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The Brauer group and the Brauer–Manin set of products of varieties

Abstract. Let X and Y be smooth and projective varieties over a field k finitely generated over $\mathbb{Q}$, and let $\overline{X}$ and $\overline{Y}$ be the varieties over an algebraic closure of k obtained from X and Y, respectively, by extension of the ground field. We show that the Galois invariant subgroup of $\mathrm{Br}(\overline{X}) \oplus \mathrm{Br}(\overline{Y})$ has finite index in the Galois invariant subgroup of $\mathrm{Br}(\overline{X \times Y})$. This implies that the cokernel of the natural map $\mathrm{Br}(X) \oplus \mathrm{Br}(Y) \to \mathrm{Br}(X \times Y)$ is finite when k is a number field. In this case we prove that the Brauer–Manin set of the product of varieties is the product of their Brauer–Manin sets.

Keywords. Brauer group, Brauer–Manin obstruction

Let k be a field with a separable closure $\bar{k}$, and $\Gamma = \mathrm{Aut}(\bar{k}/k)$. For an algebraic variety X over k we write $\overline{X}$ for the variety over $\bar{k}$ obtained from X by extending the ground field. Let $\mathrm{Br}(X)$ be the cohomological Brauer–Grothendieck group $\mathrm{H}^2_{\mathrm{\acute{e}t}}(X, \mathbf{G}_\mathrm{m})$ (see [4]). The group $\mathrm{Br}(\overline{X})$ is naturally a Galois module. The image of the natural homomorphism $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})$ lies in $\mathrm{Br}(\overline{X})^\Gamma$; the kernel of this homomorphism is denoted by $\mathrm{Br}_1(X)$, so that $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is a subgroup of $\mathrm{Br}(\overline{X})^\Gamma$. Recall that $\mathrm{Br}(X)$ and $\mathrm{Br}(\overline{X})$ are torsion abelian groups whenever X is smooth (see [4, II, Prop. 1.4]).

Theorem A. *Let k be a field finitely generated over $\mathbb{Q}$. Let X and Y be smooth, projective and geometrically integral varieties over k. Then the cokernel of the natural injective map*

$$\mathrm{Br}(\overline{X})^\Gamma \oplus \mathrm{Br}(\overline{Y})^\Gamma \to \mathrm{Br}(\overline{X \times Y})^\Gamma$$

is finite.

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See Theorem 3.1 for an analogue in finite characteristic. The proof uses the results of Faltings and the second named author on Tate's conjecture for abelian varieties.

Let k be a field finitely generated over its prime subfield. We proved in our previous paper [15] that $\mathrm{Br}(\overline{X})^{\Gamma}$ is finite when X is an abelian variety and $\mathrm{char}(k) \neq 2$, or X is a K3 surface and $\mathrm{char}(k) = 0$. As a corollary we deduce that if Z is a smooth and projective variety over k such that $\overline{Z}$ is birationally equivalent to a product of curves, abelian varieties and K3 surfaces, then the groups $\mathrm{Br}(\overline{Z})^{\Gamma}$ and $\mathrm{Br}(Z)/\mathrm{Br}_1(Z)$ are finite.

The following result easily follows from Theorem A (see Section 4).

Theorem B. *Let k be a field finitely generated over $\mathbb{Q}$. Let X and Y be smooth, projective and geometrically integral varieties over k. Assume that $(X \times Y)(k) \neq \emptyset$ or $\mathrm{H}^3(k, \bar{k}^*) = 0$. Then the cokernel of the natural map*

$$\mathrm{Br}(X) \oplus \mathrm{Br}(Y) \to \mathrm{Br}(X \times Y)$$

is finite.

Now let k be a number field. In this case $\mathrm{H}^3(k, \bar{k}^*) = 0$ (see [9, Cor. I.4.21]), so by Theorem B the Brauer group $\mathrm{Br}(X \times Y)$ is generated, modulo the image of $\mathrm{Br}(X) \oplus \mathrm{Br}(Y)$, by finitely many elements. The following result shows that these elements do not give any new Brauer–Manin conditions on the adelic points of $X \times Y$ besides those already given by the elements of $\mathrm{Br}(X) \oplus \mathrm{Br}(Y)$. For the definition of the Brauer–Manin set $X(\mathbb{A}_k)^{\mathrm{Br}}$ we refer to [14, Section 5.2].

Theorem C. *Let X and Y be smooth, projective, geometrically integral varieties over a number field k. Then*

$$(X \times Y)(\mathbb{A}_k)^{\mathrm{Br}} = X(\mathbb{A}_k)^{\mathrm{Br}} \times Y(\mathbb{A}_k)^{\mathrm{Br}}.$$

The key topological fact behind our proof of Theorem C is this: for any path-connected non-empty CW-complexes X and Y, and any commutative ring R with 1 there is a canonical isomorphism

$$\mathrm{H}^2(X \times Y, R) = \mathrm{H}^2(X, R) \oplus \mathrm{H}^2(Y, R) \oplus \left(\mathrm{H}^1(X, R) \otimes_R \mathrm{H}^1(Y, R)\right).$$

See Proposition 2.2 for this exercise in algebraic topology. (This formula does not generalise to the third cohomology group, see Remark 2.3.) The proof of Theorem C uses Theorem 2.6 that gives a similar result for the étale cohomology of connected varieties over $\bar{k}$.

T. Schlank and Y. Harpaz, using étale homotopy of Artin and Mazur, recently proved a statement similar to our Theorem C where the Brauer–Manin set is replaced by the *étale* Brauer–Manin set. In their result the varieties X and Y do not need to be proper (see [12, Cor. 1.3]).

1. Preliminaries

1.1. Notation and conventions. In this paper 'almost all' means 'all but finitely many'. If B is an abelian group, we write B_{tors} for the torsion subgroup of B. Let $B/_{\mathrm{tors}} := B/B_{\mathrm{tors}}$. If ℓ is a prime, then $B(\ell)$ is the subgroup of B_{tors} consisting of the elements

whose order is a power of ℓ, and $B(\text{non-}\ell)$ is the subgroup of B_{tors} consisting of the elements whose order is *not* divisible by ℓ. If m is a positive integer, then B_m is the kernel of multiplication by m in B.

1.2. Tate modules. Let us recall some useful elementary statements that are due to Tate [17, 19]. Let B be an abelian group. The projective limit of the groups B_{ℓ^n} (where the transition maps are multiplications by ℓ) is called the *ℓ-adic Tate module* of B, and is denoted by $T_\ell(B)$. This limit carries a natural structure of a $\mathbb{Z}_\ell$-module; there is a natural injective map $T_\ell(B)/\ell \hookrightarrow B_\ell$. One may easily check that $T_\ell(B)_\ell = 0$, and hence $T_\ell(B)$ is torsion-free.

Let us assume that B_ℓ is finite. Then all the B_{ℓ^n} are obviously finite, and $T_\ell(B)$ is finitely generated by Nakayama's lemma. Therefore, $T_\ell(B)$ is isomorphic to $\mathbb{Z}_\ell^r$ for some non-negative integer $r \leq \dim_{\mathbb{F}_\ell}(B_\ell)$. Moreover, $T_\ell(B) = 0$ if and only if $B(\ell)$ is finite. We denote by $V_\ell(B)$ the $\mathbb{Q}_\ell$-vector space $T_\ell(B) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Clearly, $V_\ell(B) = 0$ if and only if $B(\ell)$ is finite.

If A is an abelian variety over a field k, and ℓ is a prime different from $\text{char}(k)$, $n = \ell^i$, then we write A_n for the kernel of multiplication by n in $A(\bar{k})$. The group A_n is a free $\mathbb{Z}/n$-module of rank $2\dim(A)$ equipped with the natural structure of a Γ-module [11]. We write $T_\ell(A)$ for $T_\ell(A(\bar{k}))$, and $V_\ell(A)$ for $V_\ell(A(\bar{k}))$. The $\mathbb{Q}_\ell$-vector space $V_\ell(A)$ has dimension $2\dim(A)$ and carries the natural structure of a Γ-module.

1.3. The Kummer sequence and the Picard variety. Let X be a smooth, projective and geometrically integral variety over k. For a positive integer n coprime to $\text{char}(k)$ we have the Kummer exact sequence of sheaves of abelian groups in étale topology:

$$0 \to \mu_n \to \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}} \to 0.$$

Recall that $\mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbf{G}_{\mathrm{m}}) = \mathrm{Pic}(\overline{X})$. Thus the Kummer sequence gives rise to an isomorphism of Γ-modules

$$\mathrm{H}^1_{\text{ét}}(\overline{X}, \mu_n) = \mathrm{Pic}(\overline{X})_n. \tag{1}$$

Let $\mathrm{Pic}^0(\overline{X})$ be the Γ-submodule of $\mathrm{Pic}(\overline{X})$ consisting of the classes of divisors algebraically equivalent to 0. By definition, the Néron–Severi group of $\overline{X}$ is the Γ-module $\mathrm{NS}(\overline{X}) = \mathrm{Pic}(\overline{X})/\mathrm{Pic}^0(\overline{X})$. The abelian group $\mathrm{NS}(\overline{X})$ is finitely generated by a theorem of Néron and Severi.

Let A be the *Picard variety* of X (see [7]). Then A is an abelian variety over k such that the Γ-module $A(\bar{k})$ is identified with $\mathrm{Pic}^0(\overline{X})$. Since the multiplication by n is a surjective endomorphism of A, we have an exact sequence of Γ-modules

$$0 \to A_n \to \mathrm{Pic}(\overline{X})_n \to \mathrm{NS}(\overline{X})_n \to 0. \tag{2}$$

Setting $n = \ell^m$, where $\ell \neq \text{char}(k)$ is a prime, we deduce from (1) and (2) a canonical isomorphism of Γ-modules

$$\mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1)) = T_\ell(A) = T_\ell(\mathrm{Pic}(\overline{X})). \tag{3}$$

In particular, this is a free $\mathbb{Z}_\ell$-module of finite rank.

