© European Mathematical Society 2014

JEMS

Rolf Farnsteiner

Jordan types for indecomposable modules of finite group schemes

Abstract. In this article we study the interplay between algebro-geometric notions related to π-points and structural features of the stable Auslander–Reiten quiver of a finite group scheme. We show that π-points give rise to a number of new invariants of the AR-quiver on the one hand, and exploit combinatorial properties of AR-components to obtain information on π-points on the other. Special attention is given to components containing Carlson modules, constantly supported modules, and endo-trivial modules.

Keywords. Jordan type, Auslander–Reiten components

0. Introduction

Let $\mathcal{G}$ be a finite group scheme over an algebraically closed field k of characteristic $p > 0$. In their recent articles [35, 36], Friedlander and Pevtsova have expounded the theories of p-points and π-points, which generalize and unify various concepts of rank varieties defined earlier. One new feature arising via this vantage point is the notion of Jordan type [37], providing new invariants for $\mathcal{G}$-modules that are finer than those given by rank varieties and support varieties. The ramifications of the additional information encoded in Jordan types are only beginning to be understood, even for the class of modules of constant Jordan type [10, 9].

Our investigations focus on two aspects that underscore the utility of this approach. Sections 1 through 4 are concerned with the behavior of Jordan types on the components of the stable Auslander–Reiten quiver $\Gamma_s(\mathcal{G})$ of the algebra of measures $k\mathcal{G}$ of $\mathcal{G}$. The results of the first three sections can be generalized to perfect fields, while those of Section 4 rest on k being algebraically closed. The last four sections employ π-points to define and study certain classes of modules and to determine their position within the AR-quiver.

Support varieties have played an important rôle in the investigation of the quiver $\Gamma_s(\mathcal{G})$. It therefore seems expedient to explore the interplay between Jordan types and the Auslander–Reiten theory of the group scheme $\mathcal{G}$. Indeed, Jordan types turn out to

R. Farnsteiner: Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, Ludewig-Meyn-Str. 4, 24098 Kiel, Germany; e-mail: rolf@math.uni-kiel.de

provide new invariants for AR-components, enabling us to discern differences between components that cannot be detected via their support varieties. On the other hand, the Jordan types of all modules belonging to an Auslander–Reiten component may often be computed from the information corresponding to one single vertex.

In dealing with the AR-quiver, our main tools are subadditive functions on stable representation quivers, whose relevant properties are provided in Section 1. As we show in Section 2, π-points of $\mathcal{G}$ define various additive functions on the so-called locally split components of the stable Auslander–Reiten quiver. By definition, the pull-backs of the almost split sequences of such components along any π-point are split exact. With the exception of some infinite tubes, all infinite AR-components are locally split, so that our methods usually apply whenever the representation-theoretic support of the ambient block of $k\mathcal{G}$ has dimension at least 2. For these components we establish new invariants, all of which arise via π-points. In particular, we discuss the number of Jordan types of a module, the dominance order on π-points associated to an AR-component, and varieties of non-maximal supports.

For infinite tubes that are not locally split, the relevant functions are eventually additive, with their departure from additivity being controlled by the Cartan matrix A_{p-1} and the structure of certain induced modules modules that are defined by π-points corresponding to closed points of the support scheme $\Pi(\mathcal{G})$. As we show in Section 4, the structure of these components as well as the associated functions are completely understood in the cases where the group scheme $\mathcal{G}$ is trigonalizable, or $\mathcal{G}$ is reduced and the relevant component contains a module with a cyclic vertex.

Applications concerning three closely related classes of $\mathcal{G}$-modules, constantly supported modules, Carlson modules and endo-trivial modules, are the subject of the following three sections. In terms of the hierarchy given by their sets of Jordan types, constantly supported modules naturally follow the modules of constant Jordan type that were investigated in [10, 9]. In Section 5 we show that examples are given by direct summands of the Carlson modules L_ζ, associated to non-nilpotent homogeneous elements of the even cohomology ring $\mathrm{H}^\bullet(\mathcal{G}, k)$. This motivates the study of the L_ζ in the following section. Aside from their fundamental importance for theoretical purposes, Carlson modules are also of interest because their structure is closely reflected by their support varieties. For Carlson modules belonging to homogeneous non-nilpotent elements, we provide a sufficient condition for their indecomposability. In classical contexts, such as the first Frobenius kernels of semi-simple algebraic groups, it follows that these modules are usually quasi-simple. In particular, the almost split sequences originating in these L_ζ have an indecomposable middle term. If $\zeta \in \mathrm{H}^{2n}(\mathcal{G}, k) \setminus \{0\}$ is nilpotent, then the module L_ζ is indecomposable, of constant Jordan type and usually quasi-simple. For elements of odd degree, the set of Jordan types of L_ζ may have two elements and we show that L_ζ is indecomposable, provided $\mathcal{G}$ possesses sufficiently many abelian unipotent subgroups of complexity ≥ 2. Quillen's dimension theorem implies that this condition is superfluous whenever $\mathcal{G}$ corresponds to a finite group.

Endo-trivial modules were introduced by Dade [15, 16], who showed that, for abelian p-groups, these modules are stably isomorphic to the syzygies of the trivial module. In our context, endo-trivial modules are examples of modules of constant Jordan type that occur

in connection with decomposable Carlson modules of non-nilpotent type. In Section 6, we investigate AR-components containing endo-trivial modules and discuss Carlson's construction [8] of such modules from the perspective of π-points.

In the final section, we enhance recent results of Carlson–Friedlander [9] by studying the Jordan types of certain finite algebraic groups of tame representation type. In particular, we determine the Jordan types of the indecomposable modules of the restricted enveloping algebra $U_0(\mathfrak{sl}(2))$ and classify its indecomposable endo-trivial modules.

1. Subadditive functions on stable translation quivers

Throughout this section, we shall be considering quivers $\Gamma := (\Gamma_0, \Gamma_1)$ without loops or multiple arrows. For such a quiver Γ, the set of arrows is given by a subset $\Gamma_1 \subseteq \Gamma_0 \times \Gamma_0$ of the Cartesian product of the set Γ_0 of vertices. We recall a few basic facts and definitions; the interested reader may consult [4, 50, 39, 40] for further details.

Given a vertex $x \in \Gamma_0$, we put

$$x^+ := \{y \in \Gamma_0; \; (x, y) \in \Gamma_1\}, \quad x^- := \{y \in \Gamma_0; \; (y, x) \in \Gamma_1\},$$

so that x^+ and x^- are the sets of successors and predecessors of the vertex x, respectively. The quiver (Γ_0, Γ_1) is referred to as *locally finite* if $x^+ \cup x^-$ is finite for every $x \in \Gamma_0$. *Henceforth all quivers are assumed to be locally finite*.

Let $\Gamma := (\Gamma_0, \Gamma_1)$ be a quiver. A *valuation* of Γ is a map $\nu : \Gamma_0 \times \Gamma_0 \to \mathbb{N}_0 \times \mathbb{N}_0$ such that $\Gamma_1 = \nu^{-1}(\mathbb{N} \times \mathbb{N})$. The triple $(\Gamma_0, \Gamma_1, \nu)$ is then referred to as a *valued quiver*. Homomorphisms of valued quivers are defined canonically.

An automorphism $\tau : \Gamma \to \Gamma$ is called a *translation* if

$$a^- = \tau(a)^+ \quad \forall a \in \Gamma_0.$$

We then refer to $(\Gamma_0, \Gamma_1, \tau)$ as a *stable translation quiver*.

If ν is a valuation of (Γ_0, Γ_1) and τ is a translation of (Γ_0, Γ_1) such that

$$\nu(\tau(b), a) = \Delta(\nu(a, b)) \quad \forall (a, b) \in \Gamma_1,$$

where $\Delta(m, n) = (n, m)$ for all $(m, n) \in \mathbb{N} \times \mathbb{N}$, then $(\Gamma_0, \Gamma_1, \nu, \tau)$ is a *valued stable translation quiver*.

By work of Riedtmann [50, Struktursatz] (see also [4, (4.15)]), a connected stable translation quiver Γ is of the form

$$\Gamma \cong \mathbb{Z}[T_\Gamma]/G,$$

where T_Γ is a directed tree and $G \subseteq \mathrm{Aut}(\mathbb{Z}[T_\Gamma])$ is an admissible group. The isomorphism class of Γ is determined by the associated undirected tree $\bar{T}_\Gamma$, the so-called *tree class* of Γ. We refer the reader to [4, (4.15.6)] for further details and just recall that a subgroup $G \subseteq \mathrm{Aut}(\Gamma)$ is *admissible* if $|G \cdot x \cap (\{y\} \cup y^+)| \leq 1$ and $|G \cdot x \cap (\{y\} \cup y^-)| \leq 1$ for all $x, y \in \Gamma_0$.

The aforementioned result rests on the following construction of the orbit quiver Γ/G associated to an admissible subgroup $G \subseteq \mathrm{Aut}(\Gamma)$:

- By defining $(\Gamma/G)_1 := \{([x], [y]) \in \Gamma_0/G \times \Gamma_0/G;$ there are $a \in [x]$ and $b \in [y]$ with $(a, b) \in \Gamma_1\}$, we endow Γ/G with the structure of a quiver.
- The map $\bar{\nu} : \Gamma_0/G \times \Gamma_0/G \to \mathbb{N}_0 \times \mathbb{N}_0$, given by $\bar{\nu}([x], [y]) = \nu(a, b)$ if $a \in [x]$ and $b \in [y]$ are such that $(a, b) \in \Gamma_1$, is a valuation, with the canonical map $\pi : \Gamma \to \Gamma/G$ being a morphism of valued quivers.
- The map $\bar{\tau} : \Gamma_0/G \to \Gamma_0/G$, $\bar{\tau}([x]) = [\tau(x)]$, is a translation such that π is a morphism of valued stable translation quivers.

The presence of subadditive functions on valued stable translation quivers significantly restricts the possible tree classes. Let $\mathrm{pr}_1 : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ be the projection onto the first coordinate.

Definition. Let $(\Gamma_0, \Gamma_1, \nu, \tau)$ be a valued stable translation quiver. A function $f : \Gamma_0 \to \mathbb{N}_0$ is said to be *subadditive* if

$$f(y) + f(\tau(y)) \geq \sum_{x \in y^-} f(x) \, \mathrm{pr}_1(\nu(x, y)) \quad \forall y \in \Gamma_0.$$

The function f is referred to as *additive* if we have equality for every $y \in \Gamma_0$.

For future reference we record the following basic properties:

Proposition 1.1. *Let* $\Gamma := (\Gamma_0, \Gamma_1, \nu, \tau)$ *be a connected valued stable translation quiver.*

(1) *If* $f : \Gamma_0 \to \mathbb{N}_0$ *is a subadditive function with* $f \circ \tau = f$. *Then either* $f = 0$, *or* $f(x) > 0$ *for every* $x \in \Gamma_0$.

(2) *Let* $f_\Gamma : \Gamma_0 \to \mathbb{N}$ *be an additive function with* $f_\Gamma \circ \tau = f_\Gamma$ *and such that every other such function is an integral multiple of* f_Γ. *Then* $f_\Gamma \circ g = f_\Gamma$ *for every* $g \in \mathrm{Aut}(\Gamma)$.

Proof. (1) We put $X := \{x \in \Gamma_0; f(x) \neq 0\}$ and $Y := \{x \in \Gamma_0; f(x) = 0\}$. It readily follows that X and Y are τ-invariant. Let $x \in X$ and $y \in Y$. If $x \in y^+$, then $\tau(x) \in \tau(y)^+ = y^-$. Thus, the existence of an arrow between X and Y implies that there are $x_1 \in X$ and $y_1 \in Y$ such that $x_1 \in y_1^-$. However, this yields

$$0 = 2f(y_1) \geq \sum_{x \in y_1^-} f(x) \, \mathrm{pr}_1(\nu(x, y_1)),$$

so that $f(x_1) = 0$, a contradiction. Since Γ is connected, it follows that $X = \Gamma_0$ or $Y = \Gamma_0$.

(2) Let $\mathcal{A}$ be the set of additive functions $f : \Gamma_0 \to \mathbb{N}$ satisfying $f \circ \tau = f$. By assumption, we have $\mathcal{A} = \mathbb{N} f_\Gamma$. The group $\mathrm{Aut}(\Gamma)$ of automorphisms of the stable valued translation quiver Γ acts on $\mathcal{A}$ via

$$(g.f)(x) := f(g^{-1}(x)) \quad \forall g \in \mathrm{Aut}(\Gamma), \, f \in \mathcal{A}, \, x \in \Gamma_0.$$

Given $g \in \mathrm{Aut}(\Gamma)$, there exists $n(g) \in \mathbb{N}$ with $g.f_\Gamma = n(g) f_\Gamma$. Consequently,

$$f_\Gamma = g^{-1}.(g.f_\Gamma) = n(g^{-1})n(g) f_\Gamma,$$

implying $n(g) = 1$ for all $g \in G$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We recall that a *valued graph* is a pair (I, d), consisting of a set I and a map $d : I \times I \to \mathbb{N}_0$, such that

(1) $d(i, i) = 0$ for all $i \in I$,
(2) $d(i, j) \neq 0 \Leftrightarrow d(j, i) \neq 0$,
(3) for each $i \in I$, the set $A(i) := \{j \in I; \ d(i, j) \neq 0\}$ is finite.

One can equally well consider the *Cartan matrix* $C(i, j) := 2\delta_{ij} - d(i, j)$ (cf. [39]). In the graphical presentation two vertices i and j are linked by a bond if $d(i, j) \neq 0$, and we endow this bond with the valuation
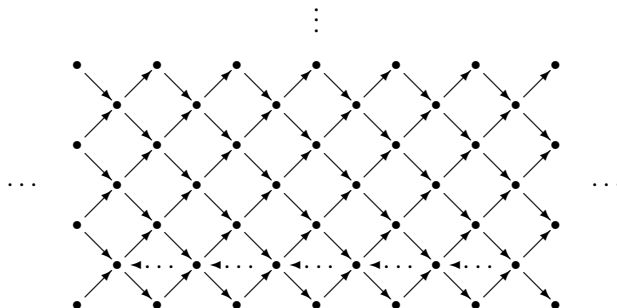
$$i \ \underline{(d(i,j), d(j,i))} \ j.$$

Let $T = (T_0, T_1)$ be a quiver. We define a stable translation quiver $\mathbb{Z}[T]$ by letting $\mathbb{Z} \times T_0$ be the set of vertices. We have arrows

$$(n, s) \to (n, t) \quad \text{and} \quad (n, t) \to (n + 1, s) \quad \forall n \in \mathbb{Z}$$

for every arrow $s \to t$ in T. The translation is given by

$$\tau : \mathbb{Z}[T] \to \mathbb{Z}[T], \quad (n, t) \mapsto (n - 1, t).$$

Below is the stable representation quiver $\mathbb{Z}[A_\infty]$, where A_∞ has $\mathbb{N}$ as set of vertices and arrows $n \to n + 1$ for every $n \in \mathbb{N}$. The dotted arrows represent the translation.

We let $\tilde{A}_{p,q}$ be the quiver whose underlying graph is the circle with $p+q$ vertices and with p consecutive clockwise oriented arrows and q consecutive counter-clockwise oriented arrows. Thus, $\tilde{A}_{2,0} \cong \tilde{A}_{0,2}$ is an oriented 2-cycle.

Lemma 1.2. *Let T be a quiver that does not contain a quiver of type $\tilde{A}_{2,0}$. Then $\langle \tau \rangle$ is an admissible subgroup of the automorphism group of the valued stable translation quiver $\mathbb{Z}[T]$.*

Proof. We put $G := \langle \tau \rangle$ and note that $G \cdot (n, x) = G \cdot (0, x)$ for all $(n, x) \in \mathbb{Z}[T]$. Let $(n_1, z_1), (n_2, z_2) \in G \cdot (0, x) \cap (\{(m, y)\} \cup (m, y)^+)$. Since $(n_1, z_1), (n_2, z_2) \in G \cdot (0, x)$, we have $z_1 = z_2 = x$. Two cases arise:

(a) $(n_1, x), (n_2, x) \in (m, y)^+$. Then we have either $n_1 = m$ and $y \to x$, or $n_1 = m + 1$ and $x \to y$. Since T does not contain $\tilde{A}_{2,0}$, the former alternative implies $n_2 = m$, while the latter yields $n_2 = m + 1$. In either case, we arrive at $(n_1, x) = (n_2, x)$, as desired.

(b) $(n_1, x) = (m, y)$. Since there is no arrow from x to x, this readily implies $(n_2, x) = (m, y)$.

The inequality $|G \cdot (0, x) \cap (\{(m, y)\} \cup (m, y)^-)| \leq 1$ follows analogously. $\square$

Definition. Let (I, d) be a valued graph. A function $f : I \to \mathbb{N}_0$ is called *subadditive* if

$$2f(j) \geq \sum_{i \in I} f(i)d(i, j) \quad \text{for every } j \in I.$$

We say that f is *additive* if equality holds for every $j \in I$.

Lemma 1.3. *Let $\Gamma = (\Gamma_0, \Gamma_1, \nu, \tau)$ be a valued stable translation quiver such that $\langle \tau \rangle \subseteq \mathrm{Aut}(\Gamma)$ is admissible.*

(1) *The function $d : \Gamma_0/\langle \tau \rangle \times \Gamma_0/\langle \tau \rangle \to \mathbb{N}_0$, given by*

$$d([x], [y]) := \begin{cases} \mathrm{pr}_1(\bar{\nu}([x], [y])) & \text{if } [x] \to [y], \\ 0 & \text{otherwise,} \end{cases}$$

endows $\Gamma_0/\langle \tau \rangle$ with the structure of a valued graph.

(2) *If Γ is connected and $f : \Gamma \to \mathbb{N}_0$ is a non-zero additive function with $f \circ \tau = f$, then there exists an additive function $\varphi : \Gamma/\langle \tau \rangle \to \mathbb{N}$ such that $\varphi([x]) = f(x)$ for every $x \in \Gamma_0$.*

(3) *If $\varphi : \Gamma/\langle \tau \rangle \to \mathbb{N}$ is an additive function, then*

$$f : \Gamma \to \mathbb{N}, \quad x \mapsto \varphi([x]),$$

is an additive function on Γ such that $f \circ \tau = f$.

Proof. (1) Since $\langle \tau \rangle$ is admissible, we have $d([x], [x]) = 0$ for all $[x] \in \Gamma_0/\langle \tau \rangle$. Let $[x] \in \Gamma_0/\langle \tau \rangle$ be given. Since the quiver $\Gamma/\langle \tau \rangle$ is locally finite, it follows that $d([x], [y]) \neq 0$ for only finitely many $[y] \in \Gamma_0/\langle \tau \rangle$. Finally, we assume that $d([x], [y]) \neq 0$. Then $[x] \to [y]$, and by definition of $\Gamma/\langle \tau \rangle$ there exist $a \in [x]$, $b \in [y]$ such that $a \to b$. Thus, $a \in b^- = \tau(b)^+$, so that $\tau(b) \to a$. This implies $[y] \to [x]$ and $d([y], [x]) \neq 0$.

(2) In view of Proposition 1.1, we have $f(\Gamma_0) \subseteq \mathbb{N}$. Hence there is a function $\varphi : \Gamma_0/\langle \tau \rangle \to \mathbb{N}$ with $\varphi([x]) = f(x)$ for every $x \in \Gamma_0$. According to (1) we obtain, for every $y \in \Gamma_0$,

$$2\varphi([y]) = f(y) + f(\tau(y)) = \sum_{x \in y^-} f(x)\,\mathrm{pr}_1(\nu(x, y))$$

$$= \sum_{[x] \in \Gamma_0/\langle \tau \rangle} \varphi([x])\,d([x], [y]), \tag{$*$}$$

so that φ is indeed additive.

(3) This follows directly from $(*)$.   □

We conclude this section by discussing certain functions of stable translation quivers Γ of tree class A_∞ that will make an appearance in the next section. If $x \in \Gamma_0$ is a vertex, then its distance to the end of Γ is referred to as the *quasi-length* $\mathrm{q}\ell(x)$ of x. Vertices of quasi-length 1 are often called *quasi-simple*. Given $\ell \geq 1$, we let $\Gamma_{(\ell)}$ be the full subquiver of Γ, whose vertices have quasi-length $\geq \ell$. A function $f : \Gamma_0 \to \mathbb{N}_0$ is called *eventually additive* if there exists $\ell \geq 1$ such that

$$f(y) + f(\tau(y)) = \sum_{x \in y^-} f(x) \quad \forall y \in (\Gamma_{(\ell)})_0.$$

The minimal ℓ with this property will be denoted $\ell(f)$.

Examples. (1) The additive functions are precisely those eventually additive functions with $\ell(f) = 1$.

(2) Every constant function $f \neq 0$ is eventually additive with $\ell(f) = 2$.

(3) The function $f : \Gamma_0 \to \mathbb{N}_0$, $x \mapsto q\ell(x) - 1$, is eventually additive with $\ell(f) = 2$.

For future reference, we record the following simple observation:

Lemma 1.4. *Let* $\Gamma = (\Gamma_0, \Gamma_1, \nu, \tau)$ *be a valued stable translation quiver of tree class* $\bar{T}_\Gamma = A_\infty$. *Let* $f : \Gamma_0 \to \mathbb{N}_0$ *be an eventually additive function such that* $f \circ \tau = f$.

(1) *Let* $x_{\ell(f)}$ *and* $x_{\ell(f)-1}$ *be two vertices of quasi-lengths* $\ell(f)$ *and* $\ell(f)-1$, *respectively. Then*

$$f(x) = (f(x_{\ell(f)}) - f(x_{\ell(f)-1}))(q\ell(x) - \ell(f)) + f(x_{\ell(f)}) \quad \forall x \in (\Gamma_{(\ell(f))})_0,$$

where we put $f(x_0) = 0$ *in case* $\ell(f) = 1$. *If* f *is bounded, then* $f|_{(\Gamma_{(\ell(f)-1)})_0}$ *is constant.*

(2) *If* f *is subadditive such that*
   (a) $f(x_1) = f(x_2)$ *for* $x_i \in \Gamma_0$ *with* $q\ell(x_i) = i$, *and*
   (b) $f(x) \geq f(x_1)$ *for all* $x, x_1 \in \Gamma_0$ *with* $q\ell(x_1) = 1$ *and* $q\ell(x) < \ell(f)$,
   then f *is constant and* $\ell(f) \leq 2$.

Proof. (1) Since f is eventually additive with $f \circ \tau = f$, the map f gives rise to a function $\varphi : A_\infty \to \mathbb{N}_0$ on the orbit graph of Γ which satisfies $\varphi(n) = f(x)$ whenever $q\ell(x) = n$. The proof of Lemma 1.3 now yields

$$2\varphi(n) = \varphi(n+1) + \varphi(n-1) \quad \forall n \geq \ell(f),$$

so that

$$\varphi(n) = \big(\varphi(\ell(f)) - \varphi(\ell(f) - 1)\big)(n - \ell(f)) + \varphi(\ell(f)) \quad \forall n \geq \ell(f).$$

(Here we define $\varphi(0) = 0$.) This readily yields the asserted formula. If f is bounded, then $\ell(f) \geq 2$ and we obtain $\varphi(\ell(f)) - \varphi(\ell(f) - 1) = 0$, whence $\varphi(n) = \varphi(\ell(f) - 1)$ for all $n \geq \ell(f) - 1$. Consequently, $f|_{(\Gamma_{(\ell(f)-1)})_0}$ is constant.

(2) We are going to show $f(x) = f(x_1)$ by induction on $n := q\ell(x)$, the cases $n = 1, 2$ being trivial. Let $n \geq 3$. Since f is subadditive and Γ has tree class A_∞, there exist vertices $y, z \in \Gamma_0$ of quasi-lengths $q\ell(y) = n - 1$ and $q\ell(z) = n - 2$ such that

$$f(x) \leq f(y) + f(\tau(y)) - f(z) = 2f(y) - f(z) = f(x_1),$$

with equality holding for $n \geq \ell(f)$. For $n < \ell(f)$, condition (b) yields the reverse inequality, so that f is constant. In particular, $\ell(f) \leq 2$. $\square$

2. Additive functions of π-points

Throughout, we let k be an algebraically closed field of characteristic $\mathrm{char}(k) = p > 0$ and consider a finite group scheme $\mathcal{G}$ over k, whose coordinate ring and algebra of measures will be denoted $k[\mathcal{G}]$ and $k\mathcal{G}$, respectively. We shall identify $\mathcal{G}$-modules and

$k\mathcal{G}$-modules and let $\mathrm{mod}\,\mathcal{G}$ be the category of finite-dimensional $\mathcal{G}$-modules. If $K : k$ is a field extension, then $\mathcal{G}_K := \mathrm{Spec}_K(k[\mathcal{G}] \otimes_k K)$ denotes the extended group, whose algebra of measures is

$$K\mathcal{G} := k\mathcal{G}_K \cong k\mathcal{G} \otimes_k K.$$

We refer the reader to [44] and [56] for background on group schemes and their representations.

Given $M \in \mathrm{mod}\,\mathcal{G}$, we write $M_K := M \otimes_k K$ for the corresponding $\mathcal{G}_K$-module. Let T be an indeterminate over k and put

$$\mathfrak{A}_{p,K} := k[T]/(T^p) \otimes_k K \cong K[T]/(T^p).$$

Following Friedlander–Pevtsova [36], we refer to a left flat algebra homomorphism $\alpha_K : \mathfrak{A}_{p,K} \to K\mathcal{G}$ as a π-point if there exists an abelian, unipotent subgroup $\mathcal{U} \subseteq \mathcal{G}_K$ such that $\mathrm{im}\,\alpha_K \subseteq K\mathcal{U}$. Like any flat algebra homomorphism, α_K gives rise to a pull-back functor $\alpha_K^* : \mathrm{mod}\,\mathcal{G}_K \to \mathrm{mod}\,\mathfrak{A}_{p,K}$ which is exact and sends projectives to projectives. Two π-points α_K and β_L are *equivalent* if

$$\alpha_K^*(M_K) \text{ projective } \Leftrightarrow \beta_L^*(M_L) \text{ projective}$$

for every $M \in \mathrm{mod}\,\mathcal{G}$. The set of equivalence classes will be denoted $\Pi(\mathcal{G})$. Given M in $\mathrm{mod}\,\mathcal{G}$, we let

$$\Pi(\mathcal{G})_M := \{[\alpha_K] \in \Pi(\mathcal{G});\ \alpha_K^*(M_K) \text{ is not projective}\}$$

be the Π-*support* of M. In view of [36, (3.4), 5(3.6)] the sets $\Pi(\mathcal{G})_M$ form the closed sets of a noetherian topology on $\Pi(\mathcal{G})$ such that

$$\dim \Pi(\mathcal{G})_M = \dim \mathcal{V}_\mathcal{G}(M) - 1,$$

where $\mathcal{V}_\mathcal{G}(M) \subseteq \mathrm{MaxSpec}(\mathrm{H}^{\mathrm{ev}}(\mathcal{G}, k))$ is the *cohomological support variety* of the $\mathcal{G}$-module M (see [5, (5.7)] for the definition that also applies in our context).

Let $\alpha_K : \mathfrak{A}_{p,K} \to K\mathcal{G}$ be a π-point. If $M \in \mathrm{mod}\,\mathcal{G}$, then $\alpha_K^*(M_K) \in \mathrm{mod}\,\mathfrak{A}_{p,K}$ uniquely decomposes as

$$\alpha_K^*(M_K) \cong \bigoplus_{i=1}^{p} \alpha_{K,i}(M)[i],$$

where $[i]$ represents the (up to isomorphism) unique indecomposable $\mathfrak{A}_{p,K}$-module of dimension i. We will interpret the right-hand side as a base change of an $\mathfrak{A}_{p,k}$-module N, that is,

$$\alpha_K^*(M_K) \cong N_K,$$

with $N = \bigoplus_{i=1}^{p} \alpha_{K,i}(M)[i] \in \mathrm{mod}\,\mathfrak{A}_{p,k}$. The isomorphism class of N is the *Jordan type of M with respect to α_K*, denoted $\mathrm{Jt}(M, \alpha_K)$.

Given a short exact sequence

$$\mathfrak{E} : \quad (0) \to M' \to M \to M'' \to (0)$$

of $\mathcal{G}_K$-modules, we write

$$\alpha_K^*(\mathfrak{E}) : \quad (0) \to \alpha_K^*(M_K') \to \alpha_K^*(M_K) \to \alpha_K^*(M_K'') \to (0).$$

If M is a non-projective indecomposable $\mathcal{G}_K$-module, then

$$\mathfrak{E}_M : \quad (0) \to N \to E \to M \to (0)$$

denotes the *almost split sequence* terminating in M. The reader is referred to [3, Chap. V] for the definition and basic properties of almost split sequences. We shall write $X \mid Y$ to indicate that X is isomorphic to a direct summand of Y.

For future reference we record a few basic properties of the functor $M \mapsto M_K$. In the sequel, Ω_Λ denotes the Heller operator of the finite-dimensional k-algebra Λ.

Lemma 2.1. *Let Λ be a finite-dimensional k-algebra, $K : k$ be a field extension, and M be a finite-dimensional Λ-module.*

(1) *If M is indecomposable, then the Λ_K-module M_K is indecomposable.*
(2) *M is projective if and only if M_K is projective.*
(3) *$\Omega_{\Lambda_K}(M_K) \cong \Omega_\Lambda(M)_K$.*

Proof. (1) Let $R := \operatorname{End}_\Lambda(M)$ and denote by J the Jacobson radical of R. Then R is local and since k is algebraically closed, we have $R/J \cong k$, whence $R_K/J_K \cong K$. As $R_K \cong \operatorname{End}_{\Lambda_K}(M_K)$, it follows that M_K is indecomposable.

(2) One implication is clear. Part (1) implies that $P \mapsto P_K$ provides a bijection between the isomorphism classes of principal indecomposable modules for Λ and Λ_K, respectively. Thus, if M_K is projective, so is M.

(3) This is a direct consequence of [45, (3.5)].   □

Definition. Let M be a $\mathcal{G}$-module, and $\alpha_K : \mathfrak{A}_{p,K} \to K\mathcal{G}$ be a π-point. We say that M is *relatively α_K-projective* if $M_K \mid (K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} \alpha_K^*(M_K))$. The $\mathcal{G}$-module M is *relatively α_K-injective* if $M_K \mid \operatorname{Hom}_{\mathfrak{A}_{p,K}}(K\mathcal{G}, \alpha_K^*(M_K))$.

Remark. Suppose that M is a $\mathcal{G}$-module such that $M_K \mid (K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} N)$ for some $\mathfrak{A}_{p,K}$-module N. Then there exists a split surjective homomorphism $\varphi : K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} N \to M_K$ of $K\mathcal{G}$-modules. Direct computation shows that the composite $\mu \circ \psi$ of the homomorphism

$$\psi : K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} N \to K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} \alpha_K^*(M_K), \quad a \otimes n \mapsto a \otimes \varphi(1 \otimes n),$$

with the canonical map

$$\mu : K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} \alpha_K^*(M_K) \to M_K, \quad a \otimes m \mapsto a.m,$$

coincides with φ. Hence μ is also split surjective. As a result, the following statements are equivalent:

- M is relatively α_K-projective.
- There exists an $\mathfrak{A}_{p,K}$-module N such that $M_K \mid (K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} N)$.
- The map $\mu : K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} \alpha_K^*(M_K) \to M_K, a \otimes m \mapsto a.m$, is split surjective.

The following subsidiary result, which elaborates on [10, (8.5)], is inspired by the modular representation theory of finite groups (see [4, (4.12.10)]).

Lemma 2.2. *Let M be a non-projective indecomposable $\mathcal{G}$-module, and let α_K : $\mathfrak{A}_{p,K} \to K\mathcal{G}$ be a π-point.*

(1) *M is relatively α_K-projective if and only if $M_K \mid (K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [i])$ for some $i \in \{1, \ldots, p-1\}$ with $\alpha_{K,i}(M) \neq 0$.*

(2) *If M is relatively α_K-projective, then M_K is cyclic, and $\Pi(\mathcal{G})_M = \{[\alpha_K]\}$.*

(3) *M is relatively α_K-projective if and only if the sequence $\alpha_K^*(\mathfrak{E}_M \otimes_k K)$ does not split.*

Proof. (1) In view of Lemma 2.1 and the above remark, this is a direct consequence of the Theorem of Krull–Remak–Schmidt.

