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Gradient regularity via rearrangements for p-Laplacian type elliptic boundary value problems

Abstract. A sharp estimate for the decreasing rearrangement of the length of the gradient of solutions to a class of nonlinear Dirichlet and Neumann elliptic boundary value problems is established under weak regularity assumptions on the domain. As a consequence, the problem of gradient bounds in norms depending on global integrability properties is reduced to one-dimensional Hardy-type inequalities. Applications to gradient estimates in Lebesgue, Lorentz, Zygmund, and Orlicz spaces are presented.

Keywords. Nonlinear elliptic equations, Dirichlet problems, Neumann problems, gradient estimates, rearrangements, Lorentz spaces, Orlicz spaces

1. Introduction and main results

The main contribution of the present paper is an estimate for the gradient of solutions to a class of nonlinear elliptic boundary value problems in domains Ω in $\mathbb{R}^n$, with $n \geq 3$. The novelty of the relevant estimate is twofold. On the one hand, it is universal in the sense that it only involves the decreasing rearrangement of the length of the gradient and the decreasing rearrangement of the datum on the right-hand side of the equation, thus allowing for a unified, simple approach to gradient regularity in a broad class of function spaces. On the other hand, it enjoys the striking property of providing a bound for a suitable nonlinear expression of the gradient with just a (sub)linear dependence on the datum. Moreover, although our results are new even for smooth domains, minimal regularity assumptions are imposed on $\partial\Omega$.

The problems under consideration consist of a quasilinear elliptic equation of the form

$$-\operatorname{div}(a(|\nabla u|)\nabla u) = f(x) \quad \text{in } \Omega \tag{1.1}$$

coupled with either the Dirichlet condition

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

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or the Neumann condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

Here, Ω is a domain (a connected open bounded set), and ν stands for the outward unit normal to $\partial\Omega$. The function f is a priori supposed to be merely in $L^1(\Omega)$.

We assume that $a : [0, \infty) \to [0, \infty)$ is of class $C^1(0, \infty)$,

$$-1 < \inf_{t>0} \frac{ta'(t)}{a(t)} \leq \sup_{t>0} \frac{ta'(t)}{a(t)} < \infty, \tag{1.4}$$

and there exist $p \in [2, n)$ and $c, C > 0$ such that

$$ct^{p-1} \leq ta(t) \leq C(t^{p-1} + 1) \quad \text{for } t > 0. \tag{1.5}$$

These assumptions correspond to standard ellipticity and growth conditions for the differential operator in (1.1). In fact, equation (1.1) is patterned on the model

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = f(x) \quad \text{in } \Omega, \tag{1.6}$$

the so called p-Laplace equation, corresponding to the choice $a(t) = t^{p-2}$ for $t > 0$.

The notion of solution u to either (1.1) & (1.2), or (1.1) & (1.3), has to be defined in a suitable, extended sense, due to the generality of our assumptions on f. Comments on this and precise definitions are given in Section 2.

Our inequality for the decreasing rearrangement $|\nabla u|^*$ of $|\nabla u|$ is contained in the following statement. As mentioned above, weak regularity is imposed on the domain Ω. We require that $\partial\Omega$ belongs to $W^2 L^{n-1,1}$, that is, Ω is locally the subgraph of a function of $n-1$ variables whose second-order distributional derivatives belong to the Lorentz space $L^{n-1,1}$. Note that this is the weakest possible integrability assumption on the second-order derivatives of a function of $n-1$ variables for its first-order derivatives to be continuous, and hence for $\partial\Omega \in C^{1,0}$.

Theorem 1.1. *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 3$, such that $\partial\Omega \in W^2 L^{n-1,1}$. Assume that $f \in L^1(\Omega)$, and let u be either a solution to the Dirichlet problem (1.1) & (1.2), or to the Neumann problem (1.1) & (1.3). Then there exists a constant $C = C(p, \Omega)$ such that*

$$|\nabla u|^*(s)^{p-1} \leq C \int_s^{|\Omega|} f^{**}(r) r^{-1/n'} dr \quad \text{for } s \in (0, |\Omega|). \tag{1.7}$$

Here, $n' = n/(n-1)$, the Hölder conjugate of n, $f^{**}(r) = (1/r) \int_0^r f^*(\rho) d\rho$ for $r \in (0, |\Omega|)$ and $|\Omega|$ denotes the Lebesgue measure of Ω.

Alternatively, our result holds for Ω just convex, without any additional regularity of $\partial\Omega$.

Theorem 1.2. *Let Ω be a convex domain in $\mathbb{R}^n$, $n \geq 3$. Then inequality (1.7) holds under the same assumptions on f and u as in Theorem 1.1.*

Either of the assumptions on Ω in the above results is essentially indispensable, as will be shown by suitable examples.

A distinctive feature of estimate (1.7) is its independence of specific function spaces. It is flexible enough to reduce any inequality between quasi-norms of $|\nabla u|$ and f depending only on the measure of their level sets (or, equivalently, on their decreasing rearrangements), called rearrangement invariant quasi-norms in the literature, to considerably simpler one-dimensional Hardy-type inequalities involving the corresponding representation quasi-norms. These implications are exhibited in Corollaries 4.1 and 4.2, which enable us to derive both well-known and new gradient estimates in customary function spaces, such as Lebesgue, Lorentz(–Zygmund), and Orlicz spaces.

A remarkable consequence of (1.7) is that it translates verbatim the linear theory of integrability of $|\nabla u|$ for solutions to homogeneous boundary value problems for the Laplace equation to the theory of integrability of $|\nabla u|^{p-1}$ for solutions to nonlinear problems involving any equation of the form (1.1). Indeed, an estimate of the form (1.7) for the gradient of solutions to the Laplace equation

$$-\Delta u = f(x) \quad \text{in } \Omega, \tag{1.8}$$

under either homogeneous Dirichlet, or Neumann boundary conditions, can be established via classical tools, provided that $\partial\Omega \in C^\infty$. Such an estimate entails that

$$|\nabla u|^*(s) \le C \int_s^{|\Omega|} f^{**}(r) r^{-1/n'} dr \quad \text{for } s \in (0, |\Omega|), \tag{1.9}$$

and follows via a representation formula for ∇u in terms of the Green function (see e.g. [MP, Section 3]), combined with a rearrangement inequality for convolutions [On]. Inequality (1.9) is recovered from (1.7), with $p = 2$, the exponent associated with the Laplace operator. Moreover, inequality (1.7) tells us that, for every $p \ge 2$, the expression $(|\nabla u|^*)^{p-1}$ admits exactly the same kind of estimate in terms of f^{**}.

Theorems 1.1 and 1.2 should also be compared with other bounds in rearrangement form available in the literature on elliptic equations. Rearrangement estimates for solutions u to Dirichlet problems go back to [Ta1, Ta2]; Neumann problems are treated in [Ci1, MS1, MS2]. These estimates hold for classes of nonsmooth elliptic operators with a more general structure than those appearing in (1.1). Moreover, no regularity at all on Ω is needed when the Dirichlet homogeneous boundary datum is imposed, and weaker—for instance Lipschitz—regularity on Ω is required in the case of Neumann boundary conditions. The relevant estimates tell us that

$$u^*(s) \le C \int_s^{|\Omega|} f^{**}(r)^{1/(p-1)} r^{-1+p'/n} dr \quad \text{for } s \in (0, |\Omega|), \tag{1.10}$$

where the constant C depends either just on n and $|\Omega|$, or on Ω, according to whether Dirichlet or Neumann conditions are imposed. The solution u is normalized in such a way that its median vanishes in the latter case.

A rearrangement estimate for $|\nabla u|$ is known for the same class of equations and domains. This estimate reads

$$|\nabla u|^*(s)^{p-1} \leq C\left(\frac{1}{s}\int_{s/2}^{|\Omega|} f^{**}(r)^{p'} r^{p'/n} dr\right)^{1/p'} \quad \text{for } s \in (0, |\Omega|), \qquad (1.11)$$

with the same dependence of the constant C as in (1.10) [ACMM] (see also [AFT] for a slightly weaker result). Suitable versions of (1.10) and (1.11) are also available for solutions to Neumann problems in irregular domains [ACMM, CM1]. Inequality (1.11) is much weaker than (1.7). Indeed, only bounds for norms of $|\nabla u|$ weaker than L^p, the natural norm associated with the nonlinearity of the problems at hand, can be deduced from (1.11). On the other hand, inequality (1.7) requires additional structure assumptions on the elliptic operator and on the regularity of the domain Ω. Earlier bounds, still in terms of f^*, for Lebesgue gradient norms which are not stronger than L^p were established in [Ma1, Ma3]; see also [Ci1, MS1, MS2, Ta1, Ta2].

