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One-parameter contractions of Lie–Poisson brackets

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Abstract. We consider contractions of Lie and Poisson algebras and the behaviour of their centres under contractions. A polynomial Poisson algebra $\mathcal{A} = \mathbb{K}[\mathbb{A}^n]$ is said to be of Kostant type if its centre $Z(\mathcal{A})$ is freely generated by homogeneous polynomials $F_1, \ldots, F_r$ that give Kostant's regularity criterion on $\mathbb{A}^n$ ($d_x F_i$ are linearly independent if and only if the Poisson tensor has maximal rank at x). If the initial Poisson algebra is of Kostant type and F_i satisfy a certain degree equality, then the contraction is also of Kostant type. The general result is illustrated by two examples. Both are contractions of a simple Lie algebra $\mathfrak{g}$ corresponding to a decomposition $\mathfrak{g} = \mathfrak{h} \oplus V$, where $\mathfrak{h}$ is a subalgebra. Here $\mathcal{A} = \mathcal{S}(\mathfrak{g}) = \mathbb{K}[\mathfrak{g}^*]$, $Z(\mathcal{A}) = \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$, and the contracted Lie algebra is a semidirect product of $\mathfrak{h}$ and an Abelian ideal isomorphic to $\mathfrak{g}/\mathfrak{h}$ as an $\mathfrak{h}$-module. In the first example, $\mathfrak{h}$ is a symmetric subalgebra, and in the second, it is a Borel subalgebra and V is the nilpotent radical of an opposite Borel.

1. Introduction

Let $\mathfrak{q}$ be a finite-dimensional Lie algebra defined over a field $\mathbb{K}$ of characteristic zero. Then the symmetric algebra $\mathcal{S}(\mathfrak{q}) = \mathbb{K}[\mathfrak{q}^*]$ carries a Poisson structure induced by the Lie bracket on $\mathfrak{q}$. In this paper, we study the algebra $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ of symmetric invariants. By a theorem of Duflo, it is isomorphic to the centre $Z\mathbf{U}(\mathfrak{q})$ of the universal enveloping algebra $\mathbf{U}(\mathfrak{q})$, and therefore is of much interest in representation theory. We can also say that $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ coincides with the Poisson centre $Z\mathcal{S}(\mathfrak{q})$ of $\mathcal{S}(\mathfrak{q})$ (for the definition of this object see Section 2), and one can employ methods of Poisson geometry to investigate this algebra.

To be more precise, the Lie algebra in question is a contraction of some other Lie algebra, whose symmetric invariants are well understood. Already contractions of simple (non-Abelian) Lie algebras provide a fairly interesting and not yet completely explored field of research. Let $\mathfrak{f} \subset \mathfrak{q}$ be a Lie subalgebra and $V \subset \mathfrak{q}$ a complementary subspace, not necessarily $\mathfrak{f}$-stable. Then one contracts $\mathfrak{q}$ to a Lie algebra $\tilde{\mathfrak{q}} = \mathfrak{f} \ltimes V$, where V is an Abelian ideal and the action of $\mathfrak{f}$ on it comes from the projection $\mathrm{pr}_V : \mathfrak{q} \to V$ along $\mathfrak{f}$. A more sophisticated description of contractions of Poisson and Lie algebras is given in Section 3.

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Suppose that $\mathfrak{g}$ is a reductive Lie algebra. Let $F_1, \ldots, F_\ell$ with $\ell = \mathrm{rk}\,\mathfrak{g}$ be homogeneous generators of $Z\mathcal{S}(\mathfrak{g})$. Then, by [K, Theorem 9], their differentials $d_\xi F_i$ at a point $\xi \in \mathfrak{g}^*$ are linearly independent if and only if $\dim \mathfrak{g}_\xi = \ell$ where $\mathfrak{g}_\xi$ the stabiliser in the coadjoint action. This is known as *Kostant's regularity criterion*. For an arbitrary Lie algebra $\mathfrak{q}$, the notion of *index*, $\mathrm{ind}\,\mathfrak{q} = \min_{\xi\in\mathfrak{q}^*} \dim \mathfrak{q}_\xi$, generalises the rank in the reductive case. A Lie algebra $\mathfrak{q}$ of index ℓ is said to be of *Kostant type* if $Z\mathcal{S}(\mathfrak{q})$ is freely generated by homogeneous polynomials $H_1, \ldots, H_\ell$ that give Kostant's regularity criterion on $\mathfrak{q}^*$. Set $\mathfrak{q}^*_{\rm sing} := \{\xi \in \mathfrak{q}^* \mid \dim \mathfrak{q}_\xi > \mathrm{ind}\,\mathfrak{q}\}$. We say that $\mathfrak{q}$ has the *"codim-2" property* or satisfies a *"codim-2" condition* if $\dim \mathfrak{q}^*_{\rm sing} \le \dim \mathfrak{q} - 2$. The importance of this condition was first noticed in [PPY] and [P07].

The decomposition $\mathfrak{q} = \mathfrak{f} \oplus V$ induces a bi-grading on $\mathcal{S}(\mathfrak{q})$. For each homogeneous $F \in \mathcal{S}(\mathfrak{q})$, let $\deg_t F$ denote its degree in V and $F^\bullet$ the bi-homogeneous component of F of bi-degree $(\deg F - \deg_t F, \deg_t F)$ in $\mathfrak{f}$ and V, respectively. In case $\mathfrak{q} = \mathfrak{g}$ is reductive and $\tilde{\mathfrak{g}}$ is the contraction of $\mathfrak{g}$ corresponding to the decomposition $\mathfrak{g} = \mathfrak{f} \oplus V$, a simplification of our main result, Theorem 3.8, can be formulated as follows.

Suppose that $\mathrm{ind}\,\tilde{\mathfrak{g}} = \ell = \mathrm{rk}\,\mathfrak{g}$. Then

- $\sum \deg_t F_i \ge \dim V$ and the polynomials $F_i^\bullet$ are algebraically independent if and only if $\sum \deg_t F_i = \dim V$.
- Moreover, if equality holds and the polynomials $F_i^\bullet$ generate $Z\mathcal{S}(\tilde{\mathfrak{g}})$ (this can be guaranteed by the "codim-2" property of $\tilde{\mathfrak{g}}$), then $\tilde{\mathfrak{g}}$ is of Kostant type.

The proof of Theorem 3.8 relies on Lemma 2.1, a statement about Poisson brackets in the algebraic setting, and good behaviour of the Poisson tensor under contractions. The resulting algebras $\tilde{\mathfrak{g}}$ are non-reductive and there is no general method for describing their symmetric invariants.

Note that Theorem 3.8 is stated and proved for an arbitrary polynomial Poisson algebra, which is not necessarily the symmetric algebra of any finite-dimensional $\mathfrak{q}$. We do not consider applications of the more general version in this paper, but hope to explore this subject in the (near) future.

Two types of contractions $\mathfrak{g} \rightsquigarrow \tilde{\mathfrak{g}}$ are studied here. In both cases it is assumed that the ground field $\mathbb{K}$ is algebraically closed. The first contraction comes from a $\mathbb{Z}_2$-grading (or symmetric decomposition) $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of $\mathfrak{g}$. It was conjectured by D. Panyushev [P07] that in this setting $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is a polynomial algebra in ℓ variables. As was shown in [P07], $\mathrm{ind}\,\tilde{\mathfrak{g}} = \ell$ and $\tilde{\mathfrak{g}}$ has the "codim-2" property. Also for many $\mathbb{Z}_2$-gradings the polynomiality of $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ was established in that paper of Panyushev. For four of the remaining cases, we construct homogeneous generators F_i such that $\sum \deg_t F_i \le \dim \mathfrak{g}_1$. Since also $\sum \deg_t F_i \ge \dim \mathfrak{g}_1$ by Theorem 3.8, we get the equality $\sum \deg_t F_i = \dim \mathfrak{g}_1$ and thereby prove that their components $F_i^\bullet$ freely generate $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$. This line of argument resembles proofs of [PPY, Theorems 4.2 & 4.4]. Our result confirms a weaker version of Panyushev's conjecture. If the restriction homomorphism $\mathbb{K}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{K}[\mathfrak{g}_1]^{\mathfrak{g}_0}$ is surjective, then $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is a polynomial algebra in ℓ variables (see Theorem 4.5).

The second contraction $\mathfrak{g} \rightsquigarrow \tilde{\mathfrak{g}}$ was recently introduced by E. Feigin [F10] and for the resulting Lie algebra, $\tilde{\mathfrak{g}}$-invariants in $\mathcal{S}(\tilde{\mathfrak{g}})$ and $\mathbb{K}[\tilde{\mathfrak{g}}]$ were studied in [PY]. Here the decomposition is $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$, where $\mathfrak{b} = \mathrm{Lie}\,B$ is a Borel subalgebra and $\mathfrak{n}^-$ is the nilpotent radical of an opposite Borel. Complementing and relying on results of [PY], we show that $\tilde{\mathfrak{g}}$ is of Kostant type (Lemma 5.2), compute its fundamental semi-invaraint (see Definition 5.4 and Theorem 5.5), and prove that the subalgebra $\mathcal{S}(\tilde{\mathfrak{g}})_{\rm si} \subset \mathcal{S}(\tilde{\mathfrak{g}})$ generated by semi-invariants of $\tilde{\mathfrak{g}}$ (Definition 4.7) is a polynomial algebra in 2ℓ variables, Theorem 5.9. If $\mathfrak{g}$ is not of type A, then $\tilde{\mathfrak{g}}$ does not have the "codim-2" property. However, the quotient map $\mathbb{K}[\tilde{\mathfrak{g}}^*] \to \mathbb{K}[\tilde{\mathfrak{g}}^*]^{\tilde{\mathfrak{g}}}$ is equidimensional and $\mathbf{U}(\tilde{\mathfrak{g}})$ is a free $Z\mathbf{U}(\tilde{\mathfrak{g}})$-module [PY].

Finally, Section 5.2 contains a few observations related to subregular orbital varieties $\mathfrak{D}_i$, which are linear subspaces of $\mathfrak{n}$ of codimension 1 forming the complement of the open B-orbit in $\mathfrak{n}$. In particular, in Proposition 5.13, we list all $\mathfrak{D}_i$ such that the stabiliser B_x is Abelian for a generic $x \in \mathfrak{D}_i$.

2. Generalities on polynomial Poisson structures

In this section, we recall a rather important equality in Poisson algebras, which has an origin in mathematical physics [OR].

Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{A}^n = \mathbb{A}^n_{\mathbb{K}}$ the n-dimensional affine space with the algebra of regular functions $\mathcal{A} = \mathbb{K}[x_1, \dots, x_n]$. Let Ω be the algebra of regular, i.e., with polynomial coefficients, differential forms on $\mathbb{A}^n$ and $\mathcal{W}$ the algebra of derivations of $\mathcal{A}$. Both are free $\mathcal{A}$-modules with bases consisting of skew-monomials in dx_i and $\partial_i = \partial_{x_i}$, respectively. In other words, $\mathcal{W}$ is a graded skew-symmetric algebra generated by polynomial vector fields on $\mathbb{A}^n$. We identify Ω^0 with $\mathcal{A}$ and regard Ω^1 as the $\mathcal{A}$-module of global sections of the cotangent bundle $T^*\mathbb{A}^n$. Let $\mathcal{W}^1$ be an $\mathcal{A}$-module generated by ∂_i with $1 \le i \le n$. We view the exterior powers $\Omega^k = \Lambda^k_{\mathcal{A}}\Omega^1$ and $\mathcal{W}^k := \Lambda^k_{\mathcal{A}}\mathcal{W}^1$ as dual $\mathcal{A}$-modules by extending the canonical non-degenerate $\mathcal{A}$-pairing $dx_i(\partial_j) = \delta_{ij}$.

