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An existence theorem for the Yamabe problem on manifolds with boundary

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Abstract. Let (M, g) be a compact Riemannian manifold with boundary. We consider the problem (first studied by Escobar in 1992) of finding a conformal metric with constant scalar curvature in the interior and zero mean curvature on the boundary. Using a local test function construction, we are able to settle most cases left open by Escobar's work. Moreover, we reduce the remaining cases to the positive mass theorem.

Keywords. Yamabe problem, manifolds with boundary

1. Introduction

The Yamabe problem, solved by Trudinger [14], Aubin [1], and Schoen [12], asserts that any Riemannian metric on a closed manifold is conformal to a metric with constant scalar curvature. Escobar [8], [9] has studied analogous questions on manifolds with boundary. To fix notation, let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ with boundary ∂M. We denote by R_g the scalar curvature of (M, g) and by κ_g the mean curvature of the boundary ∂M. There are two natural ways to extend the Yamabe problem to manifolds with boundary:

(a) Find a metric $\tilde{g}$ in the conformal class of g such that $R_{\tilde{g}}$ is constant and $\kappa_{\tilde{g}} = 0$.
(b) Find a metric $\tilde{g}$ in the conformal class of g such that $R_{\tilde{g}} = 0$ and $\kappa_{\tilde{g}}$ is constant.

The boundary value problem (a) was first proposed by Escobar [8]. The boundary value problem (b) is studied in [9] and [11].

In this paper, we focus on the boundary value problem (a). The solvability of (a) is equivalent to the existence of a critical point of the *Yamabe functional*, defined by

$$E_g(u) = \frac{\int_M \big(\frac{4(n-1)}{n-2}\, |du|_g^2 + R_g u^2\big)\, d\mathrm{vol}_g + \int_{\partial M} 2\kappa_g u^2\, d\sigma_g}{\big(\int_M u^{2n/(n-2)}\, d\mathrm{vol}_g\big)^{(n-2)/n}},$$

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where u is a smooth positive function on M. Moreover, the *Yamabe constant* is defined as

$$Y(M, \partial M, g) = \inf_{0 < u \in C^\infty(M)} E_g(u).$$

It is well known that $Y(M, \partial M, g)$ is invariant under a conformal change of the metric g. Moreover, $Y(M, \partial M, g) \leq Y(S^n_+, \partial S^n_+)$, where $Y(S^n_+, \partial S^n_+)$ denotes the Yamabe constant of the hemisphere S^n_+ equipped with the standard metric.

Proving the existence of a minimizer for the functional E_g is a difficult problem, as E_g does not satisfy the Palais–Smale condition. The following existence result was established by Escobar [8].

Theorem 1.1 (J. Escobar [8]). *If $Y(M, \partial M, g) < Y(S^n_+, \partial S^n_+)$, then there exists a metric $\tilde{g}$ in the conformal class of g such that $R_{\tilde{g}}$ is constant and $\kappa_{\tilde{g}}$ is equal to* 0.

Theorem 1.1 should be compared with Aubin's existence theorem for the Yamabe problem on manifolds without boundary (cf. [1]).

In dimension $3 \leq n \leq 5$, Escobar showed that $Y(M, \partial M, g) < Y(S^n_+, \partial S^n_+)$ unless M is conformally equivalent to the hemisphere S^n_+. In dimension $n \geq 6$, Escobar was able to verify this inequality under the assumption that ∂M is not umbilic.

Therefore, it remains to consider the case that $n \geq 6$ and ∂M is umbilic. For abbreviation, we put $d = [(n-2)/2]$. As in [4], we denote by $\mathcal{Z}$ the set of all $p \in M$ such that

$$\limsup_{x \to p} d(p, x)^{2-d} |W_g(x)| = 0,$$

where W_g denotes the Weyl tensor of (M, g). In other words, a point $p \in M$ belongs to $\mathcal{Z}$ if and only if $D^m W_g(p) = 0$ for all $m \in \{0, 1, \ldots, d-2\}$. Note that the set $\mathcal{Z}$ is invariant under a conformal change of the metric.

The following is the main result of this paper:

Theorem 1.2. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 6$ with umbilic boundary ∂M. Moreover, let $p \in \partial M$ be an arbitrary point on the boundary of M. If $p \notin \mathcal{Z}$, then $Y(M, \partial M, g) < Y(S^n_+, \partial S^n_+)$. Consequently, there exists a metric $\tilde{g}$ in the conformal class of g such that $R_{\tilde{g}}$ is constant and $\kappa_{\tilde{g}}$ is equal to* 0.

In case $p \in \mathcal{Z}$, we are able to show that $Y(M, \partial M, g) < Y(S^n_+, \partial S^N_+)$, provided that a certain asymptotically flat manifold has positive ADM mass (see Theorem 4.4 below for a precise statement). Thus, the solvability of the Yamabe problem on $(M, \partial M, g)$ can be reduced to the positive mass theorem.

We now give an outline of the proof of Theorem 1.2. By a theorem of Marques [11], we may work in conformal Fermi coordinates around p. We define

$$u_\varepsilon(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2} \right)^{(n-2)/2}. \tag{1.1}$$

The function u_ε satisfies

$$\Delta u_\varepsilon = -n(n-2) u_\varepsilon^{(n+2)/(n-2)}$$

and

$$u_\varepsilon \partial_i \partial_k u_\varepsilon - \frac{n}{n-2} \partial_i u_\varepsilon \, \partial_k u_\varepsilon = \frac{1}{n} \Big(u_\varepsilon \Delta u_\varepsilon - \frac{n}{n-2} |du_\varepsilon|^2 \Big) \delta_{ik}.$$

These identities reflect the fact that the metric $u_\varepsilon^{4/(n-2)} \delta_{ik}$ is Einstein.

We then consider a sum of the form $u_\varepsilon + w$, where u_ε is given by (1.1) and w is a correction term. This function is only defined in a small neighborhood of the point p. In order to extend the test function to all of M, we glue the function $u_\varepsilon + w$ to the Green's function of the conformal Laplacian with pole at p.

In order to show that the resulting test function has Yamabe energy less than $Y(S^n_+, \partial S^n_+)$, we make extensive use of techniques developed in [4] (see also [5], [6], [7]). In [4], these techniques were used to prove a convergence theorem for the parabolic Yamabe flow in dimension $n \geq 6$. The convergence of the Yamabe flow in dimension $3 \leq n \leq 5$ was shown in [3].

2. Auxiliary results

In this section, we consider the halfspace $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n \geq 0\}$. Moreover, we assume that $H_{ik}(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n_+$ which satisfies the following conditions:

- At each point $x \in \mathbb{R}^n_+$, we have $H_{in}(x) = 0$ for all $i \in \{1, \ldots, n\}$.
- At each point $x \in \partial \mathbb{R}^n_+$, we have $\sum_{k=1}^n H_{ik}(x) x_k = 0$ for all $i \in \{1, \ldots, n\}$.
- At each point $x \in \partial \mathbb{R}^n_+$, we have $\partial_n H_{ik}(x) = 0$ for all $i, k \in \{1, \ldots, n\}$.

Finally, we assume that the components $H_{ik}(x)$ are polynomials of the form

$$H_{ik}(x) = \sum_{2 \leq |\alpha| \leq d} h_{ik,\alpha} x^\alpha,$$

where the sum is taken over all multi-indices α of length $2 \leq |\alpha| \leq d$.

As in [4], we define

$$A_{ik} = \sum_{m=1}^n \partial_i \partial_m H_{mk} + \sum_{m=1}^n \partial_m \partial_k H_{im} - \Delta H_{ik} - \frac{1}{n-1} \sum_{m,p=1}^n \partial_m \partial_p H_{mp} \, \delta_{ik}$$

and

$$\begin{aligned} Z_{ijkl} &= \partial_i \partial_k H_{jl} - \partial_i \partial_l H_{jk} - \partial_j \partial_k H_{il} + \partial_j \partial_l H_{ik} \\ &\quad + \frac{1}{n-2} (A_{jl} \delta_{ik} - A_{jk} \delta_{il} - A_{il} \delta_{jk} + A_{ik} \delta_{jl}). \end{aligned}$$

Note that

$$\partial_l Z_{ijkl} = \frac{n-3}{n-2} (\partial_i A_{jk} - \partial_j A_{ik}).$$

Lemma 2.1. *We have $A_{in}(x) = 0$ for all $x \in \partial \mathbb{R}^n_+$ and all $i \in \{1, \ldots, n-1\}$.*

Proof. Note that $\partial_n H_{im}(x) = 0$ for all $x \in \partial\mathbb{R}^n_+$ and all $i, m \in \{1, \dots, n-1\}$. If we differentiate this identity in tangential direction, we obtain

$$A_{in}(x) = \sum_{m=1}^{n-1} \partial_m \partial_n H_{im}(x) = 0$$

for all $x \in \partial\mathbb{R}^n_+$ and all $i \in \{1, \dots, n-1\}$.

Lemma 2.2. *Assume that $Z_{ijkl}(x) = 0$ for all $x \in \partial\mathbb{R}^n_+$ and all $i, j, k, l \in \{1, \dots, n\}$. Then $H_{ik}(x) = A_{ik}(x) = 0$ for all $x \in \partial\mathbb{R}^n_+$ and all $i, k \in \{1, \dots, n-1\}$.*

Proof. We define

$$\begin{aligned}\hat{A}_{ik}(x) = &\sum_{m=1}^{n-1} \partial_i \partial_m H_{mk}(x) + \sum_{m=1}^{n-1} \partial_m \partial_k H_{im}(x) \\ &- \sum_{m=1}^{n-1} \partial_m \partial_m H_{ik}(x) - \frac{1}{n-2} \sum_{m,p=1}^{n} \partial_m \partial_p H_{mp}(x)\delta_{ik}\end{aligned}$$

for all $x \in \partial\mathbb{R}^n_+$ and all $i, k \in \{1, \dots, n-1\}$. By assumption, we have

$$\partial_n \partial_n H_{ik}(x) + \frac{1}{n-2}(A_{ik}(x) + A_{nn}(x)\delta_{ik}) = Z_{inkn}(x) = 0$$

for all $x \in \partial\mathbb{R}^n_+$ and all $i, k \in \{1, \dots, n-1\}$. This implies

$$\begin{aligned}\hat{A}_{ik}(x) &= A_{ik}(x) + \partial_n \partial_n H_{ik}(x) - \frac{1}{(n-1)(n-2)} \sum_{m,p=1}^{n-1} \partial_m \partial_p H_{mp}(x)\, \delta_{ik} \\ &= A_{ik}(x) + \partial_n \partial_n H_{ik}(x) + \frac{1}{n-2} A_{nn}(x)\delta_{ik} = \frac{n-3}{n-2} A_{ik}(x)\end{aligned}$$

for all $x \in \partial\mathbb{R}^n_+$ and all $i, k \in \{1, \dots, n-1\}$. Hence, we obtain

$$\begin{aligned}&\partial_i \partial_k H_{jl}(x) - \partial_i \partial_l H_{jk}(x) - \partial_j \partial_k H_{il}(x) + \partial_j \partial_l H_{ik}(x) \\ &\quad = -\frac{1}{n-2}(A_{jl}(x)\delta_{ik} - A_{jk}(x)\delta_{il} - A_{il}(x)\delta_{jk} + A_{ik}(x)\delta_{jl}) \\ &\quad = -\frac{1}{n-3}(\hat{A}_{jl}(x)\delta_{ik} - \hat{A}_{jk}(x)\delta_{il} - \hat{A}_{il}(x)\delta_{jk} + \hat{A}_{ik}(x)\delta_{jl})\end{aligned}$$

for all $x \in \partial\mathbb{R}^n_+$ and all $i, j, k, l \in \{1, \dots, n-1\}$. Using Proposition 7 in [4], we conclude that $H_{ik}(x) = 0$ for all $x \in \partial\mathbb{R}^n_+$ and all $i, k \in \{1, \dots, n-1\}$. This implies $A_{ik}(x) = \frac{n-2}{n-3}\hat{A}_{ik}(x) = 0$ for all $x \in \partial\mathbb{R}^n_+$ and all $i, k \in \{1, \dots, n-1\}$.

