbb{Z}_\ell[[G]])$ if and only if

$$\mathbb{Z}_\ell[[G]]_S \otimes_{\mathbb{Z}_\ell[[G]]} \varprojlim_{J\in\mathfrak{I}_{\mathbb{Z}_\ell[[G]]}} P^\bullet_J$$

is acyclic.

From the localisation theorem in [WY92] we conclude that in this case,

$$\mathrm{K}_i(\mathbf{PDG}^{\mathrm{cont},w_H}(\mathbb{Z}_\ell[[G]])) = \mathrm{K}_i(\mathbb{Z}_\ell[[G]], \mathbb{Z}_\ell[[G]]_S)$$

is the relative K-group and that the functor

$$\begin{aligned} w_H\mathbf{PDG}^{\mathrm{cont}}(\mathbb{Z}_\ell[[G]]) &\to \mathbf{P}(\mathbb{Z}_\ell[[G]]_S),\\ (P^\bullet_J)_{J\in\mathfrak{I}_{\mathbb{Z}_\ell[[G]]}} &\mapsto \mathbb{Z}_\ell[[G]]_S \otimes_{\mathbb{Z}_\ell[[G]]} \varprojlim_{J\in\mathfrak{I}_{\mathbb{Z}_\ell[[G]]}} P^\bullet_J,\end{aligned}$$

induces isomorphisms

$$\mathrm{K}_i(w_H\mathbf{PDG}^{\mathrm{cont}}(\mathbb{Z}_\ell[[G]])) = \begin{cases}\mathrm{K}_i(\mathbb{Z}_\ell[[G]]_S) & \text{if } i > 0,\\ \operatorname{im} \mathrm{K}_0(\mathbb{Z}_\ell[[G]]) \to \mathrm{K}_0(\mathbb{Z}_\ell[[G]]_S) & \text{if } i = 0\end{cases}$$

(see also [Wit08, Prop. 5.3.4]).

We need this more explicit description of $\mathrm{K}_1(w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]]))$ only in the following situation.

Lemma 4.2. *Let* Λ *be a commutative adic* $\mathbb{Z}_\ell$*-algebra,* $H = 1$ *and* $G = \Gamma_{k\ell^\infty}$ *with* k *prime to* ℓ*. Set*

$$S = \{f \in \Lambda[[\Gamma_{k\ell^\infty}]] \colon [f] \in \Lambda/\mathrm{Jac}(\Lambda)[[\Gamma_{k\ell^\infty}]] \text{ is a nonzerodivisor}\}$$

Then

$$\mathrm{K}_1(w_1\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[\Gamma_{k\ell^\infty}]])) = \mathrm{K}_1(\Lambda[[\Gamma_{k\ell^\infty}]]_S) = \Lambda[[\Gamma_{k\ell^\infty}]]_S^\times.$$

Proof. We will show that a strictly perfect complex $P^\bullet$ of $\Lambda[[\Gamma_{k\ell^\infty}]]$-modules is perfect as complex of Λ-modules if and only if its cohomology groups are S-torsion. Then the localisation theorem in [WY92] implies that

$$\mathrm{K}_n(w_1\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[\Gamma_{k\ell^\infty}]])) = \mathrm{K}_n(\Lambda[[\Gamma_{k\ell^\infty}]]_S)$$

for $n \geq 1$. Since $\Lambda[[\Gamma_{k\ell^\infty}]]_S$ is clearly a commutative semilocal ring, the determinant map induces an isomorphism

$$\mathrm{K}_1(\Lambda[[\Gamma_{k\ell^\infty}]]_S) \cong \Lambda[[\Gamma_{k\ell^\infty}]]_S^\times$$

[CR90b, Theorem 40.31].

Recall that a commutative adic ring is always noetherian and that a bounded complex of flat modules over a noetherian ring is perfect if and only if its cohomology modules are finitely generated. Hence, it suffices to show that a finitely generated $\Lambda[[\Gamma_{\ell k^\infty}]]$-module M is finitely generated as a Λ-module if and only if it is S-torsion. This is in turn a direct consequence of the structure theory for finitely generated modulues over the classical Iwasawa algebra. □

For general G, H and Λ, it is not difficult to prove that the first K-group of $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ agrees with the corresponding localised K_1-group defined in [FK06, Def. 1.3.2]:

$$\mathrm{K}_1(w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])) = \mathrm{K}_1(\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]]), \mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])),$$

but we will make no use of this.

Next, let Λ and Λ' be two adic $\mathbb{Z}_\ell$-algebras and G, G', H, H' be virtual pro-ℓ-groups. Assume that H and H' are closed subgroups of G and G', respectively. We want to investigate under which circumstances the Waldhausen exact functor

$$\Psi_{K^\bullet} \colon \mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]]) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda'[[G']])$$

for an object $K^\bullet$ in $\Lambda[[G]]^{\mathrm{op}}\text{-}\mathbf{SP}(\Lambda'[[G']])$ restricts to a functor

$$\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]]) \to \mathbf{PDG}^{\mathrm{cont},w_{H'}}(\Lambda'[[G']]).$$

Note that if this is the case, then $\Psi_{K^\bullet}$ also extends to a functor

$$w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]]) \to w_{H'}\mathbf{PDG}^{\mathrm{cont}}(\Lambda'[[G']]).$$

Both functors will again be denoted by $\Psi_{K^\bullet}$.

For any compact ring Ω, we let

$$M \hat{\otimes}_\Omega N = \varprojlim_{U,V} M/U \otimes_\Omega N/V$$

denote the *completed tensor product* of the compact right Ω-module M with the compact left Ω-module N. Here, U and V run through the open submodules of M and N, respectively. Note that the completed tensor product $M \hat{\otimes}_\Omega N$ agrees with the usual tensor product $M \otimes_\Omega N$ if either M or N is finitely presented.

Definition 4.3. We call a compact Ω-module P *compact-flat* if the completed tensor product with P preserves continuous injections of compact modules.

If the compact ring Ω is noetherian, then P is compact-flat precisely if it is flat, but in general, the two notions do not need to coincide.

Lemma 4.4. *Let Λ be an adic $\mathbb{Z}_\ell$-algebra, and H a closed subgroup of G such that both G and H are virtual pro-ℓ-groups. Then any finitely generated, projective $\Lambda[[G]]$-module is compact-flat as a $\Lambda[[H]]$-module.*

Proof. It suffices to prove the lemma for the finitely generated, projective $\Lambda[[G]]$-module $\Lambda[[G]]$. Then the statement follows since for every n and every open normal subgroup U in G, $\Lambda/\mathrm{Jac}^n(\Lambda)[G/U]$ is flat as a $\Lambda/\mathrm{Jac}^n(\Lambda)[H/H \cap U]$-module. □

Lemma 4.5. *Let Λ be an adic ring and $P^\bullet$ a strictly bounded complex of compact-flat left Λ-modules. Then $P^\bullet$ is a perfect complex of Λ-modules if and only if $\Lambda/\mathrm{Jac}(\Lambda) \otimes_\Lambda P^\bullet$ has finite cohomology groups.*

Proof. Assume that $Q^\bullet \xrightarrow{\sim} P^\bullet$ is a quasi-isomorphism with $Q^\bullet$ strictly perfect. Then the quasi-isomorphism

$$\Lambda/\mathrm{Jac}(\Lambda) \otimes_\Lambda Q^\bullet = \Lambda/\mathrm{Jac}(\Lambda) \hat{\otimes}_\Lambda Q^\bullet \xrightarrow{\sim} \Lambda/\mathrm{Jac}(\Lambda) \hat{\otimes}_\Lambda P^\bullet$$

shows that $\Lambda/\mathrm{Jac}(\Lambda) \hat{\otimes}_\Lambda P^\bullet$ has finite cohomology groups. Since $\Lambda/\mathrm{Jac}(\Lambda)$ is finite and $\mathrm{Jac}(\Lambda)$ is finitely generated, we can replace the completed tensor product by the usual tensor product.

Conversely, assume that $\Lambda/\mathrm{Jac}(\Lambda) \otimes_\Lambda P^\bullet$ has finite cohomology groups. Without loss of generality we may suppose that $P^k = 0$ for $k < 0$ and $k > n$ with some $n \geq 0$. By assumption, $\Lambda/\mathrm{Jac}(\Lambda) \otimes_\Lambda \mathrm{H}^n(P)$ is finite. By the topological Nakayama lemma we conclude that the compact module $\mathrm{H}^n(P)$ is finitely generated. We proceed by induction on n to prove the perfectness of $P^\bullet$. If $n = 0$, we see that P^0 is finitely generated. Using the lifting of idempotents in Λ, we conclude that P^0 is also projective. If $n > 0$, we may choose a homomorphism $f\colon \Lambda^k[-n] \to P^\bullet$ such that $\mathrm{H}^n(f)$ is surjective. The cone of this morphism then satisfies the induction hypothesis. Since the category of perfect complexes is closed under extensions we conclude that $P^\bullet$ is perfect. □

We can now state the following criterion:

Proposition 4.6. *Let Λ and Λ' be two adic $\mathbb{Z}_\ell$-algebras and G, G', H, H' be virtual pro-ℓ-groups. Assume that H and H' are closed subgroups of G and G', respectively. Suppose that $K^\bullet$ is a complex in $\Lambda[[G]]^{\mathrm{op}}$-$\mathbf{SP}(\Lambda'[[G']])$ such that there exists a complex $L^\bullet$ in $\Lambda[[H]]^{\mathrm{op}}$-$\mathbf{SP}(\Lambda'[[H']])$ and a quasi-isomorphism of complexes of $\Lambda'[[H']]$-$\Lambda[[G]]$-bimodules*

$$L^\bullet \hat{\otimes}_{\Lambda[[H]]} \Lambda[[G]] \xrightarrow{\sim} K^\bullet.$$

Then

$$\Psi_{K^\bullet}\colon \mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]]) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda'[[G']])$$

restricts to

$$\Psi_{K^\bullet}\colon \mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]]) \to \mathbf{PDG}^{\mathrm{cont},w_{H'}}(\Lambda'[[G']]).$$

Proof. According to Proposition 3.7 it suffices to consider a strictly perfect complex $P^\bullet$ of $\Lambda[[G]]$-modules which is also perfect as a complex of $\Lambda[[H]]$-modules. Hence, there exists a quasi-isomorphism $Q^\bullet \xrightarrow{\sim} P^\bullet$ of complexes of $\Lambda[[H]]$-modules with $Q^\bullet$ strictly perfect. According to Lemma 4.4, each P^n is compact-flat as a $\Lambda[[H]]$-module. Therefore, there exists a quasi-isomorphism of complexes of $\Lambda'[[H']]$-modules

$$(L \otimes_{\Lambda[[H]]} Q)^\bullet \xrightarrow{\sim} (L \hat{\otimes}_{\Lambda[[H]]} P)^\bullet \xrightarrow{\sim} (K \otimes_{\Lambda[[G]]} P)^\bullet.$$

Since $(L \otimes_{\Lambda[[H]]} Q)^\bullet$ is strictly perfect as a complex of $\Lambda'[[H']]$-modules, we see that $\Psi_{K^\bullet} P^\bullet$ is in $\mathbf{PDG}^{\mathrm{cont},w_{H'}}(\Lambda'[[G']])$. □

Proposition 4.7. *The following complexes $K^\bullet$ in $\Lambda[[G]]^{\mathrm{op}}$-$\mathbf{SP}(\Lambda'[[G']])$ satisfy the hypotheses of Proposition 4.6:*