Again, by surjectivity of multiplication by ℓ^m on $A(\bar{k}) = \mathrm{Pic}^0(\overline{X})$ we obtain from the Kummer sequence the following exact sequence of Γ-modules:

$$0 \to \mathrm{NS}(\overline{X})/\ell^m \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_{\ell^m}) \to \mathrm{Br}(\overline{X})_{\ell^m} \to 0. \tag{4}$$

Passing to the projective limit in m gives rise to the well known exact sequence

$$0 \to \mathrm{NS}(\overline{X}) \otimes \mathbb{Z}_\ell \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1)) \to T_\ell(\mathrm{Br}(\overline{X})) \to 0. \tag{5}$$

It shows that the torsion subgroup of $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))$ coincides with $\mathrm{NS}(\overline{X})(\ell)$ (cf. [15, Sect. 2.2]). In particular, if ℓ does not divide the order of the torsion subgroup of $\mathrm{NS}(\overline{X})$, then $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))$ is a free $\mathbb{Z}_\ell$-module of finite rank.

1.4. Products of varieties. Let X and Y be smooth, projective and geometrically integral varieties over k. We have the natural projection maps

$$\pi_X : X \times Y \to X, \quad \pi_Y : X \times Y \to Y.$$

We denote by the same symbols the projections $\overline{X} \times \overline{Y} \to \overline{X}$ and $\overline{X} \times \overline{Y} \to \overline{Y}$. Fixing $\bar{k}$-points $x_0 \in X(\bar{k})$ and $y_0 \in Y(\bar{k})$, we define closed embeddings

$$q_{y_0} : \overline{X} = \overline{X} \times y_0 \hookrightarrow \overline{X} \times \overline{Y}, \quad q_{x_0} : \overline{Y} = x_0 \times \overline{Y} \hookrightarrow \overline{X} \times \overline{Y}.$$

Then $\pi_X q_{y_0} = \mathrm{id}_{\overline{X}}$ and $\pi_Y q_{x_0} = \mathrm{id}_{\overline{Y}}$. On the other hand,

$$\pi_Y q_{y_0}(\overline{X}) = y_0 \subset \overline{Y}, \quad \pi_X q_{x_0}(\overline{Y}) = x_0 \subset \overline{X}.$$

Let $\mathcal{F}$ be an étale sheaf defined by a commutative k-group scheme (see [8, Cor. II.1.7]). For example, $\mathcal{F}$ can be the sheaf defined by the multiplicative group $\mathbf{G}_{\mathrm{m}}$, or by the finite k-groups $\mathbb{Z}/n$ or μ_n, where n is not divisible by the characteristic of k. The induced map $\pi_X^* : \mathrm{H}^i_{\text{ét}}(\overline{X}, \mathcal{F}) \to \mathrm{H}^i_{\text{ét}}(\overline{X} \times \overline{Y}, \mathcal{F})$ is a homomorphism of Γ-modules, whereas $q_{x_0}^* : \mathrm{H}^i_{\text{ét}}(\overline{X} \times \overline{Y}, \mathcal{F}) \to \mathrm{H}^i_{\text{ét}}(\overline{Y}, \mathcal{F})$ is *a priori* only a homomorphism of abelian groups. If $x_0 \in X(k)$, then $q_{x_0}^*$ is also a homomorphism of Γ-modules.

The next proposition easily follows from the definitions and the above considerations.

Proposition 1.5. *For any $i \geq 1$ we have the following statements.*

(i) *The induced maps*

$$\pi_X^* : \mathrm{H}^i_{\text{ét}}(\overline{X}, \mathcal{F}) \to \mathrm{H}^i_{\text{ét}}(\overline{X} \times \overline{Y}, \mathcal{F}), \quad \pi_Y^* : \mathrm{H}^i_{\text{ét}}(\overline{Y}, \mathcal{F}) \to \mathrm{H}^i_{\text{ét}}(\overline{X} \times \overline{Y}, \mathcal{F})$$

are injective homomorphisms of Γ-modules.

(ii) *The induced maps $q_{y_0}^*$ and $q_{x_0}^*$ define isomorphisms of abelian groups*

$$q_{y_0}^* : \pi_X^*(\mathrm{H}^i_{\text{ét}}(\overline{X}, \mathcal{F})) \to \mathrm{H}^i_{\text{ét}}(\overline{X}, \mathcal{F}), \quad q_{x_0}^* : \pi_Y^*(\mathrm{H}^i_{\text{ét}}(\overline{Y}, \mathcal{F})) \to \mathrm{H}^i_{\text{ét}}(\overline{Y}, \mathcal{F}).$$

(iii) *The subgroup* $\pi_X^*(H_{\text{ét}}^i(\overline{X}, \mathcal{F}))$ *lies in the kernel of*

$$q_{x_0}^* : H_{\text{ét}}^i(\overline{X} \times \overline{Y}, \mathcal{F}) \to H_{\text{ét}}^i(\overline{Y}, \mathcal{F}),$$

and similarly $\pi_Y^*(H^i(\overline{Y}, \mathcal{F}))$ *lies in the kernel of*

$$q_{y_0}^* : H_{\text{ét}}^i(\overline{X} \times \overline{Y}, \mathcal{F}) \to H_{\text{ét}}^i(\overline{X}, \mathcal{F}).$$

Hence $\pi_X^*(H_{\text{ét}}^i(\overline{X}, \mathcal{F})) \cap \pi_Y^*(H_{\text{ét}}^i(\overline{Y}, \mathcal{F})) = 0.$

(iv) *The map* $(a, b) \mapsto \pi_X^*(a) + \pi_Y^*(b)$ *defines an injective homomorphism of* Γ-*modules*

$$H_{\text{ét}}^i(\overline{X}, \mathcal{F}) \oplus H_{\text{ét}}^i(\overline{Y}, \mathcal{F}) \to H_{\text{ét}}^i(\overline{X} \times \overline{Y}, \mathcal{F}).$$

1.6. Picard groups of products of varieties. We identify $\text{Pic}(\overline{X}) \oplus \text{Pic}(\overline{Y})$ with its image in $\text{Pic}(\overline{X} \times \overline{Y})$. It is well known that

$$\text{Pic}^0(\overline{X} \times \overline{Y}) = \text{Pic}^0(\overline{X}) \oplus \text{Pic}^0(\overline{Y}).$$

Let A be the Picard variety of X, and let B be the Picard variety of Y. The dual abelian variety A^t of A is the *Albanese variety* of X. When $X(k) \neq \emptyset$, the choice of a point $x_0 \in X(k)$ defines a morphism $\text{Alb}_{x_0} : X \to A^t$ that sends x_0 to 0. The pair (A^t, Alb_{x_0}) can be characterized by the universal property that any morphism from X to an abelian variety A' that sends x_0 to 0 is the composition of Alb_{x_0} and a morphism of abelian varieties $A^t \to A'$. See [11], [7] for more details. It is clear that the Albanese variety of $\overline{X}$ is $\overline{A^t}$.

Proposition 1.7. *We have a commutative diagram of* Γ-*modules with exact rows and columns, where the exact sequence in the bottom row is split:*

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
A(\bar{k}) \oplus B(\bar{k}) & = A(\bar{k}) \oplus B(\bar{k}) \\
\downarrow & & \downarrow \\
0 \to \text{Pic}(\overline{X}) \oplus \text{Pic}(\overline{Y}) \to & \text{Pic}(\overline{X} \times \overline{Y}) & \to \text{Hom}(\overline{B^t}, \overline{A}) \to 0 \\
\downarrow & & \downarrow & \quad || \\
0 \to \text{NS}(\overline{X}) \oplus \text{NS}(\overline{Y}) \to & \text{NS}(\overline{X} \times \overline{Y}) & \to \text{Hom}(\overline{B^t}, \overline{A}) \to 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

If $(X \times Y)(k) \neq \emptyset$, *then the exact sequence in the middle row is also split.*

Proof. Choose a $\bar{k}$-point (x_0, y_0) in $X \times Y$, and let P_{x_0, y_0} be the kernel of the group homomorphism

$$\text{Pic}(\overline{X} \times \overline{Y}) \to \text{Pic}(\overline{X}) \oplus \text{Pic}(\overline{Y}), \qquad L \mapsto (q_{y_0}^* L, q_{x_0}^* L).$$

Let N_{x_0, y_0} be the image of P_{x_0, y_0} in $\text{NS}(\overline{X} \times \overline{Y})$. By Proposition 1.5 the intersection of P_{x_0, y_0} with $\text{Pic}(\overline{X}) \oplus \text{Pic}(\overline{Y})$ inside $\text{Pic}(\overline{X} \times \overline{Y})$ is zero, hence the natural surjective map $P_{x_0, y_0} \to N_{x_0, y_0}$ is an isomorphism of abelian groups.

