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Asymptotic properties of ground states of scalar field equations with a vanishing parameter

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Abstract. We study the leading order behaviour of positive solutions of the equation

$$-\Delta u + \varepsilon u - |u|^{p-2}u + |u|^{q-2}u = 0, \qquad x \in \mathbb{R}^N,$$

where $N \geq 3$, $q > p > 2$ and $\varepsilon > 0$ is a small parameter. We give a complete characterization of all possible asymptotic regimes as a function of p, q and N. The behaviour of solutions depends on whether p is less than, equal to or greater than the critical Sobolev exponent $2^* = \frac{2N}{N-2}$. For $p < 2^*$ the solution asymptotically coincides with the solution of the equation in which the last term is absent. For $p > 2^*$ the solution asymptotically coincides with the solution of the equation with $\varepsilon = 0$. In the most delicate case $p = 2^*$ the asymptotic behaviour of the solutions is given by a particular solution of the critical Emden–Fowler equation, whose choice depends on ε in a nontrivial way.

Keywords. Critical Sobolev exponent, subcritical, critical and supercritical nonlinearity, Pohožaev identity, asymptotic behaviour

1. Introduction

1.1. Setting of the problem

This paper deals with positive solutions of the scalar field equation

$$-\Delta u + \varepsilon u - |u|^{p-2}u + |u|^{q-2}u = 0 \quad \text{in } \mathbb{R}^N, \tag{P_ε}$$

where $N \geq 3$, $q > p > 2$ and $\varepsilon > 0$. Specifically, we are interested in the case where ε is a small parameter, with all other parameters fixed. Our goal is to understand the behaviour of ground state solutions of (P_ε) for $\varepsilon \ll 1$. By a *ground state* solution of (P_ε) we understand a positive weak solution $u_\varepsilon \in H^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ of (P_ε). These solutions are critical points (saddles) of the energy

$$\mathcal{E}_\varepsilon(u) := \int_{\mathbb{R}^N} \left(\frac{1}{2}|\nabla u|^2 + \frac{\varepsilon}{2}|u|^2 - \frac{1}{p}|u|^p + \frac{1}{q}|u|^q \right) dx. \tag{1.1}$$

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Existence and uniqueness of ground state solutions of (P_ε) with $\varepsilon > 0$ is well known. Existence goes back to Strauss [27, Example 2] and Berestycki and Lions [5, Example 2]. Note that by strict convexity of the integrand in $\mathcal{E}_\varepsilon(u)$ for large $|u|$ every weak solution of (P_ε) is essentially bounded, and so by elliptic regularity these are classical solutions of (P_ε) that decay uniformly to zero as $|x| \to \infty$. Then the classical Gidas–Ni–Nirenberg symmetry result [14, Theorem 2] implies that every ground state solution of (P_ε) is spherically symmetric about some point. The uniqueness of a spherically symmetric ground state is rather delicate and was proved only quite recently by Serrin and Tang [25, Theorem 4(ii)]. The following theorem summarizes all the above results.

Theorem A ([27, 5, 14, 25]). *Let $N \geq 3$ and $q > p > 2$. There exists $\varepsilon_* > 0$ such that (P_ε) has no ground state solutions for $\varepsilon \geq \varepsilon_*$, while for every $\varepsilon \in (0, \varepsilon_*)$ it admits a unique ground state solution $u_\varepsilon \in C^\infty(\mathbb{R}^N)$ such that $u_\varepsilon(x)$ is a decreasing function of $|x|$ and there exists $C_\varepsilon > 0$ such that*

$$\lim_{|x|\to\infty} |x|^{(N-1)/2} e^{\sqrt{\varepsilon}|x|} u_\varepsilon(x) = C_\varepsilon > 0.$$

Furthermore, every ground state solution of (P_ε) is a translate of u_ε.

We note that the threshold value ε_* in Theorem 1.1 is simply the smallest value of $\varepsilon > 0$ for which the energy $\mathcal{E}_\varepsilon$ is nonnegative and can be easily computed explicitly.