(1) *Assume $G = G'$, $H = H'$. For any complex $P^\bullet$ in $\Lambda[[G]]^{\mathrm{op}}$-$\mathbf{SP}(\Lambda')$ let $K^\bullet$ be the complex $\Lambda'[[G]] \otimes_{\Lambda'} P^\bullet$ in $\Lambda[[G]]^{\mathrm{op}}$-$\mathbf{SP}(\Lambda'[[G]])$ with the right G-operation given by the diagonal action on both factors. This applies in particular for any complex $P^\bullet$ in Λ^{op}-$\mathbf{SP}(\Lambda')$ equipped with the trivial G-operation.*
(2) *Assume that G' is an open subgroup of G and set $H' = H \cap G'$. Let $\Lambda = \Lambda'$ and let $K^\bullet$ be the complex concentrated in degree 0 given by the $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule $\Lambda[[G]]$.*
(3) *Assume $\Lambda = \Lambda'$. Let $\alpha\colon G \to G'$ be a continuous homomorphism such that α maps H to H' and induces a bijection of the sets $H \setminus G$ and $H' \setminus G'$. Let $K^\bullet$ be the $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule $\Lambda[[G']]$.*

Proof. In the first example, one may choose $L^\bullet = \Lambda'[[H]] \otimes_{\Lambda'} P^\bullet$ with the diagonal right operation of H. The isomorphism $L^\bullet \hat{\otimes}_{\Lambda[[H]]} \Lambda[[G]] \to K^\bullet$ is then induced by $h \otimes p \otimes g \mapsto hg \otimes pg$ for $h \in H$, $g \in G$ and $p \in P^n$. In the second example, $L^\bullet = \Lambda[[H]]$ will do the job. In the last example, choose $L^\bullet = \Lambda[[H']]$. The inclusion $\Lambda[[H']] \subset \Lambda[[G']]$ and the continuous ring homomorphism $\Lambda[[G]] \to \Lambda[[G']]$ induced by α give rise to a morphism of $\Lambda[[H']]$-$\Lambda[[G]]$-bimodules

$$f\colon \Lambda[[H']] \hat{\otimes}_{\Lambda[[H]]} \Lambda[[G]] \to \Lambda[[G']].$$

Let U' be any open normal subgroup of G', $U = \alpha^{-1}(U')$ its preimage under α. Then, as $\Lambda[H'/H' \cap U']$-modules, $\Lambda[H'/H' \cap U'] \otimes_{\Lambda[H/H \cap U]} \Lambda[G/U]$ is freely generated by a choice of coset representatives of $UH \setminus G$, and $\Lambda[G'/U']$ is freely generated by the images of these representatives under α. Hence, we conclude that f is an isomorphism. □

Generalising [CFK+05, Lemma 2.1], we can give a useful characterisation of the complexes in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$ if we further assume that H is normal in G. Under this condition, we find an open pro-ℓ-subgroup K in H which is normal in G (take for example the intersection of all ℓ-Sylow subgroups of G with H).

Proposition 4.8. *Let Λ be an adic ring. Assume that H is a closed subgroup of G and that both G and H are virtual pro-ℓ-groups. Let furthermore K be an open pro-ℓ-subgroup of H which is normal in G. For a complex $P^\bullet = (P^\bullet_J)_{J\in\mathfrak{I}_{\Lambda[[G]]}}$ in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$, the following assertions are equivalent:*

(1) *$P^\bullet$ is in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$,*
(2) *$\Psi_{\Lambda/\mathrm{Jac}(\Lambda)[[G/K]]}(P^\bullet)$ is in $\mathbf{PDG}^{\mathrm{cont},w_{H/K}}(\Lambda/\mathrm{Jac}(\Lambda)[[G/K]])$,*
(3) *$\Psi_{\Lambda/\mathrm{Jac}(\Lambda)[[G/K]]}(P^\bullet)$ has finite cohomology groups.*

Proof. Assume that $P^\bullet$ is a strictly perfect complex of $\Lambda[[G]]$-modules. It is a strictly bounded complex of compact-flat $\Lambda[[H]]$-modules by Lemma 4.4. Proposition 3.2 implies that

$$\Lambda[[H]]/\mathrm{Jac}(\Lambda[[H]]) \otimes_{\Lambda[[H]]} P^\bullet = R/\mathrm{Jac}(R) \otimes_R \Lambda/\mathrm{Jac}(\Lambda)[[G/K]] \otimes_{\Lambda[[G]]} P^\bullet$$

for the finite ring $R = \Lambda/\mathrm{Jac}(\Lambda)[H/K]$. Now the equivalences in the statement of Proposition 4.8 are an immediate consequence of Lemma 4.5. □

We can use similar arguments to prove the following result, which can be combined with Proposition 4.6.

Proposition 4.9. *Let H be a closed subgroup of G such that both G and H are virtual pro-ℓ-groups. Assume that H' is an open subgroup of H. Then*

$$\mathbf{PDG}^{\mathrm{cont},w_{H'}}(\Lambda[[G]]) = \mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]]).$$

Proof. Since $\Lambda[[H]]$ is a finitely generated free $\Lambda[[H']]$ module, it is clear that every perfect complex of $\Lambda[[H]]$-modules is also perfect as a complex of $\Lambda[[H']]$-modules. For the other implication we may shrink H' and assume that it is pro-ℓ, normal and open in H. By Proposition 3.2 we conclude that

$$\Lambda[[H']]/\mathrm{Jac}(\Lambda[[H']]) = \Lambda/\mathrm{Jac}(\Lambda),$$
$$\Lambda[[H]]/\mathrm{Jac}(\Lambda[[H]]) = (\Lambda/\mathrm{Jac}(\Lambda))[H/H']/\mathrm{Jac}((\Lambda/\mathrm{Jac}(\Lambda))[H/H']).$$

Let $P^\bullet$ be a strictly perfect complex of $\Lambda[[G]]$-modules which is also perfect as a complex of $\Lambda[[H']]$-modules. Then Lemma 4.5 implies that the complexes

$$\Lambda/\mathrm{Jac}(\Lambda) \otimes_{\Lambda[[H']]} P^\bullet \cong \Lambda/\mathrm{Jac}(\Lambda)[H/H'] \otimes_{\Lambda[[H]]} P^\bullet$$

have finite cohomology groups and that $P^\bullet$ is perfect as a complex of $\Lambda[[H]]$-modules. □

5. Perfect complexes of adic sheaves

We let $\mathbb{F}$ denote a finite field of characteristic p, with $q = p^\nu$ elements. Furthermore, we fix an algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$.

For any scheme X in the category $\mathbf{Sch}_\mathbb{F}$ of $\mathbb{F}$-schemes of finite type and any adic ring Λ we introduced in [Wit08] a Waldhausen category $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$ of perfect complexes of adic sheaves on X. Below, we will recall the definition.

Definition 5.1. Let R be a finite ring and X be a scheme in $\mathbf{Sch}_\mathbb{F}$. A complex $\mathcal{F}^\bullet$ of étale sheaves of left R-modules on X is called *strictly perfect* if it is strictly bounded and each $\mathcal{F}^n$ is constructible and flat. A complex is called *perfect* if it is quasi-isomorphic to a strictly perfect complex. It is *DG-flat* if for each geometric point of X, the complex of stalks is DG-flat.

Definition 5.2. Let X be a scheme in $\mathbf{Sch}_\mathbb{F}$ and let Λ be an adic ring. The *category of perfect complexes of adic sheaves* $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$ is the following Waldhausen category. The objects of $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$ are inverse systems $(\mathcal{F}^\bullet_I)_{I \in \mathfrak{I}_\Lambda}$ such that:

(1) for each $I \in \mathfrak{I}_\Lambda$, $\mathcal{F}^\bullet_I$ is a perfect and DG-flat complex of étale sheaves of Λ/I-modules on X,
(2) for each $I \subset J \in \mathfrak{I}_\Lambda$, the transition morphism

$$\varphi_{IJ} : \mathcal{F}^\bullet_I \to \mathcal{F}^\bullet_J$$

of the system induces an isomorphism

$$\Lambda/J \otimes_{\Lambda/I} \mathcal{F}^\bullet_I \xrightarrow{\sim} \mathcal{F}^\bullet_J.$$

The weak equivalences are given by quasi-isomorphisms. The cofibrations are injections with cokernel in $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$.

If $\Lambda = \mathbb{Z}_\ell$, then the subcategory of complexes concentrated in degree 0 of $\mathbf{PDG}^{\mathrm{cont}}(X, \mathbb{Z}_\ell)$ corresponds precisely to the exact category of flat constructible ℓ-adic sheaves on X in the sense of [Gro77, Exposé VI, Definition 1.1.1]. In this sense, we recover the classical theory.

If $f : Y \to X$ is a morphism of schemes, we define a Waldhausen exact functor

$$f^* : \mathbf{PDG}^{\mathrm{cont}}(X, \Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(Y, \Lambda), \qquad (\mathcal{F}^\bullet_I)_{I \in \mathfrak{I}_\Lambda} \mapsto (f^*\mathcal{F}^\bullet_I)_{I \in \mathfrak{I}_\Lambda}.$$

We will also need a Waldhausen exact functor that computes higher direct images with proper support. For the purposes of this article it suffices to use the following construction.

Definition 5.3. Let $f : X \to Y$ be a morphism of separated schemes in $\mathbf{Sch}_\mathbb{F}$. Then there exists a factorisation $f = p \circ j$ with $j : X \hookrightarrow X'$ an open immersion and $p : X' \to Y$ a proper morphism. Let $\mathrm{G}^\bullet_{X'}\mathcal{K}$ denote the Godement resolution of a complex $\mathcal{K}^\bullet$ of abelian étale sheaves on X'. Define

$$\mathrm{R}\, f_! : \mathbf{PDG}^{\mathrm{cont}}(X, \Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(Y, \Lambda), \qquad (\mathcal{F}^\bullet_I)_{I \in \mathfrak{I}_\Lambda} \mapsto (p_* \mathrm{G}^\bullet_{X'} j_! \mathcal{F}_I)_{I \in \mathfrak{I}_\Lambda}.$$

Note that this definition depends on the particular choice of the compactification $f = p \circ j$. However, all possible choices will induce the same homomorphisms

$$\mathrm{K}_n(\mathrm{R}\, f_!)\colon \mathrm{K}_n(\mathbf{PDG}^{\mathrm{cont}}(X,\Lambda)) \to \mathrm{K}_n(\mathbf{PDG}^{\mathrm{cont}}(Y,\Lambda)),$$

and this is all we need.

Definition 5.4. Let X be a separated scheme in $\mathbf{Sch}_{\mathbb{F}}$ and write $h\colon X \to \operatorname{Spec}\mathbb{F}$ for the structure map, and $s\colon \operatorname{Spec}\overline{\mathbb{F}} \to \operatorname{Spec}\mathbb{F}$ for the map induced by the embedding into the algebraic closure. We define the Waldhausen exact functors

$$\mathrm{R}\,\Gamma_c(\overline{X},-), \mathrm{R}\,\Gamma_c(X,-)\colon \mathbf{PDG}^{\mathrm{cont}}(X,\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda)$$

to be the composition of

$$\mathrm{R}\,h_!\colon \mathbf{PDG}^{\mathrm{cont}}(X,\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(\operatorname{Spec}\mathbb{F},\Lambda)$$

with the section functors

$$\begin{aligned}\mathbf{PDG}^{\mathrm{cont}}(\operatorname{Spec}\mathbb{F},\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda), \quad (\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda} &\mapsto (\Gamma(\operatorname{Spec}\overline{\mathbb{F}}, s^*\mathcal{F}_I^\bullet))_{I\in\mathfrak{I}_\Lambda},\\ (\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda} &\mapsto (\Gamma(\operatorname{Spec}\mathbb{F}, \mathcal{F}_I^\bullet))_{I\in\mathfrak{I}_\Lambda},\end{aligned}$$

respectively.

Definition 5.5. We let $\mathfrak{F}_{\mathbb{F}} \in \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ denote the *geometric Frobenius* of $\mathbb{F}$, i.e. if $\mathbb{F}$ has q elements and $x \in \overline{\mathbb{F}}$, then $\mathfrak{F}_{\mathbb{F}}(x) = x^{1/q}$.

Clearly, $\mathfrak{F}_{\mathbb{F}}$ operates on $\mathrm{R}\,\Gamma_c(\overline{X},\mathcal{F}^\bullet)$.