Proposition 2.3. *Assume that $Z_{ijkl}(x) = 0$ for all $x \in \mathbb{R}^n_+$ and all $i, j, k, l \in \{1, \dots, n\}$. Then $H_{ik}(x) = 0$ for all $x \in \mathbb{R}^n_+$ and all $i, k \in \{1, \dots, n-1\}$.*

Proof. Without loss of generality, we may assume that $H_{ik}(x)$ is homogeneous of degree $d' \geq 2$. By assumption, $Z_{ijkl}(x) = 0$ for all $x \in \mathbb{R}^n_+$ and all $i, j, k, l \in \{1, \ldots, n\}$. This implies

$$\partial_i A_{jk}(x) - \partial_j A_{ik}(x) = \frac{n-2}{n-3} \sum_{l=1}^n \partial_l Z_{ijkl}(x) = 0$$

for all $x \in \mathbb{R}^n_+$ and all $i, j, k \in \{1, \ldots, n\}$. We next define

$$\varphi(x) = \frac{1}{d'(d'-1)} \sum_{i,k=1}^n A_{ik}(x) x_i x_k$$

for all $x \in \mathbb{R}^n_+$. Clearly, $\varphi(x)$ is a homogeneous polynomial of degree d'. Moreover,

$$\partial_k \varphi(x) = \frac{1}{d'-1} \sum_{i=1}^n A_{ik}(x) x_i$$

for all $x \in \mathbb{R}^n_+$ and all $k \in \{1, \ldots, n\}$. This implies

$$\partial_i \partial_k \varphi(x) = A_{ik}(x)$$

for all $x \in \mathbb{R}^n_+$ and all $i, k \in \{1, \ldots, n\}$. Using the identity $\sum_{i=1}^{n-1} H_{ii}(x) = 0$, we obtain

$$\partial_n \partial_n \varphi(x) = A_{nn}(x) = -\frac{1}{n-1} \sum_{i=1}^{n-1} A_{ii}(x) = -\frac{1}{n-1} \sum_{i=1}^{n-1} \partial_i \partial_i \varphi(x)$$

for all $x \in \mathbb{R}^n_+$. By Lemma 2.1, $\partial_n \varphi(x) = \frac{1}{d'-1} \sum_{i=1}^{n-1} A_{in}(x) x_i = 0$ for all $x \in \partial \mathbb{R}^n_+$. Moreover, it follows from Lemma 2.2 that $\varphi(x) = \frac{1}{d'(d'-1)} \sum_{i,k=1}^{n-1} A_{ik}(x) x_i x_k = 0$ for all $x \in \partial \mathbb{R}^n_+$. Putting these facts together, we conclude that $\varphi(x) = 0$ for all $x \in \mathbb{R}^n_+$.

Therefore, $A_{ik}(x) = \partial_i \partial_k \varphi(x) = 0$ for all $x \in \mathbb{R}^n_+$ and all $i, k \in \{1, \ldots, n\}$. This implies

$$\partial_n \partial_n H_{ik}(x) = Z_{inkn}(x) - \frac{1}{n-2} (A_{ik}(x) + A_{nn}(x) \delta_{ik}) = 0$$

for all $x \in \mathbb{R}^n_+$ and all $i, k \in \{1, \ldots, n-1\}$. On the other hand, $H_{ik}(x) = \partial_n H_{ik}(x) = 0$ for all $x \in \partial \mathbb{R}^n_+$ and all $i, k \in \{1, \ldots, n-1\}$. It follows that $H_{ik}(x) = 0$ for all $x \in \mathbb{R}^n_+$ and all $i, k \in \{1, \ldots, n-1\}$. This completes the proof of Proposition 2.3.

For each $r > 0$, we denote by $U_r \subset \mathbb{R}^n_+$ the open ball of radius $r/4$ centered at the point $(0, \ldots, 0, 3r/2)$. Moreover, let $u_\varepsilon : \mathbb{R}^n_+ \to \mathbb{R}$ be defined by (1.1).

Proposition 2.4. *There exists a constant K_1, depending only on n, such that*

$$\sum_{2 \leq |\alpha| \leq d} \sum_{i,k=1}^n |h_{ik,\alpha}|^2 r^{2|\alpha|-4+n} \leq K_1 \int_{U_r} \sum_{i,j,k,l=1}^n |Z_{ijkl}(x)|^2 \, dx$$

for all $r > 0$.

Proof. It follows from Proposition 2.3 that the assertion holds for $r = 1$. The general case follows by scaling.

Proposition 2.5. *Let V be a smooth vector field on $\mathbb{R}^n_+$. Moreover, let*

$$T_{ik} = H_{ik} - \partial_i V_k - \partial_k V_i + \frac{2}{n} \operatorname{div} V \, \delta_{ik}$$

and

$$\begin{aligned} Q_{ik,l} &= u_\varepsilon \partial_l T_{ik} - \frac{2}{n-2} \partial_i u_\varepsilon \, T_{kl} - \frac{2}{n-2} \partial_k u_\varepsilon \, T_{il} \\ &\quad + \frac{2}{n-2} \sum_{p=1}^n \partial_p u_\varepsilon \, T_{ip} \delta_{kl} + \frac{2}{n-2} \sum_{p=1}^n \partial_p u_\varepsilon \, T_{kp} \delta_{il}. \end{aligned}$$

Then there exists a constant K_2, depending only on n, such that

$$\sum_{2 \leq |\alpha| \leq d} \sum_{i,k=1}^n |h_{ik,\alpha}|^2 \varepsilon^{n-2} r^{2|\alpha|+2-n} \leq K_2 \int_{(B_{2r}(0) \setminus B_r(0)) \cap \mathbb{R}^n_+} \sum_{i,k,l=1}^n |Q_{ik,l}(x)|^2 \, dx$$

for all $r \geq \varepsilon$.

Proof. In [4], the first author showed that

$$\begin{aligned} \frac{1}{4} \sum_{i,j,k,l=1}^n |Z_{ijkl}|^2 &= \sum_{i,j,k,l=1}^n \partial_j (u_\varepsilon^{-1} Q_{ik,l}) Z_{ijkl} \\ &\quad + \frac{2}{n-2} \sum_{i,j,k,l=1}^n u_\varepsilon^{-2} \partial_k u_\varepsilon \, Q_{il,j} Z_{ijkl} \end{aligned}$$

(cf. [4, p. 555]). Let us fix a smooth cut-off function $\eta : \mathbb{R}^n \to [0, 1]$ such that $\eta(x) = 1$ for $x \in U_1$ and $\eta(x) = 0$ for $x \notin (B_2(0) \setminus B_1(0)) \cap \mathbb{R}^n_+$. In particular, we have $\eta(x) = 0$ for all $x \in \partial \mathbb{R}^n_+$. Integration by parts gives

$$\begin{aligned} &\int_{\mathbb{R}^n_+} \frac{1}{4} \sum_{i,j,k,l=1}^n |Z_{ijkl}(x)|^2 \eta(x/r) \, dx \\ &= - \int_{\mathbb{R}^n_+} \sum_{i,j,k,l=1}^n u_\varepsilon(x)^{-1} Q_{ik,l}(x) \partial_j [Z_{ijkl}(x) \eta(x/r)] \, dx \\ &\quad + \int_{\mathbb{R}^n_+} \frac{2}{n-2} \sum_{i,j,k,l=1}^n u_\varepsilon(x)^{-2} \partial_k u_\varepsilon(x) \, Q_{il,j}(x) Z_{ijkl}(x) \eta(x/r) \, dx. \end{aligned}$$

Using Hölder's inequality, we obtain

$$\begin{aligned} \int_{U_r} \sum_{i,j,k,l=1}^n |Z_{ijkl}(x)|^2 \, dx &\leq K_3 \varepsilon^{-(n-2)/2} r^{n-3} \Big(\sum_{2 \leq |\alpha| \leq d} \sum_{i,k=1}^n |h_{ik,\alpha}|^2 r^{2|\alpha|-4+n} \Big)^{1/2} \\ &\quad \cdot \Big(\int_{(B_{2r} \setminus B_r(0)) \cap \mathbb{R}^n_+} \sum_{i,k,l=1}^n |Q_{ik,l}(x)|^2 \, dx \Big)^{1/2} \end{aligned}$$

for all $r \geq \varepsilon$. Here, K_3 is a positive constant that depends only on n. On the other hand, it follows from Proposition 2.4 that

$$\sum_{2 \leq |\alpha| \leq d} \sum_{i,k=1}^n |h_{ik,\alpha}|^2 r^{2|\alpha|-4+n} \leq K_1 \int_{U_r} \sum_{i,j,k,l=1}^n |Z_{ijkl}(x)|^2 \, dx.$$

Putting these facts together, the assertion follows.

Corollary 2.6. *Let V be a smooth vector field on $\mathbb{R}^n_+$. Moreover, let*

$$T_{ik} = H_{ik} - \partial_i V_k - \partial_k V_i + \frac{2}{n} \operatorname{div} V \, \delta_{ik}$$

and

$$\begin{aligned} Q_{ik,l} &= u_\varepsilon \partial_l T_{ik} - \frac{2}{n-2} \partial_i u_\varepsilon \, T_{kl} - \frac{2}{n-2} \partial_k u_\varepsilon \, T_{il} \\ &\quad + \frac{2}{n-2} \sum_{p=1}^n \partial_p u_\varepsilon \, T_{ip} \delta_{kl} + \frac{2}{n-2} \sum_{p=1}^n \partial_p u_\varepsilon \, T_{kp} \delta_{il}. \end{aligned}$$

Then there exists a constant K_4, depending only on n, such that

$$\begin{aligned} &\sum_{2 \leq |\alpha| \leq d} \sum_{i,k=1}^n |h_{ik,\alpha}|^2 \varepsilon^{n-2} \int_{B_\delta(0) \cap \mathbb{R}^n_+} (\varepsilon + |x|)^{2|\alpha|+2-2n} \, dx \\ &\qquad \leq K_4 \int_{B_\delta(0) \cap \mathbb{R}^n_+} \sum_{i,k,l=1}^n |Q_{ik,l}(x)|^2 \, dx \end{aligned}$$

for all $\delta \geq 2\varepsilon$.

3. The main estimate

We now describe the construction of the test function. Let (M, g) be a compact Riemannian manifold of dimension $n \geq 6$ with umbilic boundary ∂M. After changing the metric conformally, we may assume that ∂M is totally geodesic.

Let us fix a point $p \in \partial M$, and let $(x_1, \dots, x_n)$ denote the Fermi coordinates around p. In these coordinates, the metric has the following properties:

- At each $x \in \mathbb{R}^n_+$, we have $g_{in}(x) = \delta_{in}$ for all $i \in \{1, \dots, n\}$.
- At each $x \in \partial\mathbb{R}^n_+$, we have $\sum_{k=1}^n g_{ik}(x) x_k = x_i$ for all $i \in \{1, \dots, n\}$.
- At each $x \in \partial\mathbb{R}^n_+$, we have $\partial_n g_{ik}(x) = 0$ for all $i, k \in \{1, \dots, n\}$.

By a theorem of Marques, there exists a system of conformal Fermi coordinates around p (see [11, Proposition 3.1]). Hence, after performing a conformal change of the metric, we may assume that $\det g(x) = 1 + O(|x|^{2d+2})$, where $d = [(n-2)/2]$.

In the next step, we write $g(x) = \exp(h(x))$, where $h(x)$ is a smooth function taking values in the space of symmetric $n \times n$ matrices. This function has the following properties:

- At each $x \in \mathbb{R}^n_+$, we have $h_{in}(x) = 0$ for all $i \in \{1, \dots, n\}$.
- At each $x \in \partial\mathbb{R}^n_+$, we have $\sum_{k=1}^n h_{ik}(x) x_k = 0$ for all $i \in \{1, \dots, n\}$.
- At each $x \in \partial\mathbb{R}^n_+$, we have $\partial_n h_{ik}(x) = 0$ for all $i, k \in \{1, \dots, n\}$.