We are interested in the asymptotic behavior of the ground states u_ε as $\varepsilon \to 0$. This question naturally arises in the study of various bifurcation problems, for which (P_ε) can be considered as a canonical normal form (see e.g. [9, 30]). Problem (P_ε) itself may also be considered as a prototypical example of a bifurcation problem for elliptic equations. In fact, our results are expected to remain valid for a broader class of scalar field equations whose nonlinearity has the leading terms in the expansion around zero which coincide with the ones in (P_ε). Let us also mention that problem (P_ε) appears in the study of nonclassical nucleation near spinodal in mesoscopic models of phase transitions [7, 22, 29], as well as in the study of the decay of false vacuum in quantum field theories [8].

In order to understand the asymptotic behaviour of u_ε as $\varepsilon \to 0$, we again note that for $u \geq 1$ the energy density in $\mathcal{E}_\varepsilon(u)$ is strictly convex. Hence we may conclude that the ground state solution u_ε in Theorem 1.1 satisfies a uniform upper bound

$$u_\varepsilon(0) \leq 1 \quad \text{for all } \varepsilon \in (0, \varepsilon_*). \tag{1.2}$$

Elliptic regularity then implies that locally over compact sets the solution u_ε converges as $\varepsilon \to 0$ to a radial solution of the limit equation

$$-\Delta u - |u|^{p-2}u + |u|^{q-2}u = 0 \quad \text{in } \mathbb{R}^N. \tag{P_0}$$

It is known that (here and everywhere below, $2^* := \frac{2N}{N-2}$):

- for $2 < p \leq 2^*$ equation (P_0) has no nontrivial finite energy solutions, which is a direct consequence of Pohožaev's identity (see Remark 5.1);
- for $p > 2^*$ equation (P_0) admits a unique radial ground state solution. Existence goes back to [5, Theorem 4] (see also [20, 21]), while uniqueness was proved in [21, 18].

Note that the natural energy space for equation (P_ε) is the usual Sobolev space $H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) \mid \|\nabla u\|_{L^2} < \infty\}$, while for $p \geq 2^*$ the limit equation (P_0) is variationally well-posed in the homogeneous Sobolev space $D^1(\mathbb{R}^N)$, defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the Dirichlet norm $\|\nabla u\|_{L^2}$. Clearly, $H^1(\mathbb{R}^N) \subsetneq D^1(\mathbb{R}^N)$ and as a consequence, no natural perturbation setting (in the spirit of the implicit function theorem) is available to analyze the family of equations (P_ε) as $\varepsilon \to 0$. In fact, a linearization of (P_0) around the ground state solution is not a Fredholm operator and has zero as the bottom of the essential spectrum in $L^2(\mathbb{R}^N)$. As a consequence, advanced Lyapunov–Schmidt type reduction methods of Ambrosetti and Malchiodi [3] are not applicable to the family of equations (P_ε).

If we introduce the *canonical* rescaling associated with the lowest order nonlinear term in (P_ε):

$$v(x) = \varepsilon^{-1/(p-2)} u(x/\sqrt{\varepsilon}), \tag{1.3}$$

then (P_ε) transforms into the equation

$$-\Delta v + v = |v|^{p-2}v - \varepsilon^{\frac{q-p}{p-2}} |v|^{q-2}v \quad \text{in } \mathbb{R}^N. \tag{R_ε}$$

The limit problem associated to (R_ε) as $\varepsilon \to 0$ has the form

$$-\Delta v + v = |v|^{p-2}v \quad \text{in } \mathbb{R}^N. \tag{R_0}$$

It is well-known that:

- for $p \geq 2^*$ equation (R_0) has no nontrivial finite energy solutions, which is a direct consequence of Pohožaev's identity [23, 5];
- for $2 < p < 2^*$ equation (R_0) admits a unique radial ground state solution. Existence goes back at least to [27], and uniqueness was proved in [17].

The advantage of the rescaling (1.3) is that at least in the range $2 < p \leq 2^*$ both (R_ε) and the limit problem (R_0) are variationally well-posed in the same Sobolev space $H^1(\mathbb{R}^N)$. Then the rescaled problem (R_ε) could be naturally seen as a small perturbation of the limit problem (R_0) and the family of ground states (v_ε) of problem (R_ε) could be rigorously interpreted as a perturbation of the ground state solution of the limit problem (R_0). This could be done e.g. by using a combination of the variational and Lyapunov–Schmidt perturbation techniques as developed by Ambrosetti, Malchiodi et al. (see [3] and further references therein).

