J. Eur. Math. Soc. 16, 695–747
DOI 10.4171/JEMS/444

Gabriel Navarro · Britta Späth

On Brauer's Height Zero Conjecture

Received May 16, 2012 and in revised form August 28, 2012

Abstract. In this paper, the unproven half of Richard Brauer's Height Zero Conjecture is reduced to a question on simple groups.

Keywords. Height Zero Conjecture, block theory, representations of finite groups

1. Introduction

The aim of this paper is to provide a reduction of the unproven half of Brauer's famous Height Zero Conjecture to a question on simple groups. Much of the research in the representation theory of finite groups in the last years has been dedicated to proving several deep conjectures that assert that certain global invariants of a finite group can be calculated locally. The so called *counting conjectures* include the McKay and the Alperin–McKay (AM) conjectures, the Alperin Weight Conjecture, and a chain of conjectures by E. C. Dade, the ultimate of these generalizing both the Alperin–McKay and the Alperin Weight conjectures. Brauer's Height Zero Conjecture, of a different more structural type, asserts that all complex irreducible characters in a p-block B with defect group D have height zero if and only if D is abelian.

In [IMN07] it was proven that if all finite simple groups are *good* then the McKay conjecture is true. This result has been recently extended to include blocks: it is proved in [Spä13] that if all finite simple groups satisfy the *inductive AM-condition*, then the Alperin–McKay conjecture is true. Since the publication of [IMN07] many simple groups have been checked to be good ([Mal07], [Mal08], [Spä12]) and it is hoped that this approach will eventually lead to a proof of the McKay and Alperin–McKay conjectures.

The "if" direction of Brauer's Height Zero Conjecture (BHZC) was reduced long time ago to quasisimple groups in [BK88], and only now the knowledge of the blocks of quasisimple groups has been enough to complete a proof of this conjecture by R. Kessar and G. Malle [KM13].

G. Navarro: Departament d'Àlgebra, Universitat de València, 46100 Burjassot, Spain; e-mail: gabriel.navarro@uv.es

B. Späth: FB Mathematik, TU Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany; e-mail: spaeth@mathematik.uni-kl.de

Mathematics Subject Classification (2010): 20C20

The "only if" part of BHZC is even more complicated. While the "if" direction, for instance, is almost trivial for p-solvable groups, the proof of the *only if* direction for this class of groups was a tour de force accomplished by D. Gluck and T. Wolf [GW84] in the 1980's.

Recently, using the new techniques introduced in [IMN07], P. H. Tiep and the first author [NT12] were able to prove the $p = 2$ maximal defect "only if" case of the conjecture. At the same time a general connection was pointed out between the Alperin–McKay conjecture and the Height Zero Conjecture. (This connection was previously known for the principal blocks, [Mur95].) But also in [NT12], a major obstacle towards completing the proof of BHZC was foreseen: the so called Gluck–Wolf theorem on relative p'-degrees over a normal subgroup (a purely character-theoretical statement involving no modular representation theory) had to be proved for every finite group in order to obtain BHZC.

Now that the general Gluck–Wolf theorem has been fully established by P. H. Tiep and the first author [NT13], the main result of this paper tells us that if all the non-abelian simple groups satisfy the inductive AM-condition, then Brauer's Height Zero Conjecture will be settled if it is proved for quasisimple groups.

But in fact, there is still more. Since G. Malle has communicated to us ([KM12]) that he and R. Kessar have proved that all universal p'-covering groups of non-abelian simple groups satisfy Brauer's Height Zero Conjecture, the following constitutes now the main result of this paper.

Theorem A. *Let G be a finite group and let p be a prime. Suppose that every non-abelian simple group involved in G satisfies the inductive AM-condition for p. Then Brauer's Height Zero Conjecture holds for G and the prime p.*

In other words, what we show in this paper is that if the Alperin–McKay Conjecture is proved via the method proposed in [Spä13], then Brauer's Height Zero Conjecture is automatically true.

Next, we are more specific. If B is a p-block of G, let $\mathrm{Irr}_0(B)$ be the set of complex irreducible height zero characters in B. If N is normal in G and $\tau \in \mathrm{Irr}(N)$, then $\mathrm{Irr}(B \mid \tau)$ is the set of irreducible characters in B that lie over τ. If b is a block of N, then G_b is the stabilizer of b in G. Theorem A will follow from the following.

Theorem B. *Let G be a finite group, and let $N \lhd G$. Let $b \in \mathrm{Bl}(N)$ have defect group D. Assume that all the non-abelian simple groups involved in N satisfy the inductive AM-condition for the prime p. Then there exists an $\mathrm{N}_{G_b}(D)$-equivariant bijection*

$$\Pi_D : \mathrm{Irr}_0(b) \to \mathrm{Irr}_0(b_1),$$

where $b_1 \in \mathrm{Bl}(\mathrm{N}_N(D))$ is the Brauer correspondent of b. Furthermore, if B is a block of G covering b, then there exists a height preserving bijection between $\mathrm{Irr}(B \mid \tau)$ and $\mathrm{Irr}(B_1 \mid \Pi_D(\tau))$ for every $\tau \in \mathrm{Irr}_0(b)$, where B_1 is the only block of $\mathrm{N}_G(D)$ that induces B and covers b_1.

We wish to remark that if N is p-solvable, then the condition on the simple groups involved in N in the previous theorem trivially holds, and therefore Theorem B is true in this case. If $N = G$, Theorem B is telling us that if all the simple groups involved in G satisfy the inductive AM-condition, then the Alperin–McKay conjecture is true (which is the main result of [Spä13]). But also, Theorem B is telling more: if all the non-abelian simple groups involved in N satisfy the inductive AM-condition for the prime p, then N satisfies the equivariant form of the Alperin–McKay conjecture.

G. R. Robinson [Rob02] pointed out that the existence of a bijection like the one in Theorem B was a consequence of Dade's Projective Conjecture (in the case where D is abelian). Dade's Projective Conjecture is considered more sophisticated than the Alperin–McKay conjecture.

This article is structured in the following way: In the first section we establish our notation and prove some of the most basic results. In the succeeding two sections we introduce the convenient definition of *block isomorphic character triples* and determine several fundamental properties of them. Afterwards we study and prove further results on the so called Dade–Glauberman–Nagao (DGN) correspondence, and construct block isomorphic character triples that are associated to DGN-correspondents. In the last two sections we assume that the inductive AM-condition holds and construct a bijection such that associated characters give block isomorphic character triples, therefore proving a strong form of Theorem B.

2. Notation and basic observations

This section essentially gathers some basic results about induced blocks and heights. We use the definition of induced blocks due to R. Brauer (see [Nav98, p. 87]). In general we use the notation of [NT89, Nav98]. All groups in this paper are finite.

Notation 2.1. Let p be a prime. Let R be the ring of algebraic integers, and let S be some localization of R at a maximal ideal containing pR. Let $\mathbb{F}$ be the residue field of S, whose characteristic is p (see [Nav98, Chapter 2] for details and exact definitions). Let $()^* : S \to \mathbb{F}$ be the associated canonical epimorphism.

For every irreducible complex character $\chi \in \mathrm{Irr}(G)$ of a finite group G, we denote by $\lambda_\chi : \mathrm{Z}(\mathbb{F}G) \to \mathbb{F}$ the associated central function. Analogously, for a p-block b of G the central function $\lambda_b : \mathrm{Z}(\mathbb{F}G) \to \mathbb{F}$ is defined as $\lambda_b = \lambda_\chi$ for any $\chi \in \mathrm{Irr}(b)$. Also, if $\chi \in \mathrm{Irr}(G)$ we denote by $\mathrm{bl}(\chi)$ the block of G containing χ. By $\mathrm{Irr}_0(G \mid D)$ we denote the irreducible characters of G with height zero lying in a block having D as a defect group. We denote by $\mathfrak{Cl}_G(x)$ the G-conjugacy class containing $x \in G$. For any subset $C \subseteq G$ we denote by C^+ the sum $\sum_{x \in C} x \in \mathbb{F}G$.

Lemma 2.2. *Let $K \lhd G$ and let $\epsilon : G \to G/K = \overline{G}$ be the canonical epimorphism. Let $\theta \in \mathrm{Irr}(K)$ and suppose that $\widetilde{\theta} \in \mathrm{Irr}(G)$ is an extension of θ. Let $\overline{\eta} \in \mathrm{Irr}(G/K)$ and let $\eta := \overline{\eta} \circ \epsilon \in \mathrm{Irr}(G)$. If $x \in G$, then*

$$\lambda_{\widetilde{\theta}\eta}(\mathfrak{Cl}_G(x)^+) = \lambda_{\widetilde{\theta}_L}(\mathfrak{Cl}_L(x)^+)\lambda_{\overline{\eta}}(\mathfrak{Cl}_{\overline{G}}(\overline{x})^+),$$

where $L/K = \mathrm{C}_{\overline{G}}(\overline{x})$ and $\overline{x} = xK$.

Proof. Recall that $\widetilde{\theta}\eta \in \mathrm{Irr}(G)$ by Corollary (6.17) of [Isa76]. We observe that $|\mathfrak{Cl}_G(x)| = |G:L|\,|L:\mathrm{C}_G(x)| = |\mathfrak{Cl}_{\overline{G}}(\overline{x})|\,|\mathfrak{Cl}_L(x)|$. Hence

$$\lambda_{\widetilde{\theta}\eta}(\mathfrak{Cl}_G(x)^+) = \left(\frac{|\mathfrak{Cl}_G(x)|\widetilde{\theta}\eta(x)}{\theta(1)\eta(1)}\right)^* = \left(\frac{|\mathfrak{Cl}_{\overline{G}}(\overline{x})||\mathfrak{Cl}_L(x)|\widetilde{\theta}\eta(x)}{\theta(1)\eta(1)}\right)^*$$
$$= \left(\frac{|\mathfrak{Cl}_L(x)|\widetilde{\theta}_L(x)}{\theta(1)}\right)^*\left(\frac{|\mathfrak{Cl}_{\overline{G}}(\overline{x})|\eta(x)}{\eta(1)}\right)^* = \lambda_{\widetilde{\theta}_L}(\mathfrak{Cl}_L(x)^+)\lambda_{\overline{\eta}}(\mathfrak{Cl}_{\overline{G}}(\overline{x})^+).$$

□

Proposition 2.3. *Let $N \lhd G$ and $H \leq G$ with $NH = G$. Write $M = N \cap H$. Let $\theta \in \mathrm{Irr}(N)$ and suppose $\widetilde{\theta} \in \mathrm{Irr}(G)$ is an extension of θ. Let $\theta' \in \mathrm{Irr}(M)$ and suppose that $\widetilde{\theta}' \in \mathrm{Irr}(H)$ is an extension of θ'. Further assume that $\mathrm{bl}(\widetilde{\theta}'_{J\cap H})^J$ is defined and $\mathrm{bl}(\widetilde{\theta}'_{J\cap H})^J = \mathrm{bl}(\widetilde{\theta}_J)$ for every $N \leq J \leq G$. Then $\mathrm{bl}(\widetilde{\theta}'\eta_H)^G$ is defined and $\mathrm{bl}(\widetilde{\theta}'\eta_H)^G = \mathrm{bl}(\widetilde{\theta}\eta)$ for every $\eta \in \mathrm{Irr}(G)$ with $N \leq \ker(\eta)$.*

Proof. By definition, it is sufficient to show

$$\lambda_{\widetilde{\theta}\eta}(\mathfrak{Cl}_G(x)^+) = \lambda_{\widetilde{\theta}'\eta_H}((\mathfrak{Cl}_G(x) \cap H)^+) \qquad \text{for every } x \in G.$$

According to Lemma 2.2, we have

$$\lambda_{\widetilde{\theta}\eta}(\mathfrak{Cl}_G(x)^+) = \lambda_{\widetilde{\theta}_L}(\mathfrak{Cl}_L(x)^+)\lambda_{\overline{\eta}}(\mathfrak{Cl}_{G/N}(xN)^+),$$

where $\overline{\eta}$ is the character η viewed in G/N, and $L/N = \mathrm{C}_{G/N}(xN)$. Since $\mathrm{bl}(\widetilde{\theta}'_{L\cap H})^L = \mathrm{bl}(\widetilde{\theta}_L)$, we have

$$\lambda_{\widetilde{\theta}\eta}(\mathfrak{Cl}_G(x)^+) = \lambda_{\widetilde{\theta}'_{L\cap H}}((\mathfrak{Cl}_L(x) \cap H)^+)\lambda_{\overline{\eta}}(\mathfrak{Cl}_{G/N}(xN)^+).$$

Hence, we may assume now that $x \in H$. Notice that

$$\mathfrak{Cl}_L(x) = \mathfrak{Cl}_G(x) \cap xN.$$

Write $\mathfrak{Cl}_G(x) \cap H = \mathfrak{Cl}_H(x_1) \cup \cdots \cup \mathfrak{Cl}_H(x_t)$, a disjoint union. As $\{yM \mid y \in \mathfrak{Cl}_H(x_j)\}$ is the conjugacy class in H/M containing xM we can choose x_j such that $x_jM = xM$. Then $\mathrm{C}_{H/M}(x_jM) = (L \cap H)/M$. Now

$$\lambda_{\widetilde{\theta}'\eta_H}((\mathfrak{Cl}_G(x) \cap H)^+) = \sum_{j=1}^{t} \lambda_{\widetilde{\theta}'\eta_H}(\mathfrak{Cl}_H(x_j)^+)$$
$$= \sum_{j=1}^{t} \lambda_{\widetilde{\theta}'_{L\cap H}}(\mathfrak{Cl}_{L\cap H}(x_j)^+)\lambda_{\overline{\eta_H}}(\mathfrak{Cl}_{H/M}(x_jM)^+),$$

where $\overline{\eta_H}$ is the character η_H viewed as a character of H/M. We conclude that

$$\lambda_{\widetilde{\theta}'\eta_H}((\mathfrak{Cl}_G(x) \cap H)^+) = \left(\sum_{j=1}^{t} \lambda_{\widetilde{\theta}'_{L\cap H}}(\mathfrak{Cl}_{L\cap H}(x_j)^+)\right)\lambda_{\overline{\eta_H}}(\mathfrak{Cl}_{H/M}(xM)^+).$$

Since $\lambda_{\overline{\eta}_H}(\mathfrak{Cl}_{H/M}(xM)^+) = \lambda_{\overline{\eta}}(\mathfrak{Cl}_{G/N}(xN)^+)$, it suffices to show that

$$\Big(\sum_{j=1}^{t} \lambda_{\widetilde{\theta}'_{L\cap H}}(\mathfrak{Cl}_{L\cap H}(x_j)^+)\Big) = \lambda_{\widetilde{\theta}'_{L\cap H}}((\mathfrak{Cl}_L(x)\cap H)^+).$$

Since $xN \cap H = xM$, we have

$$\begin{aligned}\mathfrak{Cl}_L(x)\cap H &= (\mathfrak{Cl}_G(x)\cap H)\cap(xN\cap H) = (\mathfrak{Cl}_H(x_1)\cup\cdots\cup\mathfrak{Cl}_H(x_t))\cap xM\\ &= \mathfrak{Cl}_{L\cap H}(x_1)\cup\cdots\cup\mathfrak{Cl}_{L\cap H}(x_t),\end{aligned}$$

which easily concludes the proof of the lemma. □

Next we prove the following relations between dominated and induced blocks. Recall that a block of a quotient of G is contained in (or dominated by) a unique block of G (see [Nav98, p. 199]).

Proposition 2.4. *Let $Z \lhd G$ and $Z \leq H \leq G$. Let $\overline{B} \in \mathrm{Bl}(G/Z)$ and let $B \in \mathrm{Bl}(G)$ be the block dominating $\overline{B}$. Let $\overline{b} \in \mathrm{Bl}(H/Z)$ and let $b \in \mathrm{Bl}(H)$ be the block of H dominating $\overline{b}$.*

(a) *If $\overline{b}^{G/Z}$ is defined and coincides with $\overline{B}$, then the block b^G is defined and coincides with B.*
(b) *Assume that Z is a p'-group. Then $\overline{b}^{G/Z}$ is defined and coincides with $\overline{B}$ if and only if b^G is defined and coincides with B.*
(c) *Assume that Z is a central p-group and $\overline{b}^{G/Z}$ is defined. If $b^G = B$, then $\overline{b}^{G/Z} = \overline{B}$.*

Proof. We notice that (a) is a generalization of Proposition 5.7 of [Dad96]. Let $x \in G$. We want to show that $\lambda_B(\mathfrak{Cl}_G(x)^+) = \lambda_b((\mathfrak{Cl}_G(x)\cap H)^+)$. It is easy to check (see [Nav98, p. 199]) that

$$\lambda_B(\mathfrak{Cl}_G(x)^+) = |\mathfrak{Cl}_G(x)\cap xZ|^*\lambda_{\overline{B}}(\mathfrak{Cl}_{G/Z}(xZ)^+).$$

In fact, if U is a union of conjugacy classes $\mathfrak{Cl}_G(x_i)$ with $\mathfrak{Cl}_{G/Z}(x_iZ) = \mathfrak{Cl}_{G/Z}(xZ)$, then

$$\lambda_B(U^+) = |U\cap xZ|^*\lambda_{\overline{B}}(\mathfrak{Cl}_{G/Z}(xZ)^+).$$

Since $\lambda_{\overline{B}}(\mathfrak{Cl}_{G/Z}(xZ)^+) = \lambda_{\overline{b}}((\mathfrak{Cl}_{G/Z}(xZ)\cap H/Z)^+)$, it is no loss of generality to assume that $x \in H$.

Now write $\mathfrak{Cl}_G(x)\cap H = \mathfrak{O}(x_1)\cup\cdots\cup\mathfrak{O}(x_s)$, where $\mathfrak{O}(x_i)$ is the union of the conjugacy classes $\mathfrak{Cl}_H(w)$ of H inside $\mathfrak{Cl}_G(x)$ such that $\mathfrak{Cl}_{H/Z}(wZ) = \mathfrak{Cl}_{H/Z}(x_iZ)$, where

$$\mathfrak{Cl}_{G/Z}(xZ)\cap H/Z = \mathfrak{Cl}_{H/Z}(x_1Z)\cup\cdots\cup\mathfrak{Cl}_{H/Z}(x_sZ)$$

is a disjoint union. Thus

$$\lambda_{\overline{B}}(\mathfrak{Cl}_{G/Z}(xZ)^+) = \lambda_{\overline{b}}(\mathfrak{Cl}_{H/Z}(x_1Z)^+) + \cdots + \lambda_{\overline{b}}(\mathfrak{Cl}_{H/Z}(x_sZ)^+).$$

Now

$$\lambda_b((\mathfrak{Cl}_G(x)\cap H)^+) = |\mathfrak{O}(x_1)\cap x_1 Z|^* \lambda_{\overline{b}}(\mathfrak{Cl}_{H/Z}(x_1 Z)^+) + \cdots + |\mathfrak{O}(x_s)\cap x_s Z|^* \lambda_{\overline{b}}(\mathfrak{Cl}_{H/Z}(x_s Z)^+).$$

Notice that $\mathfrak{O}(x_i)\cap x_i Z = \mathfrak{Cl}_G(x)\cap x_i Z$. Since conjugation maps $\mathfrak{Cl}_G(x)\cap xZ$ to $\mathfrak{Cl}_G(x)\cap x^g Z$ bijectively, we have $|\mathfrak{O}(x_i)\cap x_i Z| = |\mathfrak{Cl}_G(x)\cap xZ|$ for all i. Now

$$\lambda_b((\mathfrak{Cl}_G(x)\cap H)^+) = |\mathfrak{Cl}_G(x)\cap xZ|^* \sum_i \lambda_{\overline{b}}(\mathfrak{Cl}_{H/Z}(x_i Z)^+),$$

proving part (a).

For the missing direction in part (b) we assume that b^G is defined, coincides with B, and Z is a p'-group. Let $y \in G$, and suppose that $\mathfrak{Cl}_G(y_1), \ldots, \mathfrak{Cl}_G(y_s)$ are all the distinct conjugacy classes of G such that $\mathfrak{Cl}_{G/Z}(y_i Z) = \mathfrak{Cl}_{G/Z}(yZ)$. Let $U = \mathfrak{Cl}_G(y_1)\cup\cdots\cup\mathfrak{Cl}_G(y_s)$, and notice that $U\cap xZ = xZ$. By the first paragraph in this proof,

$$\lambda_{\overline{B}}(\mathfrak{Cl}_{G/Z}(yZ)^+) = |Z|^{*-1}\sum_i \lambda_B(\mathfrak{Cl}_G(y_i)^+).$$

In the same way,

$$\lambda_{\overline{b}}((\mathfrak{Cl}_{G/Z}(yZ)\cap H/Z)^+) = |Z|^{*-1}\sum_i \lambda_b((\mathfrak{Cl}_G(y_i)\cap H)^+)^*.$$

Since $b^G = B$, this implies $\overline{b}^G = \overline{B}$.

In the situation of (c), we have $b^G = B$ and $\overline{C} := \overline{b}^{G/Z}$ is defined. Let $C \in \mathrm{Bl}(G)$ be the block dominating $\overline{b}^{G/Z}$. By applying (a), we have $b^G = B = C$. By [Nav98, Theorem (9.10)], C dominates a unique block of G/Z. Hence $\overline{B}$ and $\overline{b}^{G/Z}$ coincide. □

We shall also need the following proposition in which we basically collect some results on heights and defect groups. If $\chi \in \mathrm{Irr}(G)$ is in the block B with defect group D, then recall that $\chi(1)_p = (|G|_p/|D|)p^{\mathrm{ht}(\chi)}$, where $\mathrm{ht}(\chi)$ denotes the height of χ. If $N \lhd G$ and $\zeta \in \mathrm{Irr}(N)$, then we denote by $\mathrm{Irr}(G\mid\zeta)$ the set of irreducible characters χ of G such that ζ is an irreducible constituent of the restriction χ_N.

Proposition 2.5. *Let $N \lhd G$.*

(a) *If $\zeta \in \mathrm{Irr}(N)$ and $\chi \in \mathrm{Irr}(G\mid\zeta)$, then $\mathrm{ht}(\chi) \geq \mathrm{ht}(\zeta)$. In particular, if χ has height zero, then ζ has height zero.*
(b) *Let $\chi \in \mathrm{Irr}(G)$ with $\chi_N \in \mathrm{Irr}(N)$ and let $\overline{\rho} \in \mathrm{Irr}(G/N)$, where $\rho \in \mathrm{Irr}(G)$ contains N in its kernel and corresponds to $\overline{\rho}$. Then some defect group of $\mathrm{bl}(\overline{\rho})$ is contained in DN/N, where D is a defect group of $\mathrm{bl}(\chi\rho)$.*
(c) *If $\chi\rho$ is a height zero character in* (b)*, then DN/N is a defect group of $\mathrm{bl}(\overline{\rho})$.*
(d) *Let $\chi \in \mathrm{Irr}(G)$ be such that χ_N is irreducible. Let D be a defect group of $\mathrm{bl}(\chi)$. Then $p \nmid |G : ND|$. Equivalently, DN/N is a Sylow p-subgroup of G/N.*
(e) *Suppose that $\chi \in \mathrm{Irr}(G)$ has height zero, $H \leq G$ and $\psi \in \mathrm{Irr}(H)$, such that $\chi = \psi^G$. Then ψ has height zero, and a defect group of the block of ψ is a defect group of the block of χ.*

(f) *If* $\chi \in \mathrm{Irr}(G)$ *has height zero and* Q *is a defect group of* $\mathrm{bl}(\chi)$*, then there exists* $\zeta \in \mathrm{Irr}(N)$ *under* χ *such that the block of the Clifford correspondent* χ_ζ *of* χ *over* ζ *has defect group* Q*. Any two of them are* $\mathrm{N}_G(Q)$*-conjugate. Also,* $Q \cap N$ *is a defect group of* $\mathrm{bl}(\zeta)$.

(g) *Suppose that* $\chi \in \mathrm{Irr}(G)$*,* $H \leq G$ *and* $\psi \in \mathrm{Irr}(H)$*, such that* $\chi = \psi^G$*. If the blocks of* ψ *and* χ *share a common defect group, then* ψ *and* χ *have the same height.*

Proof. Part (a) is Lemma 2.2 of [Mur96]. Part (b) is exactly Corollary 1.5(i.b) of [Mur96].

We consider the situation in (c). By (a) and our hypothesis, χ_N has height zero, and hence $|D \cap N|\chi(1)_p = |N|_p$. As $\chi\rho$ has height zero, we also know that $|D|\chi(1)_p\overline{\rho}(1)_p = |G|_p$. The group $Q := DN/N$ contains a defect group of $\mathrm{bl}(\overline{\rho})$ according to (b) and satisfies

$$|Q|\overline{\rho}(1)_p = |G/N|_p$$

by the above. Hence Q is a defect group of $\mathrm{bl}(\overline{\rho})$. This proves (c).

In (d) we have $\mathrm{ht}(\chi) \geq \mathrm{ht}(\chi_N)$ by (a). By straightforward computations using the definition of height, this implies $|G/N|_p \leq |D/(D \cap N)|$.

To prove (e), using Corollary (6.2) and Lemma (4.13) of [Nav98], we know that a defect group Q of ψ is contained in a defect group D of χ. Now use the formula $\chi(1) = |G : H|\psi(1)$ and the definition of height.

Now for (f), let θ be any irreducible constituent of χ_N and let $\psi \in \mathrm{Irr}(G_\theta)$ be the Clifford correspondent of χ over θ. By the previous part, every defect group Q_1 of the block of ψ satisfies $Q_1 = Q^g$ for some $g \in G$. Then the Clifford correspondent $\chi_\zeta := \psi^{g^{-1}}$ of χ over $\zeta := \theta^{g^{-1}}$ has defect group Q. If $\nu \in \mathrm{Irr}(N)$ is any other such constituent, then $\zeta^x = \nu$, Q^x and Q are defect groups of the block of $(\chi_\zeta)^x$, and therefore $Q^{xz} = Q$ for some $z \in G_\nu$. This proves that ν and ζ are $\mathrm{N}_G(Q)$-conjugate. By Theorem (9.26) of [Nav98], $Q \cap N$ is a defect group of $\mathrm{bl}(\zeta)$.

Part (g) follows from the definition of height using that $|G : H|\psi(1) = \chi(1)$. □

3. Block isomorphic character triples

A *character triple* (G, N, θ) consists of a normal subgroup $N \lhd G$ and a G-invariant $\theta \in \mathrm{Irr}(N)$. Isomorphisms of character triples (see [Isa76]) play a key role in the character theory of finite groups. In the recent work [IMN07] and [Spä13], some special types of character triple isomorphisms have been needed. In this section we formalize all this and explore this direction further. This is essential for constructing the bijection of Theorem B.

We follow the definition and notation of character triple isomorphisms given by Definition (11.23) of [Isa76], although we shall also assume that all of our character triple isomorphisms are *strong* in the sense of Exercise (11.13) of [Isa76]. Next we recall the definition of the latter. Let (G, N, θ) be a character triple. Let $\mathrm{Char}(G \mid \theta)$ be the set of characters of G whose irreducible constituents are in $\mathrm{Irr}(G \mid \theta)$. For $g \in G$, $N \leq J \leq G$ and $\tau \in \mathrm{Char}(J \mid \theta)$, we define $\tau^{gN} \in \mathrm{Char}(J^g \mid \theta)$ by $\tau^g(x^g) = \tau(x)$ for every $x \in J$. We observe that τ^{gN} is well-defined. We say that the isomorphism

$$(\iota, \sigma) : (G, N, \theta) \to (H, M, \theta')$$

is *strong* if

$$\sigma_J(\tau)^{\iota(\overline{g})} = \sigma_{J^{\overline{g}}}(\tau^{\overline{g}}) \quad \text{for every } \overline{g} \in G/N,\ N \leq J \leq G \text{ and } \tau \in \mathrm{Char}(J \mid \theta).$$

Projective representations are essential to construct character triple isomorphisms. A map $\mathcal{P} : G \to \mathrm{GL}_n(\mathbb{C})$ is a *projective representation* if there exists a function $\alpha : G \times G \to \mathbb{C}^\times$ (the *factor set*) such that

$$\mathcal{P}(g_1)\mathcal{P}(g_2) = \alpha(g_1, g_2)\mathcal{P}(g_1 g_2) \quad \text{for every } g_1, g_2 \in G.$$

Although we recall the most important properties of projective representations in the following theorem, we refer the reader to Chapter 3.5 of [NT89] for further properties and definitions.

Theorem 3.1. *Let (G, N, θ) be a character triple.*

(a) *Given an ordinary representation $\mathcal{D}$ affording θ, there exists a projective representation $\mathcal{P}$ of G with $\mathcal{P}_N = \mathcal{D}$ and such that its factor set α is trivial on $(G \times N) \cup (N \times G)$. Hence*

$$\mathcal{P}(ng) = \mathcal{D}(n)\mathcal{P}(g) \quad \text{and} \quad \mathcal{P}(gn) = \mathcal{P}(g)\mathcal{D}(n) \quad \text{for every } g \in G \quad \text{and} \quad n \in N.$$

Furthermore, $\mathcal{P}$ can be chosen so that the values of α are roots of unity.

(b) *Let $\mathcal{P}$ be as in* (a) *and let $\mathcal{P}'$ be any projective representation of G with $(\mathcal{P})_N = (\mathcal{P}')_N$. Then $\mathcal{P}'$ also satisfies* (a) *if and only if there exists a map $\xi : G/N \to \mathbb{C}^\times$ with $\mathcal{P}(g) = \mathcal{P}'(g)\xi(gN)$ for every $g \in G$. The values of ξ are finite roots of unity and $\xi(N) = 1$.*

(c) *Let $\mathcal{P}$ be as in* (a). *Let* $\mathrm{Proj}(G/N \mid \alpha^{-1})$ *be the set of all projective representations of G/N with factor set α^{-1}, and let* $\mathrm{Rep}(G \mid \theta)$ *be the set of representations of G affording a character in* $\mathrm{Char}(G \mid \theta)$. *Then the map $\mathcal{Q} \mapsto \mathcal{Q} \otimes \mathcal{P}$ defines an injection*

$$\mathrm{Proj}(G/N \mid \alpha^{-1}) \to \mathrm{Rep}(G \mid \theta).$$

Also, $\mathcal{Q} \otimes \mathcal{P}$ and $\mathcal{Q}' \otimes \mathcal{P}$ are similar if and only if $\mathcal{Q}$ and $\mathcal{Q}'$ are (see [Nav98, p. 172] *for a definition of similar projective representations). The above map gives a bijection between the similarity classes of projective representations and the ones of representations.*

Proof. This follows from Theorems 3.5.7–3.5.9 of [NT89]. For the first part, see also the construction of $\mathcal{P}$ given in the proof of Theorem (8.28) of [Nav98]. □

We call a projective representation $\mathcal{P}$ with the properties described in (a) a *projective representation of G associated to θ*. If α is the factor set associated with $\mathcal{P}$, it is easy to check that $\alpha(g_1 n_1, g_2 n_2) = \alpha(g_1, g_2)$ for every $g_i \in G$ and $n_i \in N$. Hence α can be seen as a function $\overline{\alpha} : G/N \times G/N \to \mathbb{C}^\times$.