(2) By (1), there exists $i \in \{1, \ldots, p-1\}$ such that $M_K \mid (K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [i])$. Consequently, M_K is cyclic, and a consecutive application of [36, (5.5)] and [10, (8.4)] implies $\emptyset \neq \Pi(\mathcal{G}_K)_{M_K} \subseteq \Pi(\mathcal{G}_K)_{K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}}[i]} = \{[\alpha_K]\}$, so that $\Pi(\mathcal{G}_K)_{M_K} = \{[\alpha_K]\}$. It readily follows that $[\alpha_K] \in \Pi(\mathcal{G})_M$. Conversely, assume that β_L is a π-point of $\mathcal{G}$ such that $[\beta_L] \in \Pi(\mathcal{G})_M$. Consider a common extension field Q of L and K. Then β_Q is a π-point of $\mathcal{G}_K$ such that $\beta_Q^*((M_K)_Q) \cong \beta_Q^*(M_Q) \cong \beta_L^*(M_L)_Q$ is not projective. Consequently, $[\beta_Q] \in \Pi(\mathcal{G}_K)_{M_K}$, so that the π-points α_K and β_Q of $\mathcal{G}_K$ are equivalent. We conclude that α_K and β_L are also equivalent, when considered as π-points of $\mathcal{G}$, whence $\Pi(\mathcal{G})_M = \{[\alpha_K]\}$.

(3) Let
$$\mathfrak{E}_M : \quad (0) \to N \to E \to M \to (0)$$
be the almost split sequence terminating in M. According to Lemma 2.1, the $\mathcal{G}_K$-module M_K is indecomposable and non-projective. Thanks to [45, (3.8)], the sequence
$$\mathfrak{E}_M \otimes_k K : \quad (0) \to N_K \to E_K \xrightarrow{\pi} M_K \to (0)$$
is the almost split sequence terminating in M_K. We consider the canonical surjection $\mu : K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} \alpha_K^*(M_K) \to M_K$, $a \otimes m \mapsto am$. If this map does not split, then $\mathfrak{E}_M \otimes_k K$ being almost split implies that there exists $\omega : K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} \alpha_K^*(M_K) \to E_K$ such that $\pi \circ \omega = \mu$. The adjoint isomorphism $\mathrm{Hom}_{K\mathcal{G}}(K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} \alpha_K^*(M_K), -) \cong \mathrm{Hom}_{\mathfrak{A}_{p,K}}(\alpha_K^*(M_K), -) \circ \alpha_K^*$ thus provides a map $\tilde{\omega} : \alpha_K^*(M_K) \to \alpha_K^*(E_K)$ with
$$\alpha_K^*(\pi) \circ \tilde{\omega} = \mathrm{id}_{\alpha_K^*(M_K)}.$$

Thus, if $\alpha_K^*(\mathfrak{E}_M \otimes_k K)$ does not split, then $M_K \mid (K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} \alpha_K^*(M_K))$, and M is relatively α_K-projective.

Let M be relatively α_K-projective. According to [3, (VI.3.6)], $\alpha_K^*(\mathfrak{E}_M \otimes_k K)$ being split entails that $\mathfrak{E}_M \otimes_k K$ is split. Since $\mathfrak{E}_M \otimes_k K$ is almost split, this cannot happen, so that $\alpha_K^*(\mathfrak{E}_M \otimes_k K)$ is non-split in that case. $\square$

Remarks. (1) Part (2) of the above lemma implies that the π-points $\alpha_K : \mathfrak{A}_{p,K} \to K\mathcal{G}$ for which a given $\mathcal{G}$-module M is relatively α_K-projective give rise to closed points $[\alpha_K] \in \Pi(\mathcal{G})$. In view of [36, (4.7)], the class $[\alpha_K]$ is therefore represented by some p-point $\beta_k : \mathfrak{A}_{p,k} \to k\mathcal{G}$.

(2) Let $\alpha_K \in \mathrm{Pt}(\mathcal{G})$ be a π-point such that $[\alpha_K]$ is not closed. Then Lemma 2.2 implies that no non-projective direct summand N of the induced module $K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [i]$ is defined over k, that is, there does not exist a $\mathcal{G}$-module M with $M_K \cong N$.

(3) Lemma 2.2 could equally well be stated in terms of relatively injective modules. In particular, the sequence $\alpha_K^*(\mathfrak{E}_M \otimes_k K)$ does not split if and only if M is relatively α_K-injective.

We let $\Gamma_s(\mathcal{G})$ denote the *stable Auslander–Reiten quiver* of the Frobenius algebra $k\mathcal{G}$. By definition, $\Gamma_s(\mathcal{G})$ is a valued stable translation quiver whose vertices are the isoclasses of the non-projective indecomposable $\mathcal{G}$-modules. There is an arrow $X \to Y$ if X is a direct summand of the middle term E of the almost split sequence $\mathfrak{E}_Y$. This arrow carries the valuation (m, m) if X occurs in E with multiplicity m (see [3, (V.1.3)]). The *Auslander–Reiten translation* $\tau_\mathcal{G}$ is given by

$$\tau_\mathcal{G} = \Omega_\mathcal{G}^2 \circ \nu_\mathcal{G} = \nu_\mathcal{G} \circ \Omega_\mathcal{G}^2,$$

where $\nu_\mathcal{G} := \mathrm{Hom}_\mathcal{G}(-, k\mathcal{G})^*$ denotes the *Nakayama functor* of mod $\mathcal{G}$. By virtue of [3, (V.1.14)], $\tau_\mathcal{G}(M)$ is the initial term of the almost split sequence terminating in M.

Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a connected component. In view of [24, (3.1)], whose proof can be easily modified to cover our context, we have

$$\Pi(\mathcal{G})_M = \Pi(\mathcal{G})_N \quad \forall M, N \in \Theta.$$

Accordingly, we shall speak of the Π-*support* $\Pi(\mathcal{G})_\Theta$ of Θ.

Definition. Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a component. Given a π-point $\alpha_K : \mathfrak{A}_{p,K} \to K\mathcal{G}$, we say that Θ is α_K-*split* if the exact sequence $\alpha_K^*(\mathfrak{E}_M \otimes_k K)$ splits for every $M \in \Theta$.

By abuse of notation, we will write $f : \Theta \to \mathbb{N}_0$ for a function f that is defined on the set of vertices of Θ. If N is an $\mathfrak{A}_{p,K}$-module, then N_{pf} denotes its projective-free part, that is, the direct sum of all non-projective indecomposable summands of N.

Proposition 2.3. *Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a component, and $\alpha_K : \mathfrak{A}_{p,K} \to K\mathcal{G}$ be a π-point.*

(1) *$\alpha_{K,i} \circ \tau_\mathcal{G} = \alpha_{K,i}$ for $i \in \{1, \ldots, p-1\}$.*
(2) *If Θ is not α_K-split, then $\Pi(\mathcal{G})_\Theta = \{[\alpha_K]\}$ and Θ is either finite or isomorphic to $\mathbb{Z}[A_\infty]/\langle \tau^n \rangle$ for some $n \in \mathbb{N}$.*
(3) *Given $m \in \{1, \ldots, p-1\}$, the function*

$$\psi_m : \Theta \to \mathbb{N}_0, \quad M \mapsto \sum_{i=1}^{m-1} i \alpha_{K,i}(M) + m \Big(\sum_{i=m}^{p-1} \alpha_{K,i}(M) \Big),$$

is subadditive with $\psi_m \circ \tau_\mathcal{G} = \psi_m$.
(4) *Suppose that $\alpha_{K,i} : \Theta \to \mathbb{N}_0$ is additive for $1 \leq i \leq p-1$. Then Θ is α_K-split.*

Proof. (1) Let $\zeta : k\mathcal{G} \to k$ be the modular function of the Hopf algebra $k\mathcal{G}$. According to [31, (1.5)], the convolution

$$\mu := \zeta * \mathrm{id}_{k\mathcal{G}}$$

is a Nakayama automorphism of the Frobenius algebra $k\mathcal{G}$, so that $\nu_\mathcal{G}$ coincides with the pull-back functor $\mu^* : \mathrm{mod}\,\mathcal{G} \to \mathrm{mod}\,\mathcal{G}$, defined by μ.

Since $(\zeta \otimes \mathrm{id}_K)|_{K\mathcal{U}} = \varepsilon_{K\mathcal{U}}$ for every unipotent subgroup $\mathcal{U} \subseteq \mathcal{G}_K$ and α_K factors through some unipotent subgroup $\mathcal{U} \subseteq \mathcal{G}_K$, we have $(\mu \otimes \mathrm{id}_K) \circ \alpha_K = \alpha_K$, whence $\alpha_K^* \circ (\mu \otimes \mathrm{id}_K)^* = \alpha_K^*$. As α_K^* is exact and sends projectives to projectives, Lemma 2.1 implies

$$\alpha_K^*(\tau_{\mathcal{G}}(M)_K) \cong \alpha_K^*((\mu \otimes \mathrm{id}_K)^*(\Omega_{\mathcal{G}}^2(M)_K)) \cong \alpha_K^*(\Omega_{\mathcal{G}}^2(M_K))$$
$$\cong \Omega_{\mathfrak{A}_{p,K}}^2(\alpha_K^*(M_K)) \oplus (\text{proj.}).$$

Observing $\Omega_{\mathfrak{A}_{p,K}}^2([j]) \cong [j]$ for $1 \leq j \leq p-1$, we conclude that

$$\Omega_{\mathfrak{A}_{p,K}}^2(\alpha_K^*(M_K)) \oplus (\text{proj.}) \cong \alpha_K^*(M_K).$$

Consequently,

$$\alpha_K^*(\tau_{\mathcal{G}}(M)_K) \oplus (\text{proj.}) \cong \alpha_K^*(M_K) \oplus (\text{proj.}),$$

so that $\alpha_{K,i}(\tau_{\mathcal{G}}(M)) = \alpha_{K,i}(M)$ for $i \in \{1, \ldots, p-1\}$.

(2) By assumption, there exists $M \in \Theta$ such that the exact sequence $\alpha_K^*(\mathfrak{E}_M \otimes_k K)$ does not split. In view of (2) and (3) of Lemma 2.2, we have $\Pi(\mathcal{G})_\Theta = \Pi(\mathcal{G})_M = \{[\alpha_K]\}$. We may now apply [24, (3.3)] to see that Θ is either finite or an infinite tube $\mathbb{Z}[A_\infty]/\langle \tau^n \rangle$.

(3) We write $\mathfrak{A}_{p,K} = K[t]$ with $t^{p-1} \neq 0 = t^p$. Let $m \in \{1, \ldots, p-1\}$. Given an $\mathfrak{A}_{p,K}$-module N, we let t_N^m be the left multiplication on N effected by t^m. Direct computation yields

$$\ker t_{\alpha_K^*(M_K)}^m = \bigoplus_{i=1}^{m-1} \alpha_{K,i}(M)[i] \oplus \Big(\sum_{i=m}^{p} \alpha_{K,i}(M) \Big)[m],$$

so that

$$\dim_K \ker t_{\alpha_K^*(M_K)}^m = \psi_m(M) + m\alpha_{K,p}(M) \quad \forall M \in \mathrm{mod}\,\mathcal{G}.$$

We denote the right-hand side by φ_m and consider a short exact sequence $(0) \to M' \to M \to M'' \to (0)$ of $\mathcal{G}$-modules. Observing the left-exactness of kernels, we obtain

$$\varphi_m(M) \leq \varphi_m(M') + \varphi_m(M'').$$

For $X \in \{M, M', M''\}$ we write $\alpha_K^*(X_K) = \alpha_K^*(X_K)_{\mathrm{pf}} \oplus P_X$, with P_X projective. Then $(P_{M'} \oplus P_{M''}) \mid P_M$, so that

$$p\alpha_{K,p}(M) \geq p\alpha_{K,p}(M') + p\alpha_{K,p}(M'').$$

We therefore obtain

$$\psi_m(M') + \psi_m(M'') = \varphi_m(M') + \varphi_m(M'') - m\alpha_{K,p}(M') - m\alpha_{K,p}(M'')$$
$$\geq \varphi_m(M) - m\alpha_{K,p}(M) = \psi_m(M).$$

Consequently, ψ_m is subadditive on Θ and (1) implies $\psi_m \circ \tau_{\mathcal{G}} = \psi_m$.

(4) Let M be an element of Θ, and consider the almost split sequence

$$(0) \to N \to E \to M \to (0)$$

terminating in M. Since each $\alpha_{K,i}$ is additive for $1 \le i \le p-1$, we obtain

$$\alpha_K^*(M_K)_{\mathrm{pf}} \oplus \alpha_K^*(N_K)_{\mathrm{pf}} \cong \alpha_K^*(E_K)_{\mathrm{pf}}.$$

For $X \in \{M, N, E\}$ we write $\alpha_K^*(X_K) = \alpha_K^*(X_K)_{\mathrm{pf}} \oplus P_X$, with P_X projective. Then $(P_N \oplus P_M) \mid P_E$ and a comparison of dimensions yields $P_E \cong P_M \oplus P_N$, so that

$$\alpha_K^*(M_K) \oplus \alpha_K^*(N_K) \cong \alpha_K^*(E_K).$$

Consequently, the exact sequence

$$(0) \to \alpha_K^*(N_K) \to \alpha_K^*(E_K) \to \alpha_K^*(M_K) \to (0)$$

splits. As a result, the component Θ is α_K-split. $\square$

We say that an AR-component $\Theta \subseteq \Gamma_s(\mathcal{G})$ is *regular* if the middle terms of the almost split sequences terminating in the vertices of Θ have no non-zero projective summands. By general theory (cf. [3, (V.5.5)]), an almost split sequence, whose middle term has the principal indecomposable module P as a direct summand, is isomorphic to the sequence

$$(0) \to \mathrm{Rad}(P) \to P \oplus (\mathrm{Rad}(P)/\mathrm{Soc}(P)) \to P/\mathrm{Soc}(P) \to (0).$$

Hence all but finitely many AR-components of $k\mathcal{G}$ are regular.

By providing the converse to Proposition 2.3(4), the following basic result indicates the utility of those π-points α_K for which a component Θ is α_K-split.

Theorem 2.4. *Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a component, and $\alpha_K : \mathfrak{A}_{p,K} \to K\mathcal{G}$ be a π-point such that Θ is α_K-split.*

(1) *For every $i \in \{1, \ldots, p-1\}$, the map*

$$\alpha_{K,i} : \Theta \to \mathbb{N}_0 \,, \qquad M \mapsto \alpha_{K,i}(M)$$

is an additive function on Θ such that $\alpha_{K,i} \circ \tau_{\mathcal{G}} = \alpha_{K,i}$.
(2) *The map*

$$\alpha_{K,<p} : \Theta \to \mathbb{N}_0, \qquad M \mapsto \dim_k M - p\alpha_{K,p}(M),$$

is an additive function on Θ such that $\alpha_{K,<p} \circ \tau_{\mathcal{G}} = \alpha_{K,<p}$.
(3) *The map*

$$\alpha_{K,p} : \Theta \to \mathbb{N}_0, \qquad M \mapsto \alpha_{K,p}(M),$$

is a subadditive function on Θ, which is additive if and only if Θ is regular.

Proof. Let M be a non-projective indecomposable $\mathcal{G}$-module, and consider the almost split sequence

$$(0) \to \tau_{\mathcal{G}}(M) \to E \to M \to (0)$$

terminating in M. Since Θ is α_K-split, the sequence

$$(0) \to \alpha_K^*(\tau_{\mathcal{G}}(M)_K) \to \alpha_K^*(E_K) \to \alpha_K^*(M_K) \to (0)$$

is split exact, so that the Theorem of Krull–Remak–Schmidt implies

$$\alpha_{K,i}(E) = \alpha_{K,i}(\tau_{\mathcal{G}}(M)) + \alpha_{K,i}(M)$$

for $1 \leq i \leq p$. Upon decomposing E into its indecomposable constituents $E = n_1 E_1 \oplus \cdots \oplus n_r E_r \oplus \text{(proj.)}$ with $E_i \not\cong E_j$ for $i \neq j$, we obtain

$$\alpha_{K,i}(E) = \sum_{j=1}^{r} n_j \alpha_{K,i}(E_j)$$

for $i \leq p - 1$, while

$$\alpha_{K,p}(E) = \sum_{j=1}^{r} n_j \alpha_{K,p}(E_j) + \alpha_{K,p}(\text{proj.}).$$

Consequently, $\alpha_{K,i}$ is an additive function on Θ for $i \leq p-1$, while $\alpha_{K,p}$ is a subadditive function, which is additive if and only if there is no middle term E such that E_K has a non-zero projective summand. Owing to Lemma 2.1, the functor $M \mapsto M_K$ sends indecomposables to indecomposables and preserves and reflects projectives. Thus, E_K has a non-zero projective summand if and only if E does. As a result, $\alpha_{p,K}$ is additive if and only if Θ is regular.

Being a sum of additive functions, $\alpha_{K,<p} = \sum_{i=1}^{p-1} i\alpha_{K,i}$ is also additive. According to Proposition 2.3(1), the additive functions of (1) and (2) are $\tau_{\mathcal{G}}$-invariant. $\qquad\square$

Setting $\max \emptyset := 0$, we record the following consequence of the proof of Theorem 2.4:

Corollary 2.5. *Let* $\Theta \cong \mathbb{Z}[A_\infty]/\langle\tau^n\rangle$ *be an infinite tube, and* $\alpha_K : \mathfrak{A}_{p,K} \to K\mathcal{G}$ *be a* π*-point with* $\Pi(\mathcal{G})_\Theta = \{[\alpha_K]\}$. *The functions* $\alpha_{K,i}, \psi_m : \Theta \to \mathbb{N}_0$ *are eventually additive for* $i, m \in \{1, \ldots, p-1\}$, *with* $\ell(\alpha_{K,i}), \ell(\psi_m) \leq 1 + \max\{\mathrm{q}\ell(M); \ M \text{ is relatively } \alpha_K\text{-projective}\}$.

Proof. Given $i \in \{1, \ldots, p-1\}$, we put $f := \alpha_{K,i}$. Owing to Lemma 2.2(1), at most finitely many $M \in \Theta$ are relatively α_K-projective. Hence setting

$$\ell := 1 + \max\{\mathrm{q}\ell(M); \ M \text{ is relatively } \alpha_K\text{-projective}\},$$

we see that no $M \in \Theta_{(\ell)}$ is relatively α_K-projective. The proof of Theorem 2.4 in conjunction with Lemma 2.2(3) now ensures that f is eventually additive with $\ell(f) \leq \ell$. By the same token, the subadditive functions ψ_m are eventually additive with $\ell(\psi_m) \leq \ell$. $\quad\square$

Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a component with $\Pi(\mathcal{G})_\Theta = \{[\alpha_K]\}$. If $\Theta \subseteq \Gamma_s(\mathcal{G})$ is finite, then [24, (3.3)] ensures that the tree class $\bar{T}_\Theta$ is a finite Dynkin diagram. Since the corresponding Cartan matrix is invertible, $f = 0$ is the only $\tau_{\mathcal{G}}$-invariant additive function on Θ.

More specifically, consider the algebra $k\mathbb{G}_{a(1)}$ associated to the first Frobenius kernel $\mathbb{G}_{a(1)}$ of the additive group $\mathbb{G}_a$. Then $k\mathbb{G}_{a(1)}$ is isomorphic to $\mathfrak{A}_{p,k}$ and $\Gamma_s(\mathbb{G}_{a(1)})$ is connected. For every $j \in \{1, \ldots, p\}$ there is an indecomposable module M_j of dimension j, and the isomorphism $\alpha_k : \mathfrak{A}_{p,k} \to k\mathbb{G}_{a(1)}$ gives $\alpha_{k,i}(M_j) = \delta_{i,j}$. This shows that, for $p \geq 3$, none of the functions $\alpha_{k,i}$ ($1 \leq i \leq p-1$) is subadditive (see Proposition 1.1). For $p = 2$, the function $\alpha_{k,1} = \psi_1$ is subadditive.

Corollary 2.6. *Suppose that* $\dim \Pi(\mathcal{G}) \geq 1$. *Let* $\Theta \cong \mathbb{Z}[A_\infty]/\langle \tau^n \rangle$ *be a periodic component of* $\Gamma_s(\mathcal{G})$ *and* α_K *be a* π-*point. If there exist* $\mathcal{G}$-*modules* $M_1, M_2 \in \Theta$ *with* $q\ell(M_j) = j$ *and* $\dim_K \alpha_K^*((M_j)_K)_{\mathrm{pf}} = p$ *for* $1 \leq j \leq 2$, *then* $\alpha_{K,i} : \Theta_{(\ell(\alpha_{K,i})-1)} \to \mathbb{N}_0$ *is constant for each* $i \in \{1, \ldots, p-1\}$ *and* Θ *contains a relatively* α_K-*projective module.*

Proof. Since Θ is a periodic component, we have $\dim \Pi(\mathcal{G})_\Theta = 0$. It follows that $\Pi(\mathcal{G}) \neq \Pi(\mathcal{G})_\Theta$, and there exists a π-point β_L such that $\beta_L^*(M_L)$ is projective for every $M \in \Theta$. Consequently, the dimension of every module $M \in \Theta$ is divisible by p, so that

$$p \mid \dim_K \alpha_K^*(M_K)_{\mathrm{pf}} = \psi_{p-1}(M)$$

for all $M \in \Theta$. Thanks to Proposition 2.3, the function ψ_{p-1} is subadditive and satisfies $\psi_{p-1} \circ \tau_\mathcal{G} = \psi_{p-1}$. Lemma 1.4(2) now yields

$$\psi_{p-1}(M) = p \quad \forall M \in \Theta.$$

In particular, each of the functions $\alpha_{K,i} : \Theta \to \mathbb{N}_0$ ($1 \leq i \leq p-1$) is bounded and a consecutive application of Corollary 2.5 and Lemma 1.4(1) now ensures that each $\alpha_{K,i}|_{\Theta_{(\ell(\alpha_{K,i})-1)}}$ is constant.

Since not all $\alpha_{K,i}$ are identically zero, we conclude that at least one $\alpha_{K,i}$ is not additive. A consecutive application of Theorem 2.4 and Lemma 2.2 now yields the existence of a vertex $M \in \Theta$ which is relatively α_K-projective. $\square$

3. Invariants of Auslander–Reiten components

In this section we show that various notions introduced in [10, 37] give rise to invariants of stable Auslander–Reiten components of $k\mathcal{G}$. In view of Theorem 2.4, we shall focus on those components $\Theta \subseteq \Gamma_s(\mathcal{G})$ that are α_K-split for every π-point α_K. Thanks to Lemma 2.2 and [24, (3.3)], the remaining components are either of the form $\mathbb{Z}[A_\infty]/\langle \tau^n \rangle$ or are of finite Dynkin type, in which case their vertices form the non-projective modules of a block $\mathcal{B} \subseteq k\mathcal{G}$ of finite representation type (see [3, (VI.1.4)]). In the following, we let $\Pi t(\mathcal{G})$ be the set of π-points of $\mathcal{G}$.

Definition. A component $\Theta \subseteq \Gamma_s(\mathcal{G})$ is called *locally split* if Θ is α_K-split for every $\alpha_K \in \Pi t(\mathcal{G})$.

3.1. The fundamental invariants

Given a component $\Theta \subseteq \Gamma_s(\mathcal{G})$, we define

$$\Pi t(\mathcal{G}, \Theta) := \{\alpha_K \in \Pi t(\mathcal{G}); \Theta \text{ is } \alpha_K\text{-split}\}.$$

Our first result provides for every infinite component $\Theta \subseteq \Gamma_s(\mathcal{G})$ a function $d^\Theta : \Pi t(\mathcal{G}, \Theta) \to \mathbb{N}_0^p$ on which our new invariants will depend.

Theorem 3.1.1. *Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be an infinite component. Then there exist an additive function $f_\Theta : \Theta \to \mathbb{N}$ and a function $d^\Theta : \Pi t(\mathcal{G}, \Theta) \to \mathbb{N}_0^p$ with*

$$\alpha_{K,i}(M_K) = d_i^\Theta(\alpha_K) f_\Theta(M) \ (1 \le i \le p-1) \quad and \quad \alpha_{K,<p}(M_K) = d_p^\Theta(\alpha_K) f_\Theta(M)$$

for every $M \in \Theta$ and every $\alpha_K \in \Pi t(\mathcal{G}, \Theta)$.

Proof. According to [24, (3.2)], the tree class $\bar{T}_\Theta$ is either a simply laced finite or infinite Dynkin diagram, a simply laced Euclidean diagram, or $\tilde{A}_{12}$. If $\bar{T}_\Theta$ is a finite Dynkin diagram, then [24, (3.3)] implies that each vertex of Θ is periodic, so that Θ is finite. Consequently, the tree class $\bar{T}_\Theta$ belongs to $\{A_\infty, A_\infty^\infty, D_\infty, \tilde{A}_{12}, \tilde{D}_n, \tilde{E}_{6,7,8}\}$, with each tree carrying the standard valuation.

For each tree class $\bar{T}_\Theta$ there exists an additive function $\varphi_{\bar{T}_\Theta} : \bar{T}_\Theta \to \mathbb{N}$ such that any other additive function on $\bar{T}_\Theta$ is an integral multiple of $\varphi_{\bar{T}_\Theta}$ (cf. [3, p. 243ff] or [40, Thm. 2, Rem. p. 328]). We define

$$f_{\mathbb{Z}[T_\Theta]} : \mathbb{Z}[T_\Theta] \to \mathbb{N}, \quad (n, x) \mapsto \varphi_{\bar{T}_\Theta}(x).$$

By Lemmas 1.2 and 1.3, $\bar{T}_\Theta$ is the orbit graph of $\mathbb{Z}[T_\Theta]$, and Lemma 1.3(3) ensures that $f_{\mathbb{Z}[T_\Theta]}$ is an additive function on $\mathbb{Z}[T_\Theta]$ such that

$$f_{\mathbb{Z}[T_\Theta]} \circ \tau = f_{\mathbb{Z}[T_\Theta]}.$$

Moreover, any other additive function $f : \mathbb{Z}[T_\Theta] \to \mathbb{N}$ with $f \circ \tau = f$ is an integral multiple of $f_{\mathbb{Z}[T_\Theta]}$.

The Riedtmann structure theorem [4, (4.15)] provides an admissible subgroup G of $\mathrm{Aut}(\mathbb{Z}[T_\Theta])$ such that

$$\Theta \cong \mathbb{Z}[T_\Theta]/G.$$

Since G is admissible, the canonical projection map $\pi : \mathbb{Z}[T_\Theta] \to \Theta$ is a covering of valued stable translation quivers, i.e., π induces bijections $x^+ \xrightarrow{\sim} \pi(x)^+$ and $x^- \xrightarrow{\sim} \pi(x)^-$ for every $x \in \mathbb{Z}[T_\Theta]_0$. Thanks to Proposition 1.1(2), we have $f_{\mathbb{Z}[T_\Theta]} \circ g = f_{\mathbb{Z}[T_\Theta]}$ for every $g \in G$. Consequently, there exists a function $f_\Theta : \Theta \to \mathbb{N}$ with

$$f_\Theta \circ \pi = f_{\mathbb{Z}[T_\Theta]}.$$

As π a covering, f_Θ is additive and any other additive function $f : \Theta \to \mathbb{N}$ is an integral multiple of f_Θ.

Let $\alpha_K \in \Pi t(\mathcal{G}, \Theta)$ be a π-point. Since Θ is α_K-split, Theorem 2.4 ensures that each $\alpha_{K,i}$ $(1 \le i \le p-1)$ is an additive function, with $\alpha_{K,i} \circ \tau_\mathcal{G} = \alpha_{K,i}$. In view of Proposition 1.1, we either have $\alpha_{K,i} = 0$ or $\alpha_{K,i}(M) \in \mathbb{N}$ for every $M \in \Theta$. We thus set $d_i^\Theta(\alpha_K) = 0$ whenever $\alpha_{K,i} = 0$.

Suppose that $\alpha_{K,i} \ne 0$. Since $\alpha_{K,i} \circ \tau_\mathcal{G} = \alpha_{K,i}$, there exists $d_i^\Theta(\alpha_K) \in \mathbb{N}$ with $\alpha_{K,i} = d_i^\Theta(\alpha_K) f_\Theta$. The same reasoning yields the existence of $d_p^\Theta(\alpha_K)$. $\square$

For future reference, we record a few special instances:

Corollary 3.1.2. *Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a component.*

(1) *If Θ has tree class $\bar{T}_\Theta \cong A_\infty$, then*

$$\alpha_{K,i}(M_K) = d_i^\Theta(\alpha_K)\,\mathrm{q}\ell(M) \quad and \quad \alpha_{K,<p}(M_K) = d_p^\Theta(\alpha_K)\,\mathrm{q}\ell(M)$$

for every $i \in \{1, \ldots, p-1\}$, every $M \in \Theta$ and every $\alpha_K \in \Pi\mathrm{t}(\mathcal{G}, \Theta)$.

(2) *If $\bar{T}_\Theta \cong \tilde{A}_{12}, A_\infty^\infty$, then*

$$\alpha_{K,i}(M_K) = d_i^\Theta(\alpha_K) \quad and \quad \alpha_{K,<p}(M_K) = d_p^\Theta(\alpha_K)$$

for every $i \in \{1, \ldots, p-1\}$, every $M \in \Theta$ and every $\alpha_K \in \Pi\mathrm{t}(\mathcal{G})$.

Proof. (1) If Θ has tree class A_∞, then $f_\Theta(M) = \mathrm{q}\ell(M)$ for every $M \in \Theta$. Since Θ is infinite, our assertion follows from Theorem 3.1.1.

(2) According to [24, (3.3)], we have $\dim \Pi(\mathcal{G})_\Theta \geq 1$, so that Θ is locally split and $\Pi\mathrm{t}(\mathcal{G}, \Theta) = \Pi\mathrm{t}(\mathcal{G})$. Since $f_\Theta \equiv 1$ for $\bar{T}_\Theta \cong \tilde{A}_{12}, A_\infty^\infty$ (cf. [40, Thm. 2, Rem. p. 328]), our assertion follows directly from Theorem 3.1.1. $\qquad\square$

3.2. Sets of Jordan types

Let M be a $\mathcal{G}$-module. Then

$$\mathrm{Jt}(M) := \{\mathrm{Jt}(M, \alpha_K); \alpha_K \in \Pi\mathrm{t}(\mathcal{G})\}$$

is a finite set, the *set of Jordan types of M*. Following [10], we say that a $\mathcal{G}$-module M has *constant Jordan type*, provided $\mathrm{Jt}(M)$ is a singleton. In that case, we call $\mathrm{Jt}(M)$ the *Jordan type* of M.

Our first application, which generalizes [10, (8.7)], provides a new invariant for locally split Auslander–Reiten components.

Corollary 3.2.1. *Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a locally split component. Then*

$$|\mathrm{Jt}(M)| = |\mathrm{im}\, d^\Theta|$$

for every $M \in \Theta$.

Proof. Let M be an element of Θ, $\alpha_K, \beta_L \in \Pi\mathrm{t}(\mathcal{G})$ be π-points. Thanks to Theorem 3.1.1, we have $\mathrm{Jt}(M, \alpha_K) = \mathrm{Jt}(M, \beta_L)$ if and only if $d^\Theta(\alpha_K) = d^\Theta(\beta_L)$. Consequently, $|\mathrm{Jt}(M)| = |\mathrm{im}\, d^\Theta|$, as desired. $\qquad\square$

In view of Corollary 3.2.1, the presence of a module M of constant Jordan type in a locally split component Θ implies that all modules of Θ have constant Jordan type. This is the content of [10, (8.7)], provided that $\dim \Pi(\mathcal{G}) \geq 1$. In the remaining case, $\Pi(\mathcal{G})$ is a singleton (cf. [10, (3.4)]), so that every $\mathcal{G}$-module has constant Jordan type (cf. [37, (4.10)]).

Let M be a $\mathcal{G}$-module, $\alpha_K : \mathfrak{A}_{p,K} \to K\mathcal{G}$ be a π-point. Then the isomorphism class of

$$\bigoplus_{i=1}^{p-1} \alpha_{K,i}(M)[i] \in \mathrm{mod}\,\mathfrak{A}_{p,K}$$

is referred to as the *stable Jordan type* $\mathrm{StJt}(M, \alpha_K)$ *of M with respect to* α_K. We let $\mathrm{StJt}(M)$ be the set of stable Jordan types of M. Obviously, a $\mathcal{G}$-module has constant Jordan type if and only if it has constant stable Jordan type.

Corollary 3.2.2. *Suppose that $\Theta \subseteq \Gamma_s(\mathcal{G})$ is a component with* $\dim \Pi(\mathcal{G})_\Theta \geq 1$ *and* $\bar{T}_\Theta \ncong A_\infty$. *If α_K is a π-point, then* $|\{\mathrm{StJt}(M, \alpha_K);\ M \in \Theta\}| = |\mathrm{im}\, f_\Theta| \leq 6$.