Let $\omega = dx_1 \wedge \dots \wedge dx_n$ be the volume form. If f and g are elements of Ω^k and Ω^{n-k}, respectively, then $f \wedge g = a\omega$ with $a \in \mathcal{A}$. We will say that in this situation $a = (f \wedge g)/\omega$ and f/ω is an element of $(\Omega^{n-k})^*$ such that $(f/\omega)(g) = a$. This defines an $\mathcal{A}$-linear map

$$\frac{1}{\omega} : \Omega^k \to (\Omega^{n-k})^* \cong \mathcal{W}^{n-k}.$$

Suppose that $\mathcal{A}$ has a Poisson structure $\{\ ,\ \} \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and let π denote the corresponding *Poisson tensor (bivector)*, the element of $\mathrm{Hom}_{\mathcal{A}}(\Omega^2, \mathcal{A})$ satisfying $\pi(df \wedge dg) = \{f, g\}$ for all $f, g \in \mathcal{A}$. (It is *not* assumed that the coefficients of π are linear functions.) In view of the duality between forms and vector fields, we may regard π as an element of $\mathcal{W}^2$. For $\xi \in \mathbb{A}^n$, π_ξ can be viewed as a skew-symmetric matrix with entries $\{x_i, x_j\}(\xi)$. The *index* of the Poisson algebra $\mathcal{A}$, denoted $\mathrm{ind}\,\mathcal{A}$, is defined as

$$\mathrm{ind}\,\mathcal{A} := n - \mathrm{rk}\,\pi, \quad \text{where} \quad \mathrm{rk}\,\pi = \max_{\xi \in \mathbb{A}^n} \mathrm{rk}\,\pi_\xi.$$

An element $a \in \mathcal{A}$ is said to be *central* if $\{a, \mathcal{A}\} = 0$. Correspondingly, the set $Z\mathcal{A} = Z(\mathcal{A}, \pi)$ of all central elements is called the *Poisson centre* of $\mathcal{A}$.

Set $\operatorname{Sing}\pi := \{\xi \in \mathbb{A}^n \mid \operatorname{rk}\pi_\xi < \operatorname{rk}\pi\}$. Clearly, $\operatorname{Sing}\pi$ is a proper Zariski closed subset of $\mathbb{A}^n$. By definition, $\pi(da \wedge db) = 0$ for all $a \in Z\mathcal{A}$ and all $b \in \mathcal{A}$. Hence the linear subspace $\{d_\xi a \mid a \in Z\mathcal{A}\}$ lies in the kernel of π_ξ and we have

$$\operatorname{tr.deg}_{\mathbb{K}} Z\mathcal{A} \le \operatorname{ind}\mathcal{A}.$$

For $g_1, \dots, g_m \in \mathcal{A}$, the *Jacobian locus* $\mathcal{J}(g_1, \dots, g_m)$ consists of all $\xi \in \mathbb{A}^n$ such that the differentials $d_\xi g_1, \dots, d_\xi g_m$ are linearly dependent. In other words, $\xi \in \mathcal{J}(g_1, \dots, g_m)$ if and only if $(dg_1 \wedge \dots \wedge dg_m)_\xi = 0$. The set $\mathcal{J}(g_1, \dots, g_m)$ is Zariski closed in $\mathbb{A}^n$ and it coincides with $\mathbb{A}^n$ if and only if $g_1, \dots, g_m$ are algebraically dependent.

Given $k \in \mathbb{N}$, we let

$$\Lambda^k\pi := \underbrace{\pi \wedge \dots \wedge \pi}_{k \text{ factors}}$$

be an element of $\mathcal{W}^{2k}$. Note that $\Lambda^k\pi \ne 0$ if and only if π_ξ contains a non-zero $2k \times 2k$-minor for some $\xi \in \mathbb{A}^n$. Therefore $\Lambda^k\pi = 0$ for $k > (\operatorname{rk}\pi)/2$ and $\Lambda^k\pi \ne 0$ for $k \le (\operatorname{rk}\pi)/2$. The following statement can be extracted from the proofs of [OR, Theorem 3.1], [PPY, Theorem 1.2], [P07, Theorem 1.2].

Lemma 2.1. *Let $\mathcal{A} = \mathbb{K}[x_1, \dots, x_n]$ be a Poisson algebra of index ℓ and let $\{F_1, \dots, F_\ell\} \subset Z\mathcal{A}$ be a set of algebraically independent polynomials. Then there are coprime $q_1, q_2 \in \mathcal{A} \setminus \{0\}$ such that*

$$q_1 \frac{dF_1 \wedge \dots \wedge dF_\ell}{\omega} = q_2 \Lambda^{(n-\ell)/2}\pi.$$

Proof. Set $\mathcal{F} = dF_1 \wedge \dots \wedge dF_\ell$. Take any $F \in Z\mathcal{A}$. Because of the inequality $\operatorname{tr.deg} Z(\mathcal{A}) \le \ell$, the polynomials $F_1, \dots, F_\ell$, and F are algebraically dependent, hence $\mathcal{F} \wedge dF = 0$. Moreover, $\pi(dF, \cdot) = 0$ and $\Lambda^{(n-\ell)/2}\pi$ is zero on $dF \wedge \Omega^{n-\ell-1}$.

Changing the ordering of the coordinates, we may assume $\mathcal{F} \wedge dx_1 \wedge \dots \wedge dx_{n-\ell} \ne 0$. Let $\xi \in \mathbb{A}^n$ be such that $\operatorname{rk}\pi_\xi = n - \ell$ and the elements $d_\xi F_i$ together with $\{dx_j \mid j \le n - \ell\}$ form a basis of $T^*_\xi\mathbb{A}^n$. Then

$$\Lambda^{n-\ell}(T^*_\xi\mathbb{A}^n) = \operatorname{Ker}(\mathcal{F}/\omega)_\xi \oplus \mathbb{K}(dx_1 \wedge \dots \wedge dx_{n-\ell}).$$

Note that here the kernel of $(\mathcal{F}/\omega)_\xi$ lies also in the kernel of $\Lambda^{(n-\ell)/2}\pi_\xi$. Next $\Lambda^{(n-\ell)/2}\pi_\xi \ne 0$. Therefore $(\mathcal{F}/\omega)_\xi$ is a non-zero scalar multiple of $\Lambda^{(n-\ell)/2}\pi_\xi$. We can conclude that $\mathcal{F}/\omega$ and $\Lambda^{(n-\ell)/2}\pi$ are proportional on an open subset of $\mathbb{A}^n$. It follows that there exist non-zero coprime $q_1, q_2 \in \mathcal{A}$ such that $q_1(\mathcal{F}/\omega) = q_2\Lambda^{(n-\ell)/2}\pi$. □

Of particular interest are situations where $q_1, q_2 \in \mathbb{K}$ for q_1, q_2 as above. This can be guaranteed by "codim-2" conditions (see e.g. [PPY, Theorem 1.2]). If $\dim \operatorname{Sing}\pi \le n-2$, then q_1 must be a scalar. If $\dim \mathcal{J}(F_1, \dots, F_\ell) \le n - 2$, then q_2 must be a scalar.

In case π is homogeneous, i.e., all the (polynomial) coefficients of π are of the same degree, we can say that $\deg \Lambda^k\pi = k \deg\pi$. Suppose that π and all the F_i's are homogeneous. Then q_1, q_2 are also homogeneous and

$$\deg q_1 - \ell + \sum_{i=1}^{\ell} \deg F_i = \deg q_2 + \frac{n-\ell}{2} \deg \pi.$$

If, for example, $2\sum \deg F_i = 2\ell + (n-\ell)\deg \pi$, then $\deg q_1 = \deg q_2$ and knowing that q_1 is constant, we also know that q_2 is a constant.

Poisson tensors of degree 1 correspond to finite-dimensional Lie algebras $\mathfrak{q}$ over $\mathbb{K}$. In this case $\mathbb{A}^n = \mathfrak{q}^*$ is the dual space of an n-dimensional Lie algebra $\mathfrak{q}$ and $\mathcal{A} = \mathcal{S}(\mathfrak{q}) = \mathbb{K}[\mathfrak{q}^*]$ is the symmetric algebra of $\mathfrak{q}$. Set $\operatorname{ind}\mathfrak{q} = \operatorname{ind}\mathcal{S}(\mathfrak{q})$ and $\mathfrak{q}^*_{\rm sing} = \operatorname{Sing}\pi$. Note that

$$\operatorname{ind}\mathfrak{q} = \min_{\gamma\in\mathfrak{q}^*} \dim\mathfrak{q}_\gamma = \dim\mathfrak{q}_\alpha \quad \text{for all } \alpha\in\mathfrak{q}^*_{\rm reg} = \mathfrak{q}^*\setminus\mathfrak{q}^*_{\rm sing}.$$

It is also worth mentioning that $Z\mathcal{S}(\mathfrak{q}) = \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$.

Suppose that $\mathfrak{g}$ is a non-Abelian reductive Lie algebra. Then $\operatorname{ind}\mathfrak{g} = \operatorname{rk}\mathfrak{g}$, the algebra $Z\mathcal{S}(\mathfrak{g})$ of symmetric invariants is freely generated by homogeneous polynomials $F_1,\dots,F_\ell$ with $\ell = \operatorname{rk}\mathfrak{g}$, and $2\sum\deg F_i = n+\ell$. Moreover, $\dim\mathfrak{g}^*_{\rm sing} = n-3$. Therefore, after a suitable renormalisation,

$$\frac{dF_1\wedge\cdots\wedge dF_\ell}{\omega} = \Lambda^{(n-\ell)/2}\pi. \tag{2.1}$$

This is known as *Kostant's regularity criterion*: $x\in\mathfrak{g}^*_{\rm reg}$ if and only if the differentials d_xF_i are linearly independent [K, Theorem 9].

Definition 2.2. Equation (2.1) is called the *Kostant equality* and we will say that a Poisson algebra $\mathcal{A}$ (or a Lie algebra $\mathfrak{q}$) is of *Kostant type* if $Z\mathcal{A}$ is generated by ℓ polynomials satisfying the Kostant equality.

Apart from reductive and Abelian Lie algebras, examples of Lie algebras of Kostant type are provided by the centralisers $\mathfrak{g}_e$ of nilpotent elements in $\mathfrak{sl}_m$ and $\mathfrak{sp}_{2m}$ [PPY], truncated seaweed (biparabolic) subalgebras of $\mathfrak{sl}_m$ and $\mathfrak{sp}_{2m}$ [J], and semidirect products related to symmetric decompositions $\mathfrak{g} = \mathfrak{g}_0\oplus\mathfrak{g}_1$ (see [P07] and Section 4 here).

Let (e,h,f) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g} = \operatorname{Lie}G$. Then $\mathbf{S}_e = e+\mathfrak{g}_f$ is the *Slodowy slice* of Ge at e, and $\mathbb{K}[\mathbf{S}_e]$ inherits a Poisson bracket from $\mathfrak{g}$ ($\cong\mathfrak{g}^*$). These Poisson algebras are of Kostant type and they are associated graded algebras of finite W-algebras (see [Pr] and [PPY, Section 2]). Note that all the above mentioned Poisson and Lie algebras of Kostant type have the "codim-2" property.

3. Contractions of Poisson tensors

We begin with a definition of a contraction in the Lie algebra setting. Let $\mathfrak{q}$ be a Lie algebra, $\mathfrak{k}\subset\mathfrak{q}$ a Lie subalgebra, and $V\subset\mathfrak{q}$ a complementary (to $\mathfrak{k}$) subspace. We do not require V to be $\mathfrak{k}$-stable. For each $t\in\mathbb{K}^\times$, let $\varphi_t:\mathfrak{q}\to\mathfrak{q}$ be a linear map multiplying vectors in V by t and vectors in $\mathfrak{f}$ by 1. These automorphisms form a one-parameter subgroup in $\mathrm{GL}(\mathfrak{q})$. Each φ_t defines a new Lie algebra structure on the same vector space $\mathfrak{q}$. The *t-commutator* $[\ ,\ ]_t$ or the *t-Poisson bracket* $\{\ ,\ \}_t$ is given by

$$\{x, y\}_t := \varphi_t^{-1}(\{\varphi_t(x), \varphi_t(y)\})$$

for $x, y \in \mathfrak{q}$. Let $\mathrm{pr}_{\mathfrak{f}}$ and pr_V be the projection on $\mathfrak{f}$ and V, respectively. Then

$$[\xi, \eta]_t = [\xi, \eta], \quad [\xi, v]_t = t\,\mathrm{pr}_{\mathfrak{f}}([\xi, v]) + \mathrm{pr}_V([\xi, v]),$$
$$[v, w]_t = t^2\,\mathrm{pr}_{\mathfrak{f}}([v, w]) + t\,\mathrm{pr}_V([v, w]),$$

for $\xi, \eta \in \mathfrak{f}$ and $v, w \in V$. We can pass to the limit $\lim_{t\to 0}[\ ,\ ]_t$ and get yet another Lie algebra structure on $\mathfrak{q}$. Let $\tilde{\mathfrak{q}}$ denote this contraction of $\mathfrak{q}$. Then $\tilde{\mathfrak{q}} = \mathfrak{f} \ltimes V$, where V is an Abelian ideal of $\tilde{\mathfrak{q}}$ and the action of $\mathfrak{f}$ on V is given by pr_V. (The reader feeling uncomfortable with taking the limit, although it can be defined in a purely algebraic setting, may assume that t takes values in $\mathbb{Q} \subset \mathbb{K}$.)