Proposition 5.6. *Let X be a separated scheme in $\mathbf{Sch}_{\mathbb{F}}$. The following sequence is exact in $\mathbf{PDG}^{\mathrm{cont}}(X,\Lambda)$:*

$$0 \to \mathrm{R}\,\Gamma_c(X,\mathcal{F}^\bullet) \to \mathrm{R}\,\Gamma_c(\overline{X},\mathcal{F}^\bullet) \xrightarrow{\mathrm{id}-\mathfrak{F}_{\mathbb{F}}} \mathrm{R}\,\Gamma_c(\overline{X},\mathcal{F}^\bullet) \to 0.$$

Proof. See [Wit08, Proposition 6.1.2]. (In fact, all that we will need later on is that the cone of $\mathrm{id}-\mathfrak{F}_{\mathbb{F}}$ is quasi-isomorphic to $\mathrm{R}\,\Gamma_c(X,\mathcal{F}^\bullet)$ shifted by one, which is a well-known consequence of the Hochschild–Serre spectral sequence.) □

The definition of $\Psi_{M^\bullet}$ extends to $\mathbf{PDG}^{\mathrm{cont}}(X,\Lambda)$ as follows.

Definition 5.7. For two adic rings Λ and Λ' we let $\Lambda^{\mathrm{op}}\text{-}\mathbf{SP}(X,\Lambda')$ the Waldhausen category of strictly bounded complexes $(\mathcal{F}_J^\bullet)_{J\in\mathfrak{I}_{\Lambda'}}$ in $\mathbf{PDG}^{\mathrm{cont}}(X,\Lambda')$ with each $\mathcal{F}_J^n$ a sheaf of Λ'/J-Λ-bimodules, constructible and flat as sheaf of Λ'/J-modules. The transition maps in the system $(\mathcal{F}_J^\bullet)_{J\in\mathfrak{I}_{\Lambda'}}$ and the boundary maps of the complexes are supposed to be compatible with the right Λ-structure.

In particular, a complex $M^\bullet$ in $\Lambda^{\mathrm{op}}\text{-}\mathbf{SP}(\Lambda')$ can be identified with the complex of constant sheaves $(\Lambda'/I \otimes_{\Lambda'} M)_{I\in\mathfrak{I}_{\Lambda'}}$ in $\Lambda^{\mathrm{op}}\text{-}\mathbf{SP}(X,\Lambda')$.

Definition 5.8. For $(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda} \in \mathbf{PDG}^{\mathrm{cont}}(X,\Lambda)$ and $\mathcal{K}^\bullet \in \Lambda^{\mathrm{op}}\text{-}\mathbf{SP}(X,\Lambda')$ we set

$$\Psi_{\mathcal{K}^\bullet}((\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}) = \Big(\varprojlim_{J\in\mathfrak{I}_\Lambda} (\mathcal{K}_I \otimes_\Lambda \mathcal{F}_J)^\bullet\Big)_{I\in\mathfrak{I}_{\Lambda'}}$$

and obtain a Waldhausen exact functor

$$\Psi_{\mathcal{K}^\bullet} \colon \mathbf{PDG}^{\mathrm{cont}}(X,\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(X,\Lambda').$$

Proposition 5.9. *Let X be a separated scheme in $\mathbf{Sch}_{\mathbb{F}}$ and let $M^\bullet$ be a complex in $\Lambda^{\mathrm{op}}\text{-}\mathbf{SP}(\Lambda')$. The natural morphisms*

$$\Psi_{M^\bullet}\,\mathrm{R}\,\Gamma_c(X,\mathcal{F}^\bullet) \to \mathrm{R}\,\Gamma_c(X,\Psi_{M^\bullet}\mathcal{F}^\bullet), \qquad \Psi_{M^\bullet}\,\mathrm{R}\,\Gamma_c(\overline{X},\mathcal{F}^\bullet) \to \mathrm{R}\,\Gamma_c(\overline{X},\Psi_{M^\bullet}\mathcal{F}^\bullet)$$

are quasi-isomorphisms.

Proof. This is straightforward. See [Wit08, Proposition 5.5.7]. □

6. Adic sheaves induced by coverings

As before, we let $\mathbb{F}$ be a finite field and X an $\mathbb{F}$-scheme of finite type. Recall that for any finite étale map $h\colon Y \to X$ and any abelian étale sheaf $\mathcal{F}$ on X, $h_!h^*\mathcal{F}$ is the sheaf associated to the presheaf

$$U \mapsto \bigoplus_{\varphi\in\mathrm{Hom}_X(U,Y)} \mathcal{F}(U)$$

with the transition maps

$$\bigoplus_{\psi\in\mathrm{Hom}_X(V,Y)} \mathcal{F}(V) \to \bigoplus_{\varphi\in\mathrm{Hom}_X(U,Y)} \mathcal{F}(U), \qquad (x_\psi) \mapsto \Big(\sum_{\varphi=\psi\circ\alpha} \mathcal{F}(\alpha)(x_\psi)\Big),$$

for $\alpha\colon U \to V$. If h is a finite principal covering with Galois group G, then the right action of G on Y induces a right action on $\mathrm{Hom}_X(U,Y)$, and hence a left action on $h_!h^*\mathcal{F}$ by permutation of the components. The stalk at a geometric point ξ of X is given by

$$(h_!h^*\mathcal{F})_\xi = \bigoplus_{\varphi\in\mathrm{Hom}_X(\xi,Y)} \mathcal{F}_\xi.$$

Since Y is finite over X, the set $\mathrm{Hom}_X(\xi,Y)$ is nonempty. The choice of any element in $\mathrm{Hom}_X(\xi,Y)$ induces an isomorphism of G-sets

$$\mathrm{Hom}_X(\xi,Y) \cong \mathrm{Hom}_Y(\xi, Y\times_X Y) \cong \mathrm{Hom}_Y\Big(\xi, \coprod_{g\in G} Y\Big) \cong G,$$

and hence a $\mathbb{Z}[G]$-isomorphism

$$(h_!h^*\mathcal{F})_\xi \cong \mathbb{Z}[G]\otimes_{\mathbb{Z}} \mathcal{F}_\xi.$$

Consider an adic $\mathbb{Z}_\ell$-algebra Λ and let $(f\colon Y \to X, G)$ be a virtual pro-ℓ-covering of X.

Definition 6.1. For $\mathcal{F}^\bullet \in \mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$ we set

$$f_! f^* \mathcal{F}^\bullet = \Big(\varprojlim_{I \in \mathfrak{I}_\Lambda} \varprojlim_{U \in \mathfrak{N}_G} \Lambda[[G]]/J \otimes_{\Lambda[[G]]} f_{U\,!} f_U^* \mathcal{F}_I^\bullet \Big)_{J \in \mathfrak{I}_{\Lambda[[G]]}}.$$

Again, we note that for each $J \in \mathfrak{I}_{\Lambda[[G]]}$, there exists an $I_0 \in \mathfrak{I}_\Lambda$ and a $U_0 \in \mathfrak{N}_G$ such that $\Lambda[[G]]/J$ is a right $\Lambda/I_0[G/U_0]$-module and

$$(f_! f^* \mathcal{F})_J^\bullet \cong \Lambda[[G]]/J \otimes_{\Lambda/I_0[G/U_0]} f_{U_0 !} f_{U_0}^* \mathcal{F}_{I_0}^\bullet.$$

Proposition 6.2. *For any complex $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$, $f_! f^* \mathcal{F}^\bullet$ is a complex in $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda[[G]])$. Moreover, the functor*

$$f_! f^* \colon \mathbf{PDG}^{\mathrm{cont}}(X, \Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(X, \Lambda[[G]])$$

is Waldhausen exact.

Proof. We note that $f_{U\,!} f_U^* \mathcal{F}_I^\bullet$ is a perfect DG-flat complex of sheaves of $\Lambda/I[G/U]$-modules. This follows since for every geometric point ξ of X and every étale sheaf of left Λ/I-modules $\mathcal{P}$ on X, we have

$$(f_{U!} f_U^* \mathcal{P})_\xi \cong \Lambda/I[G/U] \otimes_\Lambda (\mathcal{P}_\xi).$$

Moreover, the functor $f_{U!} f_U^*$ is exact as a functor from the abelian category of sheaves of Λ/I-modules to the abelian category of sheaves of $\Lambda/I[G/U]$-modules, and for $V \subset U$ and $J \subset I$, we have a natural isomorphism of functors

$$\Lambda/I[G/U] \otimes_{\Lambda/J[G/V]} f_{V!} f_V^* \cong f_{U!} f_U^*.$$

These observations suffice to deduce the assertion. □

Sometimes, the following alternative description of the functor $f_! f^*$ is useful.

Proposition 6.3. *The sheaf $(f_! f^* \Lambda)$ is in Λ^{op}-$\mathbf{SP}(X, \Lambda[[G]])$ and for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$, there exists a natural isomorphism*

$$\Psi_{f_! f^* \Lambda}(\mathcal{F}^\bullet) \cong f_! f^* \mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda[[G]])$.

Proof. One easily reduces to the case that Λ and G are finite. Then the isomorphism is provided by the well-known projection formula:

$$f_! f^* \Lambda \otimes_\Lambda \mathcal{F}^\bullet \cong f_!(f^* \Lambda \otimes_\Lambda f^* \mathcal{F}^\bullet) \cong f_! f^* \mathcal{F}^\bullet. \quad □$$

In the following three propositions we formulate various base change compatibilities of our construction.

Proposition 6.4 (Change of the base scheme). *Let $a \colon X' \to X$ be a morphism of separated schemes in $\mathbf{Sch}_{\mathbb{F}}$ and write $(f' \colon Y' \to X', G)$ for the principal covering obtained from $(f \colon Y \to X, G)$ by base change. Then*

(1) *For any* $\mathcal{F}^\bullet$ *in* $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$ *there is a natural isomorphism*

$$f'_! f'^* a^* \mathcal{F}^\bullet \cong a^* f_! f^* \mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\mathrm{cont}}(X', \Lambda[[G]])$.

(2) *For any* $\mathcal{F}^\bullet$ *in* $\mathbf{PDG}^{\mathrm{cont}}(X', \Lambda)$ *there is a natural quasi-isomorphism*

$$f_! f^* \,\mathrm{R}\, a_! \mathcal{F}^\bullet \xrightarrow{\sim} \mathrm{R}\, a_! f'_! f'^* \mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda[[G]])$.

Proof. The first assertion follows from the proper base change theorem in a very trivial case. For the second assertion, we use the projection formula to see that the natural morphism

$$\Psi_{f_! f^* \Lambda} \,\mathrm{R}\, a_! \mathcal{F}^\bullet \to \mathrm{R}\, a_! \Psi_{a^* f_! f^* \Lambda} \mathcal{F}^\bullet$$

is a quasi-isomorphism and then the first assertion to identify $a^* f_! f^* \Lambda$ with $f'_! f'^* \Lambda$. □

Proposition 6.5 (Change of the group). *Let* $\mathcal{F}^\bullet$ *be a complex in* $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$.

(1) *Let H be a closed normal subgroup of G. Then there exists a natural isomorphism*

$$\Psi_{\Lambda[[G/H]]} f_! f^* \mathcal{F}^\bullet \xrightarrow{\sim} (f_H)_! (f_H)^* \mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda[[G/H]])$.

(2) *Let U be an open subgroup of G, let* $f_U : Y_U \to X$ *denote the natural projection map, and view* $\Lambda[[G]]$ *as a* $\Lambda[[U]]$-$\Lambda[[G]]$-*bimodule. Then there exists a natural quasi-isomorphism*

$$\Psi_{\Lambda[[G]]} f_! f^* \mathcal{F}^\bullet \xrightarrow{\sim} (\mathrm{R}(f_U)_!)((f^U)_!(f^U)^*) f_U^* \mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda[[U]])$.