Proof. According to [24, (3.2), (3.3)], the tree class of Θ belongs to $\{\tilde{A}_{12}, \tilde{D}_n, \tilde{E}_{6,7,8}, A_\infty^\infty, D_\infty\}$. It follows from Theorem 3.1.1 that $|\{\mathrm{StJt}(M, \alpha_K);\ M \in \Theta\}|$ is equal to the cardinality of $\mathrm{im}\, f_\Theta$. According to [40, Thm.2, Rem. p. 328] this number is 1 for $\bar{T}_\Theta = \tilde{A}_{12}, A_\infty^\infty$; 2 for $\bar{T}_\Theta = \tilde{D}_n, D_\infty$; 3 for $\bar{T}_\Theta = \tilde{E}_6$; 4 for $\bar{T}_\Theta = \tilde{E}_7$; and 6 for $\bar{T}_\Theta = \tilde{E}_8$. $\square$

Remark. If $\mathcal{G} = G_r$ is a Frobenius kernel of a reductive group G, then the support of every component $\Theta \subseteq \Gamma_s(G_r)$ of tree class $\bar{T}_\Theta \neq A_\infty$ has dimension $\dim \Pi(G_r)_\Theta \geq 1$ and the trees $\tilde{E}_{6,7,8}$ do not occur (cf. [22, (4.1)]). Therefore $|\{\mathrm{StJt}(M, \alpha_K);\ M \in \Theta\}| \leq 2$ for every component with $\bar{T}_\Theta \ncong A_\infty$.

Let $\alpha_K \in \Pi\mathrm{t}(\mathcal{G})$ be a π-point and $M \in \mathrm{mod}\, \mathcal{G}$ be a $\mathcal{G}$-module. Then

$$\mathrm{supp}_{\alpha_K}(M) := \{i \in \{1, \ldots, p-1\}; \alpha_{K,i}(M) \neq 0\}$$

is the α_K-*support* of M. The following immediate consequence of Theorem 3.1.1 provides useful invariants for locally split components, enabling us to distinguish components having the same π-supports (cf. Corollary 8.1.2 below).

Corollary 3.2.3. *Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a component. If $\alpha_K \in \Pi\mathrm{t}(\mathcal{G}, \Theta)$, then*

$$\mathrm{supp}_{\alpha_K}(M) = \{i \in \{1, \ldots, p-1\}; d_i^\Theta(\alpha_K) \neq 0\}$$

for every $M \in \Theta$. $\square$

3.3. Dominance ordering and non-maximal supports

Let α_K and β_L be π-points and $M \in \mathrm{mod}\, \mathcal{G}$ be a $\mathcal{G}$-module. In [37], the authors introduce a relation by setting

$$\alpha_K \trianglelefteq_M \beta_L :\Leftrightarrow \dim_K \mathrm{im}\, t_{\alpha_K^*(M_K)}^m \leq \dim_K \mathrm{im}\, t_{\beta_L^*(M_K)}^m\ \forall m \in \{1, \ldots, p\}.$$

The proof of Proposition 2.3 readily yields

$$\alpha_K \trianglelefteq_M \beta_L :\Leftrightarrow \sum_{i=j}^{p}(i-j)\alpha_{K,i}(M_K) \leq \sum_{i=j}^{p}(i-j)\beta_{L,i}(M_L),\ 1 \leq j \leq p.$$

In view of [14, (6.2.2)], this relation corresponds to the usual dominance ordering on the partitions of $\dim_k M$, associated to $\alpha_K^*(M_K)$ and $\beta_L^*(M_L)$, respectively. More precisely,

$$\alpha_K \sim_M \beta_L :\Leftrightarrow \mathrm{Jt}(M, \alpha_K) = \mathrm{Jt}(M, \beta_L)$$

defines an equivalence relation on $\Pi t(\mathcal{G})$ and $\trianglelefteq_M$ is a partial ordering on the set of equivalence classes.[1]

Our next result shows that the relation defined by an indecomposable module is an invariant of its stable AR-component.

Proposition 3.3.1. *Let* $\Theta \subseteq \Gamma_s(\mathcal{G})$ *be a locally split component. If* α_K *and* β_L *are* π-*points such that* $\alpha_K \trianglelefteq_{M_0} \beta_L$ *for some* $M_0 \in \Theta$, *then* $\alpha_K \trianglelefteq_M \beta_L$ *for all* $M \in \Theta$.

Proof. According to Theorem 3.1.1, we can find $d_i^\Theta(\alpha_K), d_i^\Theta(\beta_L) \in \mathbb{N}_0$ such that

$$\gamma_i(M) = d_i^\Theta(\gamma) f_\Theta(M) \ (1 \le i \le p-1) \ \text{and} \ \gamma_{<p}(M) = d_p^\Theta(\gamma) f_\Theta(M) \ (\gamma \in \{\alpha_K, \beta_L\}),$$

for all $M \in \Theta$. For any $\mathcal{G}$-module $M \in \Theta$, we have $\gamma_p(M) = \frac{1}{p}(\dim_k M - d_p^\Theta(\gamma) f_\Theta(M))$ and thus

$$\alpha_K \trianglelefteq_M \beta_L \ \Leftrightarrow \ \sum_{i=j}^{p-1} (i-j) d_i^\Theta(\alpha_K) - \frac{p-j}{p} d_p^\Theta(\alpha_K) \le \sum_{i=j}^{p-1} (i-j) d_i^\Theta(\beta_L) - \frac{p-j}{p} d_p^\Theta(\beta_L)$$

for $1 \le j \le p$. Since the right-hand side does not depend on M, our assertion follows. $\square$

Let M be a $\mathcal{G}$-module. We say that $\alpha_K \in \Pi t(\mathcal{G})$ is *maximal* for $\trianglelefteq_M$ if $\beta_L \sim_M \alpha_K$ for every $\beta_L \in \Pi t(\mathcal{G})$ with $\alpha_K \trianglelefteq_M \beta_L$. In [37], the authors introduce the variety of non-maximal supports. Given a $\mathcal{G}$-module M, we write

$$\widetilde{\Pi}(\mathcal{G})_M := \{x \in \Pi(\mathcal{G}); \text{ there exists } \alpha_K \in x \text{ that is not maximal for } \trianglelefteq_M\}.$$

By [37, (5.2)], this set is a closed subspace of $\Pi(\mathcal{G})_M$, which coincides with $\Pi(\mathcal{G})_M$ if and only if $\Pi(\mathcal{G})_M \neq \Pi(\mathcal{G})$.

Theorem 3.3.2. *Let* $\Theta \subseteq \Gamma_s(\mathcal{G})$ *be a component such that* $\Pi(\mathcal{G})_\Theta = \Pi(\mathcal{G})$. *Then* $\widetilde{\Pi}(\mathcal{G})_M = \widetilde{\Pi}(\mathcal{G})_N$ *for* $M, N \in \Theta$.

Proof. We first assume that $\dim \Pi(\mathcal{G}) \ge 1$, so that Θ is locally split. Suppose that x is not in $\widetilde{\Pi}(\mathcal{G})_M$ and let $\alpha_K \in x$ be a π-point. Then α_K is maximal for $\trianglelefteq_M$. If $\beta_L \trianglerighteq_N \alpha_K$ is another π-point, then Proposition 3.3.1 implies $\beta_L \trianglerighteq_M \alpha_K$, so that $\text{Jt}(M, \beta_L) = \text{Jt}(M, \alpha_K)$. As a result, $d^\Theta(\beta_L) = d^\Theta(\alpha_K)$, giving $\text{Jt}(N, \beta_L) = \text{Jt}(N, \alpha_K)$. We conclude that α_K is maximal for $\trianglelefteq_N$, whence $x \notin \widetilde{\Pi}(\mathcal{G})_N$. Consequently, $\widetilde{\Pi}(\mathcal{G})_N \subseteq \widetilde{\Pi}(\mathcal{G})_M$, implying our assertion.

If $\dim \Pi(\mathcal{G}) = 0$, then [10, (3.4)] ensures that $\Pi(\mathcal{G})$ is a singleton. We may now apply [37, (4.10)] to see that every $\mathcal{G}$-module has constant Jordan type. Consequently, we have $\widetilde{\Pi}(\mathcal{G})_M = \emptyset = \widetilde{\Pi}(\mathcal{G})_N$ in that case. $\square$

---

[1] Note that our formula above differs from the one given in [10, §1], where the authors set $\alpha_K \trianglelefteq_M \beta_L \ :\Leftrightarrow \ \sum_{i=j}^p i\alpha_{K,i}(M_K) \le \sum_{i=j}^p i\beta_{L,i}(M_L)$ for $1 \le j \le p$. The choice $p = 3$ and $(\alpha_3, \alpha_2, \alpha_1) = (2, 0, 1)$, $(\beta_3, \beta_2, \beta_1) = (1, 2, 0)$ defines two partitions $\underline{\alpha}$ and $\underline{\beta}$ of 7 such that $\underline{\alpha} \ge \underline{\beta}$ with respect to our definition, while the one of [10] leads to $\underline{\alpha} \not\ge \underline{\beta}$.

4. Components containing relatively projective modules

Throughout this section, we are working over an algebraically closed ground field k. We shall study stable AR-components $\Theta \subseteq \Gamma_s(\mathcal{G})$ which are not locally split. According to Lemma 2.2 these are precisely those components that contain a relatively α_K-projective $\mathcal{G}$-module for some $\alpha_K \in \Pi t(\mathcal{G})$.

4.1. Quasi-simple relatively projective modules

In order to record detailed information on the functions $\alpha_{K,i} : \Theta \to \mathbb{N}_0$ for infinite components $\Theta \subseteq \Gamma_s(\mathcal{G})$ that are not α_K-split, we introduce the tridiagonal $(p \times p)$-matrix

$$
A := \begin{pmatrix}
2 & -1 & 0 & \ldots & 0 & 0 \\
-1 & 2 & -1 & \ldots & 0 & 0 \\
0 & -1 & 2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 2 & -1 \\
0 & 0 & 0 & \ldots & -1 & 1
\end{pmatrix}
$$

whose coefficients are denoted a_{ij}. Note that the $((p-1) \times (p-1))$-principal minor of this matrix is just the Cartan matrix of the Dynkin diagram A_{p-1}.

By Proposition 2.3, every infinite component $\Theta \subseteq \Gamma_s(\mathcal{G})$ which is not locally split is of the form $\Theta \cong \mathbb{Z}[A_\infty]/\langle \tau^n \rangle$, so that we can speak of the quasi-length $q\ell(M)$ of a module belonging to Θ.

Proposition 4.1.1. *Let $\alpha_K : \mathfrak{A}_{p,K} \to K\mathcal{G}$ be a π-point, and $\Theta \subseteq \Gamma_s(\mathcal{G})$ be an infinite component such that*

(a) *Θ is not α_K-split, and*
(b) *if $N \in \Theta$ is relatively α_K-projective, then $q\ell(N) = 1$.*

Then there exist a vector $(n_1, \ldots, n_{p-1}) \in \mathbb{N}_0^{p-1} \setminus \{0\}$ and a quasi-simple module $M \in \Theta$ such that

$$
\alpha_{K,i}(X) = \left(\alpha_{K,i}(M) - \sum_{j=1}^{p-1} a_{ij} n_j \right) q\ell(X) + \sum_{j=1}^{p-1} a_{ij} n_j, \quad 1 \le i \le p-1,
$$

for every $X \in \Theta$.

Proof. Given $i \in \{1, \ldots, p\}$, we put $N_i := K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [i]$. Conditions (a) and (b) in conjunction with Lemma 2.2 ensure the existence of a quasi-simple module $M \in \Theta$ that is relatively α_K-projective. We define $n_i \in \mathbb{N}_0$ to be the multiplicity of M_K in N_i, that is,

$$
N_i \cong n_i M_K \oplus N_i' \quad \text{with} \quad M_K \nmid N_i'.
$$

Thanks to Lemma 2.2(1), at least one of the n_i is not zero. Since M_K is not projective, we have $n_p = 0$.

We consider the almost split sequence

$$
\mathfrak{E}_M : \quad (0) \to \tau_\mathcal{G}(M) \to E \to M \to (0)
$$

terminating in M. Since k is algebraically closed, [45, (3.8)] ensures that

$$\mathfrak{E}_M \otimes_k K : \quad (0) \to \tau_{\mathcal{G}}(M)_K \to E_K \to M_K \to (0)$$

is an almost split sequence terminating in M_K. Let $\varphi : N_i \to M_K$ be a homomorphism and write $\varphi = (\varphi_1, \ldots, \varphi_{n_i}, \varphi')$ with φ_t in $\mathrm{End}_{\mathcal{G}_K}(M_K)$ for $1 \leq t \leq n_i$ and φ' in $\mathrm{Hom}_{\mathcal{G}_K}(N_i', M_K)$. Then φ is split surjective if and only if $\varphi_t \notin \mathrm{Rad}(\mathrm{End}_{\mathcal{G}_K}(M_K))$ for some $t \in \{1, \ldots, n_i\}$. Since $\mathfrak{E}_M \otimes_k K$ is almost split, we obtain an exact sequence

$$(0) \to \mathrm{Hom}_{\mathcal{G}_K}(N_i, \tau_{\mathcal{G}}(M)_K) \to \mathrm{Hom}_{\mathcal{G}_K}(N_i, E_K)$$
$$\to n_i \, \mathrm{Rad}(\mathrm{End}_{\mathcal{G}_K}(M_K)) \oplus \mathrm{Hom}_{\mathcal{G}_K}(N_i', M_K) \to (0)$$

for every $i \in \{1, \ldots, p\}$. As k is algebraically closed, the right-hand term above has dimension $\dim_K \mathrm{Hom}_{\mathcal{G}_K}(N_i, M_K) - n_i$ and Frobenius reciprocity implies

$$\dim_K \mathrm{Hom}_{\mathfrak{A}_{p,K}}([i], \alpha_K^*(E_K))$$
$$= \dim_K \mathrm{Hom}_{\mathfrak{A}_{p,K}}([i], \alpha_K^*(M_K)) + \dim_K \mathrm{Hom}_{\mathfrak{A}_{p,K}}([i], \alpha_K^*(\tau_{\mathcal{G}}(M)_K)) - n_i.$$

From the formula

$$\dim_K \mathrm{Hom}_{\mathfrak{A}_{p,K}}([s], [t]) = \min\{s, t\} \quad \forall s, t \in \{1, \ldots, p\}$$

we get

$$\sum_{\ell=1}^{p} \min\{i, \ell\} \alpha_{K,\ell}(E) = \sum_{\ell=1}^{p} \min\{i, \ell\}(\alpha_{K,\ell}(M) + \alpha_{K,\ell}(\tau_{\mathcal{G}}(M))) - n_i, \quad 1 \leq i \leq p.$$

Let $B := (\min\{i, \ell\})_{1 \leq i, \ell \leq p} \in \mathrm{Mat}_p(\mathbb{Z})$ and $n = (n_1, \ldots, n_p)^{\mathrm{tr}} \in \mathbb{Z}^p$. The above identities then amount to

$$Bx = By - n,$$

where

$$x := (\alpha_{K,1}(E), \ldots, \alpha_{K,p}(E))^{\mathrm{tr}},$$
$$y := \big(\alpha_{K,1}(M) + \alpha_{K,1}(\tau_{\mathcal{G}}(M)), \ldots, \alpha_{K,p}(M) + \alpha_{K,p}(\tau_{\mathcal{G}}(M))\big)^{\mathrm{tr}}.$$

Direct computation reveals that B is invertible with inverse $B^{-1} = A$. Consequently, $x = y - An$, whence

$$\alpha_{K,i}(E) = \alpha_{K,i}(M) + \alpha_{K,i}(\tau_{\mathcal{G}}(M)) - \sum_{j=1}^{p-1} a_{ij} n_j \quad \text{for } 1 \leq i \leq p. \tag{$*$}$$

Since $\mathrm{q}\ell(M) = 1$, we have

$$E \cong M_2 \oplus P,$$

where $M_2 \in \Theta$ has quasi-length 2 and P is indecomposable projective or (0). Hence $(*)$ implies

$$\alpha_{K,i}(M_2) = \alpha_{K,i}(M) + \alpha_{K,i}(\tau_{\mathcal{G}}(M)) - \sum_{j=1}^{p-1} a_{ij} n_j \tag{$**$}$$

for $1 \leq i \leq p - 1$.

Now suppose that $i \leq p - 1$. Proposition 2.3(1) yields $\alpha_{K,i} \circ \tau_{\mathcal{G}} = \alpha_{K,i}$, while Corollary 2.5 and condition (b) ensure that $\ell(\alpha_{K,i}) \leq 2$. Thus, if $\sum_{j=1}^{p-1} a_{ij} n_j = 0$, then $\alpha_{K,i}$ is additive, and our assertion follows from Lemma 1.4. Alternatively, $\ell(\alpha_{K,i}) = 2$, and Lemma 1.4 implies

$$\alpha_{K,i}(X) = (\alpha_{K,i}(M_2) - \alpha_{K,i}(M))(q\ell(X) - 2) + \alpha_{K,i}(M_2)$$
$$= \left(\alpha_{K,i}(M) - \sum_{j=1}^{p-1} a_{ij} n_j\right) q\ell(X) + \sum_{j=1}^{p-1} a_{ij} n_j$$

for every $X \in \Theta$. $\square$

Remarks. (1) Suppose that $\Theta \cong \mathbb{Z}[A_\infty]/\langle \tau \rangle$ is a homogeneous tube. If $E = M_2 \oplus P$, with $P \neq (0)$, then $M \cong P/\operatorname{Soc}(P)$ and $M \cong \tau_{\mathcal{G}}(M) \cong \operatorname{Rad}(P)$ (see [3, (V.5.5)]). Thus, $\operatorname{Rad}(P) \cong P/\operatorname{Soc}(P)$ and the block $\mathcal{B} \subseteq k\mathcal{G}$ containing P is a Nakayama algebra. Hence Θ is finite, a contradiction. It now follows from $(\ast)$ that the formula of Proposition 4.1.2 also holds for $i = p$.

(2) The foregoing result can be stated more formally by letting $n_j(M, \alpha_K)$ be the multiplicity of M_K as a summand of $K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]$. As noted in [24, (3.2)], there is an isomorphism

$$\tau_{\mathcal{G}_K}(K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]) \oplus (\text{proj.}) \cong K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j],$$

whence $n_j(M, \alpha_K) = n_j(\tau_{\mathcal{G}}(M), \alpha_K)$ for $1 \leq j \leq p - 1$. Thus, Proposition 4.1.1 holds for *every* quasi-simple module $M \in \Theta$.

Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be an infinite component that is not α_K-split for some π-point α_K in $\Pi\mathrm{t}(\mathcal{G})$. The utility of the above result crucially depends on

(a) all relatively α_K-projective modules $M \in \Theta$ being quasi-simple, and
(b) the knowledge of the multiplicity n_j of M_K as a summand of $K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]$ for $1 \leq j \leq p - 1$.

Proposition 4.1.3 below addresses these issues for relatively α_K-projective modules whose tops contain one-dimensional modules. We begin with the following easy observation:

Lemma 4.1.2. *Let $\alpha_K \in \Pi\mathrm{t}(\mathcal{G})$ be a π-point and $j \in \{1, \ldots, p\}$. If S is a simple $\mathcal{G}$-module, then S_K is a simple $\mathcal{G}_K$-module that occurs in $\operatorname{Top}_{\mathcal{G}_K}(K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j])$ with multiplicity $\dim_K \ker t^j_{\alpha_K^*(S_K)}$.*

Proof. Since k is algebraically closed, we have $\operatorname{End}_{\mathcal{G}}(S) \cong k$, whence $\operatorname{End}_{\mathcal{G}_K}(S_K) \cong K \otimes_k k \cong K$. Hence S_K is simple, and Frobenius reciprocity implies

$$\operatorname{Hom}_{\mathcal{G}_K}(K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j], S_K) \cong \operatorname{Hom}_{\mathfrak{A}_{p,K}}([j], \alpha_K^*(S_K)).$$

Since $\dim_K \operatorname{End}_{\mathcal{G}_K}(S_K) = 1$, the dimension of the former space counts the multiplicity of S_K in $\operatorname{Top}_{\mathcal{G}_K}(K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j])$. Writing $\alpha_K^*(S_K) \cong \bigoplus_{i=1}^{p} \alpha_{K,i}(S)[i]$, we obtain $\dim_K \operatorname{Hom}_{\mathfrak{A}_{p,K}}([j], \alpha_K^*(S_K)) = \sum_{i=1}^{p} \min\{i, j\} \alpha_{K,i}(S)$. As observed in the proof of Proposition 2.3, the latter number coincides with $\dim_K \ker t^j_{\alpha_K^*(S_K)}$. $\square$

Proposition 4.1.3. *Suppose that $p \geq 3$, and let $\mathcal{G}$ be a finite group scheme of infinite representation type. Let M be a non-projective, indecomposable, relatively α_K-projective $\mathcal{G}$-module such that $\mathrm{Top}_{\mathcal{G}}(M) \oplus \mathrm{Soc}_{\mathcal{G}}(M)$ contains a one-dimensional submodule.*

(1) *The component $\Theta \subseteq \Gamma_s(\mathcal{G})$ containing M is isomorphic to $\mathbb{Z}[A_\infty]/\langle \tau^m \rangle$ for some $m \geq 1$.*

(2) *The module M is quasi-simple.*

(3) *There exists a non-zero vector $(n_1, \ldots, n_{p-1}) \in \{0, 1\}^{p-1}$ such that*

$$\alpha_{K,i}(X) = \Big(\alpha_{K,i}(M) - \sum_{j=1}^{p-1} a_{ij} n_j\Big) \mathrm{q}\ell(X) + \sum_{j=1}^{p-1} a_{ij} n_j, \quad 1 \leq i \leq p-1,$$

for every $X \in \Theta$.

Proof. (1) By assumption, there exists an algebra homomorphism $\lambda : k\mathcal{G} \to k$ such that the one-dimensional $\mathcal{G}$-module k_λ associated to λ occurs as a summand of the module $\mathrm{Top}_{\mathcal{G}}(M) \oplus \mathrm{Soc}_{\mathcal{G}}(M)$.

In view of Proposition 2.3, the component Θ is either a finite or an infinite tube. In the former case, Auslander's Theorem [3, (VII.2.1)] implies that Θ contains all non-projective indecomposable modules of the block $\mathcal{B}_M$ containing M. Thus, $\mathcal{B}_M$ has finite representation type and k_λ belongs to $\mathcal{B}_M$. The convolution $\mathrm{id}_{k\mathcal{G}} * \lambda$ is an automorphism of $k\mathcal{G}$ that sends the block $\mathcal{B}_M$ onto the principal block $\mathcal{B}_0(\mathcal{G})$ of $k\mathcal{G}$. Consequently, $\mathcal{B}_0(\mathcal{G})$ is representation-finite, and [28, (3.1)] implies that $k\mathcal{G}$ enjoys the same property, a contradiction.

(2) We first assume that $\mathrm{Top}_{\mathcal{G}}(M)$ contains a one-dimensional module. Since M is relatively α_K-projective, Lemma 2.2 provides a $j \in \{1, \ldots, p-1\}$ such that M_K is a direct summand of $K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]$. Then $K_\lambda \cong k_\lambda \otimes_K K$ is a direct summand of $\mathrm{Top}_{\mathcal{G}_K}(M_K)$ (cf. [45, (3.5)]), and Lemma 4.1.2 shows that K_λ occurs in $\mathrm{Top}(K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j])$ with multiplicity 1. Accordingly, M_K is the unique indecomposable direct summand of $K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]$ such that $K_\lambda \mid \mathrm{Top}_{\mathcal{G}_K}(M_K)$.

As observed in the preceding remarks, we have

$$\tau_{\mathcal{G}_K}(K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]) \oplus (\text{proj.}) \cong K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j],$$

so that $\tau_{\mathcal{G}_K}(M_K)$ is also a direct summand of $K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]$. We now define $\ell := \min\{n \geq 1; \tau_{\mathcal{G}}^n(M) \cong M\}$. Since $\tau_{\mathcal{G}}^\ell|_\Theta$ is an automorphism of Θ preserving quasi-lengths, we conclude that $\tau_{\mathcal{G}}^\ell|_\Theta \cong \mathrm{id}_\Theta$. Consequently, $\Theta \cong \mathbb{Z}[A_\infty]/\langle \tau^\ell \rangle$.

Suppose that the $\mathcal{G}$-module k_λ belongs to Θ. Then k_λ has complexity ≤ 1, and the formula $\mathrm{cx}_{\mathcal{G}}(M \otimes_k k_\lambda) \leq \mathrm{cx}_{\mathcal{G}}(k_\lambda)$ implies that every $\mathcal{G}$-module has complexity ≤ 1. By general theory (cf. [56, (6.8)]),

$$\mathcal{G} = \mathcal{G}^0 \rtimes \mathcal{G}_{\mathrm{red}}$$

is the semidirect product of an infinitesimal normal subgroup $\mathcal{G}^0$ and a reduced group $\mathcal{G}_{\mathrm{red}}$. Thus, the complexities of the trivial modules for the subgroups $\mathcal{G}_{\mathrm{red}}$ and $\mathcal{G}^0$ are also bounded by 1.

Suppose that $\mathrm{cx}_{\mathcal{G}_{\mathrm{red}}}(k) = 1$. Then $\mathcal{G}_{\mathrm{red}}$ is not linearly reductive and there exists a closed subgroup $\mathcal{P} \subseteq \mathcal{G}_{\mathrm{red}}$ such that $\mathcal{P}(k) \cong \mathbb{Z}/(p)$. If $\mathrm{cx}_{\mathcal{G}^0}(k) = 1$, then the proof of [28, (3.1)] ensures the existence of a closed subgroup $\mathcal{U} \subseteq \mathcal{G}^0$ such that $\mathcal{U} \cong \mathbb{G}_{a(1)}$ is isomorphic to the first Frobenius kernel of the additive group and with $\mathcal{P}$ acting trivially on $\mathcal{U}$. Consequently,

$$2 = \mathrm{cx}_{\mathcal{U} \times \mathcal{P}}(k) \leq \mathrm{cx}_{\mathcal{G}}(k),$$

a contradiction. We conclude that $\mathrm{cx}_{\mathcal{G}^0}(k) = 0$. Hence the normal subgroup $\mathcal{G}^0 \trianglelefteq \mathcal{G}$ is linearly reductive, and [23, (1.1)] provides a normal subgroup $\mathcal{N} \trianglelefteq \mathcal{G}_{\mathrm{red}}$ such that

(a) $p \nmid \mathrm{ord}(\mathcal{N}(k))$, and
(b) the principal block $\mathcal{B}_0(\mathcal{G})$ is isomorphic to $\mathcal{B}_0(\mathcal{G}_{\mathrm{red}}/\mathcal{N})$.

In particular, the trivial module of the finite group $\mathcal{G}(k)/\mathcal{N}(k)$ has complexity 1 and is thus periodic (cf. [5, (5.10.4)]). Owing to [13, (XII.11.6)], this implies that the Sylow-p-subgroups of $\mathcal{G}(k)/\mathcal{N}(k)$ are either cyclic, or generalized quaternion. In the former case, Higman's Theorem [42, Thm. 4] ensures that $\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G}_{\mathrm{red}}/\mathcal{N})$ has finite representation type, and [28, (3.1)] yields a contradiction. Alternatively, $p = 2$, which contradicts our current assumption.

Consequently, $\mathrm{cx}_{\mathcal{G}_{\mathrm{red}}}(k) = 0$, and the group $\mathcal{G}_{\mathrm{red}}$ is linearly reductive. A consecutive application of [28, (2.7)] and [28, (3.1)] now shows that $\mathcal{G}$ is representation-finite, a contradiction.

Thus, we have $k_\lambda \notin \Theta$, and the defining property of almost split sequences implies that the function $X \mapsto \dim_k \mathrm{Hom}_{\mathcal{G}}(\tau_{\mathcal{G}}^n(X), k_\lambda)$ is additive on Θ for every $n \geq 0$. Since $\Theta \cong \mathbb{Z}[A_\infty]/\langle \tau^\ell \rangle$, the map

$$d_\lambda : \Theta \to \mathbb{N}, \quad X \mapsto \sum_{n=0}^{\ell-1} \dim_k \mathrm{Hom}_{\mathcal{G}}(\tau_{\mathcal{G}}^n(X), k_\lambda),$$

is a $\tau_{\mathcal{G}}$-invariant, additive function. By Lemma 1.4, there exists $r \in \mathbb{N}$ with

$$d_\lambda(X) = r\, \mathrm{q}\ell(X) \quad \forall X \in \Theta.$$

Owing to [45, (2.5), (3.6)], the modules $\tau_{\mathcal{G}_K}^n(M_K)$, $0 \leq n \leq \ell - 1$, are pairwise non-isomorphic summands of $K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]$. Consequently, $d_\lambda(M) = 1$, so that $\mathrm{q}\ell(M) = 1$ and M is quasi-simple.

If $\mathrm{Soc}_{\mathcal{G}}(M)$ contains a one-dimensional module, then observing that $k\mathcal{G}$ is a Frobenius algebra, we conclude that the top of the injective hull $E(M)$ of M contains a one-dimensional module. Consequently, $\mathrm{Top}_{\mathcal{G}}(\Omega_{\mathcal{G}}^{-1}(M))$ has a one-dimensional constituent. The isomorphism

$$\Omega_{\mathcal{G}_K}^{-1}(K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [i]) \oplus (\mathrm{proj.}) \cong K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [p - i] \qquad (*)$$

ensures that $\Omega_{\mathcal{G}}^{-1}(M)$ is relatively α_K-projective. As a result, the module $\Omega_{\mathcal{G}}^{-1}(M)$ is quasi-simple. Since $\Omega_{\mathcal{G}}^{-1}$ induces an automorphism of the quiver $\Gamma_s(\mathcal{G})$, the module M enjoys the same property.

(3) Now let $N \in \Theta$ be an arbitrary relatively α_K-projective $\mathcal{G}$-module. Since $\Theta \cong \mathbb{Z}[A_\infty]/\langle \tau^m \rangle$ and M is quasi-simple, there exist a surjection $\tau_{\mathcal{G}}^\ell(N) \to M$ as well as

an injection $M \to \tau_{\mathcal{G}}^s(M)$ for some $\ell, s \in \{0, \ldots, m-1\}$. As $\tau_{\mathcal{G}}^\ell(N)$ and $\tau_{\mathcal{G}}^s(N)$ are also relatively α_K-projective, part (2) ensures that $\tau_{\mathcal{G}}^\ell(N)$ or $\tau_{\mathcal{G}}^s(N)$ are quasi-simple. Thus, N is also quasi-simple.

Suppose that $k_\lambda \subseteq \text{Top}_{\mathcal{G}}(M)$. Let $j \in \{1, \ldots, p-1\}$. If n_j is the multiplicity of M_K in $K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]$, then the one-dimensional module K_λ occurs in $\text{Top}_{\mathcal{G}_K}(K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j])$ with multiplicity $m_j \geq n_j$. Lemma 4.1.2 now shows that m_j is bounded by 1.

If $k_\lambda \subseteq \text{Soc}_{\mathcal{G}}(M)$, then $n_j(\Omega_{\mathcal{G}}^{-1}(M), \alpha_K) \in \{0, 1\}$. In view of (∗), we have $n_j(M, \alpha_K) = n_{p-j}(\Omega_{\mathcal{G}}^{-1}(M), \alpha_K)$, so that $n_j(M, \alpha_k) \in \{0, 1\}$. The assertion thus follows from Proposition 4.1.1. □

Remarks. (1) The proof shows that Proposition 4.1.3 also holds for $p = 2$ as long as the Sylow 2-subgroup of $\mathcal{G}(k)$ is not generalized quaternion.

(2) Let $\alpha_k \in \Pi t(\mathcal{G})$ be a π-point that is defined over k, and denote by $P(k)$ the projective cover of the trivial $\mathcal{G}$-module k. The canonical map

$$\mu : k\mathcal{G} \otimes_{\mathfrak{A}_{p,k}} k \to k, \quad a \otimes 1 \mapsto \varepsilon(a),$$

affords an $\mathfrak{A}_{p,k}$-linear splitting $\gamma : k \to k\mathcal{G} \otimes_{\mathfrak{A}_{p,k}} k, s \mapsto 1 \otimes s$. If

$$k\mathcal{G} \otimes_{\mathfrak{A}_{p,k}} k = P(k) \oplus X,$$

then Lemma 4.1.2 implies that k is not a top composition factor of X, whence $\mu(X) = (0)$. Consequently, the trivial $\mathfrak{A}_{p,k}$-module k is a direct summand of the projective module $\alpha_k^*(P(k))$, a contradiction. As a result, $k\mathcal{G} \otimes_{\mathfrak{A}_{p,k}} k$ possesses a non-projective, indecomposable constituent with k being a composition factor of its top. The arguments of Lemma 4.3.1 below show that this applies to every one-dimensional $\mathcal{G}$-module.

4.2. Special types of π-points

In this section we shall provide another two criteria ensuring that certain relatively projective modules are quasi-simple. We fix a π-point $\alpha_K \in \Pi t(\mathcal{G})$ such that $[\alpha_K] \in \Pi(\mathcal{G})$ is closed.