The distinction between the asymptotic behaviours of the solutions of problem (P_ε) as $\varepsilon \to 0$ depending on the value of p as compared to 2^* was first pointed out in [22]. There it was also observed that the asymptotic behaviour of the ground states u_ε for $p = 2^*$ is not controlled by the solution set structure of either (P_0) or (R_0). Formal asymptotic analysis of [22] explains that, in fact, three different asymptotic regimes have to be distinguished in (P_ε): the *subcritical case* $2 < p < 2^*$, the *supercritical case* $p > 2^*$ and the most delicate *critical case* $p = 2^*$.

In this work, using an adaptation of the constrained minimization techniques developed by H. Berestycki and P.-L. Lions [5], we provide a complete analysis of these three asymptotic regimes. The analysis confirms and extends the ideas introduced in [22] and gives a full characterization of the asymptotic behaviour of ground state solutions of (P_ε) for $\varepsilon \to 0$.

Notations

For $\varepsilon \ll 1$ and $f(\varepsilon), g(\varepsilon) \geq 0$, whenever there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$ the respective condition holds, we write:

- $f(\varepsilon) \lesssim g(\varepsilon)$ if there exists $C > 0$ independent of ε such that $f(\varepsilon) \leq Cg(\varepsilon)$;
- $f(\varepsilon) \sim g(\varepsilon)$ if $f(\varepsilon) \lesssim g(\varepsilon)$ and $g(\varepsilon) \lesssim f(\varepsilon)$;
- $f(\varepsilon) \simeq g(\varepsilon)$ if $f(\varepsilon) \sim g(\varepsilon)$ and $\lim_{\varepsilon \to 0} f(\varepsilon)/g(\varepsilon) = 1$.

We also use the standard notations $f = O(g)$ and $f = o(g)$, bearing in mind that $f \geq 0$ and $g \geq 0$. As usual, C, c, c_1 etc. denote generic positive constants independent of ε.

2. Main results

2.1. Subcritical case $2 < p < 2^*$

Since in the subcritical case the limit equation (P_0) has no ground state solutions, in view of (1.2) the family of ground states u_ε must converge to zero, locally over compact subsets of $\mathbb{R}^N$. To describe the asymptotic behaviour of u_ε we use the rescaling (1.3) which transforms (P_ε) into equation (R_ε). For $2 < p < 2^*$, let $v_0(x)$ denote the unique radial ground state solution of the limit equation (R_0). It is well-known that $v_0 \in C^\infty(\mathbb{R}^N)$, $v_0(x)$ is a decreasing function of $|x|$ and

$$\lim_{|x| \to \infty} |x|^{(N-1)/2} e^{|x|} u_\varepsilon(x) = C_0 > 0$$

(cf. [5]). The advantage of the rescaling (1.3) is that both (R_ε) and the limit problems (R_0) are variationally well-posed in the Sobolev space $H^1(\mathbb{R}^N)$. Note however that (R_0) is translation invariant and hence the radial ground state $v_0(x)$ is not an isolated solution. As a consequence, an Implicit Function Theorem argument is not directly applicable to (R_ε). Nevertheless, it is known that the linearization operator $-\Delta + 1 - (p-1)v_0^{p-2}$ of (R_0) around the ground state v_0 is a Fredholm operator in $H^1(\mathbb{R}^N)$ (see [3, Lemma 4.1]). Then perturbation techniques of [3] could be easily adapted in order to show that for all sufficiently small $\varepsilon > 0$ equation (R_ε) admits a radial ground state $v_\varepsilon(x)$ which converges to $v_0(x)$ as $\varepsilon \to 0$. Rescaling back to the original variable and taking into account the uniqueness of the radial ground state of (P_ε) we arrive at the following (folklore) result.

Theorem 2.1. *Let* $2 < p < 2^*$. *As* $\varepsilon \to 0$, *the rescaled family of ground states*

$$v_\varepsilon(x) := \varepsilon^{-1/(p-2)} u_\varepsilon(x/\sqrt{\varepsilon})$$

converges to $v_0(x)$ *in* $H^1(\mathbb{R}^N)$, $L^q(\mathbb{R}^N)$ *and* $C^2(\mathbb{R}^N)$. *In particular,*

$$u_\varepsilon(0) \simeq \varepsilon^{1/(p-2)} v_0(0).$$

In the last section of this work we provide a short alternative proof of this result based only upon variational methods which are developed in the main part of this paper and without explicit references to perturbation techniques.