Proof. One reduces to the case that Λ and G are finite and that $\mathcal{F}^\bullet = \Lambda$. The first morphism is induced by the natural map $f_! f^* \Lambda \to (f_H)_! (f_H)^* \Lambda$ and is easily checked to be an isomorphism by looking at the stalks. The second morphism is the composition of the isomorphism $f_! f^* \Lambda \cong f_{U!} f^U_! f^{U*} f_U^* \Lambda$ with the functorial morphism $f_{U!} \to \mathrm{R}\, f_{U!}$, the latter being a quasi-isomorphism since f_U is finite. □

Definition 6.6. Let Λ and Λ' be two adic $\mathbb{Z}_\ell$-algebras and let $\mathcal{K}^\bullet$ be in $\Lambda[[G]]^{\mathrm{op}}$-$\mathbf{SP}(X, \Lambda')$.

(1) We will write $\mathcal{K}[[G]]^{\delta\bullet}$ for the complex $\Psi_{\Lambda'[[G]]}\mathcal{K}^\bullet$ in $\Lambda[[G]]^{\mathrm{op}}$-$\mathbf{SP}(X, \Lambda'[[G]])$ with the right $\Lambda[[G]]$-structure given by the diagonal right operation of G.
(2) We will write $\widetilde{\mathcal{K}}^\bullet$ for the complex $\Psi_{\mathcal{K}^\bullet} f_! f^* \Lambda$ in Λ^{op}-$\mathbf{SP}(X, \Lambda')$.

Proposition 6.7 (Compatibility with tensor products). *Let* $\mathcal{K}^\bullet$ *be in* $\Lambda[[G]]^{\mathrm{op}}$-$\mathbf{SP}(X, \Lambda')$. *For every* $\mathcal{F}^\bullet$ *in* $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$ *there exists a natural isomorphism*

$$\Psi_{\mathcal{K}[[G]]^{\delta\bullet}} f_! f^* \mathcal{F}^\bullet \cong f_! f^* \Psi_{\widetilde{\mathcal{K}}^\bullet} \mathcal{F}^\bullet$$

Proof. One easily reduces to the case that Λ' and Λ are finite rings and that G is a finite group. Moreover, it suffices to consider a sheaf $\mathcal{K}$ of $\Lambda[G]$-Λ-bimodules viewed as a complex in $\Lambda[G]^{\mathrm{op}}$-$\mathbf{SP}(X, \Lambda')$ which is concentrated in degree 0. In view of Proposition 6.3 we may also assume that $\mathcal{F}^\bullet = \Lambda$. We begin by proving two special cases.

Case 1. Assume that G operates trivially on $\mathcal{K}$. Then

$$\Psi_{\mathcal{K}[[G]]^\delta} f_! f^* \Lambda \cong \mathcal{K} \otimes_\Lambda f_! f^* \Lambda \cong f_! f^* \mathcal{K}$$

by the projection formula. On the other hand,

$$\Psi_{\widetilde{\mathcal{K}}} \Lambda \cong \mathcal{K} \otimes_{\Lambda[G]} f_! f^* \Lambda \cong \mathcal{K},$$

and therefore $\Psi_{\mathcal{K}[[G]]^\delta} f_! f^* \Lambda \cong f_! f^* \Psi_{\widetilde{\mathcal{K}}} \Lambda$.

Case 2. Assume that $\Lambda' = \Lambda[G]$ and that $\mathcal{K}$ is the constant sheaf $\Lambda[G]$. Let $U \to X$ be finite étale and consider the homomorphism

$$u\colon \bigoplus_{\psi \in \mathrm{Hom}_X(U,Y)} \Lambda \to \bigoplus_{\psi \in \mathrm{Hom}_X(U,Y)} \bigoplus_{\phi \in \mathrm{Hom}_X(U,Y)} \Lambda, \qquad (a_\psi) \mapsto (a_\psi \delta_{\psi,\phi}),$$

with

$$\delta_{\psi,\phi} = \begin{cases} 1 & \text{if } \psi = \phi, \\ 0 & \text{else.} \end{cases}$$

For $g \in G$ we have

$$u(g(a_\psi)) = (g, g) u(a_\psi).$$

Hence, u induces a $\Lambda[G][G]$-homomorphism

$$\Psi_{\Lambda[G][G]^\delta}(f_! f^* \Lambda) = \Lambda[G][G]^\delta \otimes_{\Lambda[G]} f_! f^* \Lambda \to f_! f^* f_! f^* \Lambda \cong f_! f^* \Psi_{\widetilde{\Lambda[G]}}(\Lambda),$$

which is seen to be an isomorphism by checking on the stalks.

To prove the general case, we let $\mathcal{K}'$ be the sheaf $\mathcal{K}$ considered as a sheaf of Λ'-$\Lambda[G][G]$-bimodules, where the operation of the second copy of G is the trivial one. Then we have an obvious isomorphism of sheaves of $\Lambda'[G]$-$\Lambda[G]$-bimodules

$$\mathcal{K}[G]^\delta \cong \mathcal{K}'[G]^\delta \otimes_{\Lambda[G][G]} \Lambda[G][G]^\delta$$

and by the two cases that we have already proved we obtain

$$\begin{aligned} f_! f^* \Psi_{\widetilde{\mathcal{K}}}(\Lambda) &\cong f_! f^* \Psi_{\widetilde{\mathcal{K}'}} f_! f^* \Lambda \cong \Psi_{\mathcal{K}'[G]^\delta} f_! f^* f_! f^* \Lambda \\ &\cong \Psi_{\mathcal{K}'[G]^\delta} \Psi_{\Lambda[G][G]^\delta} f_! f^* \Lambda \cong \Psi_{\mathcal{K}[G]^\delta} f_! f^* \Lambda \end{aligned}$$

as desired. □

The significance of the construction in 6.6(2) is partly explained by the following version of the well-known equivalence of finite representations of the fundamental group with locally constant étale sheaves on a connected scheme X.

Proposition 6.8. *Let Λ and Λ' be two adic $\mathbb{Z}_\ell$-algebras. Assume that X is connected and that x is a geometric point of X. Let $(f\colon Y \to X, G)$ be a virtual pro-ℓ-subcovering of the universal covering $(\widetilde{X} \to X, \pi_1^{\text{ét}}(X,x))$. The functor*

$$\Lambda[[G]]^{\text{op}}\text{-}\mathbf{SP}(\Lambda') \to \Lambda^{\text{op}}\text{-}\mathbf{SP}(X,\Lambda'), \qquad P^\bullet \mapsto \widetilde{P}^\bullet,$$

identifies $\Lambda[[G]]^{\text{op}}\text{-}\mathbf{SP}(\Lambda')$ with a full subcategory $\mathbf{C}$ of $\Lambda^{\text{op}}\text{-}\mathbf{SP}(X,\Lambda')$. The objects of $\mathbf{C}$ are systems $(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ of strictly bounded complexes of sheaves of Λ'/I-Λ-bimodules such that for each n, $\mathcal{F}_I^n$ is constructible and flat as sheaf of Λ'/I-modules and there exists an open normal subgroup U of G such that $f_U^*\mathcal{F}_I^n$ is a constant sheaf.

Proof. We may assume that Λ' is finite. Clearly, $\widetilde{P}^\bullet$ is an object of $\mathbf{C}$ for every complex $P^\bullet$ in $\Lambda^{\text{op}}\text{-}\mathbf{SP}(\Lambda')$. To construct the inverse functor, we fix a compatible family of geometric points $(x \to Y_U)_{U\in\mathfrak{N}_G}$. If $\mathcal{F}^\bullet$ is in $\mathbf{C}$, there exists an open normal subgroup U of G such that the induced morphism $\mathcal{F}(Y_U)^\bullet \to \mathcal{F}_x^\bullet$ is an isomorphism. Turn $\mathcal{F}_x^\bullet$ into a complex in $\Lambda[[G]]^{\text{op}}\text{-}\mathbf{SP}(\Lambda')$ by considering the contragredient of the left action of G on $\mathcal{F}(Y_U)^\bullet$. One then checks that this is an inverse to the functor $P^\bullet \mapsto \widetilde{P}^\bullet$. □

Remark 6.9. Extending Definition 5.2 to arbitrary compact rings, one can also prove a corresponding statement for the full universal covering. On the other hand, if Λ and Λ' are adic rings, it follows as in [Wit08, Theorem 5.6.5] that for every $K^\bullet$ in $\Lambda[[\pi_1^{\text{ét}}(X,x)]]^{\text{op}}$-$\mathbf{SP}(\Lambda')$ there exists a factor group G of $\pi_1^{\text{ét}}(X,x)$ such that G is a virtual pro-ℓ-group and such that $K^\bullet$ also lies in $\Lambda[[G]]^{\text{op}}\text{-}\mathbf{SP}(\Lambda')$.

Remark 6.10. Proposition 6.8 implies in particular that for any virtual pro-ℓ-subcovering $(f\colon Y \to X, G)$ of the universal covering, the sheaf $\mathcal{M}(G)$ of the introduction corresponds to $f_! f^*\mathbb{Z}_\ell$.

7. The cyclotomic Γ-covering

Let X be a separated scheme in $\mathbf{Sch}_{\mathbb{F}}$. For any complex $\mathcal{F}^\bullet = (\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\text{cont}}(X,\Lambda)$, we write

$$\mathrm{H}_c^i(X,\mathcal{F}^\bullet) = \mathrm{H}^i\Big(\varprojlim_{I\in\mathfrak{I}_\Lambda} \mathrm{R}\,\Gamma_c(X,\mathcal{F}_I^\bullet)\Big)$$

for the i-th hypercohomology module of the complex $\mathrm{R}\,\Gamma_c(X,\mathcal{F}^\bullet)$.

Proposition 7.1. *Let $(f\colon X_{\ell^\infty} \to X, \Gamma_{\ell^\infty})$ be the cyclotomic Γ_{ℓ^∞}-covering of X. For all $i \in \mathbb{Z}$ and any complex $\mathcal{F}^\bullet = (\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\text{cont}}(X,\Lambda)$, we have*

$$\mathrm{H}_c^i(X, f_! f^*\mathcal{F}^\bullet) \cong \varprojlim_n \mathrm{H}_c^{i-1}(\overline{X},\mathcal{F}^\bullet)/(\mathrm{id} - \mathfrak{F}_{\mathbb{F}}^{\ell^n})\,\mathrm{H}_c^{i-1}(\overline{X},\mathcal{F}^\bullet)$$

as Λ-modules.

Proof. Write $f_n = f_{\Gamma_{\ell^\infty}^n}$. Since

$$\mathrm{R}^1 \varprojlim_{J\in\mathfrak{I}_{\Lambda[[\Gamma_{\ell^\infty}]]}} M_J = 0$$

for any inverse system $(M_J)_{J\in\mathfrak{I}_{\Lambda[[\Gamma_{\ell^\infty}]]}}$ of $\Lambda[[\Gamma_{\ell^\infty}]]$-modules with surjective transition maps and since the cohomology groups $\mathrm{H}^i_c(X, f_{n!}f_n^*\mathcal{F}_I^\bullet)$ are finite for $I \in \mathfrak{I}_\Lambda$, we conclude that

$$\mathrm{H}^i_c(X, f_! f^*\mathcal{F}^\bullet) \cong \varprojlim_{I\in\mathfrak{I}_\Lambda} \varprojlim_n \mathrm{H}^i_c(X, f_{n!}f_n^*\mathcal{F}_I^\bullet)$$

(see also [Wit08, Proposition 5.3.2]).

