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Barenblatt solutions and asymptotic behaviour for a nonlinear fractional heat equation of porous medium type

This paper is dedicated to Grisha Barenblatt, maestro and friend, for his 85th birthday

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Abstract. We establish the existence, uniqueness and main properties of the fundamental solutions for the fractional porous medium equation introduced in [51]. They are self-similar functions of the form $u(x,t) = t^{-\alpha} f(|x|t^{-\beta})$ with suitable α and β. As a main application of this construction, we prove that the asymptotic behaviour of general solutions is represented by such special solutions. Very singular solutions are also constructed. Among other interesting qualitative properties of the equation we prove an Aleksandrov reflection principle.

Keywords. Nonlinear fractional diffusion, fundamental solutions, very singular solutions, asymptotic behaviour

Contents

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1. Introduction

Our main goal is to determine the existence, uniqueness and main properties of the solution of the equation

$$\partial_t u + (-\Delta)^s (u^m) = 0, \qquad x \in \mathbb{R}^N,\ t > 0, \tag{1.1}$$

with fractional exponent $s \in (0,1)$, and $m > 0$. This equation, or rather family of equations, is one of the standard models of nonlinear diffusion involving long-distance effects in the form of fractional Laplacian operators, which are the most representative nonlocal operators of elliptic type. We recall that the fractional Laplacian operator is a kind of differentiation operator of order $2s$, for arbitrary $s \in (0,1)$, that can be conveniently defined through its Fourier transform symbol, which is $|\xi|^{2s}$, so that for $s = 1$ we recover the standard Laplacian. A major difference between the standard and the fractional Laplacian is best seen in the stochastic point of view, and it consists in taking into account long-range interactions in the latter, which explains features that we will see below, like enhanced propagation with the appearance of fat tails at long distances (such tails are to be compared with the typical exponentially small tails of the standard diffusion, or the compactly supported solutions of porous medium flows). This is known as anomalous diffusion. There is a wide literature on the subject, both for its relevance to analysis, PDEs, potential theory, stochastic processes, and for the growing number of applications in mechanics and other applied fields. For basic information see [1, 23, 49, 60, 61, 65].

We consider the Cauchy problem for (1.1) taking as initial data a Dirac delta,

$$u(x,0) = M\delta(x), \qquad M > 0. \tag{1.2}$$

Solutions with such data are called fundamental solutions in linear theory, and we will keep that name though their relevance is different in the nonlinear context. We use the concept of continuous and nonnegative weak solution introduced in [51], [52], for which there is a well-developed theory when the datum is a function in $L^1(\mathbb{R}^N)$.

Putting $s = 1$ we recover the standard Porous Medium Equation (PME), where the question under discussion is well-known (see the comments in the next sections). The exponent m varies in principle in the range $m > 1$, but the methods extend to the linear case $m = 1$ and even to the fast diffusion range (FDE) $m < 1$ on the condition that $m > m_c = \max\{(N-2s)/N, 0\}$. Such type of restriction on m from below carries over from PME-FDE theory [63].

The main result of this paper is that there exists a unique fundamental solution of problem (1.1)–(1.2):

Theorem 1.1. *For every choice of parameters $s \in (0,1)$ and $m > m_c$ where $m_c = \max\{(N-2s)/N, 0\}$, and for every $M > 0$, equation (1.1) admits a unique fundamental solution; it is a nonnegative and continuous weak solution for $t > 0$ and takes the initial data (1.2) as a trace in the sense of Radon measures. The solution has the self-similar form*

$$u_M^*(x,t) = t^{-\alpha} f(|x| t^{-\beta}) \tag{1.3}$$

for suitable α and β that can be calculated in terms of N and s in a dimensional way, precisely

$$\alpha = \frac{N}{N(m-1)+2s}, \qquad \beta = \frac{1}{N(m-1)+2s}. \tag{1.4}$$

The profile function $f(r)$, $r \geq 0$, is a bounded and Hölder continuous function, it is positive everywhere, monotone, and goes to zero at infinity.

Let us point out that f is actually a smooth function by the results of [53], but this property is not discussed here. Let us also stress that the initial data are taken in the weak sense of measures,

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(x,t)\phi(x)\,dx = M\phi(0) \tag{1.5}$$

for all $\phi \in C_b(\mathbb{R}^N)$, the space of continuous and bounded functions in $\mathbb{R}^N$. We will call these self-similar solutions of problem (1.1)–(1.2) with given $M > 0$ the *Barenblatt solutions* of the fractional diffusion model by analogy with the PME and other prominent studied cases. The form of the exponents explains the already mentioned restriction on m from below. The result is proved in Sections 3 to 9.

We then prove that the asymptotic behaviour as $t \to \infty$ of the class of nonnegative weak solutions of (1.1) with finite mass (i.e., $\int u(x,t)\,dx < \infty$) is given in first approximation by the family $u_M^*(x,t)$, i.e., the Barenblatt solutions are the attractors in that class of solutions. See Section 10, Theorem 10.1. This result is a clear example of the important role that the fundamental solutions and their properties can play in applications. Because of this, we devote Section 12 to studying the behaviour of the profile $f(r)$ for large r. This asymptotic behaviour is an important tool in works like [19, 59].

In the limit $M \to \infty$ we obtain a special solution with a fixed isolated singularity at $x = 0$ that has separate-variables form; we call it a very singular solution (VSS) by analogy with the standard Fast Diffusion Equation. Such solutions exist precisely in the range $m_c < m < m_1 = N/(N+2s)$ (see Theorems 13.1, 13.2). The result represents a marked difference with the case $s = 1$ where VSS exist in the larger range $m_c < m < 1$. It is another manifestation of the long-range interactions of the fractional Laplacian, which fails some of the purely local estimates of the standard FDE with the classical Laplacian operator; indeed, extrapolation of the standard estimates would justify the existence of a VSS in cases where there is none. Let us recall that the class of VSS has played a role in the study of nonlinear parabolic equations since the seminal paper of Brezis, Peletier and Terman [21].

We then devote one section to constructing eternal solutions in the critical exponent case $m = (N-2s)/N$. By *eternal* we mean that they are defined for all times $-\infty < t < \infty$. The solutions we construct have exponential type of self-similarity.

The study of evolution equations with fractional Laplacian operators implies the need to develop in this setting tools that have been successful in the standard and parabolic theory. One of them is the Aleksandrov reflection principle, which we state and prove in the elliptic and parabolic settings in Section 15.

Two appendices and a section of comments and extensions close the paper.

2. Motivation and historical perspective

The question of finding fundamental solutions of elliptic and parabolic equations is one of paramount importance in the study of linear elliptic and parabolic equations (see e.g. [33, 38]). In nonlinear theory their role is not so apparent, but it has been shown that they can be an important tool in the existence and regularity theory, and very important in the description of asymptotic behaviour.

Generalizing the classical heat equation, a prominent case at hand in evolution theories is the Porous Medium Equation (PME), $\partial_t u - \Delta(u^m) = 0$, $m > 1$, introduced in the last century in connection with a number of physical applications and extensively studied as a prototype of nonlinear diffusive evolution with interesting analysis and geometry (free boundaries). Fundamental solutions were discovered in the 1950's by Zel'dovich and Kompanyeets [68] and Barenblatt [6], and later by Pattle [54]; Barenblatt gave their complete description in his studies. They have usually been called Barenblatt solutions in the literature. This discovery was so to say the starting point of the rigorous mathematical theory that has been gradually developed since then. Their role in the asymptotic behaviour of general solutions of the PME was established in papers by Kamin [41, 42], Friedman and Kamin [34] and the author [62]. They have also played an important role in the existence and regularity theory developed by many authors. The monograph [64] contains a detailed exposition of the mathematical theory of the PME.

The surprising relation between existence and uniqueness of fundamental solutions and precise asymptotic behaviour relies on the existence of a scaling group under which the solutions of the PME are invariant. It implies that a fundamental solution is in fact a self-similar function. This is what we call a Barenblatt solution. We will see below how the scaling group works in the case of equation (1.1). Self-similarity plays a major role in our understanding of fundamental processes in mathematics and mechanics, as explained by Prof. Barenblatt in his books [9, 10].

Following the analysis of Barenblatt solutions for the PME, other equations have been explored. To name but a few, let us mention first the Fast Diffusion Equation, which is $\partial_t u - \Delta(u^m) = 0$ with $m < 1$. Though it looks formally the same, its qualitative theory differs in a marked way. The role of the Barenblatt solutions remains basically unchanged only for $m > (N-2)/N$, while for $m < (N-2)/N$ the whole functional setting changes abruptly (cf. [63] and [16, 18]). Barenblatt solutions have been constructed for the p-Laplacian equation, $\partial_t u - \Delta_p(u) = 0$ [7]; in this case they exist in the range $2N/(N+1) < p < \infty$ and their role and properties are quite similar (loosely speaking) to the porous medium case. These solutions have also been used extensively in the regularity theory [31] and in the asymptotic behaviour [46]. The ideas can be extended to the doubly-nonlinear diffusion equation $\partial_t u - \Delta_p(u^m) = 0$, as proposed in [7] (see [2] for recent results), and to other models, not necessarily of diffusive type. Among such models we mention conservation laws [50].

In some cases fundamental solutions have similarity exponents that cannot be calculated from dimensional considerations, giving rise to so-called anomalous exponents; this is also referred to as self-similarity of the second kind. Then the initial distribution is not a Dirac delta but a more general mass distribution, in other words some singularity located at $x = 0$, $t = 0$. We say that it is a *solution with an isolated singularity*, or a *source-type solution*, but not a fundamental solution. There is a wide literature on solutions with isolated singularities, both in elliptic and parabolic problems; a particular attention has been given to the so-called very singular solutions [12, 21, 43, 44, 67]. Again, the influence of Prof. Barenblatt has been felt in the form of interesting models from the physical sciences. One of them is the Barenblatt equation of elasto-plastic filtration, proposed in [9]; its source-type solutions have been studied in [11, 45] (see also [39]). They correspond to self-similarity of the second kind, with anomalous exponents. Another model is the equation of turbulent bursts [8] treated mathematically in [47], [36] and [40], where self-similarity with anomalous exponents is also found as asymptotic behaviour of a wide class of solutions. Anomalous exponents are discussed in [4] in connection with focussing problems, and they are found in the Fast Diffusion Equation with the critical exponent [48, 55, 63], but in these cases they do not correspond to source-type solutions.

Nonlinear diffusion with nonlocal operators. There are many other instances of the paradigm we are discussing. Recently there has been a remarkable interest in mathematical models of diffusion involving long-range effects represented by fractional Laplacian operators and other singular integral operators. This responds to a serious motivation from the physical sciences and has produced so far quite interesting mathematical developments. Let us add some comments to the information on fractional Laplacian operators that opened the paper. For functional analysis considerations, it is usually best to consider the inverse operator $(-\Delta)^{-s}$, which happens to be the integral operator associated to the Riesz kernels, as we will often do in this paper. Also, of special interest is the case $N = 1$ where the operator $(-\Delta)^s$ is the natural realization of the concept of derivative of a fractional order, $(\partial_x)^{2s}$ (up to a constant factor), in the form of a positive symmetric operator.

Regarding nonlinear evolution models, the author has been involved in the analysis of two of such models, called fractional porous medium equations. One of them is equation (1.1) that we study here, while the other model uses the equation

$$\partial_t u - \nabla \cdot (u\nabla(-\Delta)^{-s}u) = 0, \quad 0 < s < 1. \tag{2.1}$$

These two models have quite different properties (see a survey of recent results in [65]). The construction of Barenblatt solutions for the latter model has been performed by Caffarelli and the author [24]. The asymptotic behaviour is obtained in [24] using sophisticated entropy and obstacle problem methods. An explicit form for the Barenblatt solutions is found in [15]:

$$U(x,t) = c_1 t^{-\alpha}(1 - c_2|x|^2 t^{-2\alpha/N})_+^{1-s} \quad \text{with} \quad \alpha = N/(N+2-2s). \tag{2.2}$$

In the limit $s \to 1$ we obtain quite interesting fundamental solutions of the nondiffusive limit equation $u_t - \nabla \cdot (u\nabla(-\Delta)^{-1}u) = 0$, of interest in superconductivity and superfluidity. The solutions take the form

$$u_M^*(x,t) = \frac{1}{t}\chi_{B_R(t)}(x), \quad R(t) = ct^{1/N}, \quad c > 0, \tag{2.3}$$

as described in [14] and [57]. Each one represents a round vortex patch that expands to fill the space as $t \to \infty$.