For any $L \in P_{x_0,y_0}$ we have $q_{y_0}^* L = 0$, hence $q_y^* L \in \mathrm{Pic}^0(\overline{X})$ for any $y \in Y(\bar{k})$. Thus N_{x_0,y_0} is the kernel of the group homomorphism

$$\mathrm{NS}(\overline{X} \times \overline{Y}) \to \mathrm{NS}(\overline{X} \times y) \oplus \mathrm{NS}(x \times \overline{Y})$$

for any $x \in X(\bar{k})$ and $y \in Y(\bar{k})$. In particular, N_{x_0,y_0} does not depend on the choice of (x_0, y_0), so we can drop x_0 and y_0, and write $N = N_{x_0,y_0}$. It follows that N is a Galois submodule of $\mathrm{NS}(\overline{X} \times \overline{Y})$, so that we have a decomposition of Γ-modules

$$\mathrm{NS}(\overline{X} \times \overline{Y}) = \mathrm{NS}(\overline{X}) \oplus \mathrm{NS}(\overline{Y}) \oplus N.$$

It remains to show that the Γ-modules N and $\mathrm{Hom}(\overline{B^t}, \overline{A})$ are canonically isomorphic.

The Poincaré sheaf $\mathcal{P}_X$ on $A^t \times A$ is a certain canonical invertible sheaf that restricts trivially to both $\{0\} \times A$ and $A^t \times \{0\}$ (see [11], [7]). Every morphism of abelian varieties $u : \overline{B^t} \to \overline{A}$ gives rise to the invertible sheaf $(\mathrm{Alb}_{x_0}, u\, \mathrm{Alb}_{y_0})^*(\mathcal{P}_X)$ on $\overline{X} \times \overline{Y}$, whose isomorphism class is in P_{x_0,y_0}. It is well known that this defines a group isomorphism

$$\mathrm{Hom}(\overline{B^t}, \overline{A}) \xrightarrow{\sim} P_{x_0,y_0}. \tag{6}$$

The Poincaré sheaf is defined over k so from (6) we deduce a canonical isomorphism of Γ-modules $\mathrm{Hom}(\overline{B^t}, \overline{A}) \xrightarrow{\sim} N$. The last statement of the proposition is clear: it is enough to choose $(x_0, y_0) \in (X \times Y)(k)$. $\qquad\square$

The following corollary is well known. See Proposition 2.2 below for a topological analogue.

Corollary 1.8. *Let n be a positive integer not divisible by* $\mathrm{char}(k)$. *Then we have a canonical decomposition of Γ-modules*

$$\mathrm{H}^1_{\text{ét}}(\overline{X} \times \overline{Y}, \mu_n) = \mathrm{H}^1_{\text{ét}}(\overline{X}, \mu_n) \oplus \mathrm{H}^1_{\text{ét}}(\overline{Y}, \mu_n). \tag{7}$$

Proof. The middle row of the diagram of Proposition 1.7 gives an isomorphism of Γ-modules $\mathrm{Pic}(\overline{X})_n \oplus \mathrm{Pic}(\overline{Y})_n$ and $\mathrm{Pic}(\overline{X} \times \overline{Y})_n$. It remains to use the canonical isomorphism (1). $\qquad\square$

Remark 1.9. The abelian group $\mathrm{Hom}(\overline{B^t}, \overline{A})$ is finitely generated and torsion-free, hence $\mathrm{H}^1(k, \mathrm{Hom}(\overline{B^t}, \overline{A}))$ is finite. It follows that the cokernel of the natural map

$$\mathrm{H}^1(k, \mathrm{Pic}(\overline{X})) \oplus \mathrm{H}^1(k, \mathrm{Pic}(\overline{Y})) \to \mathrm{H}^1(k, \mathrm{Pic}(\overline{X} \times \overline{Y}))$$

and the kernel of the natural map

$$\mathrm{H}^2(k, \mathrm{Pic}(\overline{X})) \oplus \mathrm{H}^2(k, \mathrm{Pic}(\overline{Y})) \to \mathrm{H}^2(k, \mathrm{Pic}(\overline{X} \times \overline{Y}))$$

are both finite.

2. Künneth decompositions

2.1. Künneth decomposition with coefficients in a field. We continue to assume that X and Y are smooth, projective and geometrically integral varieties over k. Let $\ell \neq \mathrm{char}(k)$ be a prime. We have the Künneth decomposition of Γ-modules

$$\mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, \mathbb{Q}_\ell) = \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \oplus \mathrm{H}^2_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell) \oplus \left(\mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \right)$$

(see [8, Cor. VI.8.13]). From (3) we have a canonical isomorphism

$$\mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1)) = V_\ell(A).$$

When n is a positive integer coprime to $\mathrm{char}(k)$, the non-degeneracy of the Weil pairing gives rise to a canonical isomorphism of Galois modules $B_n = \mathrm{Hom}(B_n^t, \mu_n)$, and hence to a canonical isomorphism

$$\mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell) = V_\ell(B)(-1) = \mathrm{Hom}_{\mathbb{Q}_\ell}(V_\ell(B^t), \mathbb{Q}_\ell).$$

Therefore we have an isomorphism of Γ-modules [18, p. 143]

$$\mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1)) = \mathrm{Hom}_{\mathbb{Q}_\ell}(V_\ell(B^t), V_\ell(A)),$$

and hence a decomposition of Galois modules

$$\mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, \mathbb{Q}_\ell(1)) = \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1)) \oplus \mathrm{H}^2_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell(1)) \oplus \mathrm{Hom}_{\mathbb{Q}_\ell}(V_\ell(B^t), V_\ell(A)). \quad (8)$$

Our next result, Theorem 2.6, is probably well known to experts; we give a proof as we could not find it in the literature. As a motivation and for the sake of completeness we present a similar result for CW-complexes (we shall not need it in the rest of the paper).

Proposition 2.2. *Let X and Y be non-empty path-connected CW-complexes. For any commutative ring R with 1 we have canonical isomorphisms of abelian groups*

$$\mathrm{H}^1(X \times Y, R) = \mathrm{H}^1(X, R) \oplus \mathrm{H}^1(Y, R)$$

and

$$\mathrm{H}^2(X \times Y, R) = \mathrm{H}^2(X, R) \oplus \mathrm{H}^2(Y, R) \oplus \left(\mathrm{H}^1(X, R) \otimes_R \mathrm{H}^1(Y, R) \right).$$

Proof. To simplify notation we write $\mathrm{H}_n(X)$ for $\mathrm{H}_n(X, \mathbb{Z})$. The universal coefficients theorem [6, Thm. 3.2] gives the following (split) exact sequence of abelian groups:

$$0 \to \mathrm{Ext}(\mathrm{H}_{n-1}(X), R) \to \mathrm{H}^n(X, R) \to \mathrm{Hom}(\mathrm{H}_n(X), R) \to 0. \quad (9)$$

Since X is non-empty and path-connected we have $\mathrm{H}_0(X) = \mathbb{Z}$ (see [6, Prop. 2.7]). This gives a canonical isomorphism

$$\mathrm{H}^1(X, R) = \mathrm{Hom}(\mathrm{H}_1(X), R). \quad (10)$$

The Künneth formula for homology is

$$0 \to \bigoplus_{i=0}^{n} \big(H_i(X) \otimes H_{n-i}(Y)\big) \to H_n(X \times Y) \to \bigoplus_{i=0}^{n-1} \mathrm{Tor}(H_i(X), H_{n-1-i}(Y)) \to 0$$

(see [6, Thm. 3.B.6]). We deduce from it canonical isomorphisms

$$H_1(X \times Y) = H_1(X) \oplus H_1(Y) \tag{11}$$

and

$$H_2(X \times Y) = H_2(X) \oplus H_2(Y) \oplus \big(H_1(X) \otimes H_1(Y)\big). \tag{12}$$

Our first isomorphism follows from (10) and (11).

The exact sequence (9) for $n = 2$ gives rise to the commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Ext}(H_1(X) \oplus H_1(Y), R) & \to & H^2(X, R) \oplus H^2(Y, R) & \to & \mathrm{Hom}(H_2(X) \oplus H_2(Y), R) & \to 0 \\
& \uparrow & & \uparrow & & \uparrow & \\
0 \to & \mathrm{Ext}(H_1(X \times Y), R) & \to & H^2(X \times Y, R) & \to & \mathrm{Hom}(H_2(X \times Y), R) & \to 0
\end{array}$$

By (11) the left vertical arrow is an isomorphism, hence the kernels of the other two vertical arrows are isomorphic. Hence, by (12), the kernel of the middle vertical map is isomorphic to $\mathrm{Hom}(H_1(X) \otimes H_1(Y), R)$, which by (10) is isomorphic to $H^1(X, R) \otimes_R H^1(Y, R)$. Moreover, $H^2(X, R)$ and $H^2(Y, R)$ are direct factors of $H^2(X \times Y, R)$, so our second isomorphism follows.   □

Remark 2.3. Let $X = \mathbb{RP}^2$. Then $H_1(X) = \mathbb{Z}/2$ and $H_n(X) = 0$ for $n \geq 2$. From the universal coefficients theorem (9) we obtain $H^1(X, \mathbb{Z}) = 0$, $H^2(X, \mathbb{Z}) = \mathbb{Z}/2$ and $H^n(X, \mathbb{Z}) = 0$ for $n \geq 3$ (cf. [6, Ex. 3.9]). Combining the calculation of homology of X^2 in [6, Ex. 3.B.3] with the universal coefficients theorem we obtain

$$H^3(X^2, \mathbb{Z}) = \mathbb{Z}/2 \neq \bigoplus_{i=0}^{3} \big(H^i(X, \mathbb{Z}) \otimes H^{3-i}(X, \mathbb{Z})\big).$$

This shows that Proposition 2.2 does not generalise to the third cohomology group, at least when $R = \mathbb{Z}$.