Moreover, for every n, there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathrm{H}^i_c(X, f_{n+1!}f_{n+1}^*\mathcal{F}_I^\bullet) & \longrightarrow & \mathrm{H}^i_c(\overline{X}, \mathcal{F}_I^\bullet) & \xrightarrow{\mathrm{id}-\mathfrak{F}_{\mathbb{F}}^{\ell^{n+1}}} & \mathrm{H}^i_c(\overline{X}, \mathcal{F}_I^\bullet) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle tr} & & \downarrow{\scriptstyle \sum_{k=0}^{\ell-1}\mathfrak{F}_{\mathbb{F}}^{k\ell^n}} & & \downarrow{\scriptstyle =} & & \\
\cdots & \longrightarrow & \mathrm{H}^i_c(X, f_{n!}f_n^*\mathcal{F}_I^\bullet) & \longrightarrow & \mathrm{H}^i_c(\overline{X}, \mathcal{F}_I^\bullet) & \xrightarrow{\mathrm{id}-\mathfrak{F}_{\mathbb{F}}^{\ell^{n}}} & \mathrm{H}^i_c(\overline{X}, \mathcal{F}_I^\bullet) & \longrightarrow & \cdots
\end{array}$$

where $\mathrm{tr}\colon f_{n+1!}f_{n+1}^*\mathcal{F}_I^\bullet \to f_{n!}f_n^*\mathcal{F}_I^\bullet$ denotes the usual trace map. Set

$$K_n = \ker\big(\mathrm{H}^i_c(\overline{X}, \mathcal{F}_I^\bullet) \xrightarrow{\mathrm{id}-\mathfrak{F}_{\mathbb{F}}^{\ell^n}} \mathrm{H}^i_c(\overline{X}, \mathcal{F}_I^\bullet)\big)$$

Since $\mathrm{H}^i_c(\overline{X}, \mathcal{F}_I^\bullet)$ is a finite group, the inclusion chain

$$K_0 \subset K_1 \subset \cdots \subset K_n \subset \cdots$$

becomes stationary. Hence, for large n, $K_n = K_{n+1}$ and the map

$$\sum_{k=0}^{\ell-1} \mathfrak{F}_{\mathbb{F}}^{k\ell^n} : K_{n+1} \to K_n$$

is multiplication by ℓ. Since K_n is annihilated by a power of ℓ, we conclude that

$$\varprojlim_n K_n = 0.$$

The equality claimed in the proposition is an immediate consequence. □

Proposition 7.2. *Let γ denote the image of $\mathfrak{F}_{\mathbb{F}}$ in $\Gamma = \Gamma_{k\ell^\infty}$ and let $(X_{k\ell^\infty} \to X, \Gamma)$ be the cyclotomic Γ-covering of X. Let $\mathcal{F}^\bullet$ be a complex in* $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$. *There exists a quasi-isomorphism*

$$\eta\colon \Psi_{\Lambda[[\Gamma]]}\,\mathrm{R}\,\Gamma_c(\overline{X}, \mathcal{F}^\bullet) \to \mathrm{R}\,\Gamma_c(\overline{X}, f_! f^*\mathcal{F}^\bullet)$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[\Gamma]])$ *such that the following diagram commutes:*

$$\begin{array}{ccc}
\Psi_{\Lambda[[\Gamma]]}\,\mathrm{R}\,\Gamma_c(\overline{X}, \mathcal{F}^\bullet) & \xrightarrow{\gamma^{-1}\otimes\mathfrak{F}_{\mathbb{F}}} & \Psi_{\Lambda[[\Gamma]]}\,\mathrm{R}\,\Gamma_c(\overline{X}, \mathcal{F}^\bullet) \\
\downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\
\mathrm{R}\,\Gamma_c(\overline{X}, f_! f^*\mathcal{F}^\bullet) & \xrightarrow{\mathfrak{F}_{\mathbb{F}}} & \mathrm{R}\,\Gamma_c(\overline{X}, f_! f^*\mathcal{F}^\bullet)
\end{array}$$

Proof. Using Proposition 6.4 we can reduce to the case $X = \operatorname{Spec} \mathbb{F}$. Moreover, it suffices to consider $\mathcal{F}^\bullet = \Lambda$. By Proposition 6.8, the sheaf $f_! f^* \Lambda$ corresponds to the $\Lambda[[\Gamma]]$-module $\Lambda[[\Gamma]]$ with the left action of the Frobenius $\mathfrak{F}_\mathbb{F}$ given by right multiplication with γ^{-1}. The assertion of the proposition is an immediate consequence. □

8. The Iwasawa main conjecture

The following theorem is the central piece of our analogue of the noncommutative Iwasawa main conjecture. It corresponds to [CFK+05, Conjecture 5.1] in conjunction with the vanishing of the μ-invariant, the complex $\mathrm{R}\,\Gamma_c(X, f_! f^* \mathcal{F}^\bullet)$ playing the role of the module $X(E/F_\infty)$.

Theorem 8.1. *Let X be a separated scheme of finite type over a finite field $\mathbb{F}$. Fix a prime ℓ and let $(f : Y \to X, G)$ be an admissible covering of X with group $G \cong H \rtimes \Gamma_{\ell^\infty}$. If Λ is an adic $\mathbb{Z}_\ell$-algebra and $\mathcal{F}^\bullet$ a complex in $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$, then $\mathrm{R}\,\Gamma_c(X, f_! f^* \mathcal{F}^\bullet)$ is in $\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])$.*

Proof. Proposition 6.5, Proposition 6.7 (for the Λ/I-Λ-bimodule Λ/I with trivial G-operation), and Proposition 5.9 imply that for any closed normal subgroup K of G and each open two-sided ideal I of Λ, there exists a quasi-isomorphism

$$\begin{aligned}\Psi_{\Lambda/I[[G/K]]}\,\mathrm{R}\,\Gamma_c(X, f_! f^* \mathcal{F}^\bullet) &\xrightarrow{\sim} \mathrm{R}\,\Gamma_c(X, \Psi_{\Lambda/I[[G/K]]} f_! f^* \mathcal{F}^\bullet) \\ &\xrightarrow{\sim} \mathrm{R}\,\Gamma_c(X, f_{K\,!} f_K^* \Psi_{\Lambda/I} \mathcal{F}^\bullet).\end{aligned}$$

Thus, by Proposition 4.8, we may assume that Λ is a finite ring and that $G \cong H \rtimes \Gamma_{\ell^\infty}$ with a finite group H. It then suffices to show that $\mathrm{R}\,\Gamma_c(X, f_! f^* \mathcal{F}^\bullet)$ has finite cohomology groups.

There exists an open subgroup $\Gamma' \subset \Gamma_{\ell^\infty}$ which lies in the centre of G. The scheme $Y_{\Gamma'}$ is separated of finite type over a finite extension $\mathbb{F}'$ of $\mathbb{F}$ and by Lemma 2.3 the principal covering $(f^{\Gamma'} : Y \to Y_{\Gamma'}, \Gamma')$ is isomorphic to the cyclotomic Γ_{ℓ^∞}-covering of $Y_{\Gamma'}$ over $\mathbb{F}'$.

By Proposition 6.5 the cohomology groups of $\mathrm{R}\,\Gamma_c(X, f_! f^* \mathcal{F}^\bullet)$ are isomorphic to those of $\mathrm{R}\,\Gamma_c(Y_{\Gamma'}, f_!^{\Gamma'} f^{\Gamma'*} f_{\Gamma'}^* \mathcal{F}^\bullet)$ which are finite by Proposition 7.1. □

Corollary 8.2. *Under the same assumptions as above,*

$$\mathrm{id} - \mathfrak{F}_\mathbb{F} \colon \ \mathrm{R}\,\Gamma_c(\overline{X}, f_! f^* \mathcal{F}^\bullet) \to \mathrm{R}\,\Gamma_c(\overline{X}, f_! f^* \mathcal{F}^\bullet)$$

is a quasi-isomorphism in $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$, and hence gives rise to an element

$$[\mathrm{id} - \mathfrak{F}_\mathbb{F}] \in \mathrm{K}_1(w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]]))$$

satisfying

$$d[\mathrm{id} - \mathfrak{F}_\mathbb{F}] = [\mathrm{R}\,\Gamma_c(X, f_! f^* \mathcal{F}^\bullet)]$$

in $\mathrm{K}_0(\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]]))$.

Proof. By Proposition 5.6, the cone of $\mathrm{id} - \mathfrak{F}_{\mathbb{F}}$ is $\mathrm{R}\,\Gamma_c(X, f_! f^* \mathcal{F}^\bullet)$ shifted by one. Hence, $\mathrm{id} - \mathfrak{F}_{\mathbb{F}}$ is a quasi-isomorphism in $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$. Theorem A.5 then implies that $d[\mathrm{id} - \mathfrak{F}_{\mathbb{F}}] = [\mathrm{R}\,\Gamma_c(X, f_! f^* \mathcal{F}^\bullet)]$. □

Definition 8.3. We write $\mathcal{L}_G(X/\mathbb{F}, \mathcal{F}^\bullet)$ for the inverse of the element $[\mathrm{id} - \mathfrak{F}_{\mathbb{F}}]$.

The element $\mathcal{L}_G(X/\mathbb{F}, \mathcal{F}^\bullet)$ may be thought of as our analogue of the noncommutative ℓ-adic L-function that is conjectured to exist in [CFK+05]. Note that the assignment $\mathcal{F}^\bullet \mapsto \mathcal{L}_G(X/\mathbb{F}, \mathcal{F}^\bullet)$ extends to a homomorphism

$$\mathrm{K}_0(\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)) \to \mathrm{K}_1(w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])).$$

Moreover, $\mathcal{L}_G(X/\mathbb{F}, \mathcal{F}^\bullet)$ enjoys the following transformation properties.

Theorem 8.4. *Consider a separated scheme X of finite type over a finite field $\mathbb{F}$. Let Λ be any adic $\mathbb{Z}_\ell$-algebra and let $\mathcal{F}^\bullet$ be a complex in $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$.*

(1) *Let Λ' be another adic $\mathbb{Z}_\ell$-algebra. For any complex $M^\bullet$ in $\Lambda[[G]]^{\mathrm{op}}$-$\mathbf{SP}(\Lambda')$, we have*

$$\Psi_{M[[G]]^{\delta\bullet}}(\mathcal{L}_G(X/\mathbb{F}, \mathcal{F}^\bullet)) = \mathcal{L}_G(X/\mathbb{F}, \Psi_{\widetilde{M}^\bullet}\mathcal{F}^\bullet)$$

in $\mathrm{K}_1(w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda'[[G]]))$.

(2) *Let H' be a closed virtual pro-ℓ-subgroup of H which is normal in G. Then*

$$\Psi_{\Lambda[[G/H']]}(\mathcal{L}_G(X/\mathbb{F}, \mathcal{F}^\bullet)) = \mathcal{L}_{G/H'}(X/\mathbb{F}, \mathcal{F}^\bullet)$$

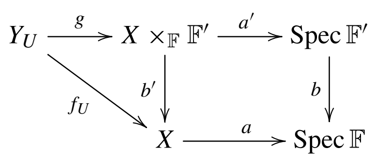
in $\mathrm{K}_1(w_{H'}\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G/H']]))$.

(3) *Let U be an open subgroup of G and let $\mathbb{F}'$ be the finite extension corresponding to the image of U in Γ_{ℓ^∞}. Then*

$$\Psi_{\Lambda[[G]]}(\mathcal{L}_G(X/\mathbb{F}, \mathcal{F}^\bullet)) = \mathcal{L}_U(Y_U/\mathbb{F}', f_U^* \mathcal{F}^\bullet)$$

in $\mathrm{K}_1(w_{H\cap U}\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[U]]))$.