Theorem 3.2. *Let (G, N, θ) and (H, M, θ') be two character triples with $NH = G$ and $N \cap H = M$. Let $\iota : G/N \to H/M$ be the canonical isomorphism. Assume that there exist two projective representations $\mathcal{P}$ and $\mathcal{P}'$ of G and H associated to θ and θ', respectively, with factor sets α and α' such that $\overline{\alpha}'(\iota(x), \iota(y)) = \overline{\alpha}(x, y)$ for all $x, y \in G/N$. Then* *there exists a strong isomorphism of character triples* (ι, σ)*, where* $\sigma_J : \mathrm{Char}(J \mid \theta) \to \mathrm{Char}(J \cap H \mid \theta')$ *is given by*

$$\sigma_J(\mathrm{tr}(\mathcal{Q} \otimes \mathcal{P}_J)) = \mathrm{tr}(\mathcal{Q} \otimes \mathcal{P}'_{J\cap H}) \quad \textit{for every } N \le J \le G,$$

where $\mathcal{Q}$ *is a projective representation of* J/N *whose factor set is the inverse of the one of* $\mathcal{P}_J$*.*

Proof. By Theorem 3.1(c), every $\gamma \in \mathrm{Irr}(J \mid \theta)$ is the trace of some representation of the form $\mathcal{Q} \otimes \mathcal{P}_J$, where $\mathcal{Q}$ is a projective representation of J/N whose factor set is the inverse of the one of $\mathcal{P}_J$. Also, $\mathcal{Q}$ is uniquely defined up to similarity. It is straightforward to check that this defines a strong isomorphism of character triples. □

We call $(\iota, \sigma) : (G, N, \theta) \to (H, M, \theta')$ as above an *isomorphism of character triples given by the projective representations* $\mathcal{P}$ and $\mathcal{P}'$.

It turns out that it is important to check the behaviour of $\mathrm{C}_G(N)$ in this type of character triple isomorphism. Recall that if $\mathcal{P}_N$ is an ordinary irreducible representation of N and $x \in \mathrm{C}_G(N)$, then $\mathcal{P}(x)$ is a scalar matrix by Schur's Lemma. If $J \le H$ and $\psi \in \mathrm{Irr}(H)$, we shall use the notation $\mathrm{Irr}(\psi_J)$ to denote the set of irreducible constituents of ψ_J.

Lemma 3.3. *Let* $(\iota, \sigma) : (G, N, \theta) \to (H, M, \theta')$ *be an isomorphism given by the projective representations* $\mathcal{P}$ *and* $\mathcal{P}'$ *with* $\mathrm{C}_G(N) \le H$*. Then the following are equivalent:*

(i) *For every* $x \in \mathrm{C}_G(N)$ *the matrices* $\mathcal{P}(x)$ *and* $\mathcal{P}'(x)$ *are associated to the same scalar.*
(ii) $\mathrm{Irr}(\psi_{\mathrm{C}_J(N)}) = \mathrm{Irr}(\sigma_J(\psi)_{\mathrm{C}_J(N)})$ *for every* $N \le J \le G$ *and* $\psi \in \mathrm{Irr}(J \mid \theta)$*.*

Proof. Assume (i). Using Theorem 3.1, suppose that $\psi \in \mathrm{Irr}(J \mid \theta)$ is afforded by a representation $\mathcal{Q} \otimes \mathcal{P}_J$ for some projective representation $\mathcal{Q}$ of J/N. Then $\psi' := \sigma_J(\psi)$ is afforded by $\mathcal{Q} \otimes \mathcal{P}'_{J\cap H}$. For $x \in \mathrm{C}_J(N)$ there exists some $\zeta_x \in \mathbb{C}$ with $\mathrm{tr}(\mathcal{P}(x)) = \theta(1)\zeta_x$ and $\mathrm{tr}(\mathcal{P}'(x)) = \theta'(1)\zeta_x$. This implies $\psi(x) = \mathrm{tr}(\mathcal{Q}(x))\theta(1)\zeta_x$ and $\psi'(x) = \mathrm{tr}(\mathcal{Q}(x))\theta'(1)\zeta_x$. This proves

$$\theta'(1)\psi_{\mathrm{C}_J(N)} = \theta(1)\psi'_{\mathrm{C}_J(N)}$$

and hence $\mathrm{Irr}(\psi_{\mathrm{C}_J(N)}) = \mathrm{Irr}(\psi'_{\mathrm{C}_J(N)})$.

Assume (ii). Let $x \in \mathrm{C}_J(N)$, and let ζ_x and ζ'_x be the scalars associated to $\mathcal{P}(x)$ and $\mathcal{P}'(x)$, respectively. If $J = \langle N, x\rangle$, then there exists an extension ψ of θ to J. Therefore $\psi' := \sigma_J(\psi)$ is an extension of θ'. Since $\mathrm{C}_J(N) \le \mathrm{Z}(J)$, we see that $\mathrm{Irr}(\psi_{\mathrm{C}_J(N)}) = \{\nu\}$ for some ν. Now $\mathrm{Irr}(\psi_{\mathrm{C}_J(N)}) = \mathrm{Irr}(\psi'_{\mathrm{C}_J(N)})$ by hypothesis, and therefore

$$\nu(x) = \frac{\psi(x)}{\theta(1)} = \zeta_x \,\mathrm{tr}(\mathcal{Q}(x)) \quad \text{and} \quad \nu(x) = \frac{\psi'(x)}{\theta(1)} = \zeta'_x \,\mathrm{tr}(\mathcal{Q}(x)),$$

where $\mathcal{Q}$ is a projective representation of J/N such that $\mathcal{Q}\otimes\mathcal{P}_J$ affords ψ and $\mathcal{Q}\otimes\mathcal{P}'_{J\cap H}$ affords ψ'. This shows $\zeta_x = \zeta'_x$. □

In the situation of Lemma 3.3, we say that (G, N, θ) and (H, M, θ') are *central isomorphic character triples* and that (ι, σ) is a *central isomorphism of character triples*. This situation is denoted by $(G, N, \theta) \sim_{\mathrm{c}} (H, M, \theta')$.

How are different central isomorphisms related?

Remark 3.4. Let (ι,σ), $(\iota,\sigma_1) : (G,N,\theta) \to (H,M,\theta')$ be isomorphisms of character triples given by projective representations.

(a) Let $\mathcal{P}_1$ be a projective representation of G associated to θ. Then there exists a projective representation $\mathcal{P}'_1$ of H associated to θ' such that (ι,σ) coincides with $(\iota,\sigma_1) : (G,N,\theta) \to (H,M,\theta')$, where (ι,σ_1) is given by $\mathcal{P}_1$ and $\mathcal{P}'_1$.
(b) There exists a unique linear character $\mu \in \mathrm{Irr}(H)$ with $M \le \ker(\mu)$ such that

$$\sigma_J(\psi) = \sigma_{1,J}(\psi)\mu_J \qquad \text{for every } N \le J \le G \text{ and } \psi \in \mathrm{Irr}(J \mid \theta).$$

(c) Assume that (ι,σ) is central. Then (ι,σ_1) is central if and only if $\mathrm{C}_G(N) \le \ker(\mu)$.

Proof. Let $\mathcal{P}$ and $\mathcal{P}'$ be projective representations associated to θ and θ' giving (ι,σ). Then there exists a matrix $X \in \mathrm{GL}_{\theta(1)}(\mathbb{C})$ such that $\mathcal{P}_1$ and $\mathcal{P}^X$ coincide on M. According to Theorem 3.1(b) there exists a function $\zeta : G/N \to \mathbb{C}^\times$ such that $\mathcal{P}_1 = \zeta\mathcal{P}^X$. The projective representations $\mathcal{P}_1$ and $\mathcal{P}'_1 := \zeta\mathcal{P}'$ give an isomorphism of character triples $(\iota,\sigma_1) : (G,N,\theta) \to (H,M,\theta')$ that coincides with (ι,σ) by straightforward computations. This proves (a).

By (a) we may assume that (ι,σ) is given by $\mathcal{P}$ and $\mathcal{P}'$, and (ι,σ_1) by $\mathcal{P}$ and $\zeta\mathcal{P}'$ for some $\zeta : H/M \to \mathbb{C}^\times$. As the factor sets of $\mathcal{P}'$ and $\zeta\mathcal{P}'$ coincide, ζ has to be a linear character. This proves (b) with $\mu := \zeta^{-1}$. Part (c) is straightforward. □

We shall often use the following lemma.

Lemma 3.5. *Suppose that $N \lhd G$, $H \le G$, and $G = NH$. Let $M = N \cap H$ and suppose that $\theta' \in \mathrm{Irr}(M)$ is H-invariant. Let D be a defect group of $\mathrm{bl}(\theta')$ and assume $\mathrm{N}_N(D) \le M$. Then:*

(a) *$H = M\mathrm{N}_G(D)$.*
(b) *If $\nu \in \mathrm{Irr}(H \mid \theta')$, then $\mathrm{bl}(\nu)^G$ is defined.*
(c) *Let $\nu \in \mathrm{Irr}(H \mid \theta')$. Then $\mathrm{bl}(\nu)$ and $\mathrm{bl}(\nu)^G$ have a common defect group Q with $Q \cap N = Q \cap M = D$. In fact, there exists a block c of $\mathrm{N}_G(D)$ such that $c^H = \mathrm{bl}(\nu)$, $c^G = \mathrm{bl}(\nu)^G = (c^H)^G$, and c, c^H and c^G have a common defect group Q. If $b_0 \in \mathrm{Bl}(\mathrm{N}_N(D) \mid D)$ is the Brauer correspondent of $\mathrm{bl}(\theta')$, then c covers b_0, and is the common Harris–Knörr correspondent of c^H and c^G.*
(d) *Let $\nu \in \mathrm{Irr}(H \mid \theta')$. Then $\mathrm{bl}(\nu)$ is the only block of H covering $\mathrm{bl}(\theta')$ that induces $\mathrm{bl}(\nu)^G$.*

Proof. Since D is a defect group of $\mathrm{bl}(\theta')$ and θ' is invariant in H, D^h is a defect group of $\mathrm{bl}(\theta')$ for every $h \in H$. By the Frattini argument, this implies $M\mathrm{N}_H(D) = H$. Thus $N\mathrm{N}_H(D) = G$ and $\mathrm{N}_G(D) = \mathrm{N}_H(D)\mathrm{N}_N(D) = \mathrm{N}_H(D)$. This proves (a).

We have $\mathrm{N}_N(D) = \mathrm{N}_M(D)$. By Brauer's first main Theorem (4.17) of [Nav98], there exists a block $b_0 \in \mathrm{Bl}(\mathrm{N}_N(D))$ with defect group D such that $b_0^M = \mathrm{bl}(\theta')$. Also $b = (b_0)^N$ has defect group D by Brauer's first main theorem. Since $\mathrm{bl}(\nu)$ covers $\mathrm{bl}(\theta')$, by the Harris–Knörr Theorem (9.28) of [Nav98] applied in H there exists a block c of $\mathrm{N}_G(D)$ covering b_0 such that c^H is defined and equals $\mathrm{bl}(\nu)$. By Theorem (4.14) of [Nav98], c^G is defined. By Exercise (4.2) of [Nav98], $\mathrm{bl}(\nu)^G$ is defined. This proves (b).

Now by Harris–Knörr, c^G and c have a common defect group, and the same happens with c and c^H. Therefore c, $\mathrm{bl}(\nu)$ and $\mathrm{bl}(\nu)^G$ have a common defect group Q inside $\mathrm{N}_G(D)$. Since c covers b_0 that has defect group D, by Theorem (9.26) of [Nav98], we have $Q \cap \mathrm{N}_N(D) = D$. Hence $\mathrm{N}_{Q\cap N}(D) = D$ and therefore $Q \cap N = D$. Since $Q \leq \mathrm{N}_G(D) \leq H$, therefore $Q \cap N = Q \cap M$. This proves (c).

Finally suppose that b_1 is a block of H covering $\mathrm{bl}(\theta')$ and inducing $\mathrm{bl}(\nu)^G = c^G$. Since b_1 covers $\mathrm{bl}(\theta')$, let $\nu_1 \in \mathrm{Irr}(b_1)$ be over θ'. Now by (c), there exists a block c_1 of $\mathrm{N}_G(D)$ such that $c_1^H = b_1, c_1^G = c^G$ and c_1 covers b_0. By uniqueness in the Harris–Knörr correspondence, we conclude that $c = c_1$ and $c^H = c_1^H$. □

Definition 3.6 (Block isomorphism of character triples). Let $(\iota, \sigma) : (G, N, \theta) \to (H, M, \theta')$ be a central isomorphism of character triples. Assume that some defect group D of $\mathrm{bl}(\theta')$ is a defect group of $\mathrm{bl}(\theta)$, and that $\mathrm{N}_N(D) \leq M$. If for every $N \leq J \leq G$ and $\psi \in \mathrm{Irr}(J \mid \theta)$ the equation

$$\mathrm{bl}(\sigma_J(\psi))^J = \mathrm{bl}(\psi)$$

holds, then we say that (ι, σ) is a *block isomorphism of character triples*. In this situation we write $(G, N, \theta) \sim_{\mathrm{b}} (H, M, \theta')$ and say that (G, N, θ) and (H, M, θ') are *block isomorphic* character triples. (Notice that by Lemma 3.5 the block $\mathrm{bl}(\sigma_J(\psi))^J$ is always defined.)

To help the reader grasp this definition, we summarize the conditions that have to be checked whenever we prove that two given character triples are block isomorphic.

Remark 3.7. Let (G, N, θ) and (H, M, θ') be two character triples. Then $(G, N, \theta) \sim_{\mathrm{b}} (H, M, \theta')$ if the following conditions are satisfied:

(i) $G = NH$ and $M = N \cap H$.
(ii) There is a common defect group D of $\mathrm{bl}(\theta)$ and of $\mathrm{bl}(\theta')$ and $\mathrm{N}_N(D) \leq M$.
(iii) There exist projective representations $\mathcal{P}$ and $\mathcal{P}'$ associated to θ and θ' with factor sets α and α' such that $\alpha(h_1, h_2) = \alpha'(h_1, h_2)$ for all $h_1, h_2 \in H$, and such that for every $x \in \mathrm{C}_G(N)$ the scalar matrices $\mathcal{P}(x)$ and $\mathcal{P}'(x)$ are associated to the same scalar.
(iv) $\mathrm{bl}(\sigma_J(\psi))^J = \mathrm{bl}(\psi)$ for every $N \leq J \leq G$ and $\psi \in \mathrm{Irr}(J \mid \theta)$, where $\sigma_J(\psi) = \mathrm{tr}(\mathcal{Q} \otimes \mathcal{P}'_{J\cap H})$, $\psi = \mathrm{tr}(\mathcal{Q} \otimes \mathcal{P}_J)$ and $\mathcal{Q}$ is a projective representation of J/N with factor set α^{-1}.

Proof. By Lemma 3.5 the first two conditions imply that $G = N\mathrm{N}_G(D)$ and $\mathrm{C}_G(N) \leq H$. The projective representations $\mathcal{P}$ and $\mathcal{P}'$ give a central isomorphism (ι, σ) of character triples, by Lemma 3.3. For every $N \leq J \leq G$ and $\psi \in \mathrm{Irr}(J \mid \theta)$ the character $\sigma_J(\psi)$ is defined as in (iv) and satisfies $\mathrm{bl}(\psi) = \mathrm{bl}(\sigma_J(\psi))^J$. □

In the next results we prove some properties of block isomorphisms of character triples.

Lemma 3.8. (a) *If $(G, N, \theta) \sim_{\mathrm{b}} (H, M, \theta')$ and $(H, M, \theta') \sim_{\mathrm{b}} (H_1, M_1, \theta'')$, then*

$$(G, N, \theta) \sim_{\mathrm{b}} (H_1, M_1, \theta'').$$

(b) *If $(G, N, \theta) \sim_{\mathrm{b}} (H, M, \theta')$, then $(J, N, \theta) \sim_{\mathrm{b}} (J \cap H, M, \theta')$ for every $N \leq J \leq G$.*

Proof. Let $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{P}''$ be projective representations of G, H and H_1 associated to θ, θ' and θ'', respectively, which determine the block isomorphisms

$$(\iota, \sigma) : (G, N, \theta) \to (H, M, \theta') \quad \text{and} \quad (\iota_1, \sigma_1) : (H, M, \theta') \to (H_1, M_1, \theta'')$$

in the sense of Theorem 3.2. The existence of such projective representations is implied by Remark 3.4(a). We have $\mathrm{C}_G(N) \leq H$ and $\mathrm{C}_H(M) \leq H_1$, and therefore $\mathrm{C}_G(N) \leq H_1$. Hence for $x \in \mathrm{C}_G(N)$ the matrices $\mathcal{P}(x)$, $\mathcal{P}'(x)$ and $\mathcal{P}''(x)$ are associated to the same scalar. Also, the factor sets of $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{P}''$ coincide via the natural isomorphisms between G/N, H/M and H_1/M_1. Hence using Lemma 3.3 and Remark 3.4(a), we easily check that the composition of (ι, σ) and (ι_1, σ_1) defines a central isomorphism

$$(G, N, \theta) \sim_c (H_1, M_1, \theta'').$$

Now, $\mathrm{bl}(\theta)$ and $\mathrm{bl}(\theta')$ have a common defect group D with $\mathrm{N}_N(D) \leq M$, and $\mathrm{bl}(\theta')$ and $\mathrm{bl}(\theta'')$ have a common defect group D_1 with $\mathrm{N}_M(D_1) \leq M_1$. Thus $D_1 = D^m$ for some $m \in M$, $\mathrm{N}_N(D^m) \leq M_1$ and D^m is a common defect group for $\mathrm{bl}(\theta)$ and $\mathrm{bl}(\theta'')$. For every $N \leq J \leq G$ and $\psi \in \mathrm{Irr}(J \mid \theta)$, write $\psi'' := \sigma_1(\sigma(\psi))$. Then the block $\mathrm{bl}(\psi'')^J$ coincides with $(\mathrm{bl}(\psi'')^{J \cap H})^J = \mathrm{bl}(\psi)$, by Exercise (4.2) of [Nav98]. Part (b) is a direct consequence of the definition of block isomorphism of character triples. □

When we have block isomorphisms of character triples, we have a good control of defect groups and relative heights, as shown in the following result. If $N \lhd G$ and $\tau \in \mathrm{Irr}(N)$, recall that G_τ is the stabilizer of τ in G. Also, if B is a block of G, then $\mathrm{Irr}(B \mid \tau)$ is the set of irreducible characters in B lying over τ. Furthermore, if $G = KH$ and $H \leq \mathrm{C}_G(K)$, then recall that for every pair of characters $\theta \in \mathrm{Irr}(H)$ and $\mu \in \mathrm{Irr}(K)$, there exists a unique $\theta \cdot \mu \in \mathrm{Irr}(G)$ that lies over θ and over μ. (We refer the reader to Section 5 of [IMN07] for notation of central products.)

Proposition 3.9. *Let $(\iota, \sigma) : (G, N, \theta) \to (H, M, \theta')$ be a block isomorphism of character triples. Suppose that $N \leq J \leq G$, and write $\sigma_J(\psi) = \psi'$ for $\psi \in \mathrm{Irr}(J \mid \theta)$.*

(a) *If $\psi \in \mathrm{Irr}(J \mid \theta)$, then*

$$\mathrm{ht}(\psi) - \mathrm{ht}(\theta) = \mathrm{ht}(\psi') - \mathrm{ht}(\theta').$$

Also, $\mathrm{bl}(\psi)$ and $\mathrm{bl}(\psi')$ have a common defect group. In fact, if D is a common defect group of θ and θ', and Q is any defect group of $\mathrm{bl}(\psi)$ with $Q \cap N = D$, then Q is a defect group of $\mathrm{bl}(\psi')$.

(b) *The map*

$$\sigma_J : \mathrm{Irr}(J \mid \theta) \to \mathrm{Irr}(J \cap H \mid \theta')$$

is an $\mathrm{N}_H(J)$-equivariant bijection with

$$(\mathrm{N}_G(J)_\tau, J, \tau) \sim_b (\mathrm{N}_H(J)_\tau, J \cap H, \tau')$$

for every $\tau \in \mathrm{Irr}(J \mid \theta)$.

(c) *Suppose $J = N\mathrm{C}_J(N)$ and let $\tau \in \mathrm{Irr}(J \mid \theta)$ be an extension of θ with $\nu \in \mathrm{Irr}(\tau_{\mathrm{C}_J(N)})$. Then τ' is the unique extension of θ' in $\mathrm{Irr}(J \cap H \mid \nu)$.*

(d) *Assume that* θ *and* θ' *have the same height. Suppose* $N \le J \lhd G$, $B \in \mathrm{Bl}(G)$ *and* $\tau \in \mathrm{Irr}(J \mid \theta)$. *Then the restriction of* σ_G *to* $\mathrm{Irr}(B \mid \tau)$ *gives a height preserving bijection between* $\mathrm{Irr}(B \mid \tau)$ *and* $\mathrm{Irr}(B' \mid \tau')$*, where* B' *is the unique block of* H *covering* $\mathrm{bl}(\tau')$ *with* $B'^G = B$.

Proof. Let D be a common defect group of $\mathrm{bl}(\theta)$ and $\mathrm{bl}(\theta')$. By Lemma 3.5(c) there exists a common defect group Q_1 of $\mathrm{bl}(\psi)$ and $\mathrm{bl}(\psi')$ such that $Q_1 \cap N = D$. Now, if Q is any other defect group of $\mathrm{bl}(\psi)$ with $Q \cap N = D$, then $Q^x = Q_1$ for some $x \in J$. Then $x \in \mathrm{N}_G(D) \le H$, by Lemma 3.5(a). Thus $x \in H \cap J$ and $Q = (Q_1)^{x^{-1}}$ is also a defect group of $\mathrm{bl}(\psi')$. Now, by Lemma (11.24) of [Isa76],

$$\frac{\psi(1)}{\theta(1)} = \frac{\psi'(1)}{\theta'(1)}.$$

This easily implies the part on heights in (a), and (a) is now proved.

As the character triples (G, N, θ) and (H, M, θ') are strongly isomorphic, the bijection σ_J is $\mathrm{N}_H(J)$-equivariant. To prove (b), we may assume that $J \lhd G$ and that τ is G-invariant by Lemma 3.8(b). Let $\mathcal{P}$ and $\mathcal{P}'$ be projective representations associated to θ and θ' determining (ι, σ), and α and α' their respective factor sets. Let $\tau \in \mathrm{Irr}(J \mid \theta)$. Then we know that τ is afforded by $\mathcal{Q} \otimes \mathcal{P}_J$, where $\mathcal{Q}$ is a projective representation of J/N. By Theorem 3.1, let $\mathcal{D}$ be a projective representation of G associated to τ with $\mathcal{D}_J = \mathcal{Q} \otimes \mathcal{P}_J$. Suppose that $\mathcal{D}$ has factor set β. Now, use the proof of Theorem (8.16) of [Nav98] to check that we can write

$$\mathcal{D}(g) = \mathcal{R}(g) \otimes \mathcal{P}(g)$$

for every $g \in G$, where $\mathcal{R}$ is a projective representation of G. This equation easily implies that $\mathcal{R}$ is a projective representation of G with factor set $\beta\alpha^{-1}$. Also, $\mathcal{R}_J = \mathcal{Q}$. Now, define $\mathcal{D}' = \mathcal{R}_H \otimes \mathcal{P}'$, which is a projective representation of H associated with τ' satisfying the conditions of Theorem 3.1(a). Now $\mathrm{C}_G(J) \le \mathrm{C}_G(N) \le H$. If $x \in \mathrm{C}_G(J)$, then $\mathcal{D}(x)$ and $\mathcal{P}(x)$ are scalar matrices, and we deduce that $\mathcal{R}(x)$ is a scalar matrix, too. Hence $\mathcal{D}(x)$ and $\mathcal{D}'(x)$ are matrices associated to the same scalar. Thus $(G, J, \tau) \sim_c (H, H \cap J, \tau')$.

For $J \le J_1 \le G$, let $\sigma_{1,J_1} : \mathrm{Char}(J_1 \mid \tau) \to \mathrm{Char}(H \cap J_1 \mid \tau')$ be the associated bijection, so that $\sigma_{1,J_1}(\psi) = \mathrm{tr}(\mathcal{X} \otimes (\mathcal{R}_{J_1} \otimes \mathcal{P}'_{J_1}))$ if $\psi = \mathrm{tr}(\mathcal{X} \otimes (\mathcal{R}_{J_1} \otimes \mathcal{P}_{J_1}))$ and $\mathcal{X}$ is a projective representation of J_1 constant on J-cosets. Now, $\mathcal{X} \otimes \mathcal{R}_{J_1}$ is a projective representation of J_1 constant on J cosets, and we see that $\sigma_{1,J_1}(\psi) = \sigma_{J_1}(\psi)$. In particular, $\mathrm{bl}(\sigma_{J_1}(\psi))^{J_1} = \mathrm{bl}(\psi)$.

Now by (a) there exists a defect group Q of $\mathrm{bl}(\tau)$ that is also a defect group of $\mathrm{bl}(\tau')$. By Theorem (9.26) of [Nav98] the group $Q \cap M$ is a defect group of $\mathrm{bl}(\theta')$. This implies that $\mathrm{N}_N(Q \cap M) \le M$ (by definition of block isomorphisms of character triples) and $\mathrm{N}_G(Q \cap M) \le H$ by Lemma 3.5. Now, since Q is a defect group of $\mathrm{bl}(\tau)$, and $\mathrm{bl}(\theta)$ is J-invariant, we also know that $Q \cap N$ is a defect group of $\mathrm{bl}(\theta)$ by Theorem (9.26) of [Nav98]. Since $\mathrm{bl}(\theta)$ and $\mathrm{bl}(\theta')$ have the same defect group, we conclude that $Q \cap N = Q \cap M$. Now we see that $\mathrm{N}_J(Q)$ is contained in $H \cap J$.

By Lemma 3.3, we deduce that $\sigma_J(\theta \cdot \nu) \in \mathrm{Irr}(J \cap H \mid \sigma_N(\theta)) \cap \mathrm{Irr}(J \cap H \mid \nu)$ for every $\nu \in \mathrm{Irr}(\mathrm{C}_J(N))$ and J with $J = N\mathrm{C}_J(N)$.

Part (d) is a consequence of (a) and (b). By (a) and the assumption that θ and θ' have the same height, σ_G preserves heights. Furthermore for every character $\psi \in \sigma_G(\mathrm{Irr}(B \mid \tau))$ the block $\mathrm{bl}(\psi)$ induces the block B. Now apply Lemma 3.5. □

Corollary 3.10. *Let $N \lhd G$, $H \leq G$, $M := N \cap H$ and suppose that $D \leq M$ is a p-subgroup such that $M\mathrm{N}_G(D) = H$. Assume that there exists an $\mathrm{N}_G(D)$-equivariant bijection*

$$\Pi : \mathrm{Irr}_0(N \mid D) \to \mathrm{Irr}_0(M \mid D)$$

such that $(G_\tau, N, \tau) \sim_{\mathrm{b}} (H_\tau, M, \tau')$ for every $\tau \in \mathrm{Irr}_0(N \mid D)$, where $\tau' = \Pi(\tau)$. Let $\tau \in \mathrm{Irr}_0(N \mid D)$ and $\tau' := \Pi(\tau) \in \mathrm{Irr}_0(M \mid D)$. Then, for every p-block $b \in \mathrm{Bl}(H)$ covering the block of τ, there exists a height preserving bijection between $\mathrm{Irr}(b^G \mid \tau)$ and $\mathrm{Irr}(b \mid \tau')$.

Proof. Let T be the stabilizer in G of $\mathrm{bl}(\tau)$ (which contains G_τ). By the Frattini argument, $T = N\mathrm{N}_T(D)$. Also, since the block of τ' induces the block of τ, Brauer's first main theorem implies that $T \cap H$ is the stabilizer of $\mathrm{bl}(\tau')$ in H. Let b_1 be the Fong–Reynolds correspondent of b over the block of τ' (see [Nav98, Theorem (9.14].) Then $(b_1)^T$ is the Fong–Reynolds correspondent of b^G over $\mathrm{bl}(\tau)$. Now, the proof of [Nav98, Theorem (9.14] tells us that induction gives height preserving bijections $\mathrm{Irr}(b_1 \mid \tau') \to \mathrm{Irr}(b \mid \tau')$ and $\mathrm{Irr}((b_1)^T \mid \tau) \to \mathrm{Irr}(b^G \mid \tau)$. Hence, we may assume that $G = T = N\mathrm{N}_G(D)$. In particular, $G = NH$. Also, by hypothesis, $G_\tau \cap H = H_{\tau'} = H_\tau$. Furthermore, by the Harris–Knörr correspondence [Nav98, Theorem (9.28)] (used for b and for b^G), b and b^G have a defect group in common. Let $B := b^G$.

Now, let $c_1, \dots, c_s$ be the different blocks of $H_{\tau'}$ that cover $\mathrm{bl}(\tau')$ and induce b. (If c_j covers $\mathrm{bl}(\tau')$, then some irreducible character η_j of c_j lies over τ'. Then $(\eta_j)^H$ is irreducible and $(c_j)^H$ is defined.) Then induction defines a bijection

$$\bigcup_{j=1}^{s} \mathrm{Irr}(c_j \mid \tau') \to \mathrm{Irr}(b \mid \tau').$$

Now, by Lemma 3.5, c_i has a defect group Q_i such that Q_i is also a defect group of $d_i = (c_i)^{G_\tau}$. We easily check using Lemma 3.5 that $d_1, \dots, d_s$ are all the different blocks of G_τ that cover $\mathrm{bl}(\tau)$ and induce B. Hence, induction defines a bijection

$$\bigcup_{j=1}^{s} \mathrm{Irr}(d_j \mid \tau) \to \mathrm{Irr}(B \mid \tau).$$

Now, by Proposition 3.9(d), σ_{G_τ} gives a height preserving bijection between $\mathrm{Irr}(d_i \mid \tau)$ and $\mathrm{Irr}(c_i \mid \tau')$. This, together with the previous paragraph, provides a bijection $\mathrm{Irr}(b \mid \tau') \to \mathrm{Irr}(B \mid \tau)$ and we only have to check that it preserves heights.

Let $\psi \in \mathrm{Irr}(B \mid \tau)$, and let $\psi_0 \in \mathrm{Irr}(d_i)$ with $\psi_0^G = \psi$. Write $\psi_0' := \sigma_{G_\tau}(\psi_0) \in \mathrm{Irr}(c_i \mid \tau')$, which has the same height as ψ_0. By Lemma (11.24) of [Isa76],

$$\left(\frac{\psi_0(1)}{\tau(1)}\right)_p = \left(\frac{\psi_0'(1)}{\tau'(1)}\right)_p.$$

The character $\psi' := (\psi'_0)^H$ is contained in the block b and has degree $|H : H_\tau|\psi'_0(1)$. We know that $|G : G_\tau| = |H : H_{\tau'}|$. Thus

$$\left(\frac{\psi(1)}{\tau(1)}\right)_p = \left(\frac{|G : G_\tau|\psi_0(1)}{\tau(1)}\right)_p = \left(\frac{|H : H_\tau|\psi'_0(1)}{\tau'(1)}\right)_p = \left(\frac{\psi'(1)}{\tau'(1)}\right)_p .$$

As τ and τ' have the same height and b and B have the same defect group, this proves that ψ and ψ' have the same height. □

When analyzing whether two character triples are block isomorphic, it is helpful to assume first that the characters extend. This will be used in Section 4 below.

Lemma 3.11. *Let (G, N, θ) and (H, M, θ') be two character triples and assume $NH = G$ and $N \cap H = M$. Assume that some defect group D of $\mathrm{bl}(\theta')$ is also a defect group of $\mathrm{bl}(\theta)$, and that $\mathrm{N}_N(D) \le M$. Assume that there exist extensions $\widetilde{\theta} \in \mathrm{Irr}(G)$ of θ and $\widetilde{\theta}' \in \mathrm{Irr}(H)$ of θ' with the following properties:*

(i) $\mathrm{Irr}(\widetilde{\theta}_{\mathrm{C}_G(N)}) = \mathrm{Irr}(\widetilde{\theta}'_{\mathrm{C}_G(N)})$, *and*
(ii) *for every $N \le J \le G$ we have* $\mathrm{bl}(\widetilde{\theta}'_{J\cap H})^J = \mathrm{bl}(\widetilde{\theta}_J)$.