Let us mention that precise bounds for the gradient of local solutions to nonlinear elliptic equations in terms of nonlinear potentials have recently been developed in the series of papers [DM3, DM5, Mi1, Mi2]; see see also [Ci7] for related results. For more classical contributions on gradient regularity in the theory of elliptic partial differential equations we refer to the monographs [BF, Gia, GT, Giu, LU], and to the references therein.

The sharpness of gradient estimates which can be derived from (1.7) is of course an interesting issue. Classical rearrangement estimates, such as (1.10) for Dirichlet problems, have an isoperimetric nature, in the sense that they can be interpreted as comparison principles between the radial symmetral of the solution to a Dirichlet problem on a domain Ω, and the (radially symmetric) solution to a symmetrized Dirichlet problem on a ball with the same measure as Ω. An inequality of this kind is optimal, since equality holds whenever Ω is a ball and f is radially decreasing. The bounds for L^p (or weaker) gradient norms for the Dirichlet problem, which we alluded to above, enjoy a similar optimality property. This is not the case, though, for Neumann problems, or, even for Dirichlet problems when pointwise rearrangement gradient inequalities such as (1.11), (1.9), or (1.7) are in question. In fact, as will be clear from the proof (see Section 3) inequality (1.7) does not rely on symmetrization arguments. Suitable isoperimetric inequalities can be used to describe the regularity of Ω, and they enter as a tool in establishing certain steps of the proof of Theorems 1.1 and 1.2. Equality cases in the isoperimetric type inequalities involved are however not preserved after combining these steps. The existence of configurations for which equality holds in a pointwise inequality of the form (1.7), or even in ensuing gradient norm inequalities, and their description, turn out to be a difficult open problem. In this connection, let us just observe that radially symmetric domains and data f do not provide an answer to that problem in general. This is shown by the explicit solution [Ci2], and by counterexamples [Ci3] in certain model linear problems in a ball or in the whole of $\mathbb{R}^n$.

As far as the regularity of Ω is concerned, the assumption $\partial\Omega \in W^2 L^{n-1,1}$ in Theorem 1.1 is essentially sharp, as will be illustrated by a couple of examples. The second example also demonstrates how the conclusion of Theorem 1.2 may fail for domains which

are very close to being convex. Both examples involve a domain Ω whose boundary contains 0, is smooth outside a neighborhood of 0, and in that neighborhood Ω coincides with

$$\{x = (x', x_n) : x_n < L|x'|\}$$

for some number $L > 0$. Here, $x' = (x_1, \ldots, x_{n-1})$. In other words, in a neighborhood of 0 the domain Ω is bounded by an inward cone, whose aperture is $\arctan(1/L)$. The shape of Ω far from 0 is immaterial.

Assume first that $2 \leq p \leq n - 1$ and that the cone in the definition of Ω is very sharp, namely that L is very large. Consider the Dirichlet problem

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.12}$$

One can easily verify that $\partial\Omega \in W^2 L^q$ for every $q < n - 1$, and, in fact, $\partial\Omega \in W^2 L^{q,1}$ for every $q < n - 1$, but $\partial\Omega \notin W^2 L^{n-1,1}$.

Now, the function f can be chosen in such a way that it is smooth, vanishes in a neighborhood of 0, and the solution u to (1.12) satisfies

$$u(x) \approx |x|^{\alpha(L)} F\big(x_n/|x|\big) \quad \text{as } x \to 0, \tag{1.13}$$

for some smooth function $F : \mathbb{R} \to \mathbb{R}$, and some exponent $\alpha(L) > 0$ such that

$$\lim_{L \to \infty} \alpha(L) = 0$$

(see [KM]). In (1.13), the notation $\approx$ means that the two sides are bounded by each other up to multiplicative constants independent of x. Thus, given any $q > n$, $|\nabla u| \notin L^q(\Omega)$, provided that L is sufficiently large, even if f is very smooth. On the contrary, if $\partial\Omega \in W^2 L^{n-1,1}$, Theorem 1.1, or its Corollary 4.1 in Section 4, entail that $|\nabla u| \in L^\infty(\Omega)$ provided that $f \in L^{n,1}(\Omega)$ (see Theorem 4.4, Section 4), and hence, in particular, if $f \in L^\infty(\Omega)$.

Suppose next that the cone in the definition of Ω is almost flat, namely that L is very small. Of course, Ω can be constructed in such a way that it is convex when $L = 0$. Consider the Neumann problem for the Laplace equation

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega, \\ \partial u/\partial v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.14}$$

One can show that there exist functions f which are smooth, vanish in a neighborhood of 0, and are such that the solution u to (1.12) satisfies

$$u(x) \approx |x|^{\beta(L)} F(x_n/|x|) \quad \text{as } x \to 0,$$

up to multiplicative constants independent of x, for some smooth function $F : \mathbb{R} \to \mathbb{R}$. Here, $\beta(L)$ is a positive exponent such that

$$\beta(L) < 1 \quad \text{if } L \text{ is sufficiently close to } 0$$

(see e.g. [KMR, Section 2.3.2]). Thus, if L is sufficiently small, there exists $q < \infty$ such that $|\nabla u| \notin L^q(\Omega)$.

An analogous conclusion holds if the Neumann condition in (1.14) is replaced with the Dirichlet condition $u = 0$ on $\partial\Omega$.

This is another example showing that the regularity assumption on $\partial\Omega$ in Theorem 1.1 cannot be essentially relaxed. Indeed, boundedness, and high integrability, of $|\nabla u|$ need not be guaranteed, even for the Laplace equation with a smooth right-hand side and $\partial\Omega$ smooth everywhere except at a single point, in a neighborhood of which $\partial\Omega$ is almost flat, and the regularity assumption $\partial\Omega \in W^2 L^{n-1,1}$ is just slightly relaxed.

The same example also demonstrates that even a mild local perturbation of convexity may affect the conclusion of Theorem 1.2.

2. Solutions

Given $p \in [1, \infty]$, we denote by $W^{1,p}(\Omega)$ the standard Sobolev space on Ω, and by $W_0^{1,p}(\Omega)$ the subspace of those functions which vanish, in the appropriate sense, on $\partial\Omega$. Their topological duals will be denoted by $(W^{1,p}(\Omega))'$ and $(W_0^{1,p}(\Omega))'$, respectively. We also set

$$W_\perp^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega) : \int_\Omega u \, dx = 0 \right\}.$$

For instance, if $p < n$, the case of interest in the present paper, then $L^{\frac{np}{np-n+p}}(\Omega) \subset (W^{1,p}(\Omega))' \subset (W_0^{1,p}(\Omega))'$, by the Sobolev embedding theorem.

Assume that $f \in L^1(\Omega) \cap (W_0^{1,p}(\Omega))'$. A function $u \in W_0^{1,p}(\Omega)$ is called a *weak solution to the Dirichlet problem* (1.1) & (1.2) if

$$\int_\Omega a(x, \nabla u) \cdot \nabla \Phi \, dx = \int_\Omega f \Phi \, dx \quad \text{for every } \Phi \in W_0^{1,p}(\Omega). \tag{2.1}$$

Analogously, if $f \in L^1(\Omega) \cap (W^{1,p}(\Omega))'$ and $\int_\Omega f(x) \, dx = 0$, then a function $u \in W^{1,p}(\Omega)$ is called a *weak solution to the Neumann problem* (1.1) & (1.3) if

$$\int_\Omega a(x, \nabla u) \cdot \nabla \Phi \, dx = \int_\Omega f \Phi \, dx \quad \text{for every } \Phi \in W^{1,p}(\Omega). \tag{2.2}$$

Standard arguments from the direct methods of the calculus of variations, based on the strict convexity, weak lower semicontinuity, and coercivity of the functional

$$J(u) = \int_\Omega \big(G(|\nabla u|) - fu\big) \, dx,$$

where $G(s) = \int_0^s a(r) r \, dr$ for $s \geq 0$, yield the following existence and uniqueness result for problems (1.1) & (1.2) and (1.1) & (1.3).

Proposition 2.1. *Let Ω be a domain in $\mathbb{R}^n$.*

(i) *Let $f \in L^1(\Omega) \cap (W_0^{1,p}(\Omega))'$. Then there exists a unique weak solution $u \in W_0^{1,p}(\Omega)$ to the Dirichlet problem* (1.1) & (1.2).

(ii) *Assume, in addition, that $\partial\Omega \in$ Lip. Let $f \in L^1(\Omega) \cap (W^{1,p}(\Omega))'$ be such that $\int_\Omega f(x)\,dx = 0$. Then there exists a unique weak solution $u \in W_\perp^{1,p}(\Omega)$ to the Neumann problem* (1.1) & (1.3).