Moreover, we have $\operatorname{tr} h(x) = O(|x|^{2d+2})$. For abbreviation, we denote by

$$H_{ik}(x) = \sum_{2 \le |\alpha| \le d} h_{ik,\alpha} x^\alpha$$

the Taylor polynomial of order d associated with the function $h_{ik}(x)$. Clearly, $H_{ik}(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n_+$. Moreover, $h_{ik}(x) = H_{ik}(x) + O(|x|^{d+1})$.

Let us fix a non-negative smooth function such that $\chi(t) = 1$ for $t \le 4/3$ and $\chi(t) = 0$ for $t \ge 5/3$. Given any $\delta > 0$, we define a cut-off function $\chi_\delta : \mathbb{R}^n \to \mathbb{R}$ by $\chi_\delta(x) = \chi(|x|/\delta)$. By Theorem A.4, there exists a smooth vector field V on $\mathbb{R}^n_+$ with the following properties:

- At each $x \in \mathbb{R}^n_+$, we have
$$\sum_{k=1}^n \partial_k \Big[u_\varepsilon^{2n/(n-2)} \Big(\chi_\delta H_{ik} - \partial_i V_k - \partial_k V_i + \frac{2}{n} \operatorname{div} V \, \delta_{ik} \Big) \Big] = 0$$
for all $i \in \{1, \dots, n\}$.
- At each $x \in \partial\mathbb{R}^n_+$, we have $V_n(x) = \partial_n V_i(x) = 0$ for all $i \in \{1, \dots, n-1\}$.

By Corollary A.6, the vector field V satisfies the estimate

$$|\partial^\beta V^{(\varepsilon,\delta)}(x)| \le C \sum_{2 \le |\alpha| \le d} \sum_{i,k=1}^n |h_{ik,\alpha}| (\varepsilon + |x|)^{|\alpha|+1-|\beta|} \tag{3.1}$$

for every multi-index β and all $x \in \mathbb{R}^n_+$. Here, C is a positive constant that depends only on n and $|\beta|$.

For abbreviation, we define

$$\begin{aligned} S_{ik} &= \partial_i V_k + \partial_k V_i - \frac{2}{n} \operatorname{div} V \, \delta_{ik}, \\ T_{ik} &= H_{ik} - S_{ik}, \\ Q_{ik,l} &= u_\varepsilon \partial_l T_{ik} - \frac{2}{n-2} \partial_i u_\varepsilon \, T_{kl} - \frac{2}{n-2} \partial_k u_\varepsilon \, T_{il} \\ &\quad + \frac{2}{n-2} \sum_{p=1}^n \partial_p u_\varepsilon \, T_{ip} \delta_{kl} + \frac{2}{n-2} \sum_{p=1}^n \partial_p u_\varepsilon \, T_{kp} \delta_{il}, \\ w &= \sum_{l=1} \partial_l u_\varepsilon \, V_l + \frac{n-2}{2n} u_\varepsilon \operatorname{div} V. \end{aligned}$$

By definition of V, we have

$$\sum_{k=1}^n \partial_k (u_\varepsilon^{2n/(n-2)} T_{ik}) = 0 \tag{3.2}$$

for all $x \in B_\delta(0) \cap \mathbb{R}^n_+$ and all $i \in \{1, \dots, n\}$. This implies

$$\sum_{k=1}^n \Big(u_\varepsilon \partial_k T_{ik} + \frac{2n}{n-2} \partial_k u_\varepsilon \, T_{ik} \Big) = 0 \tag{3.3}$$

for all $x \in B_\delta(0) \cap \mathbb{R}^n_+$ and all $i \in \{1, \dots, n\}$. The following result was established in [4]:

Proposition 3.1 (S. Brendle [4]). *There exists a smooth vector field ξ on $\mathbb{R}^n_+$ such that*

$$\begin{aligned}
&\frac{1}{4} u_\varepsilon^2 \sum_{i,k,l=1}^n \partial_l H_{ik} \, \partial_l H_{ik} - \frac{1}{2} u_\varepsilon^2 \sum_{i,k,l=1}^n \partial_k H_{ik} \, \partial_l H_{il} \\
&\qquad - 2u_\varepsilon \sum_{i,k,l=1}^n \partial_k u_\varepsilon \, H_{ik} \partial_l H_{il} - \frac{2(n-1)}{n-2} \sum_{i,k,l=1}^n \partial_k u_\varepsilon \, \partial_l u_\varepsilon \, H_{ik} H_{il} \\
&\qquad - 2u_\varepsilon w \sum_{i,k=1}^n \partial_i \partial_k H_{ik} + \frac{8(n-1)}{n-2} \sum_{i,k=1}^n \partial_i u_\varepsilon \, \partial_k w \, H_{ik} \\
&\qquad - \frac{4(n-1)}{n-2} |dw|^2 + \frac{4(n-1)}{n-2} n(n+2) u_\varepsilon^{4/(n-2)} w^2 \\
&\quad = \frac{1}{4} \sum_{i,k,l=1}^n Q_{ik,l} Q_{ik,l} + 2u_\varepsilon^{2n/(n-2)} \sum_{i,k=1}^n T_{ik} T_{ik} + \operatorname{div} \xi
\end{aligned}$$

for all $x \in B_\delta(0) \cap \mathbb{R}^n_+$.

The vector field ξ can be expressed in terms of the tensor H_{ik} and the vector field V (cf. [4, Section 2]). In the next step, we show that ξ is tangential along $\partial \mathbb{R}^n_+$. To that end, we need the following lemma:

Lemma 3.2. *At each $x \in B_\delta(0) \cap \partial \mathbb{R}^n_+$, we have*

$$S_{in}(x) = T_{in}(x) = 0 \quad \text{and} \quad \partial_n S_{ik}(x) = \partial_n T_{ik}(x) = 0$$

for all $i, k \in \{1, \dots, n-1\}$. Moreover, $\partial_n S_{nn}(x) = \partial_n T_{nn}(x) = 0$ and $\partial_n w(x) = 0$.

Proof. By assumption, we have $V_n(x) = \partial_n V_i(x) = 0$ for all $x \in \partial \mathbb{R}^n_+$. This implies $S_{in}(x) = T_{in}(x) = 0$ for all $i \in \{1, \dots, n-1\}$. It follows that

$$\sum_{k=1}^{n-1} \Big(u_\varepsilon(x) \partial_k T_{kn}(x) + \frac{2n}{n-2} \partial_k u_\varepsilon(x) \, T_{kn}(x) \Big) = 0.$$

Using (3.3), we obtain

$$u_\varepsilon(x) \partial_n T_{nn}(x) + \frac{2n}{n-2} \partial_n u_\varepsilon(x) \, T_{nn}(x) = 0.$$

This implies $\partial_n T_{nn}(x) = 0$, hence $\partial_n S_{nn}(x) = 0$. Consequently, we have $\partial_n \partial_n V(x) = 0$. From this, the assertion follows easily.

Lemma 3.3. *We have $\xi_n(x) = 0$ for all $x \in B_\delta(0) \cap \partial \mathbb{R}^n_+$.*

Proof. The vector field ξ satisfies

$$\begin{aligned}
\xi_n &= -2\sum_{k=1}^n u_\varepsilon w\,\partial_k H_{nk} + 2\sum_{i=1}^n \partial_i(u_\varepsilon w)\,H_{in} \\
&\quad + \frac{1}{2}u_\varepsilon^2 \sum_{i,k=1}^n \partial_n S_{ik}\,H_{ik} - u_\varepsilon^2 \sum_{i,l=1}^n \partial_l S_{il}\,H_{in} - 2u_\varepsilon \sum_{i,l=1}^n \partial_l u_\varepsilon\,S_{il}\,H_{in} \\
&\quad + u_\varepsilon w \sum_{k=1}^n \partial_k S_{nk} - \sum_{i=1}^n \partial_i(u_\varepsilon w)\,S_{in} \\
&\quad - \frac{1}{4}u_\varepsilon^2 \sum_{i,k=1}^n \partial_n S_{ik}\,S_{ik} + \frac{1}{2}u_\varepsilon^2 \sum_{i,l=1}^n \partial_l S_{il}\,S_{in} + u_\varepsilon \sum_{i,l=1}^n \partial_l u_\varepsilon\,S_{il}\,S_{in} \\
&\quad + \frac{4(n-1)}{n-2}\sum_{i=1}^n \partial_i u_\varepsilon\,w S_{in} - \frac{4(n-1)}{n-2} w\,\partial_n w + \frac{2}{n-2}u_\varepsilon \sum_{i,k=1}^n \partial_k u_\varepsilon\,T_{ik}T_{in}
\end{aligned}$$

(see [4, Section 2]). Using Lemma 3.2, we conclude that $\xi_n(x) = 0$ for all $x \in B_\delta(0) \cap \partial\mathbb{R}^n_+$.

Proposition 3.4. *We have*

$$\begin{aligned}
&\int_{B_\delta(0)\cap\mathbb{R}^n_+} \bigg[\frac{1}{4}u_\varepsilon^2 \sum_{i,k,l=1}^n \partial_l H_{ik}\,\partial_l H_{ik} - \frac{1}{2}u_\varepsilon^2 \sum_{i,k,l=1}^n \partial_k H_{ik}\,\partial_l H_{il}\bigg] \\
&\quad - \int_{B_\delta(0)\cap\mathbb{R}^n_+} \bigg[2u_\varepsilon \sum_{i,k,l=1}^n \partial_k u_\varepsilon\,H_{ik}\,\partial_l H_{il} + \frac{2(n-1)}{n-2}\sum_{i,k,l=1}^n \partial_k u_\varepsilon\,\partial_l u_\varepsilon\,H_{ik}H_{il}\bigg] \\
&\quad - \int_{B_\delta(0)\cap\mathbb{R}^n_+} \bigg[2u_\varepsilon w \sum_{i,k=1}^n \partial_i\partial_k H_{ik} - \frac{8(n-1)}{n-2}\sum_{i,k=1}^n \partial_i u_\varepsilon\,\partial_k w\,H_{ik}\bigg] \\
&\quad - \int_{B_\delta(0)\cap\mathbb{R}^n_+} \bigg[\frac{4(n-1)}{n-2}|dw|^2 - \frac{4(n-1)}{n-2}n(n+2)u_\varepsilon^{4/(n-2)}w^2\bigg] \\
&\geq 2\lambda \sum_{2\leq|\alpha|\leq d}\sum_{i,k=1}^n |h_{ik,\alpha}|^2 \varepsilon^{n-2} \int_{B_\delta(0)\cap\mathbb{R}^n_+} (\varepsilon+|x|)^{2|\alpha|+2-2n}\,dx \\
&\quad - C \sum_{2\leq|\alpha|\leq d}\sum_{i,k=1}^n |h_{ik,\alpha}|^2 \delta^{2|\alpha|+2-n}\varepsilon^{n-2}.
\end{aligned}$$

for $\delta \geq 2\varepsilon$. Here, λ and C are positive constants that depend only on n.