Remark 2.2. For $p \geq 2^*$ Pohožaev's identity implies that (R_0) has no nontrivial solutions in $H^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$. In fact, it is known that $v_0(0) \to \infty$ as $p \uparrow 2^*$ (see [12, 13, 11]). More specifically (see [13, Corollary 1]), if $\delta := 2^* - p$, then for $\delta \downarrow 0$,

$$v_0(0) \simeq \beta_N \begin{cases} \delta^{-(N-2)/4}, & N \geq 5, \\ \delta^{-1/2}|\log \delta|^{1/2}, & N = 4, \\ \delta^{-1/2}, & N = 3, \end{cases}$$

for some explicit constants $\beta_N > 0$. This suggests that for $p = 2^*$ rescaling (1.3) fails to capture the behaviour of the ground states u_ε and a different approach is needed to handle the critical and supercritical cases. Note also that the asymptotic behaviour of ground states of "slightly" subcritical elliptic problems in the context of bounded domains was studied in [4, 6, 16, 24].

2.2. *Supercritical case $p > 2^*$*

In contrast to the subcritical case, for $p > 2^*$ the limit equation (P_0) admits a unique radial ground state solution $u_0(x) > 0$. It is known that $u_0 \in D^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$, $u_0(x)$ is a decreasing function of $|x|$ and

$$\lim_{|x|\to\infty} |x|^{N-2} u_0(x) = C_0 > 0$$

(see [5, Theorem 4] or [20, 21] for the existence, and [21, 18] for the uniqueness proofs). However, as was already mentioned, the linearization operator $-\Delta - (p-1)u_0^{p-2}$ of (P_0) around the ground state u_0 is not Fredholm and has zero as the bottom of the essential spectrum in $L^2(\mathbb{R}^N)$. As a consequence, standard perturbation methods are not applicable to (P_0). Using a direct analysis of the family of constrained minimizations problem associated to (P_ε), we prove the following.

Theorem 2.3. *Let $p > 2^*$. As $\varepsilon \to 0$, the family of ground states u_ε converges to u_0 in $D^1(\mathbb{R}^N)$, $L^q(\mathbb{R}^N)$ and $C^2(\mathbb{R}^N)$. In particular,*

$$u_\varepsilon(0) \simeq u_0(0).$$

In addition, $\varepsilon\|u_\varepsilon\|_2^2 \to 0$.

Remark 2.4. For $p = 2^*$ Pohožaev's identity implies that (P_0) has no nontrivial solutions in $D^1(\mathbb{R}^N)$. In fact, it is not difficult to show that $u_0(0) \to 0$ as $p \downarrow 2^*$. Moreover, if $\delta := p - 2^*$, then for $\delta \downarrow 0$ we prove

$$\delta^{1/(q-2^*)} \lesssim u_0(0) \lesssim \delta^{1/(q+N)},$$

and, provided that $q > \frac{N(N+2)}{2(N-2)}$,

$$u_0(0) \sim \delta^{1/(q-2^*)}.$$

See Section 5.4 for further details.

2.3. *Critical case $p = 2^*$*

In the critical case both the unrescaled limit equation (P_0) and the "canonically" rescaled equation (R_0) have no nontrivial finite energy solutions. We are going to show that after a suitable rescaling the correct limit equation for (P_ε) is in fact given by the critical Emden–Fowler equation

$$-\Delta U = U^{2^*-1} \quad \text{in } \mathbb{R}^N. \tag{R_*}$$

It is well-known that the radial ground states of (R_*) are given by the function

$$U_1(x) := \left(1 + \frac{|x|^2}{N(N-2)}\right)^{-(N-2)/2}, \tag{2.1}$$

and the family of its rescalings

$$U_\lambda(x) := \lambda^{-(N-2)/2} U_1(x/\lambda), \qquad \lambda > 0. \tag{2.2}$$

Our main result in this work is the following.