When f is so poorly integrable that $f \notin (W_0^{1,p}(\Omega))'$—a mere function in $L^1(\Omega)$ in the worst case—the definition of weak solution for the Dirichlet problem (1.1) & (1.2) is meaningless. The same drawback occurs for the Neumann problem (1.1) & (1.3) if $f \notin (W^{1,p}(\Omega))'$. In this case, solutions can only be defined in a generalized sense. Several notions of solutions have been introduced in the literature in this connection [BG, B–V, DaA, DM, LM, M1, M2], which, a posteriori, turn out to be equivalent. Simple examples show that gradients of these solutions do not belong to $L^p(\Omega)$ in general, but just enjoy weaker integrability properties.

We shall work with notions of generalized solutions to Dirichlet and Neumann problems which are based upon the use of sequences of solutions to approximating problems [DaA, DM]. Precise definitions are as follows.

Let $f \in L^1(\Omega)$. A function $u \in W_0^{1,p-1}(\Omega)$ is called an *approximable solution* to problem (1.1) & (1.2) if there exists a sequence $\{f_k\} \subset L^1(\Omega) \cap (W_0^{1,p}(\Omega))'$ such that

$$f_k \to f \quad \text{in } L^1(\Omega),$$

and the sequence of weak solutions $\{u_k\} \subset W_0^{1,p}(\Omega)$ to problem (1.1) & (1.2), with f replaced with f_k, satisfies

$$u_k \to u \quad \text{a.e. in } \Omega.$$

Assume now that $f \in L^1(\Omega)$ and $\int_\Omega f(x)\,dx = 0$. A function $u \in W_\perp^{1,p-1}(\Omega)$ is called an *approximable solution* to problem (1.1) & (1.3) if there exists a sequence $\{f_k\} \subset L^1(\Omega) \cap (W^{1,p}(\Omega))'$ such that $\int_\Omega f_k(x)\,dx = 0$ for $k \in \mathbb{N}$,

$$f_k \to f \quad \text{in } L^1(\Omega),$$

and the sequence $\{u_k\} \subset W_\perp^{1,p}(\Omega)$ of weak solutions to problem (1.1) & (1.3), with f replaced with f_k, satisfies

$$u_k \to u \quad \text{a.e. in } \Omega.$$

Approximate solutions are solutions in the distributional sense. In fact, they can be regarded as distinguished members in the class of distributional solutions, which need not be unique, as shown by classical examples [Se]. An existence and uniqueness result for approximate solutions is the content of the next theorem. In particular, it turns out that the weak solution and the approximate solution agree, whenever the former is well defined.

Theorem 2.2. *Let Ω be a domain in $\mathbb{R}^n$.*

(i) *Let $f \in L^1(\Omega)$. Then there exists a unique approximable solution $u \in W_0^{1,p-1}(\Omega)$ to* (1.1) & (1.2), *and*

$$\int_\Omega a(x, \nabla u) \cdot \nabla \Phi \, dx = \int_\Omega f \Phi \, dx \quad \text{for every } \Phi \in C_0^\infty(\Omega). \qquad (2.3)$$

Moreover, if $\{u_k\}$ is a sequence of approximating solutions for u, then

$$\nabla u_k \to \nabla u \quad \text{a.e. in } \Omega \qquad (2.4)$$

(up to subsequences).

(ii) *Assume, in addition, that $\partial\Omega \in \mathrm{Lip}$. Let $f \in L^1(\Omega)$ be such that $\int_\Omega f(x) \, dx = 0$. Then there exists a unique approximable solution $u \in W_\perp^{1,p-1}(\Omega)$ to* (1.1) & (1.3), *and*

$$\int_\Omega a(x, \nabla u) \cdot \nabla \Phi \, dx = \int_\Omega f \Phi \, dx \quad \text{for every } \Phi \in C^\infty(\Omega). \qquad (2.5)$$

Moreover, if $\{u_k\}$ is a sequence of approximating solutions for u, then

$$\nabla u_k \to \nabla u \quad \text{a.e. in } \Omega \qquad (2.6)$$

(up to subsequences).

Part (i) of the statement of Proposition 2.2 is by now standard: see e.g. [AM, DaA, DM]. Part (ii) can be established via analogous arguments; in particular, it follows as a special instance of the results of [ACMM], where the case of possibly irregular domains is also analyzed.

3. The rearrangement estimate

This section is devoted to a (unified) proof of Theorems 1.1 and 1.2. Unless otherwise stated, in what follows the term solution either to the Dirichlet problem (1.1) & (1.2), or to the Neumann problem (1.1) & (1.3), has to be understood in the sense of approximable solution, as defined in Section 2.

We begin by recalling a few definitions and basic properties of rearrangements and Lorentz spaces to be exploited in our proof. Let u be a measurable function in Ω. The *distribution function* $\mu_u : [0, \infty) \to [0, \infty)$ of u is defined as

$$\mu_u(t) = |\{x \in \Omega : |u(x)| \geq t\}| \quad \text{for } t \geq 0. \qquad (3.1)$$

The *decreasing rearrangement* $u^* : [0, |\Omega|] \to [0, \infty]$ of u is given by

$$u^*(s) = \sup\{t \geq 0 : \mu_u(t) \geq s\}| \quad \text{for } s \in [0, |\Omega|], \qquad (3.2)$$

and its *increasing rearrangement* $u_* : [0, |\Omega|] \to [0, \infty]$ by

$$u_*(s) = u^*(|\Omega| - s) \quad \text{for } s \in [0, |\Omega|].$$

A basic property of rearrangements is the *Hardy–Littlewood inequality*

$$\int_0^{|\Omega|} u^*(s)v_*(s)\,ds \le \int_\Omega |u(x)v(x)|\,dx \le \int_0^{|\Omega|} u^*(s)v^*(s)\,ds \tag{3.3}$$

for any measurable functions u and v in Ω.

Given $q \in (1, \infty]$ and $k \in (0, \infty]$, or $q = 1$ and $k \in (0, 1]$, the *Lorentz space* $L^{q,k}(\Omega)$ is defined as the set of all measurable functions u on Ω for which the expression

$$\|u\|_{L^{q,k}(\Omega)} = \|s^{1/q-1/k}u^*(s)\|_{L^k(0,|\Omega|)} \tag{3.4}$$

is finite. In particular,

$$L^{q,q}(\Omega) = L^q(\Omega)$$

for every $q \in [1, \infty]$. Moreover, $L^{q,k_1}(\Omega) \subsetneq L^{q,k_2}(\Omega)$ if $k_1 < k_2$, and, $L^{q_1,k_1}(\Omega) \subsetneq L^{q_2,k_2}(\Omega)$ if $q_1 > q_2$ and k_1, k_2 are admissible exponents in $(0, \infty]$.

If $q > 1$, then

$$\|s^{1/q-1/k}u^*(s)\|_{L^k(0,|\Omega|)} \approx \|s^{1/q-1/k}u^{**}(s)\|_{L^k(0,|\Omega|)}, \tag{3.5}$$

up to multiplicative constants depending on q and k. Moreover, if either $q > 1$ and $k \in [1, \infty]$, or $q = k = 1$, or $q = k = \infty$, then $L^{q,k}(\Omega)$ is in fact a Banach function space, up to equivalent norms. The space $L^{q,\infty}(\Omega)$, with $q > 1$, is also called a *Marcinkiewicz*, or *weak-Lebesgue*, space.

The outline of the proof of Theorems 1.1 and 1.2 is as follows. Our point of departure consists of two gradient estimates which are borderline, in a sense, and involve Lorentz norms. The former amounts to a bound for $\|\nabla u\|_{L^\infty(\Omega)}$, the strongest possible norm of $|\nabla u|$ in the present framework, in terms of a Lorentz norm of f, which easily follows from recent results of [CM2].

Theorem 3.1. *Let Ω be a domain in $\mathbb{R}^n$, $n \ge 3$. Assume that either $\partial\Omega \in W^2L^{n-1,1}$, or Ω is convex. Let $f \in L^{n,1}(\Omega)$, and let u be either the solution to (1.1) & (1.2) or, under the compatibility condition $\int_\Omega f(x)\,dx = 0$, to (1.1) & (1.3). Then*

$$\|\nabla u\|_{L^\infty(\Omega)} \le C\|f\|_{L^{n,1}(\Omega)}^{1/(p-1)} \tag{3.6}$$

for some constant $C = C(p, \Omega)$.