Proof. We consider the identity in Proposition 3.1 and integrate over $B_\delta(0) \cap \mathbb{R}^n_+$. By Corollary 2.6, we have

$$\begin{aligned}
&\int_{B_\delta(0)\cap\mathbb{R}^n_+} \sum_{i,k,l=1}^n Q_{ik,l}Q_{ik,l} \\
&\qquad\qquad \geq 8\lambda \sum_{2\leq|\alpha|\leq d}\sum_{i,k=1}^n |h_{ik,\alpha}|^2 \varepsilon^{n-2} \int_{B_\delta(0)\cap\mathbb{R}^n_+} (\varepsilon+|x|)^{2|\alpha|+3-2n}\,dx,
\end{aligned}$$

where $\lambda = 1/(8K_4)$ is a positive constant that depends only on n. Moreover, it follows from 3.1 that

$$|\xi(x)| \leq C\varepsilon^{n-2} \sum_{2\leq|\alpha|\leq d} \sum_{i,k=1}^{n} |h_{ik,\alpha}|(\varepsilon+|x|)^{2|\alpha|+3-2n}$$

for all $x \in \mathbb{R}^n_+$. Using Lemma 3.3 and the divergence theorem, we obtain

$$\begin{aligned}\int_{B_\delta(0)\cap\mathbb{R}^n_+} \operatorname{div}\xi &= \int_{\partial B_\delta(0)\cap\mathbb{R}^n_+} \sum_{i=1}^{n} \frac{x_i}{|x|}\xi_i - \int_{B_\delta(0)\cap\partial\mathbb{R}^n_+} \xi_n \\ &\leq C\varepsilon^{n-2} \sum_{2\leq|\alpha|\leq d} \sum_{i,k=1}^{n} |h_{ik,\alpha}|\delta^{2|\alpha|+2-n}.\end{aligned}$$

Putting these facts together yields the assertion.

Finally, we need the following estimate for the scalar curvature R_g.

Proposition 3.5. *The scalar curvature R_g satisfies the estimates*

$$\Big|R_g - \sum_{i,k=1}^{n} \partial_i\partial_k H_{ik}\Big| \leq C \sum_{2\leq|\alpha|\leq d} \sum_{i,k=1}^{n} |h_{ik,\alpha}|\,|x|^{|\alpha|} + C|x|^{d-1} \tag{3.4}$$

and

$$\begin{aligned}&\Big|R_g - \sum_{i,k=1}^{n} \partial_i\partial_k h_{ik} + \sum_{i,k,l=1}^{n} \partial_k(H_{ik}\partial_l H_{il}) - \frac{1}{2}\sum_{i,k,l=1}^{n} \partial_k H_{ik}\,\partial_l H_{il} + \frac{1}{4}\sum_{i,k,l=1}^{n} \partial_l H_{ik}\,\partial_l H_{ik}\Big| \\ &\leq C \sum_{2\leq|\alpha|\leq d} \sum_{i,k=1}^{n} |h_{ik,\alpha}|^2|x|^{2|\alpha|} + C \sum_{2\leq|\alpha|\leq d} \sum_{i,k=1}^{n} |h_{ik,\alpha}|\,|x|^{|\alpha|+d-1} + C|x|^{2d} \end{aligned}\tag{3.5}$$

if $|x|$ is sufficiently small.

Proof. This follows easily from [5, Proposition 25] (see also [4, Corollary 12], where geodesic normal coordinates are considered).

Our goal is to estimate the Yamabe energy of $u_\varepsilon + w$. To that end, we proceed in several steps:

Proposition 3.6. *There exist positive constants λ, C, δ_0 such that*

$$\begin{aligned}&\int_{B_\delta(0)\cap\mathbb{R}^n_+} \Big(\frac{4(n-1)}{n-2}|d(u_\varepsilon+w)|_g^2 + R_g(u_\varepsilon+w)^2\Big) \\ &\leq \int_{B_\delta(0)\cap\mathbb{R}^n_+} \frac{4(n-1)}{n-2}|du_\varepsilon|^2 + \int_{B_\delta(0)\cap\mathbb{R}^n_+} \frac{4(n-1)}{n-2} n(n+2)u_\varepsilon^{4/(n-2)}w^2 \\ &\quad + \int_{\partial B_\delta(0)\cap\mathbb{R}^n_+} \sum_{i,k=1}^{n} \frac{x_i}{|x|}(u_\varepsilon^2\partial_k h_{ik} - 2u_\varepsilon\partial_k u_\varepsilon\, h_{ik}) \\ &\quad - \lambda \sum_{2\leq|\alpha|\leq d} \sum_{i,k=1}^{n} |h_{ik,\alpha}|^2\varepsilon^{n-2} \int_{B_\delta(0)\cap\mathbb{R}^n_+} (\varepsilon+|x|)^{2|\alpha|+2-2n}\,dx \\ &\quad + C \sum_{2\leq|\alpha|\leq d} \sum_{i,k=1}^{n} |h_{ik,\alpha}|\delta^{|\alpha|+2-n}\varepsilon^{n-2} + C\delta^{2d+4-n}\varepsilon^{n-2}\end{aligned}$$

if $0 < 2\varepsilon \leq \delta \leq \delta_0$. *The constant* λ *depends only on* n. *The constants* C, δ_0 *depend on the underlying manifold* (M, g).

Proof. Let us write

$$\begin{aligned}
&\frac{4(n-1)}{n-2}|d(u_\varepsilon + w)|_g^2 + R_g(u_\varepsilon + w)^2 \\
&\qquad = \frac{4(n-1)}{n-2}|du_\varepsilon|^2 + \frac{4(n-1)}{n-2}n(n+2)u_\varepsilon^{4/(n-2)}w^2 \\
&\qquad\quad + \frac{8(n-1)}{n-2}J^{(1)} + J^{(2)} + J^{(3)} + J^{(4)} + J^{(5)} + J^{(6)} + J^{(7)},
\end{aligned}$$

where

$$\begin{aligned}
J^{(1)} &= \sum_{i=1}^n \partial_i u_\varepsilon \partial_i w, \\
J^{(2)} &= -\frac{4(n-1)}{n-2}\sum_{i,k=1}^n \partial_i u_\varepsilon\, \partial_k u_\varepsilon\, h_{ik} + u_\varepsilon^2 \sum_{i,k=1}^n \partial_i\partial_k h_{ik}, \\
J^{(3)} &= -u_\varepsilon^2 \sum_{i,k,l=1}^n \partial_k(H_{ik}\partial_l H_{il}) - 2u_\varepsilon \sum_{i,k,l=1}^n \partial_k u_\varepsilon H_{ik}\, \partial_l H_{il}, \\
J^{(4)} &= -\frac{1}{4}\sum_{i,k,l=1}^n u_\varepsilon^2 \partial_l H_{ik}\, \partial_l H_{ik} + \frac{1}{2}\sum_{i,k,l=1}^n u_\varepsilon^2 \partial_k H_{ik}\, \partial_l H_{il} \\
&\quad + 2u_\varepsilon \sum_{i,k,l=1}^n \partial_k u_\varepsilon\, H_{ik}\partial_l H_{il} + \frac{2(n-1)}{n-2}\sum_{i,k,l=1}^n \partial_k u_\varepsilon\, \partial_l u_\varepsilon\, H_{ik} H_{il} \\
&\quad + 2u_\varepsilon w \sum_{i,k=1}^n \partial_i\partial_k H_{ik} - \frac{8(n-1)}{n-2}\sum_{i,k=1}^n \partial_i u_\varepsilon\, \partial_k w\, H_{ik} \\
&\quad + \frac{4(n-1)}{n-2}|dw|^2 - \frac{4(n-1)}{n-2}n(n+2)u_\varepsilon^{4/(n-2)}w^2, \\
J^{(5)} &= \frac{4(n-1)}{n-2}\sum_{i,k=1}^n \Big[g^{ik} - \delta_{ik} + h_{ik} - \frac{1}{2}\sum_{l=1}^n H_{il}H_{kl}\Big]\partial_i u_\varepsilon\, \partial_k u_\varepsilon \\
&\quad + \Big[R_g - \sum_{i,k=1}^n \partial_i\partial_k h_{ik} + \sum_{i,k,l=1}^n \partial_k(H_{ik}\, \partial_l H_{il}) \\
&\quad - \frac{1}{2}\sum_{i,k,l=1}^n \partial_k H_{ik}\, \partial_l H_{il} + \frac{1}{4}\sum_{i,k,l=1}^n \partial_l H_{ik}\, \partial_l H_{ik}\Big]u_\varepsilon^2, \\
J^{(6)} &= \frac{8(n-1)}{n-2}\sum_{i,k=1}^n (g^{ik} - \delta_{ik} + H_{ik})\partial_i u_\varepsilon\, \partial_k w + 2\Big[R_g - \sum_{i,k=1}^n \partial_i\partial_k H_{ik}\Big]u_\varepsilon w, \\
J^{(7)} &= R_g w^2 + \frac{4(n-1)}{n-2}\sum_{i,k=1}^n (g^{ik} - \delta_{ik})\partial_i w\, \partial_k w.
\end{aligned}$$

It follows from the divergence theorem that

$$\begin{aligned}\int_{B_\delta(0)\cap\mathbb{R}^n_+} J^{(1)} &= \int_{B_\delta(0)\cap\mathbb{R}^n_+} \sum_{i=1}^n \partial_i \bigg[\partial_i u_\varepsilon w + \frac{(n-2)^2}{2} u_\varepsilon^{2n/(n-2)} V_i \bigg] \\ &= \int_{\partial B_\delta(0)\cap\mathbb{R}^n_+} \sum_{i=1}^n \frac{x_i}{|x|} \bigg[\partial_i u_\varepsilon w + \frac{(n-2)^2}{2} u_\varepsilon^{2n/(n-2)} V_i \bigg] \\ &\leq C \sum_{2\leq|\alpha|\leq d} \sum_{i,k=1}^n |h_{ik,\alpha}| \delta^{|\alpha|+2-n} \varepsilon^{n-2}.\end{aligned}$$

We next observe that

$$\begin{aligned}J^{(2)} - \sum_{i,k=1}^n \partial_i (u_\varepsilon^2 \partial_k h_{ik} - 2u_\varepsilon \partial_k u_\varepsilon \, h_{ik}) &= 2 \sum_{i,k=1}^n \bigg(u_\varepsilon \partial_i \partial_k u_\varepsilon - \frac{n}{n-2} \partial_i u_\varepsilon \, \partial_k u_\varepsilon \bigg) h_{ik} \\ &= \frac{2}{n} \bigg(u_\varepsilon \Delta u_\varepsilon - \frac{n}{n-2} |du_\varepsilon|^2 \bigg) \operatorname{tr} h \\ &\leq C \varepsilon^{n-2} (\varepsilon + |x|)^{2d+4-2n}.\end{aligned}$$

Using the divergence theorem, we obtain

$$\begin{aligned}\int_{B_\delta(0)\cap\mathbb{R}^n_+} J^{(2)} &\leq \sum_{i,k=1}^n \partial_i (u_\varepsilon^2 \partial_k h_{ik} - 2u_\varepsilon \partial_k u_\varepsilon \, h_{ik}) + C\delta^{2d+4-n} \varepsilon^{n-2} \\ &\leq \int_{\partial B_\delta(0)\cap\mathbb{R}^n_+} \sum_{i,k=1}^n \frac{x_i}{|x|} (u_\varepsilon^2 \partial_k h_{ik} - 2u_\varepsilon \partial_k u_\varepsilon \, h_{ik}) + C\delta^{2d+4-n} \varepsilon^{n-2}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\int_{B_\delta(0)\cap\mathbb{R}^n_+} J^{(3)} &= -\int_{B_\delta(0)\cap\mathbb{R}^n_+} \sum_{i,k,l=1}^n \partial_k (u_\varepsilon^2 H_{ik} \partial_l H_{il}) \\ &= -\int_{\partial B_\delta(0)\cap\mathbb{R}^n_+} \sum_{i,k,l=1}^n \frac{x_k}{|x|} u_\varepsilon^2 H_{ik} \partial_l H_{il} \\ &\leq C \sum_{2\leq|\alpha|\leq d} \sum_{i,k=1}^n |h_{ik,\alpha}|^2 \delta^{2|\alpha|+2-n} \varepsilon^{n-2}.\end{aligned}$$

Using Proposition 3.4, we obtain

$$\begin{aligned}\int_{B_\delta(0)\cap\mathbb{R}^n_+} J^{(4)} \leq &-2\lambda \sum_{2\leq|\alpha|\leq d} \sum_{i,k=1}^n |h_{ik,\alpha}|^2 \varepsilon^{n-2} \int_{B_\delta(0)\cap\mathbb{R}^n_+} (\varepsilon + |x|)^{2|\alpha|+2-2n} \, dx \\ &+ C \sum_{2\leq|\alpha|\leq d} \sum_{i,k=1}^n |h_{ik,\alpha}|^2 \delta^{2|\alpha|+2-n} \varepsilon^{n-2}.\end{aligned}$$