Proposition 4.2.1. *Let M be a non-projective indecomposable relatively α_K-projective module which is contained in an infinite component $\Theta \subseteq \Gamma_s(\mathcal{G})$. If either*

(a) *$K\mathcal{G}\alpha_K(t)$ is an ideal of $K\mathcal{G}$, or*
(b) *there exists an abelian unipotent normal subgroup $\mathcal{U} \subseteq \mathcal{G}_K$ of complexity $\text{cx}_{\mathcal{U}}(K) = 1$ such that $\text{im}\,\alpha_K \subseteq K\mathcal{U}$,*

then M is quasi-simple.

Proof. Assuming (a), we have $\alpha_K(t)K\mathcal{G} \subseteq K\mathcal{G}\alpha_K(t)$, whence $\alpha_K(t)^i K\mathcal{G} \subseteq K\mathcal{G}\alpha_K(t)^i$ for all $1 \leq i \leq p-1$. Thus, $I_i := K\mathcal{G}\alpha_K(t)^i$ is an ideal of $K\mathcal{G}$. Since M is relatively α_K-projective, there exists $i \in \{1, \ldots, p-1\}$ such that $M_K \mid K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [i]$. Since the homomorphism α_K factors through an abelian unipotent subgroup, it is also right flat and the natural map $K\mathcal{G} \to K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [i]$, $a \mapsto a \otimes 1$, induces an isomorphism $K\mathcal{G}/I_i \cong K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [i]$. Thus, M_K is isomorphic to a principal indecomposable $(K\mathcal{G}/I_i)$-module and hence has a simple top. Consequently, M enjoys the same property (cf. [45, (3.3)]).

Assuming (b), we infer that M_K is a direct summand of $K\mathcal{G}\otimes_{K\mathcal{U}}N$ for some indecomposable $K\mathcal{U}$-module N. Since $\mathrm{cx}_{\mathcal{U}}(K)=1$, it follows that $K\mathcal{U}\cong K[X]/(X^{p^n})$ as well as $N\cong K[X]/(X^i)$ for some $i\in\{1,\dots,p^n-1\}$. Thus, $K\mathcal{G}\otimes_{K\mathcal{U}}N\cong K\mathcal{G}/K\mathcal{G}x^i$, with x being the canonical generator of $K\mathcal{U}$. As $\mathcal{U}$ is normal, the left ideal $K\mathcal{G}K\mathcal{U}^{\dagger}=K\mathcal{G}x$ is an ideal of $K\mathcal{G}$, and the arguments employed above guarantee that $\mathrm{Top}_{\mathcal{G}}(M)$ is simple.

Recall that the indecomposable $\mathcal{G}$-module $\Omega_{\mathcal{G}}^{-1}(M)$ is also relatively α_K-projective. Consequently, $\Omega_{\mathcal{G}}^{-1}(M)$ has a simple top, implying that M has a simple socle.

Since $\tau_{\mathcal{G}}^{\ell}(M)$ is relatively α_K-projective for all $\ell\in\mathbb{Z}$ and Θ is infinite, we may apply [19, (1.2)] to conclude that M is quasi-simple. □

Corollary 4.2.2. *Suppose that* $\mathrm{im}\,\alpha_K$ *is contained in the center* $\mathfrak{Z}(K\mathcal{G})$ *of* $K\mathcal{G}$. *Let* M *be a non-projective indecomposable relatively* α_K-*projective module which is contained in an infinite component* $\Theta\subseteq\Gamma_s(\mathcal{G})$. *Then there exist* $j\in\{1,\dots,p-1\}$ *and* $m,n\in\mathbb{N}$ *with* $m\geq 2n$ *such that*

$$\alpha_{K,i}(X)=\begin{cases}(m-2n)\,\mathrm{q}\ell(X)+2n & \text{for } i=j,\\ n(\mathrm{q}\ell(X)-1) & \text{for } i=j-1,\,j+1,\\ 0 & \text{for } i\neq j,\,j+1,\,j-1,\end{cases}$$

for every $X\in\Theta$ *and* $i\in\{1,\dots,p-1\}$.

Proof. By assumption, $K\mathcal{G}\alpha_K(t)$ is an ideal of $K\mathcal{G}$. Let $j\in\{1,\dots,p-1\}$ be such that $M_K\mid K\mathcal{G}\otimes_{\mathfrak{A}_{p,K}}[j]$. Since $\alpha_K(t)$ is central, we obtain $\alpha_K^*(K\mathcal{G}\otimes_{\mathfrak{A}_{p,K}}[j])\cong\left(\frac{\dim_k k\mathcal{G}}{p}\right)[j]$. Hence there exists $m\in\mathbb{N}$ with

$$\alpha_K^*(M_K)\cong m[j].$$

This readily implies $n_i(M,\alpha_K)=n\delta_{ij}$, $1\leq i\leq p-1$, for some $n\in\mathbb{N}$. The asserted formula now follows by applying Propositions 4.2.1 and 4.1.1 consecutively. Since $\alpha_{K,j}(X)\geq 0$ for all $X\in\Theta$, we also conclude that $m\geq 2n$. □

4.3. Trigonalizable group schemes

Throughout this section we consider a trigonalizable finite group scheme $\mathcal{G}$. By definition, all simple $\mathcal{G}$-modules are one-dimensional. The Theorem of Lie–Kolchin ensures that the Frobenius kernels of the smooth connected solvable algebraic groups belong to this class.

By general theory (cf. [17, IV, §2,3.5]), $\mathcal{G}=\mathcal{U}\rtimes\mathcal{D}$ is a semidirect product of a unipotent normal subgroup $\mathcal{U}$ and a diagonalizable factor $\mathcal{D}$. Thus, the coordinate ring $k[\mathcal{D}]=kX(\mathcal{D})$ is the group algebra of the finite group $X(\mathcal{D})$ of characters of $\mathcal{D}$. If $K:k$ is a field extension, then

$$K[\mathcal{D}_K]\cong k[\mathcal{D}]\otimes_k K\cong KX(\mathcal{D}),$$

so that $\mathcal{D}_K$ is diagonalizable with character group $X(\mathcal{D}_K)=X(\mathcal{D})$.

Since $\mathcal{U}$ is unipotent, the canonical restriction map $X(\mathcal{G})\to X(\mathcal{D})$ is an isomorphism. Given $\lambda\in X(\mathcal{G})$, we let k_λ be the one-dimensional $\mathcal{G}$-module defined by λ. Thus,

$\{k_\lambda; \ \lambda \in X(\mathcal{G})\}$ is a complete set of representatives for the isomorphism classes of simple $\mathcal{G}$-modules. By the above, these observations also apply to the group $\mathcal{G}_K$ given by an extension field K of k. We will henceforth identify $X(\mathcal{G}_K)$ with $X(\mathcal{G})$.

Lemma 4.3.1. *Let $\alpha_K \in \Pi\mathrm{t}(\mathcal{G})$ be a π-point, and $j \in \{1, \ldots, p\}$.*

(1) $\mathrm{Top}_{\mathcal{G}_K}(K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]) \cong \bigoplus_{\lambda \in X(\mathcal{G})} K_\lambda$.
(2) $(K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]) \otimes_K K_\lambda \cong K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]$ *for all* $\lambda \in X(\mathcal{G})$.
(3) $\tau_{\mathcal{G}_K}(K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]) \cong K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]$ *for* $1 \le j \le p - 1$.

Proof. (1) This follows directly from Lemma 4.1.2.

(2) Since $\mathcal{G}/\mathcal{U}$ is diagonalizable, every π-point α_K of $\mathcal{G}$ factors through $\mathcal{U}$. Define $V[j] := K\mathcal{U} \otimes_{\mathfrak{A}_{p,K}} [j]$, so that

$$K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j] \cong K\mathcal{G} \otimes_{K\mathcal{U}} V[j].$$

The tensor identity [44, (I.3.6)] then yields

$$(K\mathcal{G} \otimes_{K\mathcal{U}} V[j]) \otimes_k K_\lambda \cong K\mathcal{G} \otimes_{K\mathcal{U}} (V[j] \otimes_K (K_\lambda|_\mathcal{U})) \cong K\mathcal{G} \otimes_{K\mathcal{U}} V[j]$$

for every character $\lambda \in X(\mathcal{G})$.

(3) Let $j \in \{1, \ldots, p-1\}$. Consider the exact sequence

$$(0) \to K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [p - j] \to K\mathcal{G} \xrightarrow{\pi} K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j] \to (0).$$

According to (1), we have $\mathrm{Top}_{\mathcal{G}_K}(K\mathcal{G}) = \mathrm{Top}_{\mathcal{G}_K}(K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j])$, so that $(K\mathcal{G}, \pi)$ is a projective cover of $K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]$. Consequently, $\Omega_{\mathcal{G}_K}(K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]) \cong K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [p-j]$, and $\Omega^2_{\mathcal{G}_K}(K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]) \cong K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]$. Since $\tau_{\mathcal{G}_K}$ is the composite of $\Omega^2_{\mathcal{G}_K}$ with the functor $M \to M \otimes_K K_\zeta$, defined by the modular function $\zeta \in X(\mathcal{G})$, part (2) yields the desired isomorphism. $\square$

Given a $\mathcal{G}$-module M, we let

$$\mathrm{Stab}_{X(\mathcal{G})}(M) := \{\lambda \in X(\mathcal{G}); \ M \otimes_k K_\lambda \cong M_K\}$$

be the *stabilizer* of M. Let K be an extension field of k. By the above observations, we have $M_K \otimes_K K_\lambda \cong (M \otimes_k k_\lambda)_K$ for every $\lambda \in X(\mathcal{G})$. It now follows from [45, (2.5)] that

$$\mathrm{Stab}_{X(\mathcal{G})}(M) = \mathrm{Stab}_{X(\mathcal{G})}(M_K).$$

Recall that a $\mathcal{G}_K$-module N is said to be *defined over k* if there exists a $\mathcal{G}$-module M such that $N \cong M_K$.

Lemma 4.3.2. *Let $M \in \mathrm{mod}\,\mathcal{G}$ be a non-projective indecomposable $\mathcal{G}$-module, and let $\alpha_K \in \Pi\mathrm{t}(\mathcal{G})$ be a π-point. If M is relatively α_K-projective, then there exists a unique $j \in \{1, \ldots, p - 1\}$ and a subset $X_M \subseteq X(\mathcal{G})$ of cardinality $[X(\mathcal{G}) : \mathrm{Stab}_{X(\mathcal{G})}(M)]$ such that*

$$K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j] \cong \bigoplus_{\lambda \in X_M} M_K \otimes_K K_\lambda.$$

In particular, $\dim_k \mathrm{Top}_{\mathcal{G}}(M) = \mathrm{ord}(\mathrm{Stab}_{X(\mathcal{G})}(M))$, *and the module* $K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]$ *is defined over k.*

Proof. By virtue of Lemma 2.2, the $\mathcal{G}_k$-module M_K is a direct summand of $K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]$ for some $j \in \{1, \ldots, p-1\}$. We decompose

$$K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j] \cong M_1 \oplus \cdots \oplus M_n$$

into its indecomposable constituents, where $M_1 \cong M_K$. In view of Lemma 4.3.1(1), the corresponding decomposition of its top yields a partition

$$X(\mathcal{G}) = \bigsqcup_{i=1}^{n} X_i,$$

where $\text{Top}_{\mathcal{G}_K}(M_i) \cong \bigoplus_{\lambda \in X_i} K_\lambda$. Note that the isoclass of M_i is uniquely determined by X_i.

Let $\lambda \in X(\mathcal{G})$. Owing to Lemma 4.3.1, the $K\mathcal{G}$-module $M_1 \otimes_K K_\lambda$ is an indecomposable direct summand of $K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]$. Hence there exists exactly one $i(\lambda) \in \{1, \ldots, n\}$ such that $M_1 \otimes_K K_\lambda \cong M_{i(\lambda)}$. It readily follows that $X_{i(\lambda)} = \lambda.X_1$. Let $X_M \subseteq X(\mathcal{G})$ be a complete set of left coset representatives for $\text{Stab}_{X(\mathcal{G})}(M)$. Since $\text{Stab}_{X(\mathcal{G})}(M).X_1 = X_1$, we obtain

$$X(\mathcal{G}) = \bigcup_{\lambda \in X(\mathcal{G})} \lambda.X_1 = \bigcup_{\lambda \in X_M} \lambda.X_1 = \bigsqcup_{\lambda \in X_M} \lambda.X_1.$$

Thus, $N := \bigoplus_{\lambda \in X_M} M_1 \otimes_K K_\lambda$ is a direct summand of $K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]$ with $\text{Top}_{\mathcal{G}_K}(N) \cong \bigoplus_{\lambda \in X(\mathcal{G})} K_\lambda \cong \text{Top}_{\mathcal{G}_K}(K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j])$, whence $K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j] \cong \bigoplus_{\lambda \in X_M} M_K \otimes_K K_\lambda$. By definition, we have $|X_M| = [X(\mathcal{G}) : \text{Stab}_{X(\mathcal{G})}(M)]$. Since

$$\frac{\dim_k k\mathcal{G}}{p} j = [X(\mathcal{G}) : \text{Stab}_{X(\mathcal{G})}(M)] \dim_k M,$$

the number j is completely determined by M.

In view of [45, (3.5)], we have $\text{Top}_{\mathcal{G}}(M)_K \cong \text{Top}_{\mathcal{G}_K}(M_K)$, whence

$$\dim_k \text{Top}_{\mathcal{G}}(M) = \dim_K \text{Top}_{\mathcal{G}_K}(M_K) = |X_1| = \text{ord}(\text{Stab}_{X(\mathcal{G})}(M_K))$$
$$= \text{ord}(\text{Stab}_{X(\mathcal{G})}(M)),$$

as desired.   □

Theorem 4.3.3. *Let M be a non-projective, indecomposable, relatively α_K-projective $\mathcal{G}$-module. Suppose that $\mathcal{G}$ is not of finite representation type.*

(1) *The component $\Theta \subseteq \Gamma_s(\mathcal{G})$ containing M is a tube of rank ℓ, where $\ell \mid \text{ord}(X(\mathcal{G}))$.*
(2) *The module M is quasi-simple.*
(3) *There exists $j \in \{1, \ldots, p-1\}$ such that*

$$\alpha_{K,i}(X) = (\alpha_{K,i}(M) - a_{ij}) \, \mathfrak{q}\ell(X) + a_{ij}, \quad 1 \leq i \leq p-1,$$

for every $X \in \Theta$.

Proof. (1) By Lemma 4.3.1(3) and [45, (3.6)], the module $\tau_{\mathcal{G}}(M)_K \cong \tau_{\mathcal{G}_K}(M_K)$ is a direct summand of $K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]$. Hence Lemma 4.3.2 provides a character $\lambda_0 \in X(\mathcal{G})$ such that $\tau_{\mathcal{G}_K}(M_K) \cong M_K \otimes_K K_{\lambda_0}$. In view of [45, (2.5)], we thus obtain

$$\tau_{\mathcal{G}}(M) \cong M \otimes_k k_{\lambda_0}.$$

Hence there exists a divisor ℓ of $\mathrm{ord}(X(\mathcal{G}))$ such that $\tau_{\mathcal{G}}^\ell(M) \cong M$ and $\tau_{\mathcal{G}}^{\ell-1}(M) \not\cong M$. As observed in the proof of Proposition 4.1.3, this implies that $\Theta \cong \mathbb{Z}[A_\infty]/\langle \tau^\ell \rangle$.

(2) Suppose there exists $\gamma \in X(\mathcal{G})$ with $k_\gamma \in \Theta$. In view of the above, the functor

$$t_{\lambda_0} : \mathrm{mod}\,\mathcal{G} \to \mathrm{mod}\,\mathcal{G}, \qquad X \mapsto X \otimes_k k_{\lambda_0},$$

is an auto-equivalence of $\mathrm{mod}\,\mathcal{G}$ that commutes with $\tau_{\mathcal{G}}$. Moreover, $t_{-\lambda_0} \circ \tau_{\mathcal{G}}$ sends Θ to Θ and fixes M. As $t_{-\lambda_0} \circ \tau_{\mathcal{G}}$ preserves the quasi-length of a module, it follows that $t_{-\lambda_0} \circ \tau_{\mathcal{G}}|_\Theta = \mathrm{id}_\Theta$, whence $\tau_{\mathcal{G}} = t_{\lambda_0}$. This implies $\tau_{\mathcal{G}}(k_\gamma) \cong k_{\lambda_0+\gamma}$.

Now let $\omega \in X(\mathcal{G})$ be an arbitrary character. Then

$$\tau_{\mathcal{G}}(k_\omega) \cong \tau_{\mathcal{G}}(k_\gamma \otimes_k k_{\omega-\gamma}) \cong \tau_{\mathcal{G}}(k_\gamma) \otimes_k k_{\omega-\gamma} \cong k_{\gamma+\lambda_0} \otimes_k k_{\omega-\gamma} \cong k_{\omega+\lambda_0}.$$

As a result, $\tau_{\mathcal{G}}$ sends simple modules to simple modules and [3, (IV.2.10)] implies that $k\mathcal{G}$ is a Nakayama algebra. In particular, $k\mathcal{G}$ is representation-finite, a contradiction.

The arguments for Proposition 4.1.3(2) now show that M is quasi-simple.

(3) By Lemma 4.3.2, there exists a unique $j \in \{1, \ldots, p-1\}$ such that M_K is a direct summand of $K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]$. Since M_K occurs in $K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [j]$ with multiplicity 1, it follows that the multiplicities n_i of M_K in $N_i := K\mathcal{G} \otimes_{\mathfrak{A}_{p,K}} [i]$ are given by

$$n_i = \delta_{ij}.$$

In view of (2), Proposition 4.1.1 implies

$$\alpha_{K,i}(X) = \left(\alpha_{K,i}(M) - \sum_{\ell=1}^{p-1} a_{i\ell} n_\ell \right) \mathsf{q}\ell(X) + \sum_{\ell=1}^{p-1} a_{i\ell} n_\ell = (\alpha_{K,i}(M) - a_{ij}) \mathsf{q}\ell(X) + a_{ij}$$

for every $X \in \Theta$ and $i \in \{1, \ldots, p-1\}$. $\qquad\square$

Remark. If $\mathcal{G}$ is an infinitesimal supersolvable group of infinite representation type, then the principal block $\mathcal{B}_0(\mathcal{G})$ is isomorphic to the algebra of measures of a trigonalizable group $\mathcal{G}'$ (cf. [28, (2.3), (2.4)]). Hence the foregoing result also holds for indecomposable relatively projective modules that belong to $\mathcal{B}_0(\mathcal{G})$.

Corollary 4.3.4. *Let $\mathcal{U}$ be a unipotent finite group scheme of infinite representation type, and $\alpha_K : \mathfrak{A}_{p,K} \to K\mathcal{U}$ be a π-point. Suppose that M is an indecomposable, relatively α_K-projective $\mathcal{U}$-module.*

(1) *The module $\mathrm{Top}_{\mathcal{U}}(M)$ is simple and $M_K \cong K\mathcal{U} \otimes_{\mathfrak{A}_{p,K}} [j]$ for some $j \in \{1, \ldots, p-1\}$.*
(2) *The component $\Theta_M \subseteq \Gamma_s(\mathcal{U})$ containing M is isomorphic to $\mathbb{Z}[A_\infty]/\langle \tau \rangle$, and M is quasi-simple.*
(3) *There exists $j \in \{1, \ldots, p-1\}$ such that*

$$\alpha_{K,i}(X) = (\alpha_{K,i}(M) - a_{ij}) \mathsf{q}\ell(X) + a_{ij}, \qquad 1 \le i \le p,$$

for every $X \in \Theta_M$.

The $U_0(\mathfrak{h})$-module $M_x := U_0(\mathfrak{h}) \otimes_{U_0(kx)} k$ is indecomposable and relatively projective with respect to the π-point $\alpha_k : \mathfrak{A}_{p,k} \to U_0(\mathfrak{h})$ that sends t to x. The Cartan–Weyl identities (cf. [52, (I.1.3)]) yield

$$xy^n = y^n x + n y^{n-1} z \qquad \forall n \geq 0,$$

so that $x.(y^n z^m \otimes 1) = n y^{n-1} z^{m+1} \otimes 1$. Consequently, for every $\ell \in \{0, \ldots, 2p-2\}$, the k-space $M_x(\ell) := \bigoplus_{n+m=\ell} k(y^n z^m \otimes 1)$ is a $U_0(kx)$-submodule of M. We thus obtain

$$\alpha_k^*(M_x) = \bigoplus_{\ell=0}^{p-1} \alpha_k^*(M_x(\ell)) \oplus \bigoplus_{\ell=p}^{2p-2} \alpha_k^*(M_x(\ell)) \cong \bigoplus_{\ell=0}^{p-1} [\ell+1] \oplus \bigoplus_{\ell=p}^{2p-2} [2p-\ell-1]$$

$$\cong \bigoplus_{\ell=1}^{p-1} 2[\ell] \oplus [p].$$

Now let $\Theta \subseteq \Gamma_s(\mathfrak{h})$ be the component containing M_x. Given $X \in \Theta$, Corollary 4.3.4 implies

$$\alpha_{k,i}(X) = \begin{cases} 2 & \text{for } i = 1, \\ (3 - \delta_{p,2}) \, \mathrm{q}\ell(X) - 1 & \text{for } i = 2, \\ (2 - \delta_{i,p}) \, \mathrm{q}\ell(X) & \text{for } 3 \leq i \leq p. \end{cases}$$

Since $z \in \mathfrak{Z}(U_0(\mathfrak{h}))$, the choice $\alpha_k(t) := z$ and $M_z := U_0(\mathfrak{h}) \otimes_{U_0(kz)} k$ leads to the formulae of (1) with $r = 2$.

(3) According to Lemma 2.2, the π-points α_K for which a given $\mathcal{G}$-module is relatively α_K-projective all belong to one equivalence class. The following example shows that the converse does not hold. Let $\mathfrak{u} := kx \oplus ky$ be the two-dimensional abelian restricted Lie algebra with trivial p-mapping. Then $M_x := U_0(\mathfrak{u}) \otimes_{U_0(kx)} k$ is an indecomposable $U_0(\mathfrak{u})$-module of dimension p. Let $\alpha_k : \mathfrak{A}_{p,k} \to U_0(\mathfrak{u})$ be the π-point given by $\alpha_k(t) := x$. Then M is relatively α_k-projective and $\alpha_k^*(M_x) = p[1]$. According to [35, (2.2)], the map $\beta_k : \mathfrak{A}_{p,k} \to U_0(\mathfrak{u})$ defined via $\beta_k(t) := x + y^{p-1}$ is a π-point such that $\beta_k \sim \alpha_k$. Since

$$\beta_k(t)^i.(y^j \otimes 1) = \begin{cases} y^j \otimes 1 & \text{for } i = 0, \\ y^{p-1} \otimes 1 & \text{for } (i,j) = (1,0), \\ 0 & \text{else,} \end{cases}$$

we see that $\beta_k(M_x) = (p-2)[1] \oplus [2]$. In view of (1), the module M_x is not relatively β_k-projective whenever $p \geq 3$.

Examples. For $p \geq 3$ we consider the restricted Lie algebra $\mathfrak{g} := kt \oplus kx \oplus ky$, whose bracket and p-map are given by

$$[t, x] = x, \ [t, y] = 2y, \ [x, y] = 0 \quad \text{and} \quad t^{[p]} = t, \ x^{[p]} = 0 = y^{[p]}.$$

Every simple $U_0(\mathfrak{g})$-module is annihilated by the p-ideal $kx \oplus ky$ and thus one-dimensional. Hence $\mathfrak{g}$ is trigonalizable.

956   Rolf Farnsteiner

(1) Suppose that $\alpha_k : \mathfrak{A}_{p,k} \to U_0(\mathfrak{g})$ is given by $\alpha_k(t) := ky$. We consider the $U_0(\mathfrak{g})$-module

$$V := U_0(\mathfrak{g}) \otimes_{U_0(ky)} k.$$

Direct computation shows that y annihilates V, so that $V \cong U_0(\mathfrak{g}/ky)$ is a $U_0(\mathfrak{g}/ky)$-module. As such it has p-indecomposable constituents, each having a simple top.

(2) Suppose that $\alpha_k(t) = x + y =: v$ and consider

$$V := U_0(\mathfrak{g}) \otimes_{U_0(kv)} k.$$

Consider the π-point β_k defined by $\beta_k(t) := x$. Since $\beta_k^*(V)$ is projective, the dimension of every constituent $M \mid V$ is a multiple of p. Lemma 4.3.2 now shows that $\dim_k M = p$ for every proper summand M. Using techniques from [25] one can then show that V is indecomposable.

Corollary 4.3.5. *Let $\mathcal{G}$ be a trigonalizable finite group scheme of infinite representation type, and M be a non-projective indecomposable $\mathcal{G}$-module. If $\alpha_K : \mathfrak{A}_{p,K} \to K\mathcal{G}$ is a π-point such that $\alpha_{K,j}(M) \leq 1$ for $1 \leq j \leq p-1$, then M is not relatively α_K-projective.*

Proof. Suppose that M is relatively α_K-projective and let Θ be the stable AR-component containing M. Theorem 4.3.3 provides $j \in \{1, \ldots, p-1\}$ such that

$$\alpha_{K,j}(X) = (\alpha_{K,j}(M) - 2)\,\mathrm{q}\ell(X) + 2 \leq 2 - \mathrm{q}\ell(X)$$

for all $X \in \Theta$. Since $\alpha_{K,j}(X) \geq 0$, this cannot happen. $\qquad\qquad\qquad\square$

Remark. Let $\mathcal{G} = \mathrm{SL}(2)_1$ be the first Frobenius kernel of $\mathrm{SL}(2)$. In Section 8 below, we shall see that, for every non-projective baby Verma module $Z(\lambda)$, the π-points α_K giving rise to elements of $\Pi(\mathrm{SL}(2)_1)_{Z(\lambda)}$ all satisfy $\alpha_{K,j}(M) \leq 1$ for $1 \leq j \leq p-1$. On the other hand, one can show that $Z(\lambda)$ is relatively α_K-projective for each of these π-points.

4.4. Modules with cyclic vertices

For finite groups, the Mackey decomposition theorem along with the theory of vertices of indecomposable modules provides better control over relatively projective modules. Let G be a finite group, and M be an indecomposable G-module. Recall that a p-subgroup $D \subseteq G$ is called a *vertex* of M if $M \mid (kG \otimes_{kD} V)$ for some D-module V and D is minimal subject to this property. All vertices of M are conjugate, and we write $D = \mathrm{vx}(M)$ (cf. [4, (3.10)]).

Theorem 4.4.1. *Suppose that $p \geq 3$. Let G be a finite group, and $\alpha_K \in \Pi\mathrm{t}(G)$ be a π-point that factors through a cyclic p-subgroup $C \subseteq G$. Let M be a non-projective indecomposable relatively α_K-projective G-module belonging to a block of infinite representation type.*

(1) *M is quasi-simple with component $\Theta \cong \mathbb{Z}[A_\infty]/\langle \tau^q \rangle$.*
(2) *If $\beta_L \in \Pi\mathrm{t}(G)$ is a π-point factoring through C, then M is relatively β_L-projective.*

(3) *There exist* $j \in \{1, \dots, p-1\}$ *and* $m, n \in \mathbb{N}$ *with* $m \geq 2n$ *such that*

$$\beta_{L,i}(X) = \begin{cases} (m - 2n)\,\mathrm{q}\ell(X) + 2n & \text{for } i = j, \\ n(\mathrm{q}\ell(X) - 1) & \text{for } i = j - 1, j + 1, \\ 0 & \text{for } i \neq j, j + 1, j - 1, \end{cases}$$

for every $X \in \Theta$ *and every* $\beta_L \in \Pi\mathrm{t}(G)$ *that factors through* C.

(4) $\mathrm{vx}(M) \cong \mathbb{Z}/(p)$.

Proof. (1) By assumption, there exists a KC-module V such that the indecomposable KG-module M_K is a direct summand of

$$KG \otimes_{\mathfrak{A}_{p,K}} \alpha_K^*(M_K) \cong KG \otimes_{KC} V.$$

As a result, the vertex $\mathrm{vx}(M_K)$ of M_K is contained in C and hence is cyclic. If the block of KG containing the indecomposable KG-module M_K has finite representation type, then [45, (2.5)] implies that the block containing M enjoys the same property, a contradiction. In view of [19, Theorem], the module M_K is quasi-simple. According to [45, (3.8)], $\mathfrak{E}_M \otimes_k K$ is the almost split sequence terminating in M_K. Hence M also has exactly one predecessor in Θ. As a result, the G-module M is also quasi-simple.

(2) Let $X \in \Theta$ be arbitrary. Since C is cyclic, we have $\dim \Pi(C) = 0$, so that every π-point of C is generic. According to [37, (4.2)], we have

$$\mathrm{Jt}(X|_C, \alpha_K) = \mathrm{Jt}(X|_C, \beta_L),$$

whence $\alpha_{K,i}(X) = \beta_{L,i}(X)$ for $1 \leq i \leq p - 1$.

By (1), every relatively α_K-projective module belonging to Θ is quasi-simple. Hence Proposition 4.1.1 provides $(n_1, \dots, n_{p-1}) \in \mathbb{N}_0^{p-1} \setminus \{0\}$ such that

$$\alpha_{K,i}(X) = \left(\alpha_{K,i}(M) - \sum_{j=1}^{p-1} a_{ij} n_j \right) \mathrm{q}\ell(X) + \sum_{j=1}^{p-1} a_{ij} n_j, \quad 1 \leq i \leq p - 1,$$

for every $X \in \Theta$. Hence this formula also holds for the functions $\beta_{L,i} : \Theta \to \mathbb{N}_0$, so that $\ell(\beta_{L,i}) \leq 2$ with equality holding for at least one $i \in \{1, \dots, p-1\}$. In view of (1) and Corollary 2.5 the module M is relatively β_L-projective.

(3) The subgroup $C \subseteq G$ contains an element g of order p, and we define $\alpha(g)_k : \mathfrak{A}_{p,k} \to kG$ via $\alpha(g)_k(t) := g - 1$. By (2), there exists $j \in \{1, \dots, p-1\}$ such that the G-module M is a direct summand of $kG \otimes_{kC_p} [j]$, where we set $C_p := \langle g \rangle$. The Mackey decomposition theorem yields

$$\alpha(g)_k^*(kG \otimes_{kC_p} [j]) \cong (kG \otimes_{kC_p} [j])|_{kC_p} \cong \ell[j] \oplus s[p],$$

so that $\alpha(g)_k^*(M) \cong m[j] \oplus t[p]$ for some $m \geq 1$. This implies in particular that the number j defined above is uniquely determined, whence $n_i(M, \alpha(g)_k) = \delta_{i,j} n$ for some $n \in \mathbb{N}$. In view of Proposition 4.1.1, the asserted formula holds for the π-point $\alpha(g)$. As noted earlier, it thus holds for any π-point β_L factoring through C.

Finally, since $\alpha(g)_{k,j}^*(X) \geq 0$ for all $X \in \Theta$, we have $m \geq 2n$.

(4) Since M is a direct summand of $kG \otimes_{kC_p} [j]$, the vertex $\mathrm{vx}(M)$ is either trivial, or isomorphic to $\mathbb{Z}/(p)$. In the former case, M is projective, a contradiction. $\qquad\square$

We can endow the truncated polynomial ring $\mathfrak{A}_{p,k}$ with the structure of a Hopf algebra such that $\mathfrak{A}_{p,k} \cong k\mathbb{Z}/(p)$. Then we have:

Corollary 4.4.2. *Let $\alpha_K \in \Pi\mathrm{t}(G)$ be a π-point, which is a homomorphism $K\mathbb{Z}/(p) \to KG$ of Hopf algebras. If M is a non-projective indecomposable relatively α_K-projective module with infinite AR-component Θ, then $\Theta \cong \mathbb{Z}[A_\infty]/\langle\tau^q\rangle$ and there exist $j \in \{1,\dots,p-1\}$ and $m, n \in \mathbb{N}$ with $m \geq 2n$ such that*

$$\alpha_{K,i}(X) = \begin{cases} (m-2n)\,\mathrm{q}\ell(X) + 2n & \text{for } i = j, \\ n(\mathrm{q}\ell(X)-1) & \text{for } i = j-1, j+1, \\ 0 & \text{for } i \neq j, j+1, j-1, \end{cases}$$

for every $X \in \Theta$. $\qquad\square$

Remark. Proposition 8.1.1 below shows that the formulae of Corollary 4.4.2 do not necessarily hold for infinitesimal group schemes G and π-points defined by Hopf algebra homomorphisms $k\mathbb{G}_{a(1)} \to kG$.

5. Constantly supported modules

Retaining the general assumptions of Section 2, we consider those G-modules that have constant Jordan type on their Π-supports. In case a G-module M has full Π-support $\Pi(G)_M = \Pi(G)$, this amounts to M being of constant Jordan type. We are thus led to the following:

Definition. A G-module M is referred to as *constantly supported* if

(a) $\Pi(G)_M \neq \Pi(G)$, and
(b) $|\mathrm{Jt}(M)| = 2$.

By virtue of Corollary 3.2.1, locally split stable Auslander–Reiten components either contain no constantly supported modules, or consist entirely of such modules.