The second borderline estimate that will be exploited is a bound for a Marcinkiewicz norm of $|\nabla u|$ via $\|f\|_{L^1(\Omega)}$, the weakest possible norm of f. In its plain form, such a bound is already available—see [B–V] (and also [AM]) for Dirichlet problems, and [ACMM] for Neumann problems. However, this is not sufficient for our aims. Indeed, our strategy involves a kind of interpolation between the two endpoint estimates as a next step. The relevant interpolation is applied to the operator which associates with any f the gradient of the solution u to the boundary value problem under consideration. This operator is clearly nonlinear, and hence standard interpolation techniques fail. As a substitute for linearity, the Hölder continuity of the operator from L^1 into the Marcinkiewicz space can be employed. That Hölder continuity, which can be regarded as a stability result, takes the following form.

Theorem 3.2. *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 3$, such that $\partial\Omega \in$ Lip. Assume that $f, g \in L^1(\Omega)$. Let u be either the solution to (1.1) & (1.2), or to (1.1) & (1.3), and let v be the solution to the same problem with f replaced with g. In the case when the Neumann boundary condition (1.3) is imposed, the compatibility assumptions $\int_\Omega f(x)\,dx = 0$ and $\int_\Omega g(x)\,dx = 0$ are also required. Then*

$$\|\nabla u - \nabla v\|_{L^{\frac{n(p-1)}{n-1},\infty}(\Omega)} \leq C\|f - g\|_{L^1(\Omega)}^{1/(p-1)} \tag{3.7}$$

for some constant $C = C(n, p, |\Omega|)$, or $C = C(p, \Omega)$, according to whether (1.2) or (1.3) is in force.

Having the endpoint results of Theorems 3.1 and 3.2 at our disposal, a nonlinear interpolation argument is implemented, which involves the so-called K-functional introduced by J. Peetre in his construction of families of intermediate function spaces. Although less standard than linear interpolation, its nonlinear counterpart has also been investigated in the literature, by Peetre himself in [Pe], and further developed, in view of applications to nonlinear PDEs, in [Tar]. A novelty in our method consists in exploiting a pointwise inequality between K-functionals, instead of deriving bounds in specific intermediate function spaces. Such a pointwise inequality eventually leads to the desired estimate (1.7).

In preparation for a proof of Theorem 3.2, we need a strong monotonicity property of the function a appearing in (1.1).

Lemma 3.3. *Assume that $a : [0, \infty) \to [0, \infty)$ is of class $C^1(0, \infty)$, and satisfies the first inequality in (1.4) and in (1.5). Then there exists a constant C such that*

$$(a(|\xi|)\xi - a(|\eta|)\eta) \cdot (\xi - \eta) \geq C|\xi - \eta|^p \quad for\ \xi, \eta \in \mathbb{R}^n. \tag{3.8}$$

Lemma 3.3 easily follows from [AM, Lemma 3.1], owing to our assumptions on the function a.

We shall also employ a differential inequality for the distribution function of Sobolev functions first established in [Ma3]. In the statement, u_+ and u_- denote the positive and the negative part of a function u, respectively, namely $u_+ = \frac{1}{2}(|u| + u)$ and $u_- = \frac{1}{2}(|u| - u)$. Moreover,

$$\text{med}(u) = \sup\{t \in \mathbb{R} : |\{u \geq t\}| \geq |\Omega|/2\},$$

the median of u.

Lemma 3.4 ([Ma3]). *Let Ω be a domain in $\mathbb{R}^n$, and let $p \in (1, \infty)$.*

(i) *We have that*

$$1 \leq \frac{1}{n\omega_n^{1/n}}(-\mu_u'(t))^{1/p'}\mu_u(t)^{-1/n'}\left(-\frac{d}{dt}\int_{\{|u|>t\}}|\nabla u|^p\,dx\right)^{1/p} \quad for\ a.e.\ t \geq 0, \tag{3.9}$$

for every $u \in W_0^{1,p}(\Omega)$. Here $\omega_n = \pi^{n/2}/\Gamma(1 + n/2)$, the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

(ii) *Assume, in addition, that $\partial\Omega \in \mathrm{Lip}$. Then there exists a constant $C = C(\Omega, p)$ such that*

$$1 \leq C(-\mu'_{u_\pm}(t))^{1/p'} \mu_{u_\pm}(t)^{-1/n'} \left(-\frac{d}{dt} \int_{\{u_\pm > t\}} |\nabla u|^p \, dx\right)^{1/p} \quad \text{for a.e. } t \geq 0, \tag{3.10}$$

for every $u \in W^{1,p}(\Omega)$ such that $\mathrm{med}(u) = 0$.

A last preliminary ingredient in view of a proof of Theorem 3.2 is a parallel stability result for the solutions themselves, which is established in the following proposition.

Proposition 3.5. *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 3$, such that $\partial\Omega \in \mathrm{Lip}$. Assume that $f, g \in L^1(\Omega)$.*

(i) *Let u be the solution to (1.1) & (1.2), and let v be the solution to the same problem with f replaced with g. Then*

$$\|u - v\|_{L^{\frac{n(p-1)}{n-p}, \infty}(\Omega)} \leq C\|f - g\|_{L^1(\Omega)}^{1/(p-1)} \tag{3.11}$$

for some constant $C = C(n, p, |\Omega|)$.

(ii) *Assume, in addition, that $\int_\Omega f(x) \, dx = 0$ and $\int_\Omega g(x) \, dx = 0$. Let u be the solution to (1.1) & (1.3), and let v be the solution to the same problem with f replaced with g. Then*

$$\|(u - v) - \mathrm{med}(u - v)\|_{L^{\frac{n(p-1)}{n-p}, \infty}(\Omega)} \leq C\|f - g\|_{L^1(\Omega)}^{1/(p-1)} \tag{3.12}$$

for some constant $C = C(p, \Omega)$.

Proof. Let us prove part (ii). By the definition of approximable solution and by Fatou's Lemma, it suffices to prove inequality (3.12) under the additional assumption that $f, g \in L^1(\Omega) \cap (W^{1,p}(\Omega))'$. Then u is the weak solution to (1.1) & (1.3), and v is the weak solution to (1.1) & (1.3) with f replaced by g. Define

$$w = (u - v) - \mathrm{med}(u - v), \tag{3.13}$$

and, given any $t, h > 0$, make use of the test function

$$\Phi = \begin{cases} 0 & \text{if } w \leq t, \\ w - t & \text{if } t < w < t + h, \\ h & \text{if } t + h \leq w, \end{cases}$$

in the definitions of weak solution for u and v. Subtracting the resulting integral equalities and dividing through by h yield

$$\frac{1}{h} \int_{\{t < w < t+h\}} (a(|\nabla u|)\nabla u - a(|\nabla v|)\nabla v) \cdot (\nabla u - \nabla v) \, dx$$
$$= \int_{\{w > t+h\}} (f - g) \, dx + \frac{1}{h} \int_{\{t < w < t+h\}} (f - g)(w - t) \, dx. \tag{3.14}$$

Owing to (3.8), the left-hand side of (3.14) is bounded from below by

$$\frac{C}{h} \int_{\{t<w<t+h\}} |\nabla w|^p \, dx. \tag{3.15}$$

Making use of this bound in (3.14), and passing to the limit as $h \to 0^+$, yield

$$-\frac{d}{dt} \int_{\{w>t\}} |\nabla w|^p \, dx \leq \int_{\{w>t\}} (f-g) \, dx \leq \|f-g\|_{L^1(\Omega)} \quad \text{for a.e. } t > 0. \tag{3.16}$$

From (3.16) and Lemma 3.4(ii), we deduce that there exists a constant $C = C(\Omega, p)$ such that

$$1 \leq C(-\mu'_{w_+}(t)) \mu_{w_+}(t)^{-p'/n'} \|f-g\|_{L^1(\Omega)}^{1/(p-1)} \quad \text{for a.e. } t \geq 0.$$

Integration in this equation tells us that

$$t \leq C \|f-g\|_{L^1(\Omega)}^{1/(p-1)} \int_0^t (-\mu'_{w_+}(\tau)) \mu_{w_+}(\tau)^{-p'/n'} \, d\tau$$

$$\leq C' \|f-g\|_{L^1(\Omega)}^{1/(p-1)} \mu_{w_+}(t)^{\frac{p-n}{n(p-1)}} \quad \text{for } t \geq 0,$$

for some constants $C = C(\Omega, p)$ and $C' = C'(\Omega, p)$. From this inequality, and the very definition of decreasing rearrangement, one can show that

$$w_+^*(s) \leq C' s^{\frac{p-n}{n(p-1)}} \|f-g\|_{L^1(\Omega)}^{1/(p-1)} \quad \text{for } s \in (0, |\Omega|/2),$$

and hence

$$\|w_+\|_{L^{\frac{n(p-1)}{n-p},\infty}(\Omega)} \leq C \|f-g\|_{L^1(\Omega)}^{1/(p-1)}$$

for some constant $C = C(\Omega, p)$. An analogous argument yields the same estimate with w_+ replaced with w_-. Inequality (3.12) follows.

