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On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions

Abstract. We prove some results on the existence and compactness of solutions of a fractional Nirenberg problem. The crucial ingredients of our proofs are the understanding of the blow up profiles and a Liouville theorem.

Contents

1. Introduction

The *Nirenberg problem* is the following: For which positive function K on the standard sphere $(\mathbb{S}^n, g_{\mathbb{S}^n})$, $n \geq 2$, does there exist a function w on $\mathbb{S}^n$ such that the scalar curvature (Gauss curvature in dimension $n = 2$) R_g of the conformal metric $g = e^w g_{\mathbb{S}^n}$ is equal to

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K on $\mathbb{S}^n$? The problem is equivalent to solving

$$-\Delta_{g_{\mathbb{S}^n}} w + 1 = K e^{2w} \quad \text{on } \mathbb{S}^2,$$

and

$$-\Delta_{g_{\mathbb{S}^n}} v + c(n) R_0 v = c(n) K v^{\frac{n+2}{n-2}} \quad \text{on } \mathbb{S}^n \text{ for } n \geq 3,$$

where $c(n) = (n-2)/(4(n-1))$, $R_0 = n(n-1)$ is the scalar curvature of $(\mathbb{S}^n, g_{\mathbb{S}^n})$ and $v = e^{\frac{n-2}{4} w}$.

The first work on this problem is by D. Koutroufiotis [69], where the solvability on $\mathbb{S}^2$ is established when K is assumed to be an antipodally symmetric function which is close to 1. Moser [83] established the solvability on $\mathbb{S}^2$ for all antipodally symmetric functions K which are positive somewhere. Without assuming any symmetry assumption on K, sufficient conditions were given in dimension $n = 2$ by Chang and Yang [32], [33], and in dimension $n = 3$ by Bahri and Coron [7]. Compactness of all solutions in dimensions $n = 2, 3$ can be found in work of Chang, Gursky and Yang [30], Han [58] and Schoen and Zhang [91]. In these dimensions, a sequence of solutions cannot blow up at more than one point. Compactness and existence of solutions in higher dimensions were studied by Li [71], [72]. The situation here is very different, as far as the compactness issues are concerned: In dimension $n \geq 4$, a sequence of solutions can blow up at more than one point, as shown in [72]. There have been many papers on the problem and related ones: see, e.g., [1, 2, 3, 6, 8, 10, 17, 18, 26, 27, 28, 34, 35, 36, 37, 39, 46, 50, 59, 61, 64, 65, 70, 78, 80, 88, 89, 98, 100, 101].

In [56], Graham, Jenne, Mason and Sparling constructed a sequence of conformally covariant elliptic operators, $\{P_k^g\}$, on Riemannian manifolds for all positive integers k if n is odd, and for $k \in \{1, \ldots, n/2\}$ if n is even. Moreover, P_1^g is the conformal Laplacian $L_g := -\Delta_g + c(n) R_g$ and P_2^g is the Paneitz operator. The construction in [56] is based on the ambient metric construction of [52]. Up to positive constants, $P_1^g(1)$ is the scalar curvature of g and $P_2^g(1)$ is the Q-curvature. The problem of prescribing Q-curvature on $\mathbb{S}^n$ was studied extensively: see, e.g., [9, 15, 42, 43, 44, 53, 96, 97].

Making use of a generalized Dirichlet to Neumann map, Graham and Zworski [57] introduced a meromorphic family of conformally invariant operators on the conformal infinity of asymptotically hyperbolic manifolds (see Mazzeo and Melrose [81]). Recently, Chang and González [29] reconciled the way of Graham and Zworski to define conformally invariant operators P_σ^g of non-integer order $\sigma \in (0, n/2)$ and the localization method of Caffarelli and Silvestre [21] for the fractional Laplacian $(-\Delta)^\sigma$ on the Euclidean space $\mathbb{R}^n$. These lead naturally to a fractional order curvature $R_\sigma^g := P_\sigma^g(1)$, which will be called σ-curvature in this paper. Fractional Yamabe problems about finding constant σ-curvatures on the conformal infinity of given asymptotically hyperbolic manifolds have been studied by González, Mazzeo and Sire [54], González and Qing [55] and Qing and Raske [87]. Related concentration-compactness results have been obtained by Palatucci and Pisante [85].

We focus on the typical example, the standard conformal spheres $(\mathbb{S}^n, [g_{\mathbb{S}^n}])$ which are the conformal infinity of the Poincaré disks $(\mathbb{B}^{n+1}, g_{\mathbb{B}^{n+1}})$. In this case, the σ-curvature can be expressed in the following explicit way. Let g be a representative in the

conformal class $[g_{\mathbb{S}^n}]$ and write $g = v^{4/(n-2\sigma)} g_{\mathbb{S}^n}$, where v is positive and smooth on $\mathbb{S}^n$. Then we have

$$P_\sigma^g(\phi) = v^{-\frac{n+2\sigma}{n-2\sigma}} P_\sigma^{g_{\mathbb{S}^n}}(\phi v) \quad \text{for any } \phi \in C^\infty(\mathbb{S}^n), \tag{1.1}$$

and hence the σ-curvature for $(\mathbb{S}^n, g)$ can be computed as

$$R_\sigma^g = v^{-\frac{n+2\sigma}{n-2\sigma}} P_\sigma^{g_{\mathbb{S}^n}}(v). \tag{1.2}$$

$P_\sigma^{g_{\mathbb{S}^n}}$, which is simply written as P_σ, is an *intertwining operator* (see, e.g., [14]) and

$$P_\sigma = \frac{\Gamma(B + 1/2 + \sigma)}{\Gamma(B + 1/2 - \sigma)}, \quad B = \sqrt{-\Delta_{g_{\mathbb{S}^n}} + \left(\frac{n-1}{2}\right)^2}, \tag{1.3}$$

where Γ is the Gamma function and $\Delta_{g_{\mathbb{S}^n}}$ is the Laplace–Beltrami operator on $(\mathbb{S}^n, g_{\mathbb{S}^n})$. Formula (1.3) goes back at least to the work of T. P. Branson [14]. The operator P_σ can be seen more concretely on $\mathbb{R}^n$ using stereographic projection. The stereographic projection from $\mathbb{S}^n \setminus \{N\}$ to $\mathbb{R}^n$ is the inverse of

$$F : \mathbb{R}^n \to \mathbb{S}^n \setminus \{N\}, \quad x \mapsto \left(\frac{2x}{1 + |x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1}\right),$$

where N is the north pole of $\mathbb{S}^n$. Then it follows from the conformal invariance of P_σ that

$$(P_\sigma(\phi)) \circ F = |J_F|^{-(n+2\sigma)/2n} (-\Delta)^\sigma (|J_F|^{(n-2\sigma)/2n} (\phi \circ F)) \quad \text{for } \phi \in C^\infty(\mathbb{S}^n), \tag{1.4}$$

where

$$|J_F| = \left(\frac{2}{1 + |x|^2}\right)^n,$$

and $(-\Delta)^\sigma$ is the fractional Laplacian operator (see, e.g., [93, p. 117]). When $\sigma \in (0, 1)$, Pavlov and Samko [86] showed that

$$P_\sigma(v)(\xi) = P_\sigma(1)v(\xi) + c_{n,-\sigma} \int_{\mathbb{S}^n} \frac{v(\xi) - v(\zeta)}{|\xi - \zeta|^{n+2\sigma}} \, d\,\mathrm{vol}_{g_{\mathbb{S}^n}}(\zeta) \tag{1.5}$$

for $v \in C^2(\mathbb{S}^n)$, where $c_{n,-\sigma} = \frac{2^{2\sigma} \sigma \Gamma(\frac{n+2\sigma}{2})}{\pi^{n/2} \Gamma(1-\sigma)}$ and $\int_{\mathbb{S}^n}$ is understood as $\lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon}$.

From (1.2), we consider the equation

$$P_\sigma(v) = c(n, \sigma) K v^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{on } \mathbb{S}^n, \tag{1.6}$$

where $c(n, \sigma) = P_\sigma(1)$, and $K > 0$ is a continuous function on $\mathbb{S}^n$. When $K = 1$, (1.6) is the Euler–Lagrange equation for a functional associated to the fractional Sobolev inequality on $\mathbb{S}^n$ (see [9]), and all positive solutions must be of the form

$$v_{\xi_0, \lambda}(\xi) = \left(\frac{2\lambda}{2 + (\lambda^2 - 1)(1 - \cos \mathrm{dist}_{g_{\mathbb{S}^n}}(\xi, \xi_0))}\right)^{(n-2\sigma)/2}, \quad \xi \in \mathbb{S}^n, \tag{1.7}$$

for some $\xi_0 \in \mathbb{S}^n$ and positive constant λ. This classification can be found in [77], [38] and [73]. In general, (1.6) may have no positive solution, since if v is a positive solution of (1.6) with $K \in C^1(\mathbb{S}^n)$ then it has to satisfy the Kazdan–Warner type condition

$$\int_{\mathbb{S}^n} \langle \nabla_{g_{\mathbb{S}^n}} K, \nabla_{g_{\mathbb{S}^n}} \xi \rangle v^{2n/(n-2\sigma)} \, d\xi = 0. \tag{1.8}$$

Consequently, if $K(\xi) = \xi_{n+1} + 2$, (1.6) has no solutions. The proof of (1.8) is provided in Appendix A.1.

In this paper and a subsequent one [66], we study (1.6) with $\sigma \in (0, 1)$, the *fractional Nirenberg problem*. A fractional Yamabe flow on $\mathbb{S}^n$ will be studied in [67]. Throughout the paper, we assume that $\sigma \in (0, 1)$ and $n \geq 2$ without otherwise stated.

Definition 1.1. For $d > 0$, we say that $K \in C(\mathbb{S}^n)$ has *flatness order greater than d* at ξ if, in some local coordinate system $\{y_1, \ldots, y_n\}$ centered at ξ, there exists a neighborhood $\mathcal{O}$ of 0 such that $K(y) = K(0) + o(|y|^d)$ in $\mathcal{O}$.

Theorem 1.2. *Let* $K \in C^{1,1}(\mathbb{S}^n)$ *be an antipodally symmetric function, i.e.,* $K(\xi) = K(-\xi)$ *for all* $\xi \in \mathbb{S}^n$, *and suppose K is positive somewhere on* $\mathbb{S}^n$. *If there exists a maximum point of K at which K has flatness order greater than* $n - 2\sigma$, *then* (1.6) *has at least one positive* C^2 *solution.*

For $2 \leq n < 2 + 2\sigma$, $K \in C^{1,1}(\mathbb{S}^n)$ has flatness order greater than $n - 2\sigma$ at every maximum point. When $\sigma = 1$, the above theorem was proved by Escobar and Schoen [49] for $n \geq 3$.

Theorem 1.3. *Suppose that* $K \in C^{1,1}(\mathbb{S}^n)$ *is a positive function such that for every critical point* ξ_0 *of K, in some geodesic normal coordinates* $\{y_1, \ldots, y_n\}$ *centered at* ξ_0, *there exist a small neighborhood* $\mathcal{O}$ *of 0 and positive constants* $\beta = \beta(\xi_0) \in (n - 2\sigma, n)$ *and* $\gamma \in (n - 2\sigma, \beta]$ *such that* $K \in C^{[\gamma], \gamma - [\gamma]}(\mathcal{O})$ *(where* $[\gamma]$ *is the integer part of* γ*) and*

$$K(y) = K(0) + \sum_{j=1}^{n} a_j |y_j|^\beta + R(y) \quad \text{in } \mathcal{O},$$

where $a_j = a_j(\xi_0) \neq 0$, $\sum_{j=1}^{n} a_j \neq 0$, *and* $R \in C^{[\beta]-1,1}(\mathcal{O})$ *satisfies* $\sum_{s=0}^{[\beta]} |\nabla^s R(y)| |y|^{-\beta+s} \to 0$ *as* $y \to 0$. *If*

$$\sum_{\xi \in \mathbb{S}^n : \nabla_{g_{\mathbb{S}^n}} K(\xi) = 0, \, \sum_{j=1}^{n} a_j(\xi) < 0} (-1)^{i(\xi)} \neq (-1)^n,$$

where

$$i(\xi) = \#\{a_j(\xi) : \nabla_{g_{\mathbb{S}^n}} K(\xi) = 0, \, a_j(\xi) < 0, \, 1 \leq j \leq n\},$$

then (1.6) *has at least one positive* C^2 *solution. Moreover, there exists a positive constant C depending only on n, σ and K such that for all positive C^2 solutions v of* (1.6),

$$1/C \leq v \leq C \quad \text{and} \quad \|v\|_{C^2(\mathbb{S}^n)} \leq C.$$

For $n = 3$, $\sigma = 1$, the existence part of the above theorem was established by Bahri and Coron [7], and the compactness part was shown in Chang, Gursky and Yang [30], and Schoen and Zhang [91]. For $n \geq 4$, $\sigma = 1$, the above theorem was proved by Li [71].

We now consider a class of functions K more general than that in Theorem 1.3, which is modified from [71].

Definition 1.4. For any real number $\beta > 1$, we say that a sequence $\{K_i\}$ of functions satisfies *condition* $(*)'_\beta$ for some sequence of constants $L(\beta, i)$ in some region Ω_i if $K_i \in C^{[\beta], \beta - [\beta]}(\Omega_i)$ satisfies

$$[\nabla^{[\beta]} K_i]_{C^{\beta - [\beta]}(\Omega_i)} \leq L(\beta, i),$$

and, if $\beta \geq 2$,

$$|\nabla^s K_i(y)| \leq L(\beta, i)|\nabla K_i(y)|^{(\beta - s)/(\beta - 1)}$$

for all $2 \leq s \leq [\beta]$, $y \in \Omega_i$, $\nabla K_i(y) \neq 0$.

Note that the function K in Theorem 1.3 satisfies condition $(*)'_\beta$.

Remark 1.5. For $1 \leq \beta_1 \leq \beta_2$, if $\{K_i\}$ satisfies $(*)'_{\beta_2}$ for some sequence of constants $\{L(\beta_2, i)\}$ in some regions Ω_i, then $\{K_i\}$ satisfies $(*)'_{\beta_1}$ for $\{L(\beta_1, i)\}$, where

$$L(\beta_1, i) = \begin{cases} L(\beta_2, i) \max\left(\max_{2 \leq s \leq [\beta_1]} \|\nabla K_i\|_{L^\infty(\Omega_i)}^{\frac{\beta_2 - s}{\beta_2 - 1} - \frac{\beta_1 - s}{\beta_1 - 1}}, \operatorname{diam}(\Omega_i)^{\beta_2 - \beta_1}\right) & \text{if } [\beta_2] = [\beta_1], \\[3mm] L(\beta_2, i) \max\left(\max_{2 \leq s \leq [\beta_1]} \|\nabla K_i\|_{L^\infty(\Omega_i)}^{\frac{\beta_2 - s}{\beta_2 - 1} - \frac{\beta_1 - s}{\beta_1 - 1}}, \|\nabla K_i\|_{L^\infty(\Omega_i)}^{\frac{\beta_2 - [\beta_1] - 1}{\beta_2 - 1}} \operatorname{diam}(\Omega_i)^{1 + [\beta_1] - \beta_1}\right) \\ \hfill \text{if } [\beta_2] > [\beta_1], \end{cases}$$

in the corresponding regions.

The following theorem gives a priori bounds of solutions in $L^{2n/(n-2\sigma)}$ norm.

Theorem 1.6. *Let $K \in C^{1,1}(\mathbb{S}^n)$ be a positive function. If there exists some constant $d > 0$ such that K satisfies $(*)'_{n-2\sigma}$ for some constant $L > 0$ in $\Omega_d := \{\xi \in \mathbb{S}^n : |\nabla_{g_{\mathbb{S}^n}} K(\xi)| < d\}$, then for every positive solution $v \in C^2(\mathbb{S}^n)$ of (1.6),*

$$\|v\|_{L^{2n/(n-2\sigma)}(\mathbb{S}^n)} \leq C, \tag{1.9}$$

where C depends only on n, σ, $\inf_{\mathbb{S}^n} K$, $\|K\|_{C^{1,1}(\mathbb{S}^n)}$, L, and d.

For $n = 3$, $\sigma = 1$, the above theorem was proved by Chang, Gursky and Yang [30] and by Schoen and Zhang [91]. For $n \geq 4$, $\sigma = 1$, it was proved by Li [71].

Denote by $H^\sigma(\mathbb{S}^n)$ the closure of $C^\infty(\mathbb{S}^n)$ under the norm

$$\int_{\mathbb{S}^n} v P_\sigma(v) \, d \operatorname{vol}_{g_{\mathbb{S}^n}}.$$

The estimate (1.9) for the solution v is equivalent to

$$\|v\|_{H^\sigma(\mathbb{S}^n)} \le C.$$

However, the estimate (1.9) is not sufficient to imply an L^∞ bound for v on $\mathbb{S}^n$. For instance,

$$\int_{\mathbb{S}^n} v_{\xi_0, \lambda}(\xi)^{2n/(n-2\sigma)} \, d\,\mathrm{vol}_{g_{\mathbb{S}^n}} = \int_{\mathbb{S}^n} d\,\mathrm{vol}_{g_{\mathbb{S}^n}},$$

but $v_{\xi_0, \lambda}(\xi_0) = \lambda^{(n-2\sigma)/2} \to \infty$ as $\lambda \to \infty$. Furthermore, a sequence of solutions v_i may blow up at more than one point, and this is the case when $\sigma = 1$ (see [72]). The following theorem shows that the latter situation does not happen when K satisfies a somewhat stronger condition.

Theorem 1.7. *Suppose that $\{K_i\} \in C^{1,1}(\mathbb{S}^n)$ is a sequence of positive functions with uniform $C^{1,1}$ norm and $1/A_1 \le K_i \le A_1$ on $\mathbb{S}^n$ for some $A_1 > 0$ independent of i. Suppose also that $\{K_i\}$ satisfying condition $(*)'_\beta$ for some constants $\beta > n-2\sigma$, $L, d > 0$ in Ω_d. Let $\{v_i\} \in C^2(\mathbb{S}^n)$ be a sequence of corresponding positive solutions of (1.6) and $v_i(\xi_i) = \max_{\mathbb{S}^n} v_i$ for some ξ_i. Then, after passing to a subsequence, either $\{v_i\}$ is bounded in $L^\infty(\mathbb{S}^n)$, or it blows up at exactly one point in the strong sense: There exists a sequence $\{\varphi_i\}$ of Möbius diffeomorphisms from $\mathbb{S}^n$ to $\mathbb{S}^n$ satisfying $\varphi_i(\xi_i) = \xi_i$ and $|\det d\varphi_i(\xi_i)|^{(n-2\sigma)/2n} = v_i^{-1}(\xi_i)$ such that*

$$\|T_{\varphi_i} v_i - 1\|_{C^0(\mathbb{S}^n)} \to 0 \quad \text{as } i \to \infty,$$

where $T_{\varphi_i} v_i := (v \circ \varphi_i)|\det d\varphi_i|^{(n-2\sigma)/2n}$.

For $n = 3$, $\sigma = 1$, the above theorem was established by Chang, Gursky and Yang [30] and by Schoen and Zhang [91]. For $n \ge 4$, $\sigma = 1$, it was proved by Li [71].

Möbius diffeomorphisms φ from $\mathbb{S}^n$ to $\mathbb{S}^n$ are those given by $\varphi = \phi \circ F$, where ϕ is a Möbius transformation from $\mathbb{R}^n \cup \{\infty\}$ to $\mathbb{R}^n \cup \{\infty\}$ generated by translations, multiplications by nonzero constants and the inversion $x \mapsto x/|x|^2$.

Our local analysis of solutions of (1.6) relies on a localization method introduced by Caffarelli and Silvestre [21] for the fractional Laplacian $(-\Delta)^\sigma$ on the Euclidean space $\mathbb{R}^n$, through which (1.6) is connected to a degenerate elliptic differential equation in one dimension higher,

$$
\begin{cases}
\mathrm{div}\left(t^{1-2\sigma} \nabla U(x,t)\right) = 0 & \forall (x,t) \in \mathbb{R}^{n+1}_+, \\
-\lim_{t\to 0} t^{1-2\sigma} \partial_t U(x,t) = K(x) U(x,0)^{\frac{n+2\sigma}{n-2\sigma}} & \forall x \in \partial\mathbb{R}^{n+1}_+.
\end{cases}
\tag{1.10}
$$

We refer to Section 2 for more details.

We would also like to remark that when $\sigma = 1/2$, this fractional Nirenberg problem is equivalent to prescribing mean curvature on $\partial\mathbb{B}^{n+1}$ and zero scalar curvature in $\mathbb{B}^{n+1}$, as studied in, e.g., [31, 45, 47, 48], where the equations are without weights, and thus elliptic. This connection can be seen from (1.10), or from [29] which is in a general setting. The proofs of Theorems 1.6 and 1.7 make use of a blow up analysis of solutions of (1.6), which is an adaptation of the analysis for $\sigma = 1$ developed in [91] and [71]. Our blow up analysis requires a Liouville type theorem. For the definitions of weak solutions and of

the space $H_{\mathrm{loc}}(t^{1-2\sigma}, \overline{\mathbb{R}^{n+1}_+})$ in the following theorem we refer to Definition 2.2 and the beginning of Section 3.

Theorem 1.8. *Let $U \in H_{\mathrm{loc}}(t^{1-2\sigma}, \overline{\mathbb{R}^{n+1}_+})$, with $U \geq 0$ in $\mathbb{R}^{n+1}_+$ and $U \not\equiv 0$, be a weak solution of*

$$\begin{cases} \mathrm{div}(t^{1-2\sigma} \nabla U(x,t)) = 0 & in \ \mathbb{R}^{n+1}_+, \\ -\lim_{t \to 0} t^{1-2\sigma} \partial_t U(x,t) = U(x,0)^{\frac{n+2\sigma}{n-2\sigma}} & on \ \mathbb{R}^n. \end{cases} \tag{1.11}$$

Then $U(x,0)$ takes the form

$$(N_\sigma c(n,\sigma) 2^{2\sigma})^{(n-2\sigma)/4\sigma} \left(\frac{\lambda}{1 + \lambda^2 |x - x_0|^2} \right)^{(n-2\sigma)/2}, \tag{1.12}$$

where $\lambda > 0$, $x_0 \in \mathbb{R}^n$, $c(n,\sigma)$ is the constant in (1.6) and $N_\sigma = 2^{1-2\sigma} \Gamma(1-\sigma)/\Gamma(\sigma)$. Moreover,

$$U(x,t) = \int_{\mathbb{R}^n} \mathcal{P}_\sigma(x-y,t) U(y,0) \, dy$$

for $(x,t) \in \mathbb{R}^{n+1}_+$, where $\mathcal{P}_\sigma(x)$ is the kernel given in (2.2).

Remark 1.9. If we replace $U(x,0)^{\frac{n+2\sigma}{n-2\sigma}}$ by $U(x,0)^p$ for $0 \leq p < \frac{n+2\sigma}{n-2\sigma}$ in (1.11), then the only nonnegative solution of (1.11) is $U \equiv 0$. Moreover, for $p < 0$, (1.11) has no positive solution. These can be seen from the proof of Theorem 1.8 with a standard modification (see, e.g., the proof of Theorem 1.2 in [25]). For $\sigma \in (1/2, 1)$ and $1 < p < \frac{n+2\sigma}{n-2\sigma}$, this result has been proved in [13].

Remark 1.10. We do not make any assumption on the behavior of U near ∞. If we assume that $U \in H(t^{1-2\sigma}, \mathbb{R}^{n+1}_+)$, the theorem in the case of $p = \frac{n+2\sigma}{n-2\sigma}$ follows from a classification theorem in [38] and [73], as explained below.

Remark 1.11. When $\sigma = 1/2$, Theorem 1.8 and Remark 1.9 can be found in [62], [63], [76], [84] and [75].

We provide more details for Remark 1.10, which connects Theorem 1.8 to earlier results. If $U \in H(t^{1-2\sigma}, \mathbb{R}^{n+1}_+)$, then $u(\cdot) := U(\cdot,0) \in L^{2n/(n-2\sigma)}(\mathbb{R}^n)$ and $(-\Delta)^\sigma u$ is well defined. Moreover, by a theorem of Caffarelli and Silvestre [21],

$$-\lim_{t \to 0} t^{1-2\sigma} \partial_t U(x,t) = N_\sigma(-\Delta)^\sigma u(x).$$

Thus, by the second line of (1.11), u satisfies

$$N_\sigma(-\Delta)^\sigma u = u^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{on } \mathbb{R}^n. \tag{1.13}$$

With a regularity assumption on u near infinity, equation (1.13) can be, modulo a positive constant multiple, rewritten as the following integral equation:

$$u(x) = \int_{\mathbb{R}^n} \frac{u(y)^{\frac{n+2\sigma}{n-2\sigma}}}{|x-y|^{n-2\sigma}} \, dy, \quad \forall x \in \mathbb{R}^n. \tag{1.14}$$

Equation (1.14) is, modulo a positive constant multiple, the Euler–Lagrange equation for maximal functions, $f := u^{\frac{n+2\sigma}{n-2\sigma}}$, for the Hardy–Littlewood–Sobolev inequality on $\mathbb{R}^n$:

$$\left\| \int_{\mathbb{R}^n} \frac{f(y)}{|\cdot - y|^\lambda} \, dy \right\|_{L^q(\mathbb{R}^n)} \leq N_{p,\lambda,n} \|f\|_{L^p(\mathbb{R}^n)}, \tag{1.15}$$

with $N_{p,\lambda,n}$ being the sharp constant and $p = 2n/(n + 2\sigma)$, $q = 2n/(n - 2\sigma)$ and $\lambda = n - 2\sigma$. Lieb [77] proved that all maximizers are, modulo a positive constant multiple, given by (1.12). Lieb also raised the question [77, p. 361] of the (essential) uniqueness of solutions of (1.14). Chen, Li and Ou [38] proved that all $L^\infty_{\text{loc}}(\mathbb{R}^n)$ solutions of (1.14) are given by (1.12), while the regularity assumption $u \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ was weakened by Li [73] to $u \in L^{2n/(n-2\sigma)}_{\text{loc}}(\mathbb{R}^n)$.