Extending φ_t to the symmetric algebra $\mathcal{S}(\mathfrak{q})$ as well as to $\mathcal{W}$ and Ω, one can say that $\pi_t = \varphi_t^{-1}(\pi)$ and $\tilde{\pi} = \lim_{t\to 0}\pi_t$. Let $\mathfrak{q}_t$ stand for the Lie algebra corresponding to π_t. Then the Poisson centre of $\mathcal{S}(\mathfrak{q}_t)$ equals $\varphi_t^{-1}(Z\mathcal{S}(\mathfrak{q}))$.

For $H \in \mathcal{S}(\mathfrak{q})$, let $\deg_t H$ be the degree in t of $\varphi_t(H)$. This means that $\varphi_t(H) = t^d H_d + t^{d-1}H_{d-1} + \cdots + H_0$, where $d = \deg_t H$, $H_i \in \mathcal{S}(\mathfrak{q})$, and $H_d \neq 0$. We will say that $H^\bullet := H_d$ is the *highest (t-) component* of H.

Lemma 3.1. *If $H \in Z\mathcal{S}(\mathfrak{q})$, then $H^\bullet$ is a central element in $\mathcal{S}(\tilde{\mathfrak{q}})$.*

Proof. Since $H \in Z\mathcal{S}(\mathfrak{q})$, its preimage $\varphi_t^{-1}(H)$ is a central element in $\mathcal{S}(\mathfrak{q}_t)$, which one can write as $\varphi_t^{-1}(H) = t^{-d}H_d + t^{1-d}H_{d-1} + \cdots + t^{-1}H_1 + H_0$. Multiplying it by t^d, we see that $\sum_{j=0}^{d} t^{d-j}H_j$ is also a central element in $\mathcal{S}(\mathfrak{q}_t)$. Passing to the limit as $t \to 0$ shows that $H_d = H^\bullet$ is an element of $Z\mathcal{S}(\tilde{\mathfrak{q}})$. □

The automorphism $\varphi_t : \mathfrak{q} \to \mathfrak{q}$ does not need to be of degree 1 in t; also, the Poisson tensor π does not need to be linear. We can consider a one-parameter family of linear automorphisms of $\mathbb{A}^n$ and the corresponding deformation of Poisson structures on it. The only important thing is that the limit $\lim_{t\to 0}\pi_t$ exists. In order to be consistent with the Lie algebra case, we identify $\mathbb{A}^n$ with $\mathbb{K}^n$. Let φ be a $\mathbb{K}$-linear automorphism of the dual space $(\mathbb{K}^n)^*$. Then φ extends to $\mathbb{K}$-linear automorphisms of $\mathbb{A}^n$, $\mathcal{A} = \mathbb{K}[\mathbb{A}^n]$, $\mathcal{W}$, and Ω.

Definition 3.2. Let π be a polynomial Poisson tensor on $\mathbb{A}^n \cong \mathbb{K}^n$. Suppose that we have a family of automorphisms φ_t given by a regular map $\mathbb{K}^\times \to \mathrm{GL}((\mathbb{K}^n)^*)$ and that the formal expression of $\pi_t := \varphi_t^{-1}(\pi)$ is an element of $\mathcal{W}^2[t]$. Then $\tilde{\pi} := \lim_{t\to 0}\pi_t$ is called a *contraction* of π. For each $H \in \mathcal{A}$, we define its *highest (t-) component* as a non-zero polynomial $H^\bullet$ such that $H^\bullet = \lim_{t\to 0} t^d\varphi_t^{-1}(H)$ for some $d =: \deg_t H$. (One readily sees the uniqueness of this d.)

Lemma 3.3. *If $\tilde{\pi}$ is a contraction of π, then $\tilde{\pi}$ is again a Poisson tensor and for each $H \in Z(\mathcal{A}, \pi)$, the polynomial $H^\bullet$ is an element of $Z(\mathcal{A}, \tilde{\pi})$.*

Proof. An element $R \in \mathcal{W}^2$ is a Poisson tensor if and only if $[R, R] = 0$. (This is a way to state the Jacobi identity.) Since $[\pi_t, \pi_t] = 0$ for all non-zero t and $\pi_t \in \mathcal{W}^2[t]$, we have $\pi_t = \tilde{\pi} + tR$, where $R \in \mathcal{W}^2[t]$, and $0 = [\pi_t, \pi_t] = [\tilde{\pi}, \tilde{\pi}] + t\tilde{R}$ with $\tilde{R} \in \mathcal{W}^2[t]$. Therefore $[\tilde{\pi}, \tilde{\pi}] = 0$.

In order to prove the second statement one repeats the argument of Lemma 3.1:

$$0 = \lim_{t\to 0}\{t^d \varphi_t^{-1}(H), a\}_t = \lim_{t\to 0}\{H^\bullet, a\}_t = \tilde{\pi}(dH^\bullet \wedge da)$$

for all $a \in \mathcal{A}$. □

Example 3.4. Suppose we have a decomposition $\mathfrak{q} = V_0 \oplus V_1 \oplus \cdots \oplus V_{m-1}$, where V_0 is a subalgebra and in general $[V_i, V_j] \subset \bigoplus_{k\le i+j} V_k$. Then one can define $\varphi_t : \mathfrak{q} \to \mathfrak{q}$ by setting $\varphi|_{V_j} = t^j\mathrm{id}$ and consider Lie algebra structures $[\ ,\ ]_t$ on $\mathfrak{q}$. Clearly, the limit as $t \to 0$ exists and the resulting Lie algebra $\tilde{\mathfrak{q}}$ has a $\mathbb{Z}$-grading with at most m non-zero components.

Contractions of Lie algebras as in Example 3.4 were studied by Panyushev [P09].

Definition 3.5. Let $\ell = \operatorname{ind}\mathfrak{q}$. We say that a set $\{H_1, \ldots, H_\ell\} \subset \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is a *good generating system* with respect to a contraction $[\ ,\ ]_t \rightsquigarrow [\ ,\ ]_{\tilde{\mathfrak{q}}}$ if the polynomials H_i generate $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ and their highest components $H_i^\bullet$ are algebraically independent.

Let $(Z\mathcal{S}(\mathfrak{q}))^\bullet$ be the algebra of highest components of $Z\mathcal{S}(\mathfrak{q})$, i.e., the algebra generated by $H^\bullet$ with $H \in Z\mathcal{S}(\mathfrak{q})$.

Lemma 3.6. *If $\{H_1, \ldots, H_\ell\} \subset \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is a good generating system, then the $H_i^\bullet$ generate $(Z\mathcal{S}(\mathfrak{q}))^\bullet$.*

Proof. Each non-zero element $g \in Z\mathcal{S}(\mathfrak{q})$ can be expressed as a polynomial P in the H_i. Suppose that $P = \sum_{\bar{s}} a_{\bar{s}} H_1^{s_1} \ldots H_\ell^{s_\ell}$ with some $\bar{s} \in \mathbb{Z}^\ell_{\ge 0}$. Define $\tilde{P}$ as a sum of those monomials (with the coefficients $a_{\bar{s}}$), where the degree in t, $s_1 \deg_t H_1 + \cdots + s_\ell \deg_t H_\ell$, is maximal. Then $\tilde{P}(H_1^\bullet, \ldots, H_\ell^\bullet)$ is a non-zero polynomial, because the elements $H_i^\bullet$ are algebraically independent, and it equals $g^\bullet$ by construction. □

3.1. Contractions and the Kostant equality

Example 3.7. Suppose that $\mathfrak{q} = \mathfrak{sl}_2$ and a contraction $\mathfrak{q} \rightsquigarrow \tilde{\mathfrak{q}}$ is defined by a decomposition $\mathfrak{sl}_2 = \mathfrak{so}_2 \oplus V$, where V is an $\mathfrak{so}_2$-invariant complement. In a standard basis $\{e, h, f\}$ the automorphism φ_t multiplies e and f by t. In the basis $\{e/t, h, f/t\}$ the Poisson tensor π_t is given by the same formula as π in the original basis. Therefore we have

$$\frac{dF}{d(e/t) \wedge dh \wedge d(f/t)} = h\partial_{e/t}\wedge\partial_{f/t} + 2\frac{e}{t}\partial_h\wedge\partial_{e/t} + 2\frac{f}{t}\partial_{f/t}\wedge\partial_h,$$

where F is a suitably normalised invariant of degree 2, explicitly $F = -\frac{h^2}{2} - \frac{2ef}{t^2}$. After removing t from denominators, the above equality modifies to

$$\frac{-t^2 h dh - 2f de - 2e df}{de \wedge dh \wedge df} = t^2 h\partial_e\wedge\partial_f + 2e\partial_h\wedge\partial_e + 2f\partial_f\wedge\partial_h.$$

In particular, for $\tilde{\mathfrak{q}}$, we have $dF^\bullet/\omega = \tilde{\pi}$.

Example 3.7 illustrates a general phenomenon. Let D_t be the degree in t of the determinant of the map $\varphi_t : (\mathbb{K}^n)^* \to (\mathbb{K}^n)^*$, where $\mathbb{K}^n$ is identified with $\mathbb{A}^n$. In the case of a linear (in t) contraction of a Lie algebra $\mathfrak{q}$, we have $D_t = \dim V$ and φ_t multiplies the canonical volume form ω by t^{D_t}.

Theorem 3.8. *Suppose we have a contraction $\pi_t \rightsquigarrow \tilde{\pi}$ of a Poisson structure π on $\mathbb{A}^n \cong \mathbb{K}^n$ given by a family of linear automorphisms $\varphi_t : (\mathbb{K}^n)^* \to (\mathbb{K}^n)^*$ with determinant t^{D_t}. Suppose further that $\operatorname{ind} \mathcal{A} = \ell$ and it stays the same under the contraction. If the Kostant equality holds for a set of polynomials $F_1, \ldots, F_\ell \in Z(\mathcal{A}, \pi)$, then*

(i) *$\sum \deg_t F_i \geq D_t$, moreover, if $\sum \deg_t F_i > D_t$, then $F_i^\bullet$ are algebraically dependent;*
(ii) *if $\sum \deg_t F_i = D_t$, then $F_i^\bullet$ are algebraically independent and satisfy the Kostant equality with $\tilde{\pi}$;*
(iii) *if we have equality in* (i) *and* $\dim \operatorname{Sing} \tilde{\pi} \leq n-2$, *and if each $F_i^\bullet$ is a homogeneous polynomial, then the $F_i^\bullet$ generate $Z(\mathcal{A}, \tilde{\pi})$.*

Proof. We are contracting, so to say, both sides in the Kostant equality. For each nonzero t, we have

$$\frac{d\varphi_t^{-1}(F_1) \wedge \cdots \wedge d\varphi_t^{-1}(F_\ell)}{\varphi_t^{-1}(\omega)} = \Lambda^{(n-\ell)/2}\pi_t$$

and therefore

$$\frac{t^{D_t} d\varphi_t^{-1}(F_1) \wedge \cdots \wedge d\varphi_t^{-1}(F_\ell)}{\omega} = \Lambda^{(n-\ell)/2}\tilde{\pi} + tR,$$

where $R \in \mathcal{W}^{n-\ell}[t]$.

If $\sum \deg_t F_i < D_t$, then letting $t \to 0$, we get zero on the left hand side. Since index remains the same under this contraction, $\Lambda^{(n-\ell)/2}\tilde{\pi} \neq 0$, and this proves the inequality $\sum \deg_t F_i \geq D_t$. Further, if $\sum \deg_t F_i > D_t$, then $t^{D_t}(dF_1^\bullet \wedge \cdots \wedge dF_\ell^\bullet)$ is either zero or tends to infinity as $t \to 0$ and therefore $dF_1^\bullet \wedge \cdots \wedge dF_\ell^\bullet$ must be zero. This completes the proof of (i).