The proof of part (i) is analogous, and even somewhat simpler. One has just to define $w = u - v$, and make use of part (i) of Lemma 3.4 instead of part (ii). We omit the details for brevity. $\qquad \square$

Proof of Theorem 3.2. As in the proof of Proposition 3.5, we limit ourselves to proving the statement for weak solutions to (1.1) & (1.3). In the spirit of an argument of [ACMM, AM], we begin by constructing a family of test functions as follows. Let w be defined as in (3.13). Given any integrable function $\zeta : (0, |\Omega|/2) \to [0, \infty)$, define $\Lambda : [0, |\Omega|/2] \to [0, \infty)$ as

$$\Lambda(r) = \int_0^r \zeta(\rho) \, d\rho \quad \text{for } r \in [0, |\Omega|/2]. \tag{3.17}$$

Moreover, given any $s \in [0, |\Omega|/2]$, define $I : [0, |\Omega|/2] \to [0, \infty)$ as

$$I(r) = \begin{cases} \Lambda(r) & \text{if } 0 \leq r \leq s, \\ \Lambda(s) & \text{if } s < r \leq |\Omega|/2, \end{cases} \tag{3.18}$$

and $\Phi : \Omega \to [0, \infty)$ as

$$\Phi(x) = \int_0^{w_+(x)} I(\mu_{w_+}(t)) \, dt \quad \text{for } x \in \Omega. \tag{3.19}$$

Since $I \circ \mu_{w_+}$ is a bounded function, the chain rule for derivatives in Sobolev spaces tells us that $\Phi \in W^{1,p}(\Omega)$, and

$$\nabla \Phi = \chi_{\{w>0\}} I(\mu_{w_+}(w))(\nabla u - \nabla v) \quad \text{a.e. in } \Omega. \tag{3.20}$$

Choosing Φ as a test function in the definitions of weak solution for u and v and subtracting the resulting equations yield

$$\int_{\{w>0\}} I(\mu_{w_+}(w_+(x)))(a(|\nabla u|)\nabla u - a(|\nabla v|)\nabla v)) \cdot (\nabla u - \nabla v) \, dx = \int_\Omega (f - g)\Phi \, dx. \tag{3.21}$$

Observe that

$$\begin{aligned}
\|\Phi\|_{L^\infty(\Omega)} &\leq \int_0^\infty I(\mu_{w_+}(t)) \, dt = \int_{(w_+)^*(s)}^\infty \Lambda(\mu_{w_+}(t)) \, dt + \int_0^{(w_+)^*(s)} \Lambda(s) \, dt \\
&= \int_{(w_+)^*(s)}^\infty \int_0^{\mu_{w_+}(t)} \zeta(\rho) d\rho \, dt + \Lambda(s)(w_+)^*(s) \\
&= \int_0^s ((w_+)^*(\rho) - (w_+)^*(s))\zeta(\rho) \, d\rho + \Lambda(s)(w_+)^*(s) \\
&= \int_0^s (w_+)^*(\rho)\zeta(\rho) \, d\rho. \tag{3.22}
\end{aligned}$$

By inequality (3.12), there exists a constant $C = C(\Omega, p)$ such that

$$\int_0^s (w_+)^*(\rho)\zeta(\rho) \, d\rho \leq C\|f - g\|_{L^1(\Omega)}^{1/(p-1)} \int_0^s \zeta(\rho)\rho^{\frac{p-n}{n(p-1)}} \, dr. \tag{3.23}$$

On the other hand, by (3.8), there exists a constant $C = C(a)$ such that

$$\begin{aligned}
&\int_{\{w>0\}} I(\mu_{w_+}(w_+(x)))(a(|\nabla u|)\nabla u - a(|\nabla v|)\nabla v)) \cdot (\nabla u - \nabla v) \, dx \\
&\quad \geq C \int_{\{w>0\}} |\nabla w|^p I(\mu_{w_+}(w_+(x))) \, dx \\
&\quad \geq C \int_0^{|\Omega|/2} |\nabla w_+|^*(r)^p (I \circ \mu_{w_+} \circ (w_+)^*)_*(r) \, dr \\
&\quad \geq C \int_0^{|\Omega|/2} |\nabla w_+|^*(r)^p I(r) \, dr \geq C \int_0^s |\nabla w_+|^*(r)^p I(r) \, dr \\
&\quad \geq C|\nabla w_+|^*(s)^p \int_0^s \int_0^r \zeta(\rho) \, d\rho \, dr \geq C|\nabla w_+|^*(s)^p \int_0^s \zeta(\rho)(s - \rho) \, d\rho. \tag{3.24}
\end{aligned}$$

Note, in particular, that the second inequality holds by the first inequality in (3.3). Owing to the arbitrariness of ζ, we deduce from (3.21)–(3.24) that

$$C|\nabla w_+|^*(s)^p \sup_\zeta \frac{\int_0^s \zeta(\rho)(s-\rho)\,d\rho}{\int_0^s \rho^{\frac{p-n}{n(p-1)}}\zeta(\rho)\,d\rho} \le \|f-g\|_{L^1(\Omega)}^{p'} \quad \text{for } s \in (0, |\Omega|/2), \quad (3.25)$$

for some constant $C = C(\Omega, a)$. Clearly,

$$\sup_\zeta \frac{\int_0^s \zeta(\rho)(s-\rho)\,d\rho}{\int_0^s \rho^{\frac{p-n}{n(p-1)}}\zeta(\rho)\,d\rho} \ge Cs^{\frac{p(n-1)}{n(p-1)}} \quad \text{for } s \in (0, |\Omega|/2), \quad (3.26)$$

for some constant $C = C(p, n)$. Thus, inequality (3.7) follows from (3.25)–(3.26), and from analogous inequalities which can be deduced with w_+ replaced with w_-. □

As mentioned above, the interpolation argument which links Theorems 3.1 and 3.2 in the proof of Theorems 1.1 and 1.2 makes use of the K-functional associated with a couple of quasi-normed spaces. In the present framework, a *quasi-normed function space* $X(\Omega)$ on Ω is a linear space of measurable functions on Ω equipped with a functional $\|\cdot\|_{X(\Omega)}$, a *quasi-norm*, enjoying the following properties:

(i) $\|u\|_{X(\Omega)} > 0$ if $u \ne 0$;
$\|\lambda u\|_{X(\Omega)} = |\lambda|\,\|u\|_{X(\Omega)}$ for every $\lambda \in \mathbb{R}$ and $u \in X(\Omega)$;
$\|u + v\|_{X(\Omega)} \le c(\|u\|_{X(\Omega)} + \|v\|_{X(\Omega)})$ for some constant $c \ge 1$ and for every $u, v \in X(\Omega)$;
(ii) $0 \le |v| \le |u|$ a.e. in Ω implies $\|v\|_{X(\Omega)} \le \|u\|_{X(\Omega)}$;
(iii) $0 \le u_k \nearrow u$ a.e. implies $\|u_k\|_{X(\Omega)} \nearrow \|u\|_{X(\Omega)}$ as $k \to \infty$;
(iv) if G is a measurable subset of Ω and $|G| < \infty$, then $\|\chi_G\|_{X(\Omega)} < \infty$;
(v) for every measurable subset G of Ω with $|G| < \infty$, there exists a constant C such that $\int_G |u|\,dx \le C\|u\|_{X(\Omega)}$ for every $u \in X(\Omega)$.

The space $X(\Omega)$ is called a *Banach function space* if (i) holds with $c = 1$. In this case, the functional $\|\cdot\|_{X(\Omega)}$ is actually a norm which renders $X(\Omega)$ a Banach space.