Since presenting the proofs of all our results requires much space, we leave the proofs of Theorem 1.2 and the existence part of Theorem 1.3 to the subsequent paper [66]. The needed ingredients for the proof of the existence part of Theorem 1.3 are all developed in this paper. With these ingredients, the existence part of Theorem 1.3 follows from a perturbation result and a degree argument which are given in [66].

The present paper is organized as follows. In Section 2 we derive some properties of solutions of fractional Laplacian equations. In particular we prove that local Schauder estimates hold for positive solutions. In Section 3, using the method of moving spheres, we establish Theorem 1.8. This Liouville type theorem and the local Schauder estimates are used in the blow up analysis of solutions of (1.6). In Section 4 we establish accurate blow up profiles of solutions of (1.6) near isolated blow up points. In fact, most of the estimates also hold for subcritical approximations to such equations as well as in bounded domains of $\mathbb{R}^n$. In Section 5, we provide $H^\sigma(\mathbb{S}^n)$ norm a priori estimates, at most one isolated simple blow up point, and $L^\infty(\mathbb{S}^n)$ norm a priori estimates for solutions of (1.6) under appropriate hypotheses on K. The proofs of Theorem 1.3, 1.6 and 1.7 are given in that section. In the Appendix we provide a Kazdan–Warner identity, Lemma 4.10 that is in the same spirit of the classical Bôcher theorem, two lemmas on maximum principles and some complements.

2. Preliminaries

2.1. A weighted Sobolev space

Let $\sigma \in (0, 1)$, $X = (x, t) \in \mathbb{R}^{n+1}$ where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then $|t|^{1-2\sigma}$ belongs to the Muckenhoupt A_2 class in $\mathbb{R}^{n+1}$, that is, there exists a positive constant C such that for every ball $\mathcal{B} \subset \mathbb{R}^{n+1}$,

$$\left(\frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} |t|^{1-2\sigma} \, dX \right) \left(\frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} |t|^{2\sigma-1} \, dX \right) \leq C.$$

Let D be an open set in $\mathbb{R}^{n+1}$. Denote by $L^2(|t|^{1-2\sigma}, D)$ the Banach space of all measurable functions U, defined on D, for which

$$\|U\|_{L^2(|t|^{1-2\sigma}, D)} := \left(\int_D |t|^{1-2\sigma} U^2 \, dX \right)^{1/2} < \infty.$$

We say that $U \in H(|t|^{1-2\sigma}, D)$ if $U \in L^2(|t|^{1-2\sigma}, D)$, and its weak derivatives ∇U exist and belong to $L^2(|t|^{1-2\sigma}, D)$. The norm of U in $H(|t|^{1-2\sigma}, D)$ is given by

$$\|U\|_{H(|t|^{1-2\sigma}, D)} := \left(\int_D |t|^{1-2\sigma} U^2(X) \, dX + \int_D |t|^{1-2\sigma} |\nabla U(X)|^2 \, dX \right)^{1/2}.$$

It is clear that $H(|t|^{1-2\sigma}, D)$ is a Hilbert space with the inner product

$$\langle U, V \rangle := \int_D |t|^{1-2\sigma} (UV + \nabla U \nabla V) \, dX.$$

Note that $C^\infty(D)$ is dense in $H(|t|^{1-2\sigma}, D)$. Moreover, if D is a domain, i.e. a bounded connected open set, with Lipschitz boundary ∂D, then there exists a bounded linear extension operator from $H(|t|^{1-2\sigma}, D)$ to $H(|t|^{1-2\sigma}, \mathbb{R}^{n+1})$ (see, e.g., [41]).

Let Ω be an open set in $\mathbb{R}^n$. Recall that $H^\sigma(\Omega)$ is the fractional Sobolev space defined as

$$H^\sigma(\Omega) := \left\{ u \in L^2(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{n/2+\sigma}} \in L^2(\Omega \times \Omega) \right\}$$

with the norm

$$\|u\|_{H^\sigma(\Omega)} := \left(\int_\Omega u^2 \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\sigma}} \, dx \, dy \right)^{1/2}.$$

Then $C^\infty(\Omega)$ is dense in $H^\sigma(\Omega)$. If Ω is a domain with Lipschitz boundary, then there exists a bounded linear extension operator from $H^\sigma(\Omega)$ to $H^\sigma(\mathbb{R}^n)$. Note that $H^\sigma(\mathbb{R}^n)$ with the norm $\| \cdot \|_{H^\sigma(\mathbb{R}^n)}$ is equivalent to the space

$$\{u \in L^2(\mathbb{R}^n) : |\xi|^\sigma \mathscr{F}(u)(\xi) \in L^2(\mathbb{R}^n)\}$$

with the norm

$$\| \cdot \|_{L^2(\mathbb{R}^n)} + \| |\xi|^\sigma \mathscr{F}(\cdot)(\xi) \|_{L^2(\mathbb{R}^n)},$$

where $\mathscr{F}$ denotes the Fourier transform operator. It is known (see, e.g., [79]) that there exists $C > 0$ depending only on n and σ such that for $U \in H(t^{1-2\sigma}, \mathbb{R}_+^{n+1}) \cap C(\overline{\mathbb{R}_+^{n+1}})$, $\|U(\cdot, 0)\|_{H^\sigma(\mathbb{R}^n)} \leq C \|U\|_{H(t^{1-2\sigma}, \mathbb{R}_+^{n+1})}$. Hence by a standard density argument, every $U \in H(t^{1-2\sigma}, \mathbb{R}_+^{n+1})$ has a well-defined trace $u := U(\cdot, 0) \in H^\sigma(\mathbb{R}^n)$.

We define $\dot{H}^\sigma(\mathbb{R}^n)$ as the closure of the set $C_c^\infty(\mathbb{R}^n)$ of compactly supported smooth functions under the norm

$$\|u\|_{\dot{H}^\sigma(\mathbb{R}^n)} = \| |\xi|^\sigma \mathscr{F}(u)(\xi) \|_{L^2(\mathbb{R}^n)}.$$

Then there exists a constant C depending only on n and σ such that

$$\|u\|_{L^{2n/(n-2\sigma)}(\mathbb{R}^n)} \leq C \|u\|_{\dot{H}^\sigma(\mathbb{R}^n)} \qquad \text{for all } u \in C_c^\infty(\mathbb{R}^n). \tag{2.1}$$

For $u \in \dot{H}^\sigma(\mathbb{R}^n)$, set

$$U(x, t) = \mathcal{P}_\sigma[u] := \int_{\mathbb{R}^n} \mathcal{P}_\sigma(x - \xi, t) u(\xi) \, d\xi, \qquad (x, t) \in \mathbb{R}_+^{n+1} := \mathbb{R}^n \times (0, \infty), \tag{2.2}$$

where

$$\mathcal{P}_\sigma(x,t) = \beta(n,\sigma) \frac{t^{2\sigma}}{(|x|^2 + t^2)^{(n+2\sigma)/2}}$$

with a constant $\beta(n,\sigma)$ such that $\int_{\mathbb{R}^n} \mathcal{P}_\sigma(x,1)\,dx = 1$. Then $U \in L^2(t^{1-2\sigma}, K)$ for any compact set K in $\overline{\mathbb{R}_+^{n+1}}$, $\nabla U \in L^2(t^{1-2\sigma}, \mathbb{R}_+^{n+1})$ and $U \in C^\infty(\mathbb{R}_+^{n+1})$. Moreover, U satisfies (see [21])

$$\mathrm{div}(t^{1-2\sigma}\nabla U) = 0 \quad \text{in } \mathbb{R}_+^{n+1}, \tag{2.3}$$

$$\|\nabla U\|_{L^2(t^{1-2\sigma}, \mathbb{R}_+^{n+1})} = N_\sigma \|u\|_{\dot{H}^\sigma(\mathbb{R}^n)}, \tag{2.4}$$

and

$$-\lim_{t \to 0} t^{1-2\sigma} \partial_t U(x,t) = N_\sigma (-\Delta)^\sigma u(x) \quad \text{in } \mathbb{R}^n \tag{2.5}$$

in the distribution sense, where N_σ is the constant in (1.12). We refer to $U = \mathcal{P}_\sigma[u]$ in (2.2) as the *extension* of $u \in \dot{H}^\sigma(\mathbb{R}^n)$.

For a domain $D \subset \mathbb{R}^{n+1}$ with boundary ∂D, we denote by $\partial' D$ the interior of $\overline{D} \cap \partial\mathbb{R}_+^{n+1}$ in $\mathbb{R}^n = \partial\mathbb{R}_+^{n+1}$, and we set $\partial'' D = \partial D \setminus \partial' D$.

Proposition 2.1. *Let $D = \Omega \times (0, R) \subset \mathbb{R}^n \times \mathbb{R}_+$ with $R > 0$ and $\partial\Omega$ Lipschitz.*

(i) *If $U \in H(t^{1-2\sigma}, D) \cap C(D \cup \partial' D)$, then $u := U(\cdot, 0) \in H^\sigma(\Omega)$, and*

$$\|u\|_{H^\sigma(\Omega)} \le C\|U\|_{H(t^{1-2\sigma}, D)},$$

where C is a positive constant depending only on n, σ, R and Ω. Hence every $U \in H(t^{1-2\sigma}, D)$ has a well-defined trace $U(\cdot, 0) \in H^\sigma(\Omega)$ on $\partial' D$. Furthermore, there exists $C_{n,\sigma} > 0$ depending only on n and σ such that

$$\|U(\cdot, 0)\|_{L^{2n/(n-2\sigma)}(\Omega)} \le C_{n,\sigma} \|\nabla U\|_{L^2(t^{1-2\sigma}, D)} \quad \text{for all } U \in C_c^\infty(D \cup \partial' D). \tag{2.6}$$

(ii) *If $u \in H^\sigma(\Omega)$, then there exists $U \in H(t^{1-2\sigma}, D)$ such that the trace of U on Ω equals u and*

$$\|U\|_{H(t^{1-2\sigma}, D)} \le C\|u\|_{H^\sigma(\Omega)},$$

where C is a positive constant depending only on n, σ, R and Ω.

Proof. The above results are well-known and here we just sketch the proofs. For (i), by the previously mentioned result on the extension operator, there exists $\tilde{U} \in H(t^{1-2\sigma}, \mathbb{R}^{n+1})$ such that $\tilde{U} = U$ in D and

$$\|\tilde{U}\|_{H(t^{1-2\sigma}, \mathbb{R}^{n+1})} \le C\|U\|_{H(t^{1-2\sigma}, D)}.$$

Hence by the above mentioned result on the trace from $H(t^{1-2\sigma}, \mathbb{R}_+^{n+1})$ to $H^\sigma(\mathbb{R}^n)$, we have

$$\|u\|_{H^\sigma(\Omega)} \le \|\tilde{U}(\cdot, 0)\|_{H^\sigma(\mathbb{R}^n)} \le C\|\tilde{U}\|_{H(t^{1-2\sigma}, \mathbb{R}_+^{n+1})} \le C\|U\|_{H(t^{1-2\sigma}, D)}.$$

For (2.6), we extend U by zero outside of $\overline{D}$ and let V be the extension of $U(\cdot, 0)$ as in (2.2). The inequality (2.6) follows from (2.1), (2.4) and

$$\|\nabla V\|_{L^2(t^{1-2\sigma}, \mathbb{R}_+^{n+1})} \le \|\nabla U\|_{L^2(t^{1-2\sigma}, \mathbb{R}_+^{n+1})},$$

where Lemma A.4 is used in the above inequality.

For (ii), since $\partial\Omega$ is Lipschitz, there exists $\tilde{u} \in H^\sigma(\mathbb{R}^n)$ such that $\tilde{u} = u$ in Ω and $\|\tilde{u}\|_{H^\sigma(\mathbb{R}^n)} \le C\|u\|_{H^\sigma(\Omega)}$. Then $U = \mathcal{P}_\sigma[\tilde{u}]$, the extension of $\tilde{u}$, satisfies (ii). $\qquad\square$

2.2. Weak solutions of degenerate elliptic equations

Let D be a domain in $\mathbb{R}_+^{n+1}$ with $\partial'D \ne \emptyset$. Let $a \in L_{\mathrm{loc}}^{2n/(n+2\sigma)}(\partial'D)$ and $b \in L_{\mathrm{loc}}^1(\partial'D)$. Consider

$$\begin{cases} \mathrm{div}(t^{1-2\sigma}\nabla U(X)) = 0 & \text{in } D, \\ -\lim_{t\to 0^+} t^{1-2\sigma}\partial_t U(x,t) = a(x)U(x,0) + b(x) & \text{on } \partial'D. \end{cases} \quad (2.7)$$

Definition 2.2. We say that $U \in H(t^{1-2\sigma}, D)$ is a *weak solution* (resp. *supersolution*, *subsolution*) of (2.7) in D if for every nonnegative $\Phi \in C_c^\infty(D \cup \partial'D)$,

$$\int_D t^{1-2\sigma}\nabla U \nabla\Phi = (\text{resp. } \ge, \le) \int_{\partial'D}(aU\Phi + b\Phi). \quad (2.8)$$

We denote $Q_R = B_R \times (0, R)$ where $B_R \subset \mathbb{R}^n$ is the ball with radius R and centered at 0.

Proposition 2.3. *Suppose $a \in L^{n/2\sigma}(B_1)$ and $b \in L^{2n/(n+2\sigma)}(B_1)$. Let $U \in H(t^{1-2\sigma}, Q_1)$ be a weak solution of (2.7) in Q_1. There exists $\delta > 0$ depending only on n and σ such that if $\|a^+\|_{L^{n/2\sigma}(B_1)} < \delta$, then there exists a constant C depending only on n, σ and δ such that*

$$\|U\|_{H(t^{1-2\sigma}, Q_{1/2})} \le C(\|U\|_{L^2(t^{1-2\sigma}, Q_1)} + \|b\|_{L^{2n/(n+2\sigma)}(B_1)}).$$

Consequently, if $a \in L^p(B_1)$ for $p > n/2\sigma$, then C depends only on n, σ, $\|a\|_{L^p(B_1)}$.

Proof. Let $\eta \in C_c^\infty(Q_1 \cup \partial'Q_1)$ be a cut-off function which is equal to 1 in $Q_{1/2}$ and is supported in $Q_{3/4}$. By a density argument, we can choose $\eta^2 U$ as a test function in (2.8). Then we have, by the Cauchy–Schwarz inequality,

$$\int_{Q_1} t^{1-2\sigma}\eta^2|\nabla U|^2\, dX \le 4\int_{Q_1} t^{1-2\sigma}|\nabla\eta|^2 U^2\, dX + 2\int_{\partial'Q_1}(a^+(\eta U)^2 + b\eta^2 U)\, dx.$$

By the Hölder inequality and Proposition 2.1,

$$\int_{\partial'Q_1} a^+(\eta U)^2\, dx \le \delta\|\eta U\|_{L^{2n/(n-2\sigma)}(\partial'Q_1)}^2 \le \delta C(n,\sigma)\|\nabla(\eta U)\|_{L^2(t^{1-2\sigma}, Q_1)}^2.$$

By Young's inequality, for all $\varepsilon > 0$,

$$\int_{\partial' Q_1} b\eta^2 U(\cdot, 0)\, dx \leq \varepsilon \|\eta U\|^2_{L^{2n/(n-2\sigma)}(\partial' Q_1)} + C(\varepsilon)\|b\|^2_{L^{2n/(n+2\sigma)}(\partial' Q_1)}$$

$$\leq \varepsilon C(n, \sigma)\|\nabla(\eta U)\|^2_{L^2(t^{1-2\sigma}, Q_1)} + C(\varepsilon)\|b\|^2_{L^{2n/(n+2\sigma)}(\partial' Q_1)}.$$

The first conclusion follows immediately if δ is sufficiently small.

If $a \in L^p(B_1)$, we can choose r small such that $\|a\|_{L^{n/2\sigma}(B_r(x_0))} < \delta$ for any ball $B_r(x_0) \subset B_1$. Then $\hat{U}(x, t) = r^{(n-2\sigma)/2}U(rx + x_0, rt)$ satisfies (2.7) with $\hat{a}(x) = r^{2\sigma}a(rx + x_0)$ and $\hat{b}(x, t) = r^{(n+2\sigma)/2}b(rx + x_0)$ in Q_1. Since $\|\hat{a}\|_{L^{n/2\sigma}(B_1)} < \delta$, applying the above result to $\hat{U}$, we have

$$\|U\|_{H(t^{1-2\sigma}, B_{1/2}\times(0,r/2))} \leq C(\|U\|_{L^2(t^{1-2\sigma}, Q_1)} + \|b\|_{L^{2n/(n+2\sigma)}(B_1)}),$$

where C depends only on n, σ, $\|a\|_{L^\infty(B_1)}$. This, together with the fact that (2.7) is uniformly elliptic in $B_1 \times (r/4, 1)$, finishes the proof. $\qquad\square$

Proposition 2.4. *Suppose that $a \in L^{n/2\sigma}(B_1)$. There exists $\delta > 0$ which depends only on n and σ such that if $\|a^+\|_{L^{n/2\sigma}(B_1)} < \delta$, then for any $b \in L^{2n/(n+2\sigma)}(B_1)$, there exists a unique solution U in $H(t^{1-2\sigma}, Q_1)$ to (2.7) with $U|_{\partial'' Q_1} = 0$.*

Proof. We consider the bilinear form

$$B[U, V] := \int_{Q_1} t^{1-2\sigma} \nabla U \nabla V\, dX - \int_{\partial' Q_1} a U V\, dx, \quad U, V \in \mathcal{A},$$

where $\mathcal{A} := \{U \in H(t^{1-2\sigma}, Q_1) : U|_{\partial'' Q_1} = 0 \text{ in the trace sense}\}$. By Proposition 2.1, it is easy to verify that $B[\cdot, \cdot]$ is bounded and coercive provided δ is sufficiently small. Therefore, the proposition follows from the Riesz representation theorem. $\qquad\square$

Lemma 2.5. *Suppose $U \in H(t^{1-2\sigma}, D)$ is a weak supersolution of (2.7) in D with $a \equiv b \equiv 0$. If $U \geq 0$ on $\partial'' D$ in the trace sense, then $U \geq 0$ in D.*

Proof. Use U^- as a test function to conclude that $U^- \equiv 0$. $\qquad\square$

The following result is a refined version of that in [94]. Such De Giorgi–Nash–Moser type theorems for degenerate equations with Dirichlet boundary conditions have been established in [51].

Proposition 2.6. *Suppose $a, b \in L^p(B_1)$ for some $p > n/2\sigma$.*

(i) *Let $U \in H(t^{1-2\sigma}, Q_1)$ be a weak subsolution of (2.7) in Q_1. Then for all $\nu > 0$,*

$$\sup_{Q_{1/2}} U^+ \leq C(\|U^+\|_{L^\nu(t^{1-2\sigma}, Q_1)} + \|b^+\|_{L^p(B_1)}),$$

where $U^+ = \max(0, U)$, and $C > 0$ depends only on n, σ, p, ν and $\|a^+\|_{L^p(B_1)}$.

(ii) *Let $U \in H(t^{1-2\sigma}, Q_1)$ be a nonnegative weak supersolution of* (2.7) *in Q_1. Then for any $0 < \mu < \tau < 1$, $0 < \nu \leq (n+1)/n$ we have*

$$\inf_{Q_\mu} U + \|b^-\|_{L^p(B_1)} \geq C\|U\|_{L^\nu(t^{1-2\sigma}, Q_\tau)},$$

   where $C > 0$ depends only on n, σ, p, ν, μ, τ and $\|a^-\|_{L^p(B_1)}$.

(iii) *Let $U \in H(t^{1-2\sigma}, Q_1)$ be a nonnegative weak solution of* (2.7) *in Q_1. Then we have the following Harnack inequality:*

$$\sup_{Q_{1/2}} U \leq C\Big(\inf_{Q_{1/2}} U + \|b\|_{L^p(B_1)}\Big), \tag{2.9}$$

   where $C > 0$ depends only on n, σ, p, $\|a\|_{L^p(B_1)}$. Consequently, there exists $\alpha \in (0, 1)$ depending only on n, σ, p, $\|a\|_{L^p(B_1)}$ such that any weak solution U of (2.7) *is in $C^\alpha(\overline{Q_{1/2}})$. Moreover,*

$$\|U\|_{C^\alpha(\overline{Q_{1/2}})} \leq C(\|U\|_{L^\infty(Q_1)} + \|b\|_{L^p(B_1)}),$$

   where $C > 0$ depends only on n, σ, p, $\|a\|_{L^p(B_1)}$.

Proof. The proofs are modifications of those in [94], where the method of Moser iteration is used. Here we only point out the changes. Let $k = \|b^+\|_{L^p(B_1)}$ if $b^+ \not\equiv 0$, otherwise let $k > 0$ be any number, and we will eventually let $k \to 0$. Define $\overline{U} = U^+ + k$ and, for $m > 0$, let

$$\overline{U}_m = \begin{cases} \overline{U} & \text{if } U < m, \\ k + m & \text{if } U \geq m. \end{cases}$$

Consider the test function

$$\phi = \eta^2(\overline{U}_m^\beta \overline{U} - k^{\beta+1}) \in H(t^{1-2\sigma}, Q_1)$$

for some $\beta \geq 0$ and some nonnegative function $\eta \in C_c^1(Q_1 \cup \partial' Q_1)$. Direct calculations show that, if we set $W = \overline{U}_m^{\beta/2}\overline{U}$, then

$$\frac{1}{1+\beta} \int_{Q_1} t^{1-2\sigma}|\nabla(\eta W)|^2 \leq 16 \int_{Q_1} t^{1-2\sigma}|\nabla\eta|^2 W^2 + 4 \int_{\partial' Q_1}\Big(a^+ + \frac{b^+}{k}\Big)\eta^2 W^2. \tag{2.10}$$

By Hölder's inequality and the choice of k, we have

$$\int_{\partial' Q_1}\Big(a^+ + \frac{b^+}{k}\Big)\eta^2 W^2 \leq (\|a^+\|_{L^p(B_1)} + 1)\|\eta^2 W^2\|_{L^{p'}(B_1)},$$

where $p' = p/(p-1) < n/(n-2\sigma)$. Choose $0 < \theta < 1$ such that $1/p' = \theta + (1-\theta)(n-2\sigma)/n$. The interpolation inequality shows that, for any $\varepsilon > 0$,

$$\|\eta^2 W^2\|_{L^{p'}(B_1)} \leq \varepsilon\|\eta W\|_{L^{2n/(n-2\sigma)}(B_1)}^2 + \varepsilon^{-(1-\theta)/\theta}\|\eta^2 W^2\|_{L^1(B_1)}.$$

By the trace embedding inequality of Proposition 2.1, there exists $C > 0$ depending only on n, σ, such that

$$\|\eta W\|^2_{L^{2n/(n-2\sigma)}(B_1)} \le C \int_{Q_1} t^{1-2\sigma} |\nabla(\eta W)|^2.$$

By Lemma 2.3 in [94], there exist $\delta, C > 0$, both depending only on n, σ, such that

$$\|\eta^2 W^2\|_{L^1(B_1)} \le \varepsilon^{1/\theta} \int_{Q_1} t^{1-2\sigma} |\nabla(\eta W)|^2 + \varepsilon^{-\delta/\theta} \int_{Q_1} t^{1-2\sigma} \eta^2 W^2.$$

By choosing ε small, the above inequalities give

$$\int_{Q_1} t^{1-2\sigma} |\nabla(\eta W)|^2 \le C(1+\beta)^{\delta/\theta} \int_{Q_1} t^{1-2\sigma} (\eta^2 + |\nabla\eta|^2) W^2,$$

where C depends only on n, σ and $\|a^+\|_{L^p(B_1)}$. Then the proof of Proposition 3.1 in [94] goes through without any change. This finishes the proof of (i) for $\nu = 2$. Hence (i) also holds for any $\nu > 0$, by standard arguments. For part (ii) we choose $k = \|b^-\|_{L^p(B_1)}$ if $b^- \not\equiv 0$, otherwise let $k > 0$ be any number, and we will eventually let $k \to 0$. We can show that there exists some $\nu_0 > 0$ for which (ii) holds, by exactly the same proof of Proposition 3.2 in [94]. Finally, by using the test function $\phi = \overline{U}^{-\beta} \eta^2$ with $\beta \in (0, 1)$ we can repeat the proof of (i) to conclude (ii) for $0 < \nu \le (n+1)/n$. Part (iii) follows from (i), (ii) and standard elliptic theory. $\square$

Remark 2.7. The Harnack inequality (2.9) without the lower order term b, has been obtained earlier in [19] using a different method.

The above proofs can be improved to yield the following result.

Lemma 2.8. *Suppose $a \in L^{n/2\sigma}(B_1)$, $b \in L^p(B_1)$ with $p > n/2\sigma$ and $U \in H(t^{1-2\sigma}, Q_1)$ is a weak subsolution of (2.7) in Q_1. There exists $\delta > 0$ which depends only on n and σ such that if $\|a^+\|_{L^{n/2\sigma}(B_1)} < \delta$, then*

$$\|U^+(\cdot, 0)\|_{L^q(\partial' Q_{1/2})} \le C(\|U^+\|_{H(t^{1-2\sigma}, Q_1)} + \|b^+\|_{L^p(B_1)}),$$

where $C > 0$ depends only on n, p, σ, δ, and $q = \min\left(\frac{2(n+1)}{n-2\sigma}, \frac{n(p-1)}{(n-2\sigma)p} \cdot \frac{2n}{n-2\sigma}\right)$.

Remark 2.9. Analogous estimates were established for $-\Delta u = au$ in [16, Theorem 2.3] and for $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = a|u|^{p-2}u$ in [4, Lemma 3.1].