It remains to estimate the terms $J^{(5)}$, $J^{(6)}$, and $J^{(7)}$. Using Proposition 3.5, we obtain the pointwise estimate

$$\begin{aligned} J^{(5)} + J^{(6)} + J^{(7)} \leq C \sum_{2 \leq |\alpha| \leq d} \sum_{i,k=1}^{n} |h_{ik,\alpha}|^2 \varepsilon^{n-2} (\varepsilon + |x|)^{2|\alpha|+4-2n} \\ + C \sum_{2 \leq |\alpha| \leq d} \sum_{i,k=1}^{n} |h_{ik,\alpha}| \varepsilon^{n-2} (\varepsilon + |x|)^{|\alpha|+d+3-2n} \\ + C \varepsilon^{n-2} (\varepsilon + |x|)^{2d+4-2n} \end{aligned}$$

for $x \in B_\delta(0) \cap \mathbb{R}^n_+$. Using Young's inequality, we deduce that

$$\begin{aligned} J^{(5)} + J^{(6)} + J^{(7)} \leq \lambda \sum_{2 \leq |\alpha| \leq d} \sum_{i,k=1}^{n} |h_{ik,\alpha}|^2 \varepsilon^{n-2} (\varepsilon + |x|)^{2|\alpha|+2-2n} \\ + C \varepsilon^{n-2} (\varepsilon + |x|)^{2d+4-2n} \end{aligned}$$

for $x \in B_\delta(0) \cap \mathbb{R}^n_+$. Integration over $B_\delta(0) \cap \mathbb{R}^n_+$ yields

$$\begin{aligned} &\int_{B_\delta(0) \cap \mathbb{R}^n_+} (J^{(5)} + J^{(6)} + J^{(7)}) \\ &\leq \lambda \sum_{2 \leq |\alpha| \leq d} \sum_{i,k=1}^{n} |h_{ik,\alpha}|^2 \varepsilon^{n-2} \int_{B_\delta(0) \cap \mathbb{R}^n_+} (\varepsilon + |x|)^{2|\alpha|+2-2n} \, dx + C \delta^{2d+4-n} \varepsilon^{n-2}. \end{aligned}$$

Putting these facts together gives the assertion.

Proposition 3.7. *If δ_0 is sufficiently small, then*

$$\begin{aligned} &\int_{B_\delta(0) \cap \mathbb{R}^n_+} \left(u_\varepsilon^2 + \frac{n+2}{n-2} w^2 \right)^{n/(n-2)} \\ &\leq \int_{B_\delta(0) \cap \mathbb{R}^n_+} (u_\varepsilon + w)^{2n/(n-2)} \\ &\quad + C \sum_{2 \leq |\alpha| \leq d} \sum_{i,k=1}^{n} |h_{ik,\alpha}|^2 \varepsilon^n \int_{B_\delta(0) \cap \mathbb{R}^n_+} (\varepsilon + |x|)^{2|\alpha|+2-2n} \, dx \\ &\quad + C \sum_{2 \leq |\alpha| \leq d} \sum_{i,k=1}^{n} |h_{ik,\alpha}| \delta^{|\alpha|-n} \varepsilon^n \end{aligned}$$

for all $0 < 2\varepsilon \leq \delta \leq \delta_0$.

Proof. The proof is analogous to the proof of [4, Proposition 14]. We omit the details.

Proposition 3.8. *If δ_0 is sufficiently small, then*

$$\begin{aligned}\int_{B_\delta(0)\cap\mathbb{R}^n_+} \frac{4(n-1)}{n-2}|du_\varepsilon|^2 + \int_{B_\delta(0)\cap\mathbb{R}^n_+} \frac{4(n-1)}{n-2} n(n+2)u_\varepsilon^{4/(n-2)} w^2 \\ \leq Y(S^n_+, \partial S^n_+)\Big(\int_{B_\delta(0)\cap\mathbb{R}^n_+} (u_\varepsilon + w)^{2n/(n-2)}\Big)^{n-2/(n)} \\ + \int_{\partial B_\delta(0)\cap\mathbb{R}^n_+} \frac{4(n-1)}{n-2} \sum_{i=1}^n \frac{x_i}{|x|} \partial_i u_\varepsilon \, u_\varepsilon \\ + C \sum_{2\leq|\alpha|\leq d} \sum_{i,k=1}^n |h_{ik,\alpha}|^2 \varepsilon^n \int_{B_\delta(0)\cap\mathbb{R}^n_+} (\varepsilon + |x|)^{2|\alpha|+2-2n}\, dx \\ + C \sum_{2\leq|\alpha|\leq d} \sum_{i,k=1}^n |h_{ik,\alpha}| \delta^{|\alpha|-n} \varepsilon^n\end{aligned}$$

for all $0 < 2\varepsilon \leq \delta \leq \delta_0$.

Proof. The proof is similar to the proof of [4, Proposition 15]. We first observe that

$$4n(n-1)\Big(\int_{\mathbb{R}^n_+} u_\varepsilon^{2n/(n-2)}\Big)^{2/n} = Y(S^n_+, \partial S^n_+).$$

Using Hölder's inequality, we obtain

$$\begin{aligned}\int_{B_\delta(0)\cap\mathbb{R}^n_+} \frac{4(n-1)}{n-2}|du_\varepsilon|^2 - \int_{\partial B_\delta(0)} \frac{4(n-1)}{n-2} \sum_{i=1}^n \frac{x_i}{|x|} \partial_i u_\varepsilon \, u_\varepsilon \\ + \int_{B_\delta(0)\cap\mathbb{R}^n_+} \frac{4(n-1)}{n-2} n(n+2) u_\varepsilon^{4/(n-2)} w^2 \\ = -\int_{B_\delta(0)\cap\mathbb{R}^n_+} \frac{4(n-1)}{n-2} \Delta u_\varepsilon \, u_\varepsilon + \int_{B_\delta(0)\cap\mathbb{R}^n_+} \frac{4(n-1)}{n-2} n(n+2) u_\varepsilon^{4/(n-2)} w^2 \\ = \int_{B_\delta(0)\cap\mathbb{R}^n_+} 4n(n-1) u_\varepsilon^{4/(n-2)} \Big(u_\varepsilon^2 + \frac{n+2}{n-2} w^2\Big) \\ \leq Y(S^n_+, \partial S^n_+)\Big(\int_{B_\delta(0)\cap\mathbb{R}^n_+} \Big(u_\varepsilon^2 + \frac{n+2}{n-2} w^2\Big)^{n/(n-2)}\Big)^{(n-2)/n}.\end{aligned}$$

Hence, the assertion follows from Proposition 3.7.

4. Proof of the main result

In this section, we construct a smooth function $v_{(\varepsilon,\delta)} : M \to \mathbb{R}$ with Yamabe energy less than $Y(S^n_+, \partial S^n_+)$. The existence of such a function is trivial when $Y(M, \partial M, g) \leq 0$.

Hence, it suffices to consider the case $Y(M, \partial M, g) > 0$. As in the previous section, we fix a boundary point $p \in \partial M$. Moreover, we denote by $G : M \setminus \{p\} \to \mathbb{R}$ the Green's function for the conformal Laplacian with Neumann boundary condition with pole at p. In other words, G satisfies

$$\frac{4(n-1)}{n-2} \Delta_g G - R_g G = 0$$

in $M \setminus \{p\}$ and $\partial_\nu G = 0$ along $\partial M \setminus \{p\}$. We assume that $G_p(x)$ is normalized so that $\lim_{x \to 0} |x|^{n-2} G(x) = 1$. With this normalization, we have

$$\big| G(x) - |x|^{2-n} \big| \leq C \sum_{2 \leq |\alpha| \leq d} \sum_{i,k=1}^n |h_{ik,\alpha}| \, |x|^{|\alpha|+2-n} + C|x|^{d+3-n}. \tag{4.1}$$

Moreover, we consider the flux integral

$$\begin{aligned} \mathcal{I}(p, \delta) &= \int_{\partial B_\delta(0) \cap \mathbb{R}^n_+} \frac{4(n-1)}{n-2} \sum_{i=1}^n \frac{x_i}{|x|} (|x|^{2-n} \partial_i G - G \partial_i |x|^{2-n}) \\ &\quad - \int_{\partial B_\delta(0) \cap \mathbb{R}^n_+} |x|^{1-2n} \sum_{i,k=1}^n x_i (|x|^2 \partial_k h_{ik} - 2n x_k h_{ik}), \end{aligned}$$

where $\delta > 0$ is sufficiently small.

We next define a function $v_{(\varepsilon,\delta)} : M \to \mathbb{R}$ by

$$v_{(\varepsilon,\delta)} = \chi_\delta (u_\varepsilon + w) + (1 - \chi_\delta) \varepsilon^{(n-2)/2} G, \tag{4.2}$$

where χ_δ is the cut-off function defined above. Our main result is an upper bound for the Yamabe energy of $v_{(\varepsilon,\delta)}$:

Proposition 4.1. *If δ_0 is sufficiently small, then*

$$\begin{aligned} &\int_M \bigg(\frac{4(n-1)}{n-2} |dv_{(\varepsilon,\delta)}|_g^2 + R_g v_{(\varepsilon,\delta)}^2 \bigg) \, d\mathrm{vol}_g \\ &\leq Y(S^n_+, \partial S^n_+) \bigg(\int_M v_{(\varepsilon,\delta)}^{2n/(n-2)} \, d\mathrm{vol}_g \bigg)^{(n-2)/n} - \varepsilon^{n-2} \mathcal{I}(p, \delta) \\ &\quad - \frac{\lambda}{2} \sum_{2 \leq |\alpha| \leq d} \sum_{i,k=1}^n |h_{ik,\alpha}|^2 \varepsilon^{n-2} \int_{B_\delta(0) \cap \mathbb{R}^n_+} (\varepsilon + |x|)^{2|\alpha|+2-2n} \, dx \\ &\quad + C \sum_{2 \leq |\alpha| \leq d} \sum_{i,k=1}^n |h_{ik,\alpha}| \delta^{|\alpha|+2-n} \varepsilon^{n-2} + C \delta^{2d+4-n} \varepsilon^{n-2} + C \delta^{-n} \varepsilon^n \end{aligned}$$

for all $0 < 2\varepsilon \leq \delta \leq \delta_0$.