Proof. Assertions (1) and (2) follow from Propositions 6.7 and 6.5, respectively, in conjunction with Propositions 4.7 and 5.9. For (3) we need the following additional reasoning. Consider the commutative diagram



There exists a chain of quasi-isomorphisms

$$\mathrm{R}\,a_!\,\mathrm{R}\,f_{U\,!} f_!^U f^{U*} f_U^* \mathcal{F}^\bullet \sim b_!\,\mathrm{R}(a' \circ g)_! f_!^U f^{U*} f_U^* \mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\mathrm{cont}}(\operatorname{Spec}\mathbb{F}, \Lambda[[U]])$. Therefore, it suffices to understand the operation of the Frobenius $\mathfrak{F}_{\mathbb{F}}$ on $\Gamma(\operatorname{Spec}\overline{\mathbb{F}}, b_!\mathcal{F})$ for any étale sheaf $\mathcal{F}$ on $\operatorname{Spec}\mathbb{F}'$. Choosing an embedding of $\mathbb{F}'$ into $\overline{\mathbb{F}}$ we obtain an isomorphism

$$\Gamma(\operatorname{Spec}\overline{\mathbb{F}}, b_!\mathcal{F}) \cong \Gamma(\operatorname{Spec}\overline{\mathbb{F}}, \mathcal{F})^{[\mathbb{F}':\mathbb{F}]}$$

under which the Frobenius $\mathfrak{F}_{\mathbb{F}}$ on the left-hand side corresponds to multiplication with the matrix

$$M = \begin{pmatrix} 0 & \dots\dots\dots & & 0 & \mathfrak{F}_{\mathbb{F}'} \\ \mathrm{id} & 0 & \dots\dots\dots & & 0 \\ 0 & \mathrm{id} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \mathrm{id} & 0 \end{pmatrix}$$

on the right-hand side. Using only elementary row and column operations one can transform $\mathrm{id} - M$ into the matrix

$$\begin{pmatrix} \mathrm{id} & 0 & \dots\dots\dots & & 0 \\ 0 & \mathrm{id} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \mathrm{id} & 0 \\ 0 & \dots & 0 & 0 & \mathrm{id} - \mathfrak{F}_{\mathbb{F}'} \end{pmatrix}.$$

Since these elementary operations have trivial image in the first K-group, we conclude

$$[\mathrm{id} - \mathfrak{F}_{\mathbb{F}} \colon \mathrm{R}\,\Gamma_c(Y_U \times_{\mathbb{F}} \overline{\mathbb{F}}, f_!^U f^{U*} f_U^* \mathcal{F}^\bullet)] = [\mathrm{id} - \mathfrak{F}_{\mathbb{F}'} \colon \mathrm{R}\,\Gamma_c(Y_U \times_{\mathbb{F}'} \overline{\mathbb{F}}, f_!^U f^{U*} f_U^* \mathcal{F}^\bullet)]$$

in $\mathrm{K}_1(w_{H\cap U}\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[U]]))$, from which the assertion follows. □

Let now Ω be a commutative adic $\mathbb{Z}_\ell$-algebra. Consider the set

$$P = \{f(T) \in \Omega[T] \colon f(0) \in \Omega^\times\}$$

in the polynomial ring $\Omega[T]$. Then $\Omega[T]_P$ is a commutative semilocal ring and the natural homomorphism of $\mathrm{K}_1(\Omega[T]_P) = \Omega[T]_P^\times$ to $\mathrm{K}_1(\Omega[[T]]) = \Omega[[T]]^\times$ is an injection. Furthermore, let k be prime to ℓ and

$$S = \{f \in \Omega[[\Gamma_{k\ell^\infty}]] \colon [f] \in \Omega/\mathrm{Jac}(\Omega)[[\Gamma_{k\ell^\infty}]] \text{ is a nonzerodivisor}\}$$

be the set in $\Omega[[\Gamma_{k\ell^\infty}]]$ that we considered in Lemma 4.2. We write γ for the image of the Frobenius $\mathfrak{F}_{\mathbb{F}}$ in $\Gamma_{k\ell^\infty}$.

Lemma 8.5. *The homomorphism*

$$\Omega[T] \to \Omega[[\Gamma_{k\ell^\infty}]], \qquad T \mapsto \gamma^{-1},$$

maps P into S.

Proof. We can replace Ω by $\Omega/\mathrm{Jac}(\Omega)$, which is a finite product of finite fields. By considering each component separately, we may assume that Ω is a finite field. Enlarging Ω if necessary, we have an isomorphism

$$\Omega[[\Gamma_{k\ell^\infty}]] \cong \prod_{\chi\colon \mathbb{Z}/k\mathbb{Z}\to\Omega^\times} \Omega[[\Gamma_{\ell^\infty}]].$$

Recall that

$$\Omega[[\Gamma_{\ell^\infty}]] \to \Omega[[T]], \qquad \gamma^{-1} \mapsto T+1,$$

is an isomorphism. Now it suffices to remark that for any nonzero polynomial $f(T) \in \Omega[T]$ and any $u \in \Omega^\times$, $f(u(T+1))$ is again a nonzero polynomial. □

Extending the classical definition, we define as in [Wit09] the L-function for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(X,\Omega)$ as the element

$$L(\mathcal{F}^\bullet, T) = \prod_{x\in X^0} [\mathrm{id} - T^{\deg(x)}\mathfrak{F}_{k(x)} \colon \Psi_{\Omega[[T]]}(\mathcal{F}^\bullet_\xi)]^{-1} \in \mathrm{K}_1(\Omega[[T]]) \cong \Omega[[T]]^\times.$$

Here, the product extends over the set X^0 of closed points of X,

$$\mathfrak{F}_{k(x)} = \mathfrak{F}_{\mathbb{F}}^{\deg(x)} \in \mathrm{Gal}(\overline{k(x)}/k(x))$$

denotes the geometric Frobenius of the residue field $k(x)$, and ξ is a geometric point over x.

We are ready to establish the link between the classical L-function and the element $\mathcal{L}_G(X/\mathbb{F},\mathcal{F}^\bullet)$. For this, it is essential to impose the additional condition that ℓ is different from the characteristic of $\mathbb{F}$.

Theorem 8.6. *Let X be a separated scheme in $\mathbf{Sch}_{\mathbb{F}}$, let ℓ be different from the characteristic of $\mathbb{F}$, and let $(f\colon Y\to X, G)$ be an ℓ-admissible principal covering containing the cyclotomic $\Gamma_{k\ell^\infty}$-covering. Furthermore, let Λ and Ω be adic $\mathbb{Z}_\ell$-algebras with Ω commutative. For every $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(X,\Lambda)$ and every $M^\bullet$ in $\Lambda[[G]]^{\mathrm{op}}$-$\mathbf{SP}(\Omega)$, we have $L(\Psi_{\widetilde{M}^\bullet}(\mathcal{F}^\bullet), T) \in \mathrm{K}_1(\Omega[T]_P)$ and*

$$\Psi_{\Omega[[\Gamma_{k\ell^\infty}]]}\Psi_{M[[G]]^{\delta\bullet}}(\mathcal{L}_G(X/\mathbb{F},\mathcal{F}^\bullet)) = L(\Psi_{\widetilde{M}^\bullet}(\mathcal{F}^\bullet), \gamma^{-1})$$

in $\mathrm{K}_1(\Omega[[\Gamma_{k\ell^\infty}]]_S)$.

Proof. By Theorem 8.4 it suffices to consider the case $G = \Gamma_{k\ell^\infty}$, $\Lambda = \Omega$, and $M^\bullet = \Omega$. Let $p\mathbf{SP}(\Omega[T])$ be the Waldhausen category of strictly perfect complexes of $\Omega[T]$-modules with quasi-isomorphisms being the morphisms which become quasi-isomorphisms in $\mathbf{SP}(\Omega[[T]])$, that means, precisely those whose cone has P-torsion cohomology groups. Then

$$\mathrm{K}_n(p\mathbf{SP}(\Omega[T])) = \mathrm{K}_n(\Omega[T]_P)$$

for $n \geq 1$ according to [WY92]. It is easy to show that there exists a strictly perfect complex $Q^\bullet$ of Ω-modules with an endomorphism f and a quasi-isomorphism $q\colon Q^\bullet \to \mathrm{R}\,\Gamma_c(\overline{X}, \mathcal{F}^\bullet)$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} Q^\bullet & \xrightarrow{q} & \mathrm{R}\,\Gamma_c(\overline{X}, \mathcal{F}^\bullet) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathfrak{F}_{\mathbb{F}}} \\ Q^\bullet & \xrightarrow{q} & \mathrm{R}\,\Gamma_c(\overline{X}, \mathcal{F}^\bullet) \end{array}$$

(see e.g. [Wit08, Lemma 3.3.2]). By the Grothendieck trace formula [Del77, Fonction L mod ℓ^n, Theorem 2.2] (or also [Wit09, Theorem 7.2]) we know that

$$L(\mathcal{F}^\bullet, T) = [\mathrm{id} - T\mathfrak{F}_{\mathbb{F}}\colon \Psi_{\Omega[[T]]}\,\mathrm{R}\,\Gamma_c(\overline{X}, \mathcal{F}^\bullet)]^{-1}$$

in $\mathrm{K}_1(\Omega[[T]])$. Hence, the above homotopy-commutative diagram implies

$$L(\mathcal{F}^\bullet, T) = [\mathrm{id} - Tf\colon \Psi_{\Omega[[T]]}Q^\bullet]^{-1}$$

in $\mathrm{K}_1(\Omega[[T]])$ and by Proposition 7.2 also

$$\mathcal{L}_{\Gamma_{k\ell^\infty}}(X/\mathbb{F}, \mathcal{F}^\bullet) = [\mathrm{id} - \gamma^{-1}f\colon Q^\bullet]^{-1}$$

in $\mathrm{K}_1(\Omega[[\Gamma_{k\ell^\infty}]]_S)$. Hence, $L(\mathcal{F}^\bullet, T)$ and $\mathcal{L}_{\Gamma_{k\ell^\infty}}(X/\mathbb{F}, \mathcal{F}^\bullet)$ are the images of the element $[\mathrm{id} - Tf\colon \Omega[T] \otimes_\Omega Q^\bullet]^{-1}$ under the homomorphisms

$$\mathrm{K}_1(\Omega[T]_P) \to \mathrm{K}_1(\Omega[[T]]), \qquad \mathrm{K}_1(\Omega[T]_P) \to \mathrm{K}_1(\Omega[[\Gamma_{k\ell^\infty}]]_S),$$

respectively. □

Remark 8.7. If Ω is noncommutative we do not expect that

$$\mathrm{K}_1(p\mathbf{SP}(\Omega[T])) \to \mathrm{K}_1(\Omega[[T]])$$

is always injective. However, the construction in the above proof identifies a canonical preimage of $L(\mathcal{F}^\bullet, T)$ in the group $\mathrm{K}_1(p\mathbf{SP}(\Omega[T]))$. Indeed, one checks that the element $[\mathrm{id} - \gamma^{-1}f\colon Q^\bullet]$ does not depend on the particular choice of $Q^\bullet$ and f. This preimage can then be used to extend Theorem 8.6 to noncommutative rings Ω.

Theorem 1.1 in the introduction is easily seen to be a special case of Theorem 8.1, Corollary 8.2, and Theorem 8.6 for $\Lambda = \mathbb{Z}_\ell$ and $(f\colon Y \to X, G)$ being a subcovering of the universal covering of a connected scheme X.

Remark 8.8. As shown in [Wit13, Cor. 3.3], one can construct for $i \geq 0$ a section

$$s\colon \mathrm{K}_i(\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])) \to \mathrm{K}_{i+1}(w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]]))$$

of the boundary homomorphism d. We can use s to define an "algebraic L-function"

$$\mathcal{L}_G^{\mathrm{alg}}(M^\bullet) = s(M^\bullet) \in \mathrm{K}_1(w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]]))$$

for any complex $M^\bullet \in \mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$, extending the definition of Burns in [Bur09]. The quotient $u(\mathcal{F}^\bullet) = \mathcal{L}_G(\mathcal{F}^\bullet)/\mathcal{L}_G^{\mathrm{alg}}(\mathrm{R}\,\Gamma_c(X, \mathcal{F}^\bullet))$ is then a well-defined element of $\mathrm{K}_1(\Lambda[[G]])$. In general, $u(\mathcal{F}^\bullet)$ will not be trivial. If for example $X = \operatorname{Spec}\mathbb{F}$ with a finite field $\mathbb{F}$ with q elements, ℓ is a prime not dividing $q(q-1)$, $G = \Gamma_{\ell^\infty}$, $\mathcal{F}^\bullet = \mathbb{Z}_\ell(1)$, then $\mathcal{L}_G(\mathbb{Z}_\ell(1)) = (1 - q^{-1}\gamma^{-1})^{-1}$ is a unit in $\mathbb{Z}_\ell[[\Gamma_{\ell^\infty}]]$. This implies that the class of $\mathrm{R}\,\Gamma_c(\operatorname{Spec}\mathbb{F}, \mathbb{Z}_\ell(1))$ in

$$\begin{aligned}\mathrm{K}_0(\mathbf{PDG}^{\mathrm{cont},w_H}(\mathbb{Z}_\ell[[\Gamma_{\ell^\infty}]])) &= \mathrm{K}_0(\mathbb{Z}_\ell[[\Gamma_{\ell^\infty}]], \mathbb{Z}_\ell[[\Gamma_{\ell^\infty}]]_{(\ell)}) \\ &= \mathbb{Z}_\ell[[\Gamma_{\ell^\infty}]]_{(\ell)}^\times/\mathbb{Z}_\ell[[\Gamma_{\ell^\infty}]]^\times\end{aligned}$$

is trivial, and hence the algebraic L-function $\mathcal{L}_G^{\mathrm{alg}}(\mathrm{R}\,\Gamma_c(\operatorname{Spec}\mathbb{F}, \mathbb{Z}_\ell(1)))$ must be trivial, too. Thus, $u(\mathbb{Z}_\ell(1)) = (1 - q^{-1}\gamma^{-1})^{-1} \neq 1$ in this case. It would be interesting to find more explicit descriptions of $u(\mathcal{F}^\bullet)$ and of its image under the natural projection map $\mathrm{K}_1(\Lambda[[G]]) \to \mathrm{K}_1(\Lambda)$.