Theorem 2.5. *Let $p = 2^*$. There exists a rescaling $\lambda_\varepsilon : (0, \varepsilon_*) \to (0, \infty)$ such that as $\varepsilon \to 0$, the rescaled family of ground states*

$$v_\varepsilon(x) := \lambda_\varepsilon^{1/(p-2)} u_\varepsilon(\lambda_\varepsilon x)$$

converges to $U_1(x)$ in $D^1(\mathbb{R}^N)$, $L^q(\mathbb{R}^N)$ and $C^2(\mathbb{R}^N)$. Moreover,

$$\lambda_\varepsilon \sim \begin{cases} \varepsilon^{-\frac{p-2}{2q-4}}, & N \geq 5, \\ \left(\varepsilon \log \frac{1}{\varepsilon}\right)^{-1/(q-2)}, & N = 4, \\ \varepsilon^{-1/(q-4)}, & N = 3. \end{cases} \tag{2.3}$$

and

$$u_\varepsilon(0) \sim \begin{cases} \varepsilon^{1/(q-2)}, & N \geq 5, \\ \left(\varepsilon \log \frac{1}{\varepsilon}\right)^{1/(q-2)}, & N = 4, \\ \varepsilon^{1/(2q-8)}, & N = 3. \end{cases} \tag{2.4}$$

Remark 2.6. Asymptotics (2.3) and (2.4) were first derived in [22] using methods of formal asymptotic expansions. Theorem 2.5, in particular, justifies the values of precise asymptotic constants found in [22].

2.4. *Outline*

The rest of the paper is organized as follows. In Section 3 we introduce a variational characterization of the ground states u_ε of the problem (P_ε) as well as some other preliminary results. In Section 4 we study the critical case $p = 2^*$ and prove Theorem 2.5. In Section 5 we consider the supercritical case $p > 2^*$ and prove Theorem 2.3. Finally, in Section 6 we revisit the subcritical case $2 < p < 2^*$ and sketch a simple variational proof of Theorem 2.1, in the spirit of our previous arguments.

3. Variational characterization of the ground states

The existence and properties of the ground state u_ε of equation (P_ε), as summarized in Theorem A, could be established in several different ways, e.g. by means of ODE techniques. Here we shall utilize a variational characterization of the ground states u_ε developed by Berestycki and Lions [5].

Given $q > p > 2$ and $\varepsilon \geq 0$ set

$$f_\varepsilon(u) := \begin{cases} 0, & u < 0, \\ u^{p-1} - u^{q-1} - \varepsilon u, & u \in [0, 1], \\ -\varepsilon, & u > 1, \end{cases} \qquad F_\varepsilon(u) := \int_0^u f_\varepsilon(s)\, ds. \tag{3.1}$$

In view of (1.2) and since we are interested only in positive solutions of (P_ε), the nonlinearity in (P_ε) may always be replaced by its bounded truncation $f_\varepsilon(u)$ from (3.1) and therefore the truncated problem is well-posed in $H^1(\mathbb{R}^N)$ even for supercritical $q > 2^*$.

For $\varepsilon > 0$, consider the constrained minimization problem

$$S_\varepsilon := \inf\left\{ \int_{\mathbb{R}^N} |\nabla w|^2\, dx \;\middle|\; w \in H^1(\mathbb{R}^N),\ 2^* \int_{\mathbb{R}^N} F_\varepsilon(w)\, dx \;= 1 \right\}. \tag{S_ε}$$

As was proved in [5, Theorem 2], there exists $\varepsilon_* > 0$ depending only on p and q such that for all $\varepsilon \in (0, \varepsilon_*)$ the minimization problem (S_ε) admits a positive radially symmetric minimizer $w_\varepsilon(x)$. Further, there exists a Lagrange multiplier $\theta_\varepsilon > 0$ such that

$$-\Delta w_\varepsilon = \theta_\varepsilon f_\varepsilon(w_\varepsilon) \quad \text{in } \mathbb{R}^N. \tag{3.2}$$

In particular, the minimizer w_ε satisfies Nehari's identity

$$\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2\, dx = \theta_\varepsilon \int f_\varepsilon(w_\varepsilon) w_\varepsilon\, dx, \tag{3.3}$$

and Pohožaev's identity (see e.g. [5, Proposition 1])

$$\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2\, dx = \theta_\varepsilon 2^* \int F_\varepsilon(w_\varepsilon)\, dx. \tag{3.4}$$