A quasi-normed function space (in particular, a Banach function space) $X(\Omega)$ is called *rearrangement invariant* (r.i., for short) if there exists a quasi-normed function space $\overline{X}(0, |\Omega|)$ on the interval $(0, |\Omega|)$, called the representation space of $X(\Omega)$, having the property that

$$\|u\|_{X(\Omega)} = \|u^*\|_{\overline{X}(0,|\Omega|)} \quad (3.27)$$

for every $u \in X(\Omega)$. Obviously, if $X(\Omega)$ is an r.i. quasi-normed space, then

$$\|u\|_{X(\Omega)} = \|v\|_{X(\Omega)} \quad \text{if } u^* = v^*. \quad (3.28)$$

Let $X_1(\Omega)$ and $X_2(\Omega)$ be quasi-normed spaces. Their *K-functional* is defined for $u \in X_1(\Omega) + X_2(\Omega)$ as

$$K(s, u; X_1(\Omega), X_2(\Omega)) = \inf_{u=u_1+u_2} (\|u_1\|_{X_1(\Omega)} + s\|u_2\|_{X_2(\Omega)}) \quad \text{for } s \in (0, |\Omega|).$$

Similarly, given a vector-valued measurable function $U : \Omega \to \mathbb{R}^m$, $m \geq 1$, such that $U \in (X_1(\Omega))^m + (X_2(\Omega))^m$, we set

$$K(s, U; (X_1(\Omega))^m, (X_2(\Omega))^m)$$
$$= \inf_{U=U_1+U_2} (\| |U_1| \|_{X_1(\Omega)} + s \| |U_2| \|_{X_2(\Omega)}) \quad \text{for } s \in (0, |\Omega|).$$

Clearly,

$$K(s, |U|; X_1(\Omega), X_2(\Omega)) \approx K(s, U; (X_1(\Omega))^m, (X_2(\Omega))^m) \quad \text{for } s \in (0, |\Omega|), \quad (3.29)$$

and for $U \in (X_1(\Omega))^m + (X_2(\Omega))^m$, up to multiplicative constants depending on m. We refer the reader to [BS] for a comprehensive treatment of r.i. spaces.

We are now in a position to accomplish the proof of our main results.

Proof of Theorems 1.1 and 1.2. Consider the Neumann problem (1.1) & (1.3). For any $f \in L^1(\Omega)$, set

$$f_\Omega = \frac{1}{|\Omega|} \int_\Omega f(x)\,dx,$$

the mean value of f over Ω. Let

$$T : L^1(\Omega) \to (L^{\frac{n(p-1)}{n-1},\infty}(\Omega))^n, \quad Tf = \nabla u,$$

where u is the solution to

$$\begin{cases} -\operatorname{div}(a(|\nabla u|)\nabla u) = f(x) - f_\Omega & \text{in } \Omega, \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.30)$$

Assume now that $f \in L^1(\Omega)$ and $f_\Omega = 0$. Any decomposition

$$f = f^0 + f^1$$

with $f^0 \in L^{n,1}(\Omega)$ and $f^1 \in L^1(\Omega)$ induces a decomposition

$$Tf = Tf^0 + (Tf - Tf^0).$$

By the definition of K-functional, Proposition 3.2 and Theorem 3.1, there exists a constant $C = C(p, \Omega)$ such that

$$K(Tf, s; (L^{\frac{n(p-1)}{n-1},\infty}(\Omega))^n, (L^\infty(\Omega))^n)$$
$$\leq \|Tf - Tf^0\|_{L^{\frac{n(p-1)}{n-1},\infty}(\Omega)} + s\|Tf^0\|_{L^\infty(\Omega)}$$
$$\leq C\|f - (f^0 - f^0_\Omega)\|_{L^1(\Omega)}^{1/(p-1)} + sC\|f^0 - f^0_\Omega\|_{L^{n,1}(\Omega)}^{1/(p-1)} \quad (3.31)$$

for $s \in (0, |\Omega|)$. Since $f_\Omega = 0$, we have $f_\Omega^0 = -f_\Omega^1$. Thus,

$$
\begin{aligned}
\|f - (f^0 &- f_\Omega^0)\|_{L^1(\Omega)}^{1/(p-1)} + s\|f^0 - f_\Omega^0\|_{L^{n,1}(\Omega)}^{1/(p-1)} \\
&\leq (\|f^1\|_{L^1(\Omega)} + \|f_\Omega^1\|_{L^1(\Omega)})^{1/(p-1)} + s(\|f^0\|_{L^{n,1}(\Omega)} + \|f_\Omega^0\|_{L^{n,1}(\Omega)})^{1/(p-1)} \\
&\leq (2\|f^1\|_{L^1(\Omega)})^{1/(p-1)} + s(2\|f^0\|_{L^{n,1}(\Omega)})^{1/(p-1)} \\
&\leq C'(\|f^1\|_{L^1(\Omega)} + s^{p-1}\|f^0\|_{L^{n,1}(\Omega)})^{1/(p-1)} \quad \text{for } s \in (0, |\Omega|), \quad (3.32)
\end{aligned}
$$

for some constant $C' = C'(p)$. Coupling (3.31) with (3.32) yields, owing to the arbitrariness of the decomposition of f,

$$
\begin{aligned}
K(Tf, s; (L^{\frac{n(p-1)}{n-1}, \infty}(\Omega))^n, &(L^\infty(\Omega))^n) \\
&\leq CK(f, s^{p-1}; L^1(\Omega), L^{n,1}(\Omega))^{1/(p-1)} \quad \text{for } s \in (0, |\Omega|). \quad (3.33)
\end{aligned}
$$

By [Ho, (4.8)] and (3.29),

$$
\begin{aligned}
K(Tf, s; (L^{\frac{n(p-1)}{n-1}, \infty}(\Omega))^n, &(L^\infty(\Omega))^n) \\
&\approx \|r^{\frac{n-1}{n(p-1)}}(Tf)^*(r)\|_{L^\infty(0, s^{\frac{n(p-1)}{n-1}})} \quad \text{for } s \in (0, |\Omega|), \quad (3.34)
\end{aligned}
$$

and, by [Ho, Theorem 4.2],

$$
K(f, s; L^1(\Omega), L^{n,1}(\Omega)) \approx \int_0^{s^{n'}} f^*(r)\, dr + s \int_{s^{n'}}^{|\Omega|} f^*(r) r^{-1/n'}\, dr \quad \text{for } s \in (0, |\Omega|), \quad (3.35)
$$

with equivalence constants depending on p and n. From (3.33)–(3.35) we deduce that

$$
\begin{aligned}
s^{\frac{n-1}{n(p-1)}}|\nabla u|^*(s) &= s^{\frac{n-1}{n(p-1)}}(Tf)^*(s) \\
&\leq C\left(\int_0^s f^*(r)dr + s^{1/n'}\int_s^{|\Omega|} f^*(r)r^{-1/n'}\, dr\right)^{1/(p-1)} \quad \text{for } s \in (0, |\Omega|), \quad (3.36)
\end{aligned}
$$

for some constant $C = C(p, \Omega)$. Hence, inequality (1.7) easily follows.

The proof of (1.7) for solutions to the Dirichlet problem (1.1) & (1.2) is completely analogous, and even simpler, since f does not have to be normalized by subtracting f_Ω. □

4. Applications

We are concerned here with gradient norm estimates which can be deduced from our main results. We begin with a general criterion which holds for arbitrary rearrangement-invariant quasi-norms.

Corollary 4.1. *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 3$, such that $\partial\Omega \in W^2 L^{n-1,1}$. Let $X(\Omega)$ and $Y(\Omega)$ be rearrangement invariant quasi-normed spaces on Ω, and let $\overline{X}(0, |\Omega|)$ and $\overline{Y}(0, |\Omega|)$, respectively, be their representation spaces. Assume that there exists a constant C such that*

$$\left\| \int_s^{|\Omega|} \varphi(r) r^{-1/n'} \, dr \right\|_{\overline{Y}(0,|\Omega|)} \leq C \|\varphi\|_{\overline{X}(0,|\Omega|)}, \tag{4.1}$$

$$\left\| s^{-1/n'} \int_0^s \varphi(r) \, dr \right\|_{\overline{Y}(0,|\Omega|)} \leq C \|\varphi\|_{\overline{X}(0,|\Omega|)}, \tag{4.2}$$

for every nondecreasing function $\varphi \in \overline{X}(0, |\Omega|)$. If $f \in X(\Omega)$, and u is either a solution to the Dirichlet problem (1.1) & (1.2) or to the Neumann problem (1.1) & (1.3), then there exists a constant C' such that

$$\| |\nabla u|^{p-1} \|_{Y(\Omega)} \leq C' \|f\|_{X(\Omega)}. \tag{4.3}$$

Of course, Corollary 4.1 has a counterpart in convex domains.

Corollary 4.2. *Let Ω be a convex domain in $\mathbb{R}^n$, $n \geq 3$. Then inequality (4.3) holds under the same assumptions on f, u, $X(\Omega)$ and $Y(\Omega)$ as in Corollary 4.1.*

Corollaries 4.1 and 4.2 immediately follow from inequality (1.7) in Theorems 1.1 and 1.2, and basic properties of rearrangement invariant quasi-norms.