If $\sum \deg_t F_i = D_t$ then the left hand side tends to $(dF_1^\bullet \wedge \cdots \wedge dF_\ell^\bullet)/\omega$ as $t \to 0$. Therefore these highest components are algebraically independent and indeed satisfy the Kostant equality with $\Lambda^{(n-\ell)/2}\tilde{\pi}$ on the right hand side.

Part (ii) implies that $\mathcal{J}(F_1^\bullet, \ldots, F_\ell^\bullet) = \operatorname{Sing} \tilde{\pi}$. Thus, if the conditions in (iii) are satisfied, then the Jacobian locus of $F_i^\bullet$ has dimension at most $n-2$. Since $\operatorname{tr.deg} Z(\mathcal{A}, \tilde{\pi}) \leq \ell$, for each $F \in Z(\mathcal{A}, \tilde{\pi})$, the polynomials $F_1^\bullet, \ldots, F_\ell^\bullet$, and F are algebraically dependent. The assumption that each $F_i^\bullet$ is homogeneous allows us to use a characteristic zero version of Skryabin's result (see [PPY, Theorem 1.1]), which states that in this situation F lies in the subalgebra generated by $F_i^\bullet$. □

4. Symmetric invariants of $\mathbb{Z}_2$-contractions

Let G be a connected reductive algebraic group defined over $\mathbb{K}$. Suppose that $\mathbb{K} = \overline{\mathbb{K}}$. Set $\mathfrak{g} = \operatorname{Lie} G$. Let σ be an involution (automorphism of oder 2) of G. At the Lie algebra level, σ induces a $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{g}^\sigma = \operatorname{Lie} G_0$ and $G_0 := G^\sigma$ is the subgroup of σ-invariant points. In this context, G_0 is said to be a *symmetric subgroup*, G/G_0 a *symmetric space* and $(\mathfrak{g}, \mathfrak{g}_0)$ a *symmetric pair*. One can contract $\mathfrak{g}$ to a semidirect product $\tilde{\mathfrak{g}} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$, where $\mathfrak{g}_1$ becomes an Abelian ideal, as described in Section 3. We will call the resulting Lie algebra $\tilde{\mathfrak{g}}$ a $\mathbb{Z}_2$-*contraction* of $\mathfrak{g}$. In this section, our main objects of interest are $\mathbb{Z}_2$-contractions of simple (non-Abelian) Lie algebras.

Set $\ell = \operatorname{ind} \mathfrak{g} = \operatorname{rk} \mathfrak{g}$. By [P07, Proposition 2.5], $\operatorname{ind} \tilde{\mathfrak{g}} = \ell$ for a $\mathbb{Z}_2$-contraction of a reductive Lie algebra. It was also conjectured in [P07] that $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is a polynomial algebra in ℓ variables. In would be sufficient to prove the conjecture for symmetric pairs with simple $\mathfrak{g}$. For many pairs it was already proved in [P07]. Here we consider four of the remaining ones. This does not cover all of them and does not prove Panyushev's conjecture.

Proposition 4.1 ([P07, Section 6]). *Suppose that $\mathfrak{g}$ is a simple non-Abelian Lie algebra. Then all symmetric pairs $(\mathfrak{g}, \mathfrak{g}_0)$ such that the polynomiality of $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is not established yet are listed below.*

Exceptional Lie algebras:

- (E_6, F_4), $(E_7, E_6 \oplus \mathbb{K})$, $(E_8, E_7 \oplus \mathfrak{sl}_2)$, *and* $(E_6, \mathfrak{so}_{10} \oplus \mathfrak{so}_2)$;
- $(E_7, \mathfrak{so}_{12} \oplus \mathfrak{sl}_2)$.

Classical Lie algebras:

- $(\mathfrak{sp}_{2n+2m}, \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2m})$ *for* $n \geq m$;
- $(\mathfrak{so}_{2\ell}, \mathfrak{gl}_\ell)$;
- $(\mathfrak{sl}_{2n}, \mathfrak{sp}_{2n})$.

The first four exceptional symmetric pairs are collected in one item, because there are no good generating systems in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ with respect to the corresponding $\mathbb{Z}_2$-contractions (see [P07, Remark 4.3]). Moreover, it is quite possible that the algebra of symmetric invariants is not freely generated for these $\tilde{\mathfrak{g}}$. These are precisely the symmetric pairs such that the restriction homomorphism $\mathbb{K}[\mathfrak{g}]^G \to \mathbb{K}[\mathfrak{g}_1]^{G_0}$ is not surjective [H]. Note that one symmetric pair, $(\mathfrak{sp}_{2n}, \mathfrak{gl}_n)$, is missing in the list in [P07, Section 6], but not in the main text of the paper. Here $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}} = \mathbb{K}[\mathfrak{g}_1^*]^{\mathrm{GL}_n}$ is a polynomial algebra.

According to [P07, Theorem 3.3], the Lie algebra $\tilde{\mathfrak{g}}$ always has the "codim-2" property, $\dim \operatorname{Sing} \tilde{\pi} \leq \dim \tilde{\mathfrak{g}} - 2$. For the pair $(E_7, \mathfrak{so}_{12} \oplus \mathfrak{sl}_2)$ and the three classical series listed in Proposition 4.1, we will construct homogeneous generators $F_i \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $\sum \deg_t F_i \leq \dim \mathfrak{g}_1$ and using Theorem 3.8 prove that Panyushev's conjecture holds for them.

For each element $x \in \mathfrak{g}_1$, we let $\mathfrak{g}_{i,x} = \mathfrak{g}_x \cap \mathfrak{g}_i$ denote its centraliser in $\mathfrak{g}_i$ $(i = 0, 1)$. Let $\mathfrak{c} \subset \mathfrak{g}_1$ be a maximal (Abelian) subalgebra consisting of semisimple elements. Any such subalgebra is called a *Cartan subspace* of $\mathfrak{g}_1$. Let $\mathfrak{l} = \mathfrak{g}_{0,\mathfrak{c}}$ be the centraliser of $\mathfrak{c}$ in $\mathfrak{g}_0$. We will need a few facts that can be found in e.g. [KR, Thm. 1 & Prop. 8]. First, all Cartan subspaces are G_0-conjugate. Second, $\mathfrak{l} = \mathfrak{g}_{0,s}$ for a generic $s \in \mathfrak{c}$ and therefore it is a reductive subalgebra. And finally, $G_0\mathfrak{c}$ is a dense subset of $\mathfrak{g}_1$.

Let $L := (G_{0,\mathfrak{c}})^\circ$ be the connected component of the identity of $G_{0,\mathfrak{c}} = \{g \in G_0 \mid gs = s \text{ for all } s \in \mathfrak{c}\}$. Using the Killing form, we identify $\mathfrak{g} \cong \mathfrak{g}^*$, $\mathfrak{g}_1 \cong \mathfrak{g}_1^*$, and $\mathfrak{g}_0 \cong \mathfrak{g}_0^*$.

Fix also the dual decomposition $\tilde{\mathfrak{g}}^* = \mathfrak{g}_0^* \oplus \mathfrak{g}_1^*$. In order to avoid confusion, let $\hat{\mathfrak{l}}$ and $\hat{\mathfrak{c}}$ denote the subspaces of $\tilde{\mathfrak{g}}^*$ arising from $\mathfrak{l}$ and $\mathfrak{c}$, respectively, under this identification. The orthogonal complements appearing below are taken with respect to the Killing form of $\mathfrak{g}$. Let $\tilde{G} = G_0 \ltimes \exp(\mathfrak{g}_1)$ be an algebraic group with Lie $\tilde{G} = \tilde{\mathfrak{g}}$. The group G_0 is not necessarily connected and therefore $\tilde{G}$ can also have several connected components. However, note that each bi-homogeneous (with respect to the decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$) component of $H \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ is an invariant of G_0 and therefore in view of Lemma 3.1, $H^\bullet \in \mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{G}}$.

Remark 4.2. In [P07], a good generating system (g.g.s.) consists of homogeneous polynomials by definition. It is possible to show that if there is a g.g.s. in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ with respect to a contraction $\mathfrak{g} \rightsquigarrow \tilde{\mathfrak{g}}$, then there are also homogeneous polynomials forming a g.g.s. We will not use this fact and therefore will not prove it. In this and the following sections, all generating systems of invariants contain only homogeneous polynomials.

Example 4.3. Take $(\mathfrak{g}, \mathfrak{g}_0) = (E_7, \mathfrak{so}_{12} \oplus \mathfrak{sl}_2)$. Then $D_t = \dim \mathfrak{g}_1 = 64$, and the generating homogeneous invariants $H_1, \ldots, H_7 \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ have degrees: 2, 6, 8, 10, 12, 14, 18. It is known that the restrictions of H_1, H_2, H_3, H_5 to $\mathfrak{g}_1^*$ generate $\mathcal{S}(\mathfrak{g}_1)^{\mathfrak{g}_0}$, independently of the choice of H_i (see [H]). We will show that there is a g.g.s. in $Z\mathcal{S}(\mathfrak{g})$ with respect to the contraction $\mathfrak{g} \rightsquigarrow \tilde{\mathfrak{g}}$.

Take any of the remaining three generators, say H_j. Assume that $H_j^\bullet \in \mathcal{S}(\mathfrak{g}_1)$. Then it can be expressed as a polynomial P in $H_i^\bullet$ with $i \in \{1, 2, 3, 5\}$ and we can replace H_j by $H_j - P(H_1, H_2, H_3, H_5)$. Or rather assume from the beginning that $\deg_t H_j < \deg H_j$.

The next step is to show that $\deg_t H_j < \deg H_j - 1$. Assume this is not the case. Restricting $H_j^\bullet$ to $\hat{\mathfrak{l}} \oplus \hat{\mathfrak{c}}$, we get either zero or an L-invariant polynomial function of bi-degree $(\deg H_j - 1, 1)$, in other words, a sum of L-invariants in $\mathcal{S}(\mathfrak{l})$ of an odd degree with coefficients from $\mathfrak{c}$. In this example $\mathfrak{l} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ (see e.g. [VO, §5.4 and Table 9 in Ref. Chapter]) and all symmetric invariants have even degrees. This shows that $H_j^\bullet$ is zero on $\hat{\mathfrak{l}} \oplus \hat{\mathfrak{c}}$. Clearly $H_j^\bullet$ is also zero on the $\tilde{G}$-saturation $\tilde{G}(\hat{\mathfrak{l}} \oplus \hat{\mathfrak{c}})$.

Consider first the action of $\exp(\mathfrak{g}_1) \subset \tilde{G}$. Note that $[\mathfrak{g}, x] = \mathfrak{g}_x^\perp$ for any $x \in \mathfrak{g}$, and hence $[\mathfrak{g}_1, x] = \mathfrak{g}_0 \cap \mathfrak{g}_x^\perp = \mathfrak{g}_0 \cap (\mathfrak{g}_{0,x})^\perp$ for any $x \in \mathfrak{g}_1$. Now let $\hat{s} \in \hat{\mathfrak{c}}$ be an element coming from some $s \in \mathfrak{c}$. Then

$$\exp(\mathfrak{g}_1)(\hat{\mathfrak{l}} \times \{\hat{s}\}) = \hat{\mathfrak{l}} \times \{\hat{s}\} + \widehat{[\mathfrak{g}_1, s]},$$

where $\widehat{[\mathfrak{g}_1, s]}$ is the annihilator of $\mathfrak{g}_{0,s}$ in $\mathfrak{g}_0^*$. Since $\mathfrak{g}_{0,s} = \mathfrak{l}$ for generic $s \in \mathfrak{c}$, the saturation $\exp(\mathfrak{g}_1)(\hat{\mathfrak{l}} \oplus \hat{\mathfrak{c}})$ is a dense subset of $\mathfrak{g}_0^* \oplus \hat{\mathfrak{c}}$. Applying G_0 to this subset, we get $\overline{\tilde{G}(\hat{\mathfrak{l}} \oplus \hat{\mathfrak{c}})} = \tilde{\mathfrak{g}}^*$ and therefore $H_j^\bullet = 0$. Since the highest t-component of a non-zero polynomial is non-zero, we get $\deg_t H_j \leq \deg H_j - 2$. Summing up,

$$\sum_{i=1}^{7} \deg_t H_i \leq \sum \deg H_i - 6 = 70 - 6 = 64 = D_t.$$

Multiplying one of the H_i by a non-zero constant, we may assume that $H_1, \dots, H_\ell$ satisfy the Kostant equality. Then, by Theorem 3.8(i),(ii), $H_i^\bullet$ are algebraically independent, which means that H_i form a good generating system.