Appendix

Let $\mathbf{wW}$ be a Waldhausen category with weak equivalences $\mathbf{w}$, and let $\mathbf{vW}$ be the same category with the same notion of cofibrations, but with a coarser notion $\mathbf{v} \subset \mathbf{w}$ of weak equivalences. We assume that $\mathbf{wW}$ is *saturated* and *extensional*, i.e.

(1) if f and g are composable and any two of the morphisms f, g and $g \circ f$ are in $\mathbf{w}$, then so is the third;
(2) if the two outer components of a morphism of exact sequences in $\mathbf{wW}$ are in $\mathbf{w}$, then so is the middle one.

We denote by $\mathbf{vW^w}$ the full subcategory of $\mathbf{vW}$ consisting of those objects A such that $0 \to A$ is in $\mathbf{w}$. With the notions of cofibrations and weak equivalences in $\mathbf{vW}$, this subcategory is again a Waldhausen category.

Under the additional assumption that there exists an appropriate notion of cylinder functors in $\mathbf{vW}$ and $\mathbf{wW}$, which we will explain below, Waldhausen's localisation theorem [TT90, Theorem 1.8.2] states that the natural inclusion functors $\mathbf{vW^w} \to \mathbf{vW} \to \mathbf{wW}$ induce a homotopy fibre sequence of the associated K-theory spaces, and hence a long exact sequence

$$\begin{aligned}\cdots \to \mathrm{K}_n(\mathbf{vW^w}) \to \mathrm{K}_n(\mathbf{vW}) \to \mathrm{K}_n(\mathbf{wW}) \xrightarrow{d} \mathrm{K}_{n-1}(\mathbf{vW^w}) \to \cdots \\ \to \mathrm{K}_1(\mathbf{vW}) \to \mathrm{K}_1(\mathbf{wW}) \xrightarrow{d} \mathrm{K}_0(\mathbf{vW^w}) \to \mathrm{K}_0(\mathbf{vW}) \to \mathrm{K}_0(\mathbf{wW}) \to 0.\end{aligned}$$

In this appendix, we will give an explicit description of the connecting homomorphism $d\colon \mathrm{K}_1(\mathbf{wW}) \to \mathrm{K}_0(\mathbf{vW^w})$ in terms of the 1-types of the Waldhausen categories, as defined in [MT07]. A similar description has also been derived in [Sta10, Theorem 4.1] (up to some obvious sign errors) using more sophisticated arguments.

We begin by recalling the definition of a cylinder functor. For any Waldhausen category $\mathbf{W}$, the *category of morphisms* $\mathbf{Mor}(\mathbf{W})$ is again a Waldhausen category with the following cofibrations and weak equivalences. A morphism $\alpha \to \beta$ in $\mathbf{Mor}(\mathbf{W})$, i.e. a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\epsilon} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A' & \xrightarrow{\epsilon'} & B' \end{array}$$

is a cofibration if both ϵ and ϵ' are cofibrations. It is a weak equivalence if both ϵ and ϵ' are weak equivalences. One checks easily that the functors

$$\begin{aligned} s\colon \mathbf{Mor}(\mathbf{W}) \to \mathbf{W}, &\quad (A \xrightarrow{\alpha} A') \mapsto A, \\ t\colon \mathbf{Mor}(\mathbf{W}) \to \mathbf{W}, &\quad (A \xrightarrow{\alpha} A') \mapsto A', \\ S\colon \mathbf{W} \to \mathbf{Mor}(\mathbf{W}), &\quad A \mapsto (A \to 0), \\ T\colon \mathbf{W} \to \mathbf{Mor}(\mathbf{W}), &\quad A \mapsto (0 \to A), \end{aligned}$$

are Waldhausen exact. Let $\mathrm{ar}\colon s \to t$ denote the natural transformation given by $\mathrm{ar}(\alpha) = \alpha$.

For two Waldhausen categories $\mathbf{W}_1$ and $\mathbf{W}_2$ we let

$$\mathbf{Fun}(\mathbf{W}_1, \mathbf{W}_2)$$

denote the *Waldhausen category of exact functors* with natural transformations as morphisms. A natural transformation $\alpha\colon F \to G$ is a cofibration if

(1) for each object C in $\mathbf{W}_1$, the morphism $\alpha(C)\colon F(C) \to G(C)$ is a cofibration,
(2) for each cofibration $C \rightarrowtail C'$ in $\mathbf{W}_1$, $G(C) \cup_{F(C)} F(C') \to G(C')$ is a cofibration.

A natural transformation $\alpha\colon F \to G$ is a weak equivalence if for each object C in $\mathbf{W}_1$, the morphism $\alpha(C)\colon F(C) \to G(C)$ is a weak equivalence.

Definition A.1. A *cylinder functor* for $\mathbf{W}$ is an exact functor

$$\mathrm{Cyl}\colon \mathbf{Mor}(\mathbf{W}) \to \mathbf{W}$$

together with natural transformations $j_1\colon s \to \mathrm{Cyl}$, $j_2\colon t \to \mathrm{Cyl}$, $p\colon \mathrm{Cyl} \to t$ such that

(1) $p \circ j_1 = \mathrm{ar}$, $p \circ j_2 = \mathrm{id}$,
(2) $j_1 \oplus j_2\colon s \oplus t \rightarrowtail \mathrm{Cyl}$ is a cofibration in $\mathbf{Fun}(\mathbf{Mor}(\mathbf{W}), \mathbf{W})$,
(3) $\mathrm{Cyl} \circ T = \mathrm{id}$ and the compositions of j_2 with T and p with T are the identity transformation on id.

A cylinder functor satisfies the *cylinder axiom* if

(4) $p\colon \mathrm{Cyl} \xrightarrow{\sim} t$ is a weak equivalence in $\mathbf{Fun}(\mathbf{Mor}(\mathbf{W}), \mathbf{W})$.

Remark A.2. The above definition of a cylinder functor is clearly equivalent to the one given in [Wal85, Definition 1.6]. Thomason claims that it is also equivalent to the one given in [TT90, Definition 1.3.1]. However, it seems at least not to be completely evident from the axioms stated there that Cyl preserves pushouts along cofibrations.

We further set

$$\begin{aligned} \mathrm{Cone} = \mathrm{Cyl}/s\colon \mathbf{Mor}(\mathbf{W}) \to \mathbf{W}, &\quad (A \xrightarrow{\alpha} A') \mapsto \mathrm{Cyl}(\alpha)/A, \\ \Sigma = \mathrm{Cone} \circ S\colon \mathbf{W} \to \mathbf{W}, &\quad A \mapsto \mathrm{Cone}(A \to 0). \end{aligned}$$

Note that $t \rightarrowtail \mathrm{Cone} \twoheadrightarrow \Sigma \circ s$ is an exact sequence in $\mathbf{Fun}(\mathbf{Mor}(\mathbf{W}), \mathbf{W})$ for any cylinder functor Cyl.

Definition A.3. A *stable quadratic module* M_* is a homomorphism of groups

$$\partial_M\colon M_1 \to M_0 \quad \text{together with a pairing} \quad \langle -, - \rangle\colon M_0 \times M_0 \to M_1$$

satisfying the following identities for any $a, b \in M_1$ and $X, Y, Z \in M_0$:

(1) $\langle \partial_M a, \partial_M b \rangle = [b, a]$,
(2) $\partial_M \langle X, Y \rangle = [Y, X]$,
(3) $\langle X, Y \rangle \langle Y, X \rangle = 1$,
(4) $\langle X, YZ \rangle = \langle X, Y \rangle \langle X, Z \rangle$.

We set $a^X = a\langle X, \partial a \rangle$ for $a \in M_1$, $X \in M_0$. Note that this defines a right action of M_0 on M_1. Furthermore, we let

$$\pi_1(M_*) = \ker \partial_M, \qquad \pi_0(M_*) = \operatorname{coker} \partial_M$$

denote the *homotopy groups* of M_*.

Assume that $f\colon M_* \to N_*$ is any morphism of stable quadratic modules such that f_0 is injective. Then $f_1(N_1)$ is a normal subgroup of $\partial_N^{-1}(f_0(N_0))$. We set

$$\pi_0(M_*, N_*) = \partial_N^{-1}(f_0(N_0))/f_1(N_1)$$

and obtain an exact sequence

$$\pi_1(M_*) \to \pi_1(N_*) \to \pi_0(M_*, N_*) \to \pi_0(M_*) \to \pi_0(N_*).$$

Muro and Tonks give the following definition of the 1-type of a Waldhausen category [MT07, Definition 1.2].

Definition A.4. Let $\mathbf{W}$ be a Waldhausen category. The *algebraic* 1*-type* $\mathcal{D}_*\mathbf{W}$ of $\mathbf{W}$ is the stable quadratic module generated by

(G0) the symbols $[X]$ for each object X in $\mathbf{W}$ in degree 0,
(G1) the symbols $[w]$ and $[\Delta]$ for each weak equivalence w and each exact sequence Δ in $\mathbf{W}$,

with ∂ given by

(R1) $\partial[\alpha] = [B]^{-1}[A]$ for $\alpha\colon A \xrightarrow{\sim} B$,
(R2) $\partial[\Delta] = [B]^{-1}[C][A]$ for $\Delta\colon A \rightarrowtail B \twoheadrightarrow C$,

and

(R3) $\langle [A],[B]\rangle = [B \rightarrowtail A\oplus B \twoheadrightarrow A]^{-1}[A \rightarrowtail A\oplus B \twoheadrightarrow B]$ for any pair of objects A, B.

Moreover, we impose the following relations:

(R4) $[0 \rightarrowtail 0 \twoheadrightarrow 0] = 1_{\mathcal{D}_1}$,

(R5) $[\beta\alpha] = [\beta][\alpha]$ for $\alpha\colon A \xrightarrow{\sim} B$, $\beta\colon B \xrightarrow{\sim} C$,

(R6) $[\Delta'][\alpha][\gamma]^{[A]} = [\beta][\Delta]$ for any commutative diagram

$$\begin{array}{cccccc}
\Delta\colon & A & \rightarrowtail & B & \twoheadrightarrow & C \\
 & {\scriptstyle\alpha}\downarrow{\scriptstyle\sim} & & {\scriptstyle\beta}\downarrow{\scriptstyle\sim} & & {\scriptstyle\gamma}\downarrow{\scriptstyle\sim} \\
\Delta'\colon & A' & \rightarrowtail & B' & \twoheadrightarrow & C'
\end{array}$$

(R7) $[\Gamma_1][\Delta_1] = [\Delta_2][\Gamma_2]^{[A]}$ for any commutative diagram

$$\begin{array}{cccccc}
 & & & \Gamma_1\colon & & \Gamma_2\colon \\
\Delta_1\colon & A & \rightarrowtail & B & \twoheadrightarrow & C \\
 & \| & & \downarrow & & \downarrow \\
\Delta_2\colon & A & \rightarrowtail & D & \twoheadrightarrow & E \\
 & \downarrow & & \downarrow & & \downarrow \\
 & 0 & \rightarrowtail & F & = & F
\end{array}$$

Muro and Tonks then prove that

$$\mathrm{K}_1(\mathbf{W}) = \pi_1(\mathcal{D}_*(\mathbf{W})), \qquad \mathrm{K}_0(\mathbf{W}) = \pi_0(\mathcal{D}_*(\mathbf{W})).$$

The following theorem gives our explicit description of the connecting homomorphism.