Proof. For abbreviation, we denote by Ω_δ the set of all points in M such that $x_1^2 + \cdots + x_n^2 < \delta^2$, where $(x_1, \dots, x_n)$ denote the Fermi coordinates around p. (In other words, Ω_δ is a coordinate ball, not a geodesic ball.) Using the divergence theorem, we obtain

$$\begin{aligned}\int_{M\setminus\Omega_\delta} &\Big(\frac{4(n-1)}{n-2}|dv_{(\varepsilon,\delta)}|_g^2 + R_g v_{(\varepsilon,\delta)}^2\Big)\, d\mathrm{vol}_g \\ &= -\int_{M\setminus\Omega_\delta}\Big(\frac{4(n-1)}{n-2}\Delta_g v_{(\varepsilon,\delta)} - R_g v_{(\varepsilon,\delta)}\Big)(v_{(\varepsilon,\delta)} - \varepsilon^{(n-2)/2}G)\, d\mathrm{vol}_g \\ &\quad - \int_{\partial\Omega_\delta}\frac{4(n-1)}{n-2}\partial_\nu v_{(\varepsilon,\delta)}\, v_{(\varepsilon,\delta)}\, d\sigma_g \\ &\quad - \int_{\partial\Omega_\delta}\frac{4(n-1)}{n-2}\varepsilon^{(n-2)/2}(v_{(\varepsilon,\delta)}\partial_\nu G - G\partial_\nu v_{(\varepsilon,\delta)})\, d\sigma_g,\end{aligned}$$

where ν denotes the outward-pointing unit normal to $\partial\Omega_\delta$. Note that

$$v_{(\varepsilon,\delta)} - \varepsilon^{(n-2)/2}G = \chi_\delta(u_\varepsilon + w - \varepsilon^{(n-2)/2}G)$$

in $M \setminus \Omega_\delta$. In particular, $v_{(\varepsilon,\delta)} - \varepsilon^{(n-2)/2}G = 0$ in $M \setminus \Omega_{2\delta}$. Using (4.1), we obtain

$$\begin{aligned}\sup_{M\setminus\Omega_\delta} &|v_{(\varepsilon,\delta)} - \varepsilon^{(n-2)/2}G| + \delta^2 \sup_{M\setminus\Omega_\delta}\Big|\frac{4(n-1)}{n-2}\Delta_g v_{(\varepsilon,\delta)} - R_g v_{(\varepsilon,\delta)}\Big| \\ &\le C\sum_{2\le|\alpha|\le d}\sum_{i,k=1}^n |h_{ik,\alpha}|\delta^{|\alpha|+2-n}\varepsilon^{(n-2)/2} + C\delta^{d+3-n}\varepsilon^{(n-2)/2} + C\delta^{-n}\varepsilon^{(n+2)/2},\end{aligned}$$

hence

$$\begin{aligned}-\int_{M\setminus\Omega_\delta} &\Big(\frac{4(n-1)}{n-2}\Delta_g v_{(\varepsilon,\delta)} - R_g v_{(\varepsilon,\delta)}\Big)(v_{(\varepsilon,\delta)} - \varepsilon^{(n-2)/2}G)\, d\mathrm{vol}_g \\ &\le C\sum_{2\le|\alpha|\le d}\sum_{i,k=1}^n |h_{ik,\alpha}|^2\delta^{2|\alpha|+2-n}\varepsilon^{n-2} + C\delta^{2d+4-n}\varepsilon^{n-2} + C\delta^{-n-2}\varepsilon^{n+2}.\end{aligned}$$

We next observe that

$$\begin{aligned}-\int_{\partial\Omega_\delta}\partial_\nu u_\varepsilon\, u_\varepsilon\, d\sigma_g &\le -\int_{\partial B_\delta(0)\cap\mathbb{R}^n_+}\sum_{i=1}^n\frac{x_i}{|x|}\partial_i u_\varepsilon\, u_\varepsilon + \int_{\partial B_\delta(0)\cap\mathbb{R}^n_+}\sum_{i=1}^n\frac{x_i}{|x|}u_\varepsilon\partial_k u_\varepsilon\, h_{ik} \\ &\quad + C\sum_{2\le|\alpha|\le d}\sum_{i,k=1}^n|h_{ik,\alpha}|^2\delta^{2|\alpha|+2-n}\varepsilon^{n-2} + C\delta^{2d+4-n}\varepsilon^{n-2},\end{aligned}$$

hence

$$\begin{aligned}-\int_{\partial\Omega_\delta}\partial_\nu v_{(\varepsilon,\delta)}\, v_{(\varepsilon,\delta)}\, d\sigma_g &\le -\int_{\partial B_\delta(0)\cap\mathbb{R}^n_+}\sum_{i=1}^n\frac{x_i}{|x|}\partial_i u_\varepsilon u_\varepsilon + \int_{\partial B_\delta(0)\cap\mathbb{R}^n_+}\sum_{i=1}^n\frac{x_i}{|x|}u_\varepsilon\partial_k u_\varepsilon\, h_{ik} \\ &\quad + C\sum_{2\le|\alpha|\le d}\sum_{i,k=1}^n|h_{ik,\alpha}|\delta^{|\alpha|+2-n}\varepsilon^{n-2} + C\delta^{2d+4-n}\varepsilon^{n-2}.\end{aligned}$$

Moreover, we have

$$-\int_{\partial\Omega_\delta}(v_{(\varepsilon,\delta)}\partial_\nu G - G\partial_\nu v_{(\varepsilon,\delta)})\,d\sigma_g \leq -\int_{\partial B_\delta(0)\cap\mathbb{R}^n_+}\sum_{i=1}^n \frac{x_i}{|x|}(u_\varepsilon\partial_i G - G\partial_i u_\varepsilon) + C\sum_{2\leq|\alpha|\leq d}|h_{ik,\alpha}|^2\delta^{2|\alpha|+2-n}\varepsilon^{(n-2)/2} + C\delta^{2d+4-n}\varepsilon^{(n-2)/2}.$$

Putting these facts together, we obtain

$$\begin{aligned}
&\int_{M\setminus\Omega_\delta}\Big(\frac{4(n-1)}{n-2}|dv_{(\varepsilon,\delta)}|_g^2 + R_g v_{(\varepsilon,\delta)}^2\Big)\,d\mathrm{vol}_g \\
&\leq -\int_{\partial\Omega_\delta}\frac{4(n-1)}{n-2}\sum_{i=1}^n\frac{x_i}{|x|}\partial_i u_\varepsilon u_\varepsilon + \int_{\partial B_\delta(0)\cap\mathbb{R}^n_+}\frac{4(n-1)}{n-2}\sum_{i=1}^n\frac{x_i}{|x|}u_\varepsilon\partial_k u_\varepsilon h_{ik} \\
&\quad - \int_{\partial B_\delta(0)\cap\mathbb{R}^n_+}\frac{4(n-1)}{n-2}\varepsilon^{(n-2)/2}\sum_{i=1}^n\frac{x_i}{|x|}(u_\varepsilon\partial_i G - G\partial_i u_\varepsilon) \\
&\quad + C\sum_{2\leq|\alpha|\leq d}\sum_{i,k=1}^n|h_{ik,\alpha}|\delta^{|\alpha|+2-n}\varepsilon^{n-2} + C\delta^{2d+4-n}\varepsilon^{n-2} + C\delta^{-n-2}\varepsilon^{n+2}.
\end{aligned}$$

On the other hand, it follows from Propositions 3.6 and 3.8 that

$$\begin{aligned}
&\int_{\Omega_\delta}\Big(\frac{4(n-1)}{n-2}|dv_{(\varepsilon,\delta)}|_g^2 + R_g v_{(\varepsilon,\delta)}^2\Big)\,d\mathrm{vol}_g \\
&\leq Y(S^n_+,\partial S^n_+)\Big(\int_{\Omega_\delta}v_{(\varepsilon,\delta)}^{2n/(n-2)}\,d\mathrm{vol}_g\Big)^{(n-2)/n} + \int_{\partial B_\delta(0)\cap\mathbb{R}^n_+}\frac{4(n-1)}{n-2}\sum_{i=1}^n\frac{x_i}{|x|}\partial_i u_\varepsilon\,u_\varepsilon \\
&\quad + \int_{\partial B_\delta(0)\cap\mathbb{R}^n_+}\sum_{i,k=1}^n\frac{x_i}{|x|}(u_\varepsilon^2\partial_k h_{ik} - 2u_\varepsilon\partial_k u_\varepsilon\,h_{ik}) \\
&\quad - \frac{\lambda}{2}\sum_{2\leq|\alpha|\leq d}\sum_{i,k=1}^n|h_{ik,\alpha}|^2\varepsilon^{n-2}\int_{B_\delta(0)\cap\mathbb{R}^n_+}(\varepsilon+|x|)^{2|\alpha|+2-2n}\,dx \\
&\quad + C\sum_{2\leq|\alpha|\leq d}\sum_{i,k=1}^n|h_{ik,\alpha}|\delta^{|\alpha|+2-n}\varepsilon^{n-2} + C\delta^{2d+4-n}\varepsilon^{n-2}.
\end{aligned}$$

If we add the last two inequalities, we obtain

$$\begin{aligned}
&\int_M\Big(\frac{4(n-1)}{n-2}|dv_{(\varepsilon,\delta)}|_g^2 + R_g v_{(\varepsilon,\delta)}^2\Big)\,d\mathrm{vol}_g \\
&\qquad\leq Y(S^n_+,\partial S^n_+)\Big(\int_{\Omega_\delta}v_{(\varepsilon,\delta)}^{2n/(n-2)}\,d\mathrm{vol}_g\Big)^{(n-2)/n} \\
&\qquad\quad + \int_{\partial B_\delta(0)\cap\mathbb{R}^n_+}\sum_{i,k=1}^n\frac{x_i}{|x|}\Big(u_\varepsilon^2\partial_k h_{ik} + \frac{2n}{n-2}u_\varepsilon\partial_k u_\varepsilon\,h_{ik}\Big)
\end{aligned}$$

$$- \int_{\partial B_\delta(0) \cap \mathbb{R}^n_+} \frac{4(n-1)}{n-2} \varepsilon^{(n-2)/2} \sum_{i=1}^n \frac{x_i}{|x|} (u_\varepsilon \partial_i G - G \partial_i u_\varepsilon)$$
$$- \frac{\lambda}{2} \sum_{2 \leq |\alpha| \leq d} \sum_{i,k=1}^n |h_{ik,\alpha}|^2 \varepsilon^{n-2} \int_{B_\delta(0) \cap \mathbb{R}^n_+} (\varepsilon + |x|)^{2|\alpha|+2-2n} \, dx$$
$$+ C \sum_{2 \leq |\alpha| \leq d} \sum_{i,k=1}^n |h_{ik,\alpha}| \delta^{|\alpha|+2-n} \varepsilon^{n-2} + C \delta^{2d+4-n} \varepsilon^{n-2} + C \delta^{-n-2} \varepsilon^{n+2}.$$

From this, the assertion follows easily.

Theorem 4.2. *Assume that $p \notin \mathcal{Z}$. Then $Y(M, \partial M, g) < Y(S^n_+, \partial S^n_+)$.*

Proof. Since $p \notin \mathcal{Z}$, we have $\sum_{2 \leq |\alpha| \leq d} \sum_{i,k=1}^n |h_{ik,\alpha}|^2 > 0$. Using Proposition 4.1, we obtain

$$\int_M \left(\frac{4(n-1)}{n-2} |dv_{(\varepsilon,\delta)}|_g^2 + R_g v_{(\varepsilon,\delta)}^2 \right) d\mathrm{vol}_g$$
$$< Y(S^n_+, \partial S^n_+) \left(\int_M v_{(\varepsilon,\delta)}^{2n/(n-2)} \, d\mathrm{vol}_g \right)^{(n-2)/n}$$

if $\varepsilon > 0$ is sufficiently small. From this, the assertion follows.

In the remainder of this section, we study the case $p \in \mathcal{Z}$. In this case, we consider the manifold $(M \setminus \{p\}, G^{4/(n-2)} g)$. This manifold is scalar flat and its boundary is totally geodesic. After doubling this manifold, we obtain an asymptotically flat manifold with zero scalar curvature.

Proposition 4.3. *Assume that $p \in \mathcal{Z}$. Then the following statements hold:*

(i) *The limit $\lim_{\delta \to 0} \mathcal{I}(p, \delta)$ exists.*
(ii) *The doubling of $(M \setminus \{p\}, G^{4/(n-2)} g)$ has a well-defined mass which equals $\lim_{\delta \to 0} \mathcal{I}(p, \delta)$ up to a positive factor.*

Proof. For abbreviation, let $\overline{g} = G^{4/(n-2)} g$. We consider the inverted coordinates $y = x/|x|^2$, where $(x_1, \ldots, x_n)$ are conformal Fermi coordinates around p. In these coordinates, the metric $\overline{g}$ is given by

$$\overline{g}\left(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_l} \right) = \left[1 + \Phi\left(\frac{y}{|y|^2} \right) \right]^{4/(n-2)}$$
$$\cdot |y|^{-4} \sum_{i,k=1}^n (|y|^2 \delta_{ij} - 2 y_i y_j)(|y|^2 \delta_{kl} - 2 y_k y_l) g_{ik}\left(\frac{y}{|y|^2} \right),$$

where $\Phi(x) = |x|^{n-2} G(x) - 1$. Using the relations $g_{ik}(x) = \delta_{ik} + h_{ik}(x) + O(|x|^{2d+2})$ and $\Phi(x) = O(|x|^{d+1})$, we obtain

$$\overline{g}\Big(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_l}\Big) = \Big[1 + \frac{4}{n-2}\,\Phi\Big(\frac{y}{|y|^2}\Big)\Big]\delta_{jl} + |y|^{-4} \sum_{i,k=1}^n (|y|^2\delta_{ij} - 2y_i y_j)(|y|^2\delta_{kl} - 2y_k y_l) h_{ik}\Big(\frac{y}{|y|^2}\Big) + O(|y|^{-2d-2}).$$

In particular, $\overline{g}(\partial/\partial y_j, \partial/\partial y_l) = \delta_{jl} + O(|y|^{-d-1})$. Hence, the doubling of $(M \setminus \{p\}, \overline{g})$ is asymptotically flat in the sense of Bartnik [2], and has a well-defined ADM mass.