In order to simplify calculations for other pairs, we prove a simple equality concerning ranks and dimensions. Recall that $\ell = \operatorname{rk}\mathfrak{g}$.

Lemma 4.4. *Let $\mathfrak{b}_{\mathfrak{l}} \subset \mathfrak{l}$ be a Borel subalgebra. Then* $\dim\mathfrak{b} = \dim\mathfrak{g}_1 + \dim\mathfrak{b}_{\mathfrak{l}}$.

Proof. Clearly the subspace $\mathfrak{l}\oplus\mathfrak{c}$ contains a maximal torus of $\mathfrak{g}$. Therefore $\ell = \operatorname{rk}\mathfrak{l}+\dim\mathfrak{c}$. It is known that the dimension of a maximal G_0-orbit in $\mathfrak{g}_1$ equals $\dim\mathfrak{g}_0 - \dim\mathfrak{l}$ on one hand, and $\dim\mathfrak{g}_1 - \dim\mathfrak{c}$ on the other (see e.g. [KR, Proposition 9]). Consequently, $\dim\mathfrak{g}_0 - \dim\mathfrak{l} = \dim\mathfrak{g}_1 - \dim\mathfrak{c}$. Therefore

$$\begin{aligned}\dim\mathfrak{b} &= (\dim\mathfrak{g} + \ell)/2 = (\dim\mathfrak{g}_0 + \dim\mathfrak{g}_1 + \ell)/2\\ &= (\dim\mathfrak{g}_1 + \dim\mathfrak{l} - \dim\mathfrak{c} + \dim\mathfrak{g}_1 + \ell)/2\\ &= \dim\mathfrak{g}_1 + (\dim\mathfrak{l} - \dim\mathfrak{c} + \operatorname{rk}\mathfrak{l} + \dim\mathfrak{c})/2 = \dim\mathfrak{g}_1 + \dim\mathfrak{b}_{\mathfrak{l}}. \qquad \square\end{aligned}$$

The following assertion was predicted by D. Panyushev.

Theorem 4.5. *Let $(\mathfrak{g}, \mathfrak{g}_0)$ be a symmetric pair with $\mathfrak{g}$ simple. Suppose that the restriction map $\mathbb{K}[\mathfrak{g}]^G \to \mathbb{K}[\mathfrak{g}_1]^{G_0}$ is surjective. Then there is a good generating system $F_1, \dots, F_\ell$ in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ such that each F_i is homogeneous and $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is freely generated by $F_i^\bullet$.*

Proof. According to [P07], there are four pairs to consider. For one of them the existence of a g.g.s. was established in Example 4.3. Our next goal is to construct good generating systems for three classical pairs listed in Proposition 4.1. We always assume that a set of generators $F_1, \dots, F_\ell$ in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ is normalised in order to satisfy the Kostant equality.

$\mathfrak{sp}_{2n}$	$\mathfrak{g}_1$
$\mathfrak{g}_1$	$\mathfrak{sp}_{2m}$

Fig. 1. Symmetric decomposition of $\mathfrak{sp}_{2n+2m}$.

For the first pair, with $\mathfrak{g} = \mathfrak{sp}_{2n+2m}$, we start with a set of generating invariants $\{H_1, \dots, H_\ell\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$, where each H_i is the sum of all principal $2i$-minors (this is also a coefficient of the characteristic polynomial). As can be readily seen from the block structure of this symmetric pair (Figure 1), $\deg_t H_i \le 4m$ for all i. To be more precise, for $1 \le i \le m$, the highest t-components $H_i^\bullet$ lie in $\mathcal{S}(\mathfrak{g}_1)$ and form a generating set in $\mathbb{K}[\mathfrak{g}_1^*]^{G_0}$. Therefore set $F_i := H_i$ for $i \le m$. Here $\mathfrak{l} = (\mathfrak{sl}_2)^m \oplus \mathfrak{sp}_{2n-2m}$ and all symmetric $\mathfrak{l}$-invariants are of even degrees. Applying the same trick as in Example 4.3, we can modify H_j with $m < j \le 2m$ to F_j in such a way that $\deg_t F_j \le 2j - 2$. The remaining generators stay as they are, $F_i = H_i$ for $i > 2m$. Summing up,

$$\sum_{i=1}^{\ell} (\deg F_i - \deg_t F_i) \ge 2m + \sum_{j=1}^{n-m} 2j = \dim \mathfrak{b}_{\mathfrak{l}}.$$

Making use of Lemma 4.4 and again of the equality $\sum \deg F_i = \dim \mathfrak{b}$, we conclude that $\sum \deg_t F_i \le \dim \mathfrak{g}_1 = D_t$.

For the second pair, with $\mathfrak{g} = \mathfrak{so}_{2\ell}$, we have $\mathfrak{l} = (\mathfrak{sl}_2)^{\ell/2}$ when ℓ is even, and $\mathfrak{l} = (\mathfrak{sl}_2)^{[\ell/2]} \oplus \mathfrak{so}_2$ when ℓ is odd. In case ℓ is even, we argue as in Example 4.3. Choose homogeneous generators $F_i \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ such that the highest components $F_1^\bullet, \ldots, F_{\ell/2}^\bullet$ form a generating set in $\mathbb{K}[\mathfrak{g}_1^*]^{G_0}$, and $\deg_t F_i \le \deg F_i - 2$ for $i > \ell/2$. Then, taking into account Lemma 4.4, we get

$$\sum_{i=1}^{\ell} \deg_t F_i \le \sum_{i=1}^{\ell} \deg F_i - \ell = \dim \mathfrak{b} - \dim \mathfrak{b}_{\mathfrak{l}} = \dim \mathfrak{g}_1 = D_t.$$

The case of ℓ odd is more interesting. We begin with a set of generating invariants $\{H_1, \ldots, H_\ell\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$, where each H_i with $i \ne \ell$ is the sum of all principal $2i$-minors and H_ℓ is the pfaffian, in particular, $\deg F_\ell = \ell$ is odd. One can realise $\mathfrak{so}_{2\ell}$ as a set of $2\ell \times 2\ell$ matrices skew-symmetric with respect to the anti-diagonal. Then elements of $\mathfrak{g}_1$ have block structure as shown in Figure 2. This implies that all bi-homogeneous (in $\mathfrak{g}_0$ and $\mathfrak{g}_1$) components of H_i with $i < \ell$ have even degrees in $\mathfrak{g}_1$ (and in $\mathfrak{g}_0$).

$$\begin{array}{|c|c|} \hline 0 & B \\ \hline C & 0 \\ \hline \end{array}$$

Fig. 2. $\mathfrak{g}_1$ for $(\mathfrak{so}_{2\ell}, \mathfrak{gl}_\ell)$. Here the matrices B and C are skew-symmetric with respect to the anti-diagonal.

The highest t-components $H_i^\bullet$ with $2i < \ell$ form a generating set in $\mathbb{K}[\mathfrak{g}_1^*]^{G_0}$. Therefore we put $F_i := H_i$ for these i. Each H_j with $\ell/2 < j < \ell$ can be modified to F_j with $\deg_t F_j \le 2j - 2$. And, finally, since $\det \xi = 0$ for all $\xi \in \mathfrak{g}_1$, we have $\deg_t H_\ell \le \ell - 1$. Set $F_\ell := H_\ell$. Then

$$\sum_{i=1}^{\ell} \deg_t F_i \le \sum_{i=1}^{\ell} \deg F_i - (\ell - 1) - 1 = \dim \mathfrak{b} - \dim \mathfrak{b}_{\mathfrak{l}} = \dim \mathfrak{g}_1 = D_t,$$

where again we have used Lemma 4.4.

For the third pair, with $\mathfrak{g} = \mathfrak{sl}_{2n}$, we have $\mathfrak{l} = (\mathfrak{sl}_2)^n$. Here everything works exactly as in Example 4.3. We take homogeneous invariants F_i with $\deg F_i = i + 1$. Then the $F_i^\bullet$

with $1 \le i < n$ form a generating set in $\mathbb{K}[\mathfrak{g}_1^*]^{G_0}$ and, modifying F_j if necessary, we can assume that $\deg_t F_j^\bullet \le \deg F_j - 2$ for $j \ge n$. In view of Lemma 4.4,

$$\sum_{i=1}^{\ell} \deg_t F_i \le \sum_{i=1}^{\ell} \deg F_i - 2n = \dim\mathfrak{b} - \dim\mathfrak{b}_{\mathfrak{l}} = \dim\mathfrak{g}_1 = D_t.$$

For all three series we have constructed $F_i \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $\sum_{i=1}^{\ell} \deg_t F_i \le D_t$. By Theorem 3.8(i),(ii), the polynomials F_i form a g.g.s. Since in addition all F_i are homogeneous here as well as in Example 4.3, the polynomials $F_i^\bullet$ generate $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ by [P07, Theorem 4.2(i)] or by Theorem 3.8(iii), if one recalls that $\tilde{\mathfrak{g}}$ has the "codim-2" property [P07, Theorem 3.3.]. □

Corollary 4.6 (cf. [P07, Theorem 4.2(ii)]). *Let $(\mathfrak{g}, \mathfrak{g}_0)$ be a symmetric pair such that Theorem 4.5 applies. Then the Lie algebra $\tilde{\mathfrak{g}}$ is of Kostant type.*

4.1. Poisson semicentre

Definition 4.7. Let $\mathfrak{q}$ be a Lie algebra. Then an element $H \in \mathcal{S}(\mathfrak{q})$ is called a *semi-invariant* if $\{\xi, H\} \in \mathbb{K}H$ for all $\xi \in \mathfrak{q}$. We let $\mathcal{S}(\mathfrak{q})_{\rm si}$ denote the $\mathbb{K}$-algebra generated by semi-invariants. This algebra is also called the *Poisson semicentre* of $\mathcal{S}(\mathfrak{q})$.

It is easy to deduce that $\{\mathcal{S}(\mathfrak{q})_{\rm si}, \mathcal{S}(\mathfrak{q})_{\rm si}\} = 0$ (see e.g. [OV, Section 2]). Recently Poisson semicentres were studied in [OV] and [JSh]. In particular, [OV] proves a degree inequality for Lie algebras $\mathfrak{q}$ such that $Z\mathcal{S}(\mathfrak{q})$ is a polynomial ring and $Z\mathcal{S}(\mathfrak{q}) = \mathcal{S}(\mathfrak{q})_{\rm si}$. Here we show that some $\mathbb{Z}_2$-contractions $\tilde{\mathfrak{g}}$ of simple Lie algebras also have the second property. If G_0 is semisimple, then $\tilde{\mathfrak{g}}$ has no non-trivial characters and clearly $Z\mathcal{S}(\tilde{\mathfrak{g}}) = \mathcal{S}(\tilde{\mathfrak{g}})_{\rm si}$.

Until the end of this section we assume that G_0 has a non-trivial connected centre. Let G_0' be the derived group of G_0 and $\mathfrak{g}_0' = [\mathfrak{g}_0, \mathfrak{g}_0]$ the derived Lie algebra. Using an elementary observation that $\mathfrak{g}_0 \oplus [\mathfrak{g}_0, \mathfrak{g}_1]$ is an ideal of $\mathfrak{g}$, one proves the equality $\tilde{\mathfrak{g}}' = \mathfrak{g}_0' \ltimes \mathfrak{g}_1$.

Recall that a symmetric space (or a symmetric pair) can be either of *tube type*, meaning $\mathbb{K}[\mathfrak{g}_1]^{\mathfrak{g}_0'} \ne \mathbb{K}[\mathfrak{g}_1]^{\mathfrak{g}_0}$, or of non-tube type. For example, $(\mathfrak{so}_{2\ell}, \mathfrak{gl}_\ell)$ is of tube type if and only if ℓ is even. There are many characterisations of symmetric pairs of tube type. If $(\mathfrak{g}, \mathfrak{g}_0)$ is of tube type, then there are more semi-invariants than symmetric $\tilde{\mathfrak{g}}$-invariants, because $\mathcal{S}(\mathfrak{g}_1)^{\mathfrak{g}_0'} \subset \mathcal{S}(\tilde{\mathfrak{g}})_{\rm si}$. That case may be worth investigating. Here we deal with symmetric spaces of non-tube type.