The linear case. Solving (1.1) in the linear case $m = 1$ is easier. As explained in [52], when $m = 1$ and $0 < s < 1$ the evolution has the integral representation

$$u(x,t) = \int_{\mathbb{R}^N} K_s(x-z,t)u_0(z)\,dz, \tag{2.4}$$

where K_s has Fourier transform $\widehat{K}_s(\xi,t) = e^{-|\xi|^{2s}t}$. This means that, for $0 < s < 1$, the kernel K_s has the form $K_s(x,t) = t^{-N/2s}F(|x|t^{-1/2s})$ for some profile function F that is positive and decreasing, and behaves at infinity like $F(r) \sim r^{-(N+2s)}$ (see [17]). When $s = 1/2$, the kernel is explicit,

$$K_{1/2}(x,t) = C_N t(x^2+t^2)^{-(N+1)/2}. \tag{2.5}$$

If $s = 1$ the function $K_1(x,t)$ is the Gaussian heat kernel. The linear model has been well studied by probabilists.

However, an integral representation of the evolution like (2.4) is not available in the nonlinear case; tools to treat the nonlinear case were developed in [51], [52]. Below, we perform the analysis of existence, uniqueness, properties and applications of Barenblatt solutions for equation (1.1) as a further step in this study. We will concentrate on the difficulties caused by the nonlinear form of the equation. Our Barenblatt solutions will play the role of the kernel K_s in the study of asymptotic behaviour. Our results below include the linear case as a particular instance. Let us recall that nonlinear techniques are quite different from linear ones. In particular, perturbative arguments have not been successful.

3. Preliminaries

Before we start the construction we need to review some basic facts.

Solving the nonlinear problem. Existence, uniqueness and the main properties of the solutions of equation (1.1) have been considered in great detail in [51, 52]. Let us state the definition of weak and strong solution, taken from [52].

Definition. A nonnegative function $u \in C((0,\infty); L^1(\mathbb{R}^N))$ is a *weak solution* of the fractional diffusion equation (1.1) if $u^m \in L^2_{\text{loc}}(0,\infty; \dot{H}^{\sigma/2}(\mathbb{R}^N))$ and

$$\int_0^\infty \int_{\mathbb{R}^N} u\frac{\partial \varphi}{\partial t}\,dx\,dt - \int_0^\infty \int_{\mathbb{R}^N} (-\Delta)^{s/2}u^m(-\Delta)^{s/2}\varphi\,dx\,dt = 0. \tag{3.1}$$

The correct space for $u^m(\cdot,t)$ is the fractional Sobolev space $\dot{H}^s(\mathbb{R}^N)$, defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with the norm

$$\|\psi\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^N} |\xi|^{2s}|\hat{\psi}|^2\,d\xi\right)^{1/2} = \|(-\Delta)^{s/2}\psi\|_2,$$

where $\hat{\psi}$ denotes the Fourier transform. We say that a weak solution u to (1.1) is a *strong solution* if $\partial_t u \in L^\infty((\tau,\infty); L^1(\mathbb{R}^N))$ for every $\tau > 0$.

It is proved in [52, Theorem 2.1] that corresponding to any initial data $u_0 \in L^1(\mathbb{R}^N)$ there exists a unique strong solution that takes on the initial data in the sense that $u(\cdot,t) \to u_0$ in $L^1(\mathbb{R}^N)$ as $t \to 0$ if $m > m_c$. Among the other interesting properties we mention that (always for $m > m_c$) these solutions are bounded and Hölder continuous for $t > 0$ and positive everywhere. Actually, papers [51, 52] deal with the theory of solutions of any sign, and show that the equation generates an ordered semigroup of contractions in the space $L^1(\mathbb{R}^N)$, but we do not need that generality here.

Note that there is a very weak version of the concept of solution where the whole fractional Laplace operator acts on the test function,

$$\int_0^\infty \int_{\mathbb{R}^N} u \frac{\partial \varphi}{\partial t}\, dx\, dt - \int_0^\infty \int_{\mathbb{R}^N} u^m (-\Delta)^s \varphi\, dx\, dt = 0. \tag{3.2}$$

In that case we only need to assume that $u, u^m \in L^1(Q)$, $Q = \mathbb{R}^N \times (0,\infty)$. This formulation will also be useful below.

Scaling. Equation (1.1) is invariant under translations in space and time, and also under the scaling group, and this fact will play a role in what follows. First observe that for every solution $u(x,t)$ and constants $A, B, C > 0$ the function

$$\widehat{u}(x,t) = Au(Bx, Ct) \tag{3.3}$$

is again a solution of (1.1) if $C = A^{m-1}B^{2s}$. This generates the whole scaling group. If moreover we impose the condition that the solutions have constant finite mass (i.e., integral in space), then $A = B^N$, which implies that the group is reduced to the one-parameter family $\widehat{u}(x,t) = B^N u(Bx, B^{N(m-1)+2s}t)$, which is usually written in terms of $\lambda = B^{N(m-1)+2s}$ as

$$(T_\lambda u)(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t), \tag{3.4}$$

with scaling exponents given by formula (1.4):

$$\alpha = \frac{N}{N(m-1)+2s}, \qquad \beta = \frac{1}{N(m-1)+2s},$$

Note that $\alpha, \beta > 0$ iff $m > m_c = (N-2s)/N$. The values of both parameters will be fixed in what follows.

Another application of scaling consists in reducing solutions to unit mass. Indeed, if $u(x,t)$ is a solution with $\int u(x,0)\, dx = M > 0$ then

$$\widehat{u}(x,t) = M^{-1} u(x, M^{-(m-1)}t) \tag{3.5}$$

is another solution with unit initial mass, $\int \widehat{u}(x,0)\, dx = 1$.

Potential equation. Take the convolution $U(x,t) = u(x,t) * \mathcal{I}_{2s}(x)$, where $\mathcal{I}_{2s}$ is the Riesz kernel

$$\mathcal{I}_{2s}(x) = C_{N,s}|x|^{-(N-2s)}, \tag{3.6}$$

that is, the kernel of the operator $(-\Delta)^{-s}$, $0 < s < 1$.[1] Use of this kernel restricts the range of application to $0 < s < 1/2$ when $N = 1$. This is a typical difficulty of potential theory that is found in the standard heat and porous medium equations in dimensions $N = 1, 2$. We will discuss the case $N = 1$, $1/2 \le s < 1$, separately in Section 9.

After this caveat, we resume the theory. We apply the kernel to a solution $u(x, t)$ and define

$$U(x, t) = C_{N,s} \int \frac{u(y, t)}{|x - y|^{N-2s}}\, dy.$$

Then $(-\Delta)^s U = u$, and using the equation we get the following equation (PE):

$$U_t = ((-\Delta)^{-s} u)_t = -(-\Delta)^{-s}(-\Delta)^s u^m = -u^m. \tag{3.7}$$

This is easy to justify since $u^m(t) \in L^2$ with $u_t = (-\Delta)^s u^m \in L^1$ for every $t > 0$, according to the theory.

The scaling group acts also on the solutions of the PE: If $A, B > 0$ are positive and U is a solution of the PE, then

$$\widehat{U}(x, t) = AU(Bx, Ct) \quad \text{with} \quad C = A^{m-1}B^{2sm} \tag{3.8}$$

is again a solution of the same equation with suitably rescaled initial data.

The idea of using the potential equation to prove uniqueness of solutions of diffusive equations with measure data goes back to Pierre's work for the standard PME [56]. This has been followed later by a number of authors in various contexts.

4. Existence of solutions with measure data

We have mentioned the existence of a good basic theory for the initial-value problem for equation (1.2) when the initial data belong to $L^1(\mathbb{R}^N)$. When the initial data are allowed to be a bounded Radon measure, the existence and uniqueness of solutions has a higher level of difficulty. Here is the basic result that we prove about existence; $\mathcal{M}^+(\mathbb{R}^N)$ is the space of bounded and nonnegative Radon measures on $\mathbb{R}^N$.

Theorem 4.1. *For every $\mu \in \mathcal{M}^+(\mathbb{R}^N)$ there exists a nonnegative and continuous weak solution of equation (1.1) in $Q = \mathbb{R}^N \times (0, \infty)$ taking initial data μ in the sense that for every $\varphi \in C_c^2(\mathbb{R}^N)$ we have*

$$\lim_{t\to 0} \int u(x, t)\varphi(x)\, dx = \int \varphi(x)\, d\mu(x). \tag{4.1}$$

Proof. (i) Existence comes from approximation of the initial data μ with a smooth mollifier sequence $\rho_\varepsilon(x)$, $\varepsilon > 0$, thus we take smooth initial data $\mu_\varepsilon = \mu * \rho_\varepsilon$. Using the results of [52] we know that the corresponding solutions u_ε exist and are bounded in $L^\infty(0, \infty; L^1(\mathbb{R}^N))$ and $L^\infty(\mathbb{R}^N \times (\tau, \infty))$, uniformly in $\varepsilon > 0$. The decay rate with time, $\|u(t)\|_\infty \le Ct^{-\alpha}$, is a main tool, and it holds uniformly for the whole sequence.

[1] For the value of $C_{N,s}$ see formula (A.2) below.

The solutions also have uniform bounds on the energy $\int_t^\infty \int |(-\Delta)^{s/2}u^m|^2\,dx\,dt$ [52, formula (8.5)].

(ii) We pass to the limit $\varepsilon \to 0$. Convergence holds in principle up to taking subsequences $\varepsilon_n \to 0$. The solutions u_ε and the limit $\overline{u}$ are uniformly C^α in space and time for all $t \geq \tau > 0$. The limit is easily proved to be a weak solution of the equation.

(iii) The initial data are taken in the sense of initial traces, since the weak formulation passes to the limit for $t \geq \tau > 0$ and the second integral term is uniformly small for small $t \in (0, \tau)$: if $L = (-\Delta)^s$ then

$$\left| \int_0^t \int_{\mathbb{R}^N} u^m L\varphi\,dx\,dt \right| \leq \int_0^t \|u(s)\|_\infty^{m-1} \left(\int_{\mathbb{R}^N} u|L\varphi|\,dx \right) ds.$$

Now we use the fact that $|L\varphi| \leq C$, $\int u\,dx \leq C$ and the decay $\|u(t)\|_\infty \leq Ct^{-\alpha}$ with $\alpha = N/(N(m-1)+2s)$. Therefore,

$$\alpha(m-1) = \frac{N(m-1)}{N(m-1)+2s} < 1 \tag{4.2}$$

and the integral in time converges at $t = 0$. This calculation is valid for the u_ε and also for the limit $\overline{u}$. See a similar argument in [57, Section 3]. So we have initial traces in the sense of distributions, and even better, since the test functions can be taken in $W^{2s,\infty}(\mathbb{R}^N)$ for instance.