Our fundamental examples are given by direct summands of the Carlson modules L_ζ. Let $(P_n, d_n)_{n\geq 0}$ be a minimal projective resolution of the trivial G-module k. Then

$$\mathrm{Hom}_G(\Omega_G^n(k), k) \to \mathrm{H}^n(G, k), \quad \hat\zeta \mapsto [\hat\zeta \circ d_n],$$

is an isomorphism. If $\zeta := [\hat\zeta \circ d_n] \neq 0$, then the *Carlson module*

$$L_\zeta := \ker \hat\zeta \subseteq \Omega_G^n(k)$$

does not depend on the choice of the representing cocycle. In view of [5, (5.9.4)], we postulate that the zero element corresponds to the modules $L_0^n := \Omega_G^n(k) \oplus \Omega_G(k)$. Being submodules of $\Omega_G^n(k) \oplus \Omega_G(k)$, L_ζ and L_0^n are projective-free, that is, they do not have non-zero projective summands.

Given $\zeta \in \mathrm{H}^n(\mathcal{G}, k) \setminus \{0\}$, we note that $L_\zeta = (0)$ if and only if $\Omega_\mathcal{G}^n(k) \cong k$. Since $\Omega_\mathcal{G}^n(k)$ has constant Jordan type $[p-1]$ for n odd, we see that, for $p \geq 3$, this degenerate case can only occur if n is even. In view of

$$\dim \Pi(\mathcal{G}) = \dim \mathcal{V}_\mathcal{G}(k) - 1 = \mathrm{cx}_\mathcal{G}(k) - 1,$$

it follows that $L_\zeta \neq (0)$ whenever $\dim \Pi(\mathcal{G}) \geq 1$.

Remark. If $\mathcal{G}$ is an infinitesimal group scheme and $L_\zeta = (0)$ for some $\zeta \in \mathrm{H}^n(\mathcal{G}, k) \setminus \{0\}$, then [28, (2.1)] implies that $\mathcal{G}$ is supersolvable. In view of [28, (2.4)], all simple modules of the principal block $\mathcal{B}_0(\mathcal{G}) \subseteq k\mathcal{G}$ are one-dimensional. Owing to [28, (2.7)], this block is a Nakayama algebra, so that [3, (IV.2.10)] implies that $\dim_k \Omega_\mathcal{G}^{2m}(k) = 1$ for all $m \geq 0$. Thus, $\mathcal{G}$ affords no non-zero Carlson modules of even degree.

The cohomology ring

$$\mathrm{H}^*(\mathcal{G}, k) := \bigoplus_{m \geq 0} \mathrm{H}^m(\mathcal{G}, k)$$

is graded commutative, so that $\mathrm{H}^\bullet(\mathcal{G}, k) := \bigoplus_{m \geq 0} \mathrm{H}^{2m}(\mathcal{G}, k)$ is a commutative subalgebra. The importance of Carlson modules resides in their support varieties often being hyperplanes of $\mathcal{V}_\mathcal{G}(k)$: The variety

$$\mathcal{V}_\mathcal{G}(L_\zeta) = Z(\zeta)$$

is the zero locus of the homogeneous element $\zeta \in \mathrm{H}^\bullet(\mathcal{G}, k)$ (see [5, (5.9.1)], [35, (4.11)]). In particular, $L_\zeta \neq (0)$ for every non-zero nilpotent element ζ of even degree.

Recall that a $\mathcal{G}$-module M is referred to as *endo-trivial* if

$$\mathrm{End}_k(M) \cong k \oplus (\mathrm{proj.}).$$

The following subsidiary result elaborates on [7, (4.1)].

Lemma 5.1. *Let $\zeta \in \mathrm{H}^{2n}(\mathcal{G}, k) \setminus \{0\}$ and write $L_\zeta \cong M_1 \oplus \cdots \oplus M_r$, with M_i indecomposable.*

(1) *If ζ is not nilpotent and $M \mid L_\zeta$ is a non-zero summand, then M is constantly supported and*

$$\mathrm{Jt}(M) = \left\{ \left(\frac{\dim_k M}{p} \right)[p], [1] \oplus [p-1] \oplus n_M[p] \right\}.$$

(2) *If ζ is not nilpotent, then*

$$\Pi(\mathcal{G})_{L_\zeta} = \bigsqcup_{i=1}^r \Pi(\mathcal{G})_{M_i}$$

is the decomposition of $\Pi(\mathcal{G})_{L_\zeta}$ into its connected components.

(3) *If ζ is nilpotent, then L_ζ has constant Jordan type $\mathrm{Jt}(L_\zeta) = \{[1] \oplus [p-1] \oplus n_\zeta[p]\}$. Moreover, L_ζ is either indecomposable or a direct sum of two endo-trivial modules.*

Proof. We begin by proving a number of auxiliary statements.

(a) *If $[\alpha_K] \in \Pi(\mathcal{G})_{L_\zeta}$, then $\mathrm{Jt}(L_\zeta, \alpha_K) = [1] \oplus [p-1] \oplus n_\zeta[p]$ for some $n_\zeta \in \mathbb{N}_0$.*

We consider the short exact sequence

$$(0) \to L_\zeta \to \Omega_{\mathcal{G}}^{2n}(k) \xrightarrow{\hat{\zeta}} k \to (0).$$

A π-point $\alpha_K \in \Pi\mathrm{t}(\mathcal{G})$ induces an exact sequence

$$(0) \to \alpha_K^*((L_\zeta)_K) \to K \oplus (\mathrm{proj.}) \xrightarrow{(f,g)} K \to (0).$$

Since $[\alpha_K] \in \Pi(\mathcal{G})_{L_\zeta}$, the module $\alpha_K^*((L_\zeta)_K)$ is not projective and we have $f = 0$. Hence

$$\alpha_K^*((L_\zeta)_K) \cong K \oplus \Omega_{\mathfrak{A}_{p,K}}(K) \oplus (\mathrm{proj.}),$$

so that

$$\mathrm{Jt}(L_\zeta, \alpha_K) = [1] \oplus [p-1] \oplus n_\zeta[p]$$

for some $n_\zeta \in \mathbb{N}_0$. $\diamond$

(b) *Let $M \mid L_\zeta$ be a non-zero summand such that $\dim_k M \equiv 0 \mod(p)$. Then*

$$\mathrm{Jt}(M, \alpha_K) = [1] \oplus [p-1] \oplus n_M[p] \quad \textit{for all } [\alpha_K] \in \Pi(\mathcal{G})_M.$$

If $\alpha_K \in \Pi\mathrm{t}(\mathcal{G})$ is a π-point with $[\alpha_K] \in \Pi(\mathcal{G})_M \subseteq \Pi(\mathcal{G})_{L_\zeta}$ (cf. [36, (3.3)]), then (a) implies

$$\alpha_K^*(M_K) \mid ([1] \oplus [p-1] \oplus n_\zeta[p]).$$

Accordingly, the projective-free part X of $\alpha_K^*(M_K)$ is a direct summand of $[1] \oplus [p-1]$. Since $[\alpha_K] \in \Pi(\mathcal{G})_M$, the $\mathfrak{A}_{p,K}$-module X is non-zero. Our assumption on M yields $\dim_K X \equiv 0 \mod(p)$, so that $X \cong [1] \oplus [p-1]$. Consequently, M has the asserted Jordan type with respect to α_K. $\diamond$

(c) *Let $i \neq j \in \{1, \dots, r\}$ be such that $\dim_k M_i, \dim_k M_j \equiv 0 \mod(p)$. Then $\Pi(\mathcal{G})_{M_i} \cap \Pi(\mathcal{G})_{M_j} = \emptyset$.*

If $[\alpha_K] \in \Pi(\mathcal{G})_{M_i} \cap \Pi(\mathcal{G})_{M_j}$, then (b) yields

$$\alpha_K^*((M_\ell)_K) \cong [1] \oplus [p-1] \oplus m_\ell[p]$$

for $\ell \in \{i, j\}$, which contradicts $\alpha_K^*((M_i)_K) \oplus \alpha_K^*((M_j)_K)$ being a direct summand of $\alpha_K^*((L_\zeta)_K)$. $\diamond$

(1) Since the element $\zeta \in \mathrm{H}^\bullet(\mathcal{G}, k)$ is not nilpotent, [36, (3.3)] implies $\Pi(\mathcal{G})_M \subseteq \Pi(\mathcal{G})_{L_\zeta} \neq \Pi(\mathcal{G})$, so that $\dim_k M \equiv 0 \mod(p)$. The result now follows from (b).

(2) Arguing as in (1), we see that $\dim_k M_i \equiv 0 \mod(p)$ for every $i \in \{1, \dots, r\}$. As a result, [36, (3.3)] in conjunction with (c) implies

$$\Pi(\mathcal{G})_{L_\zeta} = \bigsqcup_{i=1}^{r} \Pi(\mathcal{G})_{M_i},$$

while [10, (3.4)] (see also [7]) ensures the connectedness of each subspace $\Pi(\mathcal{G})_{M_i}$.

(3) Since $\zeta \neq 0$ is nilpotent, we have $\mathcal{V}_{\mathcal{G}}(L_\zeta) = \mathcal{V}_{\mathcal{G}}(k) \neq \{0\}$. Thanks to [36, (3.4), (3.6)], this implies $\Pi(\mathcal{G})_{L_\zeta} = \Pi(\mathcal{G}) \neq \emptyset$. By (a), the module L_ζ has constant Jordan type $\mathrm{Jt}(L_\zeta) = \{[1] \oplus [p-1] \oplus n_\zeta[p]\}$. Let α_K be a π-point. Each of the modules $\alpha_K^*((M_i)_K)$ is a direct summand of $[1] \oplus [p-1] \oplus n_\zeta[p]$, whence $\dim_k M_i \equiv 1, -1, 0 \mod(p)$.

Suppose there exists $i \in \{1, \ldots, r\}$ such that $\dim_k M_i \equiv \ell \mod(p)$ for $\ell \in \{1, p-1\}$. Then $\Pi(\mathcal{G})_{M_i} = \Pi(\mathcal{G})$, and there exists $n_i \in \mathbb{N}$ such that $\alpha_K^*((M_i)_K) = [\ell] \oplus n_i[p]$ for every π-point $\alpha_K \in \Pi\mathrm{t}(\mathcal{G})$. There also exists $j \in \{1, \ldots, r\}$ such that $\alpha_K^*((M_j)_K) = [p-\ell] \oplus n_j[p]$ for all $\alpha_K \in \Pi\mathrm{t}(\mathcal{G})$. Consequently, all other summands of L_ζ are projective, so that L_ζ being projective-free yields $r = 2$. We may now apply [10, (5.6)] to see that each summand is endo-trivial.

Alternatively, $\dim_k M_i \equiv 0 \mod(p)$ for every $i \in \{1, \ldots, r\}$. According to (c), this implies

$$\Pi(\mathcal{G}) = \Pi(\mathcal{G})_{L_\zeta} = \bigsqcup_{i=1}^{r} \Pi(\mathcal{G})_{M_i},$$

which, in view of $\Pi(\mathcal{G})$ being connected (cf. [10, (3.4)]), yields $r = 1$. □

Remarks. (1) The above result shows in particular that, for a non-nilpotent element $\zeta \in \mathrm{H}^{2n}(\mathcal{G}, k)$, the indecomposable summands of L_ζ occur with multiplicity one. By the same token, the $\mathcal{G}$-module $M_i \otimes_k M_j$ is projective whenever $i \neq j$ (cf. [36, (3.2)]).

(2) We shall see in Theorem 6.3.1 that the second alternative in Lemma 5.1(3) does not occur.

Theorem 5.2. *Suppose that $\zeta \in \mathrm{H}^{2n}(\mathcal{G}, k)$ is a non-nilpotent element. Let M be an indecomposable summand of L_ζ, $\Theta \subseteq \Gamma_s(\mathcal{G})$ be the component containing M.*

(1) *If Θ is locally split, then every $N \in \Theta$ is constantly supported and*

$$\mathrm{Jt}(N) = \left\{ \left(\frac{\dim_k N}{p} \right)[p], \, f_\Theta(N)[1] \oplus f_\Theta(N)[p-1] \oplus m_N[p] \right\}.$$

(2) *If $\bar{T}_\Theta \cong A_\infty$ and Θ is locally split, then M is quasi-simple.*

Proof. (1) Let α_K be a π-point with $[\alpha_K] \in \Pi(\mathcal{G})_\Theta$. Lemma 5.1 implies $d_i^\Theta(\alpha_K) = \delta_{i,1} + \delta_{i,p-1}$ for $1 \leq i \leq p-1$, so that the assertion follows from Theorem 3.1.1.

(2) This follows directly from (1), Lemma 5.1 and Corollary 3.1.2. □

Corollary 5.3. *Suppose that $\dim \Pi(\mathcal{G}) \geq 2$. For every non-nilpotent homogeneous element $\zeta \in \mathrm{H}^\bullet(\mathcal{G}, k)$ there exists a component $\Theta_\zeta \subseteq \Gamma_s(\mathcal{G})$ such that*

$$\mathrm{Jt}(M) = \left\{ \left(\frac{\dim_k M}{p} \right)[p], \, f_{\Theta_\zeta}(M)[1] \oplus f_{\Theta_\zeta}(M)[p-1] \oplus n_M[p] \right\}$$

for every $M \in \Theta_\zeta$.

Proof. Let $\zeta \in \mathrm{H}^{2n}(\mathcal{G}, k)$ be a non-nilpotent element. According to [36, (3.4), (3.6)], we have

$$\dim \Pi(\mathcal{G})_{L_\zeta} = \dim \mathcal{V}_{\mathcal{G}}(L_\zeta) - 1 \geq \dim \mathcal{V}_{\mathcal{G}}(k) - 2 = \dim \Pi(\mathcal{G}) - 1 \geq 1.$$

Lemma 5.1 provides an indecomposable summand $M_\zeta \mid L_\zeta$ such that $\dim \Pi(\mathcal{G})_{M_\zeta} \geq 1$ and

$$\mathrm{Jt}(M_\zeta) = \left\{ \left(\frac{\dim_k M_\zeta}{p} \right)[p],\ [1] \oplus [p-1] \oplus n_{M_\zeta}[p] \right\}.$$

Thus, letting $\Theta_\zeta \subseteq \Gamma_s(\mathcal{G})$ be the stable AR-component containing M_ζ, our assertion is a consequence of Theorem 5.2.   □

We conclude this section with an application concerning trigonalizable group schemes.

Corollary 5.4. *Suppose that $p \geq 3$. Let $\mathcal{G}$ be a trigonalizable finite group scheme of infinite representation type, and $\zeta \in \mathrm{H}^{2n}(\mathcal{G}, k)$ be a non-nilpotent element. Then every component $\Theta \subseteq \Gamma_s(\mathcal{G})$ containing an indecomposable direct summand of L_ζ is locally split.*

Proof. Let $X \mid L_\zeta$ be an indecomposable direct summand such that $X \in \Theta$. If $\alpha_K : \mathfrak{A}_{p,K} \to K\mathcal{G}$ is a π-point such that Θ is not α_K-split, then Theorem 4.3.3 implies that $\Theta \cong \mathbb{Z}[A_\infty]/\langle \tau^m \rangle$ for some $m \geq 1$. By the same token, every relatively α_K-projective vertex of Θ is quasi-simple. Given such a module M, there is $j \in \{1, \ldots, p-1\}$ with

$$\alpha_{K,i}(X) = (\alpha_{K,i}(M) - a_{ij})\, \mathrm{q}\ell(X) + a_{ij}, \qquad 1 \leq i \leq p-1.$$

In view of Lemma 5.1 and Corollary 4.3.5, the module X is not relatively α_K-projective, so that $\mathrm{q}\ell(X) \geq 2$. Specializing $i = j$, we obtain

$$\delta_{j,1} + \delta_{j,p-1} - 2 = \mathrm{q}\ell(X)(\alpha_{K,j}(M) - 2).$$

Since $p \geq 3$, the left-hand side belongs to $\{-2, -1\}$, whence $\mathrm{q}\ell(X) = 2$, as well as $j \notin \{1, p-1\}$. Consequently,

$$1 = (\alpha_{K,i}(M) - a_{ij})2 + a_{ij} \quad \text{for } i \in \{1, p-1\},$$

so that $a_{1,j} \neq 0 \neq a_{p-1,j}$. Since $j \neq \{1, p-1\}$, we obtain $2 = j = p-2$, a contradiction.   □

Remark. Let $\mathcal{G} = \mathrm{SL}(2)_1$ be the first Frobenius kernel of $\mathrm{SL}(2)$. If $\zeta \in \mathrm{H}^{2n}(\mathrm{SL}(2)_1, k)$ is a non-zero element, then ζ is not nilpotent and each indecomposable summand of L_ζ belongs to a homogeneous tube. We shall see in Section 8 that none of these components is locally split and that each quasi-simple $\mathrm{SL}(2)_1$-module has dimension p. On the other hand, the examples succeeding Theorem 6.1.5 show that certain Carlson modules have indecomposable summands of dimension $2p$. Thus, Theorem 5.2 may fail for components of tree class A_∞ that are not locally split.

6. Indecomposable Carlson modules

In view of the foregoing results, the question when a Carlson module L_ζ is indecomposable arises. In this section, we provide indecomposability criteria for Carlson modules. Depending on properties of ζ, these are established by means of support varieties or Jordan types.

6.1. Carlson modules of non-nilpotent elements

Here we study the case where $\zeta \in H^\bullet(\mathcal{G}, k) \setminus \{0\}$ is a non-nilpotent homogeneous element of even degree.

Theorem 6.1.1. *Suppose that m is a positive integer such that the variety $\mathcal{V}_\mathcal{G}(k) \subseteq \mathbb{A}^m$ is equidimensional of dimension $n \geq (m+3)/2$. If $\zeta \in H^\bullet(\mathcal{G}, k)$ is a non-nilpotent homogeneous element of positive degree, then L_ζ is indecomposable.*

Proof. In view of Lemma 5.1, it suffices to show that $\mathrm{Proj}(Z(\zeta)) = \mathrm{Proj}(\mathcal{V}_\mathcal{G}(L_\zeta)) \cong \Pi(\mathcal{G})_{L_\zeta}$ is connected.

Let $X, Y \subseteq Z(\zeta)$ be closed, conical subsets such that $Z(\zeta) = X \cup Y$ and $X \cap Y = \{0\}$. We denote by $\mathrm{Irr}(\zeta)$ the set of irreducible components of $Z(\zeta)$. By general theory (cf. [46, (I.§8)]), every $Z \in \mathrm{Irr}(\zeta)$ has dimension $\geq n-1$, and the equality

$$Z = (Z \cap X) \cup (Z \cap Y)$$

implies that $\mathrm{Irr}(\zeta) = I(X) \cup I(Y)$, where the subsets $I(X), I(Y) \subseteq \mathrm{Irr}(\zeta)$ contain those irreducible components lying inside X and Y, respectively. The assumption $I(X), I(Y) \neq \emptyset$ provides $Z, Z' \in \mathrm{Irr}(\zeta)$ with $Z \cap Z' = \{0\}$. The affine dimension theorem [41, (I.7.1)] then implies

$$0 = \dim Z \cap Z' \geq 2n - 2 - m \geq m + 3 - 2 - m = 1,$$

a contradiction. Hence we may assume that $I(Y) = \emptyset$, whence $\mathrm{Irr}(\zeta) = I(X)$ and $X = Z(\zeta)$. Consequently, the variety $\mathrm{Proj}(Z(\zeta))$ is connected. $\square$

Remark. Let $n_\mathcal{G}$ be the minimum of the dimensions of the irreducible components of $\mathcal{V}_\mathcal{G}(k)$. Then the conclusion of Theorem 6.1.1 also holds if $n_\mathcal{G} \geq (m+3)/2$. For a finite group G, the invariant n_G coincides with the saturation rank $\mathrm{srk}(G)$ of G, which we consider in Section 6.4.

Corollary 6.1.2. *Let $\mathcal{U}$ be an abelian unipotent group scheme with $\dim \mathcal{V}_\mathcal{U}(k) \geq 3$. Then the Carlson module L_ζ associated to a homogeneous non-nilpotent element $\zeta \in H^\bullet(\mathcal{U}, k)$ is indecomposable.*

Proof. Let $r := \dim \mathcal{V}_\mathcal{U}(k)$ be the dimension of the support variety of $\mathcal{U}$. In view of [56, (14.4)], the ring $H^*(\mathcal{U}, k) \cong \mathrm{Ext}^*_{k\mathcal{U}}(k, k)$ is isomorphic to the cohomology ring of an abelian p-group. Thanks to [4, (3.5.5), (3.5.6)], the support variety $\mathcal{V}_\mathcal{U}(k) \cong \mathbb{A}^r$ is irreducible. Since $r \geq (r+3)/2$ for $r \geq 3$, it follows from Theorem 6.1.1 that the module L_ζ is indecomposable. $\square$

Remark. The foregoing results fail for group schemes whose support varieties have dimension ≤ 2: see our discussion below concerning restricted Lie algebras.

We now turn to infinitesimal groups and their rank varieties. Let $\mathcal{G}$ be an infinitesimal k-group. Given $r \in \mathbb{N}$, the authors of [53] introduce the scheme of infinitesimal 1-parameter subgroups of $\mathcal{G}$ of height $\leq r$, whose variety of k-rational points is

$$V_r(\mathcal{G}) := \mathrm{Hom}(\mathbb{G}_{a(r)}, \mathcal{G}).$$

Here $\mathbb{G}_{a(r)} := \mathrm{Spec}_k(k[X]/(X^{p^r}))$ is the r-th Frobenius kernel of the additive group $\mathbb{G}_a$. We set $x := X + (X^{p^r})$, so that $\{x^i; 0 \le i \le p^r - 1\}$ is a basis of $k[X]/(X^{p^r})$. If $\{\delta_0, \ldots, \delta_{p^r-1}\} \subseteq k\mathbb{G}_{a(r)}$ denotes the dual basis, then, setting $u_i := \delta_{p^i}$, we obtain a canonical set of generators $\{u_0, \ldots, u_{r-1}\}$ of the algebra $k\mathbb{G}_{a(r)}$. We consider $A_r := k[u_{r-1}] \subseteq k\mathbb{G}_{a(r)}$, the p-dimensional subalgebra generated by u_{r-1}. Given a $\mathcal{G}$-module M, Suslin et al. [54, §6] define the *r-th rank variety* of M via

$$V_r(\mathcal{G})_M := \{\alpha \in V_r(\mathcal{G}); \alpha^*(M)|_{A_r} \text{ is not projective}\}.$$

Suppose that $\mathcal{G}$ has height $\le r$, that is, $\mathcal{G}$ coincides with its r-th Frobenius kernel $\mathcal{G}_r$. Thanks to [54, (5.2), (6.8)], there exists a morphism $\Psi : V_r(\mathcal{G}) \to \mathcal{V}_{\mathcal{G}}(k)$ which is a homeomorphism such that

$$\Psi(V_r(\mathcal{G})_M) = \mathcal{V}_{\mathcal{G}}(M)$$

for every $M \in \mathrm{mod}\,\mathcal{G}$. We thus obtain the following analogue of Theorem 6.1.1 for rank varieties:

Corollary 6.1.3. *Let $\mathcal{G}$ be an infinitesimal group of height $\le r$. Suppose that m is a positive integer such that $V_r(\mathcal{G}) \subseteq \mathbb{A}^m$ is equidimensional of dimension $n \ge (m+3)/2$. If $\zeta \in \mathrm{H}^{\bullet}(\mathcal{G}, k)$ is a non-nilpotent homogeneous element of positive degree, then L_ζ is indecomposable.*

Proof. We denote by $\psi : \mathrm{H}^{\bullet}(\mathcal{G}, k)/\sqrt{(0)} \to k[V_r(\mathcal{G})]$ the comorphism of Ψ. Owing to [53, (1.14)], the map ψ is a homomorphism of k-algebras which multiplies degrees by $p^r/2$. Let $\eta \in k[V_r(\mathcal{G})]$ be the image of the residue class $\bar{\zeta}$ of ζ under ψ. Then we have

$$Z(\eta) = \Psi^{-1}(Z(\bar{\zeta})) = \Psi^{-1}(\mathcal{V}_{\mathcal{G}}(L_\zeta)) = V_r(\mathcal{G})_{L_\zeta}.$$

As before, it suffices to show that $\mathrm{Proj}(Z(\eta))$ is connected, and the arguments of the proof of Theorem 6.1.1 yield our assertion.   □

In the special case where our group $\mathcal{G}$ has height ≤ 1, the foregoing result can be formulated in the language of restricted Lie algebras. Given a restricted Lie algebra $(\mathfrak{g}, [p])$, we let

$$\mathcal{V}_{\mathfrak{g}} := \{x \in \mathfrak{g}; x^{[p]} = 0\}$$

be the *nullcone* of $\mathfrak{g}$. If M is a $U_0(\mathfrak{g})$-module, then

$$\mathcal{V}_{\mathfrak{g}}(M) := \{x \in \mathcal{V}_{\mathfrak{g}}; M|_{U_0(kx)} \text{ is not projective}\} \cup \{0\}$$

is the *rank variety* of M (see [33, 34]). Thanks to [53, (1.6)], we obtain:

Corollary 6.1.4. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Suppose that m is a positive integer such that $\mathcal{V}_{\mathfrak{g}} \subseteq \mathbb{A}^m$ is equidimensional of dimension $n \ge (m+3)/2$. If $\zeta \in \mathrm{H}^{\bullet}(U_0(\mathfrak{g}), k)$ is a non-nilpotent homogeneous element of positive degree, then L_ζ is indecomposable.*

In many cases of interest, the nullcone $\mathcal{V}_{\mathfrak{g}}$ is known to be irreducible. Let $\mathfrak{g} := \mathrm{Lie}(G)$ be the Lie algebra of a smooth reductive group G. If p is good for G, then [47, (6.3.1)] ensures the irreducibility of $\mathcal{V}_{\mathfrak{g}}$. Moreover, there is a formula expressing $\dim \mathcal{V}_{\mathfrak{g}}$ in terms of the root system of G. For the case where G is simple and simply connected explicit formulae may be found in [11]. Recall that the *rank* $\mathrm{rk}(G)$ of G is the dimension of any maximal torus $T \subseteq G$. By way of example, we provide the following result:

Theorem 6.1.5. *Suppose that $\mathfrak{g} := \mathrm{Lie}(G)$ is the Lie algebra of a semi-simple, simply connected algebraic group $G \not\cong \mathrm{SL}(2) \times \mathrm{SL}(2)$ of rank $\mathrm{rk}(G) \geq 2$, whose Coxeter number h satisfies $p \geq h$. Let $\zeta \in \mathrm{H}^{\bullet}(U_0(\mathfrak{g}), k)$ be a non-nilpotent homogeneous element of positive degree.*

(1) *The Carlson module L_{ζ} is indecomposable.*
(2) *$L_{\zeta} \in \mathbb{Z}[A_{\infty}]$ is quasi-simple.*

Proof. In virtue of our present assumption, [47, (6.3.1)] implies that $\mathcal{V}_{\mathfrak{g}}$ is irreducible of dimension

$$\dim \mathcal{V}_{\mathfrak{g}} = \dim_k \mathfrak{g} - \mathrm{rk}(G).$$

The latter number is $\geq (\dim_k \mathfrak{g} + 3)/2$ in case $\mathrm{rk}(G) \geq 3$. For groups of rank 2, a case-by-case analysis yields the same conclusion, unless G is of type $A_1 \times A_1$. Since $\mathcal{V}_{\mathfrak{g}} \subseteq \mathfrak{g}$, Corollary 6.1.4 ensures the indecomposability of L_{ζ}.

Since $\dim \mathcal{V}_{\mathfrak{g}} = \dim_k \mathfrak{g} - \mathrm{rk}(G) \geq 3\,\mathrm{rk}(G) - \mathrm{rk}(G) \geq 4$, we see that $\dim \mathcal{V}_{\mathfrak{g}}(L_{\zeta}) \geq 3$. Thanks to [22, (2.2)], the module L_{ζ} thus belongs to a stable AR-component of type $\mathbb{Z}[A_{\infty}]$, and Theorem 5.2 shows that L_{ζ} is actually quasi-simple. $\square$

Remarks. (1) If $\mathfrak{g}$ is as in Theorem 6.1.5 with $p > h$, then $\mathrm{H}^*(U_0(\mathfrak{g}), k) = \mathrm{H}^{\bullet}(U_0(\mathfrak{g}), k)$ is a reduced k-algebra (see [2] or [32]), so that the above result actually describes all Carlson modules of the Lie algebra $\mathfrak{g}$.

(2) The results of [11] give rise to refinements of the above theorem. For instance, if $G = \mathrm{SL}(n)(k)$ and $p \geq 3$, [11, (3.1)] implies that

$$\dim \mathcal{V}_{\mathfrak{sl}(n)} \geq \frac{n^2 + 2}{2},$$

unless $n = 4$ and $p = 3$. Hence Theorem 6.1.5 holds for $\mathrm{SL}(n)(k)$ for $p \geq 3$ and $(n, p) \neq (4, 3)$.

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. The connection between rank varieties and support varieties is conveyed by the *Hochschild map*

$$\Phi : S(\mathfrak{g}^*) \to \mathrm{H}^{\bullet}(U_0(\mathfrak{g}), k)$$

between the algebra $S(\mathfrak{g}^*)$ of polynomial functions on $\mathfrak{g}$ (whose variables are given degree 2) and the even cohomology ring $\mathrm{H}^{\bullet}(U_0(\mathfrak{g}), k)$ (cf. [43, p. 571]). Given $f \in S(\mathfrak{g}^*)$, Friedlander and Parshall [34, p. 560] have shown that

$$\mathcal{V}_{\mathfrak{g}}(L_{\Phi(f)}) = Z(f) \cap \mathcal{V}_{\mathfrak{g}},$$

where $Z(f) \subseteq \mathfrak{g}$ denotes the set of zeros of the polynomial function f.

If $\mathfrak{g} := \mathrm{Lie}(G)$ is the Lie algebra of an algebraic group G, then $G \times k^{\times}$ acts on $\mathfrak{g}^*$ via

$$(g, \alpha).\eta := \alpha\eta \circ \mathrm{Ad}(g^{-1}) \quad \forall (g, \alpha) \in G \times k^{\times}, \ \eta \in \mathfrak{g}^*.$$

Here Ad denotes the adjoint representation of G on $\mathfrak{g}$. The G-invariance of $\mathcal{V}_{\mathfrak{g}}$ readily yields

$$Z((g, \alpha).\eta) \cap \mathcal{V}_{\mathfrak{g}} = \mathrm{Ad}(g)(Z(\eta) \cap \mathcal{V}_{\mathfrak{g}}).$$

Examples. (1) For $p \geq 3$ we consider the Lie algebra $\mathfrak{sl}(2)$ with its standard basis $\{e, h, f\}$, so that

$$\mathcal{V}_{\mathfrak{sl}(2)} = \{ah + be + cf \in \mathfrak{sl}(2); \ a^2 + bc = 0\}$$

is a two-dimensional irreducible variety. Thus, the pair $(\mathcal{V}_{\mathfrak{g}}, \mathfrak{g})$ violates the condition $\dim \mathcal{V}_{\mathfrak{g}} \geq (\dim_k \mathfrak{g} + 3)/2$.

Let $\eta \in \mathfrak{g}^* \setminus \{0\}$. By the proof of [27, (2.2)], the group $\mathrm{SL}(2)(k) \times k^{\times}$ acts on $\mathfrak{sl}(2)^* \setminus \{0\}$ with two orbits, whose representatives η_1, η_2 have kernels $kh \oplus ke$ and $ke \oplus kf$, respectively. Thus, $\mathcal{V}_{\mathfrak{sl}(2)}(L_{\Phi(\eta_1)}) = ke$ while $\mathcal{V}_{\mathfrak{sl}(2)}(L_{\Phi(\eta_2)}) = ke \cup kf$. In view of Lemma 5.1 and the observations above, the $3p$-dimensional Carlson modules $L_{\Phi(\eta)}$ of degree 2 are either indecomposable, or decomposable with two constituents (of dimensions p and $2p$, respectively). Moreover, since the Chevalley–Eilenberg cohomology groups $\mathrm{H}^i(\mathfrak{sl}(2), k)$ vanish for $i = 1, 2$, the exact sequence of [43, p. 575] implies that the Hochschild map induces an isomorphism $\mathfrak{sl}(2)^* \cong \mathrm{H}^2(U_0(\mathfrak{sl}(2)), k)$. Hence all Carlson modules of degree 2 are of the form indicated above.

(2) Let $\mathfrak{u} := kx \oplus ky$ be the two-dimensional abelian restricted Lie algebra with trivial p-map. Let $f : \mathfrak{u} \to k$ be the polynomial map given by $f(ax + by) = ab$. Then $\mathrm{Proj}(\mathcal{V}_{\mathfrak{u}}(L_{\Phi(f)}))$ consists of two points and $L_{\Phi(f)}$ is decomposable.