2.4. The type of a torsor. After this digression into algebraic topology we return to smooth, projective and geometrically integral varieties X and Y over a field k. We now introduce some notation. Let S_X be the finite commutative k-group of multiplicative type whose Cartier dual $\hat{S}_X := \mathrm{Hom}_{\bar{k}\text{-gr.}}(S_X, \mathbf{G}_m)$ is

$$\hat{S}_X = H^1_{\text{ét}}(\overline{X}, \mu_n) = \mathrm{Pic}(\overline{X})_n.$$

Then we have a canonical identification

$$\mathrm{Hom}(S_X, \mathbb{Z}/n) = H^1_{\text{ét}}(\overline{X}, \mathbb{Z}/n). \tag{13}$$

For any finite $\bar{k}$-group scheme G of multiplicative type annihilated by n we have a canonical isomorphism, functorial in X and G:

$$\tau_G : \mathrm{H}^1_{\text{ét}}(\overline{X}, G) \xrightarrow{\sim} \mathrm{Hom}(\hat{G}, \mathrm{Pic}(\overline{X})) = \mathrm{Hom}(\hat{G}, \hat{S}_X)$$

(see [14, Cor. 2.3.9]; it is enough to check this for $G = \mu_m$, where m is an integer dividing n). It can be defined via the natural pairing

$$\mathrm{H}^1_{\text{ét}}(\overline{X}, G) \times \hat{G} \to \mathrm{H}^1_{\text{ét}}(\overline{X}, \mu_n) = \hat{S}_X \qquad (14)$$

(see [14, Section 2.3]). If $Z/\overline{X}$ is a torsor under G, then the associated homomorphism $\tau_G(Z) : \hat{G} \to \hat{S}_X$ is called the *type* of $Z/\overline{X}$. If we take $G = S_X$, then there exists a torsor $\mathcal{T}_{\overline{X}}/\overline{X}$ under S_X, unique up to isomorphism, whose type is the identity map. Thus there is a well defined class $[\mathcal{T}_{\overline{X}}] \in \mathrm{H}^1_{\text{ét}}(\overline{X}, S_X)$. This class can be used to describe τ_G^{-1} explicitly. For $\varphi \in \mathrm{Hom}(\hat{G}, \hat{S}_X)$ let $\hat{\varphi} \in \mathrm{Hom}(S_X, G)$ be the homomorphism that corresponds to φ under the identification

$$\mathrm{Hom}(\hat{G}, \hat{S}_X) = \mathrm{Hom}(S_X, G).$$

The functoriality of τ_G in G implies that $\tau_G^{-1}(\varphi)$ is the push-forward $\hat{\varphi}_*[\mathcal{T}_{\overline{X}}]$, which can also be defined as the class of the $\overline{X}$-torsor $(\mathcal{T}_{\overline{X}} \times_k G)/S_X$.

If we take $G = S_X$ in (14) we obtain a natural pairing

$$\mathrm{H}^1_{\text{ét}}(\overline{X}, S_X) \times \hat{S}_X \to \mathrm{H}^1_{\text{ét}}(\overline{X}, \mu_n) = \hat{S}_X.$$

The definition of $\mathcal{T}_{\overline{X}}$ implies that pairing with the class $[\mathcal{T}_{\overline{X}}]$ gives the identity map on $\hat{S}_X$. After a twist we obtain a natural pairing

$$\mathrm{H}^1_{\text{ét}}(\overline{X}, S_X) \times \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Z}/n) \to \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Z}/n);$$

moreover, pairing with $[\mathcal{T}_{\overline{X}}]$ gives the identity map on $\mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Z}/n)$.

Remark 2.5. There is a natural cup-product map

$$\mathrm{H}^1_{\text{ét}}(\overline{X}, S_X) \otimes \mathrm{H}^1_{\text{ét}}(\overline{Y}, S_Y) \to \mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, S_X \otimes S_Y).$$

Let us denote the image of $[\mathcal{T}_{\overline{X}}] \otimes [\mathcal{T}_{\overline{Y}}]$ by $[\mathcal{T}_{\overline{X}}] \cup [\mathcal{T}_{\overline{Y}}]$. From (13) we obtain a natural pairing

$$\mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, S_X \otimes S_Y) \times \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbb{Z}/n) \to \mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, \mathbb{Z}/n).$$

Since pairing with $[\mathcal{T}_{\overline{X}}]$ induces identity on $\mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Z}/n)$, and similarly for Y, we see that pairing with $[\mathcal{T}_{\overline{X}}] \cup [\mathcal{T}_{\overline{Y}}]$ gives the cup-product map

$$\cup : \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbb{Z}/n) \to \mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, \mathbb{Z}/n).$$

Theorem 2.6. *Let n be a positive integer coprime to* $\mathrm{char}(k)$. *Then the homomorphism of Γ-modules*

$$\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}/n) \oplus \mathrm{H}^2_{\text{ét}}(\overline{Y}, \mathbb{Z}/n) \oplus \left(\mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbb{Z}/n)\right) \to \mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, \mathbb{Z}/n)$$

given by π_X^* on the first summand, by π_Y^* on the second summand, and by the cup-product on the third summand, is an isomorphism.

Proof. It is enough to establish this decomposition at the level of abelian groups. Choose $x_0 \in X(\bar{k})$, $y_0 \in Y(\bar{k})$. Using the notation of Section 1.4 we define

$$\mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, \mathbb{Z}/n)_{\text{prim}} = \mathrm{Ker}\left[(q^*_{y_0}, q^*_{x_0}) : \mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, \mathbb{Z}/n) \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}/n) \oplus \mathrm{H}^2_{\text{ét}}(\overline{Y}, \mathbb{Z}/n)\right].$$

The étale (or Zariski) sheaf $R^q\pi_{X*}(\mathbb{Z}/n)$ is the constant sheaf associated with the finite abelian group $\mathrm{H}^q_{\text{ét}}(\overline{Y}, \mathbb{Z}/n)$. Thus we have the Leray spectral sequence

$$E_2^{p,q} = \mathrm{H}^p_{\text{ét}}(\overline{X}, \mathrm{H}^q_{\text{ét}}(\overline{Y}, \mathbb{Z}/n)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(\overline{X} \times \overline{Y}, \mathbb{Z}/n). \tag{15}$$

We have seen in Proposition 1.5 that the maps π_X^* and π_Y^* make the abelian groups $\mathrm{H}^m_{\text{ét}}(\overline{X}, \mathbb{Z}/n)$ and $\mathrm{H}^m_{\text{ét}}(\overline{Y}, \mathbb{Z}/n)$ direct summands of $\mathrm{H}^m_{\text{ét}}(\overline{X} \times \overline{Y}, \mathbb{Z}/n)$, for all $m \geq 1$. By the standard theory of spectral sequences this gives a canonical isomorphism

$$\beta : \mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, \mathbb{Z}/n)_{\text{prim}} \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbb{Z}/n)).$$

Taking $G = \hat{S}_Y$ in (14) we get an isomorphism $\tau_{\hat{S}_Y} : \mathrm{H}^1_{\text{ét}}(\overline{X}, \hat{S}_Y) \xrightarrow{\sim} \mathrm{Hom}(S_Y, \hat{S}_X)$. Using (13), after a twist we obtain an isomorphism

$$\tau : \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbb{Z}/n)) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbb{Z}/n).$$

This gives some isomorphism as in the statement of the theorem. To complete the proof we need to check that for any $x \in \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Z}/n)$ and any $y \in \mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbb{Z}/n)$ we have

$$x \cup y = \beta^{-1}\tau^{-1}(x \otimes y).$$

We have seen above that $\tau^{-1}(x \otimes y)$ is the push-forward of the class $[\mathcal{T}_{\overline{X}}]$ by the map $S_X \to \mathrm{Hom}(S_Y, \mathbb{Z}/n)$ defined by

$$x \otimes y \in \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbb{Z}/n) = \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/n).$$

In other words, $\tau^{-1}(x \otimes y)$ is obtained by pairing $[\mathcal{T}_{\overline{X}}]$ with $x \otimes y$. In view of Remark 2.5, in order to finish the proof, it remains to check that β^{-1} can be described via the pairing

$$\mathrm{H}^1_{\text{ét}}(\overline{X}, \mathrm{Hom}(S_Y, \mathbb{Z}/n)) \times \mathrm{H}^1_{\text{ét}}(\overline{Y}, S_Y) \to \mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, \mathbb{Z}/n),$$

namely, as pairing with $[\mathcal{T}_{\overline{Y}}]$. This calculation is more or less standard (cf. [14, Thm. 4.1.1] or, more recently, [5, Thm. 1.4]).