Proof of Lemma 2.8. We start from (2.10), where we choose $\beta = \min\left(\frac{2}{n}, \frac{2(2\sigma p - n)}{(n-2\sigma)p}\right)$. By the Hölder inequality and Proposition 2.1,

$$\int_{\partial' Q_1} \left(a^+ + \frac{b^+}{k}\right) \eta^2 W^2 \le \delta \|\eta^2 W^2\|_{L^{n/(n-2\sigma)}(B_1)} + \|\eta^2 W^2\|_{L^{p'}(B_1)}$$

$$\le C(n, \sigma)\delta \int_{Q_1} t^{1-2\sigma} |\nabla(\eta W)|^2 + C_{n,\sigma,p} \|\overline{U}\|_{H(t^{1-2\sigma}, Q_1)}.$$

By Poincaré's inequality of [51], we have

$$\int_{Q_1} t^{1-2\sigma} |\nabla\eta|^2 W^2 \le C_{n,\sigma,p} \|\overline{U}\|_{H(t^{1-2\sigma}, Q_1)}.$$

If δ is sufficiently small, then the above together with (2.10) implies that

$$\int_{Q_1} t^{1-2\sigma} |\nabla(\eta W)|^2 \le C_{n,\sigma,p} \|\overline{U}\|_{H(t^{1-2\sigma}, Q_1)}.$$

Hence it follows from the Hölder inequality and Proposition 2.1 that, by letting $m \to \infty$,

$$\|\overline{U}(\cdot, 0)\|_{L^q(\partial' Q_{1/2})} \le C_{n,\sigma,p} \int_{Q_1} t^{1-2\sigma} |\nabla(\eta W)|^2 \le C_{n,\sigma,p} \|\overline{U}\|_{H(t^{1-2\sigma}, Q_1)}.$$

This finishes the proof. $\qquad\square$

Corollary 2.10. *Suppose that* $K \in L^\infty(B_1)$, $U \in H(t^{1-2\sigma}, Q_1)$ *with* $U \ge 0$ *in* Q_1 *satisfies, for some* $1 \le p \le (n+2\sigma)/(n-2\sigma)$,

$$\begin{cases} \operatorname{div}(t^{1-2\sigma} \nabla U(X)) = 0 & in\ Q_1, \\ -\lim_{t\to 0^+} t^{1-2\sigma} \partial_t U(x,t) = K(x) U(x,0)^p & on\ \partial' Q_1. \end{cases}$$

Then

 (i) $U \in L^\infty_{\mathrm{loc}}(Q_1 \cup \partial' Q_1)$, *and hence* $U(\cdot, 0) \in L^\infty_{\mathrm{loc}}(B_1)$.
 (ii) *There exist* $C > 0$ *and* $\alpha \in (0,1)$ *depending only on* n, σ, p, $\|u\|_{L^\infty(B_{3/4})}$ *and* $\|K\|_{L^\infty(B_{3/4})}$ *such that* $U \in C^\alpha(\overline{Q_{1/2}})$ *and*

$$\|U\|_{H(t^{1-2\sigma}, Q_{1/2})} + \|U\|_{C^\alpha(\overline{Q_{1/2}})} \le C.$$

Note that the regularity of solutions of $-\Delta u = u^{\frac{n+2}{n-2}}$ was proved by Trudinger in [95]. When $\sigma = 1/2$, the above regularity result was proved in [40].

Proof of Corollary 2.10. By Proposition 2.1, $U(\cdot, 0) \in H^\sigma(B_1) \subset L^{2n/(n-2\sigma)}(B_1)$. Thus $U(\cdot, 0)^{p-1} \in L^{n/2\sigma}(B_1)$. Then part (i) follows from Lemma 2.8 and Proposition 2.6. Part (ii) follows from Propositions 2.3 and 2.6. $\qquad\square$

2.3. Local Schauder estimates

Let Ω be a domain in $\mathbb{R}^n$, $a \in L^{2n/(n+2\sigma)}_{\mathrm{loc}}(\Omega)$ and $b \in L^1_{\mathrm{loc}}(\Omega)$. We say $u \in \dot{H}^\sigma(\mathbb{R}^n)$ is a *weak solution* of

$$(-\Delta)^\sigma u = au + b \quad in\ \Omega$$

if for any $\phi \in C^\infty(\mathbb{R}^n)$ supported in Ω,

$$\int_{\mathbb{R}^n} (-\Delta)^{\sigma/2} u (-\Delta)^{\sigma/2} \phi = \int_\Omega (au\phi + b\phi).$$

Then by (2.5), $u \in \dot{H}^\sigma(\mathbb{R}^n)$ is a weak solution of

$$(-\Delta)^\sigma u = \frac{1}{N_\sigma}(au + b) \quad in\ B_1$$

if and only if $U = \mathcal{P}_\sigma[u]$, the extension of u defined in (2.2), is a weak solution of (2.7) in Q_1.

For $\alpha \in (0, 1)$, $C^\alpha(\Omega)$ denotes the standard Hölder space over the domain Ω. For simplicity, we use $C^\alpha(\Omega)$ to denote $C^{[\alpha],\alpha-[\alpha]}(\Omega)$ when $1 < \alpha \notin \mathbb{N}$ (the set of positive integers).

In this part, we shall prove the following local Schauder estimates for nonnegative solutions of a fractional Laplace equation.

Theorem 2.11. *Suppose $a, b \in C^\alpha(B_1)$ with $0 < \alpha \notin \mathbb{N}$. Let $u \in \dot{H}^\sigma(\mathbb{R}^n)$ and $u \geq 0$ in $\mathbb{R}^n$ be a weak solution of*

$$(-\Delta)^\sigma u = au + b \quad \text{in } B_1.$$

Suppose that $2\sigma + \alpha$ is not *an integer. Then $u \in C^{2\sigma+\alpha}(B_{1/2})$. Moreover,*

$$\|u\|_{C^{2\sigma+\alpha}(B_{1/2})} \leq C\Big(\inf_{B_{3/4}} u + \|b\|_{C^\alpha(B_{3/4})}\Big), \tag{2.11}$$

where $C > 0$ depends only on n, σ, α, $\|a\|_{C^\alpha(B_{3/4})}$.

Remark 2.12. If we replace the assumption $u \geq 0$ in $\mathbb{R}^n$ by $u \geq 0$ in B_1, estimate (2.11) may fail (see [68]). Without the sign assumption on u, (2.11) holds with $\inf_{B_{3/4}} u$ replaced by $\|u\|_{L^\infty(\mathbb{R}^n)}$, which is proved in [22], [23] and [24] in a much more general setting of fully nonlinear nonlocal equations.

The following proposition will be used in the proof of Theorem 2.11.

Proposition 2.13. *Let $a, b \in C^k(B_1)$, and $U \in H(t^{1-2\sigma}, Q_1)$ be a weak solution of (2.7) in Q_1, where k is a positive integer. Then*

$$\sum_{i=0}^{k} \|\nabla_x^i U\|_{L^\infty(Q_{1/2})} \leq C(\|U\|_{L^2(t^{1-2\sigma}, Q_1)} + \|b\|_{C^k(B_1)}),$$

where $C > 0$ depends only on n, σ, k, $\|a\|_{C^k(B_1)}$.

Proof. We know from Proposition 2.6 that U is Hölder continuous in $\overline{Q_{8/9}}$. Let $h \in \mathbb{R}^n$ with $|h|$ sufficiently small. Denote $U^h(x, t) = (U(x + h, t) - U(x, t))/|h|$. Then U^h is a weak solution of

$$\begin{cases} \operatorname{div}(t^{1-2\sigma} \nabla U^h(X)) = 0 & \text{in } Q_{8/9}, \\ -\lim_{t \to 0^+} t^{1-2\sigma} \partial_t U^h(x, t) = a(x + h)U^h + a^h U + b^h & \text{on } \partial' Q_{8/9}. \end{cases} \tag{2.12}$$

By Propositions 2.3 and 2.6,

$$\begin{aligned} \|U^h\|_{H(t^{1-2\sigma}, Q_{2/3})} + \|U^h\|_{C^\alpha(\overline{Q_{2/3}})} &\leq C(\|U^h\|_{L^2(t^{1-2\sigma}, Q_{3/4})} + \|b\|_{C^1(B_1)}) \\ &\leq C(\|\nabla U\|_{L^2(t^{1-2\sigma}, Q_{4/5})} + \|b\|_{C^1(B_1)}) \\ &\leq C(\|U\|_{L^2(t^{1-2\sigma}, Q_1)} + \|b\|_{C^1(B_1)}) \end{aligned}$$

for some $\alpha \in (0, 1)$ and positive constant $C > 0$ depending only on n, σ, $\|a\|_{C^1(B_1)}$. Hence $\nabla_x U \in H(t^{1-2\sigma}, Q_{2/3}) \cap C^\alpha(\overline{Q_{2/3}})$, and it is a weak solution of

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla(\nabla_x U)) = 0 & \text{in } Q_{2/3}, \\ -\lim_{t \to 0^+} t^{1-2\sigma}\partial_t(\nabla_x U) = a\nabla_x U + U\nabla_x a + \nabla_x b & \text{on } \partial' Q_{2/3}. \end{cases}$$

Then this proposition follows immediately from Propositions 2.3 and 2.6 for $k = 1$. We can continue this procedure for $k = 2, 3, \ldots$ (by induction). □

To prove Theorem 2.11 we first obtain Schauder estimates for solutions of the equation

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U(X)) = 0 & \text{in } Q_R, \\ -\lim_{t \to 0^+} t^{1-2\sigma}\partial_t U(x, t) = g(x) & \text{on } \partial' Q_R. \end{cases} \tag{2.13}$$

Theorem 2.14. *Let $U \in H(t^{1-2\sigma}, Q_2)$ be a weak solution of (2.13) with $R = 2$ and $g \in C^\alpha(B_2)$ for some $0 < \alpha \notin \mathbb{N}$. If $2\sigma + \alpha$ is not an integer, then $U(\cdot, 0)$ is in $C^{2\sigma+\alpha}(B_{1/2})$. Moreover,*

$$\|U(\cdot, 0)\|_{C^{2\sigma+\alpha}(B_{1/2})} \le C(\|U\|_{L^\infty(Q_2)} + \|g\|_{C^\alpha(B_2)}),$$

where $C > 0$ depends only on n, σ, α.

This theorem together with Proposition 2.6 implies the following

Theorem 2.15. *Let $U \in H(t^{1-2\sigma}, Q_1)$ be a weak solution of (2.7) with $D = Q_1$ and $a, b \in C^\alpha(B_1)$ for some $0 < \alpha \notin \mathbb{N}$. If $2\sigma + \alpha$ is not an integer, then $U(\cdot, 0)$ is in $C^{2\sigma+\alpha}(B_{1/2})$. Moreover,*

$$\|U(\cdot, 0)\|_{C^{2\sigma+\alpha}(B_{1/2})} \le C(\|U\|_{L^\infty(Q_1)} + \|b\|_{C^\alpha(B_1)}),$$

where $C > 0$ depends only on n, σ, α, $\|a\|_{C^\alpha(B_1)}$.

Proof. From Proposition 2.6, U is Hölder continuous in $\overline{Q_{3/4}}$. Theorem 2.15 follows from bootstrap arguments by applying Theorem 2.14 with $g(x) := a(x)U(x, 0) + b(x)$. □

Proof of Theorem 2.14. Our arguments are in the spirit of those in [20] and [74]. We denote by C various constants that depend only on n and σ. Let $\rho = 1/2$, $Q_k = Q_{\rho^k}(0)$, $\partial' Q_k = B_k$, $k = 0, 1, 2, \ldots$. (Note that we have abused notation a bit. Only in this proof we write Q_k, B_k for Q_{ρ^k}, B_{ρ^k}.) We also denote $M = \|g\|_{C^\alpha(B_2)}$. From Proposition 2.6 we already know that U is Hölder continuous in $\overline{Q_0}$. First, we assume that $\alpha \in (0, 1)$.

Step 1. We consider the case of $2\sigma + \alpha < 1$. Let W_k be the unique weak solution (guaranteed by Proposition 2.4) of

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla W_k(X)) = 0 & \text{in } Q_k, \\ -\lim_{t \to 0^+} t^{1-2\sigma}\partial_t W_k(x, t) = g(0) - g(x) & \text{on } \partial' Q_k, \\ W_k(X) = 0 & \text{on } \partial'' Q_k. \end{cases} \tag{2.14}$$

Let $U_k = W_k + U$ in Q_k and $h_{k+1} = U_{k+1} - U_k$ in Q_{k+1}. Then

$$\|W_k\|_{L^\infty(Q_k)} \le CM\rho^{(2\sigma+\alpha)k}. \tag{2.15}$$

Indeed, (2.15) follows by applying Lemma 2.5 to the equation of $\rho^{-2\sigma k}W_k(\rho^k x) \pm (t^{2\sigma} - 3)M\rho^{\alpha k}$ in Q_0. Hence by this weak maximum principle again we have

$$\|h_{k+1}\|_{L^\infty(Q_k)} \le CM\rho^{(2\sigma+\alpha)k}.$$

By Proposition 2.13, we have, for $i = 0, 1, 2, 3$,

$$\|\nabla_x^i h_{k+1}\|_{L^\infty(Q_{k+2})} \le CM\rho^{(2\sigma+\alpha-i)k}. \tag{2.16}$$

Similarly, applying Proposition 2.13 to U_0, we obtain

$$\|\nabla_x^i U_0\|_{L^\infty(Q_2)} \le C(\|U_0\|_{L^\infty(Q_1)} + M) \le C(\|U\|_{L^\infty(Q_0)} + M). \tag{2.17}$$

For any given point z near 0, we have

$$|U(z, 0) - U(0, 0)|$$
$$\le |U_k(0, 0) - U(0, 0)| + |U(z, 0) - U_k(z, 0)| + |U_k(z, 0) - U_k(0, 0)|$$
$$= I_1 + I_2 + I_3.$$

Let k be such that $\rho^{k+4} \le |z| \le \rho^{k+3}$. By (2.15),

$$I_1 + I_2 \le CM\rho^{(2\sigma+\alpha)k} \le CM|z|^{2\sigma+\alpha}.$$

For I_3, by (2.16) and (2.17),

$$I_3 \le |U_0(z, 0) - U_0(0, 0)| + \sum_{j=1}^{k} |h_j(z, 0) - h_j(0, 0)|$$

$$\le C|z|\left(\|\nabla_x U_0\|_{L^\infty(Q_{k+3})} + \sum_{j=1}^{k} \|\nabla_x h_j\|_{L^\infty(Q_{k+3})}\right)$$

$$\le C|z|\left(\|U\|_{L^\infty(Q_0)} + M + M\sum_{j=1}^{k} \rho^{(2\sigma+\alpha-1)j}\right)$$

$$\le C|z|\left(\|U\|_{L^\infty(Q_0)} + M(1 + |z|^{2\sigma+\alpha-1})\right).$$

Thus, for $2\sigma + \alpha < 1$, we have

$$|U(z, 0) - U(0, 0)| \le C(M + \|U\|_{L^\infty(Q_0)})|z|^{2\sigma+\alpha},$$

which finishes the proof of Step 1.

Step 2. For $1 < 2\sigma + \alpha < 2$, the arguments in Step 1 imply that

$$\|\nabla_x U(\cdot, 0)\|_{L^\infty(B_1)} \le C(\|U\|_{L^\infty(Q_0)} + M). \tag{2.18}$$

Applying (2.18) to the equation of W_k we have, together with (2.15),

$$\|\nabla_x W_k(\cdot, 0)\|_{L^\infty(B_{k+1})} \le CM\rho^{(2\sigma+\alpha-1)k}.$$

By (2.16) and (2.17),

$$|\nabla_x U_k(z,0) - \nabla_x U_k(0,0)|$$

$$\leq |\nabla_x U_0(z,0) - \nabla_x U_0(0,0)| + \sum_{j=1}^{k} |\nabla_x h_j(z,0) - \nabla_x h_j(0,0)|$$

$$\leq C|z|\Big(\|\nabla_x^2 U_0\|_{L^\infty(Q_{k+3})} + \sum_{j=1}^{k} \|\nabla_x^2 h_j\|_{L^\infty(Q_{k+3})}\Big)$$

$$\leq C|z|\Big(\|U\|_{L^\infty(Q_0)} + M + M\sum_{j=1}^{k} \rho^{(2\sigma-2+\alpha)j}\Big)$$

$$\leq C|z|\big(\|U\|_{L^\infty(Q_0)} + M(1 + |z|^{2\sigma+\alpha-2})\big).$$

Hence,

$$|\nabla_x U(z,0) - \nabla_x U(0,0)| \leq |\nabla_x W_k(0,0)| + |\nabla_x W_k(z,0)| + |\nabla_x U_k(z,0) - \nabla_x U_k(0,0)|$$

$$\leq CM\rho^{(2\sigma+\alpha-1)k} + C|z|\big(\|U\|_{L^\infty(Q_0)} + M(1 + |z|^{2\sigma+\alpha-2})\big)$$

$$\leq C(M + \|U\|_{L^\infty(Q_0)})|z|^{2\sigma+\alpha-1},$$

which finishes the proof of Step 2.

Step 3. For $2\sigma + \alpha > 2$, the arguments in Step 2 imply that

$$\|\nabla_x^2 U(\cdot,0)\|_{L^\infty(B_1)} \leq C(\|U\|_{L^\infty(Q_0)} + M). \tag{2.19}$$

Applying (2.19) to the equation of W_k we have, together with (2.15),

$$\|\nabla_x^2 W_k(\cdot,0)\|_{L^\infty(B_{k+1})} \leq CM\rho^{(2\sigma+\alpha-2)k}.$$

By (2.16) and (2.17),

$$|\nabla_x^2 U_k(z,0) - \nabla_x^2 U_k(0,0)| \leq |\nabla_x^2 U_0(z,0) - \nabla_x^2 U_0(0,0)| + \sum_{j=1}^{k} |\nabla_x^2 h_j(z,0) - \nabla_x^2 h_j(0,0)|$$

$$\leq C|z|\Big(\|\nabla_x^3 U_0\|_{L^\infty(Q_{k+3})} + \sum_{j=1}^{k} \|\nabla_x^3 h_j\|_{L^\infty(Q_{k+3})}\Big)$$

$$\leq C|z|\Big(\|U\|_{L^\infty(Q_0)} + M + M\sum_{j=1}^{k} \rho^{(2\sigma+\alpha-3)k}\Big)$$

$$\leq C|z|\big(\|U\|_{L^\infty(Q_0)} + M(1 + |z|^{2\sigma+\alpha-3})\big).$$

Hence,

$$|\nabla_x^2 U(z,0) - \nabla_x^2 U(0,0)|$$

$$\leq |\nabla_x^2 W_k(0,0)| + |\nabla_x^2 W_k(z,0)| + |\nabla_x^2 U_k(z,0) - \nabla_x U_k(0,0)|$$

$$\leq CM\rho^{(2\sigma+\alpha-2)k} + C|z|\big(\|U\|_{L^\infty(Q_0)} + M(1 + |z|^{2\sigma+\alpha-3})\big)$$

$$\leq C(M + \|U\|_{L^\infty(Q_0)})|z|^{2\sigma+\alpha-2},$$

which finishes the proof of Step 3. This finishes the proof of Theorem 2.14 for $\alpha \in (0, 1)$.

For $\alpha > 1$, we may apply ∇_x to (2.13) $[\alpha]$ times, as in the proof of Proposition 2.13, and repeat the above three steps. Theorem 2.14 is proved. $\qquad\square$

Proof of Theorem 2.11. Since $u \in \dot{H}^\sigma(\mathbb{R}^n)$ is nonnegative, its extension U is nonnegative in $\mathbb{R}_+^{n+1}$ and $U \in H(t^{1-2\sigma}, Q_1)$ is a weak solution of (2.7) in Q_1. The theorem follows immediately from Theorem 2.15 and Proposition 2.6. $\qquad\square$

Remark 2.16. Another way to show Theorem 2.11 is the following. Let $u \in \dot{H}^\sigma(\mathbb{R}^n)$ and $u \geq 0$ in $\mathbb{R}^n$ be a solution of

$$(-\Delta)^\sigma u = g \quad \text{in } B_1,$$

where $g \in C^\alpha(B_1)$. Let η be a nonnegative smooth cut-off function supported in B_1 and equal to 1 in $B_{7/8}$. Let $v \in \dot{H}^\sigma(\mathbb{R}^n)$ be the solution of

$$(-\Delta)^\sigma v = \eta g \quad \text{in } \mathbb{R}^n,$$

where ηg is considered as a function defined in $\mathbb{R}^n$ and supported in B_1, i.e., v is a Riesz potential of ηg:

$$v(x) = \frac{\Gamma\left(\frac{n-2\sigma}{2}\right)}{2^{2\sigma}\pi^{n/2}\Gamma(\sigma)} \int_{\mathbb{R}^n} \frac{\eta(y)g(y)}{|x-y|^{n-2\sigma}}\, dy.$$

Then if $2\sigma + \alpha$ and α are not integers, we have (see, e.g., [93])

$$\|v\|_{C^{2\sigma+\alpha}(B_{1/2})} \leq C(\|v\|_{L^\infty(\mathbb{R}^n)} + \|\eta g\|_{C^\alpha(\mathbb{R}^n)}) \leq C\|g\|_{C^\alpha(B_1)}.$$

Let $w = u - v$; it belongs to $\dot{H}^\sigma(\mathbb{R}^n)$ and satisfies

$$(-\Delta)^\sigma w = 0 \quad \text{in } B_{7/8}.$$

Let $W = \mathcal{P}_\sigma[w]$ be the extension of w, and $\tilde{W} = W + \|v\|_{L^\infty(\mathbb{R}^n)} \geq 0$ in $\mathbb{R}_+^{n+1}$. Notice that $\tilde{W}$ is a nonnegative weak solution of (2.7) with $a \equiv b \equiv 0$ and $D = Q_1$. By Propositions 2.13 and 2.6, we have

$$\|w + \|v\|_{L^\infty(\mathbb{R}^n)}\|_{C^{2\sigma+\alpha}(B_{1/2})} \leq C\|\tilde{W}\|_{L^2(t^{1-2\sigma}, Q_{7/8})} \leq C \inf_{Q_{3/4}} \tilde{W}$$

$$\leq C\left(\inf_{Q_{3/4}} u + \|v\|_{L^\infty(\mathbb{R}^n)}\right).$$

Hence

$$\|u\|_{C^{2\sigma+\alpha}(B_{1/2})} \leq \|v\|_{C^{2\sigma+\alpha}(B_{1/2})} + \|w\|_{C^{2\sigma+\alpha}(B_{1/2})} \leq C\left(\inf_{B_{3/4}} u + \|g\|_{C^\alpha(B_1)}\right).$$

Using bootstrap arguments as in the proof of Theorem 2.15, we conclude the proof of Theorem 2.11.

Remark 2.17. In fact, our proofs also yield the following. If we only assume that a, b, g are in $L^\infty(B_1)$, and let U, u be those of Theorems 2.14 and 2.11 respectively, then the estimates

$$\|U(\cdot, 0)\|_{C^{2\sigma}(B_{1/2})} \leq C_1(\|U\|_{L^\infty(Q_1)} + \|g\|_{L^\infty(B_1)}),$$

$$\|u\|_{C^{2\sigma}(B_{1/2})} \leq C_2\Big(\inf_{B_{3/4}} u + \|b\|_{L^\infty(B_{3/4})}\Big)$$

hold provided $\sigma \neq 1/2$, where $C_1 > 0$ depends only on n, σ, α, and $C_2 > 0$ depends only on $n, \sigma, \alpha, \|a\|_{L^\infty(B_{3/4})}$. For $\sigma = 1/2$, we have the following log-Lipschitz property: for any $y_1, y_2 \in B_{1/4}$, $y_1 \neq y_2$,

$$\frac{|U(y_1, 0) - U(y_2, 0)|}{|y_1 - y_2|} \leq C_1\big(\|U\|_{L^\infty(Q_1)} - \|g\|_{L^\infty(B_1)} \log |y_1 - y_2|\big),$$

$$\frac{|u(y_1) - u(y_2)|}{|y_1 - y_2|} \leq -C_2 \log |y_1 - y_2|\Big(\inf_{B_{3/4}} u + \|b\|_{L^\infty(B_{3/4})}\Big),$$

where $C_1 > 0$ depends only on n, σ, and $C_2 > 0$ depends only on $n, \sigma, \|a\|_{L^\infty(B_{3/4})}$.