Since $\operatorname{tr} h = O(|x|^{2d+2})$, it follows that

$$\sum_{j,l=1}^n y_j \frac{\partial}{\partial y_j} \overline{g}\Big(\frac{\partial}{\partial y_l}, \frac{\partial}{\partial y_l}\Big) = -\frac{4n}{n-2} \sum_{j=1}^n \frac{y_j}{|y|^2} (\partial_j \Phi)\Big(\frac{y}{|y|^2}\Big) + O(|y|^{-2d-2}).$$

Moreover, we have

$$\sum_{j,l=1}^n y_j \frac{\partial}{\partial y_l} \overline{g}\Big(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_l}\Big) = -\frac{4}{n-2} \sum_{i=1}^n \frac{y_i}{|y|^2} (\partial_i \Phi)\Big(\frac{y}{|y|^2}\Big) - \sum_{i,k=1}^n \frac{y_i}{|y|^2} (\partial_k h_{ik})\Big(\frac{y}{|y|^2}\Big) + 2n \sum_{i,k=1}^n \frac{y_i y_k}{|y|^2} h_{ik}\Big(\frac{y}{|y|^2}\Big) + O(|y|^{-2d-2}),$$

where $\partial_i \Phi(x) = \frac{\partial}{\partial x_i} \Phi(x)$. Putting these facts together, we obtain

$$\begin{aligned} &\sum_{j,l=1}^n y_j \frac{\partial}{\partial y_l} \overline{g}\Big(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_l}\Big) - \sum_{j,l=1}^n y_j \frac{\partial}{\partial y_j} \overline{g}\Big(\frac{\partial}{\partial y_l}, \frac{\partial}{\partial y_l}\Big) \\ &\quad = \frac{4(n-1)}{n-2} \sum_{i=1}^n \frac{y_i}{|y|^2} (\partial_i \Phi)\Big(\frac{y}{|y|^2}\Big) \\ &\qquad - \sum_{i,k=1}^n \frac{y_i}{|y|^2} (\partial_k h_{ik})\Big(\frac{y}{|y|^2}\Big) + 2n \sum_{i,k=1}^n \frac{y_i y_k}{|y|^2} h_{ik}\Big(\frac{y}{|y|^2}\Big) + O(|y|^{-2d-2}). \end{aligned}$$

This implies

$$\begin{aligned} &\int_{\partial B_{1/\delta}(0)\cap\mathbb{R}^n_+} \sum_{j,l=1}^n \frac{y_j}{|y|} \frac{\partial}{\partial y_l} \overline{g}\Big(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_l}\Big) - \int_{\partial B_{1/\delta}(0)\cap\mathbb{R}^n_+} \sum_{j,l=1}^n \frac{y_j}{|y|} \frac{\partial}{\partial y_j} \overline{g}\Big(\frac{\partial}{\partial y_l}, \frac{\partial}{\partial y_l}\Big) \\ &\quad = \int_{\partial B_\delta(0)\cap\mathbb{R}^n_+} \frac{4(n-1)}{n-2} |x|^{3-2n} \sum_{i=1}^n x_i \partial_i \Phi(x) \\ &\qquad - \int_{\partial B_\delta(0)\cap\mathbb{R}^n_+} |x|^{3-2n} \sum_{i,k=1}^n x_i (\partial_k h_{ik})(x) + \int_{\partial B_\delta(0)\cap\mathbb{R}^n_+} 2n|x|^{1-2n} \sum_{i,k=1}^n x_i x_k h_{ik}(x) \\ &\qquad + O(\delta^{2d+4-n}) \\ &\quad = \mathcal{I}(p, \delta) + O(\delta^{2d+n-4}). \end{aligned}$$

As $\delta \to 0$, the left hand side converges to a positive multiple of the ADM mass. From this, the assertion follows.

Theorem 4.4. *Assume that $p \in \mathcal{Z}$. If $\lim_{\delta\to 0} \mathcal{I}(p,\delta)$ is positive, then $Y(M, \partial M, g) < Y(S^n_+, \partial S^n_+)$.*

Proof. Since $p \in \mathcal{Z}$, we have $\sum_{2\le|\alpha|\le d} \sum_{i,k=1}^n |h_{ik,\alpha}|^2 = 0$. By Proposition 4.1, we can find positive real numbers δ_0 and C such that

$$\int_M \left(\frac{4(n-1)}{n-2}|dv_{(\varepsilon,\delta)}|_g^2 + R_g v_{(\varepsilon,\delta)}^2\right) d\mathrm{vol}_g$$
$$\le Y(S^n_+, \partial S^n_+)\left(\int_M v_{(\varepsilon,\delta)}^{2n/(n-2)}\, d\mathrm{vol}_g\right)^{(n-2)/n} - \varepsilon^{n-2}\mathcal{I}(p,\delta) + C\delta^{2d+4-n}\varepsilon^{n-2} + C\delta^{-n}\varepsilon^n$$

whenever $0 < 2\varepsilon \le \delta \le \delta_0$. Since $\lim_{\delta\to 0} \mathcal{I}(p,\delta)$ is positive, we can find $\delta \in (0, \delta_0]$ such that $\mathcal{I}(p,\delta) > C\delta^{2d+4-n}$. In the next step, we choose $\varepsilon \in (0, \delta/2]$ small enough so that $\mathcal{I}(p,\delta) > C\delta^{2d+4-n} + C\delta^{-n}\varepsilon^2$. For this choice of ε and δ, we have

$$\int_M \left(\frac{4(n-1)}{n-2}|dv_{(\varepsilon,\delta)}|_g^2 + R_g v_{(\varepsilon,\delta)}^2\right) d\mathrm{vol}_g < Y(S^n_+, \partial S^n_+)\left(\int_M v_{(\varepsilon,\delta)}^{2n/(n-2)}\, d\mathrm{vol}_g\right)^{(n-2)/n}.$$

This completes the proof.

Appendix. An elliptic system on $\mathbb{R}^n_+$

In this section, we describe the construction of the vector field V. In the following, we consider the hemisphere S^n_+, equipped with the round metric of constant sectional curvature 4. We denote by $\mathcal{X}$ the space of all vector fields V on S^n_+ such that V is of class H^1 and $\langle V, \nu\rangle = 0$ along ∂S^n_+. Moreover, we denote by $\mathcal{Y}$ the space of all trace-free symmetric two-tensors on S^n_+ of class L^2. We next define a linear operator $\mathcal{D} : \mathcal{X} \to \mathcal{Y}$ by

$$\mathcal{D}V = \widehat{\mathscr{L}}_V g = \mathscr{L}_V g - \frac{2}{n}(\mathrm{div}_g V) g.$$

In other words, $\mathcal{D}$ is the conformal Killing operator.

Lemma A.1. *We have*

$$\|\nabla V\|^2_{L^2(S^n_+)} \le \|\mathcal{D}V\|^2_{L^2(S^n_+)} + 4(n-1)\|V\|^2_{L^2(S^n_+)}$$

for all $V \in \mathcal{X}$.

Proof. Without loss of generality, we may assume that V is smooth. By definition of $\mathcal{D}$, we have

$$\|\mathcal{D}V\|^2_{L^2(S^n_+)} = \int_{S^n_+} \left[\nabla_i V^k\, \nabla^i V_k + \nabla_i V^k\, \nabla_k V^i - \frac{2}{n}(\mathrm{div}_g V)^2\right] d\mathrm{vol}_g.$$

Integration by parts yields

$$\begin{aligned}\int_{S^n_+} \nabla_i V^k \, \nabla_k V^i \, d\text{vol}_g &= -\int_{S^n_+} V^k \nabla_i \nabla_k V^i \, d\text{vol}_g \\ &= -\int_{S^n_+} V^k \nabla_k \nabla_i V^i \, d\text{vol}_g - \int_{S^n_+} \text{Ric}_{ik} V^i V^k \, d\text{vol}_g \\ &= \int_{S^n_+} (\text{div}_g \, V)^2 \, d\text{vol}_g - 4(n-1) \int_{S^n_+} |V|^2 \, d\text{vol}_g.\end{aligned}$$

Putting these facts together, we obtain

$$\|\mathcal{D}V\|^2_{L^2(S^n_+)} + 4(n-1)\|V\|^2_{L^2(S^n_+)} = \|\nabla V\|^2_{L^2(S^n_+)} + \frac{n-2}{n} \|\text{div}_g \, V\|^2_{L^2(S^n_+)}.$$

From this, the assertion follows.

It follows from Lemma A.1 and Rellich's theorem that $\ker \mathcal{D}$ is finite-dimensional. We now consider the subspace

$$\mathcal{X}_0 = \{V \in \mathcal{X} : \langle V, W \rangle_{L^2(S^n_+)} = 0 \text{ for all } W \in \ker \mathcal{D}\}.$$

Lemma A.2. *We have*

$$\|V\|^2_{L^2(S^n_+)} + \|\nabla V\|^2_{L^2(S^n_+)} \leq K \|\mathcal{D}V\|^2_{L^2(S^n_+)}$$

for all $V \in \mathcal{X}_0$. Here, K is a positive constant that depends only on n.

Proof. Suppose that the assertion is false. Then we can find a sequence of vector fields $V^{(\nu)} \in \mathcal{X}_0$ such that

$$\|V^{(\nu)}\|^2_{L^2(S^n_+)} + \|\nabla V^{(\nu)}\|^2_{L^2(S^n_+)} = 1 \tag{A.1}$$

for all ν and $\|\mathcal{D}V^{(\nu)}\|_{L^2(S^n_+)} \to 0$ as $\nu \to \infty$. After passing to a subsequence, we may assume that the sequence $V^{(\nu)}$ converges weakly to a vector field $W \in \mathcal{X}_0$. Then $\mathcal{D}W = 0$. Since $W \in \mathcal{X}_0$, we conclude that $W = 0$. This implies $\|V^{(\nu)}\|_{L^2(S^n_+)} \to 0$ as $\nu \to \infty$. Using Lemma A.1, we obtain $\|\nabla V^{(\nu)}\|_{L^2(S^n_+)} \to 0$ as $\nu \to \infty$. This contradicts (A.1).

Proposition A.3. *Given any $h \in \mathcal{Y}$, there exists a unique vector field $V \in \mathcal{X}_0$ such that $\langle h - \mathcal{D}V, \mathcal{D}W \rangle_{L^2(S^n_+)} = 0$ for all $W \in \mathcal{X}$. The vector field V satisfies the estimate*

$$\|V\|^2_{L^2(S^n_+)} + \|\nabla V\|^2_{L^2(S^n_+)} \leq K \|h\|^2_{L^2(S^n_+)}. \tag{A.2}$$

Proof. It follows from Lemma A.2 that the operator $\mathcal{D} : \mathcal{X}_0 \to \mathcal{Y}$ has closed range. Hence, we can find a vector field $V \in \mathcal{X}_0$ such that $\|h - \mathcal{D}V\|^2_{L^2(S^n_+)}$ is minimal. Then $\langle h - \mathcal{D}V, \mathcal{D}W \rangle_{L^2(S^n_+)} = 0$ for all $W \in \mathcal{X}_0$. This proves the existence statement.