Let us write $\mathcal{D}(Z)$ for the bounded derived category of étale sheaves of abelian groups on a variety Z. Let $\mathbf{R}\pi_{X*} : \mathcal{D}(\overline{X} \times \overline{Y}) \to \mathcal{D}(\overline{X})$ be the derived functor of π_X. Let $\rho : \overline{Y} \to \mathrm{Spec}(\bar{k})$ be the structure morphism, and let $\mathbf{R}\rho_* : \mathcal{D}(\overline{Y}) \to \mathcal{D}(\mathrm{Ab})$ be the corresponding derived functor to the bounded derived category of the category of abelian groups Ab. Each of these derived categories has the canonical truncation functors $\tau_{\leq m}$. We need to recall the definition of the type map of a group G of multiplicative type (see [14, Section 2.3]). This is the composite map

$$\mathrm{H}^1_{\text{ét}}(\overline{Y}, G) \to \mathrm{Ext}^1(\hat{G}, \tau_{\leq 1}\mathbf{R}\rho_*\mathbf{G}_{\mathrm{m}}) \to \mathrm{Hom}(\hat{G}, \mathrm{Pic}(\overline{Y})). \tag{16}$$

The Hom- and Ext-groups without subscript are taken in Ab or $\mathcal{D}(\mathrm{Ab})$. The second map in (16) is induced by the obvious exact triangle in $\mathcal{D}(\mathrm{Ab})$

$$\bar{k}^* \to \tau_{\leq 1}\mathbf{R}\rho_*\mathbf{G}_{\mathrm{m}} \to (\mathrm{Pic}(\overline{Y}))[-1],$$

where we used the facts that $\mathrm{H}^0_{\text{ét}}(\overline{Y}, \mathbf{G}_{\mathrm{m}}) = \bar{k}^*$, since $\overline{Y}$ is reduced and connected, and $\mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbf{G}_{\mathrm{m}}) = \mathrm{Pic}(\overline{Y})$. To define the first map in (16) consider the local-to-global spectral sequence of Ext-groups

$$E_2^{p,q} = \mathrm{H}^p_{\text{ét}}(\overline{Y}, \mathcal{E}xt_{\overline{Y}}^q(\hat{G}, \mathbf{G}_{\mathrm{m}})) \Rightarrow \mathrm{Ext}_{\overline{Y}}^{p+q}(\hat{G}, \mathbf{G}_{\mathrm{m}}).$$

It completely degenerates since $\mathcal{E}xt_{\overline{Y}}^q(\hat{G}, \mathbf{G}_{\mathrm{m}}) = 0$ for $q \geq 1$, thus giving an isomorphism $\mathrm{H}^q_{\text{ét}}(\overline{Y}, G) \xrightarrow{\sim} \mathrm{Ext}_{\overline{Y}}^q(\hat{G}, \mathbf{G}_{\mathrm{m}})$ [14, Lemma 2.3.7]. It remains to use the identities

$$\mathrm{Ext}_{\overline{Y}}^q(\hat{G}, \mathbf{G}_{\mathrm{m}}) = \mathrm{Ext}^q(\hat{G}, \mathbf{R}\rho_*\mathbf{G}_{\mathrm{m}}) = \mathrm{Ext}^q(\hat{G}, \tau_{\leq q}\mathbf{R}\rho_*\mathbf{G}_{\mathrm{m}})$$

stemming from the fact that $\mathbf{R}\mathrm{Hom}_{\overline{Y}}(\rho^*\hat{G}, \cdot) = \mathbf{R}\mathrm{Hom}(\hat{G}, \mathbf{R}\rho_*(\cdot))$. When G is annihilated by n, the image of the type map lies in $\mathrm{Hom}(\hat{G}, \mathrm{Pic}(\overline{Y})_n)$, and thus τ_G can be written as the composition of the maps

$$\mathrm{H}^1_{\text{ét}}(\overline{Y}, G) \to \mathrm{Ext}^1(\hat{G}, \tau_{\leq 1}\mathbf{R}\rho_*\mu_n) \to \mathrm{Hom}(\hat{G}, \mathrm{Pic}(\overline{Y})_n).$$

We claim that these maps fit into the following commutative diagram of pairings:

$$\begin{array}{ccccc}
\mathrm{H}^1_{\text{ét}}(\overline{X}, \hat{G}) \times & \mathrm{H}^1_{\text{ét}}(\overline{Y}, G) & \to & \mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, \mu_n) \\
\| & \downarrow & & \| \\
\mathrm{H}^1_{\text{ét}}(\overline{X}, \hat{G}) \times & \mathrm{Ext}^1_{\overline{Y}}(\hat{G}, \mu_n) & \to & \mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, \mu_n) \\
\| & \| & & \uparrow \\
\mathrm{H}^1_{\text{ét}}(\overline{X}, \hat{G}) \times \mathrm{Ext}^1(\hat{G}, \tau_{\leq 1}\mathbf{R}\rho_*\mu_n) & \to & \mathrm{H}^2_{\text{ét}}(\overline{X}, \tau_{\leq 1}\mathbf{R}\pi_{X*}\mu_n) \\
\| & \downarrow & & \downarrow \\
\mathrm{H}^1_{\text{ét}}(\overline{X}, \hat{G}) \times & \mathrm{Hom}(\hat{G}, \mathrm{Pic}(\overline{Y})_n) & \to & \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathrm{Pic}(\overline{Y})_n)
\end{array}$$

The first two pairings are compatible by [8, Prop. V.1.20]. The two lower pairings are natural, and the compatibility of the rest of the diagram is clear.

Now take $G = S_Y$, so that $\hat{G} = \mathrm{Pic}(\overline{Y})_n$. By pairing with $[\mathcal{T}_{\overline{Y}}]$, after a twist we obtain a map

$$\gamma : \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathrm{H}^1(\overline{Y}, \mathbb{Z}/n)) \to \mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, \mathbb{Z}/n),$$

which factors through the injective map

$$\mathrm{H}^2_{\text{ét}}(\overline{X}, \tau_{\leq 1}\mathbf{R}\pi_{X*}(\mathbb{Z}/n)) \to \mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, \mathbb{Z}/n).$$

Since $\bar{k}$ is separably closed we have $\mathrm{H}^1_{\text{ét}}(y_0, G) = \mathrm{H}^1_{\text{ét}}(\bar{k}, G) = 0$, so that $q^*_{y_0}\gamma = 0$. A similar argument gives $q^*_{x_0}\gamma = 0$, thus $\mathrm{Im}(\gamma) \subset \mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, \mathbb{Z}/n)_{\mathrm{prim}}$. By the standard theory of spectral sequences the map β is obtained from the right hand downward map in the diagram (after a twist). Since the type of $\mathcal{T}_{\overline{Y}}$ is the identity in $\mathrm{Hom}(\mathrm{Pic}(\overline{Y})_n, \mathrm{Pic}(\overline{Y})_n)$, the commutativity of the diagram implies that $\beta\gamma = \mathrm{id}$. □

Corollary 2.7. *Let n be a positive integer coprime to* $\mathrm{char}(k)$, $|\mathrm{NS}(\overline{X})_{\mathrm{tors}}|$ *and* $|\mathrm{NS}(\overline{Y})_{\mathrm{tors}}|$. *Then we have a canonical decomposition of Γ-modules*

$$\mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, \mu_n) = \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_n) \oplus \mathrm{H}^2_{\text{ét}}(\overline{Y}, \mu_n) \oplus \mathrm{Hom}(B^t_n, A_n). \tag{17}$$

Proof. For any prime ℓ dividing n we have $\mathrm{NS}(\overline{X})(\ell) = 0$ and $\mathrm{NS}(\overline{Y})(\ell) = 0$. Thus from the isomorphism (1) and the exact sequence (2) we obtain canonical isomorphisms

$$\mathrm{H}^1_{\text{ét}}(\overline{X}, \mu_{\ell^m}) = A_{\ell^m}, \qquad \mathrm{H}^1_{\text{ét}}(\overline{Y}, \mu_{\ell^m}) = B_{\ell^m},$$

for any $m \geq 1$. From the non-degeneracy of the Weil pairing we deduce a canonical isomorphism of Γ-modules

$$\mathrm{H}^1(\overline{Y}, \mathbb{Z}/\ell^m) = B_{\ell^m}(-1) = \mathrm{Hom}(B^t_{\ell^m}, \mathbb{Z}/\ell^m).$$

We conclude that the Galois modules $\mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbb{Z}/\ell^m) \otimes \mathrm{H}^1_{\text{ét}}(\overline{X}, \mu_{\ell^m})$ and $\mathrm{Hom}(B^t_{\ell^m}, A_{\ell^m})$ are canonically isomorphic. Hence, after a twist by μ_n, the isomorphism of Theorem 2.6 can be written as (17). □

2.8. First Chern classes. Let ℓ be a prime different from $\mathrm{char}(k)$. Tensoring (5) with $\mathbb{Q}_\ell$ we obtain the following exact sequence of Γ-modules

$$0 \to \mathrm{NS}(\overline{X}) \otimes \mathbb{Q}_\ell \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1)) \to V_\ell(\mathrm{Br}(\overline{X})) \to 0, \tag{18}$$

The injective maps from (18) and (4) are both called *the first Chern class* maps (see, e.g., [8, VI.9]):

$$c_1 : \mathrm{NS}(\overline{X}) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1)), \qquad \bar{c}_1 : \mathrm{NS}(\overline{X})/\ell^m \hookrightarrow \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_{\ell^m}).$$

Proposition 1.7 gives a natural isomorphism of Galois modules

$$\mathrm{NS}(\overline{X} \times \overline{Y}) = \mathrm{NS}(\overline{X}) \oplus \mathrm{NS}(\overline{Y}) \oplus \mathrm{Hom}(\overline{B^t}, \overline{A}).$$

Since the maps c_1 and $\bar{c}_1$ are functorial in X, we see that the map

$$c_1 : \mathrm{NS}(\overline{X} \times \overline{Y}) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, \mathbb{Q}_\ell(1))$$

forms obvious commutative diagrams with the maps

$$c_1 : \mathrm{NS}(\overline{X}) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1)), \quad c_1 : \mathrm{NS}(\overline{Y}) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{H}^2_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell(1)),$$

$$c_1 : \mathrm{Hom}(\overline{B^t}, \overline{A}) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{Hom}_{\mathbb{Q}_\ell}(V_\ell(B^t), V_\ell(A)).$$