Then $(G, N, \theta) \sim_{\mathrm{b}} (H, M, \theta')$.

Note that by Lemma 3.5(a) we have $\mathrm{N}_G(D) \le H$ and we can restrict $\widetilde{\theta}'$ to $\mathrm{C}_G(N)$ in (i).

Proof. We check the conditions described in Remark 3.7. Let $\mathcal{P}$ and $\mathcal{P}'$ be ordinary representations of G and H affording $\widetilde{\theta}$ and $\widetilde{\theta}'$ in the sense of the definition after Theorem 3.1. Hence $\mathcal{P}$ and $\mathcal{P}'$ are projective representations associated to θ and θ'. The isomorphism (ι, σ) of character triples given by these projective representations in Theorem 3.2 is the following: if $N \le J \le G$, then $\sigma_J : \mathrm{Char}(J \mid \theta) \to \mathrm{Char}(H \cap J \mid \theta')$ satisfies

$$\widetilde{\theta}_J \rho \mapsto \widetilde{\theta}'_{H\cap J}\rho$$

for every $\rho \in \mathrm{Irr}(J)$ with $N \le \ker(\rho)$. Then by hypothesis and Proposition 2.3,

$$\mathrm{bl}(\sigma_J(\psi))^J = \mathrm{bl}(\psi) \quad \text{ for every } \psi \in \mathrm{Irr}(J \mid \theta).$$

By assumption $\mathrm{Irr}(\widetilde{\theta}_{\mathrm{C}_G(N)}) = \mathrm{Irr}(\widetilde{\theta}'_{\mathrm{C}_G(N)})$. Furthermore $\mathrm{Irr}(\widetilde{\theta}'_{\mathrm{C}_G(N)}) = \{\nu\}$ for some linear character $\nu \in \mathrm{Irr}(\mathrm{C}_G(N))$. Let $\psi \in \mathrm{Irr}(J \mid \theta)$ and $\rho \in \mathrm{Irr}(J)$ be such that $N \le \ker(\rho)$ and $\psi = \widetilde{\theta}_J\rho$. Then $\mathrm{Irr}(\psi_{\mathrm{C}_J(N)})$ and $\mathrm{Irr}(\sigma_J(\psi)_{\mathrm{C}_J(N)})$ are the sets of constituents of $\nu_{\mathrm{C}_J(N)}\rho_{\mathrm{C}_J(N)}$, and therefore the two sets coincide. The proof is complete by Lemma 3.3. □

In the next two rather technical (but necessary) results we analyze the behaviour of block isomorphic character triples with respect to certain quotients.

Lemma 3.12. *Let $Z \lhd G$, $N \lhd G$ and $H \le G$ be such that $Z \le M := H \cap N$. Let $\overline{G} := G/Z$, $\overline{N} := N/Z$, $\overline{H} := H/Z$ and $\overline{M} := M/Z$. Suppose that $(\overline{G}, \overline{N}, \overline{\theta}) \sim_{\mathrm{b}} (\overline{H}, \overline{M}, \overline{\theta}')$, where $\overline{\theta} \in \mathrm{Irr}(\overline{N})$ and $\overline{\theta}' \in \mathrm{Irr}(\overline{M})$. Let $\theta \in \mathrm{Irr}(N)$ and $\theta' \in \mathrm{Irr}(M)$ be the lifts of $\overline{\theta}$ and $\overline{\theta}'$. Assume that the blocks $\mathrm{bl}(\theta)$ and $\mathrm{bl}(\theta')$ have a common defect group D such that $\overline{D} = DZ/Z$ is a defect group of both $\mathrm{bl}(\overline{\theta})$ and $\mathrm{bl}(\overline{\theta}')$. Then $(G, N, \theta) \sim_{\mathrm{b}} (H, M, \theta')$. In particular, this happens if both θ and θ' are characters of height zero.*

Proof. Let $\epsilon : G \to \overline{G}$ be the canonical epimorphism. As $\mathrm{bl}(\theta)$ covers the principal block of Z the group $D \cap Z$ is a Sylow p-subgroup of Z. This implies that D is a Sylow p-subgroup of DZ. Hence $\mathrm{N}_G(DZ) = \mathrm{N}_G(D)Z$.

Since $(\overline{G}, \overline{N}, \overline{\theta}) \sim_b (\overline{H}, \overline{M}, \overline{\theta}')$ we have $\overline{G} = \overline{N}\,\overline{H}$, and, using Lemma 3.5, $\overline{H} = \overline{M}\mathrm{N}_{\overline{G}}(\overline{D})$. This implies $G = NH$ and $H = M\mathrm{N}_G(D)$. In particular, $\mathrm{C}_G(N) \leq \mathrm{C}_G(D) \leq H$ and $\mathrm{N}_N(D) \leq M$.

Let $\overline{\mathcal{P}}$ and $\overline{\mathcal{P}}'$ be projective representations of $\overline{G}$ and $\overline{H}$ associated to $\overline{\theta}$ and $\overline{\theta}'$ giving the block isomorphism of character triples

$$(\bar{\iota}, \overline{\sigma}) : (\overline{G}, \overline{N}, \overline{\theta}) \to (\overline{H}, \overline{M}, \overline{\theta}').$$

Then the factor sets of $\overline{\mathcal{P}}$ and $\overline{\mathcal{P}}'$ coincide and for every element $\overline{x} \in \mathrm{C}_{\overline{G}}(\overline{N})$ the scalar matrices $\overline{\mathcal{P}}(x)$ and $\overline{\mathcal{P}}'(x)$ are associated to the same scalar.

The projective representations of G and H defined by

$$\mathcal{P}(x) = \overline{\mathcal{P}}(\epsilon(x)) \text{ and } \mathcal{P}'(h) = \overline{\mathcal{P}}'(\epsilon(h)) \quad \text{for every } x \in G \text{ and } h \in H$$

are associated to θ and θ'. By hypothesis, they have equal factor sets. Since $\mathrm{C}_G(N)Z/Z \leq \mathrm{C}_{\overline{G}}(\overline{N})$, the matrices $\mathcal{P}(x)$ and $\mathcal{P}'(x)$ are associated to the same scalars for $x \in \mathrm{C}_G(N)$. By Lemma 3.3, we have shown that the projective representations $\mathcal{P}$ and $\mathcal{P}'$ determine a central isomorphism of character triples $(\iota, \sigma) : (G, N, \theta) \to (H, M, \theta')$. In order to show that (ι, σ) is also a block isomorphism of character triples it is sufficient to check that

$$\mathrm{bl}(\sigma_J(\psi))^J = \mathrm{bl}(\psi) \quad \text{for every } N \leq J \leq G \text{ and } \psi \in \mathrm{Irr}(J \mid \theta).$$

For $N \leq J \leq G$ and $\psi \in \mathrm{Irr}(J \mid \theta)$ let $\overline{J} := J/Z$ and $\overline{\psi} \in \mathrm{Irr}(\overline{J} \mid \overline{\theta})$ be such that ψ is the lift of $\overline{\psi}$. By the definition of (ι, σ) the character $\sigma_J(\psi)$ is a lift of $\sigma_{\overline{J}}(\overline{\psi})$. Then the block $\mathrm{bl}(\overline{\psi})$ is contained in $\mathrm{bl}(\psi)$ and $\mathrm{bl}(\overline{\sigma}_{\overline{J}}(\overline{\psi}))$ is contained in $\mathrm{bl}(\sigma_J(\psi))$. In such a situation Proposition 2.4(a) implies $\mathrm{bl}(\sigma_J(\psi))^J = \mathrm{bl}(\psi)$.

Finally, if θ and θ' have height zero, then we use Proposition 2.5(c) to finish the proof. □

A kind of converse of this result holds under additional assumptions on Z. Recall that if $H \leq G$, $Z \lhd G$ and $\theta \in \mathrm{Irr}(H)$ contains $H \cap Z$ in its kernel, then θ uniquely determines a character $\overline{\theta} \in \mathrm{Irr}(HZ/Z)$.

Lemma 3.13. *Suppose* $(G, N, \theta) \sim_b (H, M, \theta')$ *and* $Z \leq H$ *with* $Z \lhd G$ *and* $N \cap Z = 1$. *Let* $\overline{G} := G/Z$, $\overline{H} := H/Z$, $\overline{M} := MZ/Z$ *and* $\overline{N} := NZ/Z$. *Let* $\overline{\theta}$ *be the character of* $\overline{N}$ *associated to* θ *and let* $\overline{\theta}'$ *be the character of* $\overline{M}$ *associated to* θ'.

(a) *Assume that either* Z *is a* p'*-group or a central* p*-group. Then* $(\overline{G}, \overline{N}, \overline{\theta}) \sim_b (\overline{H}, \overline{M}, \overline{\theta}')$.
(b) *Assume that* Z *is central in* G. *Then* $(\overline{G}, \overline{N}, \overline{\theta}) \sim_b (\overline{H}, \overline{M}, \overline{\theta}')$.

Proof. For the proof of (a) let $\mathcal{P}$ and $\mathcal{P}'$ be the projective representations associated to θ and θ' that define the block isomorphism (ι, σ) between character triples. Now, consider the representation $\overline{\mathcal{X}}$ of $\overline{N}$ defined by $\overline{\mathcal{X}}(Zn) = \mathcal{P}(n)$ for $n \in N$. This representation affords $\overline{\theta} \in \mathrm{Irr}(\overline{N})$, and by Theorem 3.1, there exists a projective representation $\overline{\mathcal{P}}$ of $\overline{G}$ associated to $\overline{\theta}$ such that $\overline{\mathcal{P}}(Zn) = \mathcal{P}(n)$ for $n \in N$. Now, if we define $\mathcal{D}(g) = \overline{\mathcal{P}}(Zg)$ we check that $\mathcal{D}$ is a projective representation of G such that $\mathcal{D}_N = \mathcal{P}_N$. According to Theorem 3.1(b) there exists a map $\zeta : G/N \to \mathbb{C}^\times$ such that

$$\overline{\mathcal{P}}(gZ) = \zeta(gN)\mathcal{P}(g) \quad \text{for every } g \in G.$$

The projective representation $\zeta_H \mathcal{P}'$ is a projective representation of H associated to θ' as well. As the factor sets of $\mathcal{P}$ and $\mathcal{P}'$ coincide via the canonical isomorphism $\iota : G/N \to H/M$, the factor sets of $\zeta\mathcal{P}$ and $\zeta_H\mathcal{P}'$ coincide as well. We see that $\zeta\mathcal{P}$ and $\zeta_H\mathcal{P}'$ give the same isomorphism (ι, σ).

Now notice that $\mathrm{C}_{G/Z}(NZ/Z) = \mathrm{C}_G(N)/Z$, by using that $Z \lhd G$ and $N \cap Z = 1$. As $\mathcal{P}(x)$ and $\mathcal{P}'(x)$ are associated to the same scalar for $x \in \mathrm{C}_G(N)$, the projective representations $\zeta\mathcal{P}$ and $\zeta_H\mathcal{P}'$ have the same property. Taking into account that the factor sets of $\zeta\mathcal{P}$ and $\zeta_H\mathcal{P}'$ are equal, this implies that $\zeta_H\mathcal{P}'$ defines uniquely a projective representation $\overline{\mathcal{P}}'$ of H/N as well.

The isomorphism of character triples $(\overline{\iota}, \overline{\sigma}) : (\overline{G}, \overline{N}, \overline{\theta}) \sim_b (\overline{H}, \overline{M}, \overline{\theta}')$, defined as the one given by $\overline{\mathcal{P}}$ and $\overline{\mathcal{P}}'$, is the restriction of (ι, σ) to the characters having Z in their kernel.

By these considerations it is clear that $(\overline{G}, \overline{N}, \overline{\theta})$ and $(\overline{H}, \overline{M}, \overline{\theta}')$ are central isomorphic character triples. Let $NZ \leq J \leq G$ and $\overline{\psi} \in \mathrm{Irr}(\overline{J} \mid \overline{\theta})$ for $\overline{J} := J/Z$. By the definition of $\overline{\sigma}$ the character $\overline{\sigma}_{\overline{J}}(\overline{\psi})$ lifts to the character $\sigma_J(\psi)$ on $J \cap H$ where $\psi \in \mathrm{Irr}(J \mid \theta)$ is the lift of $\overline{\psi}$. According to Proposition 2.4(b) and (c) the equality $\mathrm{bl}(\sigma_J(\psi))^J = \mathrm{bl}(\psi)$ implies $\overline{b}^{J/Z} = \overline{B}$, where $\overline{b}$ is the block of $(J \cap H)/Z$ contained in $\mathrm{bl}(\sigma_J(\psi))$ and $\overline{B}$ the block of $\overline{G}$ dominating $\mathrm{bl}(\psi)$. According to Lemma 5.8.6 of [NT89] the characters $\overline{\sigma}_{J/Z}(\overline{\psi})$ and $\overline{\psi}$ belong to $\overline{b}$ and $\overline{B}$, respectively. Finally, notice that because of the isomorphism $N \to NZ/Z$, if D is a common defect group of $\mathrm{bl}(\theta)$ and $\mathrm{bl}(\theta')$, then DZ/Z is a defect group of $\mathrm{bl}(\overline{\theta})$ and of $\mathrm{bl}(\overline{\theta}')$. Also $\mathrm{N}_{\overline{N}}(\overline{D}) = \mathrm{N}_N(D)Z/Z$. This implies $(\overline{G}, \overline{N}, \overline{\theta}) \sim_b (\overline{H}, \overline{M}, \overline{\theta}')$ in the situation of (a).

For the proof of (b) let Z_p be the Sylow p-subgroup of Z. Then we apply part (a) to G with respect to Z_p, and afterwards to G/Z_p with respect to the subgroup Z/Z_p. □

To end this section, we prove the following statement which is often used in the context of Clifford correspondence.

Theorem 3.14 (Irreducible induction giving block isomorphic character triples). *Let $N \lhd G$, $H \leq G$ and $M = H \cap N$ be such that $G = NH$. Let $G_1 \leq G$, $H_1 := H \cap G_1$, $N_1 := N \cap G_1$ and $M_1 := M \cap G_1$ be such that $H_1 M = H$ and $G_1 N = G$. Suppose that $(G_1, N_1, \theta_1) \sim_b (H_1, M_1, \theta_1')$. Assume that for every J with $N \leq J \leq G$ induction gives a bijection between $\mathrm{Irr}(J \cap G_1 \mid \theta_1)$ and $\mathrm{Irr}(J \mid \theta)$, as well as between $\mathrm{Irr}(J \cap H_1 \mid \theta_1')$ and $\mathrm{Irr}(J \cap H \mid \theta')$, where $\theta = \theta_1^N$ and $\theta' = (\theta_1')^M$. Suppose that $\mathrm{ht}(\theta) - \mathrm{ht}(\theta_1) = \mathrm{ht}(\theta') - \mathrm{ht}(\theta_1')$ and assume that for some defect group Q of $\mathrm{bl}(\theta')$ the group $\mathrm{N}_N(Q)$ is contained in M. Then*

$$(G, N, \theta_1^N) \sim_b (H, M, (\theta_1')^M).$$

Proof. We argue by using Remark 3.7. We have $G = NH$ and $M = N \cap H$.

Let D be a common defect group of $\mathrm{bl}(\theta_1)$ and $\mathrm{bl}(\theta'_1)$. The block $\mathrm{bl}(\theta'_1)^M$ is defined and coincides with $\mathrm{bl}(\theta')$, by Corollary (6.2) of [Nav98]. Also by Lemma (4.13) of [Nav98] there exists a defect group Q of $\mathrm{bl}(\theta')$ with $D \leq Q$. As θ_1^N is irreducible the block $\mathrm{bl}(\theta_1)^N$ is defined. The block $\mathrm{bl}(\theta'_1)^M$ is defined as well. Hence by Lemma 5.3.4 of [NT89] the block $\mathrm{bl}(\theta')^N$ is defined and coincides with $\mathrm{bl}(\theta'_1)^N = \mathrm{bl}(\theta)$. Also, some defect group $\widetilde{Q}$ of $\mathrm{bl}(\theta)$ contains Q, that is, $Q \leq \widetilde{Q}$.

By the definition of height we have $|Q|\theta'(1)_p = |M|_p p^{\mathrm{ht}(\theta')}$ and $|\widetilde{Q}|\theta(1)_p = |N|_p p^{\mathrm{ht}(\theta)}$. Also, $|D|\theta'_1(1)_p = |M_1|_p p^{\mathrm{ht}(\theta')}$ and $|D|\theta'(1)_p = |N_1|_p p^{\mathrm{ht}(\theta')}$. Since $\theta'(1) = |M : M_1|\theta'_1(1)$ this implies

$$|Q : D| = \frac{|M|_p p^{\mathrm{ht}(\theta')}}{|M : M_1|_p \theta'_1(1)_p} \frac{\theta'_1(1)_p}{|M_1|_p p^{\mathrm{ht}(\theta'_1)}} = p^{\mathrm{ht}(\theta')-\mathrm{ht}(\theta'_1)}$$

and analogously $|\widetilde{Q} : D| = p^{\mathrm{ht}(\theta)-\mathrm{ht}(\theta_1)}$. Since $\mathrm{ht}(\theta) - \mathrm{ht}(\theta_1) = \mathrm{ht}(\theta') - \mathrm{ht}(\theta'_1)$ these equations prove $\widetilde{Q} = Q$. Hence θ and θ' have a common defect group Q and it satisfies $\mathrm{N}_N(Q) \leq M$.

Let $\mathcal{P}_1$ and $\mathcal{P}'_1$ be projective representations of G_1 and H_1 respectively associated to θ_1 and θ'_1 such that they define a block isomorphism of character triples

$$(\iota_1, \sigma_1) : (G_1, N_1, \theta_1) \to (H_1, M_1, \theta'_1),$$

where $\iota_1 : G_1/N_1 \to H_1/M_1$ is the canonical isomorphism.

Using the construction of (10.1) of [CR90] we can define a projective representation $\mathcal{Q}$ of G, induced from the projective representation $\mathcal{P}$. Let $n_1, \ldots, n_s \in N$ be representatives of the N_1-cosets in N. By our hypotheses, $n_1, \ldots, n_s$ are representatives of the G_1-cosets in G. Let $\widehat{\mathcal{P}}_{i,j}$ be the map on G defined by

$$\widehat{\mathcal{P}}_{i,j}(x) = \begin{cases} \mathcal{P}(n_i^{-1}xn_j), & n_i^{-1}xn_j \in G_1, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\mathcal{Q} : G \to \mathrm{GL}_{\theta(1)}(\mathbb{C})$ is defined as

$$\mathcal{Q}(x) = \begin{pmatrix} \widehat{\mathcal{P}}_{1,1}(x) & \ldots & \widehat{\mathcal{P}}_{1,s}(x) \\ \vdots & & \vdots \\ \widehat{\mathcal{P}}_{s,1}(x) & \ldots & \widehat{\mathcal{P}}_{s,s}(x) \end{pmatrix}.$$

Straightforward computations show that the factor sets of $\mathcal{Q}$ and $\mathcal{P}$ coincide via the isomorphism $G/N \cong G_1/N_1$, and hence $\mathcal{Q}$ is a projective representation of G associated to $\theta = \theta_1^N$. Analogously we obtain a projective representation $\mathcal{Q}'$ of H associated to $\theta' = (\theta'_1)^M$. As $\mathcal{P}$ and $\mathcal{P}'$ have the same factor sets, the factor sets of $\mathcal{Q}$ and $\mathcal{Q}'$ coincide.

Next we prove that $\mathrm{C}_G(N) \leq \mathrm{C}_{G_1}(N_1)$. Let $x \in \mathrm{C}_G(N)$. Every extension $\widetilde{\theta}$ of $\theta = \theta_1^N$ to $\langle N, x\rangle$ has a non-zero value on x since $x \in \mathrm{Z}(\langle N, x\rangle)$. For $\widetilde{\theta}$ there exists a character $\widetilde{\theta}_1$ of $\langle N, x\rangle \cap G_1$ with $(\widetilde{\theta}_1)_{N_1} = \theta_1$ and $\widetilde{\theta}_1^{\langle N,x\rangle} = \widetilde{\theta}$. By the definition of induced characters, x

has to be $\langle N, x\rangle$-conjugate to an element in $\langle N, x\rangle \cap G_1$. This implies $x \in G_1$ and hence $\mathrm{C}_G(N) \leq \mathrm{C}_{G_1}(N_1)$. By the definition of $\mathcal{P}$ and $\mathcal{P}'$ the matrices $\mathcal{P}(x)$ and $\mathcal{P}'(x)$ are associated to the same scalar. By the construction $\mathcal{Q}$ and $\mathcal{Q}'$ then have the same property. According to Lemma 3.3 this proves that $\mathcal{Q}$ and $\mathcal{Q}'$ determine a central isomorphism of character triples $(\iota, \sigma) : (G, N, \theta) \to (H, M, \theta')$, where $\iota : G/N \to H/M$ is the canonical isomorphism. Because of the definition of $\mathcal{Q}$ and $\mathcal{Q}'$ the bijections satisfy

$$\sigma_J(\psi_1^J) = (\sigma_{1,J_1}(\psi_1))^{J\cap H}$$

for every $\psi_1 \in \mathrm{Irr}(J_1 \mid \theta_1)$, $N \leq J \leq G$ and $J_1 := J \cap G_1$.

Let $N \leq J \leq G$, $\psi \in \mathrm{Irr}(J \mid \theta)$ and $\psi_1 \in \mathrm{Irr}(J \cap G_1 \mid \theta_1)$ with $\psi_1^J = \psi$. Then the block $\mathrm{bl}(\psi)$ coincides by the definition of σ with $\mathrm{bl}(\psi_1)^J$ by Lemma 5.3.1(ii) of [NT89]. As (σ_1, ι_1) gives a block isomorphism between character triples we have $\mathrm{bl}(\psi_1')^J = \mathrm{bl}(\psi_1)$, where $\psi_1' := \sigma_{1,J_1}(\psi_1)$. By Lemma 5.3.1(ii) of [NT89] the character $\psi' = \sigma_J(\psi) = (\sigma_{1,J_1}(\psi_1))^{J\cap H}$ belongs to the block $\mathrm{bl}(\psi_1')^{J\cap H}$, hence $\mathrm{bl}(\psi')^J = \mathrm{bl}(\psi_1)^J = \mathrm{bl}(\psi)$. Hence the isomorphism (ι, σ) has all the required properties. We conclude that $(G, N, \theta) \sim_b (H, M, \theta')$. □

4. Construction of block isomorphic character triples

In this section we go deeper into constructing block isomorphic character triples. The strategy is the following. First we analyse when two character triples (G, N, θ) and (H, M, θ') are block isomorphic in the easiest case where θ and θ' extend to characters of G and H, respectively. Then, using the theory of projective representations, we will analyze the general case. Let us first state in our language the standard tool that allows us to go to the extending case from the general case. If we have a canonical epimorphism $\epsilon : \widehat{G} \to G$ with kernel Z, a *Z-section* of ϵ is any map $\mathrm{rep} : G \to \widehat{G}$ such that $\epsilon \circ \mathrm{rep} = \mathrm{id}_G$ with $\mathrm{rep}(1) = 1$.

Theorem 4.1. *Let (G, K, ξ) be a character triple. Let $\mathcal{P}$ be a projective representation of G associated to ξ. Then there is a group $\widehat{G}$, a surjective homomorphism $\epsilon : \widehat{G} \to G$ with finite cyclic central kernel Z and a Z-section* $\mathrm{rep} : G \to \widehat{G}$ *of ϵ with the following properties:*

(a) *$\widehat{K} = K_0 \times Z$, where $\widehat{K} = \epsilon^{-1}(K)$, K_0 is isomorphic to K via the isomorphism $\epsilon_{K_0} : K_0 \to K$ and $K_0 \lhd \widehat{G}$. Also, the action of $\widehat{G}$ on K_0 coincides with the action of G on K via ϵ.*

(b) *The character $\xi_0 := \xi \circ \epsilon_{K_0} \in \mathrm{Irr}(K_0)$ extends to $\widehat{G}$. There exists a representation $\mathcal{D}$ of $\widehat{G}$ with*

$$\mathcal{D}(\mathrm{rep}(g)) = \mathcal{P}(g) \quad \textit{for every } g \in G$$

and this representation affords an extension $\widetilde{\xi}_0 \in \mathrm{Irr}(\widehat{G})$ of ξ_0. The Z-section rep : *$G \to \widehat{G}$ satisfies*

$$\mathrm{rep}(k) \in K_0 \quad \textit{and} \quad \mathrm{rep}(kg) = \mathrm{rep}(k)\,\mathrm{rep}(g) \quad \textit{for every } k \in K \textit{ and } g \in G.$$

(c) *The unique irreducible constituent ν of $(\widetilde{\xi}_0)_Z$ is faithful.*

(d) *If $K \leq N \leq G$ and $\widehat{N} = \epsilon^{-1}(N)$, then $\epsilon(\mathrm{C}_{\widehat{G}}(\widehat{N})) = \mathrm{C}_G(N)$.*

Suppose now that $(G, K, \xi) \sim_{\mathrm{c}} (H, M, \xi')$. *Let* $\mathcal{P}'$ *be a projective representation of* H *associated to* ξ' *such that the isomorphism of character triples determined by* $\mathcal{P}$ *and* $\mathcal{P}'$ *is central. Then:*

(e) $\widehat{M} = M_0 \times Z$, *where* $\widehat{M} = \epsilon^{-1}(M)$, $M_0 := K_0 \cap \epsilon^{-1}(M)$.

(f) *The character* $\xi'_0 := \xi' \circ \epsilon_{M_0} \in \mathrm{Irr}(M_0)$ *extends to* $\widehat{H} := \epsilon^{-1}(H)$. *There exists a representation* $\mathcal{D}'$ *of* $\widehat{H}$ *with*

$$\mathcal{D}'(\mathrm{rep}(h)) = \mathcal{P}'(h) \quad \text{for every } h \in H$$

and this representation affords an extension $\widetilde{\xi}'_0 \in \mathrm{Irr}(\widehat{H})$ *of* ξ'_0.

(g) $\{\nu\} = \mathrm{Irr}((\widetilde{\xi}'_0)_Z) = \mathrm{Irr}((\widetilde{\xi}_0)_Z)$.

(h) *The central isomorphism of character triples* $(\iota, \sigma) : (G, K, \xi) \to (H, M, \xi')$ *given by* $\mathcal{P}$ *and* $\mathcal{P}'$ *lifts to a central isomorphism of character triples* $(\widehat{\iota}, \widehat{\sigma}) : (\widehat{G}, K_0, \xi_0) \to (\widehat{H}, M_0, \xi'_0)$ *given by* $\mathcal{D}$ *and* $\mathcal{D}'$ *and further for every* $K \leq J \leq G$ *and* $\psi \in \mathrm{Irr}(J \mid \xi)$ *we have*

$$\sigma_J(\psi) \circ \epsilon = \widehat{\sigma}_{\widehat{J}}(\psi \circ \epsilon_{\widehat{J}}),$$

where $\widehat{J} := \epsilon^{-1}(J)$.

(i) *If* $(\widehat{G}, K_0, \xi_0) \sim_{\mathrm{b}} (\widehat{H}, M_0, \xi'_0)$ *via the isomorphism* $(\widehat{\iota}, \widehat{\sigma})$, *then* $(G, K, \xi) \sim_{\mathrm{b}} (H, M, \xi')$ *via* (ι, σ).

Proof. The construction of $\widehat{G}$ that we use can be found in the proof of Theorem (8.28) of [Nav98] and in the proof of Lemma (11.28) of [Isa76]. We fix a finite subgroup Z of $\mathbb{C}^\times$ that contains all the values of the factor set α associated with $\mathcal{P}$. We define the group $\widehat{G} = \{(g, z) \mid g \in G,\ z \in Z\}$, where multiplication is given by

$$(g_1, z_1)(g_2, z_2) = (g_1 g_2, z_1 z_2 \alpha(g_1, g_2)).$$

Let $\epsilon : \widehat{G} \to G$ be the epimorphism given by $(g, z) \mapsto g$, with kernel $1 \times Z \leq Z(\widehat{G})$, which we will identify with Z. We define $\mathrm{rep} : G \to \widehat{G}$ by $g \mapsto (g, 1)$. Then $K_0 := \{(k, 1) \mid k \in K\}$ is a normal subgroup of $\widehat{G}$ that is isomorphic to K via ϵ_{K_0}. Let $\xi_0 := \xi \circ \epsilon_{K_0} \in \mathrm{Irr}(K_0)$. The map $\mathcal{D}$ defined on $\widehat{G}$ by

$$\mathcal{D}((g, z)) = z\mathcal{P}(g) \quad \text{for every } z \in Z \quad \text{and} \quad g \in G,$$

is an irreducible representation of $\widehat{G}$. Let $\widetilde{\xi}_0 \in \mathrm{Irr}(\widehat{G} \mid \xi_0)$ be the character afforded by $\mathcal{D}$, an extension of ξ_0. This proves the first three statements.

If $U \leq G$, let us denote $\widehat{U} := \epsilon^{-1}(U) = \{(u, z) \mid u \in U, z \in Z\} \leq \widehat{G}$.

If $c \in \mathrm{C}_G(N) \leq \mathrm{C}_G(K)$, then the matrix $\mathcal{P}(c)$ is scalar by Schur's Lemma. Using this and that $\mathcal{P}(c) = \mathcal{P}(n^{-1}cn)$ for $n \in N$, we see that $\epsilon(\mathrm{C}_{\widehat{G}}(\widehat{N})) = \mathrm{C}_G(N)$, as claimed in (d).

Now, we prove the remaining parts, starting with (e). The factor set α' of $\mathcal{P}'$ coincides with α via ι. In particular the values of α' are also in Z, and therefore the subgroup $\widehat{H} := \epsilon^{-1}(H)$ of $\widehat{G}$ provides a central extension of H satisfying all the requirements, now for H, of the first three statements. In fact, the map $\mathcal{D}'$ defined on $\widehat{H}$ by

$$\mathcal{D}'((h, z)) = z\mathcal{P}'(h) \quad \text{for every } z \in Z \text{ and } h \in H$$

is an irreducible representation of $\widehat{H}$. If $\widetilde{\xi}_0' \in \mathrm{Irr}(\widehat{H} \mid \xi_0')$ is the character afforded by $\mathcal{D}'$, then it is an extension of ξ_0'. Also, $\{\nu\} = \mathrm{Irr}((\widetilde{\xi}_0')_Z) = \mathrm{Irr}((\widetilde{\xi}_0)_Z)$. We see that $\epsilon^{-1}(M) = M_0 \times Z$.

Now, using (d), we have $\mathrm{C}_{\widehat{G}}(K_0) = \mathrm{C}_{\widehat{G}}(\widehat{K}) = \widehat{\mathrm{C}_G(K)} \leq \widehat{H}$, because $\mathrm{C}_G(K) \leq H$, by hypothesis. Also it is clear that $\mathcal{D}(x,z)$ and $\mathcal{D}'(x,z)$ for $(x,z) \in \widehat{\mathrm{C}_G(K)}$ are associated to the same scalar (because $\mathcal{P}(x)$ and $\mathcal{P}'(x)$ are). Thus we see that $(\widehat{G}, K_0, \xi_0) \sim_{\mathrm{c}} (\widehat{H}, M_0, \xi_0')$.