The following observation helps to treat semidirect products (cf. [R] or [P07′, Proposition 5.5]).

Lemma 4.8. *Let $\mathfrak{q} = \mathfrak{f} \ltimes V$ be a semidirect product of a Lie algebra $\mathfrak{f}$ and an Abelian ideal V. Take $x = a + b \in \mathfrak{q}^*$ with $a(V) = 0 = b(\mathfrak{f})$. Let $\mathfrak{a} = \mathrm{Ann}(\mathfrak{f}{\cdot}b)$ be a subspace of $(V^*)^* = V$. Then $\mathfrak{a} \lhd \mathfrak{q}_x$ and $\mathfrak{q}_x/\mathfrak{a} \cong (\mathfrak{f}_b)_{\tilde{a}}$, where $\tilde{a}$ is the restriction of a to $\mathfrak{f}_b$.*

Proof. Note that $V{\cdot}b$ is zero on V, because $[V, V] = 0$, and therefore $\mathfrak{q}_x \subset \mathfrak{f}_b \ltimes V$. It is also quite clear that $V{\cdot}b \subset \mathrm{Ann}(\mathfrak{f}_b \oplus V)$. Hence $\mathfrak{q}_x \subset (\mathfrak{f}_b)_{\tilde{a}} \ltimes V$. For dimensional reasons, $V \cdot b = \mathrm{Ann}(\mathfrak{f}_b \oplus V)$, and for each $\xi \in (\mathfrak{f}_b)_{\tilde{a}}$, there is $\eta \in V$ such that $\eta{\cdot}b = \xi{\cdot}a$. It remains to notice that $\mathfrak{q}_x \cap V = \mathrm{Ann}(\mathfrak{f}{\cdot}b)$. □

Proposition 4.9. *Suppose that* $(\mathfrak{g}, \mathfrak{g}_0)$ *is a symmetric pair of non-tube type. Then* $\mathcal{S}(\tilde{\mathfrak{g}})_{\mathrm{si}} = Z\mathcal{S}(\tilde{\mathfrak{g}})$.

Proof. Here we consider the connected groups $\tilde{G}^\circ$ and $(\tilde{G}')^\circ = (G_0')^\circ \ltimes \exp(\mathfrak{g}_1)$. Each character of $\tilde{G}^\circ$ is trivial on $(\tilde{G}')^\circ$, hence $\mathcal{S}(\tilde{\mathfrak{g}})_{\mathrm{si}} \subset \mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}'}$. (In fact, $\mathcal{S}(\tilde{\mathfrak{g}})_{\mathrm{si}} = \mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}'}$.) Next we take $H \in \mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}'}$ and show that it is an invariant of $\tilde{G}^\circ$.

Since $\mathfrak{g}_0'$ is semisimple, $\mathbb{K}(\mathfrak{g}_1)^{\mathfrak{g}_0'}$ is the quotient field of $\mathbb{K}[\mathfrak{g}_1]^{\mathfrak{g}_0'}$. By Rosenlicht's theorem, generic orbits of an algebraic group, in our case $(G_0')^\circ$, are separated by rational invariants. Therefore the equality $\mathbb{K}[\mathfrak{g}_1]^{\mathfrak{g}_0'} = \mathbb{K}[\mathfrak{g}_1]^{\mathfrak{g}_0}$ implies that G_0° and $(G_0')^\circ$ have the same generic orbits in $\mathfrak{g}_1$ and $\mathfrak{g}_1^*$. At the Lie algebra level this means that $\mathfrak{g}_0 = \mathfrak{g}_0' + \mathfrak{g}_{0,b}$ for generic $b \in \mathfrak{g}_1$.

Suppose that $x = \hat{a} + \hat{b} \in \tilde{\mathfrak{g}}^*$, where $\hat{a}$ and $\hat{b}$ correspond to generic $a \in \mathfrak{l}$ and $b \in \mathfrak{c}$, respectively. In view of Lemma 4.8 and the fact that $\mathfrak{l} = \mathfrak{g}_{0,b}$ is reductive, $\tilde{\mathfrak{g}}_x = \mathfrak{l}_a \ltimes ([\mathfrak{g}_0, b])^\perp$, where $([\mathfrak{g}_0, b])^\perp = \{\eta \in \mathfrak{g}_1 \mid [b, \eta] \in \mathfrak{g}_0^\perp\}$ (the orthogonal complement is taken with respect to the Killing form of $\mathfrak{g}$).

Since $\mathfrak{g}_0 = \mathfrak{g}_0' + \mathfrak{l}$ and $\mathfrak{l}_a$ contains the centre of $\mathfrak{l}$, we also have $\mathfrak{g}_0 = \mathfrak{g}_0' + \mathfrak{l}_a$ and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}' + \tilde{\mathfrak{g}}_x$. This leads to the equalities $\tilde{\mathfrak{g}}'{\cdot}x = \tilde{\mathfrak{g}}{\cdot}x$ and $\dim \tilde{G}'x = \dim \tilde{G}x$. In addition, $(\tilde{G}')^\circ$ is a normal subgroup of $\tilde{G}^\circ$. Consequently, $(\tilde{G}')^\circ x = \tilde{G}^\circ x$. This equality holds on an open subset of $\tilde{G}^\circ(\hat{\mathfrak{l}} \oplus \hat{\mathfrak{c}})$, which is a dense subset of $\tilde{\mathfrak{g}}^*$, because $\overline{\tilde{G}(\hat{\mathfrak{l}} \oplus \hat{\mathfrak{c}})} = \tilde{\mathfrak{g}}^*$, as we know from Example 4.3, and $\tilde{\mathfrak{g}}^*$ is irreducible. Thus, H is constant on a generic $\tilde{G}^\circ$-orbit and hence $H \in Z\mathcal{S}(\tilde{\mathfrak{g}})$. □

5. Applications to E. Feigin's contraction

In this section, $\mathfrak{g} = \mathrm{Lie}\, G$ is a simple Lie algebra of rank ℓ, $B \subset G$ is a Borel subgroup, and $\mathfrak{b} = \mathrm{Lie}\, B$ is a Borel subalgebra. We keep the assumption that $\mathbb{K} = \overline{\mathbb{K}}$. Fix a decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$, where $\mathfrak{n}^-$ is the nilpotent radical of an opposite Borel, and consider a one-parameter contraction of $\mathfrak{g}$ given by this decomposition. For the resulting Lie algebra $\tilde{\mathfrak{g}}$, we have $\tilde{\mathfrak{g}} = \mathfrak{b} \ltimes \mathfrak{n}^-$, where $\mathfrak{n}^-$ is an Abelian ideal. This contraction was recently introduces by E. Feigin [F10]. His motivation came from some problems in representation theory [FFL]. Degenerations of flag varieties of $\mathfrak{g}$ related to the contraction $\mathfrak{g} \rightsquigarrow \tilde{\mathfrak{g}}$ were further studied in [F11] and [FFiL].

Let $\{\alpha_1, \ldots, \alpha_\ell\}$ be the set of simple roots and e_i, f_i the corresponding elements of the Chevalley basis. Set $\mathfrak{g}_{\mathrm{reg}} = \{x \in \mathfrak{g} \mid \dim \mathfrak{g}_x = \ell\}$, where $\mathfrak{g}_x$ is the stabiliser in the adjoint representation, $\mathfrak{n}_{\mathrm{reg}} := \mathfrak{n} \cap \mathfrak{g}_{\mathrm{reg}}$. If $x \in \mathfrak{n}_{\mathrm{reg}}$, then $\mathfrak{n}_x = \mathfrak{b}_x = \mathfrak{g}_x$ and Bx is a dense open orbit in $\mathfrak{n}$. Hence $\mathfrak{n}_{\mathrm{reg}}$ is a single B-orbit. The complement of this orbit was described by Kostant [K, Theorem 4]: $\mathfrak{n} \setminus \mathfrak{n}_{\mathrm{reg}} = \bigcup_{i=1}^{\ell} \mathfrak{D}_i$, where each $\mathfrak{D}_i$ is a linear subspace of dimension $\dim \mathfrak{n} - 1$ orthogonal to f_i. We will also need an interpretation of $\mathfrak{D}_i$ as closures of orbital varieties. It is a classical fact that for each nilpotent orbit $Ge \subset \mathfrak{g}$, all irreducible components of $\overline{Ge \cap \mathfrak{n}}$ are of dimension $\frac{1}{2}\dim Ge$. In particular, $\mathfrak{n} \setminus \mathfrak{n}_{\rm reg} = \overline{\mathcal{O}^{\rm sub}} \cap \mathfrak{n}$, where $\mathcal{O}^{\rm sub}$ is the unique nilpotent G-orbit in $\mathfrak{g}$ of dimension $\dim\mathfrak{g} - \ell - 2$.

Making further use of the Killing form (,) of $\mathfrak{g}$, we identify $(\mathfrak{n}^-)^*$ with the nilpotent radical $\mathfrak{n} \subset \mathfrak{b}$ and fix the dual decomposition $\tilde{\mathfrak{g}}^* = \mathfrak{b}^* \oplus \mathfrak{n}^{\rm ab}$, where ${}^{\rm ab}$ indicates that $\mathfrak{n}^{\rm ab}$ is the space of linear functions on an Abelian ideal. Let also $\mathfrak{n}^{\rm ab}_{\rm reg}$ be a subset of $\mathfrak{n}^{\rm ab} \subset \tilde{\mathfrak{g}}^*$ corresponding to $\mathfrak{n}_{\rm reg}$. We identify $\mathfrak{D}_i$ with subsets of $\mathfrak{n}^{\rm ab}$ using the same letters for them.

Then next statement was first proved in [PY].

Lemma 5.1 (cf. Lemma 4.8). *We have* $\operatorname{ind}\tilde{\mathfrak{g}} = \ell$.

Proof. Clearly, $\operatorname{rk}\pi$ cannot get larger after a contraction, therefore $\operatorname{ind}\tilde{\mathfrak{g}} \ge \ell$. On the other hand, take $x \in \mathfrak{n}^{\rm ab}_{\rm reg}$ and extend it to a linear function on $\tilde{\mathfrak{g}}^*$ by putting $x(\mathfrak{b}) = 0$. Then $\tilde{\mathfrak{g}}_x = \mathfrak{b}_x = \mathfrak{n}_x$ and it has dimension ℓ. Thus $\operatorname{ind}\tilde{\mathfrak{g}} = \ell$. □

Another result of [PY], Theorem 3.3, states that $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is freely generated by some polynomials $\widehat{P_i}$ (with $1 \le i \le \ell$). The construction of these polynomials $\widehat{P_i}$ starts with a system of homogeneous generators F_i of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ with $\deg F_i \le \deg F_{i+1}$. It is also shown that $\widehat{P_i} = F_i^\bullet$ [PY, Theorem 3.9]. We assume that the F_i are normalised to satisfy the Kostant equality.

Lemma 5.2. *Let F_i be as above. Then $F_i^\bullet$ satisfy the Kostant equality with $\tilde{\pi}$ and therefore $\tilde{\mathfrak{g}}$ is a Lie algebra of Kostant type. Moreover,* $\deg_t F_i = \deg F_i - 1$ *for all i.*

Proof. Recall that $\mathcal{S}(\mathfrak{n}^-)^{\mathfrak{b}} = \mathbb{K}$. If $\deg_t F_i = \deg F_i$, that is, $F_i^\bullet \in \mathcal{S}(\mathfrak{n}^-)$, then also $F_i^\bullet \in \mathcal{S}(\mathfrak{n}^-)^{\mathfrak{b}}$, a contradiction. Hence $\deg_t F_i \le \deg F_i - 1$ for each i and

$$\sum \deg_t F_i \le \dim\mathfrak{b} - \ell = \dim\mathfrak{n} = D_t.$$

By Theorem 3.8(i),(ii), $\deg_t F_i = \deg F_i - 1$, the polynomials $F_i^\bullet$ are algebraically independent and satisfy the Kostant equality with $\tilde{\pi}$. Since, according to [PY, Section 3], the $F_i^\bullet$ generate $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$, the Lie algebra $\tilde{\mathfrak{g}}$ is of Kostant type. □

Actually, the bi-degrees of $F_i^\bullet$ with respect to the decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$ have already been found in [PY].