We next assume that $V \in \mathcal{X}_0$ satisfies $\langle h - \mathcal{D}V, \mathcal{D}W\rangle_{L^2(S^n_+)} = 0$ for all $W \in \mathcal{X}_0$. This implies $\langle h - \mathcal{D}V, \mathcal{D}V\rangle = 0$, hence $\|\mathcal{D}V\|^2_{L^2(S^n_+)} \leq \|h\|^2_{L^2(S^n_+)}$. Thus, we conclude that

$$\|V\|^2_{L^2(S^n_+)} + \|\nabla V\|^2_{L^2(S^n_+)} \leq K\|\mathcal{D}V\|^2_{L^2(S^n_+)} \leq K\|h\|^2_{L^2(S^n_+)}$$

by Lemma A.2. In particular, if $h = 0$, then $V = 0$. From this, the uniqueness statement follows.

In the next step, we consider the stereographic projection from S^n_+ to $\mathbb{R}^n_+ \cup \{\infty\}$. The metric g can be written in the form $g_{ik} = u^{4/(n-2)}\delta_{ik}$, where

$$u(x) = \left(\frac{1}{1+|x|^2}\right)^{(n-2)/2}.$$

Theorem A.4. *Let h be a trace-free symmetric two-tensor on $\mathbb{R}^n_+$. Assume that h is smooth and has compact support. Then there exists a smooth vector field V on $\mathbb{R}^n_+$ with the following properties:*

- *At each $x \in \mathbb{R}^n_+$, we have*

$$\sum_{k=1}^n \partial_k\left[u^{2n/(n-2)}\left(h_{ik} - \partial_i V_k - \partial_k V_i + \frac{2}{n}\,\text{div}\,V\,\delta_{ik}\right)\right] = 0$$

  for all $i \in \{1, \dots, n\}$.
- *At each $x \in \partial\mathbb{R}^n_+$, we have $V_n(x) = \partial_n V_i(x) - h_{in}(x) = 0$ for all $i \in \{1, \dots, n-1\}$.*

Moreover, the vector field V satisfies

$$\int_{\mathbb{R}^n_+} u(x)^{(2(n+2))/(n-2)}|V(x)|^2\,dx \leq K\int_{\mathbb{R}^n_+} u(x)^{2n/(n-2)}|h(x)|^2\,dx. \tag{A.3}$$

Proof. By Proposition A.3, there exists a smooth vector field $V \in \mathcal{X}_0$ such that

$$\int_{\mathbb{R}^n_+} \langle u^{4/(n-2)}h - \mathcal{D}V, \mathcal{D}W\rangle_g\,d\text{vol}_g = 0$$

for all $W \in \mathcal{X}$. This implies

$$\int_{\mathbb{R}^n_+} u^{2n/(n-2)}\sum_{i,k=1}^n\left(h_{ik} - \partial_i V_k - \partial_k V_i + \frac{2}{n}\,\text{div}\,V\,\delta_{ik}\right)\partial_k W_i\,dx = 0 \tag{A.4}$$

for all $W \in \mathcal{X}$. Since $V \in \mathcal{X}_0$, we have $V_n(x) = 0$ for $x \in \partial\mathbb{R}^n_+$.

By assumption, h is smooth. Using general regularity results for elliptic systems (cf. [10], [13]), we conclude that V is smooth. Using (A.4), we obtain

$$\sum_{k=1}^n \partial_k\left[u^{2n/(n-2)}\left(h_{ik} - \partial_i V_k - \partial_k V_i + \frac{2}{n}\,\text{div}\,V\,\delta_{ik}\right)\right] = 0$$

for all $x \in \mathbb{R}^n_+$ and all $i \in \{1, \dots, n\}$. Moreover, at each $x \in \partial\mathbb{R}^n_+$, we have $\partial_n V_i(x) - h_{in}(x) = 0$ for $i \in \{1, \dots, n-1\}$. Finally, the estimate (A.3) follows immediately from (A.2).

Proposition A.5. *Fix a real number σ such that $1 < \sigma < n-2$. Let h be a trace-free symmetric two-tensor on $\mathbb{R}^n_+$ which is smooth and has compact support. Moreover, let V be the vector field constructed in Theorem A.4. Finally, assume that*

$$\sup_{r \geq 1} r^{-2\sigma-n-2} \int_{(B_{2r}(0) \setminus B_r(0)) \cap \mathbb{R}^n_+} |V(x)|^2 \, dx < \infty.$$

Then there exists a constant C, depending only on n and σ, such that

$$\begin{aligned} \sup_{r \geq 1} r^{-2\sigma-n-2} \int_{(B_{2r}(0) \setminus B_r(0)) \cap \mathbb{R}^n_+} |V(x)|^2 \, dx \\ &\leq C \int_{\mathbb{R}^n_+} (1+|x|^2)^{-n-2} |V(x)|^2 \, dx \\ &\quad + C \sup_{r \geq 1} r^{-2\sigma-n} \int_{(B_{2r}(0) \setminus B_r(0)) \cap \mathbb{R}^n_+} |h(x)|^2 \, dx. \end{aligned} \tag{A.5}$$

Proof. We extend V and h to $\mathbb{R}^n$ by reflection. More precisely, we define a vector field $\tilde{V}$ on $\mathbb{R}^n$ by

$$\begin{aligned} \tilde{V}_i(x_1, \ldots, x_{n-1}, x_n) &= \tilde{V}_i(x_1, \ldots, x_{n-1}, -x_n) = V_i(x_1, \ldots, x_{n-1}, x_n), \\ \tilde{V}_n(x_1, \ldots, x_{n-1}, x_n) &= -\tilde{V}_n(x_1, \ldots, x_{n-1}, -x_n) = V_n(x_1, \ldots, x_{n-1}, x_n), \end{aligned}$$

for all $x \in \mathbb{R}^n_+$ and all $i \in \{1, \ldots, n-1\}$. Similarly, we define a trace-free symmetric two-tensor $\tilde{h}$ on $\mathbb{R}^n$ by

$$\begin{aligned} \tilde{h}_{ik}(x_1, \ldots, x_{n-1}, x_n) &= \tilde{h}_{ik}(x_1, \ldots, x_{n-1}, -x_n) = h_{ik}(x_1, \ldots, x_{n-1}, x_n), \\ \tilde{h}_{in}(x_1, \ldots, x_{n-1}, x_n) &= -\tilde{h}_{in}(x_1, \ldots, x_{n-1}, -x_n) = h_{in}(x_1, \ldots, x_{n-1}, x_n), \\ \tilde{h}_{nk}(x_1, \ldots, x_{n-1}, x_n) &= -\tilde{h}_{nk}(x_1, \ldots, x_{n-1}, -x_n) = h_{nk}(x_1, \ldots, x_{n-1}, x_n), \\ \tilde{h}_{nn}(x_1, \ldots, x_{n-1}, x_n) &= \tilde{h}_{nn}(x_1, \ldots, x_{n-1}, -x_n) = h_{nn}(x_1, \ldots, x_{n-1}, x_n), \end{aligned}$$

for all $x \in \mathbb{R}^n_+$ and all $i, k \in \{1, \ldots, n-1\}$.

Since $V \in \mathcal{X}$, we have $V_n(x) = 0$ for all $x \in \partial \mathbb{R}^n_+$. Consequently, $\tilde{V}$ is a vector field on S^n of class H^1. We claim that

$$\int_{\mathbb{R}^n} u^{2n/(n-2)} \sum_{i,k=1}^n \Big(\tilde{h}_{ik} - \partial_i \tilde{V}_k - \partial_k \tilde{V}_i + \frac{2}{n} \operatorname{div} \tilde{V} \, \delta_{ik} \Big) \partial_k \tilde{W}_i \, dx = 0 \tag{A.6}$$

for all vector fields $\tilde{W}$ on S^n of class H^1. In order to prove (A.6), we fix a vector field $\tilde{W}$ of class H^1. We then define a vector field W on S^n_+ by

$$\begin{aligned} W_i(x_1, \ldots, x_{n-1}, x_n) &= \tilde{W}_i(x_1, \ldots, x_{n-1}, x_n) + \tilde{W}_i(x_1, \ldots, x_{n-1}, -x_n), \\ W_n(x_1, \ldots, x_{n-1}, x_n) &= \tilde{W}_n(x_1, \ldots, x_{n-1}, x_n) - \tilde{W}_n(x_1, \ldots, x_{n-1}, -x_n), \end{aligned}$$

for all $x \in \mathbb{R}^n_+$ and all $i \in \{1, \ldots, n-1\}$. Clearly, $W \in \mathcal{X}$. Therefore, we have

$$\int_{\mathbb{R}^n_+} u^{2n/(n-2)} \sum_{i,k=1}^n \Big(h_{ik} - \partial_i V_k - \partial_k V_i + \frac{2}{n} \operatorname{div} V \, \delta_{ik}\Big) \partial_k W_i \, dx = 0$$

by definition of V. From this, the identity (A.6) follows easily.

We now complete the proof of Proposition A.5. Using [4, Proposition 23], we obtain

$$\begin{aligned}
&\sup_{r \geq 1} r^{-2\sigma-n-2} \int_{B_{2r}(0) \setminus B_r(0)} |\tilde{V}(x)|^2 \, dx \\
&\quad \leq C \int_{\mathbb{R}^n_+} (1+|x|^2)^{-n-2} |\tilde{V}(x)|^2 \, dx + C \sup_{r \geq 1} r^{-2\sigma-n} \int_{B_{2r}(0) \setminus B_r(0)} |\tilde{h}(x)|^2 \, dx.
\end{aligned}$$

Here, C is a positive constant that depends only on σ and n. (In [4], this result was stated in the special case that $\tilde{V}$ and $\tilde{h}$ are smooth, but the proof only requires that $\tilde{h}$ belongs to L^2 and $\tilde{V}$ is of class H^1.) From this the assertion follows.

Corollary A.6. *Consider a trace-free symmetric two-tensor of the form*

$$h_{ik}(x) = \chi(|x|/\rho) \sum_{2 \leq |\alpha| \leq d} h_{ik,\alpha} x^\alpha,$$

where $d = [(n-2)/2]$, $\rho \geq 1$, and $\chi : \mathbb{R} \to \mathbb{R}$ is a fixed cut-off function satisfying $\chi(t) = 0$ for $t \geq 2$. Let V be the vector field constructed in Theorem A.4. Then, for every multi-index β, we have

$$|\partial^\beta V(x)|^2 \leq C \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}|^2 (1+|x|^2)^{|\alpha|+1-|\beta|} \tag{A.7}$$

for all $x \in \mathbb{R}^n_+$. Here, C is positive constant which depends on n and $|\beta|$, but not on ρ.

Proof. Without loss of generality, we may assume that

$$h_{ik}(x) = \chi(|x|/\rho) \sum_{|\alpha| = d'} h_{ik,\alpha} x^\alpha,$$

where $2 \leq d' \leq d$. Since $d' < n/2$, we have

$$\int_{\mathbb{R}^n_+} (1+|x|^2)^{-n} |h(x)|^2 \, dx \leq C \sum_{|\alpha|=d'} |h_{ik,\alpha}|^2$$

for some uniform constant C. Using (A.3), we obtain

$$\int_{\mathbb{R}^n_+} (1+|x|^2)^{-n-2} |V(x)|^2 \, dx \leq C \sum_{|\alpha|=d'} |h_{ik,\alpha}|^2.$$

We now apply Proposition A.5 with $\sigma = d'$. This yields

$$\sup_{r\geq 1} r^{-2d'-n-2} \int_{\{r\leq |x|\leq 2r\}} |V(x)|^2\,dx \leq C \sum_{|\alpha|=d'} |h_{ik,\alpha}|^2.$$

Using elliptic estimates, we conclude that

$$|\partial^\beta V(x)|^2 \leq C \sum_{|\alpha|=d'} |h_{ik,\alpha}|^2 (1+|x|^2)^{d'+1-|\beta|}$$

for every multi-index β.

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