Next we have

Lemma 2.18 ([19, Lemma 4.5]). *Let* $g \in C^\alpha(B_1)$ *for some* $\alpha \in (0, 1)$ *and* $U \in L^\infty(Q_1)$ $\cap H(t^{1-2\sigma}, Q_1)$ *be a weak solution of* (2.13). *Then there exists* $\beta \in (0, 1)$ *depending only on* n, σ, α *such that* $t^{1-2\sigma} \partial_t U \in C^\beta(\overline{Q_{1/2}})$. *Moreover, there exists a positive constant* $C > 0$ *depending only on* n, σ *and* β *such that*

$$\|t^{1-2\sigma} \partial_t U\|_{C^\beta(\overline{Q_{1/2}})} \leq C(\|U\|_{L^\infty(Q_1)} + \|g\|_{C^\alpha(B_1)}).$$

Proposition 2.19. *Suppose that* $K \in C^1(B_1)$, *and* $U \in H(t^{1-2\sigma}, Q_1)$ *with* $U \geq 0$ *in* Q_1 *is a weak solution of*

$$\begin{cases} \operatorname{div}(t^{1-2\sigma} \nabla U) = 0 & \text{in } Q_1, \\ -\lim_{t \to 0} t^{1-2\sigma} \partial_t U(x, t) = K(x) U^p(x, 0) & \text{on } \partial' Q_1, \end{cases} \tag{2.20}$$

where $1 \leq p \leq \frac{n+2\sigma}{n-2\sigma}$. *Then there exist* $C > 0$ *and* $\alpha \in (0, 1)$, *both depending only on* n, $\sigma, p, \|U\|_{L^\infty(Q_1)}, \|K\|_{C^1(Q_1)}$, *such that* $\nabla_x U$ *and* $t^{1-2\sigma} \partial_t U$ *are in* $C^\alpha(\overline{Q_{1/2}})$ *and*

$$\|\nabla_x U\|_{C^\alpha(\overline{Q_{1/2}})} + \|t^{1-2\sigma} \partial_t U\|_{C^\alpha(\overline{Q_{1/2}})} \leq C.$$

Proof. We use C and α to denote various positive constants with dependence specified as in the proposition, which may vary from line to line. By Corollary 2.10, we know that $U \in L^\infty_{\text{loc}}(Q_1 \cup \partial' Q_1)$ and

$$\|U\|_{C^\alpha(\overline{Q_{8/9}})} \leq C.$$

With the above, we may apply Theorem 2.15 to obtain $U(\cdot, 0) \in C^{1,\sigma}(\overline{B_{7/8}})$ and

$$\|U(\cdot, 0)\|_{C^{1,\sigma}(\overline{B_{7/8}})} \leq C.$$

Hence we may differentiate (2.20) with respect to x (which can be justified from the proof of Proposition 2.13) and apply Proposition 2.6 to $\nabla_x U$ to obtain

$$\|\nabla_x U\|_{C^\alpha(\overline{Q_{1/2}})} \leq C.$$

Finally, we can apply Lemma 2.18 to obtain

$$\|t^{1-2\sigma}\partial_t U\|_{C^\alpha(\overline{Q_{1/2}})} \leq C. \qquad \square$$

3. Proof of Theorem 1.8

We first introduce some notations. We write $U \in L^\infty_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+})$ if $U \in L^\infty(\overline{Q_R})$ for any $R > 0$. Similarly, we write $U \in H_{\mathrm{loc}}(t^{1-2\sigma}, \overline{\mathbb{R}^{n+1}_+})$ if $U \in H(t^{1-2\sigma}, \overline{Q_R})$ for any $R > 0$. In the following $\mathcal{B}_R(X)$ is the ball in $\mathbb{R}^{n+1}$ with radius R and center X, $\mathcal{B}_R^+(X)$ is $\mathcal{B}_R(X) \cap \mathbb{R}^{n+1}_+$, and $B_R(x)$ is the ball in $\mathbb{R}^n$ with radius R and center x. We also write $\mathcal{B}_R, \mathcal{B}_R^+, B_R$ for $\mathcal{B}_R(0), \mathcal{B}_R^+(0), B_R(0)$ respectively.

We start with a lemma which is a version of the strong maximum principle.

Proposition 3.1. *Suppose* $U \in H(t^{1-2\sigma}, D_\varepsilon) \cap C(\mathcal{B}_1^+ \cup B_1 \setminus \{0\})$ *with* $U > 0$ *in* $\mathcal{B}_1^+ \cup B_1 \setminus \{0\}$ *is a weak supersolution of* (2.7) *with* $a \equiv b \equiv 0$ *and* $D = D_\varepsilon := \mathcal{B}_1^+ \setminus \overline{\mathcal{B}_\varepsilon^+}$ *for any* $0 < \varepsilon < 1$. *Then*

$$\liminf_{(x,t) \to 0} U(x,t) > 0.$$

Proof. For any $\delta > 0$, let

$$V_\delta = U + \frac{\delta}{|(x,t)|^{n-2\sigma}} - \min_{\partial''\mathcal{B}_{0.8}^+} U.$$

Then V is also a weak supersolution in $D_{\delta^{2/(n-2\sigma)}}$. Applying Lemma 2.5 to V_δ in $D_{\delta^{2/(n-2\sigma)}}$ for sufficiently small δ, we have $V_\delta \geq 0$ in $D_{\delta^{2/(n-2\sigma)}}$. For any $(x,t) \in \mathcal{B}_{0.8}^+ \setminus \{0\}$, we have $\lim_{\delta \to 0} V_\delta(x,t) \geq 0$, i.e., $U(x,t) \geq \min_{\partial''\mathcal{B}_{0.8}^+} U$. $\qquad \square$

The proof of Theorem 1.8 uses the method of moving spheres and is inspired by [76], [75] and [25]. For each $x \in \mathbb{R}^n$ and $\lambda > 0$, we write $\overline{X} = (x,0)$, and define

$$U_{\overline{X},\lambda}(\xi) := \left(\frac{\lambda}{|\xi - \overline{X}|}\right)^{n-2\sigma} U\left(\overline{X} + \frac{\lambda^2(\xi - \overline{X})}{|\xi - \overline{X}|^2}\right), \quad \xi \in \overline{\mathbb{R}^{n+1}_+} \setminus \{\overline{X}\}, \qquad (3.1)$$

the Kelvin transformation of U with respect to the ball $\mathcal{B}_\lambda(\overline{X})$. We point out that if U is a solution of (1.11), then $U_{\bar{x},\lambda}$ is a solution of (1.11) in $\overline{\mathbb{R}^{n+1}_+} \setminus \overline{\mathcal{B}_\varepsilon^+}$, for every $\bar{x} \in \partial\mathbb{R}^{n+1}_+$, $\lambda > 0$, and $\varepsilon > 0$.

By Corollary 2.10 any nonnegative weak solution U of (1.11) belongs to $L^\infty_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+})$, and hence by Proposition 2.6, U is Hölder continuous and positive in $\overline{\mathbb{R}^{n+1}_+}$. By Theorem 2.14, $U(\cdot, 0)$ is smooth in $\mathbb{R}^n$. From classical elliptic theory, U is smooth in $\mathbb{R}^{n+1}_+$.

Lemma 3.2. *For any $x \in \mathbb{R}^n$, there exists a positive constant $\lambda_0(x)$ such that for any $0 < \lambda < \lambda_0(x)$,*

$$U_{\overline{X},\lambda}(\xi) \leq U(\xi) \quad \text{in } \mathbb{R}^{n+1}_+ \setminus \mathcal{B}^+_\lambda(\overline{X}). \tag{3.2}$$

Proof. Without loss of generality we may assume that $x = 0$ and write $U_\lambda = U_{0,\lambda}$.

Step 1. We show that there exist $0 < \lambda_1 < \lambda_2$, which may depend on x, such that

$$U_\lambda(\xi) \leq U(\xi), \quad \forall\, 0 < \lambda < \lambda_1,\, \lambda < |\xi| < \lambda_2.$$

For every $0 < \lambda < \lambda_1 < \lambda_2$ and $\xi \in \partial'' \mathcal{B}_{\lambda_2}$, we have $\lambda^2 \xi / |\xi|^2 \in \mathcal{B}^+_{\lambda_2}$. Thus we can choose $\lambda_1 = \lambda_1(\lambda_2)$ small such that

$$U_\lambda(\xi) = \left(\frac{\lambda}{|\xi|}\right)^{n-2\sigma} U\left(\frac{\lambda^2 \xi}{|\xi|^2}\right) \leq \left(\frac{\lambda_1}{\lambda_2}\right)^{n-2\sigma} \sup_{\overline{\mathcal{B}^+_{\lambda_2}}} U \leq \inf_{\partial'' \mathcal{B}^+_{\lambda_2}} U \leq U(\xi).$$

Hence $U_\lambda \leq U$ on $\partial''(\mathcal{B}^+_{\lambda_2} \setminus \mathcal{B}^+_\lambda)$ for all $\lambda_2 > 0$ and $0 < \lambda < \lambda_1(\lambda_2)$.

We will show that $U_\lambda \leq U$ on $\mathcal{B}^+_{\lambda_2} \setminus \mathcal{B}^+_\lambda$ if λ_2 is small and $0 < \lambda < \lambda_1(\lambda_2)$. Since U_λ satisfies (1.11) in $\mathcal{B}^+_{\lambda_2} \setminus \overline{\mathcal{B}^+_{\lambda_1}}$, we have

$$\begin{cases} \operatorname{div}(t^{1-2\sigma} \nabla(U_\lambda - U)) = 0 & \text{in } \mathcal{B}^+_{\lambda_2} \setminus \overline{\mathcal{B}^+_\lambda}, \\ \lim_{t \to 0} t^{1-2\sigma} \partial_t (U_\lambda - U) = U(x,0)^{\frac{n+2\sigma}{n-2\sigma}} - U_\lambda(x,0)^{\frac{n+2\sigma}{n-2\sigma}} & \text{on } \partial'(\mathcal{B}^+_{\lambda_2} \setminus \overline{\mathcal{B}^+_\lambda}). \end{cases} \tag{3.3}$$

Let $(U_\lambda - U)^+ := \max(0, U_\lambda - U)$, which is 0 on $\partial''(\mathcal{B}^+_{\lambda_2} \setminus \mathcal{B}^+_\lambda)$. Hence, by a density argument, we can use $(U_\lambda - U)^+$ as a test function in the definition of weak solution of (3.3). We will make use of the narrow domain technique from [11]. With the help of the mean value theorem, we have

$$\int_{\mathcal{B}^+_{\lambda_2} \setminus \mathcal{B}^+_\lambda} t^{1-2\sigma} |\nabla(U_\lambda - U)^+|^2 = \int_{B_{\lambda_2} \setminus B_\lambda} (U_\lambda(x,0)^{\frac{n+2\sigma}{n-2\sigma}} - U(x,0)^{\frac{n+2\sigma}{n-2\sigma}})(U_\lambda - U)^+$$

$$\leq C \int_{B_{\lambda_2} \setminus B_\lambda} ((U_\lambda - U)^+)^2 U_\lambda^{4\sigma/(n-2\sigma)}$$

$$\leq C \left(\int_{B_{\lambda_2} \setminus B_\lambda} ((U_\lambda - U)^+)^{2n/(n-2\sigma)}\right)^{(n-2\sigma)/n} \left(\int_{B_{\lambda_2} \setminus B_\lambda} U_\lambda^{2n/(n-2\sigma)}\right)^{2\sigma/n}$$

$$\leq C \left(\int_{\mathcal{B}^+_{\lambda_2} \setminus \mathcal{B}^+_\lambda} t^{1-2\sigma} |\nabla(U_\lambda - U)^+|^2\right) \left(\int_{B_{\lambda_2}} U^{2n/(n-2\sigma)}\right)^{2\sigma/n},$$

where Proposition 2.1 is used in the last inequality and C is a positive constant depending only on n and σ. We fix λ_2 small such that

$$C \left(\int_{B_{\lambda_2}} U^{2n/(n-2\sigma)}\right)^{2\sigma/n} < 1/2.$$

Then $\nabla(U_\lambda - U)^+ = 0$ in $\mathcal{B}_{\lambda_2}^+ \setminus \mathcal{B}_\lambda^+$. Since $(U_\lambda - U)^+ = 0$ on $\partial''(\mathcal{B}_{\lambda_2}^+ \setminus \mathcal{B}_\lambda^+)$, $(U_\lambda - U)^+ = 0$ in $\mathcal{B}_{\lambda_2}^+ \setminus \mathcal{B}_\lambda^+$. We conclude that $U_\lambda \le U$ on $\mathcal{B}_{\lambda_2}^+ \setminus \mathcal{B}_\lambda^+$ for $0 < \lambda < \lambda_1 := \lambda_1(\lambda_2)$.

Step 2. We show that there exists $\lambda_0 \in (0, \lambda_1)$ such that for $0 < \lambda < \lambda_0$

$$U_\lambda(\xi) \le U(\xi), \quad |\xi| > \lambda_2, \, \xi \in \mathbb{R}_+^{n+1}.$$

Let $\phi(\xi) = (\lambda_2/|\xi|)^{n-2\sigma} \inf_{\partial'' \mathcal{B}_{\lambda_2}} U$, which satisfies

$$\begin{cases} \mathrm{div}(t^{1-2\sigma} \nabla \phi) = 0 & \text{in } \mathbb{R}_+^{n+1} \setminus \mathcal{B}_{\lambda_2}^+, \\ -\lim_{t \to 0} t^{1-2\sigma} \partial_t \phi(x, t) = 0 & \text{on } \mathbb{R}^n \setminus \overline{B_{\lambda_2}}, \end{cases}$$

and $\phi(\xi) \le U(\xi)$ on $\partial'' \mathcal{B}_{\lambda_2}$. By the weak maximum principle (Lemma 2.5),

$$U(\xi) \ge \left(\frac{\lambda_2}{|\xi|} \right)^{n-2\sigma} \inf_{\partial'' \mathcal{B}_{\lambda_2}} U, \quad \forall \, |\xi| > \lambda_2, \, \xi \in \mathbb{R}_+^{n+1}.$$

Let $\lambda_0 = \min(\lambda_1, \lambda_2(\inf_{\partial'' \mathcal{B}_{\lambda_2}} U / \sup_{\mathcal{B}_{\lambda_2}} U)^{1/(n-2\sigma)})$. Then for any $0 < \lambda < \lambda_0$ and $|\xi| \ge \lambda_2$, we have

$$U_\lambda(\xi) \le \left(\frac{\lambda}{|\xi|} \right)^{n-2\sigma} U\left(\frac{\lambda^2 \xi}{|\xi|^2} \right) \le \left(\frac{\lambda_0}{|\xi|} \right)^{n-2\sigma} \sup_{\mathcal{B}_{\lambda_2}} U \le \left(\frac{\lambda_2}{|\xi|} \right)^{n-2\sigma} \inf_{\partial'' \mathcal{B}_{\lambda_2}} U \le U(\xi).$$

Lemma 3.2 is proved. □

With Lemma 3.2 we can define, for all $x \in \mathbb{R}^n$,

$$\bar{\lambda}(x) = \sup\{\mu > 0 : U_{\overline{X}, \lambda} \le U \text{ in } \mathbb{R}_+^{n+1} \setminus \mathcal{B}_\lambda^+, \, \forall \, 0 < \lambda < \mu\}.$$

By Lemma 3.2, $\bar{\lambda}(x) \ge \lambda_0(x)$.

Lemma 3.3. *If $\bar{\lambda}(x) < \infty$ for some $x \in \mathbb{R}^n$, then $U_{\overline{X}, \bar{\lambda}(x)} \equiv U$.*

Proof. Without loss of generality we assume that $x = 0$ and write $U_\lambda = U_{0,\lambda}$ and $\bar{\lambda} = \bar{\lambda}(0)$. By the definition of $\bar{\lambda}$, $U_{\bar{\lambda}} \ge U$ in $\overline{\mathcal{B}_{\bar{\lambda}}^+} \setminus \{0\}$, and therefore, for all $0 < \varepsilon < \bar{\lambda}$,

$$\begin{cases} \mathrm{div}(t^{1-2\sigma} \nabla(U_\lambda - U)) = 0 & \text{in } \mathcal{B}_\lambda^+ \setminus \overline{\mathcal{B}_\varepsilon^+}, \\ -\lim_{t \to 0} t^{1-2\sigma} \partial_t(U_\lambda - U) \ge 0 & \text{on } \partial'(\mathcal{B}_\lambda^+ \setminus \overline{\mathcal{B}_\varepsilon^+}). \end{cases} \quad (3.4)$$

Arguing towards a contradiction, if $U_{\bar{\lambda}}$ does not identically equal U, applying the Harnack inequality (Proposition 2.6) to (3.4), we have $U_{\bar{\lambda}} > U$ in $\overline{\mathcal{B}_{\bar{\lambda}}} \setminus \{\{0\} \cup \partial'' \mathcal{B}_{\bar{\lambda}}\}$, and in view of Proposition 3.1,

$$\liminf_{\xi \to 0}(U_{\bar{\lambda}}(\xi) - U(\xi)) > 0.$$

Hence there exist $\varepsilon_1, c > 0$ such that $U_{\bar{\lambda}}(\xi) > U(0) + c$ for $0 < |\xi| < \varepsilon_1$. Choose ε_2 small such that

$$\left(\frac{\bar{\lambda}}{\bar{\lambda} + \varepsilon_2} \right)^{n-2\sigma} (U(0) + c) > U(0) + \frac{c}{2}.$$

Thus for all $0 < |\xi| < \varepsilon_1$ and $\bar{\lambda} < \lambda < \bar{\lambda} + \varepsilon_2$,

$$U_\lambda(\xi) = \left(\frac{\bar{\lambda}}{\lambda}\right)^{n-2\sigma} U_{\bar{\lambda}}\left(\frac{\bar{\lambda}^2\xi}{\lambda^2}\right) \geq \left(\frac{\bar{\lambda}}{\bar{\lambda}+\varepsilon_2}\right)^{n-2\sigma} (U(0)+c) \geq U(0) + c/2.$$

Choose ε_3 small such that for all $0 < |\xi| < \varepsilon_3$, $U(0) > U(\xi) - c/4$. Hence for all $0 < |\xi| < \varepsilon_3$ and $\bar{\lambda} < \lambda < \bar{\lambda} + \varepsilon_2$,

$$U_\lambda(\xi) > U(\xi) + c/4.$$

For δ small, which will be fixed later, denote $K_\delta = \{\xi \in \mathbb{R}^{n+1}_+ : \varepsilon_3 \leq |\xi| \leq \bar{\lambda} - \delta\}$. Then there exists $c_2 = c_2(\delta)$ such that

$$U_{\bar{\lambda}}(X) - U(X) > c_2 \quad \text{in } K_\delta.$$

By the uniform continuity of U on compact sets, there exists $\varepsilon_4 \leq \varepsilon_2$ such that for all $\bar{\lambda} < \lambda < \bar{\lambda} + \varepsilon_4$,

$$U_\lambda - U_{\bar{\lambda}} > -c_2/2 \quad \text{in } K_\delta.$$

Hence

$$U_\lambda - U > c_2/2 \quad \text{in } K_\delta.$$

Now let us focus on the region $\{\xi \in \mathbb{R}^{n+1}_+ : \bar{\lambda} - \delta \leq |\xi| \leq \lambda\}$. Using the narrow domain technique as that in Lemma 3.2, we can choose δ small (notice that we can choose ε_4 as small as we want) such that

$$U_\lambda \geq U \quad \text{in } \{\xi \in \mathbb{R}^{n+1}_+ : \bar{\lambda} - \delta \leq |\xi| \leq \lambda\}.$$

In conclusion, there exists ε_4 such that for all $\bar{\lambda} < \lambda < \bar{\lambda} + \varepsilon_4$,

$$U_\lambda \geq U \quad \text{in } \{\xi \in \mathbb{R}^{n+1}_+ : 0 < |\xi| \leq \lambda\},$$

which contradicts the definition of $\bar{\lambda}$. $\qquad\square$

Proof of Theorem 1.8. It follows from Lemma 3.3 and arguments similar to those in [75] that:

(i) Either $\bar{\lambda}(x) = \infty$ for all $x \in \mathbb{R}^n$, or $\bar{\lambda}(x) < \infty$ for all $x \in \mathbb{R}^n$ [75, Lemma 2.3].
(ii) If for all $x \in \mathbb{R}^n$, $\bar{\lambda}(x) = \infty$ then $U(x, t) = U(0, t)$ for all $(x, t) \in \mathbb{R}^{n+1}_+$ [75, Lemma 11.3].
(iii) If $\bar{\lambda}(x) < \infty$ for all $x \in \mathbb{R}^n$, then by [75, Lemma 11.1],

$$u(x) := U(x, 0) = a\left(\frac{\lambda}{1 + \lambda^2|x - x_0|^2}\right)^{(n-2\sigma)/2}, \tag{3.5}$$

where $\lambda > 0$, $a > 0$ and $x_0 \in \mathbb{R}^n$.

We claim that (ii) never happens, since this would imply, using (1.11), that

$$U(x, t) = U(0) - U(0)^{\frac{n+2\sigma}{n-2\sigma}} \frac{t^{2\sigma}}{2\sigma},$$

which contradicts the positivity of U. Thus (iii) holds.

We are only left to show that $V := U - \mathcal{P}_\sigma[u] \equiv 0$ where $u(x)$ is given in (3.5) and belongs to $\dot{H}^\sigma(\mathbb{R}^n)$. Notice that V satisfies

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla V) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ V = 0 & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases}$$

By Lemma 3.3, we know that $V_{\bar{\lambda}}$ can be extended to be a smooth function near 0. Multiplying the above equation by V and integrating by parts leads to $\int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma}|\nabla V|^2 = 0$. Hence $V \equiv 0$.

Finally, $a = (N_\sigma c(n, \sigma)2^{2\sigma})^{(n-2\sigma)/4\sigma}$ follows from (1.4) with $\phi = 1$ and (2.5). □

4. Local analysis near isolated blow up points

The analysis in this section and the next adapts the blow up analysis developed in [91] and [71] to give accurate blow up profiles for solutions of degenerate elliptic equations. For $\sigma = 1/2$, similar results have been proved in [60] and [48], where equations are elliptic.

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a domain, let $\tau_i \geq 0$ satisfy $\lim_{i\to\infty} \tau_i = 0$, let $p_i = (n + 2\sigma)/(n - 2\sigma) - \tau_i$, and let $K_i \in C^{1,1}(\Omega)$ satisfy, for some constants $A_1, A_2 > 0$,

$$1/A_1 \leq K_i(x) \leq A_1 \quad \text{for all } x \in \Omega, \qquad \|K_i\|_{C^{1,1}(\Omega)} \leq A_2. \tag{4.1}$$

Let $u_i \in L^\infty(\Omega) \cap \dot{H}^\sigma(\mathbb{R}^n)$ with $u_i \geq 0$ in $\mathbb{R}^n$ satisfy

$$(-\Delta)^\sigma u_i = c(n, \sigma)K_i u_i^{p_i} \quad \text{in } \Omega. \tag{4.2}$$

We say that $\{u_i\}$ *blows up* if $\|u_i\|_{L^\infty(\Omega)} \to \infty$ as $i \to \infty$.

Definition 4.1. Suppose that $\{K_i\}$ satisfies (4.1) and $\{u_i\}$ satisfies (4.2). We say a point $\bar{y} \in \Omega$ is an *isolated blow up point* of $\{u_i\}$ if there exist $0 < \bar{r} < \operatorname{dist}(\bar{y}, \Omega)$, $\overline{C} > 0$, and a sequence y_i tending to $\bar{y}$, such that y_i is a local maximum point of u_i, $u_i(y_i) \to \infty$ and

$$u_i(y) \leq \overline{C}|y - y_i|^{-2\sigma/(p_i-1)} \quad \text{for all } y \in B_{\bar{r}}(y_i).$$

Let $y_i \to \bar{y}$ be an isolated blow up point of u_i, and define

$$\bar{u}_i(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r(y_i)} u_i, \quad r > 0, \tag{4.3}$$

and

$$\overline{w}_j(r) = r^{2\sigma/(p_i-1)}\bar{u}_i(r), \quad r > 0.$$

Definition 4.2. We say $y_i \to \bar{y} \in \Omega$ is an *isolated simple blow up point* if $y_i \to \bar{y}$ is an isolated blow up point such that for some $\rho > 0$ (independent of i), $\overline{w}_i$ has precisely one critical point in $(0, \rho)$ for large i.

In this section, we are mainly concerned with the profile of blow up of $\{u_i\}$. And under certain conditions, we can show that isolated blow up points have to be simple.

Let $u_i \in C^2(\Omega) \cap \dot{H}^\sigma(\mathbb{R}^n)$ with $u_i \geq 0$ in $\mathbb{R}^n$ satisfy (4.2) with K_i satisfying (4.1). Without loss of generality, we assume throughout this section that $B_2 \subset \Omega$ and $y_i \to 0$ is an isolated blow up point of $\{u_i\}$ in Ω. Let $U_i = \mathcal{P}_\sigma[u_i]$ be the extension of u_i (see (2.2)). Then we have

$$\begin{cases} \text{div}(t^{1-2\sigma} \nabla U_i) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ -\lim_{t \to 0} t^{1-2\sigma} \frac{\partial U_i(x,t)}{\partial t} = c_0 K_i(x) U_i(x, 0)^{p_i} & \text{for any } x \in \Omega, \end{cases} \tag{4.4}$$

where $c_0 = N_\sigma c(n, \sigma)$ with $N_\sigma = 2^{1-2\sigma} \Gamma(1 - \sigma)/\Gamma(\sigma)$.

Lemma 4.3. *Suppose that $u_i \in C^2(\Omega) \cap \dot{H}^\sigma(\mathbb{R}^n)$ with $u_i \geq 0$ in $\mathbb{R}^n$ satisfies (4.2) with $\{K_i\}$ satisfying (4.1), and $y_i \to 0$ is an isolated blow up point of $\{u_i\}$, i.e., for some positive constants A_3 and $\bar{r}$ independent of i,*

$$|y - y_i|^{2\sigma/(p_i-1)} u_i(y) \leq A_3 \quad \text{for all } y \in B_{\bar{r}} \subset \Omega. \tag{4.5}$$

Denote $U_i = \mathcal{P}_\sigma[u_i]$ and $Y_i = (y_i, 0)$. Then for any $0 < r < \frac{1}{3}\bar{r}$, we have the following Harnack inequality:

$$\sup_{\mathcal{B}_{2r}^+(Y_i) \setminus \overline{\mathcal{B}_{r/2}^+(Y_i)}} U_i \leq C \inf_{\mathcal{B}_{2r}^+(Y_i) \setminus \overline{\mathcal{B}_{r/2}^+(Y_i)}} U_i,$$

where C is a positive constant depending only on n, σ, A_3, $\bar{r}$ and $\sup_i \|K_i\|_{L^\infty(B_{\bar{r}}(y_i))}$.