(iv) The previous argument as $t \to 0$ works for $m \geq 1$. For $m_c < m < 1$, we use good test functions and Hölder estimates:

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}^N} Lu^m \varphi\,dx\,dt \right| &\leq \left(\int_0^t \int_{\mathbb{R}^N} u\,dx\,dt \right)^m \left(\int_0^t \int_{\mathbb{R}^N} (L\varphi)^{1/(1-m)}\,dx\,dt \right)^{1-m} \\ &\leq Ct^{1-m}. \end{aligned} \tag{4.3}$$

□

Remark. The question of uniqueness of the class of solutions with a Radon measure as initial data is nontrivial and is not going to be answered here, except for the case where μ is a Dirac delta, which is the purpose of this paper. For the standard porous medium equation it is dealt with in works by Pierre, and Dahlberg and Kenig [56, 29, 64].

5. Approximate data for the fundamental solution

1. In our study of the solutions with Dirac deltas as initial data, we will use some monotone approximations. The simplest approach is to take a smooth convolution kernel ρ in $C_c^\infty(\mathbb{R}^N)$, $\rho \geq 0$, $\int \rho\,dx = 1$. We also take ρ to be radial, i.e., a function of $|x|$, and supported in the ball of radius 1. Put $\rho_\varepsilon = \varepsilon^{-N}\rho(x/\varepsilon)$, and define the regularization $V_\varepsilon = \rho_\varepsilon * V$ of the Riesz kernel $V(x) = \mathcal{I}_{2s}(x)$ as

$$V_\varepsilon(x) = C_1 \int \frac{\rho_\varepsilon(y)}{|x-y|^{N-2s}}\,dy, \tag{5.1}$$

where C_1 is the constant $C_{N,s}$ of (3.6). Note the scaling

$$V_{\varepsilon\mu}(x) = V_\varepsilon(x/\mu)/\mu^{N-2s}$$

that allows us to reduce the calculations to the case $\varepsilon = 1$. Clearly $V_\varepsilon \to V$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. We want to examine further the approximation of V by V_ε.

• We claim that

$$V_\varepsilon(x) \geq V(x) \quad \forall |x| \geq \varepsilon.$$

Proof. It is enough to prove this for $\varepsilon = 1$. Using spherical coordinates we write $y = r\sigma$, $|\sigma| = 1$, $\int_{|x|\leq 1} dx = \omega_N$, and then

$$V_1(x) = C_1 \int_0^R \rho(r)\, dr \int_{|y|=r} \frac{1}{|x-y|^{N-2s}} r^{N-1}\, d\sigma = N\omega_N C_1 \int_0^R \rho(r) r^{N-1} \Phi(r)\, dr$$

where $\Phi(r) := \fint_{|y|=r} |x-y|^{-N+2s}\, d\sigma$. So it remains to study the properties of the last integral. This is where we use the fact that for $z \neq 0$,

$$\Delta V(z) = (N-2s)(2-2s)V(z)/|z|^2 \geq 0,$$

so the integrand is subharmonic in $B_r(x)$ if $r < |x|$. We conclude that

$$\Phi(r) = \fint_{|y|=r} \frac{1}{|x-y|^{N-2s}}\, d\sigma \geq \Phi(0) = \frac{1}{|x|^{N-2s}},$$

and the result is true. In fact, Evans [33, p. 26] proves that

$$\Phi'(r) = \frac{r}{N} \fint_{B_r(0)} \Delta V(x-y)\, dy = \frac{K}{r^{N-1}} \int_{B_r(0)} \Delta V(x-y)\, dy,$$

hence $\Phi'(r) > 0$ for all $r < |x|$. Since $\rho(r) = 0$ for $r \geq 1$ we are done.

• We now estimate the error for large $|x| \geq 2$. In that situation we have

$$\Phi'(r) \leq c_2 r |\Delta V(x)|, \qquad \Phi(r) - \Phi(0) \leq c_3 r^2 V(x) |x|^{-2},$$

so that

$$V_1(x) - V(x) \leq c_4 \left(\int_0^1 \rho(r) r^{N+1}\, dr \right) V(x)|x|^{-2} = c_5 V(x)|x|^{-2}.$$

Using now the rescaling rule, we get interesting results for V_ε.

Proposition 5.1. *There is a constant $K > 0$ depending on N and s such that*

$$V(x) \leq V_\varepsilon(x) \leq V(x)\left(1 + K\frac{\varepsilon^2}{|x|^2}\right) \tag{5.2}$$

for all $\varepsilon > 0$ and all $|x| \geq 2\varepsilon$. There is another constant C such that

$$V_\varepsilon(x) \leq C V(x) \tag{5.3}$$

for all $\varepsilon > 0$ and all x. Finally, $V_\varepsilon - V \to 0$ in $L^1(\mathbb{R}^N)$.

For the last part, note that for x away from zero, $|V_\varepsilon(x) - V(x)| \le C\varepsilon^2|x|^{-N-2+2s}$, which is integrable at infinity, hence the L^1 norm in that region is small if $\varepsilon \to 0$. Near zero $|V_\varepsilon(x) - V(x)| \le CV(x)$, which is also integrable, hence uniformly small in a small ball.

2. The following construction supplies an alternative approach that may also be useful. We consider $\hat{V}_n(x) = \inf\{V(x), n\}$ for $n \ge 1$. Note that this is just a rescaling of $\hat{V}_1(x)$ of the form $\hat{V}_n(x) = n\hat{V}_1(n^{1/(N-2s)}x)$. Clearly, $\hat{V}_n(x)$ increases to $V(x)$ as $n \to \infty$, $\hat{V}_n - V \to 0$ in $L^1(\mathbb{R}^N)$.

Proposition 5.2. *Set $\phi_n(x) := (-\Delta)^s \hat{V}_n(x)$. Then $\int \phi_n(x)\,dx = \int (-\Delta)^s V\,dx = 1$ and $\phi_n(x) > 0$ everywhere in $\mathbb{R}^N$. The sequence ϕ_n is a positive approximation sequence to the Dirac delta.*

Proof. (i) We concentrate on ϕ_1. On the one hand, $\hat{V}_1$ has a flat plateau in a ball $|x| \le R$ where it attains the maximum value 1, and by a simple calculation with the explicit formula for the operator we get $(-\Delta)^s \hat{V}_1(x) > 0$, and ϕ_1 is bounded and continuous for $|x| < R$. In order to calculate the value of $(-\Delta)^s \hat{V}_1(x)$ at the points $|x| > R$ where $\hat{V}_1(x) = V(x)$ we define

$$W(x) = V(x) - \hat{V}_1(x) = (V(x) - 1)_+.$$

It is also easy to see that $(-\Delta)^s W < 0$ at the points $|x| \ge R$ since W attains there its minimum, $W(x) = 0$. The integral defining $(-\Delta)^s W$ is bounded for $|x| > R$ and there is an asymptotic estimate $|(-\Delta)^s W| \sim C|x|^{-(N+2s)}$ as $|x| \to \infty$. Finally, by using integration with respect to a suitable cutoff function we prove that $(-\Delta)^s W$ is integrable and $\int (-\Delta)^s W(x)\,dx = 0$.

Hence, $\phi_1(x) = (-\Delta)^s \hat{V}_1(x) > 0$ also at those points. We also get the estimate

$$\phi_1(x) \sim C(|x|^{-(n+2s)}) \quad \text{as } |x| \to \infty.$$

Moreover, $\phi_1(|x|)$ is continuous, bounded and smooth but for a possible asymptote at $|x| = R$. Also, $\int \phi_1(x)\,dx = \int (-\Delta)^s V\,dx = 1$.

(ii) Consider now $\phi_n(x) := (-\Delta)^s \hat{V}_n(x)$. It follows that ϕ_n is positive everywhere in $\mathbb{R}^N$, and $\phi_n(x) = n^{N/(N-2s)}\phi_1(xn^{1/(N-2s)})$. This means that $\int \phi_n(x)\,dx = \int \phi_1(x)\,dx$ and thus ϕ_n is a suitable approximation of the Dirac delta. □

6. Special construction of a fundamental solution

In the existence part we use approximation of the special initial data $M\delta(x)$ with a smooth mollifier sequence of the form $M\rho_\varepsilon(x) = M\varepsilon^{-N}\rho(x/\varepsilon)$. Without loss of regularity we fix $M = 1$ here (using mass scaling, formula (3.5)). It is then clear that the scaling group (formula (3.4)) transforms the solution u_ε with data ρ_ε into the solution $T_\lambda u_\varepsilon$ with data $\rho_{\lambda\varepsilon}$. By uniqueness of solutions we conclude that

$$T_\lambda u_\varepsilon = u_{\lambda\varepsilon}. \tag{6.1}$$

This will be used below.

The potential equation, continuity. The corresponding solutions U_ε of the potential equation (with initial data V_ε) are regular in space, since $(-\Delta)^s U_\varepsilon = u_\varepsilon \in C^\alpha_x$. Since

$$\partial_t U_\varepsilon = -u_\varepsilon^m, \tag{6.2}$$

they are also $C^{1,\alpha}$ in time uniformly for $t \ge \tau > 0$. Moveover, they are monotone in time for all $t > 0$.

Before examining the limit, we will establish another important property, the continuity of the evolution orbit $U(\cdot, t)$ in $L^1(\mathbb{R}^N)$. In fact, for every $0 < t' < t$ we have

$$\int |U(t') - U(t)|\,dx \le \int_{t'}^{t} \int |U_t|\,dx\,dt = \int_{t'}^{t} \int |u^m|\,dx\,dt < \infty.$$

The worst case happens for $t' = 0$ and we have proved above that the last integral is $O(t^{2s\alpha})$, which goes to zero as $t \to 0$. This calculation is valid for the approximate solutions U_ε, which are classical, and also for their limits.

We have assumed $m \ge 1$. For $m < 1$ using an argument like (4.3) we have

$$\int (U(x,0) - U(x,t))\phi\,dx \le Ct. \tag{6.3}$$

So in this case we have a uniform continuity control in $L^1_{\rm loc}(\mathbb{R}^N)$. The rest is similar.

Convergence for the potential equation. In the previous section we have proved a lemma that says that $U_\varepsilon(x,0) \le CV(x)$, therefore $CV(x)$ is a uniform upper bound for the initial data, and in view of (6.2) also for the solutions.

In view of internal regularity, we can pass to the limit along a subsequence $\varepsilon_n \to 0$ to obtain a limit function $U^*(x,t)$ that is a solution of the potential equation, with $0 \le U^* \le CV$.

As for the initial trace, in view of the convergence of $V_\varepsilon(x) = U_\varepsilon(x,0)$ to V in $L^1(\mathbb{R}^N)$ (Proposition 5.1), and the uniform continuity of the orbits $U_\varepsilon(t)$ in $L^1(\mathbb{R}^N)$, the limit also has continuous time-increments in L^1, and takes the initial data V in the sense that

$$\|U^*(t) - V\|_1 \to 0 \quad \text{as } t \to 0. \tag{6.4}$$

Since $\partial_t U^* \le 0$, $U^*(x,t)$ converges monotonically to $V(x)$ as $t \to 0$ for fixed x.

It is clear that for positive times, $(-\Delta)^s U^*(x,t)$ must coincide with $\overline{u}$, the limit of $u_\varepsilon(x,t) = (-\Delta)^s U_\varepsilon(x,t)$ along ε_n (in other words, the operator is closed). Since $\overline{u}(t) \to \delta$ in $\mathcal{D}'$, this also implies that $U^*(t) \to V$ in $\mathcal{D}'$, i.e., $U^*(x,0) = V(x)$.

The fact that the limit $U^*(x,t)$ takes the initial data with uniform convergence away from zero is a consequence of the monotone convergence theorem for continuous functions. Around zero the convergence holds in the Marcinkiewicz space $M^p(\mathbb{R}^N)$, $p = N/(N-2s)$, hence in $L^q(\mathbb{R}^N)$, $1 < q < p$.

Moreover, the convergence as $\varepsilon \to 0$ is uniform for $t \ge \tau > 0$.

We have concluded that U^* is a solution of the PE corresponding to a fundamental solution of the original equation (1.1), and it takes the initial data $V(x)$ in a strong sense. We now have to solve the question of uniqueness of this limit U^*.