Let us notice that the couple of conditions (4.1)–(4.2) is equivalent to

$$\left\| \int_s^{|\Omega|} r^{-1-1/n'} \int_0^r \varphi(\rho) \, d\rho \, dr \right\|_{\overline{Y}(0,|\Omega|)} \leq C \|\varphi\|_{\overline{X}(0,|\Omega|)} \tag{4.4}$$

for every nondecreasing function $\varphi \in \overline{X}(0, |\Omega|)$. It is in fact condition (4.4) that immediately follows from (1.7); the equivalence of (4.4) to (4.1)–(4.2) is a consequence of an application of Fubini's theorem in the integral appearing on the right-hand side of (1.7).

Inequality (4.4) is stronger, in general, than just (4.2), since, if $\varphi : (0, |\Omega|) \to [0, \infty)$ is nonincreasing, then

$$\varphi(s) \leq \frac{1}{s} \int_0^s \varphi(r) \, dr \quad \text{for } s > 0. \tag{4.5}$$

However, inequalities (4.4) and (4.2) are equivalent in the case when the quasi-norm in $X(\Omega)$ satisfies

$$\left\| \frac{1}{s} \int_0^s \varphi(r) dr \right\|_{\overline{X}(0,|\Omega|)} \leq C \|\varphi\|_{\overline{X}(0,|\Omega|)} \tag{4.6}$$

for some constant C and for every $\varphi \in \overline{X}(0, |\Omega|)$. Thus, if $X(\Omega)$ satisfies (4.6), then the sole inequality (4.2) implies the gradient estimate (4.3). The r.i. Banach function spaces $X(\Omega)$ making inequality (4.6) true can be characterized in terms of their upper Boyd index $I(X)$. One has $I(X) \in [0, 1]$ for every r.i. Banach function space $X(\Omega)$. It turns out that (4.6) holds if and only if $I(X) < 1$ [BS, Theorem 5.15]. Recall that the definition

of $I(X)$ relies upon that of the dilation operator. The dilation operator $D_\delta : \overline{X}(0, |\Omega|) \to \overline{X}(0, |\Omega|)$ is defined for $\delta > 0$ and $\varphi \in \overline{X}(0, |\Omega|)$ as

$$D_\delta \varphi(s) = \begin{cases} \varphi(s\delta) & \text{if } s\delta \in (0, |\Omega|), \\ 0 & \text{otherwise,} \end{cases}$$

and is bounded whenever $X(\Omega)$ is an r.i. Banach function space [BS, Chapter 3, Prop. 5.11]. Its norm is denoted by $\|D_\delta\|$. The *upper Boyd index* $I(X)$ of $X(\Omega)$ is given by

$$I(X) = \lim_{\delta \to 0^+} \frac{\log \|D_\delta\|}{\log(1/\delta)}.$$

We now establish explicit gradient bounds for either Lebesgue, Lorentz, or Orlicz norms. The results will be stated under the assumptions on Ω of Theorem 1.1 and Corollary 4.1, but the same statements, with the same proofs, hold under the assumptions of Theorem 1.2 and Corollary 4.2. The conclusions that are derived recover various estimates available in the literature, and yield new results in a unified framework. Let us recall a few basic definitions about these function spaces.

Lorentz spaces extend Lebesgue spaces, and have been introduced in Section 3. Lorentz–Zygmund spaces, a further extension of Lorentz spaces, will come into play in certain borderline situations. If either $q \in (1, \infty]$, $k \in (0, \infty]$, $\beta \in \mathbb{R}$, or $q = 1$, $k \in (0, 1]$, $\beta \in [0, \infty)$, the *Lorentz–Zygmund space* $L^{q,k}(\log L)^\beta(\Omega)$ is defined as the set of all measurable functions u on Ω making the expression

$$\|u\|_{L^{q,k}(\log L)^\beta(\Omega)} = \|s^{1/q-1/k}(1 + \log(|\Omega|/s))^\beta u^*(s)\|_{L^k(0,|\Omega|)} \tag{4.7}$$

finite. If $k \geq 1$ and the weight multiplying $u^*(s)$ on the right-hand side of (4.7) is nonincreasing, then the functional $\|u\|_{L^{q,k}(\log L)^\beta(\Omega)}$ is actually a norm, and $L^{q,k}(\log L)^\beta(\Omega)$ is an r.i. Banach function space equipped with this norm. Otherwise, this functional is only a quasi-norm. For certain values of the parameters q, k and β, it is however equivalent to an r.i. norm obtained on replacing u^* by u^{**} in the definition. A detailed analysis of Lorentz–Zygmund spaces can be found in [OP].

Orlicz spaces generalize Lebesgue spaces in a different direction. Let $A : [0, \infty) \to [0, \infty]$ be a *Young function*, that is, a convex function, vanishing at 0, which is neither identically equal to 0, nor to ∞. The *Orlicz space* $L^A(\Omega)$ associated with A is the r.i. space of those measurable functions u on Ω such that the *Luxemburg norm*

$$\|u\|_{L^A(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega A(|u(x)|/\lambda) \, dx \leq 1\right\}$$

is finite. The Orlicz spaces $L^A(\Omega)$ and $L^B(\Omega)$ coincide, with equivalent norms, if and only if the Young functions A and B are equivalent near infinity, in the sense that there exist positive constants c and t_0 such that $B(t/c) \leq A(t) \leq B(ct)$ for $t \geq t_0$.

In the special case when

$$A(t) \text{ is equivalent to } t^q (\log(1 + t))^\alpha \text{ near infinity,}$$

where either $q > 1$ and $\alpha \in \mathbb{R}$, or $q = 1$ and $\alpha \geq 0$, the space $L^A(\Omega)$ is called a *Zygmund space*, and is denoted by $L^q (\log L)^\alpha (\Omega)$. If

$$A(t) \text{ is equivalent to } e^{t^\beta} - 1 \text{ near infinity,}$$

for some $\beta > 0$, we denote $L^A(\Omega)$ by $\exp L^\beta(\Omega)$. Similarly, we use the notation $\exp(\exp L^\beta)(\Omega)$ for the Orlicz space associated with a Young function

$$A(t) \text{ equivalent to } e^{e^{t^\beta}} - e \text{ near infinity.}$$

Our first result concerns gradient estimates in classical Lebesgue spaces. In the statements below, C denotes a constant independent of u and f.

Theorem 4.3. *Let Ω, p and u be as in Theorem 1.1. Assume that $f \in L^q(\Omega)$.*

(i) *If $q = 1$, then for every $\sigma < \frac{n(p-1)}{n-1}$,*

$$\|\nabla u\|_{L^\sigma(\Omega)} \leq C \|f\|_{L^1(\Omega)}^{1/(p-1)}. \tag{4.8}$$

(ii) *If $1 < q < n$, then*

$$\|\nabla u\|_{L^{\frac{qn(p-1)}{n-q}}(\Omega)} \leq C \|f\|_{L^q(\Omega)}^{1/(p-1)}. \tag{4.9}$$

(iii) *If $q = n$, then for every $\sigma < \infty$,*

$$\|\nabla u\|_{L^\sigma(\Omega)} \leq C \|f\|_{L^n}^{1/(p-1)}. \tag{4.10}$$

(iv) *If $q > n$, then*

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C \|f\|_{L^q(\Omega)}^{1/(p-1)}. \tag{4.11}$$

Theorem 4.3 overlaps with various contributions, including [ACMM, AM, B–V, BG, DMOP, Di, Ma1, Ma3, Li, Ta1, Ta2].

A proof of Theorem 4.3 makes use of Corollary 4.1 and of one-dimensional Hardy-type inequalities in Lebesgue spaces [Ma6, Section 1.3.2]. Theorem 4.3 can also be derived from the more general, sharper, estimates in Lorentz and Lorentz–Zygmund spaces which are the object of the next result.

Theorem 4.4. *Let Ω, p and u be as in Theorem 1.1. Assume that $f \in L^{q,k}(\Omega)$.*

(i) *If $q = 1$ and $0 < k \leq 1$, then*

$$\|\nabla u\|_{L^{\frac{n(p-1)}{n-1},\infty}(\Omega)} \leq C \|f\|_{L^{1,k}(\Omega)}^{1/(p-1)}.$$

(ii) *If $1 < q < n$ and $0 < k \leq \infty$, then*

$$\|\nabla u\|_{L^{\frac{qn(p-1)}{n-q},k(p-1)}(\Omega)} \leq C \|f\|_{L^{q,k}(\Omega)}^{1/(p-1)}.$$

(iii) *If $q = n$ and $k > 1$, then*

$$\|\nabla u\|_{L^{\infty,k(p-1)}(\log L)^{-1/(p-1)}(\Omega)} \le C\|f\|_{L^{n,k}(\Omega)}^{1/(p-1)}.$$

(iv) *If either $q = n$ and $k \le 1$, or $q > n$ and $0 < k \le \infty$, then*

$$\|\nabla u\|_{L^{\infty}(\Omega)} \le C\|f\|_{L^{q,k}(\Omega)}^{1/(p-1)}.$$

Various cases of Theorem 4.4 are known, possibly under stronger assumptions on Ω—see e.g. [ACMM, AFT, AM, B–V, CM2]. Local gradient estimates in Lorentz spaces are established [DM3, DM5, Mi1].