Finally suppose that $(\widehat{G}, K_0, \xi_0) \sim_{\mathrm{b}} (\widehat{H}, M_0, \xi_0')$. Then $(G, K, \xi) \sim_{\mathrm{b}} (H, M, \xi')$ by Lemma 3.13(b). This proves that (ι, σ) is a block isomorphism of character triples. □

When checking if two character triples are block isomorphic, usually the most complicated task is to verify the equality on induced blocks between character correspondents. In the next results, ending up in Theorem 4.4, we try to facilitate that. First we start with the case where the characters extend, and later we shall use Theorem 4.1 to prove the general case. For a finite group G we denote by G^0 the set of p-regular elements in G.

Lemma 4.2. *Let $N \lhd G$ and $H \leq G$ with $NH = G$. Write $M = N \cap H$. Let $\theta \in \mathrm{Irr}(N)$ and assume that $\widetilde{\theta} \in \mathrm{Irr}(G)$ is an extension of θ. Let $\theta' \in \mathrm{Irr}(M)$ and assume that $\widetilde{\theta}' \in \mathrm{Irr}(H)$ is an extension of θ'. Suppose that D is a defect group of $\mathrm{bl}(\theta')$ and $\mathrm{bl}(\theta)$ with $\mathrm{N}_N(D) \leq M$.*

(a) *Suppose that $x \in G^0$ with $D \in \mathrm{Syl}_p(\mathrm{C}_N(x))$. If $J_1 := \langle N, x\rangle$, then*

$$\lambda_{\widetilde{\theta}_{J_1}}(\mathfrak{Cl}_{J_1}(x)^+) = \left(\frac{|N|_{p'}\widetilde{\theta}(x)}{|\mathrm{C}_N(x)|_{p'}\, p^{\mathrm{ht}(\theta)}\theta(1)_{p'}}\right)^*.$$

In particular, $\widetilde{\theta}(x)/p^{\mathrm{ht}(\theta)}$ is a local integer. Also, $|\mathrm{C}_J(x)|_{p'} \equiv |\mathrm{C}_{J\cap H}(x)|_{p'} \bmod p$ for every $N \leq J \leq G$.

(b) $\mathrm{bl}(\widetilde{\theta}'_{J\cap H})^J = \mathrm{bl}(\widetilde{\theta}_J)$ *for every $N \leq J \leq G$ if and only if*

$$\left(\frac{|N|_{p'}\widetilde{\theta}(x)}{p^{\mathrm{ht}(\theta)}\theta(1)_{p'}}\right)^* = \left(\frac{|M|_{p'}\widetilde{\theta}'(x)}{p^{\mathrm{ht}(\theta')}\theta'(1)_{p'}}\right)^* \tag{4.1}$$

for every $x \in H^0$ with $D \in \mathrm{Syl}_p(\mathrm{C}_N(x))$.

Proof. By Lemma 3.5, we know that $\mathrm{N}_G(D) \leq H$. Let $x \in G^0$ be such that $D \in \mathrm{Syl}_p(\mathrm{C}_N(x))$, and let $J_1 := \langle N, x\rangle$. Notice that $x \in H$. Since J_1/N is a p'-group, we have $D \in \mathrm{Syl}_p(\mathrm{C}_{J_1}(x))$. Also notice that $|\mathrm{C}_{J_1}(x)|/|\mathrm{C}_N(x)| = |J_1/N|$. By the definition of height, $|D|\theta(1)_p = |N|_p p^{\mathrm{ht}(\theta)}$. Now

$$\begin{aligned}\lambda_{\widetilde{\theta}_{J_1}}(\mathfrak{Cl}_{J_1}(x)^+)^* &= \left(\frac{|J_1|\widetilde{\theta}(x)}{|\mathrm{C}_{J_1}(x)|\theta(1)}\right)^* = \left(\frac{|J_1|_p|J_1|_{p'}\widetilde{\theta}(x)}{|\mathrm{C}_{J_1}(x)|_{p'}|D|\theta(1)}\right)^* = \\ &= \left(\frac{|N|_p|J_1/N|\,|N|_{p'}\widetilde{\theta}(x)}{|\mathrm{C}_N(x)|_{p'}|J_1/N|\,|D|\theta(1)}\right)^* = \left(\frac{|N|_{p'}\widetilde{\theta}(x)}{|\mathrm{C}_N(x)|_{p'}\, p^{\mathrm{ht}(\theta)}\theta(1)_{p'}}\right)^*.\end{aligned}$$

Now suppose that $N \leq J \leq G$ and let $Q \in \mathrm{Syl}_p(\mathrm{C}_J(x))$ be such that $D = Q \cap \mathrm{C}_N(x)$. Hence $\mathrm{N}_{\mathrm{C}_J(x)}(Q) \leq \mathrm{N}_G(D) \leq H$. Therefore $\mathrm{N}_{\mathrm{C}_J(x)}(Q) = \mathrm{N}_{\mathrm{C}_{J\cap H}(x)}(Q)$. By Sylow theory we know that $|\mathrm{C}_J(x) : \mathrm{N}_{\mathrm{C}_J(x)}(Q)| \equiv 1 \bmod p$, and $|\mathrm{C}_{J\cap H}(x) : \mathrm{N}_{\mathrm{C}_{J\cap H}(x)}(Q)| \equiv 1 \bmod p$. This shows

$$|\mathrm{C}_J(x)|_{p'} \equiv |\mathrm{C}_{J\cap H}(x)|_{p'} \bmod p.$$

This proves (a).

Now, suppose that $x \in H^0$ with $D \in \mathrm{Syl}_p(\mathrm{C}_N(x))$. Set $J_1 := \langle N, x\rangle$. By (a),

$$\lambda_{\widetilde{\theta}_{J_1}}(\mathfrak{Cl}_{J_1}(x)^+) = \left(\frac{|N|_{p'}\widetilde{\theta}(x)}{|\mathrm{C}_N(x)|_{p'}\, p^{\mathrm{ht}(\theta)}\theta(1)_{p'}}\right)^*.$$

Since $D \in \mathrm{Syl}_p(\mathrm{C}_M(x))$, we apply (a) to $\widetilde{\theta}'$ to get

$$\lambda_{\widetilde{\theta}'_{J_1\cap H}}(\mathfrak{Cl}_{J_1\cap H}(x)^+)^* = \left(\frac{|M|_{p'}\widetilde{\theta}'(x)}{|\mathrm{C}_M(x)|_{p'}\, p^{\mathrm{ht}(\theta')}\theta'(1)_{p'}}\right)^*.$$

By (a), we have

$$|\mathrm{C}_N(x)|_{p'} \equiv |\mathrm{C}_M(x)|_{p'} \bmod p.$$

Hence the equation in part (b) is equivalent to

$$\lambda_{\widetilde{\theta}_{J_1}}(\mathfrak{Cl}_{J_1}(x)^+) = \lambda_{\widetilde{\theta}'_{J_1\cap H}}(\mathfrak{Cl}_{J_1\cap H}(x)^+). \tag{4.2}$$

Let $N \leq J \leq G$. Now suppose that $x \in J^0$ is such that $Q \in \mathrm{Syl}_p(\mathrm{C}_J(x))$, $Q \cap N = D$ and $QN/N \in \mathrm{Syl}_p(J/N)$. Hence $|\mathrm{C}_J(x)|_p = |Q|$ and $|Q/D|_p = |J/N|_p$. Thus $|J|_p = |Q|\,|N|_p/|D|$. We get

$$\begin{aligned}
\lambda_{\widetilde{\theta}_J}(\mathfrak{Cl}_J(x)^+) &= \left(\frac{|J|\widetilde{\theta}(x)}{|\mathrm{C}_J(x)|\theta(1)}\right)^* = \left(\frac{|J|_p}{|Q|\theta(1)_p}\cdot\frac{|J|_{p'}\widetilde{\theta}(x)}{|\mathrm{C}_J(x)|_{p'}\theta(1)_{p'}}\right)^*\\
&= \left(\frac{|J|_{p'}\widetilde{\theta}(x)}{|\mathrm{C}_J(x)|_{p'}\, p^{\mathrm{ht}(\theta)}\theta(1)_{p'}}\right)^* = \left(\frac{|J/N|_{p'}}{|\mathrm{C}_J(x)|_{p'}}\right)^*\left(\frac{|N|_{p'}\widetilde{\theta}(x)}{p^{\mathrm{ht}(\theta)}\theta(1)_{p'}}\right)^*\\
&= \left(\frac{|J/N|_{p'}}{|\mathrm{C}_J(x)|_{p'}/|\mathrm{C}_N(x)|_{p'}}\right)^*\lambda_{\widetilde{\theta}_{J_1}}(\mathfrak{Cl}_{J_1}(x)^+)^*, \quad \text{where } J_1 = \langle N, x\rangle.
\end{aligned}$$

Since $Q \leq \mathrm{N}_G(D) \leq H$, $Q \cap M = D$ and $QM/M \in \mathrm{Syl}_p((J\cap H)/M)$, analogously we also have

$$\lambda_{\widetilde{\theta}'_{J\cap H}}(\mathfrak{Cl}_{J\cap H}(x)^+) = \left(\frac{|(J\cap H)/M|_{p'}}{|\mathrm{C}_{J\cap H}(x)|_{p'}/|\mathrm{C}_M(x)|_{p'}}\right)^*\lambda_{\widetilde{\theta}'_{J_1\cap H}}(\mathfrak{Cl}_{J_1\cap H}(x)^+).$$

By the second part of (a) we have

$$|\mathrm{C}_{J\cap H}(x)|_{p'} \equiv |\mathrm{C}_J(x)|_{p'} \bmod p.$$

Together with (4.2) this proves that for every $x \in J^0$ such that $Q \in \mathrm{Syl}_p(\mathrm{C}_J(x))$, $Q \cap N = D$ and $QN/N \in \mathrm{Syl}_p(J/N)$, we have

$$\lambda_{\widetilde{\theta}_J}(\mathfrak{Cl}_J(x)^+) = \lambda_{\widetilde{\theta}'_{J\cap H}}(\mathfrak{Cl}_{J\cap H}(x)^+)$$

if and only if

$$\lambda_{\widetilde{\theta}_{J_1}}(\mathfrak{Cl}_{J_1}(x)^+) = \lambda_{\widetilde{\theta}'_{J_1\cap H}}(\mathfrak{Cl}_{J_1\cap H}(x)^+)$$

where $J_1 := \langle N, x\rangle$.

Now we prove that the second statement implies the first in (b). Suppose that Q is a defect group of $\mathrm{bl}(\widetilde{\theta}_J)$ which we can choose such that $Q \cap N = D$ by Theorem (9.26) of [Nav98]. By Proposition 2.5(d), QN/N is a Sylow p-subgroup of J/N. By the definition of defect groups and defect classes, there exists $x_0 \in J^0$ with $Q \in \mathrm{Syl}_p(\mathrm{C}_J(x_0))$ and

$$0 \neq \lambda_{\widetilde{\theta}_J}(\mathfrak{Cl}_J(x_0)^+) = \lambda_{\widetilde{\theta}'_{J\cap H}}(\mathfrak{Cl}_{J\cap H}(x_0)^+).$$

By the Min-Max Theorem (4.4) of [Nav98] this implies that some defect group $\widetilde{Q}$ of $\mathrm{bl}(\widetilde{\theta}'_{J\cap H})$ is contained in Q. By Proposition 2.5(d) the groups Q/D and $\widetilde{Q}/D$ are isomorphic to a Sylow p-subgroup of J/N. This proves that $\widetilde{Q} = Q$. Let $c, c' \in \mathrm{Bl}(\mathrm{N}_J(Q))$ be the Brauer correspondents of $\mathrm{bl}(\widetilde{\theta}_J)$ and $\mathrm{bl}(\widetilde{\theta}'_{J\cap H})$. Notice that $\mathrm{N}_G(Q) \leq \mathrm{N}_G(D) \leq H$. For every $x \in J^0$ with $Q \in \mathrm{Syl}_p(\mathrm{C}_J(x))$, we have $\mathfrak{Cl}_J(x) \cap \mathrm{C}_J(Q) = \mathfrak{Cl}_{J\cap H}(x) \cap \mathrm{C}_J(Q)$ by Lemma (4.16) of [Nav98]. Then λ_c and $\lambda_{c'}$ coincide on all p-regular classes of $\mathrm{C}_J(Q)$ with defect group Q, again by Lemma (4.16) of [Nav98]. As additionally c and c' have the same defect group Q, the blocks c and c' coincide by Exercise (4.5) of [Nav98]. This implies

$$\mathrm{bl}(\widetilde{\theta}'_{J\cap H})^J = \mathrm{bl}(\widetilde{\theta}_J).$$

Finally, we prove that the second statement implies the first in (b). Let $x \in H^0$ with $D \in \mathrm{Syl}_p(\mathrm{C}_N(x))$, and let $J_1 := \langle N, x\rangle$, so that J_1/N is a p'-group. By hypothesis, we have $\mathrm{bl}(\widetilde{\theta}'_{J_1\cap H})^{J_1} = \mathrm{bl}(\widetilde{\theta}_{J_1})$. By Lemma 3.5, D is a common defect group of $\mathrm{bl}(\widetilde{\theta}'_{J_1\cap H})$ and $\mathrm{bl}(\widetilde{\theta}_{J_1})$. Now, let c be the block of $\mathrm{N}_{J_1}(D)$ which is the Brauer correspondent of $\mathrm{bl}(\widetilde{\theta}'_{J_1\cap H})$ and $\mathrm{bl}(\widetilde{\theta}_{J_1})$. Then $X = \mathfrak{Cl}_{J_1}(x) \cap \mathrm{C}_{J_1}(D) = \mathfrak{Cl}_{J_1\cap H}(x) \cap \mathrm{C}_{J_1}(D)$ by Lemma (4.16) of [Nav98]. Now clearly $\lambda_{\widetilde{\theta}_{J_1}}(\mathfrak{Cl}_{J_1}(x)^+) = \lambda_{\widetilde{\theta}'_{J_1\cap H}}(\mathfrak{Cl}_{J_1\cap H}(x)^+)$, since both coincide with $\lambda_c(X^+)$. □

Lemma 4.3. *Let (G, N, θ) be a character triple and let $\mathcal{P}$ be a projective representation of G associated to θ. Let D be a defect group of* $\mathrm{bl}(\theta)$. *Then* $\mathrm{tr}(\mathcal{P}(x))/p^{\mathrm{ht}(\theta)}$ *is a local integer for every $x \in G$ with $D \in \mathrm{Syl}_p(\mathrm{C}_N(x))$.*

Proof. Let $x \in G$ with $D \in \mathrm{Syl}_p(\mathrm{C}_N(x))$. As the factor set of $\mathcal{P}$ has finite order, the matrix $\mathcal{P}(x)$ has finite order as well. For a representation $\widetilde{\mathcal{D}}$ of $\langle N, x\rangle$ with $\widetilde{\mathcal{D}}_N = \mathcal{P}_N$ the matrices $\widetilde{\mathcal{D}}(x)$ and $\mathcal{P}(x)$ differ by a scalar ζ. As both have finite order, this scalar ζ is a finite root of unity. Now apply Lemma 4.2(a) to the character $\widetilde{\theta}$ afforded by $\widetilde{\mathcal{D}}$, with $H = G = \langle N, x\rangle$. □

Using Lemma 4.2 one can give a characterization of block isomorphic character triples, similar to the one given in Lemma 3.3 for central isomorphic character triples.

Theorem 4.4. *Let (G, N, θ) and (H, M, θ') be two character triples with $H \leq G$, $NH = G$ and $N \cap H = M$. Suppose there is a common defect group D of both $\mathrm{bl}(\theta)$ and $\mathrm{bl}(\theta')$ with $\mathrm{N}_N(D) \leq M$. Let $\mathcal{P}$ and $\mathcal{P}'$ be projective representations associated to θ and θ'. Then the following two statements are equivalent:*

(i) *The projective representations $\mathcal{P}$ and $\mathcal{P}'$ of G and H satisfy:*

  (a) *the factor sets of $\mathcal{P}$ and $\mathcal{P}'$ coincide on H,*
  (b) *for every $x \in \mathrm{C}_G(N)$ the matrices $\mathcal{P}(x)$ and $\mathcal{P}'(x)$ are scalar matrices associated with the same scalar, and*
  (c) $\left(\frac{|N|_{p'}\,\mathrm{tr}(\mathcal{P}(x))}{p^{\mathrm{ht}(\theta)}\theta(1)_{p'}}\right)^* = \left(\frac{|M|_{p'}\,\mathrm{tr}(\mathcal{P}'(x))}{p^{\mathrm{ht}(\theta')}\theta'(1)_{p'}}\right)^*$ *for every $x \in H^0$ with $D \in \mathrm{Syl}_p(\mathrm{C}_N(x))$.*

(ii) $(G, N, \theta) \sim_{\mathrm{b}} (H, M, \theta')$ *via an isomorphism of character triples determined by $\mathcal{P}$ and $\mathcal{P}'$.*

Proof. By Lemma 3.5, notice that $H = M\mathrm{N}_G(D)$. In particular, $D \in \mathrm{Syl}_p(\mathrm{C}_N(x))$ if and only if $D \in \mathrm{Syl}_p(\mathrm{C}_M(x))$, and $\mathrm{C}_G(N) \leq H$. Note that by Lemma 4.3 the elements of $\mathbb{F}$ occurring in the equation of 4.4(i)(c) are well-defined.

First we prove that (i) implies (ii). By Lemma 3.3 the projective representations $\mathcal{P}$ and $\mathcal{P}'$ determine a central isomorphism $(\iota, \sigma) : (G, N, \theta) \to (H, M, \theta')$ of character triples. By assumption, θ and θ' have a common defect group D with $M \geq \mathrm{N}_N(D)$.

Now, by Theorem 4.1 the projective representation $\mathcal{P}$ with factor set α determines a group $\widehat{G} = \{(g, z) \mid g \in G, z \in Z\}$, where $Z \lhd \widehat{G}$ is a cyclic subgroup of $\mathbb{C}^\times$ containing the values of the factor set of $\mathcal{P}$. We see that N is naturally isomorphic to $N_0 = \{(n, 1) \mid n \in N\} \lhd \widehat{G}$, and let $\theta_0 \in \mathrm{Irr}(N_0)$ be the character corresponding to θ. Also, let $\widetilde{\theta}_0 \in \mathrm{Irr}(\widehat{G})$ be the extension of $\theta_0 \in \mathrm{Irr}(N_0)$ afforded by the representation of $\widehat{G}$ given by

$$\mathcal{D}((g, z)) = z\mathcal{P}(g) \quad \text{for every } g \in G, z \in Z.$$

Also, it is easy to check that

$$(n, z)^{(g, z_1)} = (n^g, z) \quad \text{for every } g \in G, z, z_1 \in Z, n \in N.$$

Since the factor set α' of $\mathcal{P}'$ coincides with α on H, we use Theorem 4.1 and we know that $(\widehat{G}, N_0, \theta_0) \sim_{\mathrm{c}} (\widehat{H}, M_0, \theta_0')$, where $\widehat{H} = \{(h, z) \mid h \in H, z \in Z\}$, $M_0 = \{(m, 1) \mid m \in M\}$ and $\theta_0' \in \mathrm{Irr}(M_0)$ is the character of M_0 corresponding to θ'. The character $\widetilde{\theta}_0'$ of $\widetilde{H}$ afforded by the representation

$$\mathcal{D}'((h, z)) = z\mathcal{P}'(h) \quad \text{for every } h \in H, z \in Z$$

extends θ_0'.

Let $(\widehat{\iota}, \widehat{\sigma}) : (\widehat{G}, N_0, \theta_0) \to (\widehat{H}, M_0, \theta_0')$ be the central isomorphism of character triples determined by $\widetilde{\theta}_0$ and $\widetilde{\theta}_0'$. By Theorem 4.1(i) it is sufficient to prove that $(\widehat{\iota}, \widehat{\sigma})$ is a block isomorphism of character triples, as this implies $(G, N, \theta) \sim_{\mathrm{b}} (H, M, \theta')$.

We do this by checking the conditions in Remark 3.7. We do have $\widehat{G} = N_0\widehat{H}$ and $N_0 \cap \widehat{H} = M_0$. Also since $D \leq M$ is a defect group of $\mathrm{bl}(\theta)$ and $\mathrm{bl}(\theta')$, by the natural isomorphism, we see that $D_0 = \{(d, 1) \mid d \in D\}$ is a common defect group of $\mathrm{bl}(\theta_0)$ and $\mathrm{bl}(\theta_0')$. Because of $\mathrm{N}_N(D) \leq M$ the group $\mathrm{N}_{N_0}(D_0) = \mathrm{N}_N(D) \times 1$ is contained in M_0.

By using Lemma 4.2(b), we only need to verify (4.1) for those $(x, z) \in \widehat{H}^0$ with $D_0 \in \mathrm{Syl}_p(\mathrm{C}_{N_0}(x, z))$. We easily check that $\mathrm{C}_{N_0}(x, z) = \mathrm{C}_N(x) \times Z$, and therefore $D \in \mathrm{Syl}_p(\mathrm{C}_N(x))$. Also, it is clear that $x \in H^0$. Now the equation in (c) multiplied by z^* gives us (4.1), and hence $\mathrm{bl}((\widetilde{\theta}_0')_{J\cap\widehat{H}})^J = \mathrm{bl}((\widetilde{\theta}_0)_J)$ for every $N_0 \leq J \leq \widehat{G}$. By Lemma 3.11 this implies that $(\widehat{\iota}, \widehat{\sigma})$ is a block isomorphism of character triples.

We now prove that (ii) implies (i). Let $\mathcal{P}$ and $\mathcal{P}'$ be projective representations of G associated to θ and θ', that give the block isomorphism $(\iota, \sigma) : (G, N, \theta) \to (H, M, \theta')$ of character triples. According to Lemma 3.3 the projective representations $\mathcal{P}$ and $\mathcal{P}'$ have the properties described in (a) and (b). It remains to verify the equation in (c) for every $x \in H^0$ with $D \in \mathrm{Syl}_p(\mathrm{C}_N(x))$. Let $J := \langle N, x\rangle$ and let $\widetilde{\theta} \in \mathrm{Irr}(J \mid \theta)$ be an extension of θ. Let $\mathcal{Q}$ be the projective representation of J/N whose factor set is the inverse of $\mathcal{P}_J$ and such that $\mathcal{Q} \otimes \mathcal{P}_J$ affords the character $\widetilde{\theta}$. Let $\widetilde{\theta}' = \sigma_J(\widetilde{\theta}) = \mathrm{tr}(\mathcal{Q} \otimes \mathcal{P}'_{J\cap H})$, which is an extension of θ'. By hypothesis, we have $\mathrm{bl}(\widetilde{\theta}) = \mathrm{bl}(\widetilde{\theta}')^J$, and the same happens for every subgroup $N \leq J_1 \leq J$. By Lemma 4.2(b), we have

$$\left(\frac{|N|_{p'}\widetilde{\theta}(x)}{p^{\mathrm{ht}(\theta)}\theta(1)_{p'}}\right)^* = \left(\frac{|M|_{p'}\widetilde{\theta}'(x)}{p^{\mathrm{ht}(\theta')}\theta'(1)_{p'}}\right)^*.$$

Now $\widetilde{\theta}(x) = \mathrm{tr}(\mathcal{P}(x))\,\mathrm{tr}(\mathcal{Q}(x))$ and $\widetilde{\theta}'(x) = \mathrm{tr}(\mathcal{P}'(x))\,\mathrm{tr}(\mathcal{Q}(x))$. As $\mathcal{Q}$ is one-dimensional and hence $\mathrm{tr}(\mathcal{Q}(x)) \neq 0$, this finishes the proof of the theorem. □

A useful observation when applying the previous theorem is the following: If $D \leq \mathrm{N}_N(D) \leq M$, then $D \in \mathrm{Syl}_p(\mathrm{C}_N(x))$ if and only if $D \in \mathrm{Syl}_p(\mathrm{C}_M(x))$. This follows from elementary group theory.

In addition to Lemma 3.13 we derive the following lemma that concerns quotients lying in the kernel of characters.

Corollary 4.5. *Suppose that* $(G, N, \theta) \sim_b (H, M, \theta')$, *where* θ *and* θ' *are characters of height zero. Let* $Z \leq \mathrm{Z}(G) \cap \ker(\theta) \cap \ker(\theta')$ *with* $\mathrm{C}_G(N)/Z = \mathrm{C}_{G/Z}(N/Z)$. *Then* $(G/Z, N/Z, \overline{\theta}) \sim_b (H/Z, M/Z, \overline{\theta}')$, *where* $\overline{\theta} \in \mathrm{Irr}(N/Z)$ *and* $\overline{\theta}' \in \mathrm{Irr}(M/Z)$ *are the associated characters of the quotients.*

Proof. If D is a common defect group of the blocks of θ and θ', then DZ/Z is the defect group of $\mathrm{bl}(\overline{\theta})$ and $\mathrm{bl}(\overline{\theta}')$ by Proposition 2.5(c). Since the Sylow p-subgroup Z_p of Z is contained in $\mathrm{O}_p(N) \leq D$, we have $DZ = D \times Z_{p'}$ and $\mathrm{N}_{G/Z}(DZ/Z) = \mathrm{N}_G(D)/Z$.

Let $\mathcal{P}$ and $\mathcal{P}'$ be the projective representations associated to θ and θ' and giving the block isomorphism $(\iota, \sigma) : (G, N, \theta) \to (H, M, \theta')$. By Theorem 3.1(a), these projective representations are constant on Z-cosets and hence define also projective representations $\overline{\mathcal{P}}$ and $\overline{\mathcal{P}}'$ associated to $\overline{\theta}$ and $\overline{\theta}'$.

As $\mathcal{P}$ and $\mathcal{P}'$ have the same factor sets, the factor sets of $\overline{\mathcal{P}}$ and $\overline{\mathcal{P}}'$ coincide. Also, since $\mathrm{C}_G(N)/Z = \mathrm{C}_{G/Z}(N/Z)$ the matrices $\overline{\mathcal{P}}(x)$ and $\overline{\mathcal{P}}'(x)$ are associated to the same scalar for every $x \in \mathrm{C}_{G/Z}(N/Z)$.

Let $yZ \in (H/Z)^0$ with $DZ/Z \in \mathrm{Syl}_p(\mathrm{C}_{N/Z}(yZ))$ and $y \in H^0 \cap yZ$. Now, DZ/Z is centralized by y, so D/Z_p is centralized by y. Since $[Z_p, y] = 1$ and y is p-regular, we find that y centralizes D by coprime action. So $D \leq \mathrm{C}_N(y)$. Since $\mathrm{C}_N(y)Z/Z \leq \mathrm{C}_{N/Z}(yZ)$, we easily conclude that $D \in \mathrm{Syl}_p(\mathrm{C}_N(y))$. Now, by Theorem 4.4,

$$\left(\frac{|N|_{p'}\,\mathrm{tr}(\mathcal{P}(y))}{\theta(1)_{p'}}\right)^* = \left(\frac{|M|_{p'}\,\mathrm{tr}(\mathcal{P}'(y))}{\theta'(1)_{p'}}\right)^*.$$

Since $\mathcal{P}(y) = \overline{\mathcal{P}}(yZ)$ and $\mathcal{P}'(y) = \overline{\mathcal{P}}'(yZ)$ we have

$$\left(\frac{|N/Z|_{p'}\,\mathrm{tr}(\overline{\mathcal{P}}(yZ))}{\theta(1)_{p'}}\right)^* = \left(\frac{|M/Z|_{p'}\,\mathrm{tr}(\overline{\mathcal{P}}'(yZ))}{\theta'(1)_{p'}}\right)^*.$$

The projective representations $\overline{\mathcal{P}}$ and $\overline{\mathcal{P}}'$ have all required properties from Theorem 4.4 and hence $(G/Z, N/Z, \overline{\theta}) \sim_{\mathrm{b}} (H/Z, M/Z, \overline{\theta}')$. □

Theorem 4.6 (Block isomorphic character triples via multiplication). *Let $N \lhd G$, $H \leq G$ with $G = NH$. Let $K \lhd G$ with $K \leq M := H \cap N$, and let $\zeta \in \mathrm{Irr}(N)$ be a G-invariant height zero character such that ζ_K is irreducible. Let $\overline{G} := G/K$, $\overline{N} := N/K$, $\overline{H} := H/K$ and $\overline{M} := M/K$. Suppose that $(\overline{G}, \overline{N}, \overline{\rho}) \sim_{\mathrm{b}} (\overline{H}, \overline{M}, \overline{\rho}')$. Let $\rho \in \mathrm{Irr}(N)$ and $\rho' \in \mathrm{Irr}(M)$ be the characters associated to $\overline{\rho}$ and $\overline{\rho}'$ containing K in their kernel. Assume that ρ, ρ' and $\tau := \zeta\rho$ have height zero. Then*

$$(G, N, \tau) \sim_{\mathrm{b}} (H, M, \tau'),$$

where $\tau' = \rho'\zeta_M$.

Proof. Suppose that $\overline{D}$ is a common defect group of the blocks of $\overline{\rho}$ and $\overline{\rho}'$ with $\mathrm{N}_{\overline{N}}(\overline{D}) \leq \overline{M}$. By Proposition 2.5(c) let D be a defect group of $\mathrm{bl}(\tau)$ such that $\overline{D} = DK/K$. Let $\epsilon : G \to \overline{G} := G/K$ be the canonical epimorphism.

As a first step in the proof we show that $\mathrm{bl}(\tau)$ and $\mathrm{bl}(\tau')$ have a common defect group. Since τ and τ' lie over ξ_K, by Theorem (9.26) of [Nav98] we can choose a defect group Q of $\mathrm{bl}(\tau')$ such that $D \cap K = Q \cap K$. According to Proposition 2.5(b) the group QK/K contains a defect group of $\mathrm{bl}(\overline{\rho}')$, hence $|QK/K| \geq |DK/K|$.

As τ has defect group D there exists an element $x \in N^0$ with $D \in \mathrm{Syl}_p(\mathrm{C}_N(x))$ such that $\lambda_\tau(\mathfrak{Cl}_N(x)^+) \neq 0$. By Lemma 2.2,

$$\lambda_\tau(\mathfrak{Cl}_N(x)^+) = \lambda_{\zeta_L}(\mathfrak{Cl}_L(x)^+)\lambda_{\overline{\rho}}(\mathfrak{Cl}_{\overline{N}}(\overline{x})^+),$$

where $L/K = \mathrm{C}_{N/K}(\overline{x})$ with $\overline{x} := xK$. Thus $\lambda_{\zeta_L}(\mathfrak{Cl}_L(x)^+) \neq 0 \neq \lambda_{\overline{\rho}}(\mathfrak{Cl}_{\overline{N}}(\overline{x})^+)$. Since $\lambda_{\zeta_L}(\mathfrak{Cl}_L(x)^+) \neq 0$, by the Min-Max Theorem (4.4) of [Nav98], $D \in \mathrm{Syl}_p(\mathrm{C}_L(x))$ contains a defect group of $\mathrm{bl}(\zeta_L)$. Hence DK/K is a Sylow p-subgroup of L/K by Proposition 2.5(d), and therefore $\mathfrak{Cl}_{\overline{N}}(\overline{x})$ has $\overline{D}$ as defect group. Since $\overline{D} \leq \mathrm{C}_{\overline{N}}(\overline{x})$, we have $\overline{x} \in \mathrm{N}_{\overline{N}}(\overline{D}) \leq \overline{M}$. Now $\mathrm{bl}(\overline{\rho}')^{\overline{N}} = \mathrm{bl}(\overline{\rho})$, these blocks have a common Brauer correspondent, and we easily check that $\lambda_{\overline{\rho}'}(\mathfrak{Cl}_{\overline{M}}(\overline{x})^+) = \lambda_{\overline{\rho}}(\mathfrak{Cl}_{\overline{N}}(\overline{x})^+) \neq 0$.