Remark 5.3. Lemma 5.2 implies that $\mathcal{J}(F_1^\bullet, \dots, F_\ell^\bullet) = \operatorname{Sing}\tilde{\pi}$. Therefore $\operatorname{Sing}\tilde{\pi}$ contains a divisor whenever $\mathfrak{g}$ is not of type A (see [PY, Th. 4.2 & Prop. 4.3]). This means that outside of type A we get curious examples of Lie algebras of Kostant type that do not have the "codim-2" property.

Next we turn our attention to the subset $\operatorname{Sing}\tilde{\pi} = \tilde{\mathfrak{g}}^*_{\rm sing}$. Lemma 4.8 implies that $\mathfrak{b}^* \times \mathfrak{n}^{\rm ab}_{\rm reg} \subset \tilde{\mathfrak{g}}^*_{\rm reg}$ and therefore $\tilde{\mathfrak{g}}^*_{\rm sing} \subset \mathfrak{b}^* \times \bigcup_{i=1}^{\ell} \mathfrak{D}_i$, where the subspaces $\mathfrak{D}_i$ are regarded as subsets of $\mathfrak{n}^{\rm ab}$.

Definition 5.4. Let $\mathfrak{q}$ be an n-dimensional Lie algebra with $\operatorname{ind}\mathfrak{q} = \ell$ and π its Lie–Poisson tensor. Then we will say that a polynomial p is a *fundamental semi-invariant* of $\mathfrak{q}$ if $\Lambda^{(n-\ell)/2}\pi = pR$ with $R \in \mathcal{W}^{n-\ell}$ (notation as in Section 2) and the zero set of R in $\mathfrak{q}^*$ has codimension greater than or equal to 2.

In [OV], *the fundamental semi-invariant* is defined as the greatest common divisor of the $\operatorname{rk}\pi \times \operatorname{rk}\pi$ (here $\operatorname{rk}\pi = n - \ell$) minors in the matrix of π. Our polynomial is a square root of that one (up to a non-zero scalar) and is a scalar multiple of the fundamental semi-invariant in the sense of [JSh, Section 4.1].

Let δ be the highest root, e_δ a highest root vector, and $r_i = [\delta : \alpha_i]$ the i-th coefficient in the decomposition of δ, i.e., $\delta = \sum r_i\alpha_i$.

As is well-known, the highest degree of a homogeneous generator, under our assumptions, $\deg F_\ell$, equals $1 + \sum r_i$. Since $F_\ell^\bullet$ has weight zero and is of degree 1 in $\mathfrak{b}$ and of degree $\deg F_\ell - 1$ in $\mathfrak{n}^-$, up to a scalar multiple we have $F_\ell^\bullet = e_\delta \prod f_i^{r_i}$. This is also proved in [PY, Lemma 4.1].

Set $p := \prod f_i^{r_i-1}$. Note that in type A we have $r_i = 1$ for all i and hence $p = 1$. Here we generalise [PY, Proposition 4.3], stating that in type A the singular set $\operatorname{Sing}\tilde{\pi}$ contains no divisors.

Theorem 5.5. *Let $\tilde{\mathfrak{g}}$ be Feigin's contraction of a simple Lie algebra $\mathfrak{g}$. Then $p = \prod f_i^{r_i-1}$ is a fundamental semi-invariant of $\tilde{\mathfrak{g}}$.*

Proof. Set $\mathcal{F} = dF_1^\bullet \wedge \cdots \wedge dF_\ell^\bullet$. Consider also the differential 1-form

$$L = \Big(\prod_{i=1}^{\ell} f_i\Big) de_\delta + e_\delta \sum_{i=1}^{\ell} r_i f_1 \ldots f_{i-1} f_{i+1} \ldots f_\ell df_i$$

and set $R := dF_1^\bullet \wedge \cdots \wedge dF_{\ell-1}^\bullet \wedge L$. Note that $dF_\ell^\bullet = apL$ with $a \in \mathbb{K}^\times$ and therefore $\mathcal{F} = apR$. In view of the Kostant equality for $F_i^\bullet$ established in Lemma 5.2, we have to show that the zero set of R contains no divisors.

Clearly, the zero set of R is contained in $\tilde{\mathfrak{g}}^*_{\text{sing}}$ and we have to prove that R is non-zero on each irreducible divisor in $\tilde{\mathfrak{g}}^*_{\text{sing}}$. As already mentioned, Lemma 4.8 together with a result of Kostant [K, Theorem 4] imply that $\tilde{\mathfrak{g}}^*_{\text{sing}} \subset \mathfrak{b}^* \times \bigcup_{i=1}^{\ell} \mathfrak{D}_i$, where the $\mathfrak{D}_i$ are the components of $\overline{\mathcal{O}^{\text{sub}} \cap \mathfrak{n}}$. Fix $i \in \{1, \ldots, \ell\}$. There is an element $e = e(i)$ in $\mathfrak{D}_i$ such that $e \in \mathcal{O}^{\text{sub}}$ and $(f_j, e) \neq 0$ for all $j \neq i$. Take this e and add to it $b \in \mathfrak{b}^*$ such that $b(e_\delta) \neq 0$, forming a linear function $x = b + e$ on $\tilde{\mathfrak{g}}$. Evaluating L at x we get $L_x = a' df_i$ with $a' \in \mathbb{K}^\times$. The goal is to prove that $d_xF_j^\bullet$ with $1 \le j < \ell$ and L_x are linearly independent. To this end we calculate $d_xF_j^\bullet$.

Each $d_xF_j^\bullet$ can be considered as an element of $\tilde{\mathfrak{g}}$ and therefore $d_xF_j^\bullet = \xi_j + \eta_j$ with $\xi_j \in \mathfrak{b}$ and $\eta_j \in \mathfrak{n}^-$. Since each $F_j^\bullet$ has degree 1 in $\mathfrak{b}$, we have $\xi_i = d_eF_i$, where e is regarded as an element of $\mathfrak{g} \cong \mathfrak{g}^*$. By e.g. [Sl, Sect. 8.3, Lemma 1] or [P07′, Theorem 10.6], the d_eF_j with $j \le \ell$ generate a subspace of dimension $\ell - 1$. For $d_xF_\ell^\bullet$ we have two possibilities: either it is zero (if $r_i > 1$), or proportional to df_i. In any case, $\xi_\ell = 0$. Therefore $\xi_1, \ldots, \xi_{\ell-1}$ are linearly independent and clearly the $d_xF_j^\bullet$ with $1 \le j < \ell$ and L_x together generate a subspace of dimension ℓ. This proves that R is non-zero on each $\mathfrak{b}^* \times \mathfrak{D}_i$. Therefore $\dim\{\xi \in \tilde{\mathfrak{g}}^* \mid R_\xi = 0\} \le \dim\tilde{\mathfrak{g}} - 2$ and we are done. □

There are three types of simple Lie algebras such that $r_i > 1$ for all i. Here $\tilde{\mathfrak{g}}^*_{\text{sing}}$ is a union of ℓ irreducible divisors and we have the following observation.

Corollary 5.6. *Suppose that* $\mathfrak{g}$ *is of type* E_8*,* F_4 *or* G_2*. Then an element* $x \in \tilde{\mathfrak{g}}^*$ *is singular if and only if* $d_x F_\ell^\bullet = 0$*.*

For a simple (non-Abelian) Lie algebra $\mathfrak{g}$, the above statement is true, independently of the type of $\mathfrak{g}$. By a result of Varadarajan [V], $\mathfrak{g}^*_{\rm sing}$ is characterised by the property $d_x F_\ell = 0$. It would be interesting to find out whether it holds for all $\tilde{\mathfrak{g}}$ or not. That task lies beyond the scope of this paper.

5.1. Proper semi-invariants

A semi-invariant is said to be *proper* if it is not an invariant. The Lie algebra $\tilde{\mathfrak{g}}$ has proper symmetric semi-invariants, for example e_δ. Therefore describing $\mathcal{S}(\tilde{\mathfrak{g}})_{\rm si}$ is an interesting task.

Set $H_i = F_i^\bullet$ for $1 \le i < \ell$; $H_i = f_j$, where $j = i - \ell + 1$ for $\ell \le i < 2\ell$; and $H_{2\ell} = e_\delta$. Clearly all these functions are semi-invariants of $\tilde{\mathfrak{g}}$. We will show that they generate $\mathcal{S}(\tilde{\mathfrak{g}})_{\rm si}$.

Let $\mathfrak{h} \subset \mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{g}$, $U \subset B$ the unipotent radical, and $\tilde{\mathfrak{g}}'$ the derived algebra of $\tilde{\mathfrak{g}}$. We have $\tilde{\mathfrak{g}}' = \mathfrak{n} \ltimes \mathfrak{n}^-$. Note that the Lie algebra $\tilde{\mathfrak{g}}'$ has only trivial characters.

Lemma 5.7. *We have* $\mathcal{S}(\tilde{\mathfrak{g}})_{\rm si} = Z\mathcal{S}(\tilde{\mathfrak{g}}')$*. In particular,* $\mathcal{S}(\tilde{\mathfrak{g}})_{\rm si}$ *is a subalgebra of* $\mathcal{S}(\tilde{\mathfrak{g}}')$*.*

Proof. Suppose that $\mathcal{S}(\tilde{\mathfrak{g}})_\lambda$ is an eigenspace of $\tilde{\mathfrak{g}}$ corresponding to a character $\lambda \in \tilde{\mathfrak{g}}^*$ and $\mathcal{S}(\tilde{\mathfrak{g}})_\lambda \ne 0$. Let $\tilde{\mathfrak{g}}^\lambda \subset \tilde{\mathfrak{g}}$ be the kernel of λ. Then, by a result of Borho et al. [BGR, Satz 6.1], $\mathcal{S}(\tilde{\mathfrak{g}})_{\rm si} \subset \mathcal{S}(\tilde{\mathfrak{g}}^\lambda)$ (see also [RV, Lemme 4.1] or [JSh, Sect. 1.2]). Since $f_1, \dots, f_\ell \in \mathfrak{n}^- \subset \tilde{\mathfrak{g}}$ are semi-invariants with weights $-\alpha_i$ $(1 \le i \le \ell)$, we conclude that $\mathcal{S}(\tilde{\mathfrak{g}})_{\rm si} \subset \mathcal{S}(\tilde{\mathfrak{g}}')$. Next, $\tilde{\mathfrak{g}}'$ has no non-trivial characters and the action of $\mathfrak{h}$ on $Z\mathcal{S}(\tilde{\mathfrak{g}}')$ is diagonalisable. Thus indeed $\mathcal{S}(\tilde{\mathfrak{g}})_{\rm si} = Z\mathcal{S}(\tilde{\mathfrak{g}}')$. □

Lemma 5.7 shows that $\tilde{\mathfrak{g}}'$ is the canonical truncation of $\tilde{\mathfrak{g}}$ in the following sense. For any algebraic finite-dimensional Lie algebra $\mathfrak{q}$, there exists a unique subalgebra $\mathfrak{a}$ such that $\mathcal{S}(\mathfrak{q})_{\rm si} = Z\mathcal{S}(\mathfrak{a})$ [BGR]. This $\mathfrak{a}$ is said to be the *truncation* of $\mathfrak{q}$.

Lemma 5.8. *Let* H_i *be as above. Then the polynomials* H_i *are algebraically independent. Moreover,* $\operatorname{ind} \tilde{\mathfrak{g}}' = 2\ell$*.*

Proof. Let $\hat{e} \subset \mathfrak{n}^{\rm ab}$ be a linear function coming from a regular nilpotent element $e \in \mathfrak{n}$. We extend it to a function on $\tilde{\mathfrak{g}}^*$ by setting $\hat{e}(\mathfrak{b}) = 0$. Then $d_{\hat{e}} F_i^\bullet = d_e F_i$. Moreover, $d_e F_\ell = e_\delta$ up to a constant. Therefore $d_{\hat{e}} F_1^\bullet, \dots, d_{\hat{e}} F_{\ell-1}^\bullet$ and $e_\delta = d e_\delta$ generate $\mathfrak{n}_e$, a subspace of dimension ℓ. The other polynomials H_j $(\ell \le j < 2\ell)$ are linearly independent elements of $\mathfrak{n}^-$. Thus the $d_{\hat{e}} H_i$ are linearly independent and the first statement is proved.