6.2. Subsidiary results

In this subsection we collect a few results that will be applied in our investigation of Carlson modules corresponding to homogeneous nilpotent elements of the cohomology ring $\mathrm{H}^*(\mathcal{G}, k)$.

Lemma 6.2.1. *Suppose that $\zeta \in \mathrm{H}^n(\mathcal{G}, k) \setminus \{0\}$ has positive degree. If $L_{\zeta} \neq (0)$, then there exists an exact sequence*

$$(0) \to \Omega_{\mathcal{G}}^{1-n}(k) \to \Omega_{\mathcal{G}}^{-n}(L_{\zeta}) \to k \to (0).$$

Proof. According to [5, (5.9.4)] (which only holds for $L_{\zeta} \neq (0)$), the element $\zeta \in \mathrm{H}^n(\mathcal{G}, k) \cong \mathrm{Ext}_{\mathcal{G}}^1(\Omega_{\mathcal{G}}^{n-1}(k), k)$ corresponds to the exact sequence

$$(0) \to k \to \Omega_{\mathcal{G}}^{-1}(L_{\zeta}) \to \Omega_{\mathcal{G}}^{n-1}(k) \to (0).$$

Application of $\Omega_{\mathcal{G}}^{1-n}$ thus yields an exact sequence

$$(0) \to \Omega_{\mathcal{G}}^{1-n}(k) \xrightarrow{\binom{f}{g}} \Omega_{\mathcal{G}}^{-n}(L_{\zeta}) \oplus P \xrightarrow{(\varphi, \psi)} k \to (0),$$

with a projective $\mathcal{G}$-module P. This gives a commutative diagram

$$
\begin{array}{ccc}
\Omega_{\mathcal{G}}^{1-n}(k) & \xrightarrow{\ g\ } & P \\
{\scriptstyle -f}\downarrow & & \downarrow{\scriptstyle \psi} \\
\Omega_{\mathcal{G}}^{-n}(L_\zeta) & \xrightarrow{\ \varphi\ } & k
\end{array}
$$

which, by virtue of [3, (I.5.7)], is both a pull-back and a push-out diagram. If $\varphi = 0$, then

$$\Omega_{\mathcal{G}}^{1-n}(k) = \ker(0, \psi) = \Omega_{\mathcal{G}}^{-n}(L_\zeta) \oplus \ker\psi.$$

Since $\Omega_{\mathcal{G}}^{1-n}(k)$ is indecomposable, we obtain $\ker\psi = (0)$ or $\Omega_{\mathcal{G}}^{-n}(L_\zeta) = (0)$. Since $L_\zeta \neq (0)$ is not injective, we may rule out the latter case, whence $\ker\psi = (0)$ and $P \cong k$. This, however, implies $\Omega_{\mathcal{G}}^{n}(k) = (0)$, a contradiction. Consequently, φ is surjective and [3, (I.5.6)] ensures the surjectivity of g. Hence P is a direct summand of $\Omega_{\mathcal{G}}^{1-n}(k)$, so that $P = (0)$. $\qquad\square$

Recall that a $\mathcal{G}$-module is endo-trivial if and only if it has constant stable Jordan type [1] or [$p-1$] (see [10, (5.6)]).

Lemma 6.2.2. *Suppose that $\zeta \in \mathrm{H}^n(\mathcal{G}, k) \setminus \{0\}$ is an element such that $L_\zeta = X \oplus Y$ is the direct sum of two indecomposable endo-trivial modules with*

$$
\mathrm{StJt}(X) = \begin{cases} [1], & n \text{ even}, \\ [p-1], & n \text{ odd}, \end{cases} \quad \text{and} \quad \mathrm{StJt}(Y) = [p-1].
$$

(1) *Let $M := \Omega_{\mathcal{G}}^{-n}(X)$ and $N := \Omega_{\mathcal{G}}^{-n}(Y)$. There exists a commutative diagram*

$$
\begin{array}{ccccccccc}
 & & & & (0) & & (0) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & V & = \!\!=\!\!= & V & & \\
 & & & & \downarrow & & \downarrow & & \\
(0) & \longrightarrow & W & \longrightarrow & \Omega_{\mathcal{G}}^{1-n}(k) & \xrightarrow{\ g\ } & N & \longrightarrow & (0) \\
 & & \| & & {\scriptstyle -f}\downarrow & & \downarrow{\scriptstyle \psi} & & \\
(0) & \longrightarrow & W & \longrightarrow & M & \xrightarrow{\ \varphi\ } & k & \longrightarrow & (0) \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (0) & & (0) & &
\end{array}
$$

with exact rows and colums.

(2) *Let $\alpha_K \in \Pi\mathrm{t}(\mathcal{G})$ be a π-point. Then $\alpha_K^*(\varphi_K)$ is split surjective or $\alpha_K^*(\psi_K)$ is split surjective.*

Proof. We shall only consider the case where n is even. The arguments for odd n are analogous.

(1) Since n is even, we have $\Omega_{\mathcal{G}}^{-n}(L_\zeta) \cong M \oplus N$, with both constituents being indecomposable endo-trivial modules of constant stable Jordan types $[1]$ and $[p-1]$, respectively. Lemma 6.2.1 furnishes an exact sequence

$$(0) \to \Omega_{\mathcal{G}}^{1-n}(k) \xrightarrow{\binom{f}{g}} M \oplus N \xrightarrow{(\varphi,\psi)} k \to (0).$$

With $V := \ker f$ and $W := \ker g$, our claim is a consequence of [3, (I.5.6)], once we know that $\varphi, \psi \neq 0$.

If $\varphi = 0$, then

$$\Omega_{\mathcal{G}}^{1-n}(k) = \ker(0, \psi) = M \oplus \ker \psi.$$

As $\Omega_{\mathcal{G}}^{1-n}(k)$ is indecomposable, this implies $\ker \psi = (0)$, so $N \hookrightarrow k$, a contradiction.

If $\psi = 0$, then

$$\Omega_{\mathcal{G}}^{1-n}(k) = \ker(\varphi, 0) = N \oplus \ker \varphi.$$

Hence $\Omega_{\mathcal{G}}^{1-n}(k) \cong N$ and $M \cong k$, so that $\Omega_{\mathcal{G}}^{-n}(L_\zeta) \cong \Omega_{\mathcal{G}}^{1-n}(k) \oplus k$. As a result, $L_\zeta \cong \Omega_{\mathcal{G}}(k) \oplus \Omega_{\mathcal{G}}^n(k)$, which contradicts $L_\zeta \subseteq \Omega_{\mathcal{G}}^n(k)$.

(2) We apply α_K^* to the exact sequence

$$(0) \to \Omega_{\mathcal{G}_K}^{1-n}(K) \to \Omega_{\mathcal{G}_K}^{-n}((L_\zeta)_K) \xrightarrow{(\varphi_K,\psi_K)} K \to (0)$$

and obtain an exact sequence

$$(0) \to [p-1] \oplus Q \xrightarrow{\binom{c}{d}} ([1] \oplus [p-1]) \oplus P \xrightarrow{(a,b)} [1] \to (0), \qquad (*)$$

with projective $\mathfrak{A}_{p,K}$-modules P, Q satisfying $\dim_K P = \dim_K Q$. Since $\mathfrak{A}_{p,K}$ is local, this implies $P \cong Q$. As a result, the middle term of $(*)$ is isomorphic to the direct sum of the extreme terms, so that $(*)$ is split exact. There thus exist $\gamma : K \to \alpha_K^*(M_K)$ and $\eta : K \to \alpha_K^*(N_K)$ such that

$$\mathrm{id}_K = (\alpha_K^*(\varphi_K), \alpha_K^*(\psi_K)) \circ \binom{\gamma}{\eta} = \alpha_K^*(\varphi_K) \circ \gamma + \alpha_K^*(\psi_K) \circ \eta.$$

This readily implies our claim. $\square$

6.3. Carlson modules of nilpotent elements of even degree

In contrast to the results of Section 6.1, the structure of the Carlson modules associated to nilpotent elements of even degree does not depend on the internal structure of the underlying group scheme $\mathcal{G}$.

Theorem 6.3.1. *Suppose that $p \geq 3$. Let $\zeta \in \mathrm{H}^{2n}(\mathcal{G}, k) \setminus \{0\}$ be nilpotent. Then L_ζ is indecomposable.*

Proof. Recall from Section 5 that $L_\zeta \neq (0)$. Assume that L_ζ is decomposable. According to Lemma 5.1, the Carlson module $L_\zeta = X \oplus Y$ is the direct sum of two endo-trivial modules of constant stable Jordan types [1] and [$p - 1$], respectively. We consider the associated commutative diagram of Lemma 6.2.2 and note that the stable Jordan types of the modules M and N coincide with those of X and Y, respectively.

Let $\alpha_K \in \Pi t(\mathcal{G})$ be a π-point. If $\alpha_K^*(\psi_K)$ is split surjective, then

$$\alpha_K^*(N_K) \cong \alpha_K^*(V_K) \oplus [1],$$

whence $[p - 1] \oplus (\text{proj.}) \cong \alpha_K^*(V_K) \oplus [1]$. Since $p \geq 3$, the trivial module [1] is not a direct summand of $[p - 1] \oplus (\text{proj.})$, and we have reached a contradiction. It now follows from Lemma 6.2.2 that $\alpha_K^*(\varphi_K)$ is split surjective. Thus,

$$[1] \oplus (\text{proj.}) \cong \alpha_K^*(W_K) \oplus [1],$$

so that $\alpha_K^*(W_K)$ is projective.

As a result, $\Pi(\mathcal{G})_W = \emptyset$, and [36, (5.3)] ensures that W is a projective $\mathcal{G}$-module. The upper row of our diagram thus splits and

$$\Omega_\mathcal{G}^{1-2n}(k) \cong W \oplus N.$$

As the Heller shift is projective-free, we see that $W = (0)$, whence $N \cong \Omega_\mathcal{G}^{1-2n}(k)$ and $M \cong k$. Consequently, $\Omega_\mathcal{G}^{-2n}(L_\zeta) \cong M \oplus N \cong k \oplus \Omega_\mathcal{G}^{1-2n}(k)$, whence $L_\zeta \cong \Omega_\mathcal{G}^{2n}(k) \oplus \Omega_\mathcal{G}(k)$, a contradiction. □

Corollary 6.3.2. *Suppose that $p \geq 3$ and let $\Theta_\zeta \subseteq \Gamma_s(\mathcal{G})$ be the stable Auslander–Reiten component containing an indecomposable Carlson module L_ζ of the nilpotent element $\zeta \in \mathrm{H}^{2n}(\mathcal{G}, k) \setminus \{0\}$.*

(1) *If $\dim \Pi(\mathcal{G}) \geq 2$, then $\Theta_\zeta \cong \mathbb{Z}[A_\infty]$, L_ζ is quasi-simple and every $N \in \Theta_\zeta$ has constant Jordan type $\mathrm{Jt}(N) = \{\mathrm{q}\ell(N)[1] \oplus \mathrm{q}\ell(N)[p - 1] \oplus n_N[p]\}$.*
(2) *If $\mathcal{G}$ is infinitesimal, or trigonalizable and representation-infinite, then Θ_ζ is locally split.*

Proof. (1) According to Lemma 5.1, the Carlson module L_ζ has constant Jordan type $\mathrm{Jt}(L_\zeta) = \{[1] \oplus [p - 1] \oplus n_\zeta[p]\}$. Thus, $\Pi(\mathcal{G})_{\Theta_\zeta} = \Pi(\mathcal{G})$, and [24, (3.3)] implies $\Theta_\zeta \cong \mathbb{Z}[A_\infty]$. Our assertion now follows from Corollary 3.1.2.

(2) If Θ_ζ is not locally split, then $\mathrm{cx}_\mathcal{G}(k) = \dim \Pi(\mathcal{G}) + 1 = 1$. If $\mathcal{G}$ is infinitesimal, then a consecutive application of [28, (2.1)] and [28, (2.4)] shows that all simple $\mathcal{G}$-modules belonging to the principal block $\mathcal{B}_0(\mathcal{G})$ of $k\mathcal{G}$ are one-dimensional. By virtue of [28, (2.7)], $\mathcal{B}_0(\mathcal{G})$ is a Nakayama algebra, and [3, (IV.2.10)] now yields $\dim_k \Omega_\mathcal{G}^{2n}(k) = 1$. Thus, $L_\zeta = (0)$, a contradiction.

In case $\mathcal{G}$ is trigonalizable and of infinite representation type, we consider a Π-point α_K such that Θ_ζ is not α_K-split. A consecutive application of Corollary 4.3.5 and Lemma 5.1 shows that the $\mathcal{G}$-module L_ζ is not α_K-projective. Theorem 4.3.3 now implies $\mathrm{q}\ell(L_\zeta) \geq 2$, and the arguments in the proof of Corollary 5.4 yield a contradiction. □

6.4. Carlson modules of odd degree

In order to address the case where L_ζ is defined by an element $\zeta \in H^*(\mathcal{G}, k)$ of odd degree, we recall that the map $\alpha_K : \mathfrak{A}_{p,K} \to K\mathcal{G}$ induces a homomorphism $\alpha_K^* : H^*(\mathcal{G}_K, K) \to H^*(\mathfrak{A}_{p,K}, K)$. The latter algebra is known: for $p > 2$ we have $H^*(\mathfrak{A}_{p,K}, K) \cong K[X, Y]/(Y^2)$ with $\deg(X) = 2$ and $\deg(Y) = 1$.

Given a field extension $K : k$, we shall identify $H^*(\mathcal{G}, k)$ with the subalgebra $H^*(\mathcal{G}, k) \otimes 1 \subseteq H^*(\mathcal{G}, k) \otimes_k K \cong H^*(\mathcal{G}_K, K)$. Thus, we can consider $\alpha_K^*(\zeta) \in H^*(\mathfrak{A}_{p,K}, K)$ for every $\zeta \in H^*(\mathcal{G}, k)$ and $\alpha_K \in \Pi t(\mathcal{G})$.

Throughout this section, we assume that $p \geq 3$.

Lemma 6.4.1. *Let n be odd and $\zeta \in H^n(\mathcal{G}, k) \setminus \{0\}$.*

(1) $\mathrm{Jt}(L_\zeta, \alpha_K) = \begin{cases} 2[p-1] \oplus m_\zeta[p] \text{ if } \alpha_K^*(\zeta) = 0, \\ [p-2] \oplus n_\zeta[p] \quad \text{if } \alpha_K^*(\zeta) \neq 0. \end{cases}$

(2) *If $|\mathrm{Jt}(L_\zeta)| = 2$, then L_ζ is indecomposable.*

(3) *If L_ζ has constant Jordan type, then L_ζ is either indecomposable or the direct sum of two endo-trivial modules of constant stable Jordan type $[p-1]$.*

Proof. (1) Recall that $\Pi(\mathcal{G})_{L_\zeta} = \Pi(\mathcal{G})$ and let $\alpha_K \in \Pi t(\mathcal{G})$ be a π-point. Since n is odd, we obtain an exact sequence

$$(0) \to \alpha_K^*((L_\zeta)_K) \xrightarrow{\binom{\varphi}{-\gamma}} [p-1] \oplus m[p] \xrightarrow{(f,g)} K \to (0),$$

where $\alpha_K^*(\zeta) \in H^n(\mathfrak{A}_{p,K}, K)$ corresponds to f. If $f = 0$, then $g \neq 0$ and

$$\alpha_K^*((L_\zeta)_K) \cong [p-1] \oplus \ker g \cong 2[p-1] \oplus m'[p].$$

Alternatively, [3, (I.5.6), (I.5.7)] provides a commutative diagram

$$
\begin{array}{ccccccccc}
(0) & \longrightarrow & \ker \gamma & \longrightarrow & \alpha_K^*((L_\zeta)_K) & \xrightarrow{\gamma} & m[p] & \longrightarrow & (0) \\
 & & \| & & \varphi \downarrow & & g \downarrow & & \\
(0) & \longrightarrow & \ker f & \longrightarrow & [p-1] & \xrightarrow{f} & K & \longrightarrow & (0)
\end{array}
$$

with exact rows. Consequently,

$$\alpha_K^*((L_\zeta)_K) \cong \ker \gamma \oplus m[p] \cong \ker f \oplus m[p] \cong [p-2] \oplus m[p],$$

as desired.

(2) Suppose that $|\mathrm{Jt}(L_\zeta)| = 2$, so that

$$\mathrm{Jt}(L_\zeta) = \{2[p-1] \oplus m_\zeta[p], [p-2] \oplus n_\zeta[p]\}.$$

Let M be a direct summand of L_ζ, so that $\dim_k M \equiv -1, -2, 0 \mod(p)$.

If $\dim_k M \equiv 0 \mod(p)$, then, for any π-point α_K, p divides the dimension of the projective-free part X of $\alpha_K^*(M)$. As $X \mid 2[p-1]$ or $X \mid [p-2]$, it follows that $X = (0)$.

Consequently, $\alpha_K^*(M_K)$ is projective and, thanks to [36, (5.3)], M is also projective, so that $M = (0)$.

If $\dim_k M \equiv -1 \mod(p)$, then $\alpha_K^*(M_K) \cong [p-1] \oplus n_M[p]$ for any $\alpha_K \in \Pi t(\mathcal{G})$. Since there exists a π-point α_K with $\alpha_K^*((L_\zeta)_K) \cong [p-2] \oplus n'_\zeta[p]$, we have reached a contradiction.

Finally, if $\dim_k M \equiv -2 \mod(p)$, then the dimension of a direct complement N of M is divisible by p. By the first step, $N = (0)$, so that $M = L_\zeta$. As a result, L_ζ is indecomposable.

(3) Owing to [10, (3.7)], every direct summand of L_ζ has constant Jordan type. Thus, if L_ζ is decomposable, then $\mathrm{Jt}(L_\zeta) = \{2[p-1] \oplus n_\zeta[p]\}$ and each indecomposable summand $M \neq L_\zeta$ of L_ζ has dimension $\dim_k M \equiv -1 \mod(p)$. Consequently, such a summand has Jordan type $\mathrm{Jt}(M) = \{[p-1] \oplus n_M[p]\}$, and there are exactly two such summands. By virtue of [10, (5.6)], each of these summands is endo-trivial. $\qquad \square$

Remarks. (1) Let n be odd and $\zeta \in \mathrm{H}^n(\mathcal{G}, k) \setminus \{0\}$. If $p = 2$, then $\mathrm{H}^*(\mathfrak{A}_{p,K}, K) \cong K[X]$, so that $\alpha_K^*(\zeta) = 0$ for every $\alpha_K \in \Pi t(\mathcal{G})$. Consequently, we have

$$\mathrm{Jt}(L_\zeta) = \{2[1] \oplus m_\zeta[2]\}$$

in that case.

(2) If $n = 1$, then $L_\zeta \subseteq \Omega_\mathcal{G}(k)$ is a submodule of the projective cover of k and hence is either zero or indecomposable.

(3) Let $\mathcal{U}$ be an abelian unipotent group scheme, and $K \supseteq k$ be an extension field of k. General theory [4, §3.5], [44, (I.4.27)] provides an isomorphism

$$\mathrm{H}^*(\mathcal{U}_K, K) \cong S(\mathrm{H}^2(\mathcal{U}_K, K)) \otimes_k \Lambda(\mathrm{H}^1(\mathcal{U}_K, K))$$

of graded-commutative K-algebras, where $S(-)$ and $\Lambda(-)$ denote the symmetric algebra and the exterior algebra of a space, whose elements are homogeneous of the given cohomological degree. Let $\alpha_K \in \Pi t(\mathcal{U})$ be a π-point. Writing $\mathrm{H}^2(\mathfrak{A}_{p,K}, K) = KX$ and $\mathrm{H}^1(\mathfrak{A}_{p,K}, K) = KY$, we see that the graded homomorphism $\alpha_K^* : \mathrm{H}^*(\mathcal{U}_K, K) \to \mathrm{H}^*(\mathfrak{A}_{p,K}, K)$ annihilates $\Lambda^j(\mathrm{H}^1(\mathcal{U}_K, K))$ for $j \geq 2$. Consequently,

$$\alpha_K^*(\mathrm{H}^{2n+1}(\mathcal{U}_K, K)) = \alpha_K^*(S^n(\mathrm{H}^2(\mathcal{U}_K, K)))\alpha_K^*(\mathrm{H}^1(\mathcal{U}_K, K))$$

for every $n \geq 0$.

Examples. (1) Let $\mathcal{U} = \mathbb{G}_{a(1)}^3$ be the product of three copies of the first Frobenius kernel of the additive group. Then we have $\dim_k \mathrm{H}^1(\mathcal{U}, k) = 3$. Let $\alpha_i : \mathfrak{A}_{p,k} \to \mathcal{U}$ be the π-points given by the three canonical embeddings $\mathbb{G}_{a(1)} \hookrightarrow \mathcal{U}$. Recall that

$$\mathrm{H}^*(\mathcal{U}, k) = \mathrm{H}^*(\mathbb{G}_{a(1)}, k)^{\otimes 3}.$$

Let $\zeta_1 = \zeta \otimes 1 \otimes 1 \in \mathrm{H}^1(\mathcal{U}, k)$ be given by $\zeta \in \mathrm{H}^1(\mathbb{G}_{a(1)}, k) \setminus \{0\}$. Then we have $\alpha_j^*(\zeta_1) = \delta_{j,1}\zeta$. Accordingly, $|\mathrm{Jt}(L_{\zeta_1})| = 2$ and L_{ζ_1} is indecomposable.

(2) Let $\mathcal{U}$ be as in (1), and consider $\zeta_2 \in \Lambda^3(\mathrm{H}^1(\mathcal{U}, k)) \setminus \{0\}$. The foregoing remarks imply

$$\alpha_K^*(\zeta_2) = 0 \quad \forall \alpha_K \in \Pi\mathrm{t}(\mathcal{U}),$$

so that L_{ζ_2} has constant Jordan type $2[p-1] \oplus n_\zeta[p]$. We shall see in Teorem 6.4.4 below that L_{ζ_2} is indecomposable.

Let $\zeta \in \mathrm{H}^n(\mathcal{G}, k) \setminus \{0\}$ be an element of odd degree such that L_ζ is decomposable. In view of Lemma 6.4.1,

$$L_\zeta \cong X \oplus Y$$

is the direct sum of two endo-trivial modules of constant stable Jordan type $[p-1]$. We may therefore consider the commutative diagram of Lemma 6.2.2.

Lemma 6.4.2. *Let $\zeta \in \mathrm{H}^n(\mathcal{G}, k) \setminus \{0\}$ be an element of odd degree $n \neq 1$ such that L_ζ is decomposable. Set $U := \ker(\psi \circ g)$.*

(1) $\Pi(\mathcal{G})_V \cap \Pi(\mathcal{G})_W = \emptyset$.

(2) *There is a split exact sequence* $(0) \to W \to U \xrightarrow{g} V \to (0)$.

(3) *There is a split exact sequence* $(0) \to V \to U \xrightarrow{f} W \to (0)$.

(4) $\Pi(\mathcal{G})_U = \Pi(\mathcal{G})_V \sqcup \Pi(\mathcal{G})_W$.

(5) *The map* $\psi \circ g : \Omega_{\mathcal{G}}^{1-n}(k) \to k$ *corresponds to a non-split extension*

$$(0) \to k \to \Omega_{\mathcal{G}}^{-1}(U) \to \Omega_{\mathcal{G}}^{-n}(k) \to (0).$$

Proof. (1) Let $\alpha_K \in \Pi\mathrm{t}(\mathcal{G})$ be a π-point such that $[\alpha_K] \in \Pi(\mathcal{G})_V \cap \Pi(\mathcal{G})_W$. Note that $M = \Omega_{\mathcal{G}}^{-1}(X)$ and $N = \Omega_{\mathcal{G}}^{-1}(Y)$ are $\mathcal{G}$-modules of constant stable Jordan type $[1]$. By Lemma 6.2.2(2), one of the maps $\alpha^*(\varphi_K)$ or $\alpha_K(\psi_K)$ is split surjective. By symmetry, we may assume that $\alpha_K^*(\varphi_K)$ is split surjective. Consequently, the sequence

$$(0) \to \alpha_K^*(W_K) \to \alpha_K^*(M_K) \to \alpha_K^*(K) \to (0)$$

splits, so that $[1] \oplus (\text{proj.}) \cong [1] \oplus \alpha_K^*(W_K)$. Hence $\alpha_K^*(W_K)$ is projective, a contradiction.

(2)&(3) We have $U = \ker(\psi \circ g) = g^{-1}(\ker \psi) = g^{-1}(V)$ as well as $U = \ker(\varphi \circ f) = f^{-1}(\ker \varphi) = f^{-1}(W)$. Hence the two sequences are exact. In view of (1), a consecutive application of [36, (3.2)] and [36, (3.3)] implies that the $\mathcal{G}$-module $V^* \otimes_k W$ is injective. Consequently, $\mathrm{Ext}_{\mathcal{G}}^1(V, W) \cong \mathrm{Ext}_{\mathcal{G}}^1(k, V^* \otimes_k W) = (0)$, so that the sequence in (2) is split exact. The corresponding result for (3) follows by interchanging the rôles of V and W.

(4) This follows from (2), (3) and [36, (3.3)].

(5) By general theory, the extension associated to $\psi \circ g$ is the lower row of the following diagram, whose left-hand square is a push-out:

$$
\begin{array}{ccccccccc}
(0) & \longrightarrow & \Omega_{\mathcal{G}}^{1-n}(k) & \longrightarrow & P & \longrightarrow & \Omega_{\mathcal{G}}^{-n}(k) & \longrightarrow & (0) \\
 & & \downarrow{\scriptstyle \psi \circ g} & & \downarrow{\scriptstyle \lambda} & & \| & & \\
(0) & \longrightarrow & k & \longrightarrow & E & \longrightarrow & \Omega_{\mathcal{G}}^{-n}(k) & \longrightarrow & (0)
\end{array}
$$

The Snake Lemma yields $U \cong \ker \lambda$ as well as $\operatorname{im} \lambda = E$. If $U = (0)$, then (2) and (3) yield $V = (0) = W$, so that Lemma 6.2.2 gives $N \cong k \cong M$. Consequently, $L_\zeta \cong \Omega_{\mathcal{G}}^n(k) \oplus \Omega_{\mathcal{G}}^n(k)$, a contradiction. Since $\Omega_{\mathcal{G}}^{1-n}(k)$ is indecomposable and $\dim_k U = \dim_k \Omega_{\mathcal{G}}^{1-n}(k) - 1$, it now follows that $\operatorname{Soc}_{\mathcal{G}}(U) = \operatorname{Soc}_{\mathcal{G}}(\Omega_{\mathcal{G}}^{1-n}(k)) = \operatorname{Soc}_{\mathcal{G}}(P)$. As a result, P is an injective hull of U, whence $E \cong \Omega_{\mathcal{G}}^{-1}(U)$. $\qquad\square$

Let $\mathcal{G}$ be a finite group scheme. We say that $\mathcal{G}$ is *linearly reductive* if the algebra $k\mathcal{G}$ is semi-simple. Following Voigt [55, (I.2.37)], we let $\mathcal{G}_{\mathrm{lr}} \trianglelefteq \mathcal{G}$ be the unique largest linearly reductive normal subgroup of $\mathcal{G}$.

We consider the set $\operatorname{Max}_{\mathrm{au}}(\mathcal{G})$ of maximal abelian unipotent subgroups of $\mathcal{G}$, as well as its subsets

$$\operatorname{Max}_{\mathrm{au}}(\mathcal{G})_\ell := \{\mathcal{U} \in \operatorname{Max}_{\mathrm{au}}(\mathcal{G}); \ \operatorname{cx}_{\mathcal{U}}(k) \geq \ell\}$$

for every $\ell \geq 1$.

If $\mathcal{H} \subseteq \mathcal{G}$ is a subgroup, then the canonical inclusion $\iota : \mathcal{H} \hookrightarrow \mathcal{G}$ induces a continuous map $\iota_{*,\mathcal{H}} : \Pi(\mathcal{H}) \to \Pi(\mathcal{G})$. Setting $\Pi(\mathcal{G})_\ell := \bigcup_{\mathcal{U} \in \operatorname{Max}_{\mathrm{au}}(\mathcal{G})_\ell} \iota_{*,\mathcal{U}}(\Pi(\mathcal{U}))$, we define the *saturation rank* of $\mathcal{G}$ via

$$\operatorname{srk}(\mathcal{G}) := \max\{\ell \geq 1; \ \Pi(\mathcal{G}) = \Pi(\mathcal{G})_\ell\}.$$

Note that $\operatorname{srk}(\mathcal{G}) \leq \operatorname{cx}_{\mathcal{G}}(k)$. If G is a finite group, then Quillen's dimension theorem [1] implies that $\operatorname{srk}(G) - 1$ is the minimum of the dimensions of the irreducible components of $\Pi(G)$. This does not hold for infinitesimal group schemes, as the example of the group $SL(2)_1$ shows.

The following lemma is a direct consequence of the proof of [10, (6.3)]:

Lemma 6.4.3. *Suppose that $n < 0$ and let*

$$\mathfrak{E} : \quad (0) \to k \to E \to \Omega_{\mathcal{G}}^{n-1}(k) \to (0)$$

be a short exact sequence of $\mathcal{G}$-modules. If $\alpha_K \in \Pi\mathrm{t}(\mathcal{G})$ factors through an abelian unipotent subgroup $\mathcal{U}$ of complexity ≥ 2, then the sequence $\alpha_K^*(\mathfrak{E} \otimes_k K)$ splits. $\qquad\square$

Theorem 6.4.4. *Suppose that $\operatorname{srk}(\mathcal{G}/\mathcal{G}_{\mathrm{lr}}) \geq 2$. If $\zeta \in \mathrm{H}^n(\mathcal{G}, k) \setminus \{0\}$ has odd degree, then L_ζ is indecomposable.*

Proof. According to [23, (1.1)], the canonical map $\pi : k\mathcal{G} \to k(\mathcal{G}/\mathcal{G}_{\mathrm{lr}})$ induces an isomorphism $\pi : \mathcal{B}_0(\mathcal{G}) \xrightarrow{\sim} \mathcal{B}_0(\mathcal{G}/\mathcal{G}_{\mathrm{lr}})$ between the corresponding principal blocks. Thus, $\pi^* : \operatorname{mod} \mathcal{G}/\mathcal{G}_{\mathrm{lr}} \to \operatorname{mod} \mathcal{G}$ induces an equivalence $\operatorname{mod} \mathcal{B}_0(\mathcal{G}/\mathcal{G}_{\mathrm{lr}}) \xrightarrow{\sim} \operatorname{mod} \mathcal{B}_0(\mathcal{G})$. The isomorphism $\pi^* : \mathrm{H}^*(\mathcal{G}/\mathcal{G}_{\mathrm{lr}}, k) \to \mathrm{H}^*(\mathcal{G}, k)$ yields

$$\pi^*(L_\zeta) \cong L_{\pi^*(\zeta)} \quad \forall \zeta \in \mathrm{H}^*(\mathcal{G}/\mathcal{G}_{\mathrm{lr}}, k).$$

Since Carlson modules belong to the principal block, it suffices to verify our assertion under the assumption that $\operatorname{srk}(\mathcal{G}) \geq 2$.

Assume that L_ζ is decomposable, so that $n \geq 3$. Lemma 6.4.2(5) provides an exact sequence

$$(0) \to k \to \Omega_{\mathcal{G}}^{-1}(U) \to \Omega_{\mathcal{G}}^{-n}(k) \to (0).$$

Let x be a point of $\Pi(\mathcal{G})$. Since $\mathrm{srk}(\mathcal{G}) \geq 2$, there exists a π-point α_K representing x which factors through an abelian unipotent subgroup $\mathcal{U} \subseteq \mathcal{G}$ with $\mathrm{cx}_{\mathcal{U}}(k) \geq 2$. Since $1 - n \leq -2$, Lemma 6.4.3 guarantees that the sequence

$$(0) \to \alpha_K^*(K) \to \alpha_K^*(\Omega_{\mathcal{G}_K}^{-1}(U)_K) \to \alpha_K^*(\Omega_{\mathcal{G}_K}^{-n}(K)) \to (0)$$

splits. Consequently,

$$\Omega_{\mathfrak{A}_{p,K}}^{-1}(\alpha_K^*(U_K)) \oplus (\mathrm{proj.}) \cong [1] \oplus [p-1] \oplus (\mathrm{proj.}),$$

so that $\Omega_{\mathfrak{A}_{p,K}}^{-1}(\alpha_K^*(U_K)) \neq (0)$. Hence the $\mathfrak{A}_{p,K}$-module $\alpha_K^*(U_K)$ is not projective, and we conclude that $x = [\alpha_K] \in \Pi(\mathcal{G})_U$. Thus, $\Pi(\mathcal{G})_U = \Pi(\mathcal{G})$, and Lemma 6.4.2(4) implies

$$\Pi(\mathcal{G}) = \Pi(\mathcal{G})_V \sqcup \Pi(\mathcal{G})_W.$$

In view of [10, (3.4)] (see also [7]) and [36, (5.5)]), it follows that one of the modules V or W is projective. As both of these spaces are submodules of $\Omega_{\mathcal{G}}^{1-n}(k)$, we obtain $V = (0)$ or $W = (0)$. In either case, Lemma 6.2.2 implies that $\Omega_{\mathcal{G}}^{-n}(L_\zeta) \cong \Omega_{\mathcal{G}}^{1-n}(k) \oplus k$, whence $L_\zeta \cong \Omega_{\mathcal{G}}(k) \oplus \Omega_{\mathcal{G}}^n(k)$, which contradicts $L_\zeta \subseteq \Omega_{\mathcal{G}}^n(k)$. $\qquad\qquad\square$

Examples. (1) Let $\mathfrak{h} := kx \oplus ky \oplus kz$ be the Heisenberg algebra with p-map defined by

$$x^{[p]} = 0 = y^{[p]}, \qquad z^{[p]} = z.$$

If $\mathcal{G}$ is the infinitesimal group corresponding to $\mathfrak{h}$, then $\mathcal{G}_{\mathrm{lr}}$ corresponds to the p-ideal kz. Hence $\mathrm{srk}(\mathcal{G}/\mathcal{G}_{\mathrm{lr}}) = 2$, while $\mathrm{srk}(\mathcal{G}) = 1$.