Let $L_1 := L \cap M$. We compute $\lambda_{\zeta_{L_1}}(\mathfrak{Cl}_{L_1}(x)^+)$. As $\overline{D}$ is a Sylow p-subgroup of L/K and since $\overline{D} \leq \overline{M}$, we deduce that D is contained in $L_1 = M \cap L$ and the integer $|L : L_1|$

is a p'-number. On the other hand, D is a Sylow p-subgroup of $\mathrm{C}_L(x)$ and $\mathrm{C}_{L_1}(x)$. This implies that $|\mathfrak{Cl}_L(x)|/|\mathfrak{Cl}_{L_1}(x)|$ is a p'-number. Now

$$\begin{aligned}\lambda_{\zeta_{L_1}}(\mathfrak{Cl}_{L_1}(x)^+) &= \left(\frac{|\mathfrak{Cl}_{L_1}(x)|}{|\mathfrak{Cl}_L(x)|}\frac{|\mathfrak{Cl}_L(x)|\zeta(x)}{\zeta(1)}\right)^* = \left(\frac{|\mathfrak{Cl}_{L_1}(x)|}{|\mathfrak{Cl}_L(x)|}\right)^*\left(\frac{|\mathfrak{Cl}_L(x)|\zeta(x)}{\zeta(1)}\right)^* \\ &= \left(\frac{|\mathfrak{Cl}_{L_1}(x)|}{|\mathfrak{Cl}_L(x)|}\right)^*\lambda_{\zeta_L}(\mathfrak{Cl}_L(x)^+) \neq 0\end{aligned}$$

Thus $\lambda_{\zeta_{L_1}}(\mathfrak{Cl}_{L_1}(x)^+)\lambda_{\overline{\rho}'}(\mathfrak{Cl}_{\overline{M}}(\overline{x})^+) \neq 0$ and by Lemma 2.2, $\lambda_{\tau'}(\mathfrak{Cl}_M(x)^+) \neq 0$. Hence, by the Min-Max Theorem, some M-conjugate of Q is contained in D. Therefore $|QK/K| \leq |DK/K|$ and we conclude that $|QK/K| = |DK/K|$. Then $|Q : Q\cap K| = |D : D\cap K|$. Since $Q \cap K = D \cap K$, we conclude that $|Q| = |D|$, and therefore D is a defect group of both $\mathrm{bl}(\tau)$ and $\mathrm{bl}(\tau')$.

Since $\mathrm{N}_{\overline{N}}(\overline{D}) \leq \overline{M}$, we have $\mathrm{N}_N(D) \leq M$.

By Theorem 4.4 there exist projective representations $\overline{\mathcal{P}}$ and $\overline{\mathcal{P}}'$ of $\overline{G}$ and $\overline{H}$ associated to ρ and ρ' with the three properties of Theorem 4.4(i), such that they define a block isomorphism $(\overline{\iota}, \overline{\sigma})$ between the character triples $(\overline{G}, \overline{N}, \overline{\rho})$ and $(\overline{H}, \overline{M}, \overline{\rho}')$.

Now let $\mathcal{P} := \overline{\mathcal{P}} \circ \epsilon$ and $\mathcal{P}' := \overline{\mathcal{P}}' \circ \epsilon_H$. Then $\mathcal{P}$ and $\mathcal{P}'$ are projective representations of G and H with the following properties:

(a) the factor sets of $\mathcal{P}$ and $\mathcal{P}'$ coincide via the canonical isomorphism $G/N \to H/M$,
(b) if $xK \in \mathrm{C}_{G/K}(N/K)$, then the matrices $\mathcal{P}(x)$ and $\mathcal{P}'(x)$ are scalar matrices with respect to the same scalar, and
(c) we have

$$\left(\frac{|\overline{N}|_{p'}\,\mathrm{tr}(\mathcal{P}(x))}{\rho(1)_{p'}}\right)^* = \left(\frac{|\overline{M}|_{p'}\,\mathrm{tr}(\mathcal{P}'(x))}{\rho'(1)_{p'}}\right)^*$$

for every $x \in H^0$ with $\overline{D} \in \mathrm{Syl}_p(\mathrm{C}_{\overline{M}}(x))$, as $\mathrm{ht}(\rho) = \mathrm{ht}(\rho') = 0$.

Now, let $\mathcal{Q}$ be a projective representation of G associated to ζ. Then $\mathcal{Q} \otimes \mathcal{P}$ and $\mathcal{Q}_H \otimes \mathcal{P}'$ are projective representations of G and H respectively associated to τ and τ', respectively, and we check next that they satisfy the three conditions stated in Theorem 4.4(i). This will finish the proof of this theorem.

First, it is straightforward to check that $\mathcal{Q} \otimes \mathcal{P}$ and $\mathcal{Q}_H \otimes \mathcal{P}'$ have factor sets that coincide on the elements of H. Also, if $x \in \mathrm{C}_G(N)$, then $xK \in \mathrm{C}_{G/K}(N/K)$ and the matrices $\mathcal{Q}(x) \otimes \mathcal{P}(x)$ and $\mathcal{Q}(x) \otimes \mathcal{P}'(x)$ are scalar matrices associated with the same scalar.

In the final step we verify the equation from Theorem 4.4(i)(c) for every $x \in H^0$ with $D \in \mathrm{Syl}_p(\mathrm{C}_N(x))$. Without loss of generality we may assume $G = \langle N, x\rangle$ and $H = \langle M, x\rangle$. Since G/N is now cyclic, we may also assume that $\mathcal{P}, \mathcal{P}'$ and $\mathcal{Q}$ are ordinary representations. Suppose that they afford characters $\widetilde{\rho} \in \mathrm{Irr}(G \mid \rho)$, $\widetilde{\rho}' \in \mathrm{Irr}(H \mid \rho')$ and $\widetilde{\zeta} \in \mathrm{Irr}(G \mid \zeta)$, respectively. Let $\widetilde{\tau} = \widetilde{\zeta}\widetilde{\rho}$ be the character afforded by $\mathcal{Q}\otimes\mathcal{P}$ and $\widetilde{\tau}' = \widetilde{\zeta}_H\widetilde{\rho}'$ the character afforded by $\mathcal{Q}_H \otimes \mathcal{P}'$. We need to check

$$\left(\frac{|N|_{p'}\widetilde{\tau}(x)}{\tau(1)_{p'}}\right)^* = \left(\frac{|M|_{p'}\widetilde{\tau}'(x)}{\tau'(1)_{p'}}\right)^*. \tag{4.3}$$

Let $\overline{x} := xK$. Assume first $\overline{D} = DK/K \in \mathrm{Syl}_p(\mathrm{C}_{\overline{N}}(\overline{x}))$. Then we multiply the equation $\left(\frac{|\overline{N}|_{p'}\widetilde{\rho}(x)}{\rho(1)_{p'}}\right)^* = \left(\frac{|\overline{M}|_{p'}\widetilde{\rho}'(x)}{\rho'(1)_{p'}}\right)^*$ from 4.4(i)(c) for $\mathcal{P}$ and $\mathcal{P}'$ with $\left(\frac{|K|_{p'}\,\mathrm{tr}(\mathcal{Q}(x))}{\zeta(1)_{p'}}\right)^*$ to obtain

$$|\mathrm{C}_N(x)|^*_{p'}\lambda_{\widetilde{\tau}}(\mathfrak{Cl}_G(x)^+)^* = \left(\frac{|N|_{p'}\widetilde{\rho}(x)\widetilde{\zeta}(x)}{\rho(1)_{p'}\zeta(1)_{p'}}\right)^* = \left(\frac{|M|_{p'}\widetilde{\rho}'(x)\widetilde{\zeta}(x)}{\rho'(1)_{p'}\zeta(1)_{p'}}\right)^*$$
$$= |\mathrm{C}_M(x)|^*_{p'}\lambda_{\widetilde{\tau}'}(\mathfrak{Cl}_H(x)^+)^*,$$

where the first and last equalities follow from Lemma 4.2(a).

Assume now that DK/K is not a Sylow p-subgroup of $\mathrm{C}_{\overline{N}}(\overline{x})$. Hence DK/K is not a Sylow p-subgroup of $\mathrm{C}_{\overline{M}}(\overline{x})$ (see the remark after Theorem 4.4). Let $L_1/K = \mathrm{C}_{\overline{H}}(\overline{x})$ and let Q be a defect group of the block of $\widetilde{\zeta}_{L_1}$. As $\widetilde{\zeta}_K$ is irreducible, the group QK/K is a Sylow p-subgroup of L_1/K by Proposition 2.5(d). Since DK/K is not a Sylow p-subgroup of $\mathrm{C}_{\overline{M}}(\overline{x})$, D contains no defect group of $\mathrm{bl}(\widetilde{\zeta}_{L_1})$. By the Min-Max Theorem [Nav98, Theorem (4.4)] this implies $\lambda_{\widetilde{\zeta}_{L_1}}(\mathfrak{Cl}_{L_1}(x)^+) = 0$, as D is a defect group of $\mathfrak{Cl}_{L_1}(x)$. According to Lemma 2.2 we know

$$\lambda_{\widetilde{\tau}'}(\mathfrak{Cl}_H(x)^+) = \lambda_{\overline{\nu}'}(\mathfrak{Cl}_{\overline{H}}(\overline{x})^+)\lambda_{\widetilde{\zeta}_{L_1}}(\mathfrak{Cl}_{L_1}(x)^+),$$

where $\overline{\nu}'$ is a character of H/K. Lemma 4.2(a) together with Lemma 2.2 shows

$$\left(\frac{|M|_{p'}\widetilde{\tau}'(x)}{\tau'(1)_{p'}}\right)^* = (|\mathrm{C}_M(x)|_{p'})^*\lambda_{\widetilde{\tau}'}(\mathfrak{Cl}_H(x)^+)$$
$$= (|\mathrm{C}_M(x)|_{p'})^*\lambda_{\overline{\nu}'}(\mathfrak{Cl}_{\overline{H}}(\overline{x})^+)\lambda_{\zeta_{L_1}}(\mathfrak{Cl}_{L_1}(x)^+) = 0.$$

Arguing in the same way, we prove $\lambda_{\widetilde{\zeta}_L}(\mathfrak{Cl}_L(x)^+) = 0$, where $L/K = \mathrm{C}_{\overline{G}}(\overline{x})$. This implies

$$\left(\frac{|N|_{p'}\widetilde{\tau}(x)}{\tau(1)_{p'}}\right)^* = (|\mathrm{C}_N(x)|_{p'})^*\lambda_{\widetilde{\tau}}(\mathfrak{Cl}_G(x)^+) = (|\mathrm{C}_N(x)|_{p'})^*\lambda_{\overline{\nu}'}(\mathfrak{Cl}_{\overline{G}}(\overline{x})^+)\lambda_{\zeta_L}(\mathfrak{Cl}_L(x)^+)$$
$$= 0,$$

where $\overline{\nu}$ is some character of $\overline{G}$. This proves the theorem. □

In our final statement of this section, we adapt the above results to the terminology that is going to be used while proving Theorem B. If Q is a p-subgroup of G, recall that $\mathrm{Irr}_0(G \mid Q)$ is the set of irreducible characters of G with height zero, lying in blocks with defect group Q. If $K \lhd G$ and $\mathcal{G} \subseteq \mathrm{Irr}(K)$, sometimes in this paper we denote by $\mathrm{Irr}_0(G \mid Q, \mathcal{G})$ the set of characters belonging to $\mathrm{Irr}(G \mid \psi) \cap \mathrm{Irr}_0(G \mid Q)$ for some $\psi \in \mathcal{G}$.

We have already remarked (see Lemma 3.8) that whenever we have block isomorphic character triples, then there are several possible choices of block isomorphisms of character triples. In the next result, and under certain circumstances, we show how to choose them in a compatible way.

Proposition 4.7. *Let $K \lhd G$ and $H \leq G$ be such that $G = KH$. Let D_0 be a p-subgroup of $M := K \cap H$ such that $H = M\mathrm{N}_G(D_0)$. Let $\mathcal{G} := \mathrm{Irr}_0(K \mid D_0)$ and $\mathcal{G}' := \mathrm{Irr}_0(M \mid D_0)$. Assume that there exists an $\mathrm{N}_G(D_0)$-equivariant bijection $\Lambda : \mathcal{G} \to \mathcal{G}'$ such that $(G_\theta, K, \theta) \sim_{\mathrm{b}} (H_\theta, M, \theta')$ for every $\theta \in \mathcal{G}$ and $\theta' := \Lambda(\theta)$.*

(a) *There exists a choice of block isomorphisms of character triples, $\{(\sigma^{(\theta)}, \iota_{G_\theta/K})\}_{\theta\in\mathcal{G}}$, such that the isomorphism*

$$(\sigma^{(\theta)}, \iota_{G_\theta/K}) : (G_\theta, K, \theta) \to (H_\theta, M, \Lambda(\theta))$$

satisfies

$$\sigma_J^{(\theta)}(\zeta)^x = \sigma_{J^x}^{(\theta^x)}(\zeta^x)$$

for every $x \in \mathrm{N}_G(D_0)$, $\theta \in \mathcal{G}$, $K \leq J \leq G_\theta$, and $\zeta \in \mathrm{Irr}(J \mid \theta)$.

(b) *Suppose $J \lhd G$ with $K \leq J$. Let $Q \leq J$ be a p-subgroup with $Q \cap K = D_0$. Then there exists an $\mathrm{N}_H(Q)$-equivariant bijection*

$$\Pi_J : \mathrm{Irr}_0(J \mid Q) \to \mathrm{Irr}_0(J \cap H \mid Q)$$

with $(G_\tau, J, \tau) \sim_{\mathrm{b}} (H_\tau, J \cap H, \tau')$ for every $\tau \in \mathcal{G}$ with $\tau' := \Pi_J(\tau)$.

Proof. We fix a complete set $\mathcal{G}_1$ of representatives of $\mathrm{N}_G(D_0)$-orbits in $\mathcal{G} := \mathrm{Irr}_0(K \mid D_0)$ and let $\mathcal{G}_1' = \Lambda(\mathcal{G}_1)$. For every $\theta_1 \in \mathcal{G}_1$ and $\theta_1' := \Lambda(\theta_1)$ there exists a block isomorphism of character triples

$$(\sigma^{(\theta_1)}, \iota_{G_{\theta_1}/K}) : (G_{\theta_1}, K, \theta_1) \to (H_{\theta_1}, M, \Lambda(\theta_1)).$$

Notice that $H_{\theta_1} = H_{\theta_1'}$ since Λ is H-equivariant. Furthermore, by Proposition 3.9 for all $x \in \mathrm{N}_{G_{\theta_1}}(D_0)$ and $N \leq J_1 \leq G_{\theta_1}$, and all $\zeta_1 \in \mathrm{Irr}(J_1 \mid \theta_1)$, we have

$$(\sigma_{J_1}^{(\theta_1)}(\zeta_1))^x = \sigma_{J_1^x}^{(\theta_1)}(\zeta_1^x). \tag{4.4}$$

Let $\theta \in \mathcal{G}$. Then there exists $x \in \mathrm{N}_G(D_0)$ and unique $\theta_1 \in \mathcal{G}_1$ such that $\theta = \theta_1^x$. Suppose that $y \in \mathrm{N}_G(D_0)$ is another element such that $\theta = \theta_1^y$. Then $xy^{-1} \in \mathrm{N}_{G_{\theta_1}}(D_0)$. Suppose that $N \leq J \leq G_\theta$ and $\zeta \in \mathrm{Irr}(J \mid \theta)$. Notice that the characters $\sigma_{J^{x^{-1}}}^{(\theta_1)}(\zeta^{x^{-1}})$ and $\sigma_{J^{y^{-1}}}^{(\theta_1)}(\zeta^{y^{-1}})$ are well-defined. Since $xy^{-1} \in \mathrm{N}_{G_{\theta_1}}(D_0)$ (and using (4.4)), we have

$$(\sigma_{J^{x^{-1}}}^{(\theta_1)}(\zeta^{x^{-1}}))^{xy^{-1}} = \sigma_{J^{y^{-1}}}^{(\theta_1)}(\zeta^{y^{-1}}).$$

Hence $(\sigma_{J^{x^{-1}}}^{(\theta_1)}(\zeta^{x^{-1}}))^x = (\sigma_{J^{y^{-1}}}^{(\theta_1)}(\zeta^{y^{-1}}))^y$. Thus we have shown that $\sigma_J^{(\theta)}(\zeta)$ is well-defined by

$$\sigma_J^{(\theta)}(\zeta) := \sigma_{J^{x^{-1}}}^{(\theta_1)}(\zeta^{x^{-1}})^x.$$

If $z \in \mathrm{N}_G(D_0)$, we have

$$\sigma_J^{(\theta)}(\zeta)^z := \sigma_{J^{x^{-1}}}^{(\theta_1)}(\zeta^{x^{-1}})^{xz} = \sigma_{J^z}^{(\theta^z)}(\zeta^z).$$

Now it is straightforward to check that part (a) holds.

From now on we consider part (b) and its assumptions, e.g. $J \lhd G$. By Proposition 3.9, for every $\theta \in \mathcal{G}$ the map $\sigma^{(\theta)}_{J_\theta} : \mathrm{Irr}(J_\theta \mid \theta) \to \mathrm{Irr}(J_\theta \cap H_\theta \mid \Lambda(\theta))$ is H_θ-equivariant, preserves heights, the blocks of corresponding characters share defect groups, and for any p-group Q with $Q \cap K = D_0$ the restriction of $\sigma^{(\theta)}_{J_\theta}$ to $\mathrm{Irr}_0(J_\theta \mid Q, \theta) \to \mathrm{Irr}_0(J_\theta \cap H_\theta \mid Q, \Lambda(\theta))$ is a bijection. Furthermore, for every $\tau_0 \in \mathrm{Irr}_0(J_\theta \mid \theta)$ the character $\tau_0' := \sigma^{(\theta)}_{J_\theta}(\tau_0)$ satisfies

$$((G_\theta)_{\tau_0}, J_\theta, \tau_0) \sim_b ((H_\theta)_{\tau_0}, J_\theta \cap H, \tau_0').$$

Further the block of τ_0' induces the block of τ_0.

Since $K \cap Q = D_0$, it follows that $Q \leq \mathrm{N}_G(Q) \leq \mathrm{N}_G(D_0) \leq H$. Now, let $\tau \in \mathrm{Irr}_0(J \mid Q)$. By Proposition 2.5(a)&(f), there exists $\theta \in \mathcal{G}$ such that the Clifford correspondent τ_0 of τ over θ satisfies $\tau_0 \in \mathrm{Irr}_0(J_\theta \mid Q, \theta)$. Let $\theta_1 \in \mathcal{G}$ and $\tau_{0,1} \in \mathrm{Irr}_0(J_{\theta_1} \mid Q, \theta_1)$ be another pair of characters with $\tau_{0,1}^J = \tau$. By Proposition 2.5(f), there exists $y \in \mathrm{N}_J(Q)$ with $\theta^y = \theta_1$ and $\tau_0^y = \tau_{0,1}$.

Now let $\psi = \sigma^{(\theta)}_{J_\theta}(\tau_0) \in \mathrm{Irr}_0(J_\theta \cap H_\theta \mid Q, \theta')$ where $\theta' := \Lambda(\theta)$. By part (a) we have

$$\sigma^{(\theta_1)}_{J_{\theta_1}}(\tau_{0,1}) = \sigma^{(\theta^y)}_{J_{\theta^y}}(\tau_0^y) = \sigma^{(\theta)}_{J_\theta}(\tau_0)^y = \psi^y,$$

as $y \in \mathrm{N}_J(Q) \leq J \cap H$. The characters $\tau' := \psi^{J \cap H}$ and $(\psi^y)^{J \cap H}$ coincide and $\tau' := \psi^{J \cap H}$ is well-defined and independent of the choice of θ.

Since $\mathrm{bl}(\psi)^J = \mathrm{bl}(\tau_0)^J = \mathrm{bl}(\tau)$, we have $\mathrm{bl}(\tau')^J = \mathrm{bl}(\tau)$. Then, every defect group of $\mathrm{bl}(\tau')$ is contained in some J-conjugate of Q (by Lemma (4.13) of [Nav98]), and we conclude that $\tau' \in \mathrm{Irr}(J \cap H \mid Q, \Lambda(\theta))$. Now by the definition of height we see that τ' is a height zero character of $J \cap H$, i.e.

$$\tau'(1)_p |Q| = |(J \cap H) : (J_\theta \cap H)|_p \psi(1)_p |Q| = |J \cap H|_p.$$

Hence we can define $\Pi_J : \mathrm{Irr}_0(J \mid Q) \to \mathrm{Irr}_0(J \cap H \mid Q)$ by $\Pi_J(\tau) = \tau'$.

From (a) and the construction of τ we see that the map is $\mathrm{N}_H(Q)$-equivariant. We observe that τ, τ_0, ψ and τ' are height zero characters. By Clifford theory, Theorem 3.14 applies with $G_1 := G_\theta$, since the assumption $\mathrm{N}_J(Q) \leq J \cap H$ holds. Hence the character triples associated to τ and τ' satisfy

$$(G_\tau, J, \tau) \sim_b (H_\tau, J \cap H, \tau').$$

For the proof of (b) it remains to check that Π_J is a bijection. By Proposition 2.5(f), for every $\tau' \in \mathrm{Irr}_0(J \cap H \mid Q)$ there exist $\theta' \in \mathcal{G}'$ and $\psi \in \mathrm{Irr}_0(J_{\theta'} \cap H \mid Q, \theta')$ with $\psi^{J \cap H} = \tau'$. Via the bijection $\sigma^{(\theta)}_J$ this character corresponds to some $\tau_0 := (\sigma^{(\theta)}_J)^{-1}(\psi) \in \mathrm{Irr}_0(J_\theta \mid Q, \theta)$ where $\theta := \Lambda^{-1}(\theta')$. As Λ is $\mathrm{N}_G(D_0)$-equivariant, the character $\tau := \tau_0^J$ is irreducible.

Now $\mathrm{bl}(\tau)$ coincides with the block $\mathrm{bl}(\tau_0)^J = \mathrm{bl}(\psi)^J$. Since $\psi^{J \cap H} = \tau'$, we have $\mathrm{bl}(\tau') = \mathrm{bl}(\psi)^{J \cap H}$. Thus $\mathrm{bl}(\tau')^J = \mathrm{bl}(\tau)$. By the Harris–Knörr Theorem [Nav98, Theorem (9.28)] these blocks are Harris–Knörr correspondents and have the same defect group. This proves that $\tau \in \mathrm{Irr}_0(J \mid Q)$ and the map Π_J is surjective.

Let $\tau_1, \tau_2 \in \mathrm{Irr}_0(J \mid Q)$ with $\Pi_J(\tau_1) = \Pi_J(\tau_2)$. Then there exist $\theta_i \in \mathcal{G}$ and $\tau_{0,i} \in \mathrm{Irr}_0(J_{\theta_i} \mid Q, \theta_i)$ such that $\tau_{0,i}^J = \tau_i$. Since $\Pi_J(\tau_1) = \Pi_J(\tau_2)$, $\sigma_{J_{\theta_1}}^{(\theta_1)}(\tau_{0,1})^{J\cap H}$ and $\sigma_{J_{\theta_2}}^{(\theta_2)}(\tau_{0,2})^{J\cap H}$ have to coincide. The characters $\Lambda(\theta_1)$ and $\Lambda(\theta_2)$ are constituents of $\Pi_J(\tau_1)_{\mathrm{N}_K(D_0)}$ and $\Pi_J(\tau_2)_{\mathrm{N}_K(D_0)}$. Hence $\Lambda(\theta_1)$ and $\Lambda(\theta_2)$ are $J \cap H$-conjugate, i.e. $\Lambda(\theta_1)^x = \Lambda(\theta_2)$ for some $x \in J \cap H$. Analogously we see

$$\sigma_{J_{\theta_1}}^{(\theta_1)}(\tau_{0,1})^x = \sigma_{J_{\theta_2}}^{(\theta_2)}(\tau_{0,2}).$$

Using (4.4) this implies

$$\sigma_{J_{\theta_1^x}}^{(\theta_1^x)}(\tau_{0,1}^x) = \sigma_{J_{\theta_2}}^{(\theta_2)}(\tau_{0,2}).$$

As $\sigma^{(\theta_2)}$ is a bijection this proves $\tau_{0,1}^x = \tau_{0,2}$ and $\tau_{0,1}^J = \tau_{0,2}^J$. Accordingly Π_J is an $\mathrm{N}_G(Q)$-equivariant bijection as required. □

5. A generalization of the Dade–Glauberman–Nagao correspondence

In [NS14] the DGN-correspondence was generalized to characters in blocks with a normal defect group. We will use this new correspondence (but only in a special case). If a group Y acts on some group X, then $\mathrm{Irr}_Y(X)$ denotes the Y-invariant irreducible characters of X. If b is a block of X, then $\mathrm{Irr}_Y(b)$ is the set of Y-invariant characters in b.

Frequently we shall assume the following hypothesis:

Hypothesis 5.1. Let $K \lhd M$ be such that M/K is a p-group. Let D_0 be a normal p-subgroup of K contained in $\mathrm{Z}(M)$. Suppose that $b \in \mathrm{Bl}(K)$ is M-invariant and has defect group D_0. Assume that B, the only block of M covering b, has defect group D. Let $L = \mathrm{N}_K(D)$.

It is easy to check that our hypotheses imply that $M = KD$ and $K \cap D = D_0$ (see [NS14]). Also $D_0 = \mathrm{O}_p(K)$.

Theorem 5.2. *Assume Hypothesis 5.1.*

(a) *If b' is the only block of L covered by B', the Brauer first main correspondent of B, then there is a natural bijection*

$$\Pi_D : \mathrm{Irr}_D(b) \to \mathrm{Irr}_D(b').$$

In fact, if $\theta \in \mathrm{Irr}_D(b)$, then

$$\theta_L = e\theta' + p\Delta + \Psi \tag{5.1}$$

where $\theta' = \Pi_D(\theta) \in \mathrm{Irr}(L)$ is such that $\theta'(1)_p = |L : D_0|_p$, p does not divide e, every irreducible constituent γ of Δ satisfies $\gamma(1)_p = |L : D_0|_p$, and every irreducible constituent γ of Ψ is such that $\gamma(1)_p < |L : D_0|_p$.

(b) *If $\theta \in \mathrm{Irr}_D(b)$, we have*

$$\theta(1)_{p'} \equiv e|K : L|_{p'}\theta'(1)_{p'} \bmod p \quad \textit{and} \quad e \equiv \pm 1 \bmod p.$$

Also, if $x \in K$ is p-regular with $D \in \mathrm{Syl}_p(\mathrm{C}_M(x))$, then $\theta(x)^* = e\theta'(x)^*$.

(c) *Suppose that $K, M \lhd G$, where G is a finite group. Then Π_D is $\mathrm{N}_{G_b}(D)$-equivariant.*

Proof. Since every $\mu \in \mathrm{Irr}(D_0)$ is M-invariant, we apply Theorem (2.8) of [NS14] to get (a) and (b). Since this bijection is natural, using (5.1) we easily check that it is $\mathrm{N}_{G_b}(D)$-equivariant. □

Definition 5.3. Suppose that K and D are subgroups of some finite group G, where D is a p-subgroup that normalizes K. Let $\theta \in \mathrm{Irr}_D(K)$. We say that the *$D$-correspondent of θ is defined* if

(a) $D \cap K = D_0 \subseteq \mathrm{Z}(KD)$, and
(b) the block $\mathrm{bl}(\theta)$ has defect group D_0 and the unique block of KD covering $\mathrm{bl}(\theta)$ has defect group D.

In this case, there is a unique $\theta' = \Pi_D(\theta) \in \mathrm{Irr}(\mathrm{N}_K(D))$ defined, which we call the *D-correspondent* of θ.

(Let us mention that in order for θ to have a D-correspondent for some p-subgroup D, it is enough to assume that D_0 is normal in K, as shown in [NS14]. We shall not need this more general result here.)

If the D-correspondent of θ is defined, then notice that $D_0 = \mathrm{O}_p(\mathrm{Z}(K))$ because $\mathrm{O}_p(K)$ is contained in every defect group of K. Also, if $D_0 = 1$, then the D-correspondent of θ is the so called DGN-correspondent. If K is a p'-group, then this is the Glauberman correspondent.

Some of the characters that we are considering in Theorem 5.2 are relative defect zero or defect zero characters.

Notation 5.4 (Defect zero characters and relative defect zero characters). Let $N \lhd G$. We define $\mathrm{dz}(G)$ to be the set of $\chi \in \mathrm{Irr}(G)$ with $\chi(1)_p = |G|_p$. For $\mu \in \mathrm{Irr}(N)$, we set

$$\mathrm{rdz}(G \mid \mu) := \left\{ \chi \in \mathrm{Irr}(G|\mu) \;\middle|\; \left(\frac{\chi(1)}{\mu(1)}\right)_p = |G : N|_p \right\}.$$

The following statement goes back to results by [Dad80, Pui86] and was recently proven in wider generality by F. Ladisch.

Theorem 5.5. *Assume Hypothesis 5.1, and assume that $K, M \lhd G$, where G is a finite group. Let $H = \mathrm{N}_G(D)$. Suppose that $\theta \in \mathrm{Irr}_D(b)$ has defect zero (that is, $D_0 = 1$), and let $\theta' := \Pi_D(\theta)$ be defined as in Theorem 5.2. Then θ extends to G if and only if θ' extends to H.*

Proof. Use [Lad10, Corollary 11.3 and Theorem 4.3]. □

Lemma 5.6. *Let (G, N, μ) be a character triple, where N is a p-group.*

(a) *There exists a natural bijection*

$$\Gamma_{G,\mu} : \mathrm{dz}(G/N) \to \mathrm{rdz}(G \mid \mu).$$

(b) *Let G_1 be a group with $N \leq G_1 \leq G$. Assume that $\mu(1) = 1$. Let $\kappa_1 \in \mathrm{dz}(G_1/N)$ and $\kappa \in \mathrm{dz}(G/N)$. Then κ is an extension of κ_1 if and only if $\Gamma_{G,\mu}(\kappa)$ is an extension of $\Gamma_{G_1,\mu}(\kappa_1)$.*

Proof. For $g \in G$ such that $g_p \in N$, there exists a canonical extension μ_g of μ to $\langle N, g\rangle$, by Corollary (8.16) of [Isa76]. Let $\widehat{\mu}$ be the map defined on the elements $g \in G$ with $g_p \in N$ such that $\widehat{\mu}(g) = \mu_g(g)$. If $\chi \in \mathrm{dz}(G/N)$, then $\Gamma_{G,\mu}(\chi)$ is defined by $\Gamma_{G,\mu}(\chi)(g) = 0$ if $g_p \notin N$, and $\Gamma_{G,\mu}(\chi)(g) = \widehat{\mu}(g)\chi(g)$ if $g_p \in N$. This a natural bijection by Theorem 2.1 of [Nav04].

For the character triple (G_1, N, μ) this defines also $\Gamma_{G_1,\mu} : \mathrm{dz}(G_1/N) \to \mathrm{rdz}(G_1 \mid \mu)$. If $\mu(1) = 1$ then $\widehat{\mu}(g) = \mu(g_p)$ whenever $g_p \in N$. Now, suppose for instance that $\Gamma_{G,\mu}(\kappa)$ is an extension of $\Gamma_{G_1,\mu}(\kappa_1)$. We prove that $\kappa(g) = \kappa_1(g)$ for $g \in G_1$. If $g_p \notin N$, both are zero; otherwise $\kappa(g)\mu(g_p) = \kappa_1(g)\mu(g_p)$, so the assertion follows. □

We use the above bijection to extend Theorem 5.5.