7. A uniqueness result for fundamental solutions

Theorem 7.1. *The fundamental solution $U(x,t)$ of the potential equation with initial data $V(x)$ is unique under the following assumptions: $U \geq 0$, U is continuous in $Q = \mathbb{R}^N \times (0,\infty)$, it has only an initial singularity at $(0,0)$ in the sense that U is bounded for $t \geq \tau > 0$, is continuous as $t \to 0$ uniformly away from $x = 0$, and $U(x,t) \to \infty$ as $(x,t) \to (0,0)$.*

Proof. Take two solutions U_1 and U_2 of the potential equation with the same initial data $V(x)$. We will not compare U_1 and U_2 but their suitable modifications. Consider $U_3 = U_1 + \delta$, $\delta > 0$, and the approximate solution $\widehat{U}_2(x,t) = U_2(x, t+\tau)$ for a small $\tau > 0$. Then U_3 is strictly larger than the continuous and bounded function $\widehat{U}_2$ at $t = 0$. The first point of contact cannot happen near infinity because of the uniform decay of U_1 and U_2 that implies strict separation of U_3 and $\widehat{U}_2$.

At any contact point (x_0, t_0), $t_0 > 0$, and since $U_3 - \widehat{U}_2 \geq 0$ up to that time, using the representation formula for $(-\Delta)^s$ we conclude that

$$(-\Delta)^s (U_3 - \widehat{U}_2) < 0, \quad \text{i.e.} \quad u_1 < \widehat{u}_2.$$

On the other hand, at this point we necessarily have

$$0 \geq \partial_t (U_3 - \widehat{U}_2) = -u_1^m + \widehat{u}_2^m, \qquad u_1^m \geq \widehat{u}_2^m.$$

This allows us to arrive at a contradiction (note that this is a typical viscosity solution type argument). We conclude that $U_1(x,t) + \delta \geq U_2(x, t+\tau)$ for all $\delta, \tau > 0$. In the limit $\delta, \tau \to 0$ we get

$$U_1(x,t) \geq U_2(x,t) \quad \text{for all } (x,t) \in Q.$$

This concludes the proof, after reversing the roles of U_1 and U_2. □

We now know that we do not need to take subsequences in the limit as $t \to 0$ of Section 6. Note that we could have restricted the existence time and consider $Q_T = \mathbb{R}^N \times (0, T)$ instead of Q, and then the conclusion is valid for $0 \leq t \leq T$.

8. Self-similarity of the Barenblatt solution

We conclude here the construction of the Barenblatt solution and the proof of Theorem 1.1 for $N > 1$.

• Let us settle first the question of self-similarity. Indeed, the initial function of the solution of the PE with data $U_0(x) = C|x|^{-(N-2s)}$ is invariant under the scaling T defined by $(TU_0)(x) = AU_0(Bx)$ if we choose $A = B^{N-2s}$. If moreover we have uniqueness of such solution, then the scaling group introduced in (3.8) will imply that

$$TU(x,t) = B^{N-2s} U(Bx, \lambda t) \quad \text{with} \quad \lambda = A^{m-1} B^{2sm} = B^{N(m-1)+2s}$$

will be a solution with the same initial data. By the claimed uniqueness $TU \equiv U$, hence for all $B > 0$, $x \in \mathbb{R}^N$ and $t > 0$ we have

$$U(x,t) = \lambda^{(N-2s)\beta} U(\lambda^{\beta} x, \lambda t),$$

where we have used the value of β given in the Introduction. Fix now $t_1 > 0$ and let $\lambda = 1/t_1$. We get

$$U(x,t) = t_1^{(N-2s)\beta} U(x t_1^{-\beta}, 1).$$

Of course, t_1 is arbitrary and can be replaced by t. Calling now $U(x,1) \equiv H(x)$ we get

$$U(x,t) = t^{-(N-2s)\beta} H(x t^{-\beta}).$$

Applying the operator $(-\Delta)^s$ we get self-similarity for the corresponding fundamental solution of (1.1):

$$u^*(x,t) = t^{-(N-2s)\beta} (-\Delta)_x^s (H(x t^{-\beta})) = t^{-\alpha} F(x t^{-\beta}).$$

This is a rather classical argument in the study of self-similarity.

• *End of proof of the Theorem.* The proof that $u^*(x,t)$ takes the data in the sense of measures (and not only as distributions) follows easily from self-similarity. The fact that $F(r)$ is nonincreasing in r is a consequence of the Aleksandrov reflection principle proved in Section 15. We know by qualitative theory that F is always positive since u is. The same argument for Hölder continuity. □

Properties of the profile F

Equation. The self-similar profile F satisfies an elliptic equation

$$(-\Delta)^s F^m = \alpha F + \beta y \cdot \nabla F = \beta \nabla \cdot (yF), \tag{8.1}$$

so that putting $s' = 1 - s$ and integrating once we have

$$\nabla (-\Delta)^{-s'} F^m = -\beta y F,$$

which in radial coordinates gives

$$L_{s'} F^m(r) = \beta \int_r^\infty r F(r)\, dr, \tag{8.2}$$

where $L_{s'}$ the radial expression of the operator $(-\Delta)^{-s'}$.

Dependence on the mass. The scaling group acts on the profiles $F_M(r)$ for different masses $M > 0$ and indeed we have

$$F_M(r) = \mu^{2s} F_1(\mu^{1-m} r), \qquad M = \mu^{N(m-1)+2s}, \tag{8.3}$$

which reduces all calculations to the case $M = 1$. Since $N(m-1)+2s > 0$ for $M > m_c$, we get $F_M(0) \to \infty$ as $M \to \infty$. For $m \geq 1$ the same result happens for all $r > 0$, $\lim_{M\to\infty} F_M(r) = \infty$. However, this last limit may be finite for $m < 1$; see Section 13 to understand when and why.

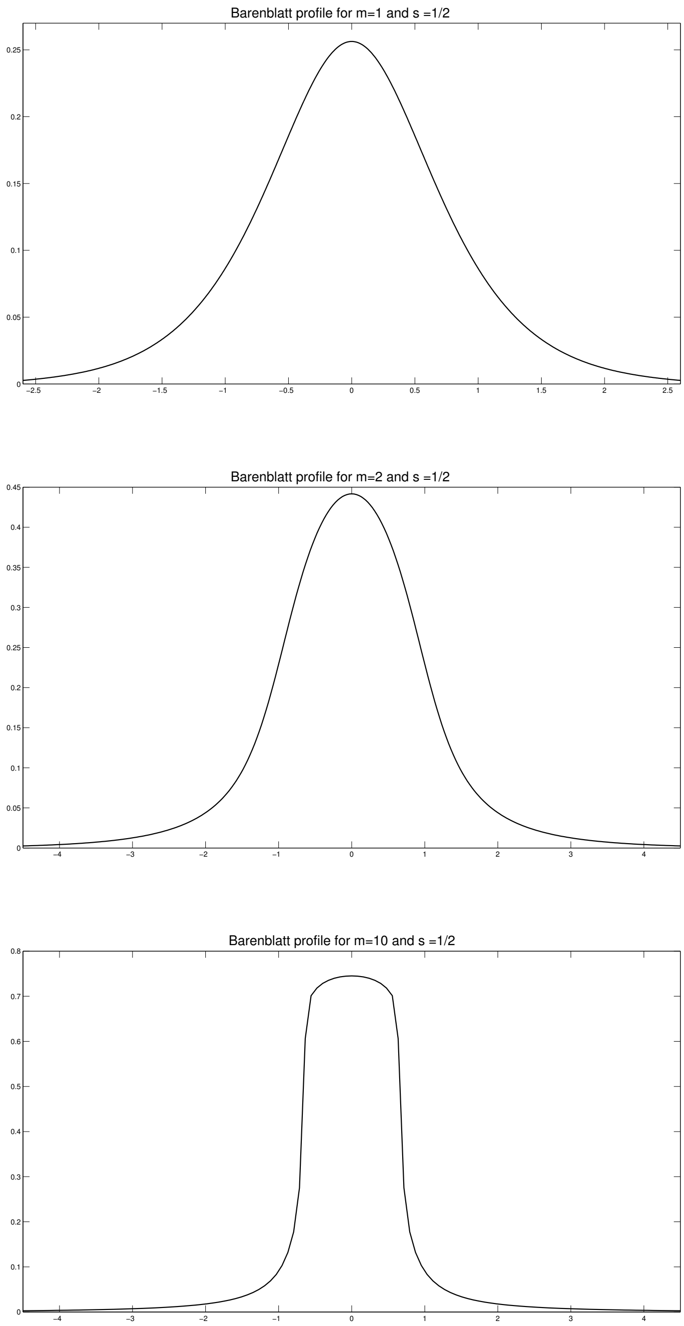

Fig. 1. Computed Barenblatt profiles for $m = 1, 2, 10$ with $s = 1/2$.

Decay at infinity. First estimate. The precise behaviour of the fundamental profiles $F(y) = F_{m,N,s}(y)$ as $y \to \infty$ is an important question in qualitative theory. It is known in the linear case of $m = 1$, since F is given by a linear kernel K that decays like $|y|^{-(N+2s)}$ (see [17]). The exact rate of decay for $m \neq 1$ is a nontrivial issue that we will discuss below. To begin with, the fact that F is monotone as a function of r and also integrable in $\mathbb{R}^N$ implies that there is a constant $C = C(F) > 0$ such that

$$F(r) \le Cr^{-N}. \tag{8.4}$$

We will use the same letter C for different positive constants as long as their value is not important in the context. We will continue with this issue in Section 12.

9. Peculiarities of one-dimensional flow

• We add here the comments that are needed to close the case $N = 1$, in the range $1/2 \le s < 1$. In that case the kernel is a function that grows at infinity and the potential approach cannot be used in the direct way as before. There are at least three natural ways of addressing this difficulty.

One of them is to consider first the problem in a bounded domain Ω with nice boundary, say a ball B_R. Then the kernel is replaced by the Green function (with zero outside conditions) and this avoids considering the divergence of the kernel as $|x| \to \infty$. Once the fundamental solution for this problem is constructed, one passes to the limit as $R \to \infty$ and then proves uniqueness.

A second approach is to replace the fractional operator $(-\Delta)^s$ by a coercive operator like $L_\varepsilon = (-\Delta)^s + \varepsilon I$, whose inverse has a nice kernel (the Bessel kernel $G_{2s}(x)$). If U solves the potential equation as before and $L_\varepsilon U = u$, then u solves $u_t = L_\varepsilon U_t = -L_\varepsilon(u^m)$, i.e.,

$$u_t + (-\Delta)^s u^m + \varepsilon u^m = 0. \tag{9.1}$$

The plan is to construct a unique fundamental solution for this equation, and then pass to the limit and prove uniqueness.

The third option is to integrate in x, $v(x,t) = \int_{-\infty}^x u(y,t)\,dy$. Then v solves

$$v_t = L((v_x)^m), \qquad L = \partial_x(-\Delta)^{s-1}. \tag{9.2}$$

This is a kind of fractional p-Laplacian operator.

• Let us develop the second method. We take $\varepsilon > 0$ and solve the equation $u_t + (-\Delta)^s u^m + \varepsilon u^m = 0$ for $x \in \mathbb{R}$ and $t > 0$ with initial data $u_0 \in L^1(\mathbb{R})$ by just copying the method used in [52] when $\varepsilon = 0$ (since the new term is dissipative, the needed estimates still hold). We obtain a solution with similar properties (except for mass conservation). Again, for fixed $\varepsilon > 0$ the maps $S_\varepsilon(t) : u_0 \mapsto u_\varepsilon(t)$ are ordered contractions in $L^1(\mathbb{R})$. If u_0 is nonnegative so is the solution u_ε for all times, and the family $\{u_\varepsilon : \varepsilon > 0\}$ of solutions is increasing as ε. In this way the standard solution u of the problem $u_t + (-\Delta)^s(u^m) = 0$, $u(x,0) = u_0$ is obtained in the limit $\varepsilon \to 0$.