Similarly, for ℓ coprime to $\mathrm{char}(k)$, $|\mathrm{NS}(\overline{X})_{\text{tors}}|$ and $|\mathrm{NS}(\overline{Y})_{\text{tors}}|$, the map

$$\bar{c}_1 : \mathrm{NS}(\overline{X} \times \overline{Y})/\ell^m \hookrightarrow \mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, \mu_{\ell^m})$$

forms similar commutative diagrams with the maps

$$\bar{c}_1 : \mathrm{NS}(\overline{X})/\ell^m \hookrightarrow \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_{\ell^m}), \quad \bar{c}_1 : \mathrm{NS}(\overline{Y})/\ell^m \hookrightarrow \mathrm{H}^2_{\text{ét}}(\overline{Y}, \mu_{\ell^m}),$$

$$\bar{c}_1 : \mathrm{Hom}(\overline{B^t}, \overline{A})/\ell^m \hookrightarrow \mathrm{Hom}(B^t_{\ell^m}, A_{\ell^m}) = \mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbb{Z}/\ell^m) \otimes \mathrm{H}^1_{\text{ét}}(\overline{X}, \mu_{\ell^m}).$$

2.9. Brauer groups of products of varieties. Let ℓ be a prime different from $\mathrm{char}(k)$. Applying (18) to X, Y and $X \times Y$, using (8) and the compatibilities from Section 2.8, we obtain the following decomposition of Galois modules:

$$V_\ell(\mathrm{Br}(\overline{X} \times \overline{Y})) = V_\ell(\mathrm{Br}(\overline{X})) \oplus V_\ell(\mathrm{Br}(\overline{Y})) \oplus \big(\mathrm{Hom}_{\mathbb{Q}_\ell}(V_\ell(B^t), V_\ell(A))/\mathrm{Hom}(\overline{B^t}, \overline{A}) \otimes \mathbb{Q}_\ell\big). \tag{19}$$

When ℓ is also coprime to $|\mathrm{NS}(\overline{X})_{\text{tors}}|$ and $|\mathrm{NS}(\overline{Y})_{\text{tors}}|$, we apply (4) to X, Y and $X \times Y$, and obtain from Corollary 2.7 and Section 2.8 the decomposition of Γ-modules

$$\mathrm{Br}(\overline{X} \times \overline{Y})_{\ell^m} = \mathrm{Br}(\overline{X})_{\ell^m} \oplus \mathrm{Br}(\overline{Y})_{\ell^m} \oplus \mathrm{Hom}(B^t_{\ell^m}, A_{\ell^m})/(\mathrm{Hom}(\overline{B^t}, \overline{A})/\ell^m). \tag{20}$$

The case when X and Y are elliptic curves was considered in [16, Prop. 3.3].

3. Proof of Theorem A

The proof of Theorem A crucially uses the following properties. Let C and D be abelian varieties over a field k finitely generated over its prime subfield. Then

(1) the Γ-modules $V_\ell(C)$ and $V_\ell(D)$ are semisimple, and the natural injective map

$$\mathrm{Hom}(C, D) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{Hom}_\Gamma(V_\ell(C), V_\ell(D))$$

is bijective;

(2) for almost all primes ℓ the Γ-modules C_ℓ and D_ℓ are semisimple, and the natural injective map

$$\mathrm{Hom}(C, D)/\ell \hookrightarrow \mathrm{Hom}_\Gamma(C_\ell, D_\ell)$$

is bijective.

Statement (1) was proved by the second named author in characteristic $p > 2$ [20, 21], and by Faltings [2, 3] in characteristic zero. In characteristic $p = 2$ this follows from the results of S. Mori [10, Thm. 2.5, pp. 244–245] (see also [26, Sect. 1]). Statement (2) was proved by the second named author in [22, Thm. 1.1], [23, Cor. 5.4.3 and Cor. 5.4.5] and [26, Cor. 2.3 and Cor. 2.7] (see also [15, Prop. 3.4], [24, Thm. 4.4], [27] and [25]).

Theorem A is a consequence of the following result.

Theorem 3.1. *Let k be a field finitely generated over its prime subfield. Let X and Y be smooth, projective and geometrically integral varieties over k.*

(i) *If* $\operatorname{char}(k) = 0$, *then* $[\operatorname{Br}(\overline{X} \times \overline{Y})/(\operatorname{Br}(\overline{X}) \oplus \operatorname{Br}(\overline{Y}))]^{\Gamma}$ *is finite.*

(ii) *If* $\operatorname{char}(k) = p$, *then the group* $[\operatorname{Br}(\overline{X} \times \overline{Y})/(\operatorname{Br}(\overline{X}) \oplus \operatorname{Br}(\overline{Y}))]^{\Gamma}$ (non-p) *is finite.*

Proof. Since $\operatorname{Br}(\overline{X} \times \overline{Y})$ is a torsion group, it is enough to prove these statements:

(a) *If* ℓ *is a prime,* $\ell \neq \operatorname{char}(k)$, *then* $V_{\ell}\big((\operatorname{Br}(\overline{X} \times \overline{Y})/(\operatorname{Br}(\overline{X}) \oplus \operatorname{Br}(\overline{Y})))^{\Gamma}\big) = 0.$

(b) *For almost all primes* ℓ *we have* $\big(\operatorname{Br}(\overline{X} \times \overline{Y})_{\ell}/(\operatorname{Br}(\overline{X})_{\ell} \oplus \operatorname{Br}(\overline{Y})_{\ell})\big)^{\Gamma} = 0.$

Let us prove (a). Using (19) we obtain

$$V_{\ell}\big((\operatorname{Br}(\overline{X} \times \overline{Y})/(\operatorname{Br}(\overline{X}) \oplus \operatorname{Br}(\overline{Y})))^{\Gamma}\big) = V_{\ell}\big(\operatorname{Br}(\overline{X} \times \overline{Y})/(\operatorname{Br}(\overline{X}) \oplus \operatorname{Br}(\overline{Y}))\big)^{\Gamma}$$
$$= \big(V_{\ell}(\operatorname{Br}(\overline{X} \times \overline{Y}))/(V_{\ell}(\operatorname{Br}(\overline{X})) \oplus V_{\ell}(\operatorname{Br}(\overline{Y})))\big)^{\Gamma}$$
$$= \big(\operatorname{Hom}_{\mathbb{Q}_{\ell}}\big(V_{\ell}(B^t), V_{\ell}(A)\big)/\operatorname{Hom}(\overline{B^t}, \overline{A}) \otimes \mathbb{Q}_{\ell}\big)^{\Gamma}.$$

By a theorem of Chevalley [1, p. 88] the semisimplicity of the Γ-modules $V_{\ell}(B^t)$ and $V_{\ell}(A)$ implies the semisimplicity of the Γ-module $\operatorname{Hom}_{\mathbb{Q}_{\ell}}(V_{\ell}(B^t), V_{\ell}(A))$. This implies

$$V_{\ell}\big((\operatorname{Br}(\overline{X} \times \overline{Y})/(\operatorname{Br}(\overline{X}) \oplus \operatorname{Br}(\overline{Y})))^{\Gamma}\big) = \operatorname{Hom}_{\Gamma}(V_{\ell}(B^t), V_{\ell}(A))/\operatorname{Hom}(B^t, A) \otimes \mathbb{Q}_{\ell} = 0,$$

thus proving (a).

Let us prove (b). By (20) it is enough to show that

$$\big(\operatorname{Hom}(B_{\ell}^t, A_{\ell})/(\operatorname{Hom}(\overline{B^t}, \overline{A})/\ell)\big)^{\Gamma} = 0.$$

Since $\operatorname{Hom}(\overline{B^t}, \overline{A})^{\Gamma} = \operatorname{Hom}(B^t, A)$, the exact sequence

$$0 \to \operatorname{Hom}(\overline{B^t}, \overline{A})^{\Gamma}/\ell \to (\operatorname{Hom}(\overline{B^t}, \overline{A})/\ell)^{\Gamma} \to \operatorname{H}^1(k, \operatorname{Hom}(\overline{B^t}, \overline{A}))$$

implies that for all but finitely many primes ℓ we have

$$(\operatorname{Hom}(\overline{B^t}, \overline{A})/\ell)^{\Gamma} = \operatorname{Hom}(B^t, A)/\ell.$$

If we further assume that $\ell > 2 \dim(A) + 2 \dim(B) - 2$, then, by a theorem of Serre [13], the semisimplicity of the Γ-modules B_{ℓ}^t and A_{ℓ} implies the semisimplicity of $\operatorname{Hom}(B_{\ell}^t, A_{\ell})$. Hence we obtain

$$\big(\operatorname{Hom}(B_{\ell}^t, A_{\ell})/(\operatorname{Hom}(\overline{B^t}, \overline{A})/\ell)\big)^{\Gamma} = \operatorname{Hom}(B_{\ell}^t, A_{\ell})^{\Gamma}/(\operatorname{Hom}(\overline{B^t}, \overline{A})/\ell)^{\Gamma}$$
$$= \operatorname{Hom}_{\Gamma}(B_{\ell}^t, A_{\ell})/(\operatorname{Hom}(B^t, A)/\ell) = 0,$$

thus proving (b).   □

Corollary 3.2. *Let k be a field finitely generated over its prime subfield. Let X and Y be smooth, projective and geometrically integral varieties over k.*

(i) *Assume* $\operatorname{char}(k) = 0$. *The group* $\operatorname{Br}(\overline{X} \times \overline{Y})^{\Gamma}$ *is finite if and only if the groups* $\operatorname{Br}(\overline{X})^{\Gamma}$ *and* $\operatorname{Br}(\overline{Y})^{\Gamma}$ *are finite.*

(ii) *Assume that* $\operatorname{char}(k)$ *is a prime* p. *The group* $\operatorname{Br}(\overline{X} \times \overline{Y})^{\Gamma}$ (non-p) *is finite if and only if the groups* $\operatorname{Br}(\overline{X})^{\Gamma}$ (non-p) *and* $\operatorname{Br}(\overline{Y})^{\Gamma}$ (non-p) *are finite.*

4. Proof of Theorem B

It is enough to prove the following statements:

(a) *The subgroup of* $\mathrm{Br}(X \times Y)$ *generated by* $\mathrm{Br}_1(X \times Y)$ *and the images of* $\mathrm{Br}(X)$ *and* $\mathrm{Br}(Y)$*, has finite index.*

(b) *The cokernel of the natural map* $\mathrm{Br}_1(X) \oplus \mathrm{Br}_1(Y) \to \mathrm{Br}_1(X \times Y)$ *is finite.*

Each of these statements formally follows from Theorem A, the functoriality of the Hochschild–Serre spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q_{\mathrm{\acute{e}t}}(\overline{X}, \mathbf{G}_{\mathrm{m}})) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{\acute{e}t}}(X, \mathbf{G}_{\mathrm{m}}) \tag{21}$$

with respect to X, and the finiteness property stated in Remark 1.9.