Theorem A.5. *Let* $\mathbf{wW}$ *be a Waldhausen category and* $\mathbf{vW}$ *the same category with a coarser notion of weak equivalences. Assume that* $\mathbf{wW}$ *is saturated and extensional and let* Cyl *be a cylinder functor for both* $\mathbf{wW}$ *and* $\mathbf{vW}$ *which satisfies the cylinder axiom for* $\mathbf{wW}$. *Then the assignment*

$$d(\Delta) = 1 \qquad \text{for every exact sequence } \Delta \text{ in } \mathbf{wW},$$

$$d(\alpha) = [\mathrm{Cone}(\alpha)]^{-1}[\mathrm{Cone}(\mathrm{id}_A)] \qquad \text{for every weak equivalence } \alpha\colon A \to A' \text{ in } \mathbf{wW}$$

defines a homomorphism $d\colon \mathcal{D}_1(\mathbf{wW}) \to K_0(\mathbf{vW}^{\mathbf{w}})$, *and the sequence*

$$\mathrm{K}_1(\mathbf{vW}) \to \mathrm{K}_1(\mathbf{wW}) \xrightarrow{d} \mathrm{K}_0(\mathbf{vW}^{\mathbf{w}}) \to \mathrm{K}_0(\mathbf{vW}) \to \mathrm{K}_0(\mathbf{wW}) \to 0$$

is exact.

Proof. We may view $\mathrm{K}_0(\mathbf{vW}^{\mathbf{w}})$ as a stable quadratic module with trivial group in degree zero and $\mathrm{K}_0(\mathbf{vW}^{\mathbf{w}})$ in degree one. By the universal property of $\mathcal{D}_*(\mathbf{wW})$ it suffices to verify the following two assertions in order to show that the homomorphism $d\colon \mathcal{D}_1(\mathbf{wW}) \to \mathrm{K}_0(\mathbf{vW}^{\mathbf{w}})$ is well-defined.

(1) For commutative diagrams

$$\begin{array}{ccccc} A & \rightarrowtail & B & \twoheadrightarrow & C \\ {\scriptstyle\alpha}\downarrow{\scriptstyle\sim} & & {\scriptstyle\beta}\downarrow{\scriptstyle\sim} & & {\scriptstyle\gamma}\downarrow{\scriptstyle\sim} \\ A' & \rightarrowtail & B' & \twoheadrightarrow & C' \end{array}$$

in **wW** we have $d(\beta) = d(\alpha)d(\gamma)$.

(2) For weak equivalences $\alpha\colon A \xrightarrow{\sim} B$, $\beta\colon B \xrightarrow{\sim} C$ in **wW** we have $d(\beta\circ\alpha) = d(\beta)d(\alpha)$.

Assertion (1) follows easily by applying the exact functors Cone and $\alpha \mapsto \mathrm{Cone}(\mathrm{id}_{s(\alpha)})$ to the exact sequence $\alpha \rightarrowtail \beta \twoheadrightarrow \gamma$ in **Mor**(**wW**).

To prove (2), we first consider a weak equivalence $\alpha\colon A \xrightarrow{\sim} B$ in **wW** between objects A and B in $\mathbf{vW^w}$. The exact sequences

$$B \rightarrowtail \mathrm{Cone}(\alpha) \twoheadrightarrow \Sigma A, \qquad B \rightarrowtail \mathrm{Cone}(\mathrm{id}_B) \twoheadrightarrow \Sigma B$$

in $\mathbf{vW^w}$ imply

$$\begin{aligned} d(\alpha)d(B \xrightarrow{\sim} 0) &= [\mathrm{Cone}(\alpha)]^{-1}[\mathrm{Cone}(\mathrm{id}_A)][\Sigma B]^{-1}[\mathrm{Cone}(\mathrm{id}_B)] \\ &= [\Sigma A]^{-1}[\mathrm{Cone}(\mathrm{id}_A)] = d(A \xrightarrow{\sim} 0). \end{aligned}$$

We obtain

$$d(\beta\circ\alpha) = d(C \xrightarrow{\sim} 0)^{-1}d(A \xrightarrow{\sim} 0) = d(\beta)d(\alpha)$$

in the special case that $\alpha\colon A \xrightarrow{\sim} B$ and $\beta\colon B \xrightarrow{\sim} C$ are weak equivalences in **wW** between objects A, B, and C in $\mathbf{vW^w}$.

Let now $\alpha\colon A \xrightarrow{\sim} B$ and $\beta\colon B \xrightarrow{\sim} C$ by arbitrary weak equivalences in **wW**. Viewing the vertical morphisms in the commutative diagram

$$\begin{array}{ccccc} A & = & A & = & A \\ \| & & {\scriptstyle\alpha}\downarrow{\scriptstyle\sim} & & {\scriptstyle\beta\circ\alpha}\downarrow{\scriptstyle\sim} \\ A & \xrightarrow[\alpha]{\sim} & B & \xrightarrow[\beta]{\sim} & C \end{array}$$

as morphisms $\mathrm{id}_A \xrightarrow{\sim} \alpha \xrightarrow{\sim} \beta\circ\alpha$ in **Mor**(**wW**) and applying the exact sequence $t \rightarrowtail \mathrm{Cone} \twoheadrightarrow \Sigma s$ we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccc} A & \rightarrowtail & \mathrm{Cone}(\mathrm{id}_A) & \twoheadrightarrow & \Sigma A \\ {\scriptstyle\alpha}\downarrow{\scriptstyle\sim} & & {\scriptstyle\alpha_*}\downarrow{\scriptstyle\sim} & & \| \\ B & \rightarrowtail & \mathrm{Cone}(\alpha) & \twoheadrightarrow & \Sigma A \\ {\scriptstyle\beta}\downarrow{\scriptstyle\sim} & & {\scriptstyle\beta_*}\downarrow{\scriptstyle\sim} & & \| \\ C & \rightarrowtail & \mathrm{Cone}(\beta\circ\alpha) & \twoheadrightarrow & \Sigma A \end{array}$$

Assertion (1) and the previously proved special case of (2) imply

$$d(\beta \circ \alpha) = d(\beta_* \circ \alpha_*) = d(\beta_*)d(\alpha_*) = d(\beta)d(\alpha).$$

This completes the proof of (2) in general. Hence, we have also proved the existence of the homomorphism $d\colon \mathcal{D}_1(\mathbf{wW}) \to \mathrm{K}_0(\mathbf{vW^w})$.

We will now prove the exactness of the sequence in the statement of the theorem. Note that $\mathcal{D}_0(\mathbf{vW^w})$ injects into $\mathcal{D}_0(\mathbf{vW})$ and that $\mathcal{D}_0(\mathbf{vW}) = \mathcal{D}_0(\mathbf{wW})$. Write $K = \pi_0(\mathcal{D}_*(\mathbf{vW}), \mathcal{D}_*(\mathbf{wW}))$, i.e. K is the cokernel of the natural homomorphism $\mathcal{D}_1(\mathbf{vW}) \to \mathcal{D}_1(\mathbf{wW})$. As explained above, the sequence of abelian groups

$$\mathrm{K}_1(\mathbf{vW}) \to \mathrm{K}_1(\mathbf{wW}) \to K \to \mathrm{K}_0(\mathbf{vW}) \to \mathrm{K}_0(\mathbf{wW}) \to 0$$

is exact.

Let $\alpha\colon A \xrightarrow{\sim} B$ be a weak equivalence in $\mathbf{vW}$. Since $\mathrm{Cone}\colon \mathbf{Mor}(\mathbf{vW}) \to \mathbf{vW}$ is an exact functor, we see that the induced morphism $\alpha_*\colon \mathrm{Cone}(\mathrm{id}_A) \xrightarrow{\sim} \mathrm{Cone}(\alpha)$ is a weak equivalence in $\mathbf{vW^w}$. Hence,

$$d(\alpha) = d(\alpha_*) = 1,$$

i.e. the homomorphism d factors through K. It remains to show that $d\colon K \to \mathrm{K}_0(\mathbf{vW^w})$ is an isomorphism.

Consider the homomorphism $h\colon \mathcal{D}_0(\mathbf{vW^w}) \to \mathcal{D}_1(\mathbf{wW})$ induced by sending an object X of $\mathbf{vW^w}$ to $[X \xrightarrow{\sim} 0]$ in $\mathcal{D}_1(\mathbf{wW})$. One checks easily that $h \circ \partial_{\mathcal{D}_*(\mathbf{vW^w})}$ agrees with the natural homomorphism $\mathcal{D}_1(\mathbf{vW^w}) \to \mathcal{D}_1(\mathbf{wW})$; hence, h induces a homomorphism $H\colon \mathrm{K}_0(\mathbf{vW^w}) \to K$.

For any weak equivalence $\alpha\colon A \xrightarrow{\sim} B$ in $\mathbf{wW}$ we have

$$\begin{aligned} H(d(\alpha)) &= [\mathrm{Cone}(\alpha) \xrightarrow{\sim} 0]^{-1}[\mathrm{Cone}(\mathrm{id}_A) \xrightarrow{\sim} 0] \\ &= [\mathrm{Cone}(\mathrm{id}_A) \xrightarrow{\alpha_*} \mathrm{Cone}(\alpha)] = [\alpha]; \end{aligned}$$

for any object X in $\mathbf{vW^w}$ we have

$$d(H(X)) = [\Sigma X]^{-1}[\mathrm{Cone}(\mathrm{id}_X)] = [X].$$

Therefore, $d\colon K \to \mathrm{K}_0(\mathbf{vW^w})$ is indeed an isomorphism with inverse H. □

Note that if Cyl also satisfies the cylinder axiom in $\mathbf{vW}$, then $[\mathrm{Cone}(\mathrm{id}_A)] = 1$ in $\mathrm{K}_0(\mathbf{vW^w})$ for every object A in $\mathbf{wW}$. This will be the case in the most common situations.

Assume that S is a left denominator set in a ring R and let $\mathbf{vW}$ be the category of strictly perfect complexes of left R-modules with the class $\mathbf{v}$ of quasi-isomorphisms as weak equivalences. Let further $\mathbf{w}$ be the class of complex morphisms which become quasi-isomorphisms after localisation with respect to S. Then the usual cylinder functor Cyl satisfies the cylinder axiom in $\mathbf{vW}$ and $\mathbf{wW}$ and the resulting long exact localisation sequence identifies with the localisation sequence

$$\begin{aligned} \cdots \to \mathrm{K}_n(R, S^{-1}R) \to \mathrm{K}_n(R) \to \mathrm{K}_n(S^{-1}R) \xrightarrow{d} \mathrm{K}_{n-1}(R, S^{-1}R) \to \cdots \\ \to \mathrm{K}_1(R) \to \mathrm{K}_1(S^{-1}R) \xrightarrow{d} \mathrm{K}_0(R, S^{-1}R) \to \mathrm{K}_0(R) \to \mathrm{K}_0(S^{-1}R) \end{aligned}$$

[WY92]. It is then easy to see that the boundary homomorphism constructed in Theorem A.5 satisfies the formula stated in [WY92, p. 2], and hence agrees with the classical boundary homomorphism in this situation.

The above theorem can also be applied to derive the description of Weiss' generalised Whitehead torsion given in [Mur08, Remark 6.3].

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