We now introduce the potential functions U_ε using the Bessel potential instead of the Riesz potential, so that $U_\varepsilon(\cdot,t) = ((-\Delta)^s + \varepsilon I)^{-1}u_\varepsilon(\cdot,t)$. The U_ε will satisfy, all of them, the potential equation $U_t = -u^m$. We prove the existence and uniqueness of a solution with data $G_{2s}(x) = ((-\Delta)^s + \varepsilon I)^{-1}M\delta(x)$ much as above. This implies that the fundamental solution $u_{M,\varepsilon}(x,t)$ of the equation $u_t + (-\Delta)^s u^m + \varepsilon u^m = 0$ exists and is unique. We leave these details to the reader.

It follows easily that $u_{M,\varepsilon}(x,t)$ is increasing as $\varepsilon \to 0$ and the limit is a fundamental solution of (1.1) for $N = 1$. We then prove that the limit is a minimal element among such possible fundamental solutions, and then conservation of mass implies that the fundamental solution must be unique. The result of Section 8 then follows by easy changes in the argument.

Note. We have collected some properties of the Bessel potentials in an appendix at the end of the paper for the reader's convenience. Note that these potentials have exponential decay as $|x| \to \infty$. Also (for $N = 1$) they are singular unbounded at $x = 0$ for $2s = \alpha < 1$, but they are bounded for $2s \geq 1$.

10. Asymptotic behaviour of general solutions

We use the methods of the monograph [64] to get the following general theorem. We recall that all the solutions considered here are nonnegative.

Theorem 10.1. *Let $u_0 = \mu \in \mathcal{M}_+(\mathbb{R}^N)$, let $M = \mu(\mathbb{R}^N)$ and let u_M^* be the self-similar Barenblatt solution with mass M. Then as $t \to \infty$ the solutions $u(x,t)$ and $u_M^*(x,t)$ are increasingly similar, more precisely*

$$\lim_{t\to\infty} \|u(\cdot,t) - u_M^*(\cdot,t)\|_1 = 0, \tag{10.1}$$

and also

$$\lim_{t\to\infty} t^\alpha |u(x,t) - u_M^*(x,t)| = 0, \qquad \alpha = N/(N(m-1)+2s), \tag{10.2}$$

uniformly in $x \in \mathbb{R}^N$. It follows that for every $p \in (1,\infty)$ we have

$$\lim_{t\to\infty} t^{(p-1)\alpha/p} \|u(\cdot,t) - u_M^*(\cdot,t)\|_p = 0. \tag{10.3}$$

Proof. **I.** Firstly, we give the proof under the assumption that $d\mu = u_0(x)\,dx$ where u_0 is a bounded function with compact support. We divide this proof into several steps.

(i) We may assume that there are $K > 0$ large enough and $\tau > 0$ such that $u_0(x) \leq u_K^*(x,\tau)$. The Maximum Principle implies then that

$$u(x,t) \leq u_K^*(x,t+\tau) \quad \text{for all } x \in \mathbb{R}^N,\ t > 0.$$

We now perform the mass-preserving scaling transformation

$$(T_\lambda u)(x,t) = \lambda^\alpha u(x\lambda^\beta, \lambda t).$$

It is easy to see that we have a similar estimate for the family $u_\lambda = T_\lambda u$:

$$(T_\lambda u)(x,t) \le u_K^*(x, t+\tau/\lambda) \quad \text{for all } x \in \mathbb{R}^N,\ t > 0.$$

The family $(T_\lambda u)$ is bounded in $L^1 \cap L^\infty \cap C^\alpha$ in any set of the form $\Omega = B_R(0) \times (1/2, 2)$, hence it converges (along a sequence $\lambda_n \to \infty$) to a solution $\widetilde{u}$ of the equation for $t > 0$. The limit satisfies the bound $\widetilde{u}(x,t) \le u_K^*(x,t)$ for all $x \in \mathbb{R}^N$, $t > 0$.

(ii) We claim that $\widetilde{u}$ takes the initial data $M\delta$ in the sense of measures. Indeed, away from zero the convergence of the limit $\widetilde{u}(x,t)$ to zero as $t \to 0$ is uniform, because it is bounded above by the tail of the self-similar Barenblatt solution $u_K^*(x,t)$. This leaves as possible initial trace a Dirac delta. It only remains to recall that the total mass of the whole family $(T_\lambda u)$ is the same, M, at all times.

(iii) At this point we pass to the corresponding solutions $U_\lambda(x,t) = (-\Delta_x)^{-s}(T_\lambda u(x,t))$. They are solutions of the Potential Equation. The limit as $\lambda_n \to \infty$, $U(x,t)$, is also a solution of the PE with the already mentioned regularity and monotonicity properties. The initial datum of $U(x,t)$ is M times $V(x)$ in a weak sense, since

$$\int (MV(x) - U(x,t))\phi(x)\,dx = M(-\Delta)^{-s}\phi(0) - \int u(x,t)(-\Delta)^{-s}\phi(x)\,dx,$$

which goes to zero since $\zeta(x) = (-\Delta)^{-s}\phi(x)$ is an acceptable test function for the convergence in the sense of measures of $u(\cdot,t)$ to $M\delta$.

On the other hand, the family $U(t)$ converges in $L^1_{\rm loc}(\mathbb{R}^N)$ and monotonically to the initial trace that is $MV(x)$, as we have shown. The properties of the uniqueness theorem 7.1 are met, so that we conclude that U is the fundamental solution $U_M^*(x,t)$. By a standard argument, this uniqueness implies that the whole family U_λ converges.

(iv) Applying $(-\Delta)^s$ to $U = U^*$ we obtain $\widetilde{u} = u_M^*$. We thus have

$$(T_\lambda u)(x,1) \to u_M^*(x,1) \quad \text{uniformly in } x \in \mathbb{R}^N.$$

It is a routine calculation to transform this expression into the convergences (10.1) and (10.2) (see for instance [64, Chapter 18]). The convergence in L^p-norm follows from simple interpolation.

II. For a general initial function $u_0 \in L^1(\mathbb{R}^N)$ we have to work a bit more. First we fix a $\delta > 0$ and truncate u_0 above and near infinity to fall into the previous case, namely

$$0 \le u_{0,\delta}(x) \le u_0(x), \qquad \|u_{0,\delta}\|_1 = M_\delta \ge M - \delta.$$

Let us examine the convergence of the rescaled versions $u_\delta^\lambda(x,1)$ to $u_M^*(x,1;M_\delta)$ at $t = 1$. Since the sequence $\{(T_\lambda u_\delta)(x,1) : \delta > 0\}$ is uniformly bounded and monotone in δ, we get the convergence of the functions $\{(T_\lambda u_\delta)(x,1) : \delta > 0\}$ to $u_M^*(x,1;M)$ in L^1 and since they are bounded, they are uniformly C^α, hence convergence is uniform on any fixed ball.

In order to examine the convergence of $(T_\lambda u_\delta)(x,1)$ outside a big ball we first observe that all the functions are bounded in $L^1 \cap L^\infty$. By regularity theory (cf. [5, 52]), they are C^α with a uniform constant and exponent, which makes them a compact set of functions, hence we can extract a uniformly convergent subsequence. On the other hand, the L^1 norm of $(T_\lambda u)(x,1)$ is small in the outer domain as a consequence of the following clever trick: fixing $\delta > 0$ small, the above L^1 norm is bounded by δ plus the norm of $(T_\lambda u_\delta)(x,1)$, and the latter is controlled in that domain by the tail of $u^*_K(x, 1+\tau/\lambda)$, with $K = K(\delta)$ large. By interpolation between the L^1 and C^α norms we get a small norm in L^∞, uniformly in δ for ε small.

Once we have the L^1 and L^∞ convergence of $u^\lambda_\delta(x,1)$ to $u^*_M(x,1;M_\delta)$, we complete the proof of this case as in the previous case.

For a measure as initial datum, we displace the origin of time to $t = \tau > 0$ and may then assume that u_0 is integrable and bounded. □

• We have a strong version of uniqueness of the self-similar solution, using the asymptotic behaviour:

Corollary 10.2. *Let $u(x,t)$ be a nonnegative weak solution of equation (1.1), and assume that u is self-similar and the mass is finite, i.e., $\int u(x,t)\,dx < \infty$ for $t > 0$. Then u is one of the Barenblatt solutions mentioned in the previous theorem, $u(x,t) = U_M(x,t)$, where $M = \int u(x,t)\,dx$, which is constant in time.*

11. Nonexistence for small $m \le m_c$

When we go below the critical exponent m_c, fundamental solutions cease to exist. This was proved for the standard Fast Diffusion Equation ($s = 1$, $0 < m < (N-2)/N$) by Brezis and Friedman [20]. Without entering into full details of the issue, we give here a simple proof of nonexistence in the fractional case, based on the existence of the scaling group (3.4). The main point to take into account is that when $m < m_c$ the similarity exponent β of formula (1.4) is negative, i.e., the transformation involves space contraction instead of expansion. Therefore, if we consider a sequence of integrable functions approximating the Dirac delta of the form

$$u_{0n}(x) = n^N u_{01}(nx) \quad \text{with} \quad \int u_{01}(x)\,dx = 1,$$

and say $u_{0n} \in C^\infty_c(\mathbb{R}^N)$, then the sequence of solutions satisfy

$$u_n(x,t) = n^N u_1(nx, n^{1/\beta} t).$$

Letting $n \to \infty$ for fixed $t > 0$ implies that $u_n(\cdot,t) \to \delta$ (thanks to the fact that $n^{1/\beta} \to 0$), and we conclude two things: first, that we do not find in the limit the expected fundamental solution, and second, that the result can be interpreted as saying that, under this evolution equation, an initial Dirac delta does not spread with time, and the "physical solution" is just $u^*_M(x,t) = M\delta(x)$.

In the critical case $m = m_c$ we have $1/\beta = 0$ and the conclusion is the same.

We can try to take the limit of the fundamental solution that exists for $m > m_c$ and prove that the solution concentrates around the origin as $m \to m_c$. This is clear in the standard FDE since the solutions are explicit; see the very peculiar case $N = 2$, $m \to 0$ in [63, Lemma 8.3].

An analysis of the behaviour of the potentials is also illuminating. We leave it to the reader.

12. Precise decay rates of the Barenblatt profiles

We resume the study of the decay rates of the Barenblatt solutions started at the end of Section 8. Notation: for positive functions $f(r)$, $g(r)$ defined for all large r we say that $f(r) \sim g(r)$ when there exist two positive constants $C_1 < C_2$ such that $C_1 g(r) \le f(r) \le C_2 g(r)$ for all $r > 0$ large enough.

• *Case* $m > 1$. We may get the exact rate as follows: since by (8.4) we have $F^m \le Cr^{-mN}$, the convolution formula yields $L_{s'}F^m \sim r^{-N+2s'}$, and this means that $\int_r^\infty rF(r)\,dr$ behaves like $r^{-N+2-2s}$ as $r \to \infty$. But this also implies that the "mass" in an annulus given by the integral $\int_r^{Cr} rF(r)\,dr$ with $C > 0$ large behaves like $r^{-N+2-2s}$ (with a different constant). Using the monotonicity of F we arrive at the conclusion that F has the decay rate

$$F(r) \sim r^{-N-2s}, \tag{12.1}$$

valid for all $m > 1$, which equals the decay rate for $m = 1$. This decay for $s < 1$ is in stark contrast with the case $s = 1$, where the profiles of the Barenblatt solutions are *compactly supported* in the same m-range, $m > 1$.