Theorem 5.7. *Assume Hypothesis 5.1, and assume that $K, M \lhd G$, where G is a finite group. Let $H = \mathrm{N}_G(D)$. Let $\theta \in \mathrm{Irr}_D(b)$ and let $\theta' = \Pi_D(\theta)$ be defined as in Theorem 5.2. Then θ' extends to H if and only if θ extends to G.*

Proof. Suppose first that $\widetilde{\theta} \in \mathrm{Irr}(G)$ extends θ. Then b is G-invariant, and therefore B is also G-invariant. Hence $G = KH$ by the Frattini argument. By Theorem 5.2, the character θ uniquely determines $\theta' \in \mathrm{Irr}(L)$. Hence θ' is H-invariant. According to Corollary (11.31) of [Isa76] it is sufficient to prove that θ' extends to Q for every prime q, where Q/L is a Sylow q-subgroup of H/L. Hence, we may assume that $MQ = G$.

First we consider the case $q = p$. Then H/L is a p-group. By Theorem 5.2 the character $\theta' \in \mathrm{Irr}(L)$ has p'-multiplicity e in θ_L. Hence there is an irreducible constituent $\tau \in \mathrm{Irr}(H)$ of $\widetilde{\theta}_H$ such that τ_L contains θ' with p'-multiplicity. Since $\tau(1)/\theta'(1)$ divides $|H : L|$ by [Isa76, (11.29)], we conclude that τ is an extension of θ' to H.

Suppose now that $q \neq p$. By hypothesis θ extends to KQ. Let $\kappa := \Gamma_{K,\mu}^{-1}(\theta) \in \mathrm{dz}(K/D_0)$, where $\mu \in \mathrm{Irr}(D_0)$ is the only irreducible constituent of θ_{D_0}. Now κ extends to KQ/D_0 by Lemma 5.6. The character κ has a D/D_0-correspondent κ', which extends to Q/D_0 by Theorem 5.5. By Lemma (2.7) of [NS14], we have $\theta' = \Gamma_{L,\mu}(\kappa')$ (see also the remark before the proof of Theorem A of [NS14]). Hence θ' extends to Q by Lemma 5.6.

The same considerations also prove that θ extends to G if θ' extends to H. □

We need the following property, which in the case where K is a p'-group is a special case of Theorem A(b) of [IN91].

Proposition 5.8. *Suppose that K and D are subgroups of some finite group G, where D is a p-subgroup that normalizes K. Suppose that $K \lhd K_1 \subseteq K\mathrm{C}_G(D)$, where K_1/K is a p'-group. Let $\theta \in \mathrm{Irr}_D(K)$ and let $\psi \in \mathrm{Irr}_D(K_1)$ be over θ.*

(a) *The D-correspondent θ' of θ is defined if and only if the D-correspondent ψ' of ψ is defined.*
(b) *Suppose that K_1/K is supersolvable and assume that θ' is defined. If $\psi_K = \theta$, then ψ' extends θ'.*

Proof. Suppose that the D-correspondent of θ is defined. We have $K_1 = K\mathrm{C}_{K_1}(D)$ so that D normalizes K_1. Also $KD \lhd K_1D$. Since K_1/K is a p'-group, we have $D \cap K_1 = D \cap K = D_0$ which is central in K_1. Also, by Theorem (9.26) of [Nav98], D_0 is a defect group of the block of ψ. Let C be the unique block of K_1D covering the block c of ψ, and let B be the unique block of KD covering the block b of θ. Since C covers b, it necessarily covers B. Now, since $K_1D/KD \cong K_1/K$ is a p'-group, by Theorem (9.26) of [Nav98], D is a defect group of C. This proves that the D-correspondent of ψ is defined. The converse is proved in the same way.

Now, write $\theta_{\mathrm{N}_K(D)} = e\theta' + p\Delta + \Psi$ as in Theorem 5.2. Also, write $\psi_{\mathrm{N}_{K_1}(D)} = d\psi' + p\Delta_0 + \Psi_0$. Since $\psi_K = \theta$, it follows that θ' is $\mathrm{N}_{K_1}(D)$-invariant and

$$d\psi'_{\mathrm{N}_K(D)} + p(\Delta_0)_{\mathrm{N}_K(D)} + (\Psi_0)_{\mathrm{N}_K(D)} = e\theta' + p\Delta + \Psi.$$

Now, since $\mathrm{N}_{K_1}(D)/\mathrm{N}_K(D)$ is a p'-group, any $\tau \in \mathrm{Irr}(\mathrm{N}_{K_1}(D))$ has D_0-relative defect zero if and only if some irreducible constituent of $\tau_{\mathrm{N}_K(D)}$ has D_0-relative defect zero. We easily deduce that θ' lies under ψ'. If K_1/K is cyclic, this implies that ψ' extends θ'. Since K_1/K is supersolvable, we can apply the same argument to a chief series from K to K_1, to deduce that $(\psi')_{\mathrm{N}_K(D)} = \theta'$. □

The extension $\widetilde{\theta}'$ from Theorem 5.7 can be chosen to have several additional properties.

Proposition 5.9. *Assume Hypothesis 5.1, and that $K, M \lhd G$, where G is a finite group. Write $H = \mathrm{N}_G(D)$. Suppose that $\theta \in \mathrm{Irr}_D(b)$ and let $\theta' \in \mathrm{Irr}(L)$ be its D-correspondent. Suppose that $\widetilde{\theta} \in \mathrm{Irr}(G)$ extends θ. Let $C \le \mathrm{C}_G(M)$ be abelian with $C \lhd G$. Then there exists an extension $\widetilde{\theta}' \in \mathrm{Irr}(H)$ of θ' with*

$$\mathrm{Irr}(\widetilde{\theta}_C) = \mathrm{Irr}(\widetilde{\theta}'_C).$$

Proof. First of all notice that $C \le \mathrm{C}_G(D) \le H$. Let $M_1 := MC$ and $K_1 := KC$. Then M_1/K_1 is a p-group.

Let $\psi := \widetilde{\theta}_{KC}$ and let P_0 be a defect group of $\mathrm{bl}(\psi)$. Now P_0/D_0 is isomorphic to a Sylow p-subgroup of KC/K by Proposition 2.5(d). As C is abelian, $K\,\mathrm{O}_p(C)/K$ is the only Sylow p-subgroup of KC/K, so $P_0K = K\,\mathrm{O}_p(C)$. Since $\mathrm{O}_p(C)$ is contained in P_0, this proves that $P_0 = D_0\,\mathrm{O}_p(C)$.

The character $\widetilde{\psi} := \widetilde{\theta}$ is then an extension of ψ to G. We determine a defect group P of $\mathrm{bl}(\widetilde{\psi}_{M_1})$. This can be chosen to contain D as $M \lhd M_1$ by Theorem (9.26) of [Nav98]. By Proposition 2.5(d) the group P/P_0 is a Sylow p-subgroup of M_1/K_1. Hence $MP = M\,\mathrm{O}_p(C)$, and by the same argument as above $P = D\,\mathrm{O}_p(C)$.

Now ψ has a P-correspondent $\psi' \in \mathrm{Irr}(\mathrm{N}_{K_1}(P))$ in the sense of Definition 5.3. Since $C \le \mathrm{Z}(K_1)$ and ψ' is an irreducible constituent of $\psi_{\mathrm{N}_{K_1}(P)}$, we have $\mathrm{Irr}(\psi_C) = \mathrm{Irr}(\psi'_C)$. By Theorem 5.7, ψ' has an extension $\widetilde{\psi}' \in \mathrm{Irr}(\mathrm{N}_{G_1}(P))$.

Since $C \le \mathrm{Z}(K_1)$ we have $\mathrm{N}_K(P) = \mathrm{N}_K(D)$. Furthermore every character of $\mathrm{N}_K(P)$ extends to $\mathrm{N}_{K_1}(P) = \mathrm{N}_K(P)C$. As $C/\mathrm{O}_p(C)$ is a p'-group, we have $|\mathrm{N}_{K_1}(P) : P_0|_p = |\mathrm{N}_K(P) : D_0|_p$. By Theorem 5.2(b) the restriction $\psi_{\mathrm{N}_{K_1}(P)}$ contains exactly one constituent ψ' with p'-multiplicity and D_0-relative defect zero. Then $\psi_{\mathrm{N}_K(P)} = \theta_{\mathrm{N}_K(P)}$ has only one character with p'-multiplicity and D_0-relative defect zero, namely $\psi'_{\mathrm{N}_K(P)}$. Hence ψ' extends θ'.

Since $\mathrm{O}_p(C)$ is normal in G, we have $\mathrm{N}_G(D) \le \mathrm{N}_G(P)$. Also every $x \in \mathrm{N}_G(P)$ stabilizes $P \cap M = D$ because $M \lhd G$. This implies $\mathrm{N}_G(P) \le \mathrm{N}_G(D)$ and we notice that $N_G(D) = N_G(P)$. We see that $\widetilde{\psi}'$ is an extension of θ' to $N_G(D)$ with the required properties. □

The DGN-correspondence behaves well with respect to kernel of characters, although this is not a complete triviality. Suppose that $E \lhd G$, write $\overline{G} = G/E$ and use the bar-convention. If $K \leq G$, then $K/(E \cap K)$ is naturally isomorphic to the subgroup KE/E of G/E. Consequently, all the characters τ of K that contain $E \cap K$ in its kernel can be seen as characters $\overline{\tau}$ of $\overline{K} = KE/E$, where $\overline{\tau}(\overline{k}) = \overline{\tau}(Ek) = \tau(k)$ for all $k \in K$.

Lemma 5.10. *Suppose that K and D are subgroups of a finite group G, where D is a p-subgroup of G that normalizes K, with $K \cap D = D_0 \leq Z(K)$. Suppose that the D-correspondent $\theta' \in \mathrm{Irr}(N_K(D))$ of $\theta \in \mathrm{Irr}(K)$ is defined, and suppose that $E \lhd G$ is such that $E \cap K \leq \ker(\theta)$. Let $\overline{G} = G/E$ and use the bar convention. Then $\overline{N_K(D)} = N_{\overline{K}}(\overline{D})$, the $\overline{D}$-correspondent of $\overline{\theta} \in \mathrm{Irr}(\overline{K})$ is defined and*

$$(\overline{\theta})' = \overline{\theta'}.$$

Proof. This is a special case of Theorem (2.11) of [NS14]. The definition of being D-defined in [NS14] does not require that D_0 is central in KD, but only normal. This extra condition is easily checked in this theorem. □

We use this lemma to generalize Proposition 5.9.

Proposition 5.11. *Assume Hypothesis 5.1, and that $K, M \lhd G$, where G is a finite group. Write $H = N_G(D)$. Suppose that $\theta \in \mathrm{Irr}_D(b)$ and let $\theta' \in \mathrm{Irr}(L)$ be its D-correspondent. Suppose that $\widetilde{\theta} \in \mathrm{Irr}(G)$ extends θ. Then there exists an extension $\widetilde{\theta}' \in \mathrm{Irr}(H)$ of θ' with*

$$\mathrm{Irr}(\widetilde{\theta}_{C_G(M)}) = \mathrm{Irr}(\widetilde{\theta}'_{C_G(M)}).$$

Proof. If $C_G(M)$ is abelian then this follows from Proposition 5.9 for $C := C_G(M)$. Let $E := [C, C] \lhd G$. Since $\widetilde{\theta}_M$ is irreducible, E is contained in the kernel of $\widetilde{\theta}$ by elementary character theory. Thus $K \cap E \leq \ker(\theta)$. Now, we apply Lemma 5.10, noticing that $\overline{H} = \overline{N_G(D)} = N_{\overline{G}}(\overline{D})$, and then Proposition 5.9 to the abelian normal subgroup $C_G(M)/E$. □

Proposition 5.12. *Assume Hypothesis 5.1, and that $K, M \lhd G$, where G is a finite group. Write $H = N_G(D)$. Suppose that $\theta \in \mathrm{Irr}_D(b)$ and let $\theta' \in \mathrm{Irr}(L)$ be its D-correspondent. Suppose that $\widetilde{\theta} \in \mathrm{Irr}(G)$ extends θ. Write $[\theta_L, \theta'] = e$. Then there exists an extension $\widetilde{\theta}' \in \mathrm{Irr}(H)$ of θ' with*

$$\widetilde{\theta}(x)^* = e(\widetilde{\theta}'(x))^*$$

for all p-regular elements $x \in G$ with $D \in \mathrm{Syl}_p(C_M(x))$ and

$$\mathrm{Irr}(\widetilde{\theta}_{C_G(M)}) = \mathrm{Irr}(\widetilde{\theta}'_{C_G(M)}).$$

Proof. According to Proposition 5.11 the character θ' has an extension $\kappa \in \mathrm{Irr}(H)$ with $\mathrm{Irr}(\widetilde{\theta}_{C_G(M)}) = \mathrm{Irr}(\kappa_{C_G(M)})$. For the proof it is sufficient to verify that there exists a linear character $\zeta \in \mathrm{Irr}(H)$ with $C_G(M)L \leq \ker(\zeta)$ such that, if $\widetilde{\theta}' := \zeta\kappa$, then

$$\widetilde{\theta}(x)^* = e(\widetilde{\theta}'(x))^*$$

for all p-regular elements $x \in \mathrm{C}_G(D)$ with $D \in \mathrm{Syl}_p(\mathrm{C}_M(x))$. In a first step we construct $\zeta_{\mathrm{C}_G(D)}$ such that the equation on the values is satisfied. Later we verify that $\mathrm{C}_G(M)L \leq \ker(\zeta)$.

Suppose that $x \in \mathrm{C}_G(D)$ is p-regular. Consider $K_x = K\langle x\rangle$ which is a normal subgroup of $K_x D$ such that $K_x D/K_x$ is a p-group. Let $L_x = \mathrm{N}_{K_x}(D)$. Then $\widetilde{(\theta)}_{K_x}$ is a D-invariant extension of θ and is contained in a block with normal defect group D_0, such that the character in $\mathrm{Irr}(\theta_{D_0})$ is G-invariant. By Proposition 5.8 the same happens for the corresponding characters:

$$((\widetilde{\theta}_{K_x})')_L = \theta'.$$

Since $(\widetilde{\theta}_{K_x})'$ is an extension of θ' to L_x, by Gallagher's Corollary (6.17) of [Isa76], there exists a unique linear character $\epsilon_x \in \mathrm{Irr}(L_x/L)$ such that

$$\epsilon_x \kappa_{L_x} = (\widetilde{\theta}_{K_x})'.$$

We define a complex function

$$\epsilon : \mathrm{C}_G(D)L \to \mathbb{C} \quad \text{by} \quad \epsilon(y) = \epsilon_{y_{p'}}(y_{p'})$$

and we prove that ϵ is a linear character of $\mathrm{C}_G(D)L$ that extends to $\mathrm{N}_G(D)$. First, we claim that $\epsilon(y^n) = \epsilon(y)^n = \epsilon(y)$ for $n \in H$ and $y \in \mathrm{C}_G(D)L$. We may assume that y is p-regular. Since the correspondence from Theorem 5.2 satisfies (5.1), we have $(\epsilon_y)^n = \epsilon_{y^n}$ and this proves the claim. In particular, ϵ is a class function that is constant on L-cosets. Now, notice that $\epsilon_y = \epsilon_{y^{-1}}$, so that

$$\epsilon(y^{-1}) = \epsilon(y)^{-1}.$$

Hence the value of the inner product on (ϵ, ϵ) is 1.

Next, we prove that ϵ is a generalized character of $\mathrm{C}_G(D)L$ by using Brauer's characterization of characters (see [Isa76, Corollary (8.12)]). Suppose that $E = P \times Q \subseteq \mathrm{C}_G(D)L$ is elementary, where P is a p-group and Q is a p'-group.

Again, $\widetilde{(\theta)}_{KQ}$ is D-invariant and extends θ. Since κ_{LQ} is an extension of θ' to $LQ = \mathrm{N}_{KQ}(D)$, by Gallagher there exists a unique linear character $\nu_Q \in \mathrm{Irr}(LQ/L)$ such that

$$\nu_Q \kappa_{LQ} = (\widetilde{\theta}_{KQ})'.$$

By the definition of ϵ we deduce $\epsilon_E = 1_P \times \nu_Q$, and this proves that ϵ is a generalized character. Since $\epsilon(1) = 1$, we deduce that ϵ is a character.

We show that ϵ extends to $\mathrm{N}_G(D)$. Again, by Corollary (11.31) of [Isa76] it is enough to show that ϵ extends to $L\mathrm{C}_G(D)Q$ for $Q/L \in \mathrm{Syl}_q(\mathrm{N}_G(D)/L)$ and every prime q. This is clear if $q = p$, since the order of ϵ is not divisible by p. So we assume that $q \neq p$. Let $Q_1 = Q \cap L\mathrm{C}_G(D)$. Since ϵ is linear, it suffices to show that ϵ_{Q_1} extends to Q.

Since $Q \leq \mathrm{N}_G(D)$, the group $G_1 := (KD)Q$ is well-defined. The groups $K_1 := KQ_1$ and $M_1 := KQ_1D$ are normal subgroups of G_1 such that $K_1 \lhd M_1$ and M_1/K_1 is a p-group. The character $\psi := \widetilde{(\theta)}_{K_1}$ has a D-correspondent $\psi' \in \mathrm{Irr}(L_1)$ with $L_1 := \mathrm{N}_{KQ_1}(D) = LQ_1$. As ψ is an extension of θ, the character ψ' is an extension of θ' by Proposition 5.8. Further as ψ extends to G_1, ψ' has an extension η to $\mathrm{N}_{G_1}(D)$, by

Theorem 5.5. By Gallagher there exists a linear character $\delta \in \mathrm{Irr}(\mathrm{N}_{G_1}(D)/L)$ with $\eta = \delta\kappa_{\mathrm{N}_{G_1}(D)}$. On the other hand, by our definition, $\psi' = \epsilon_{Q_1}\kappa_{Q_1}$. This implies $\delta_{Q_1} = \epsilon_{Q_1}$ and δ_Q is an extension of ϵ_{Q_1}. Thus ϵ has an extension to $\mathrm{N}_G(D)$.

The character $\kappa\epsilon$ satisfies by definition the equation

$$\kappa(x)\epsilon(x) = \Pi_D(\widetilde{\theta}_{K_x})(x)$$

for every p-regular element $x \in \mathrm{C}_G(D)$. If the p-regular element $x \in \mathrm{C}_G(D)$ satisfies additionally that $D \in \mathrm{Syl}_p(\mathrm{C}_M(x))$, then Theorem 5.2(b) shows

$$\Pi_D(\widetilde{\theta}_{K_x})(x)^* = e\widetilde{\theta}(x)^*,$$

where e is the multiplicity of θ' in θ_L, and hence

$$\kappa(x)^*\epsilon(x)^* = e^*\widetilde{\theta}(x)^*.$$

For $C := \mathrm{C}_G(M)$ it remains to verify $\mathrm{Irr}(\widetilde{\theta}_C) = \mathrm{Irr}((\kappa\epsilon)_C)$. As κ is an extension of an irreducible character of θ the set $\mathrm{Irr}((\kappa\epsilon)_C)$ consists of only one linear character ν', and analogously $\mathrm{Irr}(\widetilde{\theta}_C) = \{\nu\}$. The kernel of ϵ contains $[C, C]$ as κ and $\kappa\epsilon$ are extensions of θ'. Further ϵ is by definition trivial on p-elements. Thus ϵ is trivial on O with $O/[C, C] = \mathrm{O}_p(C/[C, C])$. By the definition of κ this character satisfies $\mathrm{Irr}(\widetilde{\theta}_C) = \mathrm{Irr}(\kappa_C)$, hence $\nu_O = \nu'_O$.

For every $\overline{y} \in \mathrm{C}_G(M)/O$ there exists a p-regular element $x \in \mathrm{C}_G(M)$ with $\overline{y} = xO$. Let L_x and K_x be defined as above. By the definition we have $\kappa_{L_x}\epsilon_{L_x} = \Pi_D(\widetilde{\theta}_{K_x})$. Also the character $\kappa_{L_x}\epsilon_{L_x}$ is covered by a block B' of L_xD that is the Brauer correspondent of $B = \mathrm{bl}(\widetilde{\theta}_{K_xD})$. According to Theorem (4.14) of [Nav98] this implies

$$\lambda_B(\mathfrak{Cl}_{K_xD}(x)^+)^* = \lambda_{B'}((\mathfrak{Cl}_{K_xD}(x) \cap L_xD)^+)^*.$$

Since $x \in \mathrm{C}_G(M)$ the element x is central in K_xD and by Theorem (9.5) of [Nav98], we have

$$\lambda_B(\mathfrak{Cl}_{K_xD}(x)^+)^* = \lambda_{\widetilde{\theta}}(\mathfrak{Cl}_{K_xD}(x)^+)^* = \left(\frac{\widetilde{\theta}(x)}{\widetilde{\theta}(1)}\right)^*,$$
$$\lambda_{B'}(\mathfrak{Cl}_{L_xD}(x)^+)^* = \lambda_{\kappa\epsilon}(\mathfrak{Cl}_{L_xD}(x)^+)^* = \left(\frac{\kappa\epsilon(x)}{\kappa(1)}\right)^*.$$

Hence

$$\frac{\kappa\epsilon(x)}{\kappa(1)} = \nu'(x) \quad \text{and} \quad \frac{\widetilde{\theta}(x)}{\widetilde{\theta}(1)} = \nu(x).$$

This implies $\nu(x) = \nu'(x)$ for every p-regular element x of $\mathrm{C}_G(M)$. Together with the fact $\nu_O = \nu'_O$, this proves $\nu = \nu'$. Thus

$$\mathrm{Irr}(\widetilde{\theta}_{\mathrm{C}_G(M)}) = \mathrm{Irr}((\kappa\epsilon)_{\mathrm{C}_G(M)}). \qquad \square$$

Finally we can prove the main results of this section.

Theorem 5.13. *Assume Hypothesis 5.1, and suppose that $K, M \lhd G$. Let $H = \mathrm{N}_G(D)$. Suppose that $\theta \in \mathrm{Irr}(b)$ is G-invariant, and let $\theta' \in \mathrm{Irr}(L)$ be the D-correspondent of θ. Then there exists an H-equivariant bijection*

$$\Pi_{\theta,D} : \mathrm{Irr}_0(M \mid \theta) \to \mathrm{Irr}_0(\mathrm{N}_M(D) \mid \theta')$$

with $(G_{\theta_1}, M, \theta_1) \sim_b (H_{\theta_1}, \mathrm{N}_M(D), \Pi_{\theta,D}(\theta_1))$ for every $\theta_1 \in \mathrm{Irr}_0(M \mid \theta)$.

Proof. By Corollary (9.18) of [Nav98], every $\xi \in \mathrm{Irr}_0(M \mid \theta)$ extends θ. Hence, using Proposition 5.7, we may assume that θ extends to M.

Now, let $\mathcal{P}$ be a projective representation of G associated to θ, and construct the group $\widehat{G}$ as in the proof of Theorem 4.1, where Z is a subgroup of $\mathbb{C}^\times$ containing the values of the factor set of $\mathcal{P}$, and use the same notation. Write $\epsilon : \widehat{G} \to G$ for the canonical epimorphism $\epsilon(g,z) = g$. Thus $K_0 := \{(k,1) \mid k \in K\}$ is a normal subgroup of $\widehat{G}$ that is isomorphic to K. Let $\theta_0 \in \mathrm{Irr}(K_0)$ be the character corresponding to θ. Let $D_{00} = \{(d,1) \mid d \in D_0\}$, which is the central defect group of the block of θ_0. The map $\mathcal{D}$ defined on $\widehat{G}$ by

$$\mathcal{D}((g,z)) = z\mathcal{P}(g) \qquad \text{for every } z \in Z \text{ and } g \in G$$

defines an irreducible representation of $\widehat{G}$. Let $\widetilde{\theta_0} \in \mathrm{Irr}(\widehat{G} \mid \theta_0)$ be the character afforded by $\mathcal{D}$, which extends θ_0. Set $\widehat{K} := K \times Z$, and notice that $\psi := \widetilde{\theta_0}_{\widehat{K}}$ is contained in a block with defect group $P_0 = D_{00} \times Z_p$, where Z_p is the Sylow p-subgroup of Z. (This follows easily since $D_{00} \times Z_p = \mathrm{O}_p(\widehat{K})$ should be contained in every defect group of $\mathrm{bl}(\psi)$, and every defect group intersects K in D_{00}.) Now let $\widehat{M} := \epsilon^{-1}(M)$. The group $\widehat{M}$ is a normal subgroup of $\widehat{G}$ and $\widehat{M}/\widehat{K} \cong M/K$ is a p-group. Notice that P_0 is central in $\widehat{M}$.

Let $B_{\widehat{M}}$ be the block of $\widehat{M}$ covering $\mathrm{bl}(\psi)$. Next we claim that we can choose a defect group P of this block such that $\epsilon(P) = D$. We know that $P\widehat{K} = \widehat{M}$ and $P \cap \widehat{K} = P_0$. Thus $|P : P_0| = |M : K| = |D : D_0|$. Also, $|\epsilon(P)| = |P/P \cap Z| = |P|/|Z_p| = |D|$. Since P is a defect group of $B_{\widehat{M}}$ there exists a p-regular $(x,z) \in \widehat{M}$ such that P is a Sylow p-subgroup of $\mathrm{C}_{\widehat{M}}(x,z)$ and $\lambda_{B_{\widehat{M}}}(\mathfrak{Cl}_{\widehat{M}}(x,z)^+) \neq 0$. Since (x,z) is p-regular, it follows that $(x,z) \in \widehat{K}$ and $x \in K$ is p-regular. Also $\mathrm{C}_{\widehat{M}}(x,z) = \widehat{\mathrm{C}_M(x)}$. Furthermore, $|\widehat{M} : \mathrm{C}_{\widehat{M}}(x,z)| = |\widehat{K} : \mathrm{C}_{\widehat{K}}(x,z)| = |K : \mathrm{C}_K(x)|$. Finally,

$$\begin{aligned} 0 \neq \lambda_{B_{\widehat{M}}}(\mathfrak{Cl}_{\widehat{M}}(x,1)^+) &= \lambda_\psi(\mathfrak{Cl}_{\widehat{M}}(x,1)^+) = \lambda_{\theta_0}(\mathfrak{Cl}_{K_0}(x)^+) = \lambda_\theta(\mathfrak{Cl}_K(x)^+) \\ &= \lambda_B(\mathfrak{Cl}_M(x)^+), \end{aligned}$$

where B is the only block of M covering b, and we deduce that D is contained in a Sylow p-subgroup of $\mathrm{C}_M(x)$ by the Min-Max Theorem. Since $\epsilon(P)$ is a Sylow p-subgroup of $\mathrm{C}_M(x)$, the claim easily follows. In particular, we conclude that $\mathrm{N}_{\widehat{M}}(P) = \widehat{\mathrm{N}_M(D)}$ and $\mathrm{N}_{\widehat{G}}(P) = \widehat{\mathrm{N}_G(D)}$.

By Theorem 5.2 the character ψ has a P-correspondent $\psi' \in \mathrm{Irr}(\widehat{L})$ with $\widehat{L} := \mathrm{N}_{\widehat{K}}(P) = \mathrm{N}_{K_0}(D) \times Z$. This character ψ' is the unique constituent of $\psi_{\widehat{L}}$ that has p'-multiplicity and P_0-relative defect zero. Since $|\widehat{L} : P_0|_p = |\mathrm{N}_K(D) : D_0|_p$, and Z is central in $\widehat{G}$, we easily check that ψ'_{L_0} coincides with θ' via the isomorphism ϵ_{L_0}.

As ψ has an extension $\widetilde{\psi} = \widetilde{\theta}_0 \in \mathrm{Irr}(\widehat{G} \mid \theta_0)$, the character ψ' has an extension $\widetilde{\psi}' \in \mathrm{Irr}(\widehat{H} \mid \psi')$ for $\widehat{H} := \mathrm{N}_{\widehat{G}}(P)$ with the properties described in Proposition 5.12, namely $\mathrm{Irr}(\widetilde{\psi}_{\mathrm{C}_{\widehat{G}}(\widehat{M})}) = \mathrm{Irr}(\widetilde{\psi}'_{\mathrm{C}_{\widehat{G}}(\widehat{M})})$ and

$$\widetilde{\psi}(x)^* = e^*(\widetilde{\psi}'(x))^*$$

for all p-regular elements $x \in \mathrm{C}_{\widehat{G}}(P)$ with $P \in \mathrm{Syl}_p(\mathrm{C}_{\widehat{M}}(x))$, where e is the multiplicity of ψ' in $\psi_{\widehat{L}}$. According to Theorem 5.2(b) the integer e also satisfies $e \equiv \pm 1 \bmod p$ and

$$\psi(1)_{p'} \equiv e^* |\widehat{K} : \widehat{L}|_{p'} \psi'(1)_{p'} \bmod p. \tag{5.2}$$

Now by Theorem 4.1(d), we have $\epsilon(\mathrm{C}_{\widehat{G}}(\widehat{M})) = \mathrm{C}_G(M)$. Also, if $x \in \mathrm{C}_G(M)$ then the scalar matrix $\mathcal{P}(x)$ is associated to the scalar $\mu((x, 1))$, where $\{\mu\} = \mathrm{Irr}(\widetilde{\psi}_{\mathrm{C}_{\widehat{G}}(\widehat{M})})$.

Now, let $\mathcal{D}'$ be a representation affording $\widetilde{\psi}'$. We define $\mathcal{P}' : H \to \mathrm{GL}_{\psi'(1)}(\mathbb{C})$ by

$$\mathcal{P}'(h) = \mathcal{D}'((h, 1)) \qquad \text{for every } h \in H$$

which defines a projective representation of H. Recall that ψ'_{L_0} coincides with θ' via ϵ_{L_0}. Hence $\mathcal{P}'_L$ affords θ'. Since the factor set α' of $\mathcal{P}'$ satisfies

$$\alpha'(h_1, h_2) = (h_1, 1)(h_2, 1)(h_1 h_2, 1)^{-1} \qquad \text{for every } h_1, h_2 \in H$$

by the definition of $\mathcal{P}'$, we see that α' is the restriction of α to $H \times H$ and hence $\mathcal{P}'$ is a projective representation associated to θ'. We can also check that the scalar matrix $\mathcal{P}'(x)$ ($x \in \mathrm{C}_G(M)$) is associated to the scalar $\mu((x, 1))$ where $\{\mu\} = \mathrm{Irr}(\widetilde{\psi}'_{\mathrm{C}_{\widehat{G}}(\widehat{M})})$.

Let $x \in G$ be a p-regular element with $D \in \mathrm{Syl}_p(\mathrm{C}_M(x))$. Then there exists $z \in Z \leq \mathrm{Z}(\widehat{G})$ such that $y := (x, z)$ is p-regular. This element centralizes P/Z_p.

The group $\langle y \rangle$ acts on P. Since $(|\langle y \rangle|, |P|) = 1$, Theorem (3.28) of [Isa08] implies that

$$P/Z_p = \mathrm{C}_{P/Z_p}(y) = \mathrm{C}_P(y) Z_p / Z_p.$$

This proves $P \leq \mathrm{C}_{\widehat{G}}((x, 1))$ and $P \in \mathrm{Syl}_p(\mathrm{C}_{\widehat{G}}((x, 1)))$. The extensions $\widetilde{\psi}$ and $\widetilde{\psi}'$ satisfy

$$\widetilde{\psi}(y)^* = e^* \widetilde{\psi}'(y)^*$$

and this implies by the definition of $\mathcal{P}$ and $\mathcal{P}'$ that

$$\mathrm{tr}(\mathcal{P}(x))^* = e^* \, \mathrm{tr}(\mathcal{P}'(x))^*.$$

For any character $\theta_1 \in \mathrm{Irr}_0(M \mid \theta)$ there exists an associated projective representation $\mathcal{R}$ of G_{θ_1} such that $\mathcal{R}_K = \mathcal{P}_K$. The projective representation $\mathcal{R}$ can be obtained from $\mathcal{P}_{G_{\theta_1}}$ by multiplying with a map $\xi : G_{\theta_1}/K \to \mathbb{C}$ with $\mathcal{R} = \xi \mathcal{P}_{G_{\theta_1}}$.