Proof. For $0 < r < \bar{r}/3$, set

$$V_i(Y) = r^{2\sigma/(p_i-1)} U_i(Y_i + rY) \quad \text{for } Y \in \mathcal{B}_3^+.$$

It is easy to see that

$$\text{div}(s^{1-2\sigma} \nabla V_i) = 0 \quad \text{in } \mathcal{B}_3^+,$$

and

$$-\lim_{s \to 0} s^{1-2\sigma} \partial_s V_i(y, s) = c_0 K(y_i + ry) V_i(y, 0)^{p_i} \quad \text{on } \partial' \mathcal{B}_3^+.$$

Since $y_i \to 0$ is an isolated blow up point of u_i,

$$V_i(y, 0) \leq A_3 |y|^{-2\sigma/(p_i-1)} \quad \text{for all } y \in B_3.$$

Lemma 4.3 follows after applying to V_i Proposition 2.6 and the standard Harnack inequality for uniformly elliptic equations in the domain $\mathcal{B}_{5/2}^+ \setminus \overline{\mathcal{B}_{1/4}^+}$. $\qquad \square$

Proposition 4.4. *Suppose that $u_i \in C^2(\Omega) \cap \dot{H}^\sigma(\mathbb{R}^n)$ with $u_i \geq 0$ in $\mathbb{R}^n$ satisfies (4.2) with $K_i \in C^{1,1}(\Omega)$ satisfying (4.1). Suppose also that $y_i \to 0$ is an isolated blow up point of $\{u_i\}$ with (4.5). Then for any $R_i \to \infty$ and $\varepsilon_i \to 0^+$, we have, after passing to a subsequence (still denoted as $\{u_i\}$, $\{y_i\}$, etc.),*

$$\|m_i^{-1}u_i(m_i^{-(p_i-1)/2\sigma} \cdot + y_i) - (1 + k_i |\cdot|^2)^{(2\sigma-n)/2}\|_{C^2(B_{2R_i}(0))} \leq \varepsilon_i,$$

$$R_i m_i^{-(p_i-1)/2\sigma} \to 0 \quad as\ i \to \infty,$$

where $m_i = u_i(y_i)$ and $k_i = K_i(y_i)^{1/\sigma}/4$.

Proof. Let

$$\phi_i(x) = m_i^{-1}u_i(m_i^{-(p_i-1)/2\sigma}x + y_i) \quad \text{for } x \in \mathbb{R}^n.$$

It follows that

$$(-\Delta)^\sigma \phi_i(x) = c(n,\sigma)K_i(m_i^{-(p_i-1)/2\sigma}x + y_i)\phi_i^{p_i},$$

$$0 < \phi_i(x) \leq A_3 |x|^{-2\sigma/(p_i-1)}, \quad |x| < \bar{r}m_i^{(p_i-1)/2\sigma}, \tag{4.6}$$

and

$$\phi_i(0) = 1, \quad \nabla\phi_i(0) = 0.$$

Let $\Phi_i = \mathcal{P}_\sigma[\phi_i]$ be the extension of ϕ_i (see (2.2)). Then Φ_i satisfies

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla\Phi_i(x,t)) = 0, \quad |(x,t)| < \bar{r}m_i^{(p_i-1)/2\sigma}, \\ -\lim_{t\to 0} t^{1-2\sigma}\partial_t\Phi_i(x,t) = N_\sigma c(n,\sigma)K_i(m_i^{-(p_i-1)/2\sigma}x + y_i)\Phi_i(x,0)^{p_i}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad |x| < \bar{r}m_i^{(p_i-1)/2\sigma}. \end{cases}$$

By the weak maximum principle, for any $0 < r < 1$, we have $1 = \phi_i(0) = \Phi_i(0,0) \geq \min_{\partial''\mathcal{B}_r} \Phi_i$. It follows from Lemma 4.3 that

$$\max_{\partial B_r} \phi_i \leq \max_{\partial''\mathcal{B}_r} \Phi_i \leq C \min_{\partial''\mathcal{B}_r} \Phi_i \leq C.$$

Thus,

$$\max_{\overline{B_1}} \phi_i \leq C$$

for some $C > 0$ depending on n, σ, A_1, A_2, A_3. This and (4.6) imply that for any $R > 1$,

$$\max_{\overline{B_R}} \phi_i \leq C(R)$$

for some $C(R) > 0$ depending on n, σ, A_1, A_2, A_3 and R. Then by Corollary 2.10 there exists some $\alpha \in (0,1)$ such that for every $R > 1$,

$$\|\Phi_i\|_{H(t^{1-2\sigma}, Q_R)} + \|\Phi_i\|_{C^\alpha(\overline{Q_R})} \leq C_1(R),$$

where α and $C_1(R)$ are independent of i. Bootstrapping using Theorem 2.11 we have, for every $0 < \beta < 2$ with $2\sigma + \beta \notin \mathbb{N}$,

$$\|\phi_i\|_{C^{2\sigma+\beta}(\overline{B_R})} \leq C_2(R,\beta),$$

where $C_2(R, \beta)$ is independent of i. Thus, after passing to a subsequence we have, for some nonnegative functions $\Phi \in H_{\mathrm{loc}}(t^{1-2\sigma}, \overline{\mathbb{R}^{n+1}}) \cap C^\alpha_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}})$ and $\phi \in C^2(\mathbb{R}^n)$,

$$\begin{cases} \Phi_i \rightharpoonup \Phi & \text{weakly in } H_{\mathrm{loc}}(t^{1-2\sigma}, \mathbb{R}^{n+1}_+), \\ \Phi_i \to \Phi & \text{in } C^{\alpha/2}_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+}), \\ \phi_i \to \phi & \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^n). \end{cases}$$

It follows that

$$\Phi(\cdot, 0) \equiv \phi, \quad \phi(0) = 1, \quad \nabla\phi(0) = 0,$$

and Φ satisfies

$$\begin{cases} \mathrm{div}(t^{1-2\sigma}\nabla\Phi) = 0 & \text{in } \mathbb{R}^{n+1}, \\ -\lim_{t\to 0} t^{1-2\sigma}\partial_t\Phi(x, t) = c_0 K\,\Phi(x, 0)^{(n+2\sigma)/(n-2\sigma)} & \text{on } \partial'\mathbb{R}^{n+1}, \end{cases}$$

where $K = \lim_{i\to\infty} K_i(y_i)$. By Theorem 1.8, we have

$$\phi(x) = \left(1 + \lim_{i\to\infty} k_i|x|^2\right)^{(2\sigma-n)/2},$$

where $k_i = K_i(y_i)^{1/\sigma}/4$. Proposition 4.4 follows immediately. $\qquad\square$

Note that since passing to subsequences does not affect our proofs, we will always choose $R_i \to \infty$ first, and then $\varepsilon_i \to 0^+$ as small as we wish (depending on R_i) and only then choose our subsequence $\{u_i\}$ to work with.

Proposition 4.5. *Under the hypotheses of Proposition 4.4, there exists some positive constant $C = C(n, \sigma, A_1, A_2, A_3)$ such that*

$$u_i(y) \geq C^{-1}m_i(1 + k_i m_i^{(p_i-1)/\sigma}|y - y_i|^2)^{(2\sigma-n)/2}, \quad |y - y_i| \leq 1.$$

In particular, for any $e \in \mathbb{R}^n$ with $|e| = 1$, we have

$$u_i(y_i + e) \geq C^{-1}m_i^{-1+((n-2\sigma)/2\sigma)\tau_i},$$

where $\tau_i = (n + 2\sigma)/(n - 2\sigma) - p_i$.

Proof. Denote $r_i = R_i m_i^{-(p_i-1)/2\sigma}$. It follows from Proposition 4.4 that $r_i \to 0$ and

$$u_i(y) \geq C^{-1}m_i R_i^{2\sigma-n} \quad \text{for all } |y - y_i| = r_i.$$

By the Harnack inequality of Lemma 4.3, we have

$$U_i(Y) \geq C^{-1}m_i R_i^{2\sigma-n} \quad \text{for all } |Y - Y_i| = r_i,$$

where $U_i = \mathcal{P}_\sigma[u_i]$ is the extension of u_i, $Y = (y, s)$ with $s \geq 0$, and $Y_i = (y_i, 0)$.

Set

$$\Psi_i(Y) = C^{-1}R_i^{2\sigma-n}r_i^{n-2\sigma}m_i(|Y-Y_i|^{2\sigma-n} - (3/2)^{2\sigma-n}), \quad r_i \le |Y-Y_i| \le 3/2.$$

Clearly, Ψ_i satisfies

$$\mathrm{div}(s^{1-2\sigma}\nabla\Psi_i) = 0 = \mathrm{div}(s^{1-2\sigma}\nabla U_i), \quad r_i \le |Y-Y_i| \le 3/2,$$
$$\Psi_i(Y) \le U_i(Y) \quad \text{on } \partial''\mathcal{B}_{r_i} \cup \partial''\mathcal{B}_{3/2},$$
$$-\lim_{s\to0^+} s^{1-2\sigma}\partial_s\Psi_i(y,s) = 0 \le -\lim_{s\to0^+} s^{1-2\sigma}\partial_s U_i(y,s), \quad r_i \le |y-y_i| \le 3/2.$$

By the weak maximum principle of Lemma 2.5 applied to $U_i - \Psi_i$, we have

$$U_i(Y) \ge \Psi_i(Y) \quad \text{for all } r_i \le |Y-Y_i| \le 3/2.$$

Therefore, Proposition 4.5 follows immediately from Proposition 4.4.   □

Lemma 4.6. *Under the hypotheses of Proposition 4.4, and in addition that $y_i \to 0$ is also an isolated simple blow up point with constant ρ, there exist $\delta_i > 0$, $\delta_i = O(R_i^{-2\sigma+o(1)})$, such that*

$$u_i(y) \le C_1 u_i(y_i)^{-\lambda_i}|y-y_i|^{2\sigma-n+\delta_i} \quad \text{for all } r_i \le |y-y_i| \le 1,$$

where $\lambda_i = (n-2\sigma-\delta_i)(p_i-1)/2\sigma - 1$ and C_1 is some positive constant depending only on n, σ, A_1, A_3 and ρ.

Proof. From Proposition 4.4, we see that

$$u_i(y) \le Cu_i(y_i)R_i^{2\sigma-n} \quad \text{for all } |y-y_i| = r_i. \tag{4.7}$$

Let $\overline{u}_i(r)$ be the average of u_i over the sphere of radius r centered at y_i. It follows from the assumption of isolated simple blow up and Proposition 4.4 that

$$r^{2\sigma/(p_i-1)}\overline{u}_i(r) \quad \text{is strictly decreasing for } r_i < r < \rho. \tag{4.8}$$

By Lemma 4.3, (4.8) and (4.7), we have, for all $r_i < |y-y_i| < \rho$,

$$|y-y_i|^{2\sigma/(p_i-1)}u_i(y) \le C|y-y_i|^{2\sigma/(p_i-1)}\overline{u}_i(|y-y_i|)$$
$$\le r_i^{2\sigma/(p_i-1)}\overline{u}_i(r_i) \le CR_i^{(2\sigma-n)/2+o(1)},$$

where $o(1)$ denotes some quantity tending to 0 as $i \to \infty$. Applying Lemma 4.3 again, we obtain

$$U_i(Y)^{p_i-1} \le O(R_i^{-2\sigma+o(1)})|Y-Y_i|^{-2\sigma} \quad \text{for all } r_i \le |Y-Y_i| \le \rho. \tag{4.9}$$

Consider the operators

$$\begin{cases} \mathfrak{L}(\Phi) = \mathrm{div}(s^{1-2\sigma}\nabla\Phi(Y)) & \text{in } \mathcal{B}_2^+, \\ L_i(\Phi) = -\lim_{s\to0^+} s^{1-2\sigma}\partial_s\Phi(y,s) - c_0 K_i u_i^{p_i-1}(y)\Phi(y,0) & \text{on } \partial'\mathcal{B}_2^+. \end{cases}$$

Clearly, $U_i > 0$ satisfies $\mathfrak{L}(U_i) = 0$ in $\mathcal{B}_2^+$ and $L_i(U_i) = 0$ on $\partial'\mathcal{B}_2^+$.

For $0 \le \mu \le n - 2\sigma$, a direct computation yields

$$\mathfrak{L}(|Y - Y_i|^{-\mu} - \varepsilon s^{2\sigma} |Y - Y_i|^{-(\mu+2\sigma)})$$
$$= s^{1-2\sigma} |Y - Y_i|^{-(\mu+2)} \left\{ -\mu(n - 2\sigma - \mu) + \frac{\varepsilon(\mu + 2\sigma)(n - \mu)s^{2\sigma}}{|Y - Y_i|^{2\sigma}} \right\}$$

and

$$L_i(|Y - Y_i|^{-\mu} - \varepsilon s^{2\sigma} |Y - Y_i|^{-(\mu+2\sigma)}) = |Y - Y_i|^{-(u+2\sigma)}(2\varepsilon\sigma - c_0 K_i u_i^{p_i-1}|Y - Y_i|^{2\sigma}).$$

It follows from (4.9) that we can choose $\varepsilon_i = O(R_i^{-2\sigma+o(1)}) > 0$, and then choose $\delta_i = O(R_i^{-2\sigma+o(1)}) > 0$ such that for $r_i < |y - y_i| < \rho$,

$$L_i(|Y - Y_i|^{-\delta_i} - \varepsilon_i s^{2\sigma} |Y - Y_i|^{-(\delta_i+2\sigma)}) \ge 0,$$
$$L_i(|Y - Y_i|^{2\sigma-n+\delta_i} - \varepsilon_i s^{2\sigma} |Y - Y_i|^{-n+\delta_i}) \ge 0,$$

and for $r_i < |Y - Y_i| < \rho$,

$$\mathfrak{L}(|Y - Y_i|^{-\delta_i} - \varepsilon_i s^{2\sigma} |Y - Y_i|^{-(\delta_i+2\sigma)}) \le 0,$$
$$\mathfrak{L}(|Y - Y_i|^{2\sigma-n+\delta_i} - \varepsilon_i s^{2\sigma} |Y - Y_i|^{-n+\delta_i}) \le 0.$$

Set $M_i = 2\max_{\partial'' \mathcal{B}_\rho^+} U_i$, $\lambda_i = (n - 2\sigma - \delta_i)(p_i - 1)/2\sigma - 1$ and

$$\Phi_i = M_i \rho^{\delta_i}(|Y - Y_i|^{-\delta_i} - \varepsilon_i s^{2\sigma} |Y - Y_i|^{-(\delta_i+2\sigma)})$$
$$+ 2Au_i(y_i)^{-\lambda_i}(|Y - Y_i|^{2\sigma-n+\delta_i} - \varepsilon_i s^{2\sigma} |Y - Y_i|^{-n+\delta_i}),$$

where $A > 1$ will be chosen later. By the choice of M_i and λ_i, we immediately have

$$\Phi_i(Y) \ge M_i \ge U_i(Y) \qquad \text{for all } |Y - Y_i| = \rho,$$
$$\Phi_i(Y) \ge AU_i(Y_i)R_i^{2\sigma-n+\delta_i} \ge AU_i(Y_i)R_i^{2\sigma-n} \quad \text{for all } |Y - Y_i| = r_i.$$

Due to (4.9), we can choose A sufficiently large such that

$$\Phi_i \ge U_i \quad \text{for all } |Y - Y_i| = r_i.$$

Therefore, applying the maximum principles of Section A.3 to $\Phi_i - U_i$ in $\mathcal{B}_\rho \setminus \overline{\mathcal{B}_{r_i}}$ yields

$$U_i \le \Phi_i \quad \text{for all } r_i \le |Y - Y_i| \le \rho.$$

For $r_i < \theta < \rho$, the same arguments as in (4.9) yield

$$\rho^{2\sigma/(p_i-1)} M_i \le C\rho^{2\sigma/(p_i-1)}\overline{u}_i(\rho) \le C\theta^{2\sigma/(p_i-1)}\overline{u}_i(\theta)$$
$$\le C\theta^{2\sigma/(p_i-1)}\{M_i \rho^{\delta_i}\theta^{-\delta_i} + Au_i(y_i)^{-\lambda_i}\theta^{2\sigma-n+\delta_i}\}.$$

Choose $\theta = \theta(n, \sigma, \rho, A_1, A_3)$ sufficiently small so that

$$C\theta^{2\sigma/(p_i-1)}\rho^{\delta_i}\theta^{-\delta_i} \le \tfrac{1}{2}\rho^{2\sigma/(p_i-1)}.$$

It follows that $M_i \le Cu_i(y_i)^{-\lambda_i}$. Then Lemma 4.6 follows from the Harnack inequality.

$\square$

Below we are going to improve the estimate in Lemma 4.6. First, we prove a Pohozaev type identity.

Proposition 4.7. *Suppose that* $K \in C^1(B_{2R})$. *Let* $U \in H(t^{1-2\sigma}, \mathcal{B}_{2R}^+)$ *with* $U \geq 0$ *in* $\mathcal{B}_{2R}^+$ *be a weak solution of*

$$
\begin{cases}
\operatorname{div}(t^{1-2\sigma} \nabla U) = 0 & \text{in } \mathcal{B}_{2R}^+, \\
-\lim_{t \to 0} t^{1-2\sigma} \partial_t U(x, t) = K(x) U^p(x, 0) & \text{on } \partial' \mathcal{B}_{2R}^+,
\end{cases}
\tag{4.10}
$$

where $p > 0$. *Then*

$$
\int_{\partial' \mathcal{B}_R^+} B'(X, U, \nabla U, R, \sigma) + \int_{\partial'' \mathcal{B}_R^+} t^{1-2\sigma} B''(X, U, \nabla U, R, \sigma) = 0,
\tag{4.11}
$$

where

$$
B'(X, U, \nabla U, R, \sigma) = \frac{n - 2\sigma}{2} K U^{p+1} + \langle X, \nabla U \rangle K U^p,
$$

$$
B''(X, U, \nabla U, R, \sigma) = \frac{n - 2\sigma}{2} U \frac{\partial U}{\partial \nu} - \frac{R}{2} |\nabla U|^2 + R \left| \frac{\partial U}{\partial \nu} \right|^2.
$$

Proof. Let $\Omega_\varepsilon = \mathcal{B}_R^+ \cap \{t > \varepsilon\}$ for small $\varepsilon > 0$. Multiplying (4.10) by $\langle X, \nabla U \rangle$ and integrating by parts in Ω_ε, we have, with the notations $\partial' \Omega_\varepsilon = $ interior of $\overline{\Omega_\varepsilon} \cap \{t = \varepsilon\}$, $\partial'' \Omega_\varepsilon = \partial \Omega_\varepsilon \setminus \partial' \Omega_\varepsilon$ and $\nu = $ unit outer normal of $\partial \Omega_\varepsilon$,

$$
-\int_{\partial' \Omega_\varepsilon} t^{1-2\sigma} \partial_t U \langle X, \nabla U \rangle + \int_{\partial'' \Omega_\varepsilon} t^{1-2\sigma} R \left| \frac{\partial U}{\partial \nu} \right|^2
$$

$$
= \int_{\Omega_\varepsilon} t^{1-2\sigma} |\nabla U|^2 + \frac{1}{2} \int_{\Omega_\varepsilon} t^{1-2\sigma} X \cdot \nabla (|\nabla U|^2)
$$

$$
= -\frac{n-2\sigma}{2} \int_{\Omega_\varepsilon} t^{1-2\sigma} |\nabla U|^2 + \frac{1}{2} \int_{\partial'' \Omega_\varepsilon} t^{1-2\sigma} R |\nabla U|^2 - \frac{1}{2} \int_{\partial' \Omega_\varepsilon} t^{2-2\sigma} |\nabla U|^2.
\tag{4.12}
$$

Multiplying (4.10) by U and integrating by parts in Ω_ε, we have

$$
\int_{\Omega_\varepsilon} t^{1-2\sigma} |\nabla U|^2 = -\int_{\partial' \Omega_\varepsilon} t^{1-2\sigma} U \partial_t U + \int_{\partial'' \Omega_\varepsilon} t^{1-2\sigma} \frac{\partial U}{\partial \nu} U.
\tag{4.13}
$$

By Corollary 2.10 and Proposition 2.19, there exists some $\alpha \in (0, 1)$ such that U, $\nabla_x U$, and $t^{1-2\sigma} \partial_t U$ belong to $C^\alpha(\overline{\mathcal{B}_r^+})$ for all $r < 2R$. With this we can let $\varepsilon \to 0$ as follows. By (4.10),

$$
-t^{1-2\sigma} \partial_t U(x, t) \to K(x) U(x, 0)^p \quad \text{uniformly in } B_{3R/2} \text{ as } t \to 0.
$$

Hence (4.11) follows by letting $\varepsilon \to 0$ in (4.12) and (4.13). $\qquad \square$

Lemma 4.8. *Under the assumptions of Lemma 4.6, we have*

$$
\tau_i = O(u_i(y_i)^{-2/(n-2\sigma)+o(1)}), \quad \text{and thus} \quad u_i(y_i)^{\tau_i} = 1 + o(1).
$$

Proof. Since U_i satisfies (4.4) and $\mathrm{div}(y - y_i) = n$, we have, using integration by parts,

$$
\frac{1}{c_0} \int_{\partial' \mathcal{B}_1^+(Y_i)} B'(Y, U_i, \nabla U_i, 1, \sigma)
$$

$$
= \frac{n - 2\sigma}{2n} \int_{\partial' \mathcal{B}_1^+(Y_i)} \mathrm{div}(y - y_i) K_i U^{p_i+1} + \frac{1}{p_i + 1} \int_{\partial' \mathcal{B}_1^+(Y_i)} \langle y - y_i, \nabla_y U_i^{p_i+1} \rangle K_i
$$

$$
= -\frac{n - 2\sigma}{2n} \int_{\partial' \mathcal{B}_1^+(Y_i)} [\langle y - y_i, \nabla_y K_i \rangle U_i^{p_i+1} + \langle y - y_i, \nabla_y U_i^{p_i+1} \rangle K_i]
$$

$$
+ \frac{n - 2\sigma}{2n} \int_{\partial B_1(y_i)} K_i U_i^{p_i+1} + \frac{1}{p_i + 1} \int_{\partial' \mathcal{B}_1^+(Y_i)} \langle y - y_i, \nabla_y U_i^{p_i+1} \rangle K_i
$$

$$
= \frac{\tau_i (n - 2\sigma)^2}{2n(2n - \tau_i(n - 2\sigma))} \int_{\partial' \mathcal{B}_1^+(Y_i)} \langle y - y_i, \nabla_y U_i^{p_i+1} \rangle K_i
$$

$$
- \frac{n - 2\sigma}{2n} \int_{\partial' \mathcal{B}_1^+(Y_i)} \langle y - y_i, \nabla_y K_i \rangle U_i^{p_i+1} + \frac{n - 2\sigma}{2n} \int_{\partial B_1(y_i)} K_i U_i^{p_i+1}
$$

and

$$
\int_{\partial' \mathcal{B}_1^+(Y_i)} \langle y - y_i, \nabla_y U_i^{p_i+1} \rangle K_i
$$

$$
= -n \int_{\partial' \mathcal{B}_1^+(Y_i)} K_i U_i^{p_i+1} - \int_{\partial' \mathcal{B}_1^+(Y_i)} \langle y - y_i, \nabla_y K_i \rangle U_i^{p_i+1} + \int_{\partial B_1(y_i)} K_i U_i^{p_i+1}.
$$

Combining the above two, together with Proposition 4.7, we conclude that

$$
\tau_i \int_{\partial' \mathcal{B}_1^+(Y_i)} U_i^{p_i+1} \le C(n, \sigma, A_1, A_2) \bigg\{ \int_{\partial' \mathcal{B}_1^+(Y_i)} |y - y_i| U_i^{p_i+1}
$$

$$
+ \int_{\partial B_1(y_i)} U_i^{p_i+1} + \int_{\partial'' \mathcal{B}_1^+(Y_i)} t^{1-2\sigma} |B''(Y, U_i, \nabla U_i, 1, \sigma)| \bigg\}. \quad (4.14)
$$

Since $U_i = u_i$ on $\partial' \mathcal{B}_1(Y_i) = B_1(y_i) \times \{0\}$, it follows from Proposition 4.5 that

$$
\int_{\partial' \mathcal{B}_1(Y_i)} U_i^{p_i+1} = \int_{B_1(y_i)} u_i^{p_i+1} \ge C^{-1} \int_{B_1(y_i)} \frac{m_i^{p_i+1}}{(1 + |m_i^{(p_i-1)/2\sigma} (y - y_i)|^2)^{(n-2\sigma)(p_i+1)/2}}
$$

$$
\ge C^{-1} m_i^{\tau_i(n/2\sigma - 1)} \int_{B_{m_i^{(p_i-1)/2\sigma}}} \frac{1}{(1 + |z|^2)^{(n-2\sigma)(p_i+1)/2}}
$$

$$
\ge C^{-1} m_i^{\tau_i(n/2\sigma - 1)}, \quad (4.15)
$$

where we have used the change of variables $z = m_i^{(p_i-1)/2\sigma} (y - y_i)$ in the second inequality.