• *Case* $m_c < m < 1$. A power-like bound from below is obtained as follows: We may start from the homogeneity estimate [13] that says that $(1-m)tu_t \le u$. In terms of the self-similar profile, this just means that $-(1-m)\beta(NF + rF'(r)) \le F$, hence

$$\frac{-rF'(r)}{F(r)} \le N + \frac{1}{(1-m)\beta} = \frac{2s}{1-m}.$$

Integration of this inequality gives the following lower bound, valid for all $r \ge 1$, all $s \in (0,1)$ and all $m > m_c(s,N)$:

$$F(r) \ge Cr^{-2s/(1-m)}. \tag{12.2}$$

Moreover, the function $J(r) := F(r)r^{-2s/(1-m)}$ is nondecreasing, so it has a limit as $r \to \infty$. The lower bound (12.2) is a good starting point since it is the exact decay rate for the standard diffusion case $s = 1$ when $m_c < m < 1$ with compactly supported initial data [37, 25].

We will obtain a similar upper bound in the range $m_c < m < m_1$ as a consequence of the VSS construction in Section 13, where the asymptotic constant $c_\infty = \lim_{r\to\infty} J(r)$ is calculated. See more details in that section. Note in passing that $2s/(1-m) > N$ precisely for $m > m_c(s,N)$. Summing up, the decay rate (12.2) is optimal for $m_c < m < m_1 = N/(N+2)$, and a very precise rate is obtained.

• However, the rate (12.2) is very far from a realistic estimate for m close to 1, since $2s/(1-m) \to \infty$, which is not, loosely speaking, an admissible decay in the situation of fractional Laplacian diffusion.

We look for a bound from above as follows. Since $F(r) \le Cr^{-N}$ we have $F^m \le Cr^{-Nm}$ and the fact that $Nm < N$ implies that $L_{s'}F^m \le Cr^{-Nm+2s'}$, hence

$$\int_r^\infty rF(r)\,dr \le Cr^{-Nm+2s'}.$$

Using the monotonicity of F this means that $F(r) \le Cr^{-mN-2s}$. But $Nm + 2s > N$ for $m > m_c$, so that we have a gain of upper bound exponent from $\gamma_0 = N$ to $\gamma_1 = mN+2s$. Proceeding iteratively as long as the whole argument we have used is justified, we get a sequence of increasing exponents γ_k given by $\gamma_{k+1} = \gamma_k m + 2s$ that converge to the fixed point of the iteration formula, i.e., $2s/(1-m)$. This is all right for $m \le m_1$. However, for $1 > m > m_1$ we have $2sm/(1-m) > N$ so that after a number of steps $F(r)^m \le r^{-m\gamma_k}$ with $\gamma_k m > N$. In that case, the line of the previous argument changes to give $L_{s'}F^m \sim r^{-N+2s'}$, and following the same line of argument yields $F(r) \sim r^{-N-2s}$ for all large r, which is the desired conclusion.

• In the limit case $m = N/(N+2s)$, the iteration from above is not interrupted and we get $F(r) \le Cr^{-N-2s+\varepsilon}$ for every $\varepsilon > 0$. Regarding the iteration from below, the starting rate is $\gamma = N + 2s$, so that the exponent of our lower bound for F^m is $mN/(N + 2s) = N$. In this case the convolution formula for $L_{s'}$ produces a logarithmic correction for $L_{r'}F^m$ that we have to take into account. The estimate is then

$$F(r) \ge cr^{-N-2s}\log(r) \qquad \text{for all large } r;\; m = m_1. \tag{12.3}$$

We may sum up the results as follows.

Theorem 12.1. *For every $m > m_1 = N/(N + 2s)$ we have the asymptotic estimate*

$$C_1 M^\sigma \le F_M(r) r^{N+2s} \le C_2 M^\sigma, \tag{12.4}$$

where $M = \int F(x)\,dx$, $C_i = C_i(m, N, s) > 0$, and $\sigma = (m - m_1)(n + 2s)\beta$. On the other hand, for $m_c < m < m_1$, there is a constant $C_\infty(m, N, s)$ such that

$$F_M(r) r^{2s/(1-m)} = C_\infty. \tag{12.5}$$

The case $m = m_1$ is borderline and has a logarithmic correction.

The analysis done before assumed $M = 1$, for $M \ne 1$ just use scaling. For $m \in (m_c, m_1)$ the fact that C_∞ does not depend on M is commented and explained in whole detail in Section 13, where the last elements needed to complete this proof are given.

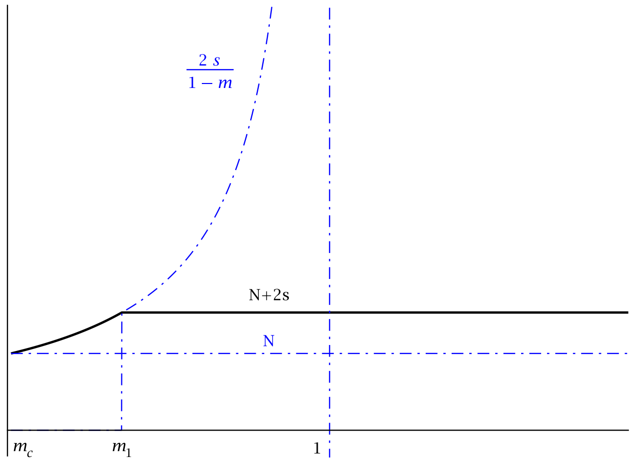

Fig. 2. Plot of decay rates of self-similar profiles. Here, $N = 3$ and $s = 0.8$.

13. Very singular solutions in the fast diffusion range

Existence. A special solution that plays an important role in the theory of the standard Fast Diffusion Equation is the so-called Very Singular Solution (VSS) [25, 63]. More precisely, for every $1 > m > m_c$ there is a global in time function with separated variables that solves the FDE away from $x = 0$ and has a standing singularity at $x = 0$ for all times. It also starts with $V(x, 0) = 0$ for $x \neq 0$.

Using the same idea, we want to find a similar solution for the fractional diffusion equation for $0 < s < 1$ in the good fast diffusion range $m_c < m < 1$. We look for solutions of the form

$$\widetilde{U}(x, t) = T(t)X(x).$$

Substitution into the equation leads to the value $T(t) = t^{1/(1-m)}$ for the time factor, while $Y = X^m$ has to satisfy

$$(-\Delta)^s Y + \frac{1}{1-m} Y^p = 0, \quad p = 1/m. \tag{13.1}$$

Now we try as solution the function $Y(x) = C^m |x|^{-\alpha}$. Under suitable conditions on α, to be discussed below, we will have

$$(-\Delta)^s Y = C^m k |x|^{-(\alpha+2s)} \tag{13.2}$$

for some negative constant $k = k(\alpha, N, s)$. It follows that equation (13.1) for Y can be satisfied only if α takes the value $\alpha(m, s) = 2s/(p-1) = 2sm/(1-m)$. We keep this value of α in this section. The constant $C > 0$ is then determined by $-C^m k(\alpha) = C/(1-m)$, so that

$$C^{1-m} = (1-m)(-k(\alpha)). \tag{13.3}$$

Therefore, we need $k(\alpha) < 0$. Let us examine the conditions on α.

First, we need the condition $\alpha < N$ to make Y locally integrable and formula (13.2) hold, which means $m < m_1 := N/(N+2s)$. Without this restriction we enter into a theory of Laplacians of nonintegrable functions, which is acceptable anyway for $s = 1$, but it does not work in the fractional case $s < 1$, as we will see below.

We need to calculate $k(\alpha)$ to know when it is negative. This is done in (A.3). Note that in the interval $m_c < m < m_1$ we have $N - 2s < \alpha < N$ so that $N < \alpha + 2s < N + 2s$. We check there that $k(\alpha) > 0$ for all $\alpha > N - 2s$.

Finally, let us check the properties of the constructed function. We remark that $\widetilde{U}$ is not a weak solution in the standard sense since $\widetilde{U}(\cdot, t)$ is not even integrable in space around the origin. However, $\widetilde{U}^m$ is locally integrable, and we can propose a generalized definition of weak solution that restricts test functions to be supported away from the origin. We then use the name *generalized weak solution with an isolated singularity.*

Theorem 13.1. *There exists a VSS for equation* (15.1) *in the range $m_c < m < m_1$ and it has the explicit form*

$$\widetilde{U}(x,t) = C t^{1/(1-m)} |x|^{-2s/(1-m)} \tag{13.4}$$

where $C = C(m, s, N)$ is given by the explicit formula (13.3)*. $\widetilde{U}$ is a weak solution in a generalized sense, more precisely, a weak solution with an isolated singularity at $x = 0$.*

In order to compute the exact value of C we must use (13.3) together with (A.3). For further reference we call the constant C_{VSS}.

VSS as limit. We now establish the relation of the VSS to the Barenblatt solutions we have already constructed.

Theorem 13.2. *The VSS* (13.4) *is the limit of the Barenblatt solutions $u_M^*(x,t)$ as the mass M goes to infinity.*

Proof. By the known comparison properties, it is clear that the sequence of Barenblatt solutions $u_M^*(x,t)$ is increasing in $M > 0$.

Next, we check that they are all bounded above by the VSS. A direct comparison of u_M^* with $\widetilde{U}$ is difficult. Instead, we compare $u(x,t)$ and $\widetilde{U}$, where u is a solution with mass M and smooth data u_0 supported in a small ball, so that $u(x,0) \le \widetilde{U}(x,1)$ for all $x \in \mathbb{R}^N$. The comparison result (viscosity style, at the possible points of contact) holds and proves that $u(x,t) \le \widetilde{U}(x,t)$ for all $t > 0$. Now we do the rescaling that is used in the proof of asymptotic behaviour in Section 10, and observe that $\widetilde{U}$ is invariant. Therefore, $T_\lambda u(x,t) \le \widetilde{U}(x,t)$. In the limit $t \to \infty$ we use the result on asymptotic convergence to get $u_M^*(x,1) \le \widetilde{U}(x,1)$. The result is obviously true for all other times with a minimal modification of the argument.

This upper bound allows us to pass to the monotone limit in the family $\{u_M^*(x,t) : M > 0\}$ and obtain a function $U^*(x,t) \le \widetilde{U}(x,t)$. This upper bound implies that U^* is a weak solution of the Fractional FDE, and it is locally bounded away from the space origin $x = 0$ for all times $t > 0$. We conclude that U^* is another possible very singular solution.

It also has the same invariance under $(T_\mu u)(x,t) = \mu^{2s/(1-m)} u(\mu x, t)$, which means that the space form must be $c(t)|x|^{2s/(1-m)}$. But this easily implies the self-similar form we have chosen, which yields $U^* = \widetilde{U}$.

Precise asymptotic behaviour. We claim that the asymptotic constant

$$c_\infty(F_M) = \lim_{r\to\infty} F_M(r) r^{2s/(1-m)} \tag{13.5}$$

for the Barenblatt solutions coincides with the constant C for VSS, which we have called C_{VSS}.

Indeed, by the ordering of the solutions we must have $c_\infty(F_M) \le C_{\mathrm{VSS}}$. On the other hand, the scaling of the Barenblatt solutions immediately implies that $c_\infty(F_M)$ is independent of M. Therefore, for every $M > 0$ we have

$$F_M(r) \le c_\infty r^{-2s/(1-m)}. \tag{13.6}$$

If we had $c_\infty(F) < C_{\mathrm{VSS}}$, then the limit of the Barenblatt solutions as $M \to \infty$ would be a very singular solution of the form (13.4) with a constant less than C_{VSS}, and this is not possible, as we have already seen.

The limit is infinite in other ranges. There are no fundamental solutions for $m \le m_c$, so that the ranges to discuss are $m \ge 1$ and $m_1 \le m < 1$. In the linear case $m = 1$ the scaling of the Barenblatt solutions is

$$u_M^*(x,t) = M u(x,t), \tag{13.7}$$

so it is clear that $\lim_{M\to\infty} u_M^*(x,t) = \infty$ everywhere in $Q = \mathbb{R}^N \times (0,\infty)$. A similar result holds for $m > 1$, in view of the scaling

$$u_M^*(x,t) = \mu^{2s} u(\mu^{1-m}x, t), \quad M = \mu^{N(m-1)+2s}. \tag{13.8}$$

Since u is monotone in $r = |x|$ and $m > 1$, we have $u_M^*(x,t) \ge \mu^{2s} u(x,t)$, and the result follows by letting $M \to \infty$.