(2) Now consider the p-map on $\mathfrak{h}$ that is defined via

$$x^{[p]} = z = y^{[p]}, \qquad z^{[p]} = 0.$$

Since $\mathcal{V}_{\mathfrak{h}} = k(x - y) \oplus kz$ is an abelian unipotent p-subalgebra of complexity 2, general theory (cf. [24, p. 68f]) implies that every p-point of $U_0(\mathfrak{h})$ is equivalent to one factoring through $U_0(\mathcal{V}_{\mathfrak{h}})$. Accordingly, $\mathrm{srk}(\mathfrak{h}) = 2$, and every Carlson module L_ζ of odd degree is indecomposable.

Corollary 6.4.5. *Suppose that $\zeta \in \mathrm{H}^n(\mathcal{G}, k) \setminus \{0\}$ has odd degree. If there exists an abelian unipotent subgroup $\mathcal{U} \subseteq \mathcal{G}$ such that $\mathrm{res}_{\mathcal{U}}(\zeta) \neq 0$, then the Carlson module L_ζ is indecomposable.*

Proof. Let $\eta \in \mathrm{H}^n(\mathcal{U}, k) \setminus \{0\}$ be such that $\eta = \mathrm{res}_{\mathcal{U}}(\zeta)$. General theory (cf. [5, p. 190]) provides a projective $\mathcal{U}$-module P such that

$$L_\zeta|_{\mathcal{U}} \cong L_\eta \oplus P.$$

If L_ζ is decomposable, then Lemma 6.4.1 shows that $L_\zeta \cong M \oplus N$, with M and N having constant stable Jordan type $[p - 1]$. Thus, $M|_{\mathcal{U}}$ and $N|_{\mathcal{U}}$ are non-projective $\mathcal{U}$-modules whose projective-free parts are summands of L_η. Consequently, L_η is decomposable, and Theorem 6.4.4 implies that $\dim \Pi(\mathcal{U}) = 0$.

In view of [56, (14.4)], there exist $r_1, \ldots, r_n \in \mathbb{N}$ such that $k\mathcal{U}$ is isomorphic to $k[X_1, \ldots, X_n]/(X_1^{p^{r_1}}, \ldots, X_n^{p^{r_n}})$. Consequently, $n = \dim \mathcal{V}_{\mathcal{U}}(k) = \dim \Pi(\mathcal{U}) + 1 = 1$, so that $k\mathcal{U}$ is a Nakayama algebra. As noted earlier, the Auslander–Reiten translation $\tau_{\mathcal{U}}$ of the local algebra $k\mathcal{U}$ coincides with the square of the Heller translate $\Omega_{\mathcal{U}}$. Thus, [3,

(IV.2.10)] yields $\Omega_{\mathcal{U}}^n(k) \cong \Omega_{\mathcal{U}}(k)$, implying that $\mathrm{Soc}_{\mathcal{U}}(\Omega_{\mathcal{U}}^n(k)) \cong k$ is simple. Hence $L_\eta \subseteq \Omega_{\mathcal{U}}^n(k)$ is indecomposable, a contradiction. $\qquad\square$

We finally address the case of finite groups:

Corollary 6.4.6. *Let G be a finite group. If $\zeta \in \mathrm{H}^n(G, k) \setminus \{0\}$ has odd degree, then L_ζ is indecomposable.*

Proof. In view of Theorem 6.4.4 and its proof, we may assume that $\mathrm{srk}(G) = 1$.

(∗) *We have $\mathrm{cx}_G(k) = 1$.*

Thanks to [36, (4.2)], we have

$$\Pi(G) = \bigcup_E \iota_{*,E}(\Pi(E)),$$

where E runs through the maximal p-elementary abelian subgroups of G. Since $\mathrm{srk}(G) = 1$, there exists a maximal p-elementary abelian subgroup $E_0 \subseteq G$ such that $\mathrm{cx}_{E_0}(k) = 1$. Consequently, $E_0 \cong \mathbb{Z}/(p)$.

Let $P \subseteq G$ be a Sylow p-subgroup of G containing E_0 and let $C(P)_p := \{x \in C(P);\ x^p = 1\}$ be the subgroup of those elements of the center $C(P)$ of P whose order is a divisor of p. Then $C(P)_p$ is a non-trivial normal subgroup of P and, given a maximal p-elementary abelian subgroup $F \subseteq P$, the group $C(P)_pF$ is p-elementary abelian, whence

$$\{e\} \subsetneq C(P)_p \subseteq F.$$

In particular, $C(P)_p = E_0$, so that $F = E_0$.

Now let E be a maximal p-elementary abelian subgroup of G. Sylow's theorem provides an element $g \in G$ such that $gEg^{-1} \subseteq P$. By the above, we conclude that $gEg^{-1} = E_0$, whence $\mathrm{rk}(E) = 1$. Quillen's dimension theorem (cf. [1]) now implies $\mathrm{cx}_G(k) = 1$, as desired. $\qquad\diamond$

Let $P \subseteq G$ be a Sylow p-subgroup. Owing to [21, (4.2.2)], the canonical restriction map

$$\mathrm{res} : \mathrm{H}^*(G, k) \to \mathrm{H}^*(P, k)$$

is injective, so that $\eta := \mathrm{res}(\zeta) \neq 0$. By (∗), we have $\mathrm{cx}_G(k) = 1$, and [13, (XII.11.6)] shows that P is cyclic. The assertion now follows from Corollary 6.4.5. $\qquad\square$

Remark. Suppose that $\mathcal{G}$ is a finite group scheme. Let $\zeta \in \mathrm{H}^n(\mathcal{G}, k) \setminus \{0\}$ be an element of arbitrary positive degree such that L_ζ is quasi-simple. Then L_ζ has exactly one successor in $\Gamma_s(\mathcal{G})$ and the middle term of the almost split sequence originating in L_ζ either is indecomposable or has a non-zero projective summand. In the latter case, [3, (V.5.5)] provides a principal indecomposable $\mathcal{G}$-module P such that $L_\zeta \cong \mathrm{Rad}(P)$. Consequently, $\Omega_{\mathcal{G}}^{-1}(L_\zeta)$ is simple. On the other hand, there is a short exact sequence

$$(0) \to k \to \Omega_{\mathcal{G}}^{-1}(L_\zeta) \to \Omega_{\mathcal{G}}^{n-1}(k) \to (0)$$

(see [5, (5.9.4)]). Thus, the left-hand arrow is an isomorphism, so that $\Omega_{\mathcal{G}}^{n-1}(k) = (0)$, a contradiction. As a result, the almost split sequence originating in L_ζ has an indecomposable middle term.

7. Endo-trivial modules

Endo-trivial modules play an important rôle in the modular representation theory of finite groups, where they occur in connection with the study of sources of simple modules. The reader may consult [12] for the classification of endo-trivial modules over p-groups.

In view of [10, (5.6)], a $\mathcal{G}$-module M is endo-trivial if and only if there exists $i \in \{1, p-1\}$ such that M has constant stable Jordan type $\mathrm{StJt}(M) = \{[i]\}$. Consequently, the $\tau_{\mathcal{G}}$-orbit of an indecomposable endo-trivial module consists entirely of endo-trivial modules.

The following immediate consequence of recent work by Dave Benson [6] characterizes endo-trivial modules as being precisely the modules of constant Jordan type with one non-projective block:

Proposition 7.1. *Let $\mathcal{G}$ be a finite group scheme possessing an abelian unipotent subgroup $\mathcal{U} \subseteq \mathcal{G}$ of complexity $\mathrm{cx}_{\mathcal{U}}(k) \geq 2$. If $M \in \mathrm{mod}\,\mathcal{G}$ has constant Jordan type $[i] \oplus n[p]$, then $i \in \{1, p-1\}$. In particular, M is endo-trivial.*

Proof. Since every π-point of $\mathcal{U}$ is also a π-point of $\mathcal{G}$, it readily follows that the $\mathcal{U}$-module $M|_{\mathcal{U}}$ has constant Jordan type $[i] \oplus n[p]$. We may therefore assume that $\mathcal{G} = \mathcal{U}$. As noted in [24, (1.6)], this implies that $k\mathcal{G} = U_0(\mathfrak{u})$ is the restricted enveloping algebra of an abelian p-unipotent restricted Lie algebra $\mathfrak{u}$. By assumption, the nullcone

$$\mathcal{V}_{\mathfrak{u}} := \{x \in \mathfrak{u}; \ x^{[p]} = 0\}$$

is a p-subalgebra of dimension ≥ 2, and $M|_{\mathcal{V}_{\mathfrak{u}}}$ is a $\mathcal{V}_{\mathfrak{u}}$-module of constant Jordan type $[i] \oplus n[p]$. Writing $r := \dim \mathcal{V}_{\mathfrak{u}}$, we have an isomorphism $U_0(\mathcal{V}_{\mathfrak{u}}) \cong k(\mathbb{Z}/(p))^r$ of k-algebras, and [6, (1.1)] yields $i \in \{1, p-1\}$. In view of [10, (5.6)], this implies that M is endo-trivial. $\qquad\square$

Remarks. (1) If G is a finite group, then Quillen's dimension theorem implies that the condition of the proposition is equivalent to $\mathrm{cx}_G(k) \geq 2$.

(2) The infinitesimal group $\mathrm{SL}(2)_1$ has complexity 2, but also has indecomposable modules of stable constant Jordan types $[2], \ldots, [p-2]$ (see Proposition 8.1.1 below).

Theorem 7.2. *Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a component containing an indecomposable endo-trivial module M_0. Suppose that $\dim \Pi(\mathcal{G}) \geq 1$.*

(1) *A $\mathcal{G}$-module M in Θ is endo-trivial if and only if $f_{\Theta}(M) = 1$.*
(2) *If $\bar{T}_{\Theta} \cong \tilde{A}_{12}, A_{\infty}^{\infty}$, then every $\mathcal{G}$-module $M \in \Theta$ is endo-trivial.*
(3) *If $\dim \Pi(\mathcal{G}) \geq 2$, then $\Theta \cong \mathbb{Z}[A_{\infty}]$.*
(4) *If $\Theta \cong \mathbb{Z}[A_{\infty}]$, then $\mathrm{q}\ell(M_0) = 1$ and $M \in \Theta$ is endo-trivial if and only if $M \cong \tau_{\mathcal{G}}^n(M_0)$ for some $n \in \mathbb{Z}$.*

Proof. (1) In view of [10, (5.6)], the module M_0 has constant Jordan type, so that

$$\dim \Pi(\mathcal{G})_{\Theta} = \dim \Pi(\mathcal{G}) \geq 1.$$

By the same token, there exist $i \in \{1, p-1\}$ and $m_0 \in \mathbb{N}_0$ such that M_0 has constant Jordan type

$$\mathrm{Jt}(M_0) = [i] \oplus m_0[p].$$

Since $\dim \Pi(\mathcal{G})_\Theta \geq 1$, Lemma 2.2 implies that the component Θ is locally split. Consequently, Theorem 3.1.1 yields $d_j^\Theta \equiv 0$ for $1 \leq j \neq i \leq p-1$ and $d_i^\Theta \equiv 1$, so that

$$\mathrm{Jt}(M, \alpha_K) = f_\Theta(M)[i] \oplus \alpha_{K,p}(M)[p] \quad \forall M \in \Theta, \ \alpha_K \in \Pi\mathrm{t}(\mathcal{G}).$$

Applying [10, (5.6)] again, we conclude that $M \in \Theta$ is endo-trivial exactly when $f_\Theta(M) = 1$.

(2) In this case, we have $f_\Theta \equiv 1$, so that (1) yields the assertion.

(3) In view of $\Pi(\mathcal{G}) = \Pi(\mathcal{G})_\Theta$, the assertion follows directly from [24, (3.3)].

(4) If $\Theta \cong \mathbb{Z}[A_\infty]$, then $f_\Theta(M) = \mathrm{q}\ell(M)$ for every $M \in \Theta$. Hence the endo-trivial modules are the quasi-simple modules. As these form the $\tau_\mathcal{G}$-orbit of M_0, our assertion follows. $\qquad\square$

Remark. Part (1) of the foregoing result implies that components of tree class $D_\infty, \tilde{D}_n$, $\tilde{E}_6, \tilde{E}_7$ and $\tilde{E}_8$ containing an endo-trivial module have 2, 4, 3, 2 and 1 $\tau_\mathcal{G}$-orbits of endo-trivial modules, respectively.

We turn to Carlson's construction [8, (4.5), (9.3), (9.4)] of endo-trivial modules, which we shall study from the vantage point of Π-supports and Jordan types.

Let $\zeta \in \mathrm{H}^{2n}(\mathcal{G}, k) \setminus \{0\}$ for some $n \geq 1$. Given a proper summand $M \mid L_\zeta$, we write $L_\zeta = M \oplus M'$ and consider the push-out diagram

$$
\begin{array}{ccc}
(0) & & (0) \\
\downarrow & & \downarrow \\
M' & =\!=\!= & M' \\
\downarrow & & \downarrow \\
(0) \longrightarrow L_\zeta \longrightarrow \Omega_\mathcal{G}^{2n}(k) \longrightarrow k \longrightarrow (0) \\
\downarrow \qquad\quad \downarrow \qquad\qquad \| \\
(0) \longrightarrow M \longrightarrow N_M \longrightarrow k \longrightarrow (0) \\
\downarrow \qquad\quad \downarrow \\
(0) \qquad\quad (0)
\end{array}
\qquad (7.1)
$$

The following application of Lemma 5.1 shows that decomposable Carlson modules often give rise to indecomposable endo-trivial modules that are not syzygies of the trivial module.

Theorem 7.3. *Suppose that $\zeta \in \mathrm{H}^\bullet(\mathcal{G}, k)$ is a non-nilpotent homogeneous element of positive degree. Let $M \mid L_\zeta$ be a proper summand.*

(1) *The $\mathcal{G}$-module N_M is endo-trivial, indecomposable, and of constant stable Jordan type* [1].
(2) *If $p \geq 3$ and $\dim \Pi(\mathcal{G}) \geq 2$, then $N_M \not\cong \Omega_{\mathcal{G}}^m(k)$ for all $m \in \mathbb{Z}$.*
(3) *If $p \geq 3$ and $\dim \Pi(\mathcal{G}) \geq 2$, then the module $N_M \in \mathbb{Z}[A_\infty]$ is quasi-simple and does not belong to the stable AR-components containing k or $\Omega_{\mathcal{G}}(k)$.*

Proof. (1) Let $\alpha_K \in \Pi\mathrm{t}(\mathcal{G})$ be a π-point. Application of the exact functor α_K^* to diagram (7.1) yields

$$
\begin{array}{ccc}
(0) & & (0) \\
\downarrow & & \downarrow \\
\alpha_K^*(M_K') & =\!=\!= & \alpha_K^*(M_K') \\
\downarrow & & \downarrow \\
(0) \longrightarrow \alpha_K^*((L_\zeta)_K) \longrightarrow K \oplus (\mathrm{proj.}) \longrightarrow K \longrightarrow (0) \\
\downarrow \qquad\qquad \downarrow \qquad\qquad \| \\
(0) \longrightarrow \alpha_K^*(M_K) \longrightarrow \alpha_K^*((N_M)_K) \longrightarrow K \longrightarrow (0) \\
\downarrow \qquad\qquad \downarrow \\
(0) \qquad\qquad (0)
\end{array}
$$

where we have used $\alpha_K^*(\Omega_{\mathcal{G}}^{2n}(k)_K) \cong \Omega_{\mathfrak{A}_{p,K}}^{2n}(K) \oplus (\mathrm{proj.}) \cong K \oplus (\mathrm{proj.})$. Lemma 5.1 ensures that at least one of the $\mathfrak{A}_{p,K}$-modules $\alpha_K^*(M_K)$ and $\alpha_K^*(M_K')$ is projective. If $\alpha_K^*(M_K)$ is projective, then the lower row is split exact and

$$\alpha_K^*((N_M)_K) \cong K \oplus (\mathrm{proj.}).$$

Alternatively, the right-hand column splits, so that

$$\alpha_K^*((N_M)_K) \oplus (\mathrm{proj.}) \cong K \oplus (\mathrm{proj.}).$$

We thus obtain $\mathrm{Jt}(N_M, \alpha_K) = [1] \oplus n[p]$ in either case, and [10, (5.6)] shows that N_M is endo-trivial.

Any indecomposable summand of N_M is either endo-trivial or projective, with exactly one summand being endo-trivial. The right-hand column of diagram (7.1) shows that the projective summands of N_M are also summands of $\Omega_{\mathcal{G}}^{2n}(k)$ and thus are equal to zero. Consequently, the module N_M is indecomposable.

(2) Since $\mathrm{Jt}(N_M) = [1] \oplus n[p]$ and $p \geq 3$, it follows that $N_M \not\cong \Omega_{\mathcal{G}}^m(k)$ whenever $m \in \mathbb{Z}$ is odd.

Assume that $N_M \cong \Omega_{\mathcal{G}}^{2m}(k)$ for some $m \in \mathbb{Z}$. As M and M' are both non-zero, we have $m \neq n, 0$, and the right-hand exact column of diagram (7.1) provides an exact sequence

$$(0) \to M' \to \Omega_{\mathcal{G}}^{2n}(k) \to \Omega_{\mathcal{G}}^{2m}(k) \to (0).$$

Lemma 5.1 in conjunction with [36, (3.2)] implies that $\emptyset = \Pi(\mathcal{G})_M \cap \Pi(\mathcal{G})_{M'} = \Pi(\mathcal{G})_{M \otimes_K M'}$, so that $M \otimes_k M'$ is projective (cf. [36, (5.3)]). Tensoring the above sequence with M thus yields an isomorphism

$$\Omega_{\mathcal{G}}^{2n}(M) \oplus (\text{proj.}) \cong \Omega_{\mathcal{G}}^{2m}(M) \oplus (\text{proj.}).$$

As M is projective-free, we obtain $\Omega_{\mathcal{G}}^{2(n-m)}(M) \cong M$. Accordingly, the $\mathcal{G}$-module M is periodic, and

$$\dim \Pi(\mathcal{G})_M = \dim \mathcal{V}_{\mathcal{G}}(M) - 1 = \operatorname{cx}_{\mathcal{G}}(M) - 1 = 0.$$

On the other hand, the lower exact row of diagram (7.1) now reads

$$(0) \to M \to \Omega_{\mathcal{G}}^{2m}(k) \to k \to (0),$$

and tensoring with M' yields

$$\Omega_{\mathcal{G}}^{2m}(M') \oplus (\text{proj.}) \cong M' \oplus (\text{proj.}).$$

Since M' has no projective summands, we conclude that M' is periodic with $\dim \Pi(\mathcal{G})_{M'} = 0$. It follows that

$$\dim \Pi(\mathcal{G}) \leq \dim \Pi(\mathcal{G})_{L_\xi} + 1 = \max\{\dim \Pi(\mathcal{G})_M, \dim \Pi(\mathcal{G})_{M'}\} + 1 \leq 1,$$

a contradiction.

(3) Since N_M is endo-trivial, we have $\Pi(\mathcal{G})_{N_M} = \Pi(\mathcal{G})$, and Theorem 7.2 ensures that $N_M \in \mathbb{Z}[A_\infty]$ is quasi-simple. By the same token, the assumption that N_M belongs to the component containing k or $\Omega_{\mathcal{G}}(k)$ implies that

$$N_M \cong \Omega_{\mathcal{G}}^m \circ \nu_{\mathcal{G}}^n(k)$$

for some $m, n \in \mathbb{Z}$. The arguments for (2) now show that this cannot happen. $\square$

Remarks. (1) Note that the module N_M belongs to the principal block $\mathcal{B}_0(\mathcal{G})$ of $k\mathcal{G}$. There are of course group schemes having endo-trivial modules that do not belong to the principal block: Any one-dimensional module is endo-trivial, so the simple modules of trigonalizable group schemes belong to this class. In view of [28, (2.4)], the algebras of measures of such groups often have more than one block.

(2) Consider the abelian restricted Lie algebra $\mathfrak{u} := kx \oplus ky$ with trivial p-map. If $f \in S(\mathfrak{u}^*)$ is a homogeneous polynomial function which is not a power of an irreducible polynomial function, then $\operatorname{Proj}(\mathcal{V}_{\mathfrak{u}}(L_\Phi(f))) = \operatorname{Proj}(Z(f))$ is not connected, so that $L_{\Phi(f)}$ is decomposable. By Dade's Theorem [15, 16] and [10, (5.6)], the endo-trivial modules of $U_0(\mathfrak{u}) \cong k(\mathbb{Z}/(p))^2$ are Heller shifts of the trivial module. Hence (2) of Theorem 7.3 may fail for groups $\mathcal{G}$ whose Π-support $\Pi(\mathcal{G})$ has dimension ≤ 1.

8. Examples

In their recent article [9], Carlson and Friedlander endow the category $\mathcal{C}(\mathcal{G})$ of modules of constant Jordan type with an exact structure and study realizability problems via $K_0(\mathcal{C}(\mathcal{G}))$. In this section we explicitly compute the Jordan types of the indecomposable modules for a few group schemes of tame representation type and in particular classify their indecomposable modules of constant Jordan type. We refer the reader to [20] for the definition of tameness. For our purposes, it suffices to know that only representation-finite and tame algebras admit (in principle) a classification of their indecomposables. As noted earlier, the stable Auslander–Reiten components belonging to blocks of finite representation type are not locally split. Hence the next more complicated class of tame algebras serves as testing ground for the invariants of AR-components defined via π-points.

   Throughout, we assume that $\mathrm{char}(k) = p \geq 3$.

8.1. The groups $\mathrm{SL}(2)_1 T_r$

Consider the infinitesimal group scheme $\mathcal{G} = \mathrm{SL}(2)_1$, that is, the first Frobenius kernel of $\mathrm{SL}(2)$. General theory provides an isomorphism $k\,\mathrm{SL}(2)_1 \cong U_0(\mathfrak{sl}(2))$ between the algebra of measures on $\mathrm{SL}(2)_1$ and the restricted enveloping algebra $U_0(\mathfrak{sl}(2))$ of the restricted Lie algebra $\mathfrak{sl}(2)$ (see [17, II, §7, no. 4]). In [49] Premet explicitly determined the indecomposable $U_0(\mathfrak{sl}(2))$-modules. We shall use the interpretation of his result within the framework of Auslander–Reiten theory (cf. [25, §4]).

   Recall that the algebra $k\,\mathrm{SL}(2)_1$ has $(p-1)/2$ non-simple blocks $\mathcal{B}_1, \ldots, \mathcal{B}_{(p-1)/2}$, with each $\mathcal{B}_i$ having two simple modules, of dimensions i and $p-i$, respectively. The stable Auslander–Reiten quiver $\Gamma_s(\mathcal{B}_i)$ is the full subquiver of $\Gamma_s(\mathcal{G})$, whose vertices belong to the block $\mathcal{B}_i$.

   Part (2) of the following result shows in particular that the stable AR-components $\Theta \subseteq \Gamma_s(\mathrm{SL}(2)_1)$ with $\dim \Pi(\mathrm{SL}(2)_1)_\Theta = 0$ are not locally split.

Proposition 8.1.1. *Let* $\Theta \subseteq \Gamma_s(\mathrm{SL}(2)_1)$ *be a component.*

(1) *If* $\dim \Pi(\mathcal{G})_\Theta = 1$, *then* $\Theta \cong \mathbb{Z}[\tilde{A}_{12}]$ *and every module belonging to* Θ *has constant Jordan type. Moreover, there exists* $s_\Theta \in \{1, \ldots, p-1\}$ *such that*

$$\mathrm{Jt}(M) = \left\{ [s_\Theta] \oplus \left(\frac{\dim_k M - s_\Theta}{p} \right)[p] \right\}$$

for every $M \in \Theta$.

(2) *If* $\dim \Pi(\mathcal{G})_\Theta = 0$, *then* $\Theta \cong \mathbb{Z}[A_\infty]/\langle \tau \rangle$ *and every module belonging to* Θ *is constantly supported. Moreover, there exists* $i_\Theta \in \{1, \ldots, (p-1)/2\}$ *such that* $\Theta \subseteq \Gamma_s(\mathcal{B}_{i_\Theta})$ *and*

$$\mathrm{Jt}(M) = \{\mathrm{q}\ell(M)[p], [i_\Theta] \oplus [p - i_\Theta] \oplus (\mathrm{q}\ell(M) - 1)[p]\}$$

for every $M \in \Theta$.

(3) *An indecomposable* $\mathrm{SL}(2)_1$-*module* M *is endo-trivial if and only if* $M \cong \Omega_{\mathcal{G}}^n(S)$ *for* $n \in \mathbb{Z}$ *and* S *simple of dimension* 1 *or* $p-1$.

Proof. By work of Drozd [18], Fischer [30] and Rudakov [51], each of the $(p-1)/2$ non-simple blocks of $k\,\mathrm{SL}(2)_1$ is Morita equivalent to the trivial extension $\mathrm{Kr}\ltimes\mathrm{Kr}^*$ of the path algebra $\mathrm{Kr}=k[\bullet\rightrightarrows\bullet]$ of the Kronecker quiver. As a result, the Auslander–Reiten quiver $\Gamma_s(\mathcal{B})$ of each such block $\mathcal{B}\subseteq k\,\mathrm{SL}(2)_1$ has two components of type $\mathbb{Z}[\tilde{A}_{12}]$, and infinitely many components of type $\mathbb{Z}[A_\infty]/\langle\tau\rangle$ (see [38, (V.3.2), (I.5.5), (I.5.6)]).

(1) Since $\dim\Pi(\mathrm{SL}(2)_1)_\Theta=1$, the above in conjunction with [24, (3.3)] implies $\Theta\cong\mathbb{Z}[\tilde{A}_{12}]$. It follows from [57, (2.4)] that such a component contains a simple $\mathrm{SL}(2)_1$-module S. As S is the restriction of an $\mathrm{SL}(2)$-module, [10, (2.5)] implies that S has constant Jordan type.

Let $\{e,h,f\}$ be the standard basis of $\mathfrak{sl}(2)$. Recall that simple $U_0(\mathfrak{sl}(2))$-modules are cyclic f-spaces (cf. [52, p. 208]). Hence S has constant Jordan type $\mathrm{Jt}(S)=[[\dim_k S]]$.

We define $s_\Theta:=\dim_k S$. Corollary 3.1.2 now yields $d_i^\Theta(\alpha_K)=\delta_{i,s(\Theta)}$ for every π-point α_K of $\mathrm{SL}(2)_1$ and $i\in\{1,\ldots,p-1\}$. We therefore conclude that $\mathrm{Jt}(M)=\left\{[s_\Theta]\oplus\left(\frac{\dim_k M-s_\Theta}{p}\right)[p]\right\}$ for every $M\in\Theta$.

(2) By the above, we have $\Theta\cong\mathbb{Z}[A_\infty]/\langle\tau\rangle$. In view of [25, (4.1.2)], there exist $g\in\mathrm{SL}(2)(k)$ and $a\in\{0,\ldots,p-2\}$ such that the unique module $M\in\mathrm{Ad}(g)^*(\Theta)$ of quasi-length s is the maximal submodule $W(sp+a)$ of dimension sp of the Weyl module $V(sp+a)$. Recall that $\mathcal{V}_{\mathfrak{sl}(2)}(W(sp+a))=ke$.

Let $\alpha_K:\mathfrak{A}_{p,K}\to U_0(\mathfrak{sl}(2))_K$ be a π-point. Then there exists an element $x\in\mathcal{V}_{\mathfrak{sl}(2)_K}$ such that $\mathrm{im}\,\alpha_K\subseteq U_0(Kx)$. We write $x=\alpha(e\otimes1)+\beta(h\otimes1)+\gamma(f\otimes1)$ with $\beta^2+\alpha\gamma=0$. If $\gamma\neq0$, then $W(sp+a)_K|_{K[x]}$ is projective, so that $\mathrm{Jt}(W(sp+a),\alpha_K)=\{s[p]\}$. Alternatively, $\beta=\gamma=0$ and $\alpha_K^*(W(sp+a)_K)\cong W(sp+a)_K|_{K[e\otimes1]}$, whence $\mathrm{Jt}(W(sp+a),\alpha_K)=\{[a+1]\oplus[p-a-1]\oplus(s-1)[p]\}$. Owing to [25, (4.1.2)], the component $\mathrm{Ad}(g)^*(\Theta)$ contains the baby Verma module with highest weight a. Consequently, $\mathrm{Ad}(g)^*(\Theta)\subseteq\Gamma_s(\mathcal{B}_i)$, where $i:=\min\{a+1,p-a-1\}$.

Since $\mathrm{Jt}(M)=\mathrm{Jt}(\mathrm{Ad}(g)^*(M))$ for any $\mathrm{SL}(2)_1$-module M, and the connected group $\mathrm{SL}(2)(k)$ acts trivially on the blocks of $k\,\mathrm{SL}(2)_1$, our assertion follows.

(3) Suppose that $M\cong\Omega_{\mathcal{G}}^n(S)$, where S is a simple $k\,\mathrm{SL}(2)_1$-module of dimension $s\in\{1,p-1\}$. According to (1) we have $\mathrm{Jt}(S)=\{[s]\}$ and [10, (5.6)] implies that S is endo-trivial. Being a Heller shift of an endo-trivial module, the module M is also endo-trivial.

Let M be an indecomposable endo-trivial module. Then $\Pi(\mathrm{SL}(2)_1)_M=\Pi(\mathrm{SL}(2)_1)$ is one-dimensional, so that M belongs to a component $\Theta\cong\mathbb{Z}[\tilde{A}_{12}]$. In view of (1) and [10, (5.6)], the simple module $S\in\Theta$ has dimension $s_\Theta\in\{1,p-1\}$ and thus belongs to the principal block of $k\,\mathrm{SL}(2)_1$. Let T be the other simple module of the principal block. The standard AR-sequence involving the projective cover $P(T)$ of T has the form

$$(0)\to\mathrm{Rad}(P(T))\to P(T)\oplus(\mathrm{Rad}(P(T))/\mathrm{Soc}(P(T)))\to P(T)/\mathrm{Soc}(P(T))\to(0)$$

(see [3, (V.5.5)]). Since $\mathrm{Rad}(P(T))/\mathrm{Soc}(P(T))\cong S\oplus S$ (cf. [48, Thm. 3]), we conclude that S and $\Omega_{\mathcal{G}}(T)$ are representatives of the two $\tau_{\mathcal{G}}$-orbits of Θ (cf. [20, (IV.3.8.3)]). As $k\,\mathrm{SL}(2)_1$ is symmetric, it follows that $\{\Omega_{\mathcal{G}}^{2n}(S),\Omega_{\mathcal{G}}^{2n+1}(T);\ n\in\mathbb{Z}\}$ is the set of vertices of Θ. Consequently, there exists a simple $k\,\mathrm{SL}(2)_1$-module S of dimension $\dim_k S$ in $\{1,p-1\}$ and $n\in\mathbb{Z}$ with $M\cong\Omega_{\mathcal{G}}^n(S)$. $\square$

Remarks. (1) Since the principal block $\mathcal{B}_0(SL(2)_1) \subseteq k\,SL(2)_1$ is Morita equivalent to the trivial extension of the Kronecker algebra, it follows that $H^n(SL(2)_1, k) = (0)$, whenever n is odd. Let n be even and $\zeta \in H^n(SL(2)_1, k) \setminus \{0\}$ be nilpotent. In view of Theorem 6.3.1, the Carlson module L_ζ is indecomposable of constant Jordan type $\mathrm{Jt}(L_\zeta) = \{[1] \oplus [p-1] \oplus n_\zeta[p]\}$ (cf. Lemma 5.1). As this contradicts Proposition 8.1.1(1), we have retrieved the well-known fact that the algebra $H^*(SL(2)_1, k) = H^\bullet(SL(2)_1, k)$ is reduced.