Proof of Theorem 4.4. By Theorem 1.1,

$$\|\nabla u\|_{L^{\frac{n(p-1)}{n-1},\infty}(|\Omega|)}^{p-1} \le C \sup_{s\in(0,|\Omega|)} s^{(n-1)/n} \int_s^{|\Omega|} f^{**}(r)r^{-1/n'}\,dr \qquad (4.12)$$

$$\le \|f\|_{L^1(\Omega)} \sup_{s>0} s^{1/n'} \int_s^{\infty} r^{-1-1/n'}\,dr = C'\|f\|_{L^1(\Omega)}$$

for some constants $C = C(\Omega)$ and $C' = C'(\Omega)$. Hence, assertion (i) follows.

By Corollary 4.1 and (3.5), part (ii) is easily reduced to the inequality

$$\left\|s^{\frac{n-q}{nq}-\frac{1}{k}} \int_s^{|\Omega|} r^{-1/n'} f^{**}(r)\,dr\right\|_{L^k(0,|\Omega|)} \le C\|s^{1/q-1/k} f^{**}(s)\|_{L^k(0,|\Omega|)} \qquad (4.13)$$

for some constant $C = C(n, q, k)$, and for every $f \in L^{q,k}(\Omega)$. Inequality (4.13) follows from a classical weighted Hardy-type inequality in Lebesgue spaces if $k \ge 1$ [Ma6, Section 1.3.2], and from a weighted Hardy-type inequality in Lebesgue spaces for nonincreasing functions if $0 < k < 1$ [CS].

Similarly, by Corollary 4.1 and (3.5), case (iii) is a consequence of the inequality

$$\left\|s^{-1/k}(1 + \log(|\Omega|/s))^{-1} \int_s^{|\Omega|} f^{**}(r)r^{-1/n'}\,dr\right\|_{L^k(0,|\Omega|)} \le C\|s^{1/n-1/k} f^{**}(s)\|_{L^k(0,|\Omega|)} \qquad (4.14)$$

for some constant $C = C(n, k, |\Omega|)$, and for every $f \in L^{n,k}(\Omega)$, which holds by standard criteria for one-dimensional Hardy inequalities [Ma6, Section 1.3.2].

Finally, by Theorem 1.1,

$$\|\nabla u\|_{L^{\infty}(\Omega)}^{p-1} \le C \int_0^{|\Omega|} f^{**}(r)r^{-1/n'}\,dr$$

$$\le C \int_0^{|\Omega|} f^*(\rho) \int_\rho^{\infty} r^{-1-1/n'}\,dr\,d\rho = Cn'\|f\|_{L^{n,1}(\mathbb{R}^n)}, \qquad (4.15)$$

where C is the constant appearing in (1.7). Hence, part (iv) follows, owing to the inclusion relations between Lorentz spaces recalled in Section 3. □

Let us next derive gradient estimates in Orlicz spaces. Their formulation requires the notions of Young conjugate and Sobolev conjugate of a Young function A.

The *Young conjugate* of A is the Young function $\widetilde{A}$ given by

$$\widetilde{A}(t) = \sup\{st - A(s) : s \geq 0\} \quad \text{for } t \geq 0.$$

The *Sobolev conjugate*, introduced in [Ci4, Ci5], of a Young function A such that

$$\int_0 \left(\frac{t}{A(t)}\right)^{1/(n-1)} dt < \infty, \tag{4.16}$$

is the Young function A_n defined as

$$A_n(t) = A(H^{-1}(t)) \quad \text{for } t \geq 0, \tag{4.17}$$

where $H : [0, \infty) \to [0, \infty)$ is given by

$$H(s) = \left(\int_0^s \left(\frac{t}{A(t)}\right)^{1/(n-1)} dt\right)^{1/n'} \quad \text{for } s \geq 0, \tag{4.18}$$

and H^{-1} is the generalized left-continuous inverse of H. Accordingly, given a Young function B such that

$$\int_0 \left(\frac{t}{\widetilde{B}(t)}\right)^{1/(n-1)} dt < \infty, \tag{4.19}$$

we denote by $(\widetilde{B})_n$ the Sobolev conjugate of $\widetilde{B}$, obtained as in (4.17)–(4.18), on replacing A with $\widetilde{B}$.

Theorem 4.5. *Let Ω, p and u be as in Theorem 1.1. Let A and B be Young functions satisfying (4.16) and (4.19), respectively. Assume that $f \in L^A(\Omega)$, and that there exist $c > 0$ and $t_0 > 0$ such that*

$$B(t) \leq A_n(ct) \quad \text{and} \quad \widetilde{A}(t) \leq \left(\widetilde{B}\right)_n(ct) \quad \text{for } t \geq t_0. \tag{4.20}$$

Let E be the Young function given by

$$E(t) = B(t^{p-1}) \quad \text{for } t \geq 0.$$

Then

$$\|\nabla u\|_{L^E(\Omega)} \leq C \|f\|_{L^A(\Omega)}^{1/(p-1)}. \tag{4.21}$$

We emphasize that assumptions (4.16) and (4.19) are, in fact, irrelevant. Indeed, the functions A and B can be replaced, if necessary, by Young functions equivalent near infinity, which satisfy (4.16) and (4.19). Such a replacement leaves the spaces $L^A(\Omega)$ and $L^B(\Omega)$ unchanged, up to equivalent norms.

Proof of Theorem 4.5. The first inequality in (4.20) ensures that

$$\left\|\int_s^{|\Omega|} \varphi(r) r^{-1/n'} dr\right\|_{L^B(0,|\Omega|)} \leq C \|\varphi\|_{L^A(0,|\Omega|)} \tag{4.22}$$

and the second inequality in (4.20) ensures that

$$\left\| s^{-1/n'} \int_0^s \varphi(r)\,dr \right\|_{L^B(0,|\Omega|)} \leq C \|\varphi\|_{L^A(0,|\Omega|)} \tag{4.23}$$

for all $\varphi \in L^A(0,|\Omega|)$. These are consequences of [Ci4, Lemma 1] and [Ci6, Lemma 2]. Hence, (4.21) follows from Corollary 4.1. □

Theorem 4.5 can be easily specialized to the case when $L^A(\Omega)$ is a Zygmund space. This is the content of our last result.

Theorem 4.6. *Let Ω, p and u be as in Theorem 1.1. Let $f \in L^q(\log L)^\alpha(\Omega)$.*

(i) *If $q = 1$ and $\alpha > 0$, then*

$$\|\nabla u\|_{L^{\frac{n(p-1)}{n-1}}(\log L)^{\frac{n\alpha}{n-1}-1}(\Omega)} \leq C \|f\|_{L(\log L)^\alpha(\Omega)}^{1/(p-1)}. \tag{4.24}$$

(ii) *If $1 < q < n$ and $\alpha \in \mathbb{R}$, then*

$$\|\nabla u\|_{L^{\frac{nq(p-1)}{n-q}}(\log L)^{\frac{n\alpha}{n-q}}(\Omega)} \leq C \|f\|_{L^q(\log L)^\alpha(\Omega)}^{1/(p-1)}. \tag{4.25}$$

(iii) *If $q = n$ and $\alpha < n - 1$, then*

$$\|\nabla u\|_{\exp L^{\frac{n(p-1)}{n-1-\alpha}}(\Omega)} \leq C \|f\|_{L^n(\log L)^\alpha(\Omega)}^{1/(p-1)}. \tag{4.26}$$

(iv) *If $q = n$ and $\alpha = n - 1$, then*

$$\|\nabla u\|_{\exp\left(\exp L^{\frac{n(p-1)}{n-1}}\right)(\Omega)} \leq C \|f\|_{L^n(\log L)^{n-1}(\Omega)}^{1/(p-1)}. \tag{4.27}$$

(v) *If either $q = n$ and $\alpha > n - 1$, or $q > n$ and $\alpha \in \mathbb{R}$, then*

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C \|f\|_{L^q(\log L)^\alpha(\Omega)}^{1/(p-1)}. \tag{4.28}$$

Special cases of Theorem 4.6 are known. In particular, some instances of case (i) can be found in [B–V, De].

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