In the remaining fast diffusion range, $m_1 \le m < 1$, the final answer is the same, an infinite limit, but the argument is not so simple. It is based on the minimum decay rate of the Barenblatt profiles, like $O(r^{-(N+2s)})$ (with logarithmic correction if $m = m_1$). In particular, such behaviour implies that

$$\lim_{r\to\infty} F_1(r) r^{2s/(1-m)} = \infty.$$

The conclusion follows by scaling: For all large $M > 0$ we put $M = \mu^{N(m-1)+2s}$ for fixed $r > 0$. Then as $M \to \infty$, we get

$$\lim_{M\to\infty} F_M(r) = \lim_{\mu\to\infty} \mu^{2s} F_1(\mu^{1-m} r) = r^{-2s/(1-m)} \lim_{z\to\infty} z^{2s} F_1(z) = \infty.$$

Remarks. (1) This last divergence result means that no VSS in the sense of the beginning of this section can exist in this range. The result represents a marked difference with the case $s = 1$, where a VSS solution exists for all $m \in (m_c, 1)$, and the Barenblatt solutions are uniformly bounded by it.

(2) The construction of a separated-variables solution with a fixed isolated singularity can be performed for $m < m_c$ but then we would get finite-time extinction profiles, in the line of [63, Chapter 5]. This construction has been performed in [66].

14. Construction of eternal solutions for $m = m_c$

We have already seen that the Barenblatt solutions, in the sense of fundamental solutions with self-similar form, do not exist for $m \le m_c := (N-2s)/N$. One wonders if there is a special way to pass to the limit $m \to m_c$ to obtain a self-similar solution, even if it cannot be a fundamental solution. The answer is yes and we obtain a family of eternal solutions, similar to the ones that can be obtained for the standard fast diffusion equation with critical exponent [63]. Let us mention that eternal solutions are a popular topic in the study of geometrical flows (cf. for instance [30] and [35]).

Theorem 14.1. *There exists a family of eternal solutions of equation* (1.1) *with exponent* $m = m_c$ *and* $0 < s \le 1$. *They have the form*

$$U(x,t) = e^{-Nct} F(|x|e^{-ct}) \tag{14.1}$$

where $F(r)$ *is a continuous, radial function with* $F(0) = a > 0$ *and* $F(r)r^N \to b > 0$ *as* $r \to \infty$.

Proof. **1.** The process is known in the standard case $s = 1$, but it will be convenient to explain it in some detail in this context, since it involves a subtle manipulation of the equations. We consider a different scaling of the flow that is good when passing to the limit $m \to m_c = (N-2)/N$ with $m > m_c$, namely

$$u(x,t) = (1+ct/\beta)^{-N\beta} v(y,\tau), \qquad y = x(1+ct/\beta)^{-\beta},\ \tau = \beta \log(1+ct/\beta).$$

This applies for $t \ge -c\beta$, where β is the self-similar exponent and $c > 0$ is a free constant. The original equation[2] $u_t = \Delta(u^m/m)$ becomes

$$cv_\tau = \Delta(v^m/m) + c\nabla\cdot(yv) = \nabla\left(v\nabla\left(-p + \frac{c}{2}y^2\right)\right), \qquad p = \frac{1}{1-m}v^{m-1}. \tag{14.2}$$

Notice that no β appears in this expression, which is important since $\beta \to \infty$ as $m \to m_c$. Equation (14.2) has the stationary solution (profile)

$$(\bar F_m(y))^{m-1} = b + \frac{c(1-m)}{2}y^2,$$

with $b > 0$ arbitrary. This holds for $1 > m > m_c$. In the limit $m \to m_c$ we have: $1/(1-m) \to N/2$, $\beta \to \infty$, $\tau(t) \to t$, $(1+ct/\beta)^{-\beta} \to e^{-ct}$, $(1+ct/\beta)^{-N\beta} \to e^{-cNt}$, so that the profile in the limit is

$$F_{m_c}(y) = \left(b + \frac{c}{N}y^2\right)^{-N/2},$$

[2] Note the slight change in the form of the equation due to the constant m, which is customary in studies of FDE. It is inessential here but we have inserted it for agreement with published information.

and the solution becomes (with $a = c/N$)

$$U_{m_c}(x,t) = e^{-cNt}(b + ax^2 e^{-2ct})^{-N/2} = (ax^2 + be^{2Nat})^{-N/2}, \tag{14.3}$$

which is the (two-parameter family of) eternal solutions obtained in [63, formula (5.36)]. We may still enlarge the family by moving the origin of space coordinates.

The scaling group $Tu(x,t) = \lambda^N \mu^{-N/2} u(\lambda x, \mu t)$, $\lambda, \mu > 0$, acts on these solutions to change the parameters a and b in the form: $T(a,b) = (a\mu, b\mu/\lambda^2)$, so the whole family of solutions comes from a single one.

Moreover, when $b \to 0$ we get the singular stationary case $U_\infty(x,t) = A|x|^{-N}$, $A = a^{-N/2} > 0$, which is not a solution at $x = 0$ since ΔU_∞^m has a delta function there. This situation reminds of the increasing convergence of Barenblatt solutions to the VSS for $m > m_c$; in that case we used the mass as parameter, here the parameter is rather $F_{m_c}(0) = b^{-N/2}$ and $a > 0$ is kept fixed. Note also that U_∞ does not depend on time, and the constant A is arbitrary.

2. We now apply these ideas to the limit $m \to m_c(s) = (N-2s)/N$ in the fractional diffusion case $0 < s < 1$. This case is more difficult since the Barenblatt solutions are not explicit. We have to avoid the vanishing of the space asymptotic constant in the limit by careful time scaling of the above type that leads to the correct renormalized equation.

The calculations start as follows: the equations for $m > m_c$ have VSS solutions in the standard scaling of the form

$$V(x) = Ct^{1/(1-m)} x^{-2s/(1-m)},$$

where C_{VSS} is given by $C^{1-m} = -(1-m)k(\alpha)$ and

$$k(\alpha) = 2^{2s} \frac{\Gamma((N-\alpha)/2)\Gamma((\alpha+2s)/2)}{\Gamma((N-\alpha-2s)/2)\Gamma(\alpha/2)}$$

with $\alpha = 2ms/(1-m)$. When $m \to m_c$ we have $\alpha \sim N - 2s$, $(N-\alpha)/2 \sim s$, $(\alpha+2s)/2 \sim N/2$, $\alpha/2 \sim N/2 - s$, and finally

$$\frac{N-\alpha-2s}{2} = \frac{N-2s}{2} - \frac{sm}{1-m} = \frac{N(1-m)-2s}{2(1-m)} = -\frac{1}{2(1-m)\beta}.$$

Therefore, $C^{1-m} \sim k_1/\beta$. Comparing the profile equations (8.1) and the one above, we see that we replace β by c so that the new constant C is given by $C^{1-m} = k_1/c$, which can be made 1 by a suitable choice of c.

3. *Construction.* We now consider the family of self-similar solutions for $m \sim m_c$ with $F_m(0) = 1$ (which means that we need to choose the mass of the solution appropriately). We also have

$$F_m(r) \le \hat{C}_m r^{-2s/(1-m)}, \qquad \lim_{r\to\infty} F_m(r) r^{2s/(1-m)} = \hat{C}_m \to 1.$$

Using the monotonicity of the profiles and C^α regularity of the family F_m we may pass to the limit $F_m(r) \to F(r)$ so that $F(0) = 1$, $F(r) \le r^{-N}$, with $F \in C^\alpha$, and $F(r)r^N$ increasing. This is an eternal solution, after some extra work that is rather easy.

We may then construct a two-parameter family of radial solutions, and then add displacement in space. The two parameters are $F(0)$ controlled by constant mass scaling, and the constant at infinity that is controlled by the other part of the scaling group. □

15. Aleksandrov's symmetry principle

The Aleksandrov–Serrin reflection method is a well-established tool to prove monotonicity of solutions of wide classes of (possibly nonlinear) elliptic and parabolic equations [3, 58]. It has been quite useful in particular in the case of the PME, as documented in [64]. Uses of the method for fractional operators are recent [26, 22]. Here we will establish a version of the principle valid for our type of problems, and then derive useful monotonicity results.

Elliptic setting. We consider the problem

$$Lu^m + u = f \tag{15.1}$$

in $\mathbb{R}^N$ with $L = (-\Delta)^s$, $m > 0$, and $f \in L^1(\mathbb{R}^N)$. We take a hyperplane H that divides $\mathbb{R}^N$ into two half-spaces Ω_1 and Ω_2 and consider the symmetry Π with respect to H that maps Ω_1 onto Ω_2. Then we have

Theorem 15.1. *Let u be the unique solution of (15.1) with data $f \in L^1(\mathbb{R}^N)$, $f \ge 0$. Under the assumption that*

$$f(x) \ge f(\Pi(x)) \quad \text{in } \Omega_1 \tag{15.2}$$

we have

$$u(x) \ge u(\Pi(x)) \quad \text{in } \Omega_1. \tag{15.3}$$

Proof. (i) Due to the translation and rotation invariance of the equation we may assume that $H = \{x \in \mathbb{R}^N : x_1 = 0\}$ and $\Omega_1 = \{x \in \mathbb{R}^N : x_1 > 0\}$. We write $x' = \Pi(x)$ where Π is the symmetry with respect to $x_1 = 0$. We take a function f such that $f(x') \le f(x)$ when $x_1 > 0$. We assume at this stage that f is bounded, continuous and integrable.

We solve the elliptic problem with data f to get a solution u and write $\widehat{u}(x) = u(x')$; that solves the problem with data $\widehat{f}(x) = f(x')$ in the whole space $\mathbb{R}^N$.

(ii) We now assume for a moment that $\widehat{u} \le u$ in Ω_1 and consider a point $x_0 \in \Omega_1$ where $\widehat{u}$ touches u from below. If such a point exists, then $u(x') \le u(x)$ for all $x \in \Omega_1$ and

$$Lu^m(x_0) = C \int \frac{u^m(x_0) - u^m(y)}{|x_0 - y|^{N+2s}}\, dy,$$
$$L\widehat{u}^m(x_0) = C \int \frac{\widehat{u}^m(x_0) - C\widehat{u}^m(y)}{|x_0 - y|^{N+2s}}\, dy = C \int \frac{u^m(x_0) - u^m(y')}{|x_0 - y|^{N+2s}}\, dy.$$

Therefore, when calculating $L\widehat{u}^m(x_0) - Lu^m(x_0)$ the terms in $u^m(x_0) = \widehat{u}^m(x_0)$ cancel out and we get

$$L\widehat{u}^m(x_0) - Lu^m(x_0) = C \int_{\Omega_1} \frac{u^m(y) - u^m(y')}{|x_0 - y|^{N+2s}}\, dy + C \int_{\Omega_2} \frac{u^m(y) - u^m(y')}{|x_0 - y|^{N+2s}}\, dy.$$

Interchanging y and y' in the last integral gives

$$L\widehat{u}^m(x_0) - Lu^m(x_0) = \int_{\Omega_1} \frac{u^m(y) - u^m(y')}{|x_0 - y|^{N+2s}}\,dy + \int_{\Omega_1} \frac{u^m(y') - u^m(y)}{|x_0 - y'|^{N+2s}}\,dy,$$

hence

$$L(\widehat{u}^m - u^m)(x_0) = \int_{\Omega_1} (u^m(y) - u^m(y'))\left(\frac{1}{|x_0 - y|^{N+2s}} - \frac{1}{|x_0 - y'|^{N+2s}}\right) dy > 0,$$

where we have used the fact that for $x_0, y \in \Omega_1$ we have $|x_0 - y| \le |x_0 - y'|$. On the other hand, at x_0 we have $f(x_0) \ge f(x_0')$. We get a contradiction with the equation unless both solutions coincide everywhere. The solution must then equal its symmetric reflection, which is impossible if f is not symmetric.