Let $\mathcal{R}' := \xi_{H_{\theta_1}} \mathcal{P}'_{H_{\theta_1}}$. Then $\mathcal{R}$ and $\mathcal{R}'$ have the same factor sets. In particular, $\mathcal{R}'$ has a factor set that is constant on $\mathrm{N}_M(D)$-cosets. This implies that $\mathcal{R}'_{\mathrm{N}_N(D)}$ affords a character η_1 that is an extension of θ'. In this way, we obtain a well-defined bijection

$$\Pi_{\theta, D} : \mathrm{Irr}_0(M \mid \theta) \to \mathrm{Irr}_0(\mathrm{N}_M(D) \mid \theta')$$

with $\theta_1 \mapsto \eta_1$. By the construction $\Pi_{\theta, D}$ is H_θ-equivariant.

In the next step we prove that $(G_{\theta_1}, M, \theta_1) \sim_b (\mathrm{N}_G(D)_{\theta_1}, \mathrm{N}_M(D), \eta_1)$. According to Theorem 4.4 it is sufficient to ensure that $\mathcal{R}$ and $\mathcal{R}'$ satisfy the conditions described there. Observe first that the groups $M \lhd G_{\theta_1}$ and $\mathrm{N}_M(D) \lhd H_{\eta_1}$ satisfy the required equations there.

We now check the properties on the defect group. Let Q be a defect group of $\mathrm{bl}(\eta_1)$. By the considerations above, $\mathrm{bl}(\theta')$ has D_0 as a defect group. By Theorem (9.26) of [Nav98] the group Q satisfies $Q \cap L = D_0$. Since $D \lhd \mathrm{N}_M(D)$, Q contains D. All this implies $Q = D$. Hence D is a defect group of $\mathrm{bl}(\theta_1)$ and of $\mathrm{bl}(\eta_1)$.

As $\mathcal{P}$ and $\mathcal{P}'$ have the same factor sets, the projective representations $\mathcal{R}$ and $\mathcal{R}'$ have the analogous property. For $x \in \mathrm{C}_G(M)$ the matrices $\mathcal{P}(x)$ and $\mathcal{P}'(x)$ were associated to the same scalar $\mu((x,1))$, and $\mathcal{R}(x)$ and $\mathcal{R}'(x)$ are scalar matrices associated to $\zeta(x)\mu((x,1))$.

It remains to check the equation from 4.4(i)(c). Let $x \in G$ be a p-regular element with $D \in \mathrm{Syl}_p(\mathrm{C}_M(x))$. The characters θ_1 and η_1 have height zero. We already know that

$$\mathrm{tr}(\mathcal{P}(x))^* = e^* \,\mathrm{tr}(\mathcal{P}'(x))^*. \tag{5.3}$$

As $\theta_1(1) = \psi(1)$ and $\eta_1(1) = \psi'(1)$, equation (5.2) implies

$$\theta_1(1)_{p'} \equiv e|K : L|_{p'}\eta_1(1)_{p'} \bmod p.$$

This proves

$$\left(\frac{|\mathrm{N}_M(D)|_{p'}}{\eta_1(1)_{p'}}\right)^* = \left(e\frac{|M|_{p'}}{\theta_1(1)_{p'}}\right)^*.$$

With (5.3), this shows

$$\left(\frac{|M|_{p'}\,\mathrm{tr}(\mathcal{P}(x))}{\theta_1(1)_{p'}}\right)^* = \left(\frac{|\mathrm{N}_M(D)|_{p'}\,\mathrm{tr}(\mathcal{P}'(x))}{\eta_1(1)_{p'}}\right)^*$$

for every p-regular element $x \in H$ with $D \in \mathrm{Syl}_p(\mathrm{C}_M(x))$. Now Theorem 4.4 implies that $(G_{\theta_1}, M, \theta_1) \sim_b (H_{\eta_1}, M, \eta_1)$. □

Corollary 5.14. *Let G be a finite group, and $K, M \lhd G$ with $K \le M$ be such that M/K is a p-group. Let $D_0 \lhd K$ be a p-subgroup with $D_0 \le \mathrm{Z}(M)$. Let D be a p-subgroup of M and let $H := \mathrm{N}_G(D)$. Then there exists an H-equivariant bijection*

$$\Pi_D : \mathrm{Irr}_0(M \mid D) \to \mathrm{Irr}_0(\mathrm{N}_M(D) \mid D)$$

with $(G_\tau, M, \tau) \sim_b (H_\tau, \mathrm{N}_M(D), \Pi_D(\tau))$ for every $\tau \in \mathrm{Irr}_0(M \mid D)$.

Proof. Let $\tau \in \mathrm{Irr}_0(M \mid D)$. Then $\theta := \tau_K$ is an irreducible character, as τ is a height zero character and M/K is a p-group. This implies that $\tau \in \mathrm{Irr}_0(M \mid \theta)$. Let $\mathbb{T}$ be the set of characters $\theta \in \mathrm{Irr}_0(K \mid D_0)$ such that the block of M covering $\mathrm{bl}(\theta)$ has D as defect group and θ extends to a character of M. Then $\mathrm{Irr}_0(M \mid D)$ can be written as

$$\mathrm{Irr}_0(M \mid D) = \bigcup_{\theta \in \mathbb{T}} \mathrm{Irr}_0(M \mid \theta).$$

Now we choose a set $\mathbb{T}_1$ inside $\mathbb{T}$ that is a complete set of representatives of H-orbits.

For $\tau \in \mathrm{Irr}_0(M \mid \theta)$ with $\theta \in \mathbb{T}_1$ we define

$$\Pi_D(\tau) = \Pi_{\theta,D}(\tau),$$

where $\Pi_{\theta,D}$ is the G_θ-invariant bijection from Theorem 5.13. Now $\Pi_{\theta,D}(\tau)$ belongs to $\mathrm{Irr}_0(\mathrm{N}_M(D) \mid \theta')$, where θ' is the D-correspondent of θ. This character belongs to $\mathrm{Irr}_0(\mathrm{N}_M(D) \mid D)$. Furthermore we have

$$(G_\tau, M, \tau) \sim_{\mathrm{b}} (H_\tau, H \cap M, \Pi_D(\tau)).$$

Every character $\tau \in \mathrm{Irr}_0(M \mid D)$ is H-conjugate to some character in $\mathrm{Irr}_0(M \mid \theta)$ with $\theta \in \mathbb{T}_1$, i.e.

$$\tau^h \in \mathrm{Irr}_0(M \mid \theta)$$

for some $h \in H$. By the definition of $\mathbb{T}_1$ the coset hH_θ is unique. We set

$$\Pi_D(\tau) := \Pi_{\theta,D}(\tau^h)^{h^{-1}}.$$

As $\Pi_{\theta,D}$ is H_θ-equivariant and as θ and hH_θ are uniquely determined by τ, we see that $\Pi_D(\tau)$ is well-defined. By the definition, Π_D satisfies

$$(G_\tau^h, M, \tau^h) \sim_{\mathrm{b}} (H_\tau^h, H \cap M, \Pi_D(\tau)^h).$$

Now, we easily check that conjugation is compatible with the relation "$\sim_{\mathrm{b}}$". This implies

$$(G_\tau, M, \tau) \sim_{\mathrm{b}} (H_\tau, H \cap M, \Pi_D(\tau)). \qquad \square$$

6. A height preserving bijection coming from the inductive AM-condition

In this section we concentrate on the proof of a height preserving bijection in the presence of semisimple normal subgroups.

Theorem 6.1 (Theorem 7.9 of [Spä13]). *Let K be a finite perfect group such that $K/\mathrm{Z}(K) \cong S^r$ for some non-abelian simple group S. Let $D_0 \not\leq \mathrm{Z}(K)$ be a radical p-subgroup of K. Assume that S satisfies the inductive AM-condition for p from Definition* 7.2 *of* [Spä13]. *Then:*

(a) *There exists an $\mathrm{Aut}(K)_{D_0}$-stable subgroup $M \lneq K$ with $\mathrm{N}_K(D_0) \leq M$.*
(b) *There exists an $\mathrm{Aut}(K)_{D_0}$-equivariant bijection $\Omega : \mathrm{Irr}_0(K \mid D_0) \to \mathrm{Irr}_0(M \mid D_0)$ with*

$$\mathrm{Irr}(\Omega(\chi)_{\mathrm{Z}(K)}) = \mathrm{Irr}(\chi_{\mathrm{Z}(K)}) \quad \textit{for every } \chi \in \mathrm{Irr}_0(K \mid D_0) \tag{6.1}$$

and

$$\Omega(\mathrm{Irr}_0(b)) = \mathrm{Irr}_0(b_1) \quad \textit{for every } b \in \mathrm{Bl}(K \mid D_0) \textit{ and } b_1 \in \mathrm{Bl}(M \mid D_0),$$

whenever b and b_1 have the same Brauer correspondent in $\mathrm{Bl}(\mathrm{N}_K(D_0))$.

(c) *Suppose $K \lhd G$ for some finite group G with $\mathrm{Z}(K) \leq \mathrm{Z}(G)$ and let $B \in \mathrm{Bl}(G)$ be such that some defect group D of B satisfies $D \cap K = D_0$. Let $\theta \in \mathrm{Irr}_0(K \mid D_0)$, $\theta' := \Omega(\theta) \in \mathrm{Irr}_0(M \mid D_0)$, and $B_1 \in \mathrm{Bl}(M\mathrm{N}_G(D_0))$ the block with $B_1^G = B$ that covers $\mathrm{bl}(\theta')$. Then*

$$|\mathrm{Irr}(G \mid \theta) \cap \mathrm{Irr}_0(B)| = |\mathrm{Irr}(M\mathrm{N}_G(D_0) \mid \theta') \cap \mathrm{Irr}_0(B_1)|.$$

Proof. By the construction of Ω given in the proof of Theorem 7.9 of [Spä13] the bijection Ω depends only on the epimorphism $\pi : \widetilde{X} \to K$, where $\widetilde{X}$ is the universal covering group of S^i. Using the bijection from Proposition 7.7 of [Spä13] we deduce that Ω is $\mathrm{Aut}(K)_{D_0}$-equivariant. The remaining parts of the statement are identical with the ones of Theorem 7.9 of [Spä13]. □

We extend this statement by including the non-height zero characters of G that lie above a height zero character of K.

Proposition 6.2. *Let K be a perfect group such that $K/\mathrm{Z}(K) \cong S^r$, where S is a non-abelian simple group satisfying the inductive AM-condition from Definition 7.2 of [Spä13]. Let D_0 be a non-central radical p-subgroup of K and let $\theta \in \mathrm{Irr}_0(K \mid D_0)$. Let M be the $\mathrm{N}_G(D_0)$-stable group with $\mathrm{N}_K(D_0) \leq M \lneq K$ and*

$$\Omega : \mathrm{Irr}_0(K \mid D_0) \to \mathrm{Irr}_0(M \mid D_0)$$

the $\mathrm{N}_G(D_0)$-equivariant bijection from Theorem 6.1. Suppose that $K \lhd G$. Then for every $\theta \in \mathrm{Irr}_0(K \mid D_0)$ and $\theta' := \Omega(\theta)$, we have

$$(G_\theta, K, \theta) \sim_b (M\mathrm{N}_G(D_0)_{\theta'}, M, \theta').$$

Proof. Let X be the universal covering group of S, $\widetilde{X} := X^r$ and $\iota : \widetilde{X} \to K$ be an epimorphism that exists as K is a perfect group and a central extension of S^r. Let $\widetilde{D}$ be the Sylow p-subgroup of $\iota^{-1}(D_0)$ and $\widetilde{M} \leq \widetilde{X}$ be the group from Proposition 7.7(a) of [Spä13]. Accordingly $\widetilde{M}$ is $\mathrm{Aut}(\widetilde{X})_{\widetilde{D}}$-stable and satisfies $\mathrm{N}_{\widetilde{X}}(\widetilde{D}) \leq \widetilde{M} \lneq \widetilde{X}$. The group $M := \iota(\widetilde{M})$ has the required properties.

By Proposition 7.7(b) of [Spä13] there also exists an $\mathrm{Aut}(\widetilde{X})_{\widetilde{D}}$-equivariant bijection

$$\widetilde{\Omega} : \mathrm{Irr}_0(\widetilde{X} \mid \widetilde{D}) \to \mathrm{Irr}_0(\widetilde{M} \mid \widetilde{D})$$

such that $\widetilde{\Omega}(\mathrm{Irr}_0(\widetilde{X} \mid \widetilde{D}) \cap \mathrm{Irr}(\widetilde{X} \mid \nu)) \subseteq \mathrm{Irr}_0(\widetilde{M} \mid \nu)$ for every $\nu \in \mathrm{Irr}(\mathrm{Z}(\widetilde{X}))$.

The bijection

$$\Omega : \mathrm{Irr}_0(K \mid D_0) \to \mathrm{Irr}_0(M \mid D_0)$$

defined by $\Omega(\chi) \circ \iota_M = \widetilde{\Omega}(\chi \circ \iota)$ for every $\chi \in \mathrm{Irr}_0(K \mid D_0)$ is $\mathrm{N}_G(D_0)$-equivariant.

Let $\theta \in \mathrm{Irr}_0(K \mid D_0)$ and assume first that θ is faithful. Without loss of generality we may assume $G_\theta = G$. Then the assumptions of Theorem 6.1 are satisfied by the groups $K \lhd G$ as $\mathrm{Z}(K) \leq \mathrm{Z}(G)$. In the following we assume without loss of generality $G_\theta = G$. In the proof of Theorem 7.9 of [Spä13] a projective representation $\mathcal{P}$ of G associated to θ is given, and a projective representation of $\mathcal{P}'$ of H associated to $\theta' := \Omega(\theta) \in \mathrm{Irr}_0(M \mid D_0)$. Using Section 3 of [Spä13], we check that the factor sets of $\mathcal{P}$ and $\mathcal{P}'$

coincide. Furthermore for every $x \in \mathrm{C}_G(K)$ the matrices $\mathcal{P}(x)$ and $\mathcal{P}'(x)$ are associated to the same scalar. According to (7.13) of [Spä13] the projective representations satisfy

$$\left(\frac{|K|_{p'}\operatorname{tr}(\mathcal{P}(x))}{\theta(1)_{p'}}\right)^* = \left(\frac{|M|_{p'}\operatorname{tr}(\mathcal{P}'(x))}{\theta'(1)_{p'}}\right)^*$$

for every p'-element $x \in M\mathrm{N}_G(D_0)$ with $D_0 \in \mathrm{Syl}_p(\mathrm{C}_M(x))$. According to Theorem 4.4 this implies

$$(G_\theta, K, \theta) \sim_b (M\mathrm{N}_G(D_0)_{\theta'}, M, \theta') \qquad \text{for every } \theta \in \mathrm{Irr}_0(K \mid D_0),$$

where $\theta' := \Omega(\theta)$.

Let $\theta \in \mathrm{Irr}_0(K \mid D_0)$ and $Z := \mathrm{Z}(K) \cap \ker(\theta)$, $\overline{G} := G/Z$, $\overline{K} := K/Z$, $\overline{M} := M/Z$, and $\overline{D}_0 := D_0Z/Z$ and $\overline{M} := M/Z$. As above, $\widetilde{\Omega}$ defines also an $\mathrm{N}_{\overline{G}}(\overline{D}_0)$-equivariant bijection $\overline{\Omega} : \mathrm{Irr}_0(\overline{K} \mid \overline{D}_0) \to \mathrm{Irr}_0(\overline{M} \mid \overline{D}_0)$. Let $\overline{\theta} \in \mathrm{Irr}(\overline{K})$ be the character associated to θ, which is also of height zero and belongs to a block with defect group $\overline{D}_0 := D_0Z/Z$ by Proposition 2.5(c). The character $\overline{\theta}_{Z(K)/Z}$ is faithful. Hence the above considerations imply

$$(\overline{G}_{\overline{\theta}}, \overline{K}, \overline{\theta}) \sim_b (\overline{M}\mathrm{N}_{\overline{G}}(\overline{D})_{\overline{\theta}'}, \overline{M}, \overline{\theta}'),$$

where $\overline{\theta}' := \overline{\Omega}(\overline{\theta})$. By Lemma 3.12, this implies

$$(G_\theta, K, \theta) \sim_b (M\mathrm{N}_G(D)_{\theta'}, M, \theta'),$$

where $\theta' \in \mathrm{Irr}(M)$ is the character associated to $\overline{\theta}'$. This character θ' coincides with $\Omega(\theta)$ by the definition of $\overline{\Omega}$. □

We next relax the assumption that $K/\mathrm{Z}(K)$ is the direct product of isomorphic non-abelian simple groups.

Corollary 6.3. *Let K be a finite group such that $K/\mathrm{Z}(K)$ is the direct product of non-abelian simple groups which satisfy the inductive AM-condition from Definition* 7.2 *of* [Spä13]. *Let D_0 be a radical non-central p-subgroup of K. Then there exist an $\mathrm{N}_G(D_0)$-stable group M with $\mathrm{N}_K(D_0) \leq M \lneq K$ and an $\mathrm{N}_G(D_0)$-equivariant bijection*

$$\Omega : \mathrm{Irr}_0(K \mid D_0) \to \mathrm{Irr}_0(M \mid D_0).$$

Suppose $K \lhd G$ for some finite group G. Then

$$(G_\theta, K, \theta) \sim_b (M\mathrm{N}_G(D_0)_{\theta'}, M, \theta')$$

for every $\theta \in \mathrm{Irr}_0(K \mid D_0)$ and $\theta' := \Omega(\theta)$.

Proof. By hypothesis, there exist non-isomorphic non-abelian simple groups $S_1, \ldots, S_j$ and integers $r_1, \ldots, r_j$ such that

$$K/\mathrm{Z}(K) \cong S_1^{r_1} \times \cdots \times S_j^{r_j}.$$

For each group S_i there exists a group K_i with $K_i \lhd K$, $[K_i, K_i] = K_i$ and $K_i/\mathrm{Z}(K_i) \cong S_i^{r_i}$. Let $D_i := D_0 \cap K_i$ and $\theta \in \mathrm{Irr}_0(K \mid D_0)$. If $D_i \not\leq \mathrm{Z}(K_i)$ then there exists an $\mathrm{N}_G(D_i)$-equivariant bijection $\Omega_i : \mathrm{Irr}_0(K_i \mid D_i) \to \mathrm{Irr}_0(M_i \mid D_i)$ with

$$(G_{\theta_i}, K_i, \theta_i) \sim_{\mathrm{b}} (H_{i,\theta_i}, M_i, \theta_i') \quad \text{for every } \theta_i \in \mathrm{Irr}_0(K_i \mid D_i) \text{ and } \theta_i' := \Omega_i(\theta_i),$$

where M_i with $K_i \gneq M_i \geq \mathrm{N}_{K_i}(D_i)$ is given by Proposition 6.2 and $H_i := M_i \mathrm{N}_G(D_i)$ (see Proposition 6.2). For i with $D_i \leq \mathrm{Z}(K_i)$ let $M_i := K_i$ and $\Omega_i : \mathrm{Irr}_0(K_i \mid D_i) \to \mathrm{Irr}_0(M_i \mid D_i)$ with $\theta_i \mapsto \theta_i' := \theta_i$ be the identity map.

Let $\theta \in \mathrm{Irr}_0(K \mid D_0)$. We can write θ as central product of characters $\theta_i \in \mathrm{Irr}_0(K_i \mid D_i)$ and a character $\nu \in \mathrm{Irr}(\mathrm{Z}(K))$. Without loss of generality we may assume $G_\theta = G$.

We observe that by Proposition 3.9(c) the block isomorphism of character triples between (G, K_1, θ_1) and (H_1, M_1, θ_1') maps $\theta_1 \cdot \theta_2$ to $\theta_1' \cdot \theta_2$ because $K_2 \leq \mathrm{C}_G(K_1)$. (See Section 5 of [IMN07] for the notation of characters of central products.) This implies that

$$(G, K_1K_2, \theta_1 \cdot \theta_2) \sim_{\mathrm{b}} (H_1, M_1K_2, \theta_1' \cdot \theta_2),$$

using Proposition 3.9(b).

Analogously, we know $(H_1, K_2, \theta_2) \sim_{\mathrm{b}} (H_2 \cap H_1, M_2, \theta_2')$ from the properties of Ω_2. This implies $(H_1, M_1K_2, \theta_1' \cdot \theta_2) \sim_{\mathrm{b}} (H_1 \cap H_2, M_1M_2, \theta_1' \cdot \theta_2')$, as above. According to Lemma 3.8(a) this implies

$$(G, K_1K_2, \theta_1 \cdot \theta_2) \sim_{\mathrm{b}} (H_1 \cap H_2, M_1M_2, \theta_1' \cdot \theta_2').$$

Successively applying this procedure we obtain

$$(G, K, \theta) \sim_{\mathrm{b}} (\mathrm{N}_G(D_0)M, M, \theta'),$$

where $M = \langle M_1, M_2, \ldots \rangle$, and $\theta' := \theta_1' \cdots \theta_j' \cdot \nu$. □

7. A bijection above height zero characters

In this section we finally prove Theorem B, which is a direct consequence of our Theorem 7.1 below and Corollary 3.10.

Recall that $\mathrm{Irr}_0(N \mid D)$ denotes the set of height zero characters of N that lie in a block with defect group D. If $Z \lhd N$ and $\nu \in \mathrm{Irr}(Z)$, then we write $\mathrm{Irr}_0(N \mid D, \nu) = \mathrm{Irr}_0(N \mid D) \cap \mathrm{Irr}(N \mid \nu)$.

Theorem 7.1. *Let $\mathcal{S}$ be a collection of finite non-abelian simple groups of order divisible by p that satisfy the inductive AM-condition* (*Definition* 7.2 *of* [Spä13]) *for the prime p. Assume that N is a finite group such that all the non-abelian simple groups involved in N of order divisible by p are in $\mathcal{S}$. Suppose that $N \lhd G$ for some finite group G, and let D be any p-subgroup of N. Then there exists an $\mathrm{N}_G(D)$-equivariant bijection*

$$\Pi_D : \mathrm{Irr}_0(N \mid D) \to \mathrm{Irr}_0(\mathrm{N}_N(D) \mid D)$$

with

$$(G_\tau, N, \tau) \sim_{\mathrm{b}} (\mathrm{N}_G(D)_{\tau'}, \mathrm{N}_N(D), \tau')$$

for every $\tau \in \mathrm{Irr}_0(N \mid D)$ and $\tau' := \Pi_D(\tau)$.

We prove Theorem 7.1 by induction on $|N : \mathrm{Z}(N)|$, and we do this in a series of intermediate results.

Remark 7.2. Without loss of generality we can assume $G = N\mathrm{N}_G(D)$.

Proof. Assume that Theorem 7.1 holds for N and $G_1 := N\mathrm{N}_G(D)$. The bijection Π_D has then the required equivariance properties since $\mathrm{N}_{G_1}(D) = \mathrm{N}_G(D)$. Now, for every $\tau \in \mathrm{Irr}_0(N \mid D)$ the group G_τ is contained in G_1. Hence Theorem 7.1 holds for the groups $N \lhd G$ if it holds for $N \lhd N\mathrm{N}_G(D)$. □

Lemma 7.3. *Let $K \lhd G$ where $K \leq N$ and $|N : K| < |N : \mathrm{Z}(N)|$. Suppose that $\zeta \in \mathrm{Irr}_0(K \mid D_0)$ is N-invariant, where $D_0 := K \cap D$. Then there exists an $\mathrm{N}_{G_\zeta}(D)$-equivariant bijection*

$$\Upsilon_{D,\zeta} : \mathrm{Irr}_0(N \mid D, \zeta) \to \mathrm{Irr}_0(K\mathrm{N}_N(D) \mid D, \zeta)$$

such that $(G_\tau, N, \tau) \sim_b (K\mathrm{N}_{G_\tau}(D), K\mathrm{N}_N(D), \Upsilon_{D,\zeta}(\tau))$ for every $\tau \in \mathrm{Irr}_0(N \mid D, \zeta)$.

Proof. Since $G_\tau \leq G_\zeta$ for every $\tau \in \mathrm{Irr}_0(N \mid D, \zeta)$, we may assume that $G_\zeta = G$.

Let $\mathcal{P}$ be a projective representation of G associated to ζ. Then $\mathcal{P}$ determines a group $\widehat{G} = \{(g, z) \mid g \in G, z \in Z\}$, where $1 \times Z \leq \mathrm{Z}(\widehat{G})$, and an epimorphism $\epsilon : \widehat{G} \to G$ given by $(g, z) \mapsto g$, as in the proof of Theorem 4.1. (We identify Z with $1 \times Z$.) Furthermore, let $K_0 \lhd \widehat{G}$ be defined as in Theorem 4.1(a). This group K_0 is isomorphic to K via ϵ_{K_0} and via this isomorphism the character ζ defines also a character $\zeta_0 \in \mathrm{Irr}(K_0)$ from Theorem 4.1(b). This character has an extension $\widetilde{\zeta} \in \mathrm{Irr}(\widehat{G})$.

Below, sometimes it is useful to identify certain subgroups U/Z of $\widehat{G}/Z$ with $\epsilon(U) \leq G$. Also, if $X \leq G$, then $\widehat{X} := \epsilon^{-1}(X) = \{(x, z) \mid z \in Z\}$ is a subgroup of $\widehat{G}$ of order $|X|\,|Z|$, and $\widehat{X}/Z \cong X$.

Now, $\epsilon^{-1}(D)/Z \cong D$ is a p-group, hence let $\widehat{D}_0$ be the unique Sylow p-subgroup of $\widehat{D} = \epsilon^{-1}(D)$. Thus $\epsilon^{-1}(D) = \widehat{D}_0 Z$. Notice that $\mathrm{N}_{\widehat{G}}(\widehat{D}) = \mathrm{N}_{\widehat{G}}(\widehat{D}_0) = \widehat{\mathrm{N}_G(D)}$.

Now, $\widehat{N}/\widehat{K} \cong N/K$, and therefore all non-abelian simple groups involved in $\widehat{N}/K_0$ are also involved in N and contained in $\mathcal{S}$. Also, $\widehat{K}/K_0 \leq \mathrm{Z}(\widehat{N}/K_0) \leq \widehat{N}/K_0$, and we conclude that $|\widehat{N}/K_0 : \mathrm{Z}(\widehat{N}/K_0)| \leq |N : K| < |N : \mathrm{Z}(N)|$. Now, we apply induction to the groups $\overline{G} := \widehat{G}/K_0$, $\overline{N} := \widehat{N}/K_0$ and $\overline{D} := \widehat{D}_0 K_0/K_0$. We conclude that there exists an $\mathrm{N}_{\overline{G}}(\overline{D})$-equivariant bijection

$$\Pi_{\overline{D}} : \mathrm{Irr}_0(\overline{N} \mid \overline{D}) \to \mathrm{Irr}_0(\mathrm{N}_{\overline{N}}(\overline{D}) \mid \overline{D}),$$

such that for every $\overline{\rho} \in \mathrm{Irr}_0(\overline{N} \mid \overline{D})$ we have

$$(\overline{G}_{\overline{\rho}}, \overline{N}_{\overline{\rho}}, \overline{\rho}) \sim_b (\mathrm{N}_{\overline{G}_{\overline{\rho}}}(\overline{D}), \mathrm{N}_{\overline{N}}(\overline{D}), \overline{\rho}'),$$

where $\overline{\rho}' := \Pi_{\overline{D}}(\overline{\rho})$.

Now, let $\tau \in \mathrm{Irr}_0(\widehat{N} \mid \widehat{D}_0, \zeta_0)$. By Corollary (6.17) of [Isa76] there exists a unique character $\rho \in \mathrm{Irr}(\widehat{N})$ with $K_0 \leq \ker(\rho)$ such that $\tau = \widetilde{\zeta}_{\widehat{N}}\rho$. Proposition 2.5(c) implies that the character $\overline{\rho} \in \mathrm{Irr}(\overline{N})$ associated to ρ has height zero and lies in a block with defect group $\overline{D} = \widehat{D}_0 K_0/K_0$, since τ is a character of height zero and lies in a block with defect group $\widehat{D}_0$. Hence $\overline{\rho} \in \mathrm{Irr}_0(\overline{N} \mid \overline{D})$. Now, $\overline{\rho}' := \Pi_{\overline{D}}(\overline{\rho})$ is defined and satisfies

$$(\overline{G}_{\overline{\rho}}, \overline{N}_{\overline{\rho}}, \overline{\rho}) \sim_b (\mathrm{N}_{\overline{G}_{\overline{\rho}}}(\overline{D}), \mathrm{N}_{\overline{N}}(\overline{D}), \overline{\rho}').$$

Let $\rho' \in \mathrm{Irr}(\mathrm{N}_{\widehat{N}}(\widehat{D}_0 K_0))$ be the lift of $\overline{\rho}'$. Theorem 4.6 shows that the character $\tau' := \widetilde{\zeta}_{\mathrm{N}_{\widehat{N}}(\widehat{D}_0 K_0)}\rho'$ satisfies

$$(\widehat{G}_\tau, \widehat{N}, \tau) \sim_b (\mathrm{N}_{\widehat{G}_\tau}(\widehat{D}_0 K_0), \mathrm{N}_{\widehat{N}}(\widehat{D}_0 K_0), \tau').$$

We define

$$\Upsilon_{\widehat{D}_0, \zeta_0} : \mathrm{Irr}_0(\widehat{N} \mid \widehat{D}_0, \zeta_0) \to \mathrm{Irr}_0(\mathrm{N}_{\widehat{N}}(\widehat{D}_0 K_0) \mid \widehat{D}_0, \zeta_0)$$

by $\tau = \widetilde{\zeta}_{\widehat{N}}\rho \mapsto \widetilde{\zeta}_{\mathrm{N}_{\widehat{N}}(\widehat{D}_0 K_0)}\rho'$. Let $\tau' = \widetilde{\zeta}_{\mathrm{N}_{\widehat{N}}(\widehat{D}_0 K_0)}\rho'$. As $\widetilde{\zeta}_{\widehat{N}}$ is $\widehat{G}$-invariant and $\Pi_{\overline{D}}$ is $\mathrm{N}_{\overline{G}}(\overline{D})$-equivariant, the map $\Upsilon_{\widehat{D}_0, \zeta_0}$ is $\mathrm{N}_{\widehat{G}}(\widehat{D}_0 K_0)$-equivariant.

Also, using the definition of block isomorphism, we also have $\mathrm{Irr}(\tau_Z) = \mathrm{Irr}(\tau'_Z)$ for every $\tau \in \mathrm{Irr}_0(\widehat{N} \mid \widehat{D}_0, \zeta_0)$. Hence $\Upsilon_{\widehat{D}_0, \zeta_0}$ maps the characters $\tau \in \mathrm{Irr}_0(\widehat{N} \mid \widehat{D}_0, \zeta_0)$ with $Z \leq \ker(\tau)$ to the characters in $\mathrm{Irr}_0(\mathrm{N}_{\widehat{N}}(\widehat{D}_0 K_0) \mid \widehat{D}_0, \zeta_0)$ which contain Z in its kernel. Now we observe that the characters in $\mathrm{Irr}_0(\widehat{N} \mid \widehat{D}_0, \zeta_0 \times 1_Z)$ correspond naturally to the characters $\mathrm{Irr}_0(N \mid D, \zeta)$. (Use that Z is central in $\widehat{N}$ and Theorems 5.8.8 and 5.8.11 of [NT89].) Hence there is a natural correspondence between $\mathrm{Irr}_0(\widehat{N} \mid \widehat{D}_0, \zeta_0 \times 1_Z)$ and $\mathrm{Irr}_0(N \mid D, \zeta)$.