By Proposition 2.19 and Lemma 4.6, it is easy to see that the last two integral terms of the right-hand side of (4.14) are $O(m_i^{-2+o(1)})$. By Proposition 4.4, we have

$$
\int_{\partial' \mathcal{B}_{r_i}(Y_i)} |Y - Y_i| U_i^{p_i+1} = \int_{B_{r_i}(y_i)} |y - y_i| u_i^{p_i+1}
$$

$$
\leq C \int_{B_{r_i}(y_i)} \frac{|y - y_i| m_i^{p_i+1}}{(1 + |m_i^{(p_i-1)/2\sigma}(y - y_i)|^2)^{(n-2\sigma)(p_i+1)/2}}
$$

$$
\leq C m_i^{-2/(n-2\sigma)+o(1)} \int_{B_{R_i}} \frac{|z|}{(1 + |z|^2)^{n+o(1)}} C m_i^{-2/(n-2\sigma)+o(1)}. \quad (4.16)
$$

By Lemma 4.6, as $R_i \to \infty$, we have

$$
\int_{\partial' \mathcal{B}_1(Y_i) \setminus \partial' \mathcal{B}_{r_i}(Y_i)} |Y - Y_i| U_i^{p_i+1} = \int_{B_1(y_i) \setminus B_{r_i}(y_i)} |y - y_i| u_i^{p_i+1}
$$

$$
\leq m_i^{-\lambda_i(p_i+1)} r_i^{n+1+(2\sigma-n+\delta_i)(p_i+1)} = o(m_i^{-2/(n-2\sigma)+o(1)}). \quad (4.17)
$$

Combining (4.14)–(4.17) and $\tau_i = o(1)$, we complete the proof. $\square$

Proposition 4.9. *Under the assumptions of Lemma* 4.6*, we have*

$$
u_i(y) \leq C u_i^{-1}(y_i) |y - y_i|^{2\sigma-n} \quad \textit{for all } |y - y_i| \leq 1.
$$

Our proof of this proposition makes use of

Lemma 4.10. *Suppose that for all* $\varepsilon \in (0, 1)$*,* $U \in H(t^{1-2\sigma}, \mathcal{B}_1^+ \setminus \overline{\mathcal{B}_\varepsilon^+})$ *with* $U > 0$ *in* $\mathcal{B}_1^+ \setminus \overline{\mathcal{B}_\varepsilon^+}$ *is a weak solution of*

$$
\begin{cases} \operatorname{div}(t^{1-2\sigma} \nabla U) = 0 & \textit{in } \mathcal{B}_1^+ \setminus \overline{\mathcal{B}_\varepsilon^+}, \\ -\lim_{t\to 0} t^{1-2\sigma} \partial_t U(x, t) = 0 & \textit{on } B_1 \setminus \overline{B_\varepsilon^+}. \end{cases} \quad (4.18)
$$

Then

$$
U(X) = A|X|^{2\sigma-n} + W(X),
$$

where A *is a nonnegative constant and* $W \in H(t^{1-2\sigma}, \mathcal{B}_1^+)$ *satisfies*

$$
\begin{cases} \operatorname{div}(t^{1-2\sigma} \nabla W) = 0 & \textit{in } \mathcal{B}_1^+, \\ -\lim_{t\to 0} t^{1-2\sigma} \partial_t W(x, t) = 0 & \textit{on } B_1. \end{cases} \quad (4.19)
$$

The proof of Lemma 4.10 is provided in Appendix A.2.

Proof of Proposition 4.9. For $|y - y_i| < r_i$, it follows from Proposition 4.4 that

$$
u_i(y) \leq C m_i \left(\frac{1}{1 + |m_i^{(p_i-1)/2\sigma}(y - y_i)|^2} \right)^{(n-2\sigma)/2}
$$

$$
\leq C m_i^{-1-\frac{n-2\sigma}{2\sigma}\tau_i} |y - y_i|^{2\sigma-n} \leq C m_i^{-1} |y - y_i|^{2\sigma-n}, \quad (4.20)
$$

where Lemma 4.8 is used in the last inequality.

Suppose $|y-y_i| \geq r_i$. Let $e \in \overline{\mathbb{R}_+^{n+1}}$ with $|e| = 1$, and set $V_i(Y) = U_i(Y_i+e)^{-1}U_i(Y)$. Then V_i satisfies

$$\begin{cases} \operatorname{div}(s^{1-2\sigma}\nabla V_i) = 0 & \text{in } \mathcal{B}_2^+, \\ -\lim_{s\to 0} s^{1-2\sigma}\partial_s V_i(y,s) = c(n,\sigma)KU_i(Y_i+e)^{p_i-1}V_i^{p_i} & \text{for } y \in B_2^+. \end{cases}$$

Note that $U_i(Y_i + e) \to 0$ by Lemma 4.6, and for any $r > 0$,

$$V_i(Y) \leq C(n,\sigma,A_1,r) \quad \text{for all } r < |y - y_i| \leq 1, \tag{4.21}$$

which follows from Lemma 4.3. It follows that $\{V_i\}$ converges to some positive function V in $C_{\text{loc}}^\infty(\mathcal{B}_{3/2}^+) \cap C_{\text{loc}}^\alpha(\overline{\mathcal{B}}_{3/2}^+ \setminus \{0\})$ for some $\alpha \in (0,1)$, and V satisfies

$$\begin{cases} \operatorname{div}(s^{1-2\sigma}\nabla V) = 0 & \text{in } \mathcal{B}_1^+, \\ -\lim_{s\to 0} s^{1-2\sigma}\partial_s V(y,s) = 0 & \text{for } y \in B_1 \setminus \{0\}. \end{cases}$$

Hence $\lim_{i\to\infty} r^{2\sigma/(p_i+1)}\bar{v}_i(r) = r^{n-2\sigma}\bar{v}(r)$, where $v(y) = V(y,0)$. Since $r_i \to 0$, and $y_i \to 0$ is an isolated simple blow up point of $\{u_i\}$, it follows from Lemma 4.3 that $r^{(n-2\sigma)/2}\overline{V}(r)$ is *almost decreasing* for all $0 < r < \rho$, i.e., there exists a positive constant C (which comes from the Harnack inequality in Lemma 4.3) such that for any $0 < r_1 \leq r_2 < \rho$,

$$r_1^{(n-2\sigma)/2}\overline{V}(r_1) \geq Cr_2^{(n-2\sigma)/2}\overline{V}(r_2).$$

Therefore, V has to have a singularity at $Y = 0$. Lemma 4.10 implies

$$V(Y) = A|Y|^{2\sigma-n} + W(Y), \tag{4.22}$$

where $A > 0$ is a constant and W is as in Lemma 4.10.

We first establish the inequality in Proposition 4.9 for $|Y - Y_i| = 1$. Namely, we prove that

$$U_i(Y_i + e) \leq CU_i^{-1}(Y_i). \tag{4.23}$$

Suppose that (4.23) does not hold. Then along a subsequence we have

$$\lim_{i\to\infty} U_i(Y_i + e)U_i(Y_i) = \infty. \tag{4.24}$$

By integration by parts (using Ω_ε and letting $\varepsilon \to 0$, as in the proof of Proposition 4.7), we obtain

$$\begin{aligned} 0 &= -\int_{\mathcal{B}_1^+} \operatorname{div}(s^{1-2\sigma}\nabla V_i) \\ &= \int_{\partial''\mathcal{B}_1^+} s^{1-2\sigma}\frac{\partial V_i}{\partial\nu} + c(n,\sigma)U_i(Y_i+e)^{-1}\int_{\partial'\mathcal{B}_1^+} KU_i^{p_i}. \end{aligned} \tag{4.25}$$

By Lemma 4.8 and similar computations to (4.16) and (4.17), we see that

$$\int_{\partial'\mathcal{B}_1^+} KU_i^{p_i} \leq CU_i(Y_i)^{-1}.$$

Due to (4.24),

$$\lim_{i \to \infty} U_i(Y_i + e)^{-1} \int_{\partial' \mathcal{B}_1^+} K U_i^{p_i} = 0.$$

A direct computation yields with (4.21) (again using Ω_ε and letting $\varepsilon \to 0$)

$$\lim_{i \to \infty} \int_{\partial'' \mathcal{B}_1^+} s^{1-2\sigma} \frac{\partial V_i}{\partial \nu} = \lim_{i \to \infty} \int_{\partial'' \mathcal{B}_1^+} s^{1-2\sigma} \frac{\partial}{\partial \nu}(A|Y|^{2\sigma-n} + W(Y))$$

$$= A(2\sigma - n) \int_{\partial'' \mathcal{B}_1^+} s^{1-2\sigma} < 0,$$

which contradicts (4.25). Thus we have proved (4.23). By Lemma 4.3, we have established the inequality in Proposition 4.9 for $\rho \leq |Y - Y_i| \leq 1$.

By a standard scaling argument, we can reduce the case of $r_i \leq |Y - Y_i| < \rho$ to $|Y - Y_i| = 1$. We refer to [71, p. 340] for details. $\qquad\square$

Propositions 4.5 and 4.9 give a clear picture of u_i near the isolated simple blow up point. By the estimates there, it is easy to deduce the following result.

Lemma 4.11. *We have*

$$\int_{|y-y_i| \leq r_i} |y - y_i|^s u_i(y)^{p_i+1} = \begin{cases} O(u_i(y_i)^{-2s/(n-2\sigma)}), & -n < s < n, \\ O(u_i(y_i)^{-2n/(n-2\sigma)} \log u_i(y_i)), & s = n, \\ o(u_i(y_i)^{-2n/(n-2\sigma)}), & s > n, \end{cases}$$

and

$$\int_{r_i < |y-y_i| \leq 1} |y - y_i|^s u_i(y)^{p_i+1} = \begin{cases} o(u_i(y_i)^{-2s/(n-2\sigma)}), & -n < s < n, \\ O(u_i(y_i)^{-2n/(n-2\sigma)} \log u_i(y_i)), & s = n, \\ O(u_i(y_i)^{-2n/(n-2\sigma)}), & s > n. \end{cases}$$

Proof. The first set of estimates follows from Proposition 4.4 and Lemma 4.8, and the second one from Proposition 4.9 and Lemma 4.8. $\qquad\square$

For later application, we replace K_i by $K_i(x)H_i(x)^{\tau_i}$ in (4.2) and consider

$$(-\Delta)^\sigma u_i(x) = c(n, \sigma)K_i(x)H_i(x)^{\tau_i}u_i(x)^{p_i} \quad \text{in } B_2, \tag{4.26}$$

where $H_i \in C^{1,1}(B_2)$ satisfies

$$A_4^{-1} \leq H_i(y) \leq A_4 \quad \text{for all } y \in B_2, \quad \text{and} \quad \|H_i\|_{C^{1,1}(B_2)} \leq A_5 \tag{4.27}$$

for some positive constants A_4 and A_5.

Lemma 4.12. *Suppose that $\{K_i\}$ satisfies (4.1) and condition $(*)'_\beta$ with $\beta < n$ for some positive constants A_1, A_2, $\{L(\beta, i)\}$, and that $\{H_i\}$ satisfies (4.27) with A_4, A_5. Let $u_i \in \dot{H}^\sigma(\mathbb{R}^n) \cap C^2(B_2)$ with $u_i \geq 0$ in $\mathbb{R}^n$ be a solution of (4.26). If $y_i \to 0$ is an isolated simple blow up point of $\{u_i\}$ with (4.5) for some positive constant A_3, then*

$$\tau_i \leq C u_i(y_i)^{-2} + C|\nabla K_i(y_i)| u_i(y_i)^{-2/(n-2\sigma)}$$
$$+ C(L(\beta, i) + L(\beta, i)^{\beta-1}) u_i(y_i)^{-2\beta/(n-2\sigma)},$$

where $C > 0$ depends only on n, σ, A_1, A_2, A_3, A_4, A_5, β and ρ.

Proof. Using Lemma 4.8 and arguing as in the proof of Lemma 4.8, we have

$$\tau_i \leq C u_i(y_i)^{-2} + C\left| \int_{B_1(y_i)} \langle y - y_i, \nabla_y(K_i H_i^{\tau_i})\rangle u_i^{p_i+1} \right|$$
$$\leq C u_i(y_i)^{-2} + C\tau_i \left| \int_{B_1(y_i)} |y - y_i| u_i^{p_i+1} \right| + C\left| \int_{B_1(y_i)} \langle y - y_i, \nabla K_i\rangle H_i^{\tau_i} u_i^{p_i+1} \right|.$$

Making use of Lemma 4.11, we have

$$\left| \int_{B_1(y_i)} \langle y - y_i, \nabla K_i\rangle H_i^{\tau_i} u_i^{p_i+1} \right|$$
$$\leq C|\nabla K_i(y_i)| \int_{B_1(y_i)} |y - y_i| u_i^{p_i+1} + C \int_{B_1(y_i)} |y - y_i| |\nabla K_i(y) - \nabla K_i(y_i)| u_i^{p_i+1}$$
$$\leq C|\nabla K_i(y_i)| u_i(y_i)^{-2/(n-2\sigma)} + C \int_{B_1(y_i)} |y - y_i| |\nabla K_i(y) - \nabla K_i(y_i)| u_i^{p_i+1}.$$

Recalling the definition of $(*)'_\beta$, a direct computation yields

$$|\nabla K_i(y) - \nabla K_i(y_i)|$$
$$\leq \left\{ \sum_{s=2}^{[\beta]} |\nabla^s K_i(y_i)| |y - y_i|^{s-1} + [\nabla^{[\beta]} K_i]_{C^{\beta-[\beta]}(B_1(y_i))} |y - y_i|^{\beta-1} \right\}$$
$$\leq CL(\beta, i) \left\{ \sum_{s=2}^{[\beta]} |\nabla K_i(y_i)|^{(\beta-s)/(\beta-1)} |y - y_i|^{s-1} + |y - y_i|^{\beta-1} \right\}. \tag{4.28}$$

By the Cauchy–Schwarz inequality, we have

$$L(\beta, i)|\nabla K_i(y_i)|^{(\beta-s)/(\beta-1)} |y - y_i|^s$$
$$\leq C\big(|\nabla K_i(y_i)| |y - y_i| + (L(\beta, i) + L(\beta, i)^{\beta-1})|y - y_i|^\beta\big). \tag{4.29}$$

Hence, by Lemma 4.11 we obtain

$$\int_{B_1(y_i)} |y - y_i| |\nabla K_i(y) - \nabla K_i(y_i)| u_i^{p_i+1}$$
$$\leq C|\nabla K_i(y_i)| u_i(y_i)^{-2/(n-2\sigma)} + C(L(\beta, i) + L(\beta, i)^{\beta-1}) u_i(y_i)^{-2\beta/(n-2\sigma)}. \tag{4.30}$$

Lemma 4.12 follows immediately. $\qquad\square$

Lemma 4.13. *Under the hypotheses of Lemma 4.12,*

$$|\nabla K_i(y_i)| \leq C u_i(y_i)^{-2} + C(L(\beta, i) + L(\beta, i)^{\beta-1}) u_i(y_i)^{-2(\beta-1)/(n-2\sigma)},$$

where $C > 0$ depends only on n, σ, A_1, A_2, A_3, A_4, A_5, β and ρ.

Proof. Choose a cut-off function $\eta \in C_c^\infty(\mathcal{B}_{1/2})$ satisfying

$$\eta(Y) = 1, \quad |Y| \leq 1/4, \quad \text{and} \quad \eta(Y) = 0, \quad |Y| \geq 1/2.$$

Let $U_i(Y)$ be the extension of $u_i(y)$, namely,

$$\begin{cases} \operatorname{div}(s^{1-2\sigma} \nabla U_i) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ -\lim_{s \to 0} s^{1-2\sigma} \partial_s U(y, s) = c_0 K_i(y) H_i^{\tau_i} U_i^{p_i} & \text{on } B_2. \end{cases} \tag{4.31}$$

Multiplying (4.31) by $\eta(Y - Y_i)\partial_{y_j} U_i(y, s)$, $j = 1, \ldots, n$, and integrating by parts over $\mathcal{B}_1^+$, we obtain

$$\begin{aligned} 0 &= \int_{\mathcal{B}_1^+} \operatorname{div}(s^{1-2\sigma} \nabla U_i) \eta \partial_{y_j} U_i \\ &= -\int_{\mathcal{B}_1^+} s^{1-2\sigma} \nabla U_i \nabla(\eta \partial_{y_j} U_i) + c_0 \int_{\partial'\mathcal{B}_1^+(Y_i)} \eta K_i H_i^{\tau_i} \partial_{y_j} U_i U_i^{p_i} \\ &= \frac{1}{2} \int_{\mathcal{B}_{1/2}^+ \setminus \mathcal{B}_{1/4}^+} s^{1-2\sigma} (|\nabla U_i|^2 \partial_{y_j} \eta - 2\nabla U_i \nabla \eta \partial_{y_j} U_i) - \frac{c_0}{p_i + 1} \int_{\partial'\mathcal{B}_1^+} \partial_{y_j}(K_i H_i^{\tau_i} \eta) U_i^{p_i+1}. \end{aligned}$$

By Proposition 4.9, we have

$$U_i(Y) \leq C U_i(Y_i)^{-1} \quad \text{for all } 1/2 \geq |Y| \geq 1/4$$

and

$$\int_{\mathcal{B}_{1/2}^+ \setminus \mathcal{B}_{1/4}^+} s^{1-2\sigma} |\nabla U_i|^2 \leq C U_i(Y_i)^{-2}.$$

Therefore, by Lemma 4.11 we conclude that

$$\left| \int_{B_1} \partial_{y_j} K_i H_i^{\tau_i} u_i^{p_i+1} \right| \leq C u_i(y_i)^{-2} + C\tau_i. \tag{4.32}$$

Hence

$$\left| \partial_j K_i(y_i) \int_{B_1} H_i^{\tau_i} u_i^{p_i+1} \right| - C u_i(y_i)^{-2} - C\tau_i \leq \int_{B_1} |\partial_j K_i(y_i) - \partial_j K_i(y)| H_i^{\tau_i} u_i^{p_i+1}.$$

Summing over j, then making use of (4.28), (4.29) and Lemma 4.11, we have

$$\begin{aligned} |\nabla K_i(y_i)| \leq{}& C u_i(y_i)^{-2} + C\tau_i + \tfrac{1}{2} |\nabla K_i(y_i)| \\ &+ C(L(\beta, i) + L(\beta, i)^{\beta-1}) u_i(y_i)^{-2(\beta-1)/(n-2\sigma)}. \end{aligned}$$

Then Lemma 4.13 follows from Lemma 4.12. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4.14. *Under the assumptions of Lemma* 4.12 *we have*

$$\tau_i \le C u_i(y_i)^{-2} + C(L(\beta, i) + L(\beta, i)^{\beta-1}) u_i(y_i)^{-2\beta/(n-2\sigma)}.$$

Proof. This follows immediately from Lemmas 4.12 and 4.13. $\qquad\square$

Corollary 4.15. *In addition to the assumptions of Lemma* 4.12, *further assume that one of the following two conditions holds:*

(i) $\beta = n - 2\sigma$ *and* $L(\beta, i) = o(1)$,
(ii) $\beta > n - 2\sigma$ *and* $L(\beta, i) = O(1)$.

Then for any $0 < \delta < 1$ *we have*

$$\lim_{i \to \infty} u_i(y_i)^2 \int_{B_\delta(y_i)} (y - y_i) \cdot \nabla(K_i H_i^{\tau_i}) u_i^{p_i+1} = 0.$$

Proof. We have

$$
\begin{aligned}
&\left| \int_{B_\delta(y_i)} (y - y_i) \cdot \nabla(K_i H_i^{\tau_i}) u_i^{p_i+1} \right| \\
&\le \left| \int_{B_\delta(y_i)} (y - y_i) \cdot \nabla K_i H_i^{\tau_i} u_i^{p_i+1} \right| + \tau_i \left| \int_{B_\delta(y_i)} (y - y_i) \cdot \nabla H_i H_i^{\tau_i-1} K_i u_i^{p_i+1} \right| \\
&\le C|\nabla K_i(y_i)| \int_{B_\delta(y_i)} |y - y_i| u_i^{p_i+1} \\
&\quad + C \int_{B_\delta(y_i)} |y - y_i| \, |\nabla K_i(y) - \nabla K_i(y_i)| u_i^{p_i+1} + \tau_i \int_{B_\delta(y_i)} |y - y_i| u_i^{p_i+1}.
\end{aligned}
$$

The corollary follows immediately from Lemma 4.13, (4.30) and Lemma 4.14. $\qquad\square$

Proposition 4.16. *Suppose that* $\{K_i\}$ *satisfies* (4.1) *and condition* $(*)'_{n-2\sigma}$ *for some positive constants* A_1, A_2, L *independent of* i, *and that* $\{H_i\}$ *satisfies* (4.27) *with* A_4, A_5. *Let* $u_i \in \dot{H}^\sigma(\mathbb{R}^n) \cap C^2(B_2)$ *be a solution of* (4.26). *If* $y_i \to 0$ *is an isolated blow up point of* $\{u_i\}$ *with* (4.5) *for some positive constant* A_3, *then* $y_i \to 0$ *is an isolated simple blow up point.*

Proof. By Proposition 4.4, $r^{2\sigma/(p_i-1)} \bar{u}_i(r)$ has precisely one critical point in the interval $0 < r < r_i$, where $r_i = R_i u_i(y_i)^{-(p_i-1)/2\sigma}$ as before. Suppose $y_i \to 0$ is not an isolated simple blow up point and let μ_i be the second critical point of $r^{2\sigma/(p_i-1)} \bar{u}_i(r)$. Then we see that

$$\mu_i \ge r_i, \qquad \lim_{i \to \infty} \mu_i = 0. \tag{4.33}$$

Without loss of generality, we assume that $y_i = 0$. Set

$$\phi_i(y) = \mu_i^{2\sigma/(p_i-1)} u_i(\mu_i y), \qquad y \in \mathbb{R}^n.$$

Clearly, ϕ_i satisfies

$$(-\Delta)^\sigma \phi_i(y) = \tilde{K}_i(y)\tilde{H}_i^{\tau_i}(y)\phi_i^{p_i}(y),$$

$$|y|^{2\sigma/(p_i-1)}\phi_i(y) \le A_3, \quad |y| < 1/\mu_i,$$

$$\lim_{i\to\infty} \phi_i(0) = \infty,$$

$$r^{2\sigma/(p_i-1)}\overline{\phi}_i(r) \text{ has precisely one critical point in } 0 < r < 1,$$

$$\frac{d}{dr}\{r^{2\sigma/(p_i-1)}\overline{\phi}_i(r)\}\bigg|_{r=1} = 0,$$

where $\tilde{K}_i(y) = K_i(\mu_i y)$, $\tilde{H}_i(y) = H_i(\mu_i y)$ and $\overline{\phi}_i(r) = |\partial B_r|^{-1}\int_{\partial B_r}\phi_i$.

Therefore, 0 is an isolated simple blow up point of ϕ_i. Let $\Phi_i(Y)$ be the extension of $\phi_i(y)$ in the upper half-space. Then Lemma 4.3, Proposition 4.9, Lemma 4.10 and elliptic theory together imply that

$$\Phi_i(0)\Phi_i(Y) \to G(Y) = A|Y|^{2\sigma-n} + W(Y) \quad \text{in } C^\alpha_{\text{loc}}(\overline{\mathbb{R}^{n+1}_+} \setminus \{0\}) \cap C^2_{\text{loc}}(\mathbb{R}^{n+1}_+)$$

and

$$\phi_i(0)\phi_i(y) \to G(y,0) = A|y|^{2\sigma-n} + W(y,0) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \tag{4.34}$$

as $i \to \infty$, where $A > 0$, $W(Y)$ satisfies

$$\begin{cases} \text{div}(s^{1-2\sigma}\nabla W) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\lim_{s\to 0} s^{1-2\sigma}\partial_s W(y,s) = 0 & \text{for } y \in \mathbb{R}^n. \end{cases}$$

Noting that $G(Y)$ is nonnegative, we have $\liminf_{|Y|\to\infty} W(Y) \ge 0$. It follows from the weak maximum principle and the Harnack inequality that $W(Y)$ is a nonnegative constant function. Since

$$\frac{d}{dr}\{r^{2\sigma/(p_i-1)}\phi_i(0)\overline{\phi}_i(r)\}\bigg|_{r=1} = \phi_i(0)\frac{d}{dr}\{r^{2\sigma/(p_i-1)}\overline{\phi}_i(r)\}\bigg|_{r=1} = 0,$$

we have, after letting $i \to \infty$ and making use of (4.34),

$$W(Y) \equiv A > 0.$$

We are going to derive a contradiction to the Pohozaev identity of Proposition 4.7, by showing that for small positive δ,

$$\limsup_{i\to\infty} \Phi_i(0)^2 \int_{\partial'\mathcal{B}^+_\delta} B'(Y, \Phi_i, \nabla\Phi_i, \delta, \sigma) \le 0, \tag{4.35}$$

and

$$\limsup_{i\to\infty} \Phi_i(0)^2 \int_{\partial''\mathcal{B}^+_\delta} s^{1-2\sigma} B''(Y, \Phi_i, \nabla\Phi_i, \delta, \sigma) < 0. \tag{4.36}$$

And thus, Proposition 4.16 will be established.

By Proposition 2.19, it is easy to see that

$$\limsup_{i \to \infty} \Phi_i(0)^2 \int_{\partial'' \mathcal{B}_\delta^+} s^{1-2\sigma} B''(Y, \Phi_i, \nabla\Phi_i, \delta, \sigma)$$

$$= \int_{\partial'' \mathcal{B}_\delta^+} s^{1-2\sigma} B''(Y, G, \nabla G, \delta, \sigma) = -\frac{(n-2\sigma)^2}{2} A^2 \int_{\partial'' \mathcal{B}_1^+} t^{1-2\sigma} < 0,$$

which shows (4.36). On the other hand, via integration by parts, we have

$$\int_{\partial' \mathcal{B}_\delta^+} B'(Y, \Phi_i, \nabla\Phi_i, \delta, \sigma) = \frac{n-2\sigma}{2} \int_{B_\delta} \tilde{K}_i \tilde{H}_i^{\tau_i} \phi_i^{p_i+1} + \int_{B_\delta} \langle y, \nabla\phi_i \rangle \tilde{K}_i \tilde{H}_i^{\tau_i} \phi_i^{p_i}$$

$$= \frac{n-2\sigma}{2} \int_{B_\delta} \tilde{K}_i \tilde{H}_i^{\tau_i} \phi_i^{p_i+1} - \frac{n}{p_i+1} \int_{B_\delta} \tilde{K}_i \tilde{H}_i^{\tau_i} \phi_i^{p_i+1}$$

$$- \frac{1}{p_i+1} \int_{B_\delta} \langle y, \nabla(\tilde{K}_i \tilde{H}_i^{\tau_i}) \rangle \phi_i^{p_i+1} + \frac{\delta}{p_i+1} \int_{\partial B_\delta} \tilde{K}_i \tilde{H}_i^{\tau_i} \phi_i^{p_i+1}$$

$$\leq - \frac{1}{p_i+1} \int_{B_\delta} \langle y, \nabla(\tilde{K}_i \tilde{H}_i^{\tau_i}) \rangle \phi_i^{p_i+1} + C\phi_i(0)^{-(p_i+1)}.$$

where Proposition 4.9 is used in the last inequality. It is easy to see that $\{\tilde{K}_i\}$ satisfies $(*)'_{n-2\sigma}$ with $L(\beta, i) = o(1)$. Therefore, (4.35) follows from Corollary 4.15. $\square$

Proposition 4.17. *Suppose the assumptions of Proposition* 4.16 *hold except condition* $(*)'_{n-2\sigma}$ *for* K_i. *Then* $|\nabla K_i(y_i)| \to 0$ *as* $i \to \infty$.