(2) Let $\Theta \subseteq \Gamma_s(SL(2)_1)$ be a component with zero-dimensional Π-support. If α_K is a π-point such that $[\alpha_K] \in \Pi(SL(2)_1)_\Theta$, then Proposition 8.1.1 shows that $\ell(\alpha_{K,i}) \leq 2$ for $1 \leq i \leq p-1$. One can show that the quasi-simple module of Θ is the only relatively α_K-projective module belonging to Θ.

Let $T \subseteq SL(2)$ be the standard maximal torus of diagonal matrices. Our next example, $\mathcal{G} = SL(2)_1 T_r$ ($r \geq 2$), the product of the first Frobenius kernel of $SL(2)$ and the r-th Frobenius kernel of T, is closely related to the previous one. In fact, $k\,SL(2)_1 T_r$ is a Galois extension of $k\,SL(2)_1$ with Galois group $\mathbb{Z}/(p^{r-1})$. According to [29, (5.5)], the groups $SL(2)_1 T_r$ are precisely the semi-simple infinitesimal groups whose principal blocks have tame representation type.

By the proof of [29, (5.5)], the algebra $k\,SL(2)_1 T_r$ has non-simple blocks $\mathcal{B}_1, \ldots, \mathcal{B}_{(p-1)/2}$ with $\mathcal{B}_i$ possessing p^{r-1} simple modules of dimensions i and $p-i$, respectively.

Corollary 8.1.2. *Let $\Theta \subseteq \Gamma_s(SL(2)_1 T_r)$ be a component.*

(1) *If* $\dim \Pi(SL(2)_1 T_r)_\Theta = 1$, *then* $\Theta \cong \mathbb{Z}[\tilde{A}_{p^{r-1}, p^{r-1}}]$, *and there exists* s_Θ *in* $\{1, \ldots, p-1\}$ *such that*

$$\mathrm{Jt}(M) = \left\{ [s_\Theta] \oplus \left(\frac{\dim_k M - s_\Theta}{p} \right)[p] \right\}$$

for every $M \in \Theta$.

(2) *If* $\dim \Pi(SL(2)_1 T_r)_\Theta = 0$, *then* $\Theta \cong \mathbb{Z}[A_\infty]/\langle \tau \rangle$, $\mathbb{Z}[A_\infty]/\langle \tau^{p^{r-1}} \rangle$, *and there exists* $i_\Theta \in \{1, \ldots, (p-1)/2\}$ *such that* $\Theta \subseteq \Gamma_s(\mathcal{B}_{i_\Theta})$ *and*

$$\mathrm{Jt}(M) = \left\{ \left(\frac{\dim_k M}{p} \right)[p], [i_\Theta] \oplus [p - i_\Theta] \oplus \left(\frac{\dim_k M}{p} - 1 \right)[p] \right\}$$

for every $M \in \Theta$.

(3) *An indecomposable* $SL(2)_1 T_r$*-module* M *is endo-trivial if and only if* $M \cong \Omega^n_{SL(2)_1 T_r}(S)$ *for* $n \in \mathbb{Z}$ *and* S *simple of dimension* 1 *or* $p-1$.

Proof. According to [29, (5.6)], the stable Auslander–Reiten quiver of each non-simple block $\mathcal{B} \subseteq k\,SL(2)_1 T_r$ has two components of type $\mathbb{Z}[\tilde{A}_{p^{r-1}, p^{r-1}}]$, four components of type $\mathbb{Z}[A_\infty]/\langle \tau^{p^{r-1}} \rangle$, and infinitely many components of type $\mathbb{Z}[A_\infty]/\langle \tau \rangle$.

Let $\alpha_K : \mathfrak{A}_{p,K} \to K\,SL(2)_1 T_r$ be a π-point. Then there exists an abelian, unipotent subgroup $\mathcal{U} \subseteq (SL(2)_1 T_r)_K$ such that we have $\mathrm{im}\,\alpha_K \subseteq K\mathcal{U}$. Since the group

$(\mathrm{SL}(2)_1 T_r)_K / (\mathrm{SL}(2)_1)_K \cong (T_{r-1})_K$ is diagonalizable, we obtain $\mathcal{U} \subseteq (\mathrm{SL}(2)_1)_K$. Accordingly, we have

$$\alpha_K^*(M_K) = \alpha_K^*(M_K|_{(\mathrm{SL}(2)_1)_K}) \qquad\qquad (*)$$

for every $\mathrm{SL}(2)_1 T_r$-module M.

(1) If $\dim \Pi(\mathrm{SL}(2)_1 T_r)_\Theta = 1$ and $M \in \Theta$, then [25, (2.1.2)] ensures that $M|_{\mathrm{SL}(2)_1}$ belongs to a component with one-dimensional Π-support. Hence Proposition 8.1.1 in conjunction with $(*)$ provides $s \in \{1, \ldots, p-1\}$ such that

$$\mathrm{Jt}(M) = \left\{ [s] \oplus \left(\frac{\dim_k M - s}{p} \right)[p] \right\}.$$

By the above, $\Theta \cong \mathbb{Z}[\tilde{A}_{p^{r-1}, p^{r-1}}]$ has tree class A_∞^∞, so that our assertion follows from Corollary 3.1.2.

(2) Assume that $\dim \Pi(\mathrm{SL}(2)_1 T_r)_\Theta = 0$. Let $M_0 \in \Theta$. Then the restriction $M_0|_{\mathrm{SL}(2)_1}$ belongs to a homogeneous tube $\mathbb{Z}[A_\infty]/\langle\tau\rangle$, so that [25, (4.1.2)], Proposition 8.1.1 and $(*)$ imply

- $\dim_k M_0 = p\, \mathrm{q}\ell(M_0|_{\mathrm{SL}(2)_1})$, and
- $\mathrm{Jt}(M_0) = \{\mathrm{q}\ell(M_0|_{\mathrm{SL}(2)_1})[p], [i] \oplus [p-i] \oplus (\mathrm{q}\ell(M_0|_{\mathrm{SL}(2)_1}) - 1)[p]\}$, where $M_0|_{\mathrm{SL}(2)_1}$ belongs to the block $\mathcal{B}_i \subseteq k\,\mathrm{SL}(2)_1$.

Since the restrictions of simple $\mathrm{SL}(2)_1 T_r$-modules are simple (cf. [29, (5.1)]), we conclude that M_0 belongs to $\mathcal{B}_i \subseteq k\,\mathrm{SL}(2)_1 T_r$. Hence any $M \in \Theta$ will yield the same data for $\mathrm{Jt}(M)$.

(3) Let M be an indecomposable, endo-trivial $\mathrm{SL}(2)_1 T_r$-module. Owing to [25, (2.1.2)], the module $M|_{\mathrm{SL}(2)_1}$ is indecomposable. Moreover, a two-fold application of [10, (5.6)] in conjunction with $(*)$ implies that $M|_{\mathrm{SL}(2)_1}$ is endo-trivial. Proposition 8.1.1 now provides $n \in \mathbb{Z}$ such that $\Omega_{\mathrm{SL}(2)_1}^n(M|_{\mathrm{SL}(2)_1})$ is a simple $\mathrm{SL}(2)_1$-module of dimension 1 or $p-1$. Since $\Omega_{\mathrm{SL}(2)_1 T_r}^n(M)|_{\mathrm{SL}(2)_1}$ is indecomposable, it follows that $\Omega_{\mathrm{SL}(2)_1 T_r}^n(M)|_{\mathrm{SL}(2)_1} \cong \Omega_{\mathrm{SL}(2)_1}^n(M|_{\mathrm{SL}(2)_1})$ is also simple. Consequently, $\Omega_{\mathrm{SL}(2)_1 T_r}^n(M)$ is simple.

If S is a simple $\mathrm{SL}(2)_1 T_r$-module of dimension 1 or $p-1$, then $S|_{\mathrm{SL}(2)_1}$ has the same properties (see [29, (5.1)]), so that (1) and [10, (5.6)] imply that S is endo-trivial. Consequently, $\Omega_{\mathrm{SL}(2)_1 T_r}^n(S)$ is also endo-trivial. $\qquad\square$

Remarks. (1) The foregoing results indicate the utility of π-points. In either case, components with one-dimensional Π-supports cannot be distinguished via their support varieties, yet their Jordan types nicely separate them (cf. Corollary 3.2.3). On the other hand, Jordan types do not reflect the ranks of the tubes occurring in Corollary 8.1.2.

(2) In each of the cases above, the indecomposable modules are either projective, constantly supported or of constant Jordan type. It is therefore possible to compute $\mathrm{Jt}(M)$ for any $\mathcal{G}$-module M whose indecomposable constituents are known.

8.2. A central extension of $\mathfrak{sl}(2)$

Our next example necessitates a simple technical preparation. Recall that $\mathfrak{A}_{p,K} = K[t]$, where $t^p = 0$. Given $j \in \{1, \ldots, p\}$ and $1 \leq \ell \leq \lfloor p/j \rfloor$, the unique ℓ-dimensional indecomposable module of the local subalgebra $K[t^j] \subseteq \mathfrak{A}_{p,K}$ will also be denoted $[\ell]$. (Here $\lfloor \ \rfloor$ denotes the floor function.)

Lemma 8.2.1. *Let $i, j \in \{1, \ldots, p\}$. Then*

$$[i]|_{K[t^j]} = \begin{cases} (j-r)[a] \oplus r[a+1] & \text{if } i = aj + r \text{ with } 0 \leq r < j \leq i, \\ i[1] & \text{otherwise.} \end{cases}$$

Proof. Let $\{v_1, \ldots, v_i\}$ be a basis of the cyclic $\mathfrak{A}_{p,K}$-module $[i]$ such that $t.v_\ell = v_{\ell+1}$ ($v_{i+1} = 0$). If $j \geq i$, then t^j acts trivially on $[i]$, whence $[i]|_{K[t^j]} = i[1]$. Alternatively, for $1 \leq \ell \leq j$ we have

$$K[t^j]v_\ell = \sum_{q=0}^{a} K v_{v_{qj+\ell}} = \begin{cases} \bigoplus_{q=0}^{a} K v_{v_{qj+\ell}}, & 1 \leq \ell \leq r, \\ \bigoplus_{q=0}^{a-1} K v_{v_{qj+\ell}}, & r+1 \leq \ell \leq j. \end{cases}$$

Hence $[i]|_{K[t^j]} = (j-r)[a] \oplus r[a+1]$. $\qquad\square$

We consider the 4-dimensional restricted Lie algebra $\mathfrak{sl}(2)_s := \mathfrak{sl}(2) \oplus kv_0$, whose bracket and p-map are given by

$$[(x, \alpha v_0), (y, \beta v_0)] = ([x, y], 0) \quad \text{and} \quad (x, \alpha v_0)^{[p]} = (x^{[p]}, \psi(x)v_0)$$

respectively, where the p-semilinear map $\psi : \mathfrak{sl}(2) \to k$ satisfies $\psi(e) = 0 = \psi(f)$ and $\psi(h) = 1$. Note that the nullcone

$$\mathcal{V}_{\mathfrak{sl}(2)_s} = (\mathcal{V}_{\mathfrak{sl}(2)} \cap \ker \psi) \times kv_0 = (ke \oplus kv_0) \cup (kf \oplus kv_0)$$

is 2-dimensional and reducible.

Since kv_0 is a unipotent p-ideal of $\mathfrak{sl}(2)_s$, the simple $U_0(\mathfrak{sl}(2)_s)$-modules are just the pull-backs of the simple $U_0(\mathfrak{sl}(2))$-modules. Moreover, the p-dimensional simple $U_0(\mathfrak{sl}(2)_s)$-module belongs to a block which is a Morita equivalent to a truncated polynomial ring $k[X]/(X^n)$ (a Nakayama algebra).

According to [27, §7], each stable AR-component $\Theta \subseteq \Gamma_s(\mathfrak{sl}(2)_s)$ with 2-dimensional support is isomorphic to $\mathbb{Z}[A_\infty^\infty]$, so that $f_\Theta \equiv 1$ (cf. Corollary 3.1.2). We shall consider two types of these components. One type contains a simple module, the others are given by certain induced modules.

Recall that $\{e, h, f\}$ is the standard basis of $\mathfrak{sl}(2)$. Let $\lambda : kh \oplus ke \oplus kv_0 \to k$ be a linear form such that $\lambda(h) \in \{0, \ldots, p-2\}$ and $\lambda(e) = 0 = \lambda(v_0)$. Observing that $\mathfrak{b}_s := kh \oplus ke \oplus kv_0$ is a p-subalgebra of $\mathfrak{sl}(2)_s$, we consider the "Verma module"

$$Z(\lambda) := U_0(\mathfrak{sl}(2)_s) \otimes_{U_0(\mathfrak{b}_s)} k_\lambda.$$

Owing to [27, (7.3)], we have $ke \oplus kv_0 \subseteq \mathcal{V}_{\mathfrak{sl}(2)_s}(Z(\lambda))$, while general results on induced modules (see for instance [37, (4.12)]) guarantee the reverse inclusion. Thus, $Z(\lambda)$ is not of constant Jordan type and belongs to a component $\Theta_{(\lambda)} \cong \mathbb{Z}[A_\infty^\infty]$.

985 Jordan types for indecomposable modules

Proposition 8.2.2. (1) *Let* $\Theta_S \subseteq \Gamma_s(\mathfrak{sl}(2)_s)$ *be the component containing the simple module* S. *If* $n := \dim_k S \leq p - 1$, *then* $\Theta_S \cong \mathbb{Z}[A_\infty^\infty]$ *and*

$$\mathrm{Jt}(M) = \left\{ (j - r)[a] \oplus r[a+1] \oplus \left(\frac{\dim_k M - n}{p} \right)[p]; \ n = aj + r, \ 0 \leq r < j \leq n \right\}$$

for every $M \in \Theta_S$. *In particular,* $|\mathrm{Jt}(M)| = n$ *for every* $M \in \Theta_S$.

(2) *Let* $\Theta_{(\lambda)} \subseteq \Gamma_s(\mathfrak{sl}(2)_s)$ *be the component containing* $Z(\lambda)$. *Then* $\Theta_{(\lambda)} \cong \mathbb{Z}[A_\infty^\infty]$ *and setting* $i := \min\{\lambda(h) + 1, \ p - \lambda(h) - 1\}$ *we have*

$$\mathrm{StJt}(M) = \{(j - r)[a] \oplus r[a+1]; \ p = aj + r, \ 1 \leq r < j \leq p\}$$
$$\cup \{i[1] \oplus (j - r)[a] \oplus r[a+1]; \ p - i = aj + r, \ 0 \leq r < j, \ i \leq j \leq p - i\}$$
$$\cup \{(j - r')[b] \oplus r'[b+1] \oplus (j - r)[a] \oplus r[a+1]; \ i = bj + r', \ p - i = aj + r,$$
$$0 \leq r, r' < j < i\}$$

for every $M \in \Theta_{(\lambda)}$.

(3) *Every* $M \in \Theta_k \cup \Theta_{\Omega_{\mathfrak{sl}(2)_s}(k)}$ *is endo-trivial.*

Proof. (1) The central, nilpotent element $v_0 \in U_0(\mathfrak{sl}(2)_s)$ acts trivially on every simple $U_0(\mathfrak{sl}(2)_s)$-module S. Let $\alpha_K : \mathfrak{A}_{p,K} \to U_0(\mathfrak{sl}(2)_s)_K$ be a π-point. Then there exists a p-unipotent subalgebra $\mathfrak{u} \subseteq (\mathfrak{sl}(2)_s)_K$ with $\mathrm{im}\, \alpha_K \subseteq U_0(\mathfrak{u})$. Hence we can find $x \in \mathfrak{sl}(2)_K$ with $x^{[p]} \in K(v_0 \otimes 1)$ and $\mathfrak{u} \subseteq Kx \oplus Kx^{[p]}$. If $x^{[p]} \neq 0$, then the condition $\alpha_K(t)^p = 0$ entails $\mathrm{im}\, \alpha_K \subseteq (v_0 \otimes 1)U_0(\mathfrak{sl}(2)_s)_K$, so that $\mathrm{Jt}(S, \alpha_K) = n[1]$. Alternatively, $x \in \mathcal{V}_{\mathfrak{sl}(2)_s}$ and $\mathfrak{u} \subseteq K(e \otimes 1) \oplus K(v_0 \otimes 1)$ or $\mathfrak{u} \subseteq K(f \otimes 1) \oplus K(v_0 \otimes 1)$. Both cases being similar, we write $\alpha_K(t) = \sum_{i,j=0}^{p-1} \gamma_{i,j}(e \otimes 1)^i(v_0 \otimes 1)^j$ ($\gamma_{0,0} = 0$) and note that $\alpha_K(t)w = (\sum_{i=1}^{p-1} \gamma_{i,0}(e \otimes 1)^i).w$ for all $w \in S_K$. Since $S|_{K[e \otimes 1]} = [n]$, Lemma 8.2.1 in conjunction with [35, (2.2)] gives

$$\mathrm{Jt}(S) = \{(j - r)[a] \oplus r[a+1]; \ 1 \leq j \leq n, \ n = aj + r, \ 0 \leq r \leq j - 1\}.$$

As $\Theta_S \cong \mathbb{Z}[A_\infty^\infty]$, Corollary 3.1.2 yields the first assertion. Moreover, each Jordan type of S is uniquely determined by $j \in \{1, \ldots, n\}$, whence $|\mathrm{Jt}(S)| = n$. Thanks to Corollary 3.2.1, this implies $|\mathrm{Jt}(M)| = n$ for every $M \in \Theta_S$.

(2) Let $\alpha_K : \mathfrak{A}_{p,K} \to U_0(\mathfrak{sl}(2)_s)_K$ be a π-point, and $\mathfrak{u} \subseteq (\mathfrak{sl}(2)_s)_K$ be a p-unipotent subalgebra through which α_K factors. Since v_0 acts trivially on $Z(\lambda)$, the arguments of part (1) imply $\mathrm{Jt}(Z(\lambda), \alpha_K) = p[1]$ whenever $\mathfrak{u}^{[p]} \neq (0)$. By the same token, the remaining cases are $\mathfrak{u} \subseteq K(e \otimes 1) \oplus K(v_0 \otimes 1)$ and $\mathfrak{u} \subseteq K(f \otimes 1) \oplus K(v_0 \otimes 1)$.

Since $Z(\lambda)|_{K[f \otimes 1]} = [p]$, Lemma 8.2.1 implies

$$\mathrm{Jt}(Z(\lambda), \alpha_K) \in \{(j - r)[a] \oplus r[a+1]; \ p = aj + r, \ 0 \leq r < j \leq p\}$$

in the latter case.

In view of $Z(\lambda)|_{K[e \otimes 1]} = [i] \oplus [p - i]$ and $i < p - i$, the assumption $\mathrm{im}\, \alpha_K \subseteq K(e \otimes 1) \oplus K(v_0 \otimes 1)$ in conjunction with Lemma 8.2.1 gives

$$\mathrm{Jt}(Z(\lambda), \alpha_K) \in \{i[1] \oplus (j - r)[a] \oplus r[a+1]; \ p - i = aj + r, \ 0 \leq r < j, \ i \leq j \leq p - i\}$$
$$\cup \{(j - r')[b] \oplus r'[b+1] \oplus (j - r)[a] \oplus r[a+1]; \ i = bj + r', \ p - i = aj + r,$$
$$0 \leq r, r' < j < i\}.$$

Owing to [35, (2.2)], all types in the sets above actually occur, so that we have in fact determined $\mathrm{Jt}(Z(\lambda))$. Since $\Theta_{(\lambda)} \cong \mathbb{Z}[A_\infty^\infty]$, Corollary 3.1.2 gives the result.

(3) Since the trivial module k and its Heller shift $\Omega_{\mathfrak{sl}(2)_s}(k)$ are endo-trivial, the assertion is a direct consequence of Theorem 7.2(2). $\qquad\square$

Remarks. (1) In Proposition 8.1.1 and Corollary 8.1.2, the indecomposable periodic modules are constantly supported. This is in general not the case. Consider the group scheme $\mathbb{G}_{a(2)}$ with algebra of measures $k\mathbb{G}_{a(2)} = k[u_0, u_1]$, where $k\mathbb{G}_{a(1)} = k[u_0]$. Then $M := k\mathbb{G}_{a(2)} \otimes_{k[u_0]} k$ is indecomposable and periodic. We consider the π-points $\alpha_k, \beta_k : \mathfrak{A}_p \to k\mathbb{G}_{a(2)}$ given by $\alpha_k(t) = u_0$ and $\beta_k(t) = u_0 + u_1^2$. Then [35, (2.2)] implies $\alpha_k \sim \beta_k$, while Lemma 8.2.1 yields

$$\mathrm{Jt}(M, \alpha_k) = p[1] \quad \text{and} \quad \mathrm{Jt}(M, \beta_k) = \left[\frac{p-1}{2}\right] \oplus \left[\frac{p+1}{2}\right],$$

so that $\left\{[p], p[1], \left[\frac{p-1}{2}\right] \oplus \left[\frac{p+1}{2}\right]\right\} \subseteq \mathrm{Jt}(M)$.

Also, since $k[u_0] = k\mathbb{G}_{a(1)} \subseteq k\mathbb{G}_{a(2)}$, the module $M|_{\mathbb{G}_{a(1)}}$ has constant Jordan type. Thus, $\mathcal{G}$-modules M whose restrictions to a subgroup $\mathcal{H} \subseteq \mathcal{G}$ are of constant Jordan type with $\Pi(\mathcal{G})_M = \Pi(\mathcal{H})$ are not necessarily constantly supported.

(2) Note that the modules belonging to the components Θ_S are usually not annihilated by v_0. In fact, direct computation shows that $\Omega^2_{\mathfrak{sl}(2)_s}(S)$ is too big to be annihilated by v_0. Thus, our method of computing $\mathrm{Jt}(S)$ does not carry over to all modules of Θ_S.

References

[1] Alperin, J., Evens, L.: Representations, resolutions and Quillen's dimension theorem. J. Pure Appl. Algebra **22**, 1–9 (1981)   Zbl 0469.20008   MR 0621284

[2] Andersen, H., Jantzen, J.: Cohomology of induced representations for algebraic groups. Math. Ann. **269**, 487–525 (1984)   Zbl 0529.20027   MR 0766011

[3] Auslander, M., Reiten, I., Smalø, S.: Representation Theory of Artin Algebras. Cambridge Stud. Adv. Math. 36, Cambridge Univ. Press (1995)   Zbl 0834.16001   MR 1476671

[4] Benson, D.: Representations and Cohomology, I. Cambridge Stud. Adv. Math. 30, Cambridge Univ. Press (1991)   Zbl 0718.20001   MR 1644252

[5] Benson, D.: Representations and Cohomology, II. Cambridge Stud. Adv. Math. 31, Cambridge Univ. Press (1991)   Zbl 0731.20001   MR 1634407

[6] Benson, D.: Modules of constant Jordan type with one non-projective block. Algebr. Represent. Theory **13**, 315–318 (2010)   Zbl 1215.20013   MR 2630123

[7] Carlson, J.: The variety of an indecomposable module is connected. Invent. Math. **77**, 291–299 (1984)   Zbl 0543.20032   MR 0752822

[8] Carlson, J.: Constructing endotrivial modules. J. Pure Appl. Algebra **206**, 83–110 (2006)   Zbl 1098.20004   MR 2220083

[9] Carlson, J., Friedlander, E.: Exact category of modules of constant Jordan type. In: Manin Festschrift, Progr. Math. 269, Birkhäuser, 267–290 (2010) Zbl 1267.20059 MR 2641174

[10] Carlson, J., Friedlander, E., Pevtsova, J.: Modules of constant Jordan type. J. Reine Angew. Math. **614**, 191–234 (2008) Zbl 1144.20025 MR 2376286

[11] Carlson, J., Lin, Z., Nakano, D., Parshall, B.: The restricted nullcone. In: Combinatorial and Geometric Representation Theory (Seoul, 2001), Contemp. Math. 325, Amer. Math. Soc., 51–75 (2003) Zbl 1059.17013 MR 1988985

[12] Carlson, J., Thévenaz, J.: The classification of endo-trivial modules. Invent. Math. **158**, 389–411 (2004) Zbl 1110.20004 MR 2096798

[13] Cartan, H., Eilenberg, S.: Homological Algebra. Princeton Math. Ser. 19, Princeton Univ. Press, Princeton (1956) Zbl 0075.24305 MR 0077480

[14] Collingwood, D., McGovern, W.: Nilpotent Orbits in Semisimple Lie Algebras. Van Nostrand Reinhold, New York (1993) Zbl 0972.17008 MR 1251060

[15] Dade, E.: Endo-permutation modules over p-groups I. Ann. of Math. **107**, 459–494 (1978) Zbl 0395.16007 MR 0496843

[16] Dade, E.: Endo-permutation modules over p-groups II. Ann. of Math. **108**, 317–346 (1978) Zbl 0404.16003 MR 0506990

[17] Demazure, M., Gabriel, P.: Groupes Algébriques I. Masson/North-Holland (1970) Zbl 0203.23401 MR 0302656

[18] Drozd, Yu.: On the representations of the Lie algebra $\mathfrak{sl}(2)$. Visn. Kiiv. Univ. Mat. i Mekh. **25**, 70–77 (1983) Zbl 0900.17002 MR 0746766

[19] Erdmann, K.: On modules with cyclic vertices in the Auslander–Reiten quiver. J. Algebra **104**, 289–300 (1986) Zbl 0602.20011 MR 0866776

[20] Erdmann, K.: Blocks of Tame Representation Type and Related Algebras. Lecture Notes in Math. 1428, Springer (1990) Zbl 0696.20001 MR 1064107

[21] Evens, L.: The Cohomology of Groups. Oxford Math. Monogr., Clarendon Press (1991) Zbl 0742.20050 MR 1144017

[22] Farnsteiner, R.: On the Auslander–Reiten quiver of an infinitesimal group. Nagoya Math. J. **160**, 103–121 (2000) Zbl 0976.16015 MR 1804139

[23] Farnsteiner, R.: Polyhedral groups, McKay quivers, and the finite algebraic groups with tame principal blocks. Invent. Math. **166**, 27–94 (2006) Zbl 1115.20035 MR 2242632

[24] Farnsteiner, R.: Support spaces and Auslander–Reiten components. In: Lie Algebras, Vertex Operator Algebras and Their Applications, Contemp. Math. 442, Amer. Math. Soc., 61–87 (2007) Zbl 1185.16023 MR 2372557

[25] Farnsteiner, R.: Group-graded algebras, extensions of infinitesimal groups, and applications. Transform. Groups **14**, 127–162 (2009) Zbl 1182.14052 MR 2480855

[26] Farnsteiner, R., Röhrle, G., Voigt, D.: Infinitesimal unipotent group schemes of complexity 1. Colloq. Math. **89**, 179–192 (2001) Zbl 0989.16013 MR 1854701

[27] Farnsteiner, R., Skowroński, A.: Classification of restricted Lie algebras with tame principal block. J. Reine Angew. Math. **546**, 1–45 (2002) Zbl 0989.17011 MR 1900992

[28] Farnsteiner, R., Voigt, D.: On cocommutative Hopf algebras of finite representation type. Adv. Math. **155**, 1–22 (2000) Zbl 0981.16030 MR 1789845

[29] Farnsteiner, R., Voigt, D.: On infinitesimal groups of tame representation type. Math. Z. **244**, 479–513 (2003) Zbl 1031.16022 MR 1992021

[30] Fischer, G.: Die Köcher der Frobeniuskerne der SL_2. Dissertation, Universität Bielefeld (1983)

[31] Fischman, D., Montgomery, S., Schneider, H.: Frobenius extensions of subalgebras of Hopf algebras. Trans. Amer. Math. Soc. **349**, 4857–4895 (1996) Zbl 0899.16020 MR 1401518

[32] Friedlander, E., Parshall, B.: Cohomology of Lie algebras and algebraic groups. Amer. J. Math. **108**, 235–253 (1986) Zbl 0601.20042 MR 0821318

[33] Friedlander, E., Parshall, B.: Geometry of p-unipotent Lie algebras. J. Algebra **109**, 25–45 (1987) Zbl 0626.17011 MR 0898334

[34] Friedlander, E., Parshall, B.: Support varieties for restricted Lie algebras. Invent. Math. **86**, 553–562 (1986) Zbl 0626.17010

[35] Friedlander, E., Pevtsova, J.: Representation-theoretic support spaces for finite group schemes. Amer. J. Math. **127**, 379–420 (2005); Erratum, ibid. **128**, 1067–1068 (2006) Zbl 1072.20009 Zbl 1098.20500 MR 2251594

[36] Friedlander, E., Pevtsova, J.: Π-supports for modules for finite group schemes over a field. Duke Math. J. **139**, 317–368 (2007) Zbl 1128.20031 MR 2352134

[37] Friedlander, E., Pevtsova, J., Suslin, A.: Generic and maximal Jordan types. Invent. Math. **168**, 485–522 (2007) Zbl 1117.14047 MR 2299560

[38] Happel, D.: Triangulated Categories in the Representation Theory of Finite Dimensional Algebras. London Math. Soc. Lecture Note Ser. 119, Cambridge Univ. Press (1988) Zbl 0635.16017 MR 0935124

[39] Happel, D., Preiser, U., Ringel, C.: Vinberg's characterization of Dynkin diagrams using sub-additive functions with application to DTr-periodic modules. In: Lecture Notes in Math. 832, Springer, 280–294 (1981) Zbl 0446.16032 MR 0607159

[40] Happel, D., Preiser, U., Ringel, C.: Binary polyhedral groups and Euclidean diagrams. Manuscripta Math. **31**, 317–329 (1980) Zbl 0436.20005 MR 0576503

[41] Hartshorne, R., Algebraic Geometry. Grad. Texts in Math. 52, Springer (1977) Zbl 0367.14001 MR 0463157

[42] Higman, D.: Indecomposable representations of characteristic p. Duke Math. J. **21**, 377–381 (1954) Zbl 0055.25503 MR 0067896

[43] Hochschild, G.: Cohomology of restricted Lie algebras. Amer. J. Math. **76**, 555–580 (1954) Zbl 0055.26505 MR 0063361

[44] Jantzen, J.: Representations of Algebraic Groups. Math. Surveys Monogr. 107, Amer. Math. Soc. (2003) Zbl 1034.20041 MR 2015057

[45] Kasjan, S.: Auslander–Reiten sequences under base field extension. Proc. Amer. Math. Soc. **128**, 2885–2896 (2000) Zbl 0974.16012 MR 1670379

[46] Mumford, D.: The Red Book of Varieties and Schemes. Lecture Notes in Math. 1358, Springer (1999) Zbl 0658.14001 MR 0971985

[47] Nakano, D., Parshall, B., Vella, D.: Support varieties for algebraic groups. J. Reine Angew. Math. **547**, 15–49 (2002) Zbl 1009.17013 MR 1900135

[48] Pollack, R.: Restricted Lie algebras of bounded type. Bull. Amer. Math. Soc. **74**, 326–331 (1968) Zbl 0175.31202 MR 0219582

[49] Premet, A.: Green ring for simple three-dimensional Lie p-algebras. Soviet Math. (Izv. VUZ) **35**, no. 10, 56–67 (1991) Zbl 0748.17015 MR 1179217

[50] Riedtmann, C.: Algebren, Darstellungsköcher, Überlagerungen und zurück. Comment. Math. Helv. **55**, 199–224 (1980) Zbl 0444.16018 MR 0576602

[51] Rudakov, A.: Reducible p-representations of a simple three-dimensional Lie p-algebra. Moscow Univ. Math. Bull. **37**, 51–56 (1982) Zbl 0516.17005 MR 0685263

[52] Strade, H., Farnsteiner, R.: Modular Lie Algebras and Their Representations. Monogr. Text-books Pure Appl. Math. 116, Dekker, New York (1988) Zbl 0648.17003 MR 0929682

[53] Suslin, A., Friedlander, E., Bendel, C.: Infinitesimal 1-parameter subgroups and cohomology. J. Amer. Math. Soc. **10**, 693–728 (1997) Zbl 0960.14023 MR 1443546

[54] Suslin, A., Friedlander, E., Bendel, C.: Support varieties for infinitesimal group schemes. J. Amer. Math. Soc. **10**, 729–759 (1997) Zbl 0960.14024 MR 1443547

[55] Voigt, D.: Induzierte Darstellungen in der Theorie der endlichen, algebraischen Gruppen. Lecture Notes in Math. 592, Springer (1977) Zbl 0374.14010 MR 0486168

[56] Waterhouse, W.: Introduction to Affine Group Schemes. Grad. Texts in Math. 66, Springer (1979) Zbl 0442.14017 MR 0547117

[57] Webb, P.: The Auslander–Reiten quiver of a finite group. Math. Z. **179**, 97–121 (1982) Zbl 0479.20008 MR 0643050