According to the index formula of Raïs [R] (cf. Lemma 4.8),

$$\operatorname{ind} \tilde{\mathfrak{g}}' = (\dim \mathfrak{n}^{\rm ab} - \dim U\hat{e}) + \operatorname{ind} \mathfrak{n}_{\hat{e}} = 2\ell.$$

Alternatively, one can use [OV, Lemma 3.7], which calculates the index of a truncated Lie algebra. In our case it reads $\operatorname{ind} \tilde{\mathfrak{g}}' = \operatorname{ind} \tilde{\mathfrak{g}} + (\dim \tilde{\mathfrak{g}} - \dim \tilde{\mathfrak{g}}')$. □

Theorem 5.9. *Let $\tilde{\mathfrak g}$ be Feigin's contraction of $\mathfrak g$. Then $\mathcal S(\tilde{\mathfrak g})_{\rm si}$ is generated by the polynomials H_i defined above.*

Proof. First, we extract the "$\mathfrak h$-part" out of $\tilde\pi$ and its powers. Let $\tilde\pi'$ be the Poisson tensor of $\tilde{\mathfrak g}'$, and ω' a volume form on $(\tilde{\mathfrak g}')^*$. Then $\tilde\pi = \tilde\pi' + R_{\mathfrak h}$, where $R_{\mathfrak h}$ is a sum $\sum[h_i, y_j]\partial_{h_i}\wedge\partial_{y_j}$ over a basis $h_1,\ldots,h_\ell$ of $\mathfrak h$ and a basis of $\tilde{\mathfrak g}'$. Since $\dim\mathfrak h = \ell$, for $k > \ell$ we have $\Lambda^k R_{\mathfrak h} = 0$. Taking into account that also $\Lambda^k\tilde\pi' = 0$ for $k > (n-3\ell)/2$ (Lemma 5.8), we get the equality

$$\Lambda^{(n-\ell)/2}\tilde\pi = (\Lambda^{(n-3\ell)/2}\tilde\pi')\wedge(\Lambda^\ell R_{\mathfrak h}).$$

Therefore a fundamental semi-invariant of $\tilde{\mathfrak g}'$ is a divisor of $p = \prod f_i^{r_i-1}$.

Set $\mathcal H := dH_1\wedge\cdots\wedge dH_{2\ell}$. By Lemma 5.8, $\mathcal H \neq 0$. Note also that by the same lemma, $\operatorname{ind}\tilde{\mathfrak g}' = 2\ell$. Therefore, applying Lemma 2.1 to $\mathcal S(\tilde{\mathfrak g}')$, we get non-zero coprime $q_1, q_2 \in \mathcal S(\tilde{\mathfrak g}')$ such that

$$q_1(\mathcal H/\omega') = q_2\Lambda^{(n-3\ell)/2}\tilde\pi'.$$

Since the polynomials q_1 and q_2 are coprime, q_1 must be a divisor of p as well. In particular, $\deg q_1 \le \deg p$. Next we compute and sum the degrees of all objects involved in the equality

$$\deg q_2 + (n-3\ell)/2 = \deg q_1 + \deg\mathcal H \le \deg p + \Big(\sum_{i=1}^{\ell-1}\deg F_i - \ell + 1\Big)$$
$$= \deg p + ((n+\ell)/2 - \deg F_\ell - \ell + 1) = (n+\ell)/2 - (\ell+1) - \ell + 1 = (n-3\ell)/2.$$

This is possible only if $q_2 \in \mathbb K$ and $q_1 = p$ (up to a scalar multiple). Thus $p(\mathcal H/\omega') = a\Lambda^{(n-3\ell)/2}\tilde\pi'$ with $a \in \mathbb K^\times$. Moreover, p is a fundamental semi-invariant of $\tilde\pi'$ and therefore the Jacobian locus $\mathcal J(H_1,\ldots,H_{2\ell})$ of H_i does not contain divisors. We have $\dim\mathcal J(H_1,\ldots,H_{2\ell}) \le \dim\tilde{\mathfrak g} - 2$ and all polynomials H_i are homogeneous. This allows us to use the characteristic zero version of a result of Skryabin (see [PPY, Theorem 1.1]), stating that here any $H \in \mathcal S(\tilde{\mathfrak g}')$ that is algebraic over a subalgebra generated by H_i is contained in that subalgebra. Since $\operatorname{tr.deg} Z\mathcal S(\tilde{\mathfrak g}') \le 2\ell$, we conclude that $Z\mathcal S(\tilde{\mathfrak g}')$ is generated by $H_1,\ldots,H_{2\ell}$. Now the result follows from Lemma 5.7. □

5.2. *Subregular orbital varieties*

Irreducible components of $\overline{\mathcal O^{\rm sub}\cap\mathfrak n}$ are called *subregular orbital varieties*. They have played a major rôle in the proof of Theorem 5.5 and we know that each of them is a linear space $\mathfrak D_i$. Every $\mathfrak D_i$ is also the nilpotent radical of a minimal parabolic subalgebra $\mathfrak p_{\alpha_i}$. An interesting question is whether B acts on $\mathfrak D_i$ with an open orbit. This problem was addressed and solved in [GHR]. It turns out that our results complement some of the arguments in [GHR].

Let $\tilde G = B\ltimes\exp(\mathfrak n^-)$ be an algebraic group with $\operatorname{Lie}\tilde G = \tilde{\mathfrak g}$. Let also $\mathcal O_i$ be an irreducible component of $\mathcal O^{\rm sub}\cap\mathfrak n$ lying in $\mathfrak D_i$. Note that $\mathcal O_i$ is a dense open subset of $\mathfrak D_i$.

Lemma 5.10. *Suppose that* $r_i = [\delta : \alpha_i] = 1$. *Then there is a dense open* B*-orbit in* $\mathfrak{D}_i$. *In the other direction, suppose that there is a dense open* B*-orbit in* $\mathfrak{D}_i$, *but* $r_i > 1$. *Then* $\mathfrak{b}_e$ *is Abelian for generic* $e \in \mathfrak{D}_i$.

Proof. According to Theorem 5.5, if $r_i = 1$, then the intersection $(\mathfrak{b}^* \times \mathfrak{D}_i) \cap \tilde{\mathfrak{g}}^*_{\text{reg}}$ is non-empty, and hence it is a non-empty open subset of $\mathfrak{b}^* \times \mathfrak{D}_i$. Therefore there is a regular x in $\mathfrak{b}^* \times \mathcal{O}_i$. Let $\hat{e}$ be the $\mathfrak{n}^{\text{ab}}$-component of this x. In other words, $x \in \mathfrak{b}^* \times \{\hat{e}\}$, where $\hat{e} \in \mathfrak{n}^{\text{ab}}$ comes from a subregular nilpotent element $e \in \mathfrak{n}$. Next we compute the codimension of $\tilde{G}x$. This can be done in the spirit of the Raïs formula for the index of a semidirect product [R] (see also [P07′, Proposition 5.5] and Lemma 4.8). And the result is that

$$\dim \tilde{\mathfrak{g}} - \dim \tilde{G}x = \dim \mathfrak{n} - (\dim \mathfrak{b} - \dim \mathfrak{b}_e) + \operatorname{ind} \mathfrak{b}_e = \dim \mathfrak{b}_e - \ell + \operatorname{ind} \mathfrak{b}_e. \quad (5.1)$$

Suppose that Be is not dense in $\mathfrak{D}_i$. Then $\dim Be < (\dim \mathfrak{n} - 1)$ and therefore $\dim \mathfrak{b}_e \geq \ell + 2$, implying $\mathfrak{b}_e = \mathfrak{g}_e$. Since $\operatorname{ind} \mathfrak{g}_e \geq \ell$ by Vinberg's inequality [P03, Corollary 1.7], we get $\dim \tilde{\mathfrak{g}} - \dim \tilde{G}x \geq (\ell + 2)$ and therefore $x \in \tilde{\mathfrak{g}}^*_{\text{sing}}$. This contradiction proves the first statement of the lemma.

Next suppose that $r_i > 1$, $\overline{Be} = \mathfrak{D}_i$, and x is a generic element in $\mathfrak{b}^* \times \{\hat{e}\}$. The equation (5.1) is valid for this x and gives $\operatorname{ind} \mathfrak{b}_e \geq \ell$, since $\dim \mathfrak{b}_e = \ell + 1$. Because the difference $\dim \mathfrak{q} - \operatorname{ind} \mathfrak{q}$ is even for every finite-dimensional Lie algebra $\mathfrak{q}$, this inequality is possible only if $\mathfrak{b}_e$ is commutative. □

In type A all r_i are equal to 1 and all components $\mathfrak{D}_i$ have open B-orbits. This result was first obtained by J. A. Vargas [Va].

Example 5.11. Suppose that $\mathfrak{g} = \mathfrak{so}_{2\ell}$ with $\ell > 3$ and $e \in \mathfrak{g}$ is a subregular nilpotent element. Then e is given by a partition $(2\ell - 3, 3)$, odd powers e^{2k+1} of the underlying matrix are elements of $\mathfrak{g}$, and $\mathfrak{g}_e$ has a basis

$$e, e^3, \ldots, e^{2\ell-5}, \xi_1, \xi_2, \xi_3, \eta$$

with the following non-trivial commutators: $[\xi_1, \xi_2] = e^{2n-5}$, $[\xi_1, \eta] = \xi_2$, and $[\xi_2, \eta] = \xi_3$. (The structure of $\mathfrak{g}_e$ is described, for example, in [Y, Section 1].) It is not difficult to see that $\mathfrak{g}_e$ does not contain a commutative subalgebra of codimension 1. In particular, if $\dim \mathfrak{b}_e = \ell + 1$, then $\mathfrak{b}_e$ is not Abelian. By Lemma 5.10, in type D there is an open B-orbit in $\mathfrak{D}_i$ if and only if $r_i = 1$.

Calculating centralisers $\mathfrak{g}_e$ of subregular nilpotent elements in type E, on GAP or by hand, one can show that $\mathfrak{g}_e$ does not contain an Abelian subalgebra of codimension 1. Together with Lemma 5.10, this fact provides an additional explanation for [GHR, Theorem 2.4(a)(i)]. That result states that for $\mathfrak{g}$ simply laced, $\mathfrak{D}_i$ contains an open B-orbit if and only if $r_i = 1$.

Remark 5.12. Actually, Theorem 2.4 of [GHR] asserts that there are a finite number of B-orbits in $\mathcal{O}_i$ if $r_i = 1$. As is explained in the Introduction of [GHR], this is equivalent to the existence of an open orbit. Note also that [GHR] proves the existence of an open B-orbit by giving its representative in each particular case. Moreover, results for the exceptional Lie algebras rely on GAP calculations of S. Goodwin [G].

Independently of the above discussion, D. Panyushev has posed a related question. When is the stabiliser B_e of a generic $e \in \mathfrak{D}_i$ Abelian? The centraliser of a nilpotent element is Abelian only when the element is regular; for a conceptual proof of this fact see [P03, Theorem 3.3]. In particular, $\mathfrak{g}_e$ is not Abelian for a subregular nilpotent element e. This implies that $\mathfrak{b}_e$ can be Abelian only if $\dim \mathfrak{b}_e < \ell + 2$ and there is a dense B-orbit in $\mathfrak{D}_i$. On the other hand, for an Abelian Lie algebra, $\operatorname{ind} \mathfrak{b}_e = \dim \mathfrak{b}_e = \ell + 1$ and therefore r_i must be larger than 1. In the simply laced case, $\mathfrak{g}_e$ does not contain Abelian subalgebras of codimension 1. For the remaining Lie algebras, [GHR, Theorem 2.4(a)(ii)] provides the following answer.

Proposition 5.13. *The stabiliser in B of a generic $e \in \mathfrak{D}_i$ is Abelian if and only if*

- $\mathfrak{g}$ *is of type B_ℓ and $i > 1$, or*
- $\mathfrak{g}$ *is of type C_ℓ with $\ell > 1$ and $i = 1$, or*
- $\mathfrak{g}$ *is of type F_4 and $i = 4$, or*
- $\mathfrak{g}$ *is of type G_2 and $i = 2$,*

in the Vinberg–Onishchik numbering of simple roots [VO].

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