Proof. Suppose that

$$|\nabla K_i(y_i)| \to d > 0. \tag{4.37}$$

Without loss of generality, we assume $y_i = 0$. There are two cases.

Case 1: 0 is an isolated simple blow up point. In this case, we argue as in the proof of Lemma 4.13 and obtain

$$\left| \int_{B_1} \nabla K_i H_i^{\tau_i} u_i^{p_i+1} \right| \leq C u_i(0)^{-2} + C\tau_i.$$

It follows from the mean value theorem, Lemma 4.8 and Lemma 4.11 that

$$|\nabla K_i(0)| \leq C \int_{B_1} |\nabla K_i(y) - \nabla K_i(0)| H_i^{\tau_i} u_i^{p_i+1} + o(1) = o(1).$$

Case 2: 0 is not an isolated simple blow up point. In this case we argue as in the proof of Proposition 4.16. The only difference is that we cannot derive (4.35) from Corollary 4.15, since condition $(*)'_{n-2\sigma}$ for K_i is not assumed. Instead, we will use (4.37) to show (4.35).

Let μ_i, ϕ_i, Φ_i, $\tilde{K}_i$ and $\tilde{H}_i$ be as in the proof of Proposition 4.16. The computation at the end of the proof of Proposition 4.16 gives

$$\int_{\partial'\mathcal{B}_\delta^+} B'(Y, \Phi_i, \nabla\Phi_i, \delta, \sigma) \le -\frac{1}{p_i+1}\int_{B_\delta}\langle y, \nabla(\tilde{K}_i\tilde{H}_i^{\tau_i})\rangle\phi_i^{p_i+1} + C\phi_i(0)^{-(p_i+1)}.$$

Now we estimate the integral term $\int_{B_\delta}\langle y, \nabla(\tilde{K}_i\tilde{H}_i^{\tau_i})\rangle\phi_i^{p_i+1}$. Using Lemma 4.8 and arguing as in the proof of Lemma 4.8, we have

$$\tau_i \le C\phi_i(0)^{-2} + C\int_{B_\delta}|y|\,|\nabla\tilde{K}_i(y)|H_i^{\tau_i}\phi_i^{p_i+1} \le C\phi_i(0)^{-2} + C\mu_i\phi_i(0)^{-2/(n-2\sigma)}.$$

By (4.32),

$$\left|\int_{B_\delta}\nabla\tilde{K}_i\tilde{H}_i^{\tau_i}\phi_i^{p_i+1}\right| \le C\phi_i(y_i)^{-2} + C\tau_i.$$

It follows that

$$|\nabla\tilde{K}_i(0)| \le C\int_{B_\delta}|\nabla\tilde{K}_i(y) - \nabla\tilde{K}_i(0)|\phi_i^{p_i+1} + C\phi_i(0)^{-2} + C\tau_i$$

$$\le C\mu_i\phi_i(0)^{-2/(n-2\sigma)} + C\phi_i(0)^{-2} + C\tau_i.$$

Since $|\nabla\tilde{K}_i(0)| = \mu_i|\nabla K_i(0)| \ge (d/2)\mu_i$, we have

$$\mu_i \le C\phi_i(0)^{-2} + C\tau_i.$$

It follows that $\tau_i \le C\phi_i(0)^{-2}$ and $\mu_i \le C\phi_i(0)^{-2}$. Therefore,

$$\left|\int_{B_\delta}\langle y, \nabla(\tilde{K}_i\tilde{H}_i^{\tau_i})\rangle\phi_i^{p_i+1}\right| \le C\phi_i(0)^{-2-2/(n-2\sigma)},$$

and (4.35) follows immediately. $\qquad\square$

5. Estimates on the sphere and proofs of main theorems

Consider

$$P_\sigma(v) = c(n,\sigma)Kv^p \quad \text{on } \mathbb{S}^n, \tag{5.1}$$

where $p \in \left(1, \frac{n+2\sigma}{n-2\sigma}\right]$ and K satisfies

$$A_1^{-1} \le K \le A_1 \quad \text{on } \mathbb{S}^n, \tag{5.2}$$

and

$$\|K\|_{C^{1,1}(\mathbb{S}^n)} \le A_2. \tag{5.3}$$

Proposition 5.1. *Let $v \in C^2(\mathbb{S}^n)$ be a positive solution of (5.1). For any $0 < \varepsilon < 1$ and $R > 1$, there exist large positive constants C_1, C_2 depending on n, σ, A_1, A_2, ε and R such that, if*

$$\max_{\mathbb{S}^n} v \geq C_1,$$

then $\frac{n+2\sigma}{n-2\sigma} - p < \varepsilon$, and there exists a finite set $\wp(v) \subset \mathbb{S}^n$ such that:

(i) *If $P \in \wp(v)$, then it is a local maximum point of v and in the stereographic projection coordinate system $\{y_1, \ldots, y_n\}$ with P as the south pole,*

$$\|v^{-1}(P)v(v^{-(p-1)/2\sigma}(P)y) - (1 + k|y|^2)^{(2\sigma-n)/2}\|_{C^2(B_{2R})} \leq \varepsilon, \qquad (5.4)$$

where $k = K(P)^{1/\sigma}/4$.

(ii) *If P_1, P_2 belonging to $\wp(v)$ are two different points, then*

$$B_{Rv(P_1)^{-(p-1)/2\sigma}}(P_1) \cap B_{Rv(P_2)^{-(p-1)/2\sigma}}(P_2) = \emptyset.$$

(iii) *$v(P) \leq C_2\{\mathrm{dist}(P, \wp(v))\}^{-2\sigma/(p-1)}$ for all $P \in \mathbb{S}^n$.*

Proof. Given Theorem 1.8, Remark 1.9 and the proof of Proposition 4.4, the proof of Proposition 5.1 is similar to that of [71, Proposition 4.1] and [91, Lemma 3.1], and is omitted here. We refer to [71] and [91] for details. $\qquad\square$

Proposition 5.2. *Assume the hypotheses of Proposition 5.1 hold. Suppose that there is some constant $d > 0$ such that K satisfies $(*)'_{n-2\sigma}$ for some L in $\Omega_d = \{P \in \mathbb{S}^n : |\nabla K(P)| < d\}$. Then, for $\varepsilon > 0$, $R > 1$ and any solution v of (5.1) with $\max_{\mathbb{S}^n} v > C_1$, we have*

$$|P_1 - P_2| \geq \delta^* > 0 \quad \text{for any } P_1, P_2 \in \wp(v), \ P_1 \neq P_2,$$

where δ^* depends only on n, σ, δ, ε, R, A_1, A_2, L_2, d.

Proof. Suppose the contrary: there exist sequences $\{p_i\}$ and $\{K_i\}$ satisfying the above assumptions, and a sequence of corresponding solutions $\{v_i\}$ such that

$$\lim_{i \to \infty} |P_{1i} - P_{2i}| = 0, \quad P_{1i}, P_{2i} \in \wp(v_i), \quad |P_{1i} - P_{2i}| = \min_{\substack{P_1, P_2 \in \wp(v_i) \\ P_1 \neq P_2}} |P_1 - P_2|. \quad (5.5)$$

Since $B_{Rv_i(P_{1i})^{-(p_i-1)/2\sigma}}(P_{1i})$ and $B_{Rv_i(P_{2i})^{-(p_i-1)/2\sigma}}(P_{2i})$ have to be disjoint, we infer from (5.5) that $v_i(P_{1i}) \to \infty$ and $v_i(P_{2i}) \to \infty$. Therefore, we can pass to a subsequence (still denoted v_i) with $R_i \to \infty$, $\varepsilon_i \to 0$ as in Proposition 4.4 (ε_i depends on R_i and can be chosen as small as we need in the following arguments) such that, for y being the stereographic projection coordinate with south pole at P_{ji}, $j = 1, 2$, we have

$$\|m_i^{-1}v_i(m_i^{-(p_i-1)/2\sigma}y) - (1 + k_{ji}|y|^2)^{(2\sigma-n)/2}\|_{C^2(B_{2R_i}(0))} \leq \varepsilon_i, \qquad (5.6)$$

where $m_i = v_i(0)$, $k_{ji} = K_i(q_{ji})^{1/\sigma}/4$, $j = 1, 2$, $i = 1, 2, \ldots$.

In the stereographic coordinates with P_{1i} being the south pole, the equation (5.1) is transformed into

$$(-\Delta)^\sigma u_i(y) = c(n,\sigma) K_i(y) H_i^{\tau_i}(y) u_i(y)^{p_i}, \quad y \in \mathbb{R}^n, \qquad (5.7)$$

where

$$u_i(y) = \left(\frac{2}{1+|y|^2}\right)^{(n-2\sigma)/2} v_i(F(y)), \quad H_i(y) = \left(\frac{2}{1+|y|^2}\right)^{(n-2\sigma)/2}, \qquad (5.8)$$

and F is the inverse of the stereographic projection. Let us still use $P_{2i} \in \mathbb{R}^n$ to denote the stereographic coordinates of $P_{2i} \in \mathbb{S}^n$ and set $\vartheta_i = |P_{2i}| \to 0$. For simplicity, we assume P_{2i} is a local maximum point of u_i: we can always reselect a sequence of points as in the proof of Proposition 5.1 to substitute for P_{2i}.

From (ii) in Proposition 5.1, there exists some constant C, depending only on n, σ, such that

$$\vartheta_i > \frac{1}{C} \max\{R_i u_i(0)^{-(p_i-1)/2\sigma}, R_i u_i(P_{2i})^{-(p_i-1)/2\sigma}\}. \qquad (5.9)$$

Set

$$w_i(y) = \vartheta_i^{2\sigma/(p_i-1)} u_i(\vartheta_i y) \quad \text{in } \mathbb{R}^n.$$

It is easy to see that w_i, which is positive in $\mathbb{R}^n$, satisfies

$$(-\Delta)^\sigma w_i(y) = c(n,\sigma) \tilde{K}_i(y) \tilde{H}_i^{\tau_i}(y) w_i(y)^{p_i} \quad \text{in } \mathbb{R}^n \qquad (5.10)$$

and

$$w_i \in C^2(\mathbb{R}^n), \quad \liminf_{|y|\to\infty} w_i(y) < \infty,$$

where $\tilde{K}_i(y) = K_i(\vartheta_i y)$ and $\tilde{H}_i(y) = H_i(\vartheta_i y)$.

By Proposition 5.1, u_i satisfies

$$u_i(y) \le C_2 |y|^{-2\sigma/(p_i-1)} \qquad \text{for all } |y| \le \vartheta_i/2,$$

$$u_i(y) \le C_2 |y - P_{2i}|^{-2\sigma/(p_i-1)} \quad \text{for all } |y - P_{2i}| \le \vartheta_i/2.$$

In view of (5.9), we therefore have

$$\lim_{i\to\infty} w_i(0) = \infty, \quad \lim_{i\to\infty} w_i(|P_{2i}|^{-1}P_{2i}) = \infty,$$

$$|y|^{2\sigma/(p_i-1)} w_i(y) \le C_2, \quad |y| \le 1/2,$$

$$\left|y - |P_{2i}|^{-1}P_{2i}\right|^{2\sigma/(p_i-1)} w_i(y) \le C_2, \quad \left|y - |P_{2i}|^{-1}P_{2i}\right| \le 1/2.$$

After passing to a subsequence if necessary, there exists a point $\overline{P} \in \mathbb{R}^n$ with $|\overline{P}| = 1$ such that $|P_{2i}|^{-1}P_{2i} \to \overline{P}$ as $i \to \infty$. Hence 0 and $\overline{P}$ are both isolated blow up points of w_i.

If $|\nabla K_i(0)| \le d/2$, then 0 is an isolated simple blow up point of w_i by condition $(*)'_{n-2\sigma}$ and Proposition 4.16. If $|\nabla K_i(0)| \ge d/2$, arguing as in the proof of Proposition

4.17 we conclude that 0 is an isolated simple blow up point of w_i. Similarly, $\overline{P}$ is also an isolated simple blow up point of w_i.

By Proposition 4.9,

$$w_i(0)w_i(y) \leq C_\varepsilon \quad \text{for all } \varepsilon \leq |y| \leq 1/2,$$

where C_ε is independent of i. Let W_i be the extension of w_i. Due to Proposition 5.1, the Harnack inequality of Lemma 4.3, and the choice of P_{1i}, P_{2i}, there exists an at most countable set $\wp \subset \mathbb{R}^n$ such that

$$\inf\{|x - y| : x, y \in \wp, \ x \neq y\} \geq 1,$$

and

$$\lim_{i \to \infty} W_i(0)W_i(Y) = G(Y) \quad \text{in } C^0_{\text{loc}}(\overline{\mathbb{R}^{n+1}_+} \setminus \wp),$$

$$G(Y) > 0 \quad \text{for } Y \in \overline{\mathbb{R}^{n+1}_+} \setminus \wp.$$

Let $\wp_1 \subset \wp$ contain those points near which G is singular. Clearly, $0, \overline{P} \in \wp_1$. Since $p_i > 1$, it follows from (5.10) that

$$\begin{cases} \text{div}(s^{1-2\sigma}\nabla G) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\lim_{s \to 0} s^{1-2\sigma}\partial_s G(y, s) = 0 & \text{for all } y \in \mathbb{R}^n \setminus \wp_1. \end{cases}$$

By Lemma 4.10 and the maximum principle, there exist positive constants N_1, N_2 and some nonnegative function H satisfying

$$\begin{cases} \text{div}(s^{1-2\sigma}\nabla H) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\lim_{s \to 0} s^{1-2\sigma}\partial_s H(y, s) = 0 & \text{for all } y \in \mathbb{R}^n \setminus \{\wp_1 \setminus \{0, \overline{P}\}\}, \end{cases}$$

such that

$$G(Y) = N_1|Y|^{2\sigma-n} + N_2|Y - \overline{P}|^{2\sigma-n} + H(Y), \quad Y \in \overline{\mathbb{R}^{n+1}_+} \setminus \{\wp_1\}.$$

Applying Proposition 2.19 to H, it is not difficult to verify (4.36) with Φ_i replaced by W_i. On the other hand, we can establish (4.35) with Φ_i replaced by W_i if $|\nabla K_i(0)| \leq d/2$, because condition $(*)'_{n-2\sigma}$ with $L = o(1)$ holds for $\tilde{K}_i$ and thus Corollary 4.15 holds. If $|\nabla K_i(0)| \geq d/2$, we can apply the argument in the proof of Proposition 4.17 to conclude that $\vartheta_i, \tau_i \leq w_i(0)^{-2}$, and hence (4.35) also holds for W_i.

Proposition 5.2 is established. □

Consider

$$\begin{aligned} P_\sigma(v) &= c(n, \sigma)K_i v_i^{p_i} \quad \text{on } \mathbb{S}^n, \\ v_i &> 0 \quad \text{on } \mathbb{S}^n, \\ p_i &= \frac{n + 2\sigma}{n - 2\sigma} - \tau_i, \quad \tau_i \geq 0, \ \tau_i \to 0. \end{aligned} \tag{5.11}$$

Theorem 5.3. *Suppose K_i satisfies the assumptions on K in Proposition* 5.2*. Let v_i be solutions of* (5.11)*. Then*

$$\|v_i\|_{H^\sigma(\mathbb{S}^n)} \leq C, \tag{5.12}$$

where $C > 0$ depends only on n, σ, A_1, A_2, L, d. Furthermore, after passing to a subsequence, either $\{v_i\}$ stays bounded in $L^\infty(\mathbb{S}^n)$, or $\{v_i\}$ has only isolated simple blow up points and the distance between any two such points is bounded below by some positive constant depending only on n, σ, A_1, A_2, L, d.

Proof. The theorem follows immediately from Propositions 5.2, 4.17, 4.16, 4.4 and Lemma 4.11. □

Proof of Theorem 1.6. This follows immediately from Theorem 5.3. □

In the next theorem, we impose a stronger condition on K_i such that $\{u_i\}$ has at most one blow up point.

Theorem 5.4. *Suppose the assumptions of Theorem* 5.3 *hold. Suppose further that either $\{K_i\}$ satisfies condition $(*)'_{n-2\sigma}$ for some sequences $L(n - 2\sigma, i) = o(1)$ in $\Omega_{d,i} = \{q \in \mathbb{S}^n : |\nabla_{g_{\mathbb{S}^n}} K_i| < d\}$, or $\{K_i\}$ satisfies condition $(*)'_\beta$ with $\beta > n - 2\sigma$ in $\Omega_{d,i}$. Then, after passing to a subsequence, either $\{v_i\}$ stays bounded in $L^\infty(\mathbb{S}^n)$, or $\{v_i\}$ has precisely one isolated simple blow up point.*

Proof. The strategy is the same as in the proof of Proposition 5.2. We assume there are two isolated blow up points. After some transformation, we can assume that they are in the same half-sphere. The condition of $\{K_i\}$ guarantees that Corollary 4.3 holds for u_i, where u_i is as in (5.8). Hence (4.35) holds for U_i, which is the extension of u_i. Moreover, (4.36) for U_i is also valid, since the distance between these blow up points is uniformly bounded below due to Proposition 5.2. □

Proof of Theorem 1.7. By Theorem 5.4, we only need to show the latter case of the theorem. After passing a subsequence, $\xi_i \to \overline{\xi}$ is the only isolated simple blow up point of v_i. For simplicity, assume that ξ_i is identical to the south pole and $K(\xi_i) = 1$. Let $F : \mathbb{R}^n \to \mathbb{S}^n$ be the inverse of the stereographic projection defined at the beginning of the paper. Define, for any $\lambda > 0$,

$$\psi_\lambda : x \mapsto \lambda x, \quad \forall x \in \mathbb{R}^n.$$

Set $\varphi_i = F \circ \psi_{\lambda_i} \circ F^{-1}$ with $\lambda_i = v_i(\xi_i)^{-2/(n-2\sigma)}$. Then $T_{\varphi_i} v_i$ satisfies

$$P_\sigma(T_{\varphi_i} v_i) = c(n, \sigma) K \circ \varphi_i T_{\varphi_i} v_i^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{on } \mathbb{S}^n.$$

For $x \in \mathbb{R}^n$, let

$$u_i(x) = \left(\frac{2}{1 + |x|^2}\right)^{(n-2\sigma)/2} v_i \circ F(x)$$

and

$$\tilde{u}_i(x) = \left(\frac{2}{1 + |x|^2}\right)^{(n-2\sigma)/2} T_{\varphi_i} v_i \circ F(x).$$

Note that

$$|\det d\varphi_i(F(x))|^{(n-2\sigma)/2n} = \left(\left(\frac{2}{1+|\lambda_i x|^2}\right)^n \lambda_i^n \left(\frac{2}{1+|x|^2}\right)^{-n}\right)^{(n-2\sigma)/2n}.$$

Hence, $\tilde{u}_i(x) = \lambda^{(n-2\sigma)/2} u_i(\lambda_i x)$ for any $x \in \mathbb{R}^n$ and $0 < u_i \leq 2^{(n-2\sigma)/2}$. Arguing as before, we see that

$$\tilde{u}_i(x) \to \left(\frac{2}{1+|x|^2}\right)^{(n-2\sigma)/2} \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n).$$

Therefore, $v_i \to 1$ in $C^2_{\text{loc}}(\mathbb{S}^n \setminus \{N\})$, where N is the north pole of $\mathbb{S}^n$.

Since $T_{\varphi_i} v_i$ is uniformly bounded near the north pole, it follows from Hölder estimates that there exists a constant $\alpha \in (0, 1)$ such that $T_{\varphi_i} v_i \to f$ in $C^\alpha(B_\delta(N))$ for a small constant $\delta > 0$ and some function $f \in C^\alpha(B_\delta(N))$. It is clear that $f = 1$. Thus, the proof is complete. $\qquad\square$

Theorem 5.5. *Suppose that $\{K_i\} \subset C^\infty(\mathbb{S}^n)$ satisfies* (5.3)*, and for some $P_0 \in \mathbb{S}^n$, $\varepsilon_0 > 0$, $A_1 > 0$ independent of i and $1 < \beta < n$,*

$$\{K_i\} \text{ is bounded in } C^{[\beta],\beta-[\beta]}(B_{\varepsilon_0}(q_0)), \qquad K_i(P_0) \geq A_1$$

and

$$K_i(y) = K_i(0) + Q_i^{(\beta)}(y) + R_i(y), \qquad |y| \leq \varepsilon_0,$$

where y is the stereographic projection coordinate with P_0 as the south pole, $Q_i^{(\beta)}(y)$ satisfies $Q_i^{(\beta)}(\lambda y) = \lambda^\beta Q_i^{(\beta)}(y)$ for all $\lambda > 0$ and $y \in \mathbb{R}^n$, and $R_i(y)$ satisfies

$$\sum_{s=0}^{[\beta]} |\nabla^s R_i(y)| \, |y|^{-\beta+s} \to 0$$

uniformly for i as $y \to 0$.

Suppose also that $Q_i^{(\beta)} \to Q^{(\beta)}$ in $C^1(\mathbb{S}^{n-1})$ and for some positive constant A_6,

$$A_6|y|^{\beta-1} \leq |\nabla Q^{(\beta)}(y)|, \qquad |y| \leq \varepsilon_0, \tag{5.13}$$

and

$$\left(\begin{matrix} \int_{\mathbb{R}^n} \nabla Q^{(\beta)}(y+y_0)(1+|y|^2)^{-n} \, dy \\ \int_{\mathbb{R}^n} Q^{(\beta)}(y+y_0)(1+|y|^2)^{-n} \, dy \end{matrix}\right) \neq 0, \qquad \forall y_0 \in \mathbb{R}^n. \tag{5.14}$$

Let v_i be positive solutions of (1.6) *with $K = K_i$. If P_0 is an isolated simple blow up point of v_i, then v_i has to have at least another blow up point.*

Proof. Suppose the contrary: P_0 is the only blow up point of v_i.

We make a stereographic projection with P_0 being the south pole to the equatorial plane of $\mathbb{S}^n$, with inverse F. Then the equation (1.6) is transformed to

$$(-\Delta)^\sigma u_i = c(n,\sigma) K_i(y) u_i^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{in } \mathbb{R}^n, \tag{5.15}$$

with

$$u_i(y) = \left(\frac{2}{1+|y|^2}\right)^{(n-2\sigma)/2} v_i(F(y)).$$

Let $y_i \to 0$ be the local maximum point of u_i. It follows from Lemma 4.13 that

$$|\nabla K_i(y_i)| = O(u_i(y_i)^{-2} + u_i(y_i)^{-2(\beta-1)/(n-2\sigma)}).$$

First, we establish

$$|y_i| = O(u_i(y_i)^{-2/(n-2\sigma)}). \tag{5.16}$$

Since we have assumed that v_i has no blow up point other than P_0, it follows from Proposition 4.9 and the Harnack inequality that $u_i(y) \leq C(\varepsilon)|y|^{2\sigma-n} u_i(y_i)^{-1}$ for $|y| \geq \varepsilon > 0$.

By Proposition A.1, we have

$$\int_{\mathbb{R}^n} \nabla K_i u_i^{2n/(n-2\sigma)} = 0. \tag{5.17}$$

It follows that for $\varepsilon > 0$ small we have

$$\left| \int_{B_\varepsilon} \nabla K_i(y+y_i) u_i(y+y_i)^{2n/(n-2\sigma)} \right| \leq C(\varepsilon) u_i(y_i)^{-2n/(n-2\sigma)}.$$

Using our hypotheses on $\nabla Q^{(\beta)}$ and R_i, we have

$$\left| \int_{B_\varepsilon} (1 + o_\varepsilon(1)) \nabla Q_i^{(\beta)}(y+y_i) u_i(y+y_i)^{2n/(n-2\sigma)} \right| \leq C(\varepsilon) u_i(y_i)^{-2n/(n-2\sigma)}.$$

Multiplying the above by $m_i^{(2/(n-2\sigma))(\beta-1)}$ with $m_i = u_i(y_i)$, we have

$$\left| \int_{B_\varepsilon} (1 + o_\varepsilon(1)) \nabla Q_i^{(\beta)}(m_i^{2/(n-2\sigma)} y + \tilde{y}_i) u_i(y+y_i)^{2n/(n-2\sigma)} \right|$$
$$\leq C(\varepsilon) u_i(y_i)^{(2/(n-2\sigma))(\beta-1-n)},$$

where $\tilde{y}_i = m_i^{2/(n-2\sigma)} y_i$. Suppose (5.16) is false, namely, $\tilde{y}_i \to +\infty$ along a subsequence. Then it follows from Proposition 4.4 (we may choose $R_i \leq |\tilde{y}_i|/4$) that

$$\left| \int_{|y| \leq R_i m_i^{-2/(n-2\sigma)}} (1 + o_\varepsilon(1)) \nabla Q_i^{(\beta)}(m_i^{2/(n-2\sigma)} y + \tilde{y}_i) u_i(y+y_i)^{2n/(n-2\sigma)} \right|$$
$$= \left| \int_{|z| \leq R_i} (1 + o_\varepsilon(1)) \nabla Q_i^{(\beta)}(z + \tilde{y}_i) \left(m_i^{-1} u_i(m_i^{-2/(n-2\sigma)} z + y_i)\right)^{2n/(n-2\sigma)} \right| \sim |\tilde{y}_i|^{\beta-1}.$$

On the other hand, it follows from Lemma 4.11 that

$$\left| \int_{R_i m_i^{-2/(n-2\sigma)} \leq |y| \leq \varepsilon} (1 + o_\varepsilon(1)) \nabla Q_i^{(\beta)}(m_i^{2/(n-2\sigma)} y + \tilde{y}_i) u_i(y + y_i)^{2n/(n-2\sigma)} \right|$$

$$\leq C \left| \int_{R_i m_i^{-2/(n-2\sigma)} \leq |y| \leq \varepsilon} (|m_i^{2/(n-2\sigma)} y|^{\beta-1} + |\tilde{y}_i|^{\beta-1}) u_i(y + y_i)^{2n/(n-2\sigma)} \right| \leq o(1)|\tilde{y}_i|^{\beta-1}.$$

It follows that

$$|\tilde{y}_i|^{\beta-1} \leq C(\varepsilon) m_i^{(2/(n-2\sigma))(\beta-1-n)},$$

which implies that

$$|y_i| \leq C(\varepsilon) m_i^{-(2/(n-2\sigma))(n/(\beta-1))} = o(m_i^{-2/(n-2\sigma)}).$$

This contradicts $\tilde{y}_i \to \infty$. Thus (5.16) holds.