Let us recall how (21) is usually applied. If $X(k) \neq \varnothing$, then the canonical map

$$E_2^{3,0} = \mathrm{H}^3(k, \bar{k}^*) \to \mathrm{H}^3_{\mathrm{\acute{e}t}}(X, \mathbf{G}_{\mathrm{m}})$$

has a section given by a k-point on X, and hence is injective. The same is obviously true if $\mathrm{H}^3(k, \bar{k}^*) = 0$. The standard theory of spectral sequences now implies that the kernel of the canonical map

$$E_2^{0,2} = \mathrm{Br}(\overline{X})^{\Gamma} \to E_2^{2,1} = \mathrm{H}^2(k, \mathrm{Pic}(\overline{X}))$$

is the image of $\mathrm{Br}(X)$ in $\mathrm{Br}(\overline{X})^{\Gamma}$.

Let us prove (a). By functoriality of the spectral sequence (21) we have the following commutative diagram with exact rows:

$$
\begin{array}{ccccc}
\mathrm{Br}(X \times Y) & \to & \mathrm{Br}(\overline{X} \times \overline{Y})^{\Gamma} & \to & \mathrm{H}^2(k, \mathrm{Pic}(\overline{X} \times \overline{Y})) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Br}(X) \oplus \mathrm{Br}(Y) & \to & \mathrm{Br}(\overline{X})^{\Gamma} \oplus \mathrm{Br}(\overline{Y})^{\Gamma} & \to & \mathrm{H}^2(k, \mathrm{Pic}(\overline{X})) \oplus \mathrm{H}^2(k, \mathrm{Pic}(\overline{Y}))
\end{array}
$$

Note that the middle vertical map here is injective. To prove (a) we must show that the image of $\mathrm{Br}(X) \oplus \mathrm{Br}(Y)$ in $\mathrm{Br}(\overline{X} \times \overline{Y})^{\Gamma}$ has finite index in the subgroup of the elements that go to 0 in $\mathrm{H}^2(k, \mathrm{Pic}(\overline{X} \times \overline{Y}))$. This follows from Theorem 3.1(i) and Remark 1.9.

To prove (b) we consider another commutative diagram with exact rows, also constructed using the functoriality of the spectral sequence (21):

$$
\begin{array}{ccccccc}
\mathrm{Br}(k) & \to & \mathrm{Br}_1(X \times Y) & \to & \mathrm{H}^1(k, \mathrm{Pic}(\overline{X} \times \overline{Y})) & & \to 0 \\
& & \uparrow & & \uparrow & & \\
& & \mathrm{Br}_1(X) \oplus \mathrm{Br}_1(Y) & \to & \mathrm{H}^1(k, \mathrm{Pic}(\overline{X})) \oplus \mathrm{H}^1(k, \mathrm{Pic}(\overline{Y})) & & \to 0
\end{array}
$$

Statement (b) follows from this diagram and Remark 1.9.

5. Proof of Theorem C

The inclusion of the left hand side into the right hand side follows from functoriality of the Brauer group. Thus we can assume that $X(\mathbb{A}_k)^{\mathrm{Br}}$ and $Y(\mathbb{A}_k)^{\mathrm{Br}}$ are not empty. Since the Brauer group of a smooth projective variety is a torsion group, to prove the opposite

inclusion it is enough to show that for any positive integer n the subgroup $\mathrm{Br}(X \times Y)_n$ is generated by the images of $\mathrm{Br}(X)_n$ and $\mathrm{Br}(Y)_n$, together with some elements that pair trivially with $X(\mathbb{A}_k)^{\mathrm{Br}} \times Y(\mathbb{A}_k)^{\mathrm{Br}}$ with respect to the Brauer–Manin pairing. The Kummer sequence gives a surjective map $\mathrm{H}^2_{\text{ét}}(X \times Y, \mu_n) \to \mathrm{Br}(X \times Y)_n$, so it suffices to show that, modulo the images of $\mathrm{H}^2_{\text{ét}}(X, \mu_n)$ and $\mathrm{H}^2_{\text{ét}}(Y, \mu_n)$, the group $\mathrm{H}^2_{\text{ét}}(X \times Y, \mu_n)$ is generated by the elements that pair trivially with $X(\mathbb{A}_k)^{\mathrm{Br}} \times Y(\mathbb{A}_k)^{\mathrm{Br}}$.

If Z/X is a torsor under a k-group of multiplicative type G annihilated by n, then the type of Z/X, as recalled in Section 2.4, is the image of the class $[Z/X]$ under the composite map

$$\mathrm{H}^1_{\text{ét}}(X, G) \to \mathrm{H}^1_{\text{ét}}(\overline{X}, G)^\Gamma \to \mathrm{Hom}_k(\hat{G}, \mathrm{Pic}(\overline{X})) = \mathrm{Hom}_k(\hat{G}, \mathrm{Pic}(\overline{X})_n).$$

Recall that S_X denotes the k-group scheme of multiplicative type that is dual to the Γ-module $\mathrm{Pic}(\overline{X})_n$.

Lemma 5.1. *If $X(\mathbb{A}_k)^{\mathrm{Br}}$ is not empty, then there exists an X-torsor under S_X whose type is the identity map on $\hat{S}_X$.*

Proof. One of the main results of the descent theory of Colliot-Thélène and Sansuc says that if $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing$, then for any homomorphism of Γ-modules $\hat{G} \to \mathrm{Pic}(\overline{X})$ there exists an X-torsor under G of this type (see [14, Cor. 6.1.3(1)]). $\qquad\square$

Let us choose one such X-torsor under S_X, and call it $\mathcal{T}_X$. (It is unique up to twisting by a k-torsor under S_X.) Then $\overline{\mathcal{T}_X}$ is isomorphic to the $\overline{X}$-torsor $\mathcal{T}_{\overline{X}}$ from Section 2. As in Remark 2.5 we form the class

$$[\mathcal{T}_X] \cup [\mathcal{T}_Y] \in \mathrm{H}^2_{\text{ét}}(X \times Y, S_X \otimes S_Y).$$

Pairing with it gives a map

$$\varepsilon : \mathrm{Hom}_k(S_X \otimes S_Y, \mu_n) = \mathrm{Hom}_k(S_X, \hat{S}_Y) \to \mathrm{H}^2_{\text{ét}}(X \times Y, \mu_n).$$

For $\varphi \in \mathrm{Hom}_k(S_X, \hat{S}_Y)$ we can write $\varepsilon(\varphi) = \varphi_*[\mathcal{T}_X] \cup [\mathcal{T}_Y]$, where $\cup$ stands for the cup-product pairing

$$\mathrm{H}^1_{\text{ét}}(X, \hat{S}_Y) \times \mathrm{H}^1_{\text{ét}}(Y, S_Y) \to \mathrm{H}^2_{\text{ét}}(X \times Y, \mu_n).$$

Remark 2.5 gives a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_k(S_X, \hat{S}_Y) & \xrightarrow{\ \varepsilon\ } & \mathrm{H}^2_{\text{ét}}(X \times Y, \mu_n) \\
\| & & \downarrow \\
\left(\mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(\overline{Y}, \mu_n)\right)^\Gamma & \xrightarrow{\ \cup\ } & \mathrm{H}^2_{\text{ét}}(\overline{X} \times \overline{Y}, \mu_n)^\Gamma
\end{array} \qquad (22)$$

It is clear that Theorem C is a consequence of Lemmas 5.2 and 5.3 below.