(iii) In order to apply this argument we need to show that the situation where the solution and its reflection are ordered happens, which is not clear. The general argument that fits the present situation applies approximations of the solution, as is typical in the proofs of maximum principles for standard elliptic equations. We argue as follows:

We take f in, say, $L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and with compact support. We use the theory of existence for data $\widehat{f} = f + C$, which is not difficult since it is similar to what is done in [51, 52] after a vertical translation of the solution. We thus get for bounded data $f_C = f + C$, $C > 0$, a solution $u_C \ge C$ and $u_C - C \in L^1(\mathbb{R}^N)$.

Under the stated hypotheses the existence and regularity theory says that $u_C \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, is continuous and $u_C \ge u_0 = u$. The regularity is obtained by a Morrey embedding theorem.

A known strong maximum principle for fractional Laplacian operators implies that these functions do not touch in $\mathbb{R}^N$, i.e., $u_C > u_0$. The argument is not very different from step (ii) above, but simpler since there is no reflection.

Let us now check the symmetrization argument on this family. We will write for more clarity $u_C = u(f + C)$, $u_0 = u(f)$. For large $C > 0$ we have $u(f) \le u(f + C)$ and $\widehat{u(f)} \le u(f + C)$ in Ω_1 ($\widehat{u(f)}$ indicates the symmetrical image of $u(f)$ as before). We use the previous argument to conclude that there is no possible point of contact for $\widehat{u(f)}$ and $u(f + C)$ inside Ω_1 and there was none at the boundary H where $\widehat{u(f)} = u(f)$.

At this point we may lower the C until we get a point of contact in one of the two comparisons. If $C > 0$ there is no contact at $x_1 = 0$ and none as $|x| \to \infty$, and we get a contradiction in the two comparisons.

Therefore, the infimum of the C is zero. Thus, there is contact as $|x| \to \infty$ and at the boundary $x_1 = 0$. But the plain comparison yields $\widehat{u(f)} \le u(f)$. The proof is finished under the stated assumptions on f.

(iv) For general $f \in L^1(\mathbb{R}^N)$ we prove the result by approximation with functions f_n as above and then use the L^1-continuity of the map $f \mapsto u(f)$ (see [52]). □

Parabolic setting. We consider the problem

$$u_t + Lu^m = 0 \tag{15.4}$$

for $x \in \mathbb{R}^N$ and $t > 0$ with initial data $u(x, 0) = u_0(x) \in L^1(\mathbb{R}^N)$, $u_0 \geq 0$. We use the same notations and conventions for the hyperplane H, half-space Ω_1 and symmetry Π. Then we have

Theorem 15.2. *Let u be the unique solution of (15.4) with initial data u_0. Under the assumption that*

$$u_0(x) \geq u_0(\Pi(x)) \quad \text{in } \Omega_1 \tag{15.5}$$

we have, for all $t > 0$,

$$u(x, t) \geq u(\Pi(x), t) \quad \text{for } x \in \Omega_1. \tag{15.6}$$

Proof. This is an easy consequence of the fact that the nonlinear operator $S : u \mapsto Lu^m$ can be defined so as to be m-accretive in $L^1(\mathbb{R}^N)$, and its resolvent $(I + \lambda S)^{-1}$ also an ordered contraction [52]. We can then solve the parabolic problem by implicit discretization in time, i.e., for times $t_0 = 0$, $t_1 = h$, $t_2 = 2h, \ldots,$ we define the approximations

$$\frac{1}{h}(u(t_n) - u(t_{n-1})) + L(u(t_n)^m) = 0.$$

We apply the conclusion of the elliptic theorem iteratively at every step to get $u(x, t_n) \geq u(\Pi(x), t_n)$ for every $x \in \Omega_1, n \geq 1$. Finally, we pass to the limit $h \to 0$ in the discretization step (using the Crandall–Liggett Theorem as in [52]) to get the solution $u(x, t)$ with the desired properties. See more details of this method in [28, 27, 64]. ◻

Application. Comparison after reflection in suitable hyperplanes allows us now to conclude that the solution with initial data supported in a small ball B_ε will be decreasing in the radial exterior direction in the annulus $\{x : |x| \geq 2\varepsilon\}$. We also conclude that it is decreasing along a cone of directions centred along the radius with aperture that goes to the flat cone when the annulus is taken $\{x : |x| \geq R\}$ with $R/\varepsilon \to \infty$. This implies that the solution with initial data a Dirac delta must be radially symmetric. All these techniques are well-known, as explained in Chapter 9 of [64], where references to further literature are given.

Appendix 1. Fractional Laplacians and potentials

- According to Stein [60, Chapter V], $(-\Delta)^{\beta/2}$ is defined by means of Fourier series

$$((-\Delta)^{\beta/2} f)\widehat{\ }(x) = (2\pi|x|)^\beta \hat{f}(x), \tag{A.1}$$

and can be used for positive and negative values of β. For $\beta = -\alpha$ negative, with $0 < \alpha < N$, we have equivalence with the Riesz potentials

$$(-\Delta)^{-\alpha/2} f = I_\alpha(f) := \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^N} \frac{f(y)}{|x - y|^{N-\alpha}}\, dy \tag{A.2}$$

(acting on functions of the class $\mathcal{S}$ for instance) with precise constant

$$\gamma(\alpha) = \pi^{N/2} 2^\alpha \Gamma(\alpha/2)/\Gamma((N - \alpha)/2).$$

Note that $\gamma(\alpha) \to \infty$ as $\alpha \to N$, but $\gamma(\alpha)/(N-\alpha)$ converges to a nonzero constant, $\pi^{N/2}2^{N-1}\Gamma(N/2)$. Note that in this notation α stands for $2s$ in our previous notation.

• The Fourier transform of the function $f(x) = |x|^{-N+\alpha}$ is $\hat{f}(\xi) = \gamma(\alpha)(2\pi)^{-\alpha}|\xi|^{-\alpha}$. The Fourier transform of $f(x) = |x|^{-\alpha}$ is $\hat{f}(\xi) = \gamma(N-\alpha)(2\pi)^{\alpha-N}|\xi|^{\alpha-N}$. We conclude that its s-Laplacian, $(-\Delta)^s|x|^{-\alpha}$, is the power $k(\alpha)|x|^{-\alpha-2s}$, with a constant factor that equals

$$k(\alpha) = \frac{\gamma(N-\alpha)}{\gamma(N-\alpha-2s)} = \frac{\gamma(N-\alpha)}{\gamma(N-\alpha-2s)}$$

so that

$$k(\alpha) = 2^{2s}\frac{\Gamma((N-\alpha)/2)\Gamma((\alpha+2s)/2)}{\Gamma((N-\alpha-2s)/2)\Gamma(\alpha/2)}. \tag{A.3}$$

When $s = 1$ we get

$$k(\alpha) = \alpha(N-\alpha-2),$$

which is positive whenever $\alpha \in (0, N-2)$. For $0 < s < 1$ this is not so clear. In any case, in the range of interest, $0 < N-2s < \alpha < n$, all the factors minus one are positive and the remaining one, $\Gamma((N-\alpha-2s)/2)$ corresponds to an argument $((N-2s)-\alpha)/2$ which is negative but larger than $-s > -1$, hence $\Gamma((N-\alpha-2s)/2) < 0$.

Appendix 2. Bessel kernels

They are usually introduced via Fourier transform,

$$\widehat{G}_\alpha(\xi) = (1+|\xi|^2)^{-\alpha/2}, \quad \xi \in \mathbb{R}^N, \alpha > 0.$$

There is an expression for the kernel of the form

$$G_\alpha(x) = c(\alpha)\int_0^\infty e^{\pi|x|^2/t}t^{(\alpha-N)/2}\,\frac{dt}{t}, \qquad x \in \mathbb{R}^N,$$

(see [60, Section V.3.1]), so that G_α is a nonnegative, radially decreasing function. Moreover, G_α is integrable and $c(\alpha)$ is a positive constant chosen so that $\|G_\alpha\|_1 = 1$. The Bessel potential of order $\alpha > 0$ of the density ρ is defined by

$$\mathcal{B}_\alpha(x) = \int_{\mathbb{R}^N} G_\alpha(x-y)\rho(y)\,dy.$$

We have the following estimates for $0 < \alpha < N$:

$$\begin{aligned} &0 < G_\alpha(x) \le C|x|^{-(N-\alpha)} && \text{if } 0 < |x| < 1,\\ &0 < G_\alpha(x) < e^{-|x|/2} && \text{if } |x| > 1, \end{aligned}$$

where $C = C(\alpha; N)$. More precisely, the kernel can be represented by means of the McDonald function:

$$G_\alpha(x) = c(\alpha, N)|x|^{(\alpha-N)/2}K_{(N-\alpha)/2}(|x|).$$

The Macdonald function with index $\nu \in \mathbb{R}$, K_ν, also called the modified Bessel function of the second kind, is given by the following formula:

$$K_\nu(r) = 2^{-1-\nu} r^\nu \int_0^\infty e^{-t} e^{-r^2/4t} t^{-1-\nu} dt, \qquad r > 0.$$

The asymptotic behaviour of K_ν is as follows:

$$K_\nu(r) \sim \frac{\Gamma(\nu)}{2} (r/2)^{-\nu} \qquad \text{as } r \to 0+, \; \nu > 0,$$

while $K_0(r) \sim \log(1/r)$ as $r \to 0+$. At infinity we have

$$K_\nu(r) \sim \frac{\sqrt{\pi}}{\sqrt{2r}} e^{-r} \qquad \text{as } r \to \infty,$$

where the notation $g(r) \sim f(r)$ means that the ratio of g and f tends to 1. For $\nu < 0$ we have $K_{-\nu}(r) = K_\nu(r)$, which determines the asymptotic behaviour for negative indices. For properties of K_ν see [32].

Comments and extensions

• The linear case was known, and our results recover the information with a completely different machinery. In fact, the main tools of the paper combine a selection of the extensive machinery developed for the study of porous medium and fast diffusion equations with the subtleties of singular operators. This combination ranges from smooth adaptation to unexpected difficulties and new types of results. The lack of explicit formulas for some important classes of solutions is a challenge in obtaining the existence of some phenomena or the description of their properties.

• One wonders if there are any explicit or semi-explicit formulas for the family of Barenblatt solutions of this paper.

• The limits $s \to 0$ and $s \to 1$ are worth examining.

• Fundamental solutions can be constructed for the equation in a bounded domain. They do not, however, play such an important role in the theory. For instance, they shed no light on the asymptotic behaviour, even in the simplest case of zero Dirichlet boundary conditions.

• The positivity of the fundamental solutions in all $\mathbb{R}^N$, and more precisely, the power-like maximal decay rate as $|x| \to \infty$, are properties shared by all nonnegative solutions of the FPME and not only the solutions that take a special self-similar form. Quantitative versions of the lower bounds on the positivity of solutions are established in a paper with Bonforte [19]. Higher regularity of positive solutions of equation (1.1) has been investigated in [53].

• There is the open problem of proving the uniqueness of solutions with measure data, not just Dirac deltas.

• In the case of the standard PME, the range $m < m_c$ is now understood: no fundamental solutions exist, but the family of Barenblatt solutions for $m > m_c$ can be continued algebraically and they exhibit interesting attraction properties [16, 18]. Not much is known to our knowledge for the corresponding fractional equations.

• New applications of these models in applied sciences would be welcome as a motivation to develop further aspects of this theory.

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