We are going to find some y_0 such that (5.14) fails.

It follows from the Kazdan–Warner condition of Proposition A.1 that

$$\int_{\mathbb{R}^n} \langle y, \nabla K_i(y + y_i)\rangle u_i(y + y_i)^{2n/(n-2\sigma)} = 0. \tag{5.18}$$

Since P_0 is an isolated simple blow up point and the only blow up point of v_i, we have for any $\varepsilon > 0$,

$$\left| \int_{B_\varepsilon} \langle y, \nabla K_i(y + y_i)\rangle u_i(y + y_i)^{2n/(n-\sigma)} \right| \leq C(\varepsilon) u_i(y_i)^{-2n/(n-2\sigma)}.$$

It follows from Lemma 4.11 and the expression of K_i that

$$\left| \int_{B_\varepsilon} \langle y, \nabla Q_i^{(\beta)}(y + y_i)\rangle u_i(y + y_i)^{2n/(n-2\sigma)} \right|$$

$$\leq C(\varepsilon) u_i(y_i)^{-2n/(n-2\sigma)} + o_\varepsilon(1) \int_{B_\varepsilon} |y| \, |y + y_i|^{\beta-1} u_i(y + y_i)^{-2n/(n-2\sigma)}$$

$$\leq C(\varepsilon) u_i(y_i)^{-2n/(n-2\sigma)} + o_\varepsilon(1) \int_{B_\varepsilon} (|y|^\beta + |y| \, |y_i|^{\beta-1}) u_i(y + y_i)^{-2n/(n-2\sigma)}$$

$$\leq C(\varepsilon) u_i(y_i)^{-2n/(n-2\sigma)} + o_\varepsilon(1) u_i(y_i)^{-2\beta/(n-2\sigma)},$$

we used (5.16) in the last inequality.

Multiplying the above by $u_i(y_i)^{2\beta/(n-2\sigma)}$, due to $\beta < n$ we obtain

$$\lim_{i \to \infty} u_i(y_i)^{2\beta/(n-2\sigma)} \left| \int_{B_\varepsilon} \langle y, \nabla Q_i^{(\beta)}(y + y_i)\rangle u_i(y + y_i)^{2n/(n-2\sigma)} \right| = o_\varepsilon(1). \tag{5.19}$$

Let $R_i \to \infty$ as $i \to \infty$. We assume that $r_i := R_i u_i(y_i)^{-2/(n-2\sigma)} \to 0$ as we did in Proposition 4.4. By Lemma 4.11, we have

$$u_i(y_i)^{2\beta/(n-2\sigma)} \left| \int_{r_i \le |y| \le \varepsilon} \langle y, \nabla Q_i^{(\beta)}(y + y_i) \rangle u_i(y + y_i)^{2n/(n-2\sigma)} \right|$$

$$\le \lim_{i \to \infty} u_i(y_i)^{2\beta/(n-2\sigma)} \left| \int_{r_i \le |y| \le \varepsilon} (|y|^\beta + |y||y_i|^{\beta-1}) u_i(y + y_i)^{2n/(n-2\sigma)} \right| \to 0 \quad (5.20)$$

as $i \to \infty$. Combining (5.19) and (5.20), we conclude that

$$\lim_{i \to \infty} u_i(y_i)^{2\beta/(n-2\sigma)} \left| \int_{B_{r_i}} \langle y, \nabla Q_i^{(\beta)}(y + y_i) \rangle u_i(y + y_i)^{2n/(n-2\sigma)} \right| = o_\varepsilon(1).$$

It follows from the change of variable $z = u_i(y_i)^{2/(n-2\sigma)} y$, applying Proposition 4.4 and then letting $\varepsilon \to 0$ that

$$\int_{\mathbb{R}^n} \langle z, \nabla Q^{(\beta)}(z + z_0) \rangle (1 + k|z|^2)^{-n} = 0, \quad (5.21)$$

where $z_0 = \lim_{i \to \infty} u_i(y_i)^{2/(n-2\sigma)} y_i$ and $k = \lim_{i \to \infty} K_i(y_i)^{1/\sigma}/4$.

On the other hand, from (5.17),

$$\int_{\mathbb{R}^n} \nabla K_i(y + y_i) u_i(y + y_i)^{2n/(n-2\sigma)} = 0. \quad (5.22)$$

Arguing as above, we will have

$$\int_{\mathbb{R}^n} \nabla Q^{(\beta)}(z + z_0)(1 + k|z|^2)^{-n} = 0. \quad (5.23)$$

It follows from (5.21) and (5.23) that

$$\int_{\mathbb{R}^n} Q^{(\beta)}(z + z_0)(1 + k|z|^2)^{-n} \, dz$$
$$= \beta^{-1} \int_{\mathbb{R}^n} \langle z + z_0, \nabla Q^{(\beta)}(z + z_0) \rangle (1 + k|z|^2)^{-n} \, dz = 0.$$

Therefore, (5.14) does not hold for $y_0 = \sqrt{k} z_0$. $\square$

Theorem 5.6. *Suppose that $K \in C^{1,1}(\mathbb{S}^n)$ and for some constant $A_1 > 0$,*

$$1/A_1 \le K(\xi) \le A_1 \quad \text{for all } \xi \in \mathbb{S}^n.$$

Suppose also that for any critical point ξ_0 of K, under the stereographic projection coordinate system $\{y_1, \ldots, y_n\}$ with ξ_0 as south pole, there exist some small neighborhood $\mathcal{O}$ of 0, a positive constant L, and $\beta = \beta(\xi_0) \in (n - 2\sigma, n)$ such that

$$\|\nabla^{[\beta]} K\|_{C^{\beta - [\beta]}(\mathcal{O})} \le L$$

and

$$K(y) = K(0) + Q^{(\beta)}_{(\xi_0)}(y) + R_{(\xi_0)}(y) \quad in \ \mathscr{O},$$

where $Q^{(\beta)}_{\xi_0} \in C^{[\beta]-1,1}(\mathbb{S}^{n-1})$ *satisfies* $Q^{(\beta)}_{\xi_0}(\lambda y) = \lambda^\beta Q^{(\beta)}_{\xi_0}(y)$ *for all* $\lambda > 0$ *and* $y \in \mathbb{R}^n$, *and for some positive constant* A_6,

$$A_6 |y|^{\beta-1} \le |\nabla Q^{(\beta)}(y)|, \quad y \in \mathscr{O},$$

and

$$\begin{pmatrix} \int_{\mathbb{R}^n} \nabla Q^{(\beta)}(y + y_0)(1 + |y|^2)^{-n} \, dy \\ \int_{\mathbb{R}^n} Q^{(\beta)}(y + y_0)(1 + |y|^2)^{-n} \, dy \end{pmatrix} \neq 0, \quad \forall y_0 \in \mathbb{R}^n,$$

and $R_{\xi_0} \in C^{[\beta]-1,1}(\mathscr{O})$ *satisfies* $\lim_{y \to 0} \sum_{s=0}^{[\beta]} |\nabla^s R|\xi_0(y)||y|^{-\beta+s} = 0$. *Then there exists a positive constant* $C \ge 1$ *depending on* n, σ, K *such that for any solution* v *of* (1.6),

$$1/C \le v \le C \quad on \ \mathbb{S}^n.$$

Proof. This follows directly from Theorems 5.4 and 5.5. □

Proof of the compactness part of Theorem 1.3. It is easy to check that, if K satisfies the condition in Theorem 1.3, then it must satisfy the condition in Theorem 5.6. Therefore, we have the lower and upper bounds of v. The C^2 norm bound of v follows immediately.

□

Appendix

A.1. A Kazdan–Warner identity

In this section we are going to show (1.8), which is a consequence of the following

Proposition A.1. *Let* $K > 0$ *be a* C^1 *function on* $\mathbb{S}^n$, *and let* v *be a positive function in* $C^2(\mathbb{S}^n)$ *satisfying*

$$P_\sigma(v) = K v^{\frac{n+2\sigma}{n-2\sigma}} \quad on \ \mathbb{S}^n. \tag{A.1}$$

Then, for any conformal Killing vector field X *on* $\mathbb{S}^n$, *we have*

$$\int_{\mathbb{S}^n} (\nabla_X K) v^{2n/(n-2\sigma)} \, dV_{g_{\mathbb{S}^n}} = 0. \tag{A.2}$$

Let $\varphi_t : \mathbb{S}^n \to \mathbb{S}^n$ be a one-parameter family of conformal diffeomorphisms (in this case they are Möbius transformations), depending smoothly on t, $|t| < 1$, and $\varphi_0 =$ identity. Then

$$X := \frac{d}{dt}(\varphi_t)^{-1}\Big|_{t=0} \quad \text{is a conformal Killing vector field on } \mathbb{S}^n. \tag{A.3}$$

Proof. The proof is standard (see, e.g., [12] for a Kazdan–Warner identity for prescribed scalar curvature problems) and we include it here for completeness. Since P_σ is a self-adjoint operator, (A.1) has a variational structure:

$$I[v] := \frac{1}{2} \int_{\mathbb{S}^n} v P_\sigma(v) \, dV_{g_{\mathbb{S}^n}} - \frac{n-2\sigma}{2n} \int_{\mathbb{S}^n} K v^{2n/(n-2\sigma)} \, dV_{g_{\mathbb{S}^n}}.$$

Let X be a conformal Killing vector field. Then there exists $\{\varphi_t\}$ satisfying (A.3). Let

$$v_t := (v \circ \varphi_t) w_t,$$

where w_t is given by

$$g_t := \varphi_t^* g_{\mathbb{S}^n} = w_t^{4/(n-2\sigma)} g_{\mathbb{S}^n}.$$

Then

$$I[v_t] = \frac{1}{2} \int_{\mathbb{S}^n} v P_\sigma(v) \, dV_{g_{\mathbb{S}^n}} - \frac{n-2\sigma}{2n} \int_{\mathbb{S}^n} K(\varphi_t^{-1}(x)) v^{2n/(n-2\sigma)} \, dV_{g_{\mathbb{S}^n}}.$$

It follows from (A.1) that

$$0 = I'[v]\left(\frac{d}{dt}\bigg|_{t=0} v_t \right) = \frac{d}{dt} I[v_t]\bigg|_{t=0} = -\frac{n-2\sigma}{2n} \int_{\mathbb{S}^n} (\nabla_X K) v^{2n/(n-2\sigma)} \, dV_{g_{\mathbb{S}^n}}. \qquad \square$$

A.2. The proof of Lemma 4.10

The classical Bôcher theorem in harmonic function theory states that a positive harmonic function u in a punctured ball $B_1 \setminus \{0\}$ must be of the form

$$u(x) = \begin{cases} -a \log |x| + h(x), & n = 2, \\ a|x|^{2-n} + h(x), & n \geq 3, \end{cases}$$

where a is a nonnegative constant and h is a harmonic function in B_1.

We are going to establish a similar result, Lemma 4.10, in our setting. Denote $\mathcal{B}_R^+ = \{X : |X| < R, t > 0\}$, $\partial'\mathcal{B}_R^+ = \{(x, t) : |x| < R\}$ and $\partial''\mathcal{B} = \partial\mathcal{B}_R^+ \setminus \partial'\mathcal{B}_R^+$.

Proof of Lemma 4.10. We adapt the proof of the Bôcher theorem given in [5]. Define

$$A[U](r) = \frac{\int_{\partial''\mathcal{B}_r^+} t^{1-2\sigma} U(x, t) \, dS_r}{\int_{\partial''\mathcal{B}_r^+} t^{1-2\sigma} \, dS_r},$$

where $r = |(x, t)| > 0$ and dS_r is the volume element of $\partial''\mathcal{B}_r$.

By direct computations we have

$$\frac{d}{dr} A[U](r) = \frac{\int_{\partial''\mathcal{B}_r^+} t^{1-2\sigma} \nabla U(x, t) \cdot \frac{(x,t)}{r} \, dS_r}{\int_{\partial''\mathcal{B}_r^+} t^{1-2\sigma} \, dS_r}.$$

Let

$$f(r) = \int_{\partial''\mathcal{B}_r^+} t^{1-2\sigma} \nabla U(x, t) \cdot \frac{(x, t)}{r} \, dS_r.$$

Since U satisfies (4.18), by integration by parts we have

$$f(r_1) = f(r_2), \quad \forall \, 0 < r_1, r_2 < 1.$$

Notice that

$$\int_{\partial'' \mathcal{B}_r^+} t^{1-2\sigma} \, dS_r = r^{n+1-2\sigma} \int_{\partial'' \mathcal{B}_1^+} t^{1-2\sigma} \, dS_1.$$

Thus there exists a constant b such that

$$\frac{d}{dr} A[U](r) = b r^{-n-1+2\sigma}.$$

So there exist constants a and b such that

$$A[U](r) = a + b r^{2\sigma - n}.$$

Since we have the Harnack inequalities for U as in the proof of Lemma 4.3, the rest of the arguments are rather similar to those in [5] and are omitted here. We refer to [5] for details. $\qquad\square$

A.3. Two lemmas on maximum principles

Lemma A.2. *There exists $\varepsilon = \varepsilon(n, \sigma)$ such that for all $|a(x)| \leq \varepsilon |x|^{-2\sigma}$, if $U \in H(t^{1-2\sigma}, Q_1)$, $U \geq 0$ on $\partial'' Q_1$, and is a supersolution of (2.7) in Q_1 with $b \equiv 0$, then*

$$U \geq 0 \quad \text{in } Q_1.$$

Proof. By a density argument, we can use U^- as a test function. Hence we have

$$\int_{Q_1} t^{1-2\sigma} |\nabla U^-|^2 \leq \int_{B_1} |a| (U^-(\cdot, 0))^2. \tag{A.4}$$

We extend U^- to be zero outside of Q_1 and still denote it by U^-. Then its trace satisfies $U^-(\cdot, 0) \in \dot{H}^\sigma(\mathbb{R}^n)$. Since

$$N_\sigma^2 \|U^-(\cdot, 0)\|_{\dot{H}^\sigma(\mathbb{R}^n)}^2 = \int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} |\nabla \mathcal{P}_\sigma * U^-(\cdot, 0)|^2 \leq \int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} |\nabla U^-|^2,$$

we have

$$N_\sigma^2 \|U^-(\cdot, 0)\|_{\dot{H}^\sigma(\mathbb{R}^n)}^2 \leq \int_{B_1} |a| (U^-(\cdot, 0))^2.$$

By Hardy's inequality (see, e.g., [99])

$$C_1(n, \sigma) \int_{\mathbb{R}^n} |x|^{-2\sigma} (U^-(\cdot, 0))^2 \leq \|U^-(\cdot, 0)\|_{\dot{H}^\sigma(\mathbb{R}^n)}^2,$$

where $C_1(n, \sigma) = 2^{2\sigma} \frac{\Gamma((n+2\sigma)/4)}{\Gamma((n-2\sigma)/4)}$ is the best constant. Hence if $\varepsilon < N_\sigma^2 C_1(n, \sigma)$, then $U^-(\cdot, 0) \equiv 0$ and hence by (A.4), $U^- \equiv 0$ in Q_1. $\qquad\square$

Lemma A.3. *Let $a \in L^\infty(B_1)$. Let $W \in C(\overline{Q_1}) \cap C^2(Q_1)$ satisfy $\nabla_x W \in C(\overline{Q_1})$, $t^{1-2\sigma} \partial_t W \in C(\overline{Q_1})$, and*

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma} \nabla W) \geq 0 & \text{in } Q_1, \\ -\lim_{t \to 0} t^{1-2\sigma} \partial_t W(x, t) \geq a(x)W(x, 0) & \text{on } \partial' Q_1, \\ W > 0 & \text{in } \overline{Q_1}. \end{cases} \tag{A.5}$$

If $U \in C(\overline{Q_1}) \cap C^2(Q_1)$ satisfies $\nabla_x U \in C(\overline{Q_1})$, $t^{1-2\sigma} \partial_t U \in C(\overline{Q_1})$, and

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma} \nabla U) \geq 0 & \text{in } Q_1, \\ -\lim_{t \to 0} t^{1-2\sigma} \partial_t U(x, t) \geq a(x)U(x, 0) & \text{on } \partial' Q_1, \\ U \geq 0 & \text{in } \partial'' Q_1. \end{cases} \tag{A.6}$$

Then $U \geq 0$ in Q_1.

Proof. Let $V = U/W$. Then

$$\begin{cases} -\operatorname{div}(t^{1-2\sigma} \nabla V) - 2t^{1-2\sigma} \dfrac{\nabla V \nabla W}{W} - \dfrac{\operatorname{div}(t^{1-2\sigma} \nabla W)V}{W} \geq 0 & \text{in } Q_1, \\ -\lim_{t \to 0} t^{1-2\sigma} \partial_t V + \dfrac{V}{W}\big(-\lim_{t \to 0} t^{1-2\sigma} \partial_t W(x, t) - a(x)W(x, 0) \big) \geq 0 & \text{on } \partial' Q_1, \\ V \geq 0 & \text{in } \partial'' Q_1. \end{cases} \tag{A.7}$$

We are going to show that $V \geq 0$ in Q_1. If not, then we choose k such that $\inf_{Q_1} v < k \leq 0$. Let

$$V_k = V - k \quad \text{and} \quad V_k^- = \max(-V_k, 0).$$

Multiplying by V_k^- the first inequality in (A.7) and integrating by parts, we have

$$\int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \leq 2 \int_{Q_1} t^{1-2\sigma} W^{-1} V_k^- \nabla V_k^- \nabla W. \tag{A.8}$$

Case 1: $1 - 2\sigma \leq 0$. Denote $\Gamma_k = \operatorname{supp}(\nabla V_k^-)$. Then by the Hölder inequality and the bounds of $\nabla_x W$ and $t^{1-2\sigma} \partial_t W$,

$$2 \int_{Q_1} t^{1-2\sigma} W^{-1} V_k^- \nabla V_k^- \nabla W \leq C \left(\int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \right)^{1/2} \left(\int_{\Gamma_k} t^{1-2\sigma} |V_k^-|^2 \right)^{1/2}.$$

Hence it follows from (A.8) that

$$\int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \leq C \int_{\Gamma_k} t^{1-2\sigma} |V_k^-|^2. \tag{A.9}$$

Since $V_k^- = 0$ on $\partial'' Q_1$, by Lemma 2.1 in [94],

$$\left(\int_{Q_1} t^{1-2\sigma} |V_k^-|^{2(n+1)/n} \right)^{n/(n+1)} \leq C \int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2. \tag{A.10}$$

By (A.9), (A.10) and the Hölder inequality,

$$\int_{\Gamma_k} t^{1-2\sigma} \geq C.$$

This yields a contradiction when $k \to \inf_{Q_1} v$, since $\nabla V = 0$ on the set where $V \equiv \inf_{Q_1} V$.

Case 2: $1 - 2\sigma > 0$. Denote $\Gamma_k = \text{supp}(V_k^-)$. Then by the Hölder inequality and the bounds of $\nabla_x W$ and $t^{1-2\sigma} \partial_t W$,

$$\int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \leq 2 \int_{Q_1} t^{1-2\sigma} W^{-1} V_k^- \nabla V_k^- \nabla W \leq C \int_{Q_1} V_k^- \nabla V_k^-$$
$$\leq C \left(\int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \right)^{1/2} \left(\int_{Q_1} t^{2\sigma-1} |V_k^-|^2 \right)^{1/2}.$$

Hence

$$\int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \leq C \int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \int_{Q_1} t^{2\sigma-1} |V_k^-|^2.$$

Since $V_k^- = 0$ on $\partial'' Q_1$, by the proof of Lemma 2.3 in [94] we have, for any $\beta > -1$,

$$\int_{Q_1} t^\beta |V_k^-|^2 \leq C(\beta) \int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2.$$

In the following we choose $\beta = \sigma - 1$. Hence,

$$\int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \int_{Q_1} t^{\sigma-1} |V_k^-|^2 \leq C \int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \int_{Q_1} t^{2\sigma-1} |V_k^-|^2,$$

i.e.

$$\int_{\Gamma_k} t^{1-2\sigma} |\nabla V_k^-|^2 \int_{\Gamma_k} t^{\sigma-1} |V_k^-|^2 \leq C \int_{\Gamma_k} t^{1-2\sigma} |\nabla V_k^-|^2 \int_{\Gamma_k} t^{2\sigma-1} |V_k^-|^2.$$

Fix $\varepsilon > 0$ sufficiently small which will be chosen later. By the strong maximum principle, $\inf_{Q_1} V$ has to be attained only on $\partial' Q_1$, then we can choose k sufficiently close to $\inf_{Q_1} V$ such that $\Gamma_k \subset B_1 \times [0, \varepsilon]$. Then

$$\varepsilon^{-\sigma} \int_{\Gamma_k} t^{2\sigma-1} |V_k^-|^2 \leq C \int_{\Gamma_k} t^{\sigma-1} |V_k^-|^2.$$

Choose ε small enough such that $\varepsilon^{-\sigma} > C + 1$. It follows that

$$\int_{\Gamma_k} t^{1-2\sigma} |\nabla V_k^-|^2 \int_{\Gamma_k} t^{2\sigma-1} |V_k^-|^2 = 0.$$

Hence one of them has to be zero, which yields a contradiction immediately. $\qquad\square$

A.4. Complements

Lemma A.4. *Let* $u \in C_c^\infty(\mathbb{R}^n)$ *and* $V(\cdot, t) = \mathcal{P}_\sigma(\cdot, t) * u(\cdot)$. *Then for any* U *in* $C_c^\infty(\mathbb{R}_+^{n+1} \cup \partial\mathbb{R}_+^{n+1})$ *with* $U(x, 0) = u(x)$,

$$\int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} |\nabla V|^2 \leq \int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} |\nabla U|^2.$$

Proof. Let $0 \leq \eta(x, t) \leq 1$, $\text{supp}(\eta) \subset \mathcal{B}_{2R}^+$, $\eta = 1$ in $\mathcal{B}_R^+$ and $|\nabla \eta| \leq 2/R$. In the end we will let $R \to \infty$ and hence we may assume that U is supported in $\overline{\mathcal{B}_{R/2}^+}$. Since $\text{div}(t^{1-2\sigma}\nabla V) = 0$, we have

$$
\begin{aligned}
0 &= \int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} \nabla V \nabla(\eta(U - V)) \\
&= \int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} \eta \nabla U \nabla V - \int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} \eta |\nabla V|^2 - \int_{\mathcal{B}_{2R}^+ \setminus \mathcal{B}_R^+} t^{1-2\sigma} V \nabla \eta \nabla V,
\end{aligned}
$$

where we used $\eta(U - V) = 0$ on the boundary of $\mathcal{B}_{2R}^+$ in the first equality.

Note that for $(x, t) \in \mathcal{B}_{2R}^+ \setminus \mathcal{B}_R^+$,

$$
\begin{aligned}
|V(x, t)| &= \beta(n, \sigma) \left| \int_{\mathbb{R}^n} \frac{t^{2\sigma}}{(|x - \xi|^2 + t^2)^{(n+2\sigma)/2}} u(\xi) \, d\xi \right| \\
&\leq \beta(n, \sigma) \int_{\mathbb{R}^n} \frac{(|x|^2 + t^2)^\sigma}{(|x|^2/4 + t^2)^{(n+2\sigma)/2}} |u(\xi)| \, d\xi \\
&\leq C(n, \sigma)(|x|^2 + t^2)^{-n/2} \|u\|_{L^1},
\end{aligned}
$$

where in the first inequality we have used that U is supported in $\overline{\mathcal{B}_{R/2}^+}$.

Direct computations yield that

$$
\begin{aligned}
\left| \int_{\mathcal{B}_{2R}^+ \setminus \mathcal{B}_R^+} t^{1-2\sigma} V \nabla \eta \nabla V \right| &\leq \left(\int_{\mathcal{B}_{2R}^+ \setminus \mathcal{B}_R^+} t^{1-2\sigma} |\nabla V|^2 \right)^{1/2} \left(\int_{\mathcal{B}_{2R}^+ \setminus \mathcal{B}_R^+} t^{1-2\sigma} V^2 |\nabla \eta|^2 \right)^{1/2} \\
&\leq \left(\int_{\mathcal{B}_{2R}^+ \setminus \mathcal{B}_R^+} t^{1-2\sigma} |\nabla V|^2 \right)^{1/2} C(n, \sigma) |u|_{L^1(\mathbb{R}^n)} (R^{n+2-2\sigma-2-2n})^{1/2} \to 0 \quad \text{as } R \to \infty,
\end{aligned}
$$

where we have used (2.4) that $\int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} |\nabla V|^2 < \infty$. Therefore,

$$\int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} |\nabla V|^2 \leq \left| \int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} \nabla U \nabla V \right|.$$

Finally, by the Hölder inequality,

$$\int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} |\nabla V|^2 \leq \int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma} |\nabla U|^2. \qquad \square$$

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