Lemma 5.2. *We have $\mathrm{H}^2_{\text{ét}}(X \times Y, \mu_n) = \pi_X^* \mathrm{H}^2_{\text{ét}}(X, \mu_n) + \pi_Y^* \mathrm{H}^2_{\text{ét}}(Y, \mu_n) + \mathrm{Im}(\varepsilon)$.*

Lemma 5.3. *For any positive integer n we have the inclusion*

$$X(\mathbb{A}_k)^{\mathrm{Br}_1(X)[n]} \times Y(\mathbb{A}_k)^{\mathrm{Br}_1(Y)[n]} \subset (X \times Y)(\mathbb{A}_k)^{\mathrm{Im}(\varepsilon)}.$$

Proof of Lemma 5.2. We use the spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(\overline{X}, \mu_n)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mu_n). \tag{23}$$

Let us make a few observations in the case when X is a smooth, projective and geometrically integral variety over a number field k such that $X(\mathbb{A}_k) \neq \emptyset$. The canonical maps

$$E_2^{p,0} = \mathrm{H}^p(k, \mu_n) \to \mathrm{H}^p_{\text{ét}}(X, \mu_n)$$

are injective for $p \geq 3$. Indeed, for such p the natural map

$$\mathrm{H}^p(k, M) \to \bigoplus_{k_v \simeq \mathbb{R}} \mathrm{H}^p(k_v, M)$$

is a bijection for any finite Galois module M (see [9, Thm. I.4.10(c)]). Next, the natural map $\mathrm{H}^p(k_v, M) \to \mathrm{H}^p_{\text{ét}}(X \times_k k_v, M)$ is injective for any p since any k_v-point of X defines a section of it. It follows that the composite map

$$\mathrm{H}^p(k, M) \to \mathrm{H}^p_{\text{ét}}(X, M) \to \bigoplus_{k_v \simeq \mathbb{R}} \mathrm{H}^p_{\text{ét}}(X \times_k k_v, M)$$

is injective, and this implies our claim. We note that

$$E_2^{2,0} = \mathrm{H}^2(k, \mu_n) \to \mathrm{H}^2_{\text{ét}}(X, \mu_n)$$

is also injective. The argument is similar once we identify $\mathrm{H}^2(k, \mu_n) = \mathrm{Br}(k)_n$ using the Kummer sequence and Hilbert's Theorem 90, and use the embedding of $\mathrm{Br}(k)$ into the direct sum of $\mathrm{Br}(k_v)$, for all completions k_v of k, provided by global class field theory, together with the existence of k_v-points on X for every place v. This implies the triviality of all the canonical maps in the spectral sequence whose target is $E_2^{p,0} = \mathrm{H}^p(k, \mu_n)$ for $p \geq 2$.

Let us write $\tilde{\mathrm{H}}^2_{\text{ét}}(X, \mu_n)$ for the quotient of $\mathrm{H}^2_{\text{ét}}(X, \mu_n)$ by the (injective) image of $\mathrm{H}^2(k, \mu_n)$. Using the above remarks we obtain from (23) the following exact sequence:

$$0 \to \mathrm{H}^1(k, \mathrm{H}^1_{\text{ét}}(\overline{X}, \mu_n)) \to \tilde{\mathrm{H}}^2_{\text{ét}}(X, \mu_n) \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_n)^{\Gamma} \to \mathrm{H}^2(k, \mathrm{H}^1_{\text{ét}}(\overline{X}, \mu_n)). \tag{24}$$

There are similar sequences for Y and $X \times Y$ linked by the maps π_X^* and π_Y^*.

Let us define

$$\mathcal{H} = \pi_X^* \mathrm{H}^2_{\text{ét}}(X, \mu_n) + \pi_Y^* \mathrm{H}^2_{\text{ét}}(Y, \mu_n) + \mathrm{Im}(\varepsilon) \subset \mathrm{H}^2_{\text{ét}}(X \times Y, \mu_n).$$

It is clear that the (injective) image of $\mathrm{H}^2(k, \mu_n)$ in $\mathrm{H}^2_{\text{ét}}(X \times Y, \mu_n)$ is contained in $\mathcal{H}$, so to prove Lemma 5.2 it is enough to prove that the natural map $\mathcal{H} \to \tilde{\mathrm{H}}^2_{\text{ét}}(X \times Y, \mu_n)$ is surjective.

By (7) the image of $H^1(k, H^1_{ét}(\overline{X} \times \overline{Y}, \mu_n)) \to \tilde{H}^2_{ét}(X \times Y, \mu_n)$ is contained in $\mathcal{H}$. In view of (24) it remains to show that every element of the kernel of the map

$$H^2_{ét}(\overline{X} \times \overline{Y}, \mu_n)^\Gamma \to H^2(k, H^1_{ét}(\overline{X} \times \overline{Y}, \mu_n))$$

comes from $\mathcal{H}$. By Theorem 2.6 and (7) this map can be written as

$$H^2_{ét}(\overline{X}, \mu_n)^\Gamma \oplus H^2_{ét}(\overline{Y}, \mu_n)^\Gamma \oplus \mathrm{Hom}_k(S_X, \hat{S}_Y) \to H^2(k, H^1_{ét}(\overline{X}, \mu_n)) \oplus H^2(k, H^1_{ét}(\overline{Y}, \mu_n)).$$

By the commutativity of the diagram (22) for any $\varphi \in \mathrm{Hom}_k(S_X, \hat{S}_Y)$, the element $\varepsilon(\varphi)$ of $H^2_{ét}(X \times Y, \mu_n)$ maps to

$$\varphi \in \mathrm{Hom}(S_X, \hat{S}_Y)^\Gamma \subset H^2_{ét}(\overline{X} \times \overline{Y}, \mu_n)^\Gamma.$$

This implies that for any $a \in H^2_{ét}(X \times Y, \mu_n)$ there exists an element $b \in \mathcal{H}$ such that the image of $a - b$ in $H^2_{ét}(\overline{X} \times \overline{Y}, \mu_n)^\Gamma$ is $\pi_X^*(x) + \pi_Y^*(y)$ for some $x \in H^2_{ét}(\overline{X}, \mu_n)^\Gamma$ and $y \in H^2_{ét}(\overline{Y}, \mu_n)^\Gamma$. From the exact sequence (24) for $X \times Y$ we see that $\pi_X^*(x) + \pi_Y^*(y)$ goes to zero in $H^2(k, H^1_{ét}(\overline{X}, \mu_n)) \oplus H^2(k, H^1_{ét}(\overline{Y}, \mu_n))$, hence x goes to zero in $H^2(k, H^1_{ét}(\overline{X}, \mu_n))$, and y goes to zero in $H^2(k, H^1_{ét}(\overline{Y}, \mu_n))$. By (24) for X we see that x is the image of some $c \in H^2_{ét}(X, \mu_n)$. Similarly, y is the image of some $d \in H^2_{ét}(Y, \mu_n)$. This proves that $a - (b + \pi_X^*(c) + \pi_Y^*(d))$ goes to zero in $H^2_{ét}(\overline{X} \times \overline{Y}, \mu_n)^\Gamma$, and hence belongs to $\mathcal{H}$. Thus $a \in \mathcal{H}$. $\qquad\square$

Proof of Lemma 5.3. Let M be a finite Γ-module such that $nM = 0$. Let M^D be the dual module $\mathrm{Hom}(M, \bar{k}^*)$. If v is a non-archimedean place of k, we write $H^1_{nr}(k_v, M)$ for the unramified subgroup of $H^1(k_v, M)$. By definition, it consists of the classes that are annihilated by the restriction to the maximal unramified extension of k_v. We write $P^1(k, M)$ for the restricted product of the abelian groups $H^1(k_v, M)$ relative to the subgroups $H^1_{nr}(k_v, M)$, where v is a non-archimedean place of k. By [9, Lemma I.4.8] the image of the diagonal map

$$H^1(k, M) \to \prod_{\text{all } v} H^1(k_v, M)$$

is contained in $P^1(k, M)$. Let us denote this image by $U^1(k, M)$.

The local pairings $H^1(k_v, M) \times H^1(k_v, M^D) \to H^2(k_v, \mu_n)$ give rise to the global Poitou–Tate pairing

$$(\ , \ ) : P^1(k, M) \times P^1(k, M^D) \to \mathbb{Z}/n.$$

It is a perfect duality of locally compact abelian groups; moreover, the subgroups $U^1(k, M)$ and $U^1(k, M^D)$ are exact annihilators of each other [9, Thm. I.4.10(b)].

Let $\varphi \in \mathrm{Hom}_k(S_X, \hat{S}_Y)$. Let $(P_v) \in X(\mathbb{A}_k)$ be an adelic point that is Brauer–Manin orthogonal to $\mathrm{Br}_1(X)[n]$, and let $(Q_v) \in Y(\mathbb{A}_k)$ be an adelic point orthogonal to $\mathrm{Br}_1(Y)[n]$. The Brauer–Manin pairing of the adelic point $(P_v \times Q_v)$ with the image of $\varepsilon(\varphi) =$

$\varphi_*[\mathcal{T}_X] \cup [\mathcal{T}_Y]$ in $\mathrm{Br}(X \times Y)$ is given by the Poitou–Tate pairing, so to prove Lemma 5.3 we need to show that

$$(\varphi_*[\mathcal{T}_X](P_v), [\mathcal{T}_Y](Q_v)) = 0, \tag{25}$$

where in the above notation $M = \hat{S}_Y$, $M^D = S_Y$. We point out that for any $a \in \mathrm{H}^1(k, \hat{S}_Y)$ we have $a \cup [\mathcal{T}_Y] \in \mathrm{Br}_1(Y)[n]$, and hence $(a, [\mathcal{T}_Y](Q_v)) = 0$. If an element of $P^1(k, S_Y)$ is orthogonal to $U^1(k, \hat{S}_Y)$, then it belongs to $U^1(k, S_Y)$. Therefore, we must have

$$[\mathcal{T}_Y](Q_v) \in U^1(k, S_Y). \tag{26}$$

Similarly, for any $b \in \mathrm{H}^1(k, S_Y)$ we have $\varphi_*[\mathcal{T}_X] \cup b \in \mathrm{Br}_1(X)[n]$, and hence $(\varphi_*[\mathcal{T}_X](P_v), b) = 0$. Since every element of $P^1(k, \hat{S}_Y)$ orthogonal to $U^1(k, S_Y)$ belongs to $U^1(k, \hat{S}_Y)$, this implies

$$\varphi_*[\mathcal{T}_X](P_v) \in U^1(k, \hat{S}_Y). \tag{27}$$

Now (26) and (27) imply (25). $\square$

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