Recall $\mathrm{N}_{\widehat{G}}(\widehat{D}) = \mathrm{N}_{\widehat{G}}(\widehat{D}_0) = \widehat{\mathrm{N}_G(D)}$ and hence $(K_0 \mathrm{N}_{\widehat{G}}(\widehat{D}))/Z = K\mathrm{N}_G(D)$.

Just as before, the set $\mathrm{Irr}_0(\mathrm{N}_{\widehat{N}}(\widehat{D}K_0) \mid \widehat{D}, \zeta_0 \times 1_Z)$ naturally corresponds to the set $\mathrm{Irr}_0(K\mathrm{N}_N(D) \mid D, \zeta_0)$. Via these correspondences, $\Upsilon_{\widehat{D}_0, \zeta_0}$ defines a bijection

$$\Upsilon_{D,\zeta} : \mathrm{Irr}_0(N \mid D, \zeta) \to \mathrm{Irr}_0(K\mathrm{N}_N(D) \mid D, \zeta).$$

If $\overline{\tau} \in \mathrm{Irr}_0(N \mid D, \zeta)$, let $\tau \in \mathrm{Irr}_0(\widehat{N} \mid \widehat{D}, \zeta_0 \times 1_Z)$ denote the corresponding element. Then

$$(\widehat{G}_\tau, \widehat{N}, \tau) \sim_b (\mathrm{N}_{\widehat{G}_\tau}(\widehat{D}K_0), \mathrm{N}_{\widehat{N}}(\widehat{D}K_0), \tau'),$$

where $\tau' := \Upsilon_{\widehat{D}, \zeta_0}(\tau)$. Let $\overline{\tau}'$ be the corresponding character in $\mathrm{N}_N(DK)$.

Finally, using that $\mathrm{C}_{\widehat{G}}(\widehat{N})/Z = \mathrm{C}_G(N)$ by Theorem 4.1(d), and Corollary 4.5, we have

$$(G_{\overline{\tau}}, N, \overline{\tau}) \sim_b (K\mathrm{N}_G(D)_{\overline{\tau}}, K\mathrm{N}_N(D), \overline{\tau}')$$

for every $\overline{\tau} \in \mathrm{Irr}_0(N \mid D, \zeta)$. □

In the proof of the following statement, for $K \lhd N$, $\tau \in \mathrm{Irr}(N)$ and $\zeta \in \mathrm{Irr}(\tau_K)$ we denote by τ_ζ the unique character in $\mathrm{Irr}(N_\zeta \mid \zeta)$ with $\tau_\zeta^N = \tau$.

Proposition 7.4. *Let $K \lhd G$ where $K \leq N$ and $|N : K| < |N : \mathrm{Z}(N)|$. Then there exists an $\mathrm{N}_G(D)$-equivariant bijection*

$$\Upsilon_{D,K} : \mathrm{Irr}_0(N \mid D) \to \mathrm{Irr}_0(K\mathrm{N}_N(D) \mid D)$$

with

$$(G_\tau, N, \tau) \sim_b (K\mathrm{N}_{G_{\tau'}}(D), K\mathrm{N}_N(D), \tau')$$

for every $\tau \in \mathrm{Irr}_0(N \mid D)$ and $\tau' := \Upsilon_{D,K}(\tau)$.

Proof. Let $D_0 = K \cap D$. Let $\{\zeta_1, \ldots, \zeta_s\}$ be a complete set of representatives of $\mathrm{N}_G(D)$-orbits on the D-invariant characters in $\mathrm{Irr}_0(K \mid D_0)$. Let $N_i = N_{\zeta_i}$ and $G_i = G_{\zeta_i}$. We have $|N_i : K| \leq |N : K| < |N : \mathrm{Z}(N)|$. By Lemma 7.3 applied to N_i, there exists a $\mathrm{N}_{G_i}(D)$-equivariant bijection

$$\Upsilon_{D,\zeta_i} : \mathrm{Irr}_0(N_i \mid D, \zeta_i) \to \mathrm{Irr}_0(K\mathrm{N}_{N_i}(D) \mid D, \zeta_i)$$

with $((G_i)_\gamma, N_i, \gamma) \sim_b (K\mathrm{N}_{(G_i)_\gamma}(D), K\mathrm{N}_{N_i}(D), \Upsilon_{D,\zeta_i}(\gamma))$ for all $\gamma \in \mathrm{Irr}(N_i \mid D, \zeta_i)$.

Let $\tau \in \mathrm{Irr}_0(N \mid D)$. By Proposition 2.5(f), let $\zeta \in \mathrm{Irr}_0(K \mid D_0)$ be under τ (where $D_0 = D \cap K$) such that the character $\tau_\zeta \in \mathrm{Irr}(N_\zeta \mid D, \zeta)$ with $\tau_\zeta^N = \tau$. We also know that if ζ' is chosen instead of ζ, then $\zeta' = \zeta^u$ for some $u \in \mathrm{N}_N(D)$. Now, $\zeta^h = \zeta_i$ for some $h \in \mathrm{N}_G(D)$, and a unique i.

We define

$$\begin{aligned}\Upsilon_{D,K}(\tau) &= (\Upsilon_{D,\zeta_i}((\tau_\zeta)^h)^{K\mathrm{N}_N(D)})^{h^{-1}} \\ &= (\Upsilon_{D,\zeta_i}((\tau_\zeta)^h)^{h^{-1}})^{K\mathrm{N}_N(D)} \in \mathrm{Irr}(K\mathrm{N}_N(D) \mid \zeta).\end{aligned}$$

(Notice that $\Upsilon_{D,\zeta_i}((\tau_\zeta)^h) \in \mathrm{Irr}_0(K\mathrm{N}_{N_i}(D) \mid D, \zeta_i)$, and therefore it induces irreducibly to $K\mathrm{N}_N(D)$ by the Clifford correspondence.)

First, we check that this is a well-defined $\mathrm{N}_G(D)$-equivariant map. Suppose that ζ^u is chosen instead of ζ for some $u \in \mathrm{N}_N(D)$, and suppose that $(\zeta^u)^{h_1} = \zeta_i$ for some $h_1 \in \mathrm{N}_G(D)$. We have to check that

$$(\Upsilon_{D,\zeta_i}((\tau_\zeta)^h)^{h^{-1}})^{K\mathrm{N}_N(D)} = (\Upsilon_{D,\zeta_i}((\tau_{\zeta^u})^{h_1})^{{h_1}^{-1}})^{K\mathrm{N}_N(D)}.$$

Now, $\tau_{\zeta^u} = (\tau_\zeta)^u$ and $h^{-1}uh_1 \in \mathrm{N}_{G_i}(D)$. Then

$$\Upsilon_{D,\zeta_i}((\tau_\zeta)^{uh_1}) = \Upsilon_{D,\zeta_i}(((\tau_\zeta)^h)^{h^{-1}uh_1}) = \Upsilon_{D,\zeta_i}((\tau_\zeta)^h)^{h^{-1}uh_1},$$

and therefore

$$\Upsilon_{D,\zeta_i}((\tau_\zeta)^{uh_1})^{h_1^{-1}u^{-1}} = \Upsilon_{D,\zeta_i}((\tau_\zeta)^h)^{h^{-1}}.$$

Since $u \in \mathrm{N}_N(D)$, both characters induce the same character of $K\mathrm{N}_N(D)$. Hence $\Upsilon_{D,K}$ is well-defined.

Now, we prove that $\Upsilon_{D,K}$ is $\mathrm{N}_G(D)$-equivariant. Let $\tau \in \mathrm{Irr}_0(N \mid D)$ and $x \in \mathrm{N}_G(D)$. Let $\zeta \in \mathrm{Irr}_0(K \mid D_0)$ be under τ such that $\tau_\zeta \in \mathrm{Irr}(N_\zeta \mid D, \zeta)$. Suppose that $\zeta^h = \zeta_i$. Now $\tau_{\zeta^x} = (\tau_\zeta)^x$, $(\zeta^x)^{x^{-1}h} = \zeta_i$ and we have

$$\begin{aligned}\Upsilon_{D,K}(\tau^x) &= (\Upsilon_{D,\zeta_i}((\tau_\zeta)^{x(x^{-1}h)})^{h^{-1}x})^{K\mathrm{N}_N(D)} = ((\Upsilon_{D,\zeta_i}((\tau_\zeta)^h)^{h^{-1}})^{K\mathrm{N}_N(D)})^x \\ &= \Upsilon_{D,K}(\tau)^x.\end{aligned}$$

Next we prove that the block of $\Upsilon_{D,K}(\tau)$ has defect group D and that $\Upsilon_{D,K}(\tau)$ has height zero. The block of $(\tau_\zeta)^h$ has defect group D, and induces $\mathrm{bl}(\tau^h)$ which also has defect group D. Now let b be the block $\Upsilon_{D,\zeta_i}((\tau_\zeta)^h)$ which also has defect group D. We know that $b^{N_i} = \mathrm{bl}((\tau_\zeta)^h)$. Also, b induces a block c of $K\mathrm{N}_N(D)$, which has defect group R containing D. Now, c^N is the block of $(\tau)^h$ that has defect group D, and therefore R is contained in some conjugate of D. We conclude that c has defect group D. By Proposition 2.5(g), we conclude that $\Upsilon_{D,K}(\tau)$ has height zero.

Now, we prove that $\Upsilon_{D,K}$ is one-to-one. Suppose that $\Upsilon_{D,K}(\tau) = \Upsilon_{D,K}(\psi)$, where $\tau, \psi \in \mathrm{Irr}_0(N \mid D)$. Let $\zeta \in \mathrm{Irr}(K)$ be such that $\tau_\zeta \in \mathrm{Irr}_0(N_\zeta \mid D, \zeta)$. We know that $\Upsilon_{D,K}(\tau) \in \mathrm{Irr}(K\mathrm{N}_N(D) \mid \zeta)$. Therefore if $\zeta' \in \mathrm{Irr}(K)$ is such that $\psi_{\zeta'} \in \mathrm{Irr}_0(N_{\zeta'} \mid D, \zeta')$, then we conclude that $\zeta' = \zeta^n$ for some $n \in \mathrm{N}_N(D)$, by Clifford's theorem applied to $\Upsilon_{D,K}(\tau) = \Upsilon_{D,K}(\psi)$ for the normal subgroup $K \lhd K\mathrm{N}_N(D)$. Now, suppose that $\zeta^h = \zeta_i$, where $h \in \mathrm{N}_G(D)$. Then $(\zeta')^{n^{-1}h} = \zeta_i$. Now

$$(\Upsilon_{D,\zeta_i}((\tau_\zeta)^h)^{K\mathrm{N}_N(D)})^{h^{-1}} = \Upsilon_{D,K}(\tau) = \Upsilon_{D,K}(\psi) = (\Upsilon_{D,\zeta_i}((\psi_{\zeta^n})^{n^{-1}h})^{K\mathrm{N}_N(D)})^{h^{-1}n}.$$

Now, using that $n \in \mathrm{N}_N(D) \leq K\mathrm{N}_N(D)$ and that h^{-1} normalizes $K\mathrm{N}_N(D)$, we obtain

$$\Upsilon_{D,\zeta_i}((\tau_\zeta)^h)^{K\mathrm{N}_N(D)} = \Upsilon_{D,\zeta_i}((\psi_{\zeta^n})^{n^{-1}h})^{K\mathrm{N}_N(D)}.$$

By the Clifford correspondence, we have

$$\Upsilon_{D,\zeta_i}((\tau_\zeta)^h) = \Upsilon_{D,\zeta_i}((\psi_{\zeta^n})^{n^{-1}h}),$$

and using that Υ_{D,ζ_i} is a bijection, $(\tau_\zeta)^h = (\psi_{\zeta^n})^{n^{-1}h}$ and $\tau_\zeta = (\psi_{\zeta^n})^{n^{-1}}$. Inducing to N, we get $\tau = \psi$.

Next we prove that $\Pi_{D,K}$ is surjective. Given $\gamma \in \mathrm{Irr}_0(K\mathrm{N}_N(D) \mid D)$, by Proposition 2.5(f), let $\zeta \in \mathrm{Irr}(K)$ be such that $\gamma_\zeta \in \mathrm{Irr}_0(K\mathrm{N}_{N_\zeta}(D) \mid D, \zeta)$. Now $\zeta^h = \zeta_i$ for some i and $h \in \mathrm{N}_G(D)$. Using that conjugation by h^{-1} is a bijection $\mathrm{Irr}_0(K\mathrm{N}_{N_i}(D) \mid D, \zeta_i) \to \mathrm{Irr}_0(K\mathrm{N}_{N_\zeta}(D) \mid D, \zeta)$, we write $\gamma_\zeta = \rho^{h^{-1}}$ for some $\rho \in \mathrm{Irr}_0(K\mathrm{N}_{N_i}(D) \mid D, \zeta_i)$. Now using that Υ_{D,ζ_i} is a bijection and that conjugation by h is a bijection $\mathrm{Irr}_0(N_\zeta \mid D, \zeta) \to \mathrm{Irr}_0(N_i \mid D, \zeta_i)$, we can write $\rho = \Upsilon_{D,\zeta_i}(\mu^h)$ for some $\mu \in \mathrm{Irr}_0(N_\zeta \mid D, \zeta)$. It is enough to check that the irreducible character $\tau = \mu^N$ lies in $\mathrm{Irr}_0(N \mid D)$. Since $\mathrm{bl}(\gamma)$ has defect group D, by Brauer's first main theorem, $\mathrm{bl}(\gamma)^N$ is defined and has defect group D. Since $\mathrm{bl}(\gamma_\zeta)^{K\mathrm{N}_N(D)} = \mathrm{bl}(\gamma)$, also $\mathrm{bl}(\gamma_\zeta)^N$ is defined and has defect group D. Since $\mathrm{bl}(\rho)^{h^{-1}} = \mathrm{bl}(\gamma_\zeta)$, it follows that $\mathrm{bl}(\gamma_\zeta)^{N_\zeta}$ is defined and equals $\mathrm{bl}(\mu)$. Hence we see that $\mathrm{bl}(\mu)^N = \mathrm{bl}(\gamma)^N$ has defect group D. Since $\tau = \mu^N$, we find that $\mathrm{bl}(\tau)$ has defect group D. By Proposition 2.5(g), τ has height zero and $\Pi_{D,K}(\tau) = \gamma$.

It remains to show that

$$(G_\tau, N, \tau) \sim_b (K\mathrm{N}_{G_{\tau'}}(D), K\mathrm{N}_N(D), \tau')$$

for every $\tau \in \mathrm{Irr}_0(N \mid D)$ and $\tau' := \Pi_{D,K}(\tau)$.

First we prove this statement in the case where for $\tau \in \mathrm{Irr}_0(N \mid D)$ there exists some $\zeta_i \in \mathrm{Irr}(K)$ such that τ has a Clifford correspondent $\tau_{\zeta_i} \in \mathrm{Irr}_0(N_{\zeta_i} \mid D, \zeta_i)$ for some i. We know that

$$((G_i)_\gamma, N_i, \gamma) \sim_b (K\mathrm{N}_{(G_i)_\gamma}(D), K\mathrm{N}_{N_i}(D), \gamma').$$

We may assume that τ is G-invariant and that τ' is $\mathrm{N}_G(D)$-invariant. Now, for every $N \leq J \leq G$ and $J_i := G_i \cap J$, by the Clifford correspondence, induction gives a bijection between $\mathrm{Irr}(J_i \mid \gamma)$ and $\mathrm{Irr}(J \mid \tau)$ as J_i is the stabilizer of ζ_i in J. Analogously for $H := K\mathrm{N}_G(D)$ induction gives a bijection between $\mathrm{Irr}(J_i \cap H \mid \gamma')$ and $\mathrm{Irr}(H \cap J \mid \tau')$. Let $M = K\mathrm{N}_N(D)$. Since τ is G-invariant, we have $G = NG_i$ by the Frattini argument. Also $H = MH_i$. All the hypotheses of Theorem 3.14 are satisfied, and we conclude that

$$(G_\tau, N, \tau) \sim_b (H_{\tau'}, K\mathrm{N}_N(D), \tau'),$$

where $\tau' := (\gamma')^{K\mathrm{N}_N(D)} = \Upsilon_{D,K}(\tau)$.

By conjugation with elements of $\mathrm{N}_G(D)$ we see that

$$(G_\tau, N, \tau) \sim_b (H_\tau, K\mathrm{N}_N(D), \Upsilon_{D,K}(\tau))$$

for all characters $\tau \in \mathrm{Irr}_0(K \mid D)$. □

Corollary 7.5. *Let $K \lhd G$ where $K \leq N$, $|N : K| < |N : \mathrm{Z}(N)|$ and $K\mathrm{N}_N(D) \neq N$. Then there exists an $\mathrm{N}_G(D)$-equivariant bijection*

$$\Pi_D : \mathrm{Irr}_0(N \mid D) \to \mathrm{Irr}_0(\mathrm{N}_N(D) \mid D)$$

with

$$(G_\tau, N, \tau) \sim_b (\mathrm{N}_G(D)_{\tau'}, \mathrm{N}_N(D), \tau')$$

for every $\tau \in \mathrm{Irr}_0(N \mid D)$ and $\tau' := \Pi_D(\tau)$.

Proof. By hypothesis, $N_1 := K\mathrm{N}_N(D)$ is a proper subgroup of N. We see that every non-abelian simple group involved in N_1 is in $\mathcal{S}$ and $|N_1/\mathrm{Z}(N_1)| < |N/\mathrm{Z}(N)|$. Let $G_1 := K\mathrm{N}_G(D)$. By induction, there exists an $\mathrm{N}_{G_1}(D)$-equivariant bijection

$$\Pi'_D : \mathrm{Irr}_0(N_1 \mid D) \to \mathrm{Irr}_0(\mathrm{N}_{N_1}(D) \mid D),$$

such that $((G_1)_\tau, N_1, \tau) \sim_b (\mathrm{N}_{G_1}(D)_{\tau'}, \mathrm{N}_{N_1}(D), \tau')$ for every $\tau \in \mathrm{Irr}_0(N_1 \mid D)$ with $\tau' := \Pi'_D(\tau)$. Note that $\mathrm{N}_{G_1}(D) = \mathrm{N}_G(D)$.

Now, by Proposition 7.4, we have an $\mathrm{N}_G(D)$-equivariant bijection

$$\Upsilon_{D,K} : \mathrm{Irr}_0(N \mid D) \to \mathrm{Irr}_0(K\mathrm{N}_N(D) \mid D)$$

with

$$(G_\tau, N, \tau) \sim_b (K\mathrm{N}_{G_{\tau'}}(D), K\mathrm{N}_N(D), \tau')$$

for every $\tau \in \mathrm{Irr}_0(N \mid D)$ and $\tau' := \Upsilon_{D,K}(\tau)$. Using Lemma 3.8(a), we check that the map $\Pi_D := \Pi'_D \circ \Upsilon_{D,K}$ is an $\mathrm{N}_G(D)$-equivariant bijection that satisfies the requirements. $\square$

In the next result, we consider the case where there exists a normal subgroup K such that $K \cap D$ is central in N.

Proposition 7.6. *Suppose that there exists a subgroup $K \lhd G$ with $\mathrm{Z}(N) \lneq K \leq N$ with $K \cap D \leq Z(N)$. Then there exists an $\mathrm{N}_G(D)$-equivariant bijection*

$$\Pi_D : \mathrm{Irr}_0(N \mid D) \to \mathrm{Irr}_0(\mathrm{N}_N(D) \mid D)$$

with

$$(G_\tau, N, \tau) \sim_{\mathrm{b}} (\mathrm{N}_G(D)_{\tau'}, \mathrm{N}_N(D), \tau')$$

for every $\tau \in \mathrm{Irr}_0(N \mid D)$ and $\tau' := \Pi_D(\tau)$.

Proof. By Corollary 7.5, we may assume that $N = K\mathrm{N}_N(D)$. By Remark 7.2, we have $G = K\mathrm{N}_G(D)$. Hence the group $M := KD$ is a normal subgroup of G and M/K is a p-group with $D_0 := K \cap D \leq \mathrm{Z}(M)$. By Corollary 5.14 there exists a bijection

$$\Lambda_D : \mathrm{Irr}_0(M \mid D) \to \mathrm{Irr}_0(\mathrm{N}_M(D) \mid D),$$

such that for every $\theta \in \mathrm{Irr}_0(M \mid D)$ and $H := \mathrm{N}_G(D)$ we have

$$(G_\theta, M, \theta) \sim_{\mathrm{b}} (H_\theta, M \cap H, \Lambda_D(\theta)).$$

We now apply Proposition 4.7(b) with $\mathcal{G} := \mathrm{Irr}_0(M \mid D)$, $\mathcal{G}' := \mathrm{Irr}_0(\mathrm{N}_M(D) \mid D)$, $Q := D$ and $J := N$. Hence there exists an $\mathrm{N}_G(D)$-equivariant bijection

$$\Pi_D : \mathrm{Irr}_0(N \mid D) \to \mathrm{Irr}_0(\mathrm{N}_N(D) \mid D),$$

such that for every $\tau \in \mathrm{Irr}_0(N \mid D)$ and $\tau' := \Pi_D(\tau)$ we have

$$(G_\tau, N, \tau) \sim_{\mathrm{b}} (H_\tau, N \cap H, \tau'). \quad \square$$

The next statement concerns the case where Proposition 7.6 cannot be applied, completing the proof of Theorem 7.1.

Proposition 7.7. *Suppose that there exists no normal subgroup $K \lhd G$ with $\mathrm{Z}(N) \lneq K \leq N$ and $K \cap D \leq Z(N)$. Then there exists an $\mathrm{N}_G(D)$-equivariant bijection*

$$\Pi_D : \mathrm{Irr}_0(N \mid D) \to \mathrm{Irr}_0(\mathrm{N}_N(D) \mid D)$$

with

$$(G_\tau, N, \tau) \sim_{\mathrm{b}} (\mathrm{N}_G(D)_{\tau'}, \mathrm{N}_N(D), \tau')$$

for every $\tau \in \mathrm{Irr}_0(N \mid D)$ and $\tau' := \Pi_D(\tau)$.

Proof. Since $\mathrm{O}_{p'}(N)\,\mathrm{Z}(N) \cap D \leq \mathrm{Z}(N)$, we know that $\mathrm{O}_{p'}(N) \leq \mathrm{Z}(N)$ by hypothesis. If $\mathrm{O}_p(N)$ is not contained in $\mathrm{Z}(N)$, then Corollary 7.5 implies $\mathrm{O}_p(N)\mathrm{N}_N(D) = N$. Since D contains $\mathrm{O}_p(N)$ (Theorem (4.8) of [Nav98]) we conclude that $\mathrm{N}_N(D) = N$ and the theorem is trivial in this case. Also, the theorem is trivial if N is abelian. Hence we may assume that the Fitting subgroup $F(N)$ of N is central and that N is non-abelian.

Now let $K := \mathrm{E}(N)$ be the layer of N (see [Isa08, p. 274] for an exact definition). Since the generalized Fitting subgroup $F^*(N) = KF(N)$ contains its own centralizer in N, we deduce that K is not contained in $\mathrm{Z}(N)$. Also $\mathrm{Z}(K) \leq \mathrm{Z}(N)$ since $\mathrm{Z}(K) \leq F(N)$. Now by hypothesis, $D_0 := D \cap K$ is not contained in $\mathrm{Z}(K)$. By Corollary 6.3 there exists an $\mathrm{N}_G(D_0)$-stable group M with $\mathrm{N}_K(D_0) \leq M \lneq K$ and an $\mathrm{N}_G(D_0)$-equivariant bijection

$$\Omega : \mathrm{Irr}_0(K \mid D_0) \to \mathrm{Irr}_0(M \mid D_0)$$

such that

$$(G_\zeta, K, \zeta) \sim_b (M\mathrm{N}_{G_\zeta}(D_0), M, \Omega(\zeta)) \qquad \text{for every } \zeta \in \mathrm{Irr}_0(K \mid D_0).$$

We can apply Proposition 4.7(b) for $\mathcal{G} := \mathrm{Irr}_0(K \mid D_0)$, $\mathcal{G}' := \mathrm{Irr}_0(M \mid D_0)$, $Q := D$ and $J := N$. Since $\mathrm{N}_G(D)$ is contained in $\mathrm{N}_G(D_0)$, we conclude that there exists an $\mathrm{N}_G(D)$-equivariant bijection

$$\Pi_D : \mathrm{Irr}_0(N \mid D) \to \mathrm{Irr}_0(M\mathrm{N}_N(D_0) \mid D)$$

such that for every $\tau \in \mathrm{Irr}_0(N \mid D)$ and $\tau' = \Pi_D(\tau')$ we have

$$(G_\tau, N, \tau) \sim_b (M\mathrm{N}_G(D_0)_\tau, M\mathrm{N}_N(D_0), \tau').$$

Now we apply the inductive hypothesis to the group $M\mathrm{N}_N(D_0) < N$. This and Remark 3.8 imply the final statement. □

We can now complete the proof of Theorem A.

Theorem 7.8. *Let G be a finite group and let p be a prime. Suppose that every non-abelian simple group involved in G satisfies the inductive AM-condition for p. Then Brauer's Height Zero Conjecture holds for G and the prime p.*

Proof. The "if" direction of Brauer's Height Zero Conjecture (BHZC) has been proved in [KM13]. The proof of the main result in [Mur11] shows that Theorem B, together with the generalized Gluck–Wolf theorem, and a proof of the "only if" direction of BHZC for quasisimple groups, implies the "only if" direction of BHZC for every finite group. Now, the generalized Gluck–Wolf theorem has been proved in [NT13], while the BHZC has been proved for quasisimple groups in [KM12]. □

Acknowledgments. We would like to thank Gunter Malle for many helpful comments leading to the present improved version of the paper. We mention that the clever path from Theorem B to Theorem A is due to Masafumi Murai in his recent paper [Mur11].

The research of the first author was partially supported by the Spanish Ministerio de Economía proyecto MTM2010-15296, and Prometeo/Generalitat Valenciana, and the research of the second author was supported by the DFG priority program, SPP 1388 and the ERC Advanced Grant 291512.

References

[BK88] Berger, T. R., Knörr, R.: On Brauer's height 0 conjecture. Nagoya Math. J. **109**, 109–116 (1988) Zbl 0637.20006 MR 0931954

[CR90] Curtis, C. W., Reiner, I.: Methods of Representation Theory. Vol. I. Wiley Classics Library, Wiley, New York (1990) (reprint of the 1981 original) Zbl 0469.20001 MR 0632548

[Dad80] Dade, E. C.: A correspondence of characters. In: The Santa Cruz Conference on Finite Groups (Santa Cruz, CA, 1979), Proc. Sympos. Pure Math. 37, Amer. Math. Soc., 401–403 (1980) Zbl 0456.20002 MR 0604612

[Dad96] Dade, E.: Counting characters in blocks with cyclic defect groups, I. J. Algebra **186**, 934–969 (1996) Zbl 0873.20006 MR 1424599

[GW84] Gluck, D., Wolf, T. R.: Brauer's height conjecture for p-solvable groups. Trans. Amer. Math. Soc. **282**, 137–152 (1984) Zbl 0543.20007 MR 0728707

[IMN07] Isaacs, I. M., Malle, G., Navarro, G.: A reduction theorem for the McKay conjecture. Invent. Math. **170**, 33–101 (2007) Zbl 1138.20010 MR 2336079

[IN91] Isaacs, I. M., Navarro, G.: Character correspondences and irreducible induction and restriction. J. Algebra **140**, 131–140 (1991) Zbl 0743.20005 MR 1114910

[Isa76] Isaacs, I. M.: Character Theory of Finite Groups. Academic Press, New York (1976) Zbl 0337.20005 MR 0460423

[Isa08] Isaacs, I. M.: Finite Group Theory. Amer. Math. Soc., Providence, RI (2008) Zbl 1169.20001 MR 2426855

[KM12] Kessar, R., Malle, G.: Brauer's height zero conjecture for universal covering groups of simple groups. Private correspondence (2012)

[KM13] Kessar, R., Malle, G.: Quasi-isolated blocks and Brauer's height zero conjecture. Ann. of Math. **178**, 321–384 (2013) Zbl 06190562 MR 3043583

[Lad10] Ladisch, F.: Character correspondences induced by magic representations. arXiv:1004.4538 (2010)

[Lar11] Laradji, A.: Irreducible characters and normal subgroups in groups of odd order. J. Algebra **336**, 242–256 (2011) Zbl 1246.20009 MR 2802541

[Mal07] Malle, G.: Height 0 characters of finite groups of Lie type. Represent. Theory **11**, 192–220 (2007) Zbl 1139.20008 MR 2365640

[Mal08] Malle, G.: The inductive McKay condition for simple groups not of Lie type. Comm. Algebra **36**, 455–463 (2008) Zbl 1153.20009 MR 2387535

[Mur95] Murai, M.: A remark on Brauer's height zero conjecture. J. Math. Kyoto Univ. **35**, 607–610 (1995) Zbl 0848.20005 MR 1365250

[Mur96] Murai, M.: Normal subgroups and heights of characters. J. Math. Kyoto Univ. **36**, 31–43 (1996) Zbl 0867.20011 MR 1381537

[Mur04] Murai, M.: A remark on the Alperin–McKay conjecture. J. Math. Kyoto Univ. **44**, 245–254 (2004) Zbl 1086.20007 MR 2081072

[Mur11] Murai, M.: On Brauer's height zero conjecture. Proc. Japan Acad. Ser. A **88**, 38–40 (2012) Zbl 06051463 MR 2908621

[NT89] Nagao, H., Tsushima, Y.: Representations of Finite Groups. Academic Press (1989) Zbl 0673.20002 MR 0998775

[Nav98] Navarro, G.: Characters and Blocks of Finite Groups. London Math. Soc. Lecture Note Ser. 250, Cambridge Univ. Press, Cambridge (1998) Zbl 0903.20004 MR 1632299

[Nav04] Navarro, G.: Actions and characters in blocks. J. Algebra **275**, 471–480 (2004) Zbl 1075.20005 MR 2052620

[NS14] Navarro, G., Späth, B.: Character correspondences in blocks with normal defect group. J. Algebra **398**, 396–406 (2014) MR 3123773

[NT12] Navarro, G., Tiep, P. H.: Brauer's height zero conjecture for the 2-blocks of maximal defect. J. Reine Angew. Math. **669**, 225–247 (2012) Zbl 06081364 MR 2980457

[NT13] Navarro, G., Tiep, P. H.: Characters of relative p'-degree over normal subgroups. Ann. of Math. **178**, 1135–1171 (2013) Zbl 06220730 MR 3092477

[Pui86] Puig, L.: Local extensions in endo-permutation modules split: a proof of Dade's theorem. In: Séminaire sur les groupes finis, Tome III, Publ. Math. Univ. Paris VII 25, 199–205 (1986) Zbl 0641.20008 MR 0907480

[Rob02] Robinson, G.: Cancellation theorems related to conjectures of Alperin and Dade. J. Algebra **249**, 196–219 (2002) Zbl 1011.20013 MR 1887992

[Spä12] Späth, B.: Inductive McKay condition in defining characteristic. Bull. London Math. Soc. **44**, 426–438 (2012) Zbl 1251.20020 MR 2966987

[Spä13] Späth, B.: A reduction theorem for the Alperin–McKay conjecture. J. Reine Angew. Math. **680**, 153–189 (2013) Zbl 06199082 MR 3100954