The latter immediately implies that

$$\theta_\varepsilon = S_\varepsilon. \tag{3.5}$$

Then a direct calculation involving (3.5) shows that the rescaled function

$$u_\varepsilon(x) := w_\varepsilon(x/\sqrt{S_\varepsilon}) \tag{3.6}$$

is the radial ground state of (P_ε), described in Theorem A. Another simple consequence of (3.4) is that (P_ε) has no nontrivial finite energy solutions for $\varepsilon \geq \varepsilon_*$.

Equivalently to (S_ε), we may seek to minimize the quotient

$$\mathcal{S}_\varepsilon(w) := \frac{\|\nabla w\|_2^2}{(2^* \int_{\mathbb{R}^N} F_\varepsilon(w)\, dx)^{(N-2)/N}}, \qquad w \in \mathcal{M}_\varepsilon,$$

where

$$\mathcal{M}_\varepsilon := \left\{ 0 \le u \in D^1(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} F_\varepsilon(w)\,dx > 0 \right\}. \tag{3.7}$$

Clearly, if we set $w_\lambda(x) := w(\lambda x)$ then $\mathcal{S}_\varepsilon(w_\lambda) = \mathcal{S}_\varepsilon(w)$ for all $\lambda > 0$, that is, $\mathcal{S}_\varepsilon$ is invariant under dilations. This implies that

$$S_\varepsilon = \inf_{w \in \mathcal{M}_\varepsilon} \mathcal{S}_\varepsilon(w). \tag{3.8}$$

In addition, since clearly $\mathcal{M}_{\varepsilon_2} \subset \mathcal{M}_{\varepsilon_1}$ for $\varepsilon_2 > \varepsilon_1 > 0$, (3.8) shows that S_ε *is a nondecreasing function of* $\varepsilon \in (0, \varepsilon_*)$.

One of the consequences of Pohožaev's identity (3.4) is an expression for the total energy of the solution

$$\mathcal{E}_\varepsilon(u_\varepsilon) = \left(\frac{1}{2} - \frac{1}{2^*}\right) S_\varepsilon^{N/2},$$

(see [5, Corollary 2]), which shows that u_ε is indeed a ground state, i.e. a nontrivial solution with the least energy.

We will be frequently using the following well-known decay and compactness properties of radial functions on $\mathbb{R}^N$.

Lemma 3.1 ([5, Lemma A.IV, Theorem A.I′]).

(1) *Let* $s \ge 1$ *and let* $u \in L^s(\mathbb{R}^N)$ *be a radial nonincreasing function. Then for every* $x \neq 0$,

$$u(x) \le C_{s,N} |x|^{-N/s} \|u\|_s, \tag{3.9}$$

where $C_{s,N} = |B_1(0)|^{-1/s}$.

(2) *Let* $u_n \in H^1(\mathbb{R}^N)$ *be a sequence of radial nondecreasing functions such that* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^N)$. *Then upon extracting a subsequence,*

$$u_n \to u \quad \textit{in } L^\infty(\mathbb{R}^N \setminus B_r(0)) \textit{ and } L^s(\mathbb{R}^N \setminus B_r(0)) \quad \forall r > 0,\ \forall s > 2^*.$$

4. Critical case $p = 2^*$

Throughout this section we always assume that $p = 2^*$. In this critical case Pohožaev's identity implies that both the limit equation (P_0) and the canonically rescaled limit equation (R_0) have no positive finite energy solutions. We are going to show that after a suitably chosen rescaling, the limit equation for (P_ε) is in fact given by the critical Emden–Fowler equation.

4.1. Critical Emden–Fowler equation

Let

$$S_* := \inf\left\{\int_{\mathbb{R}^N} |\nabla w|^2\,dx \;\middle|\; w \in D^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |w|^p\,dx = 1\right\} \tag{S_*}$$

be the optimal constant in the Sobolev inequality

$$\int_{\mathbb{R}^N} |\nabla w|^2\,dx \ge S_*\left(\int_{\mathbb{R}^N} |w|^p\,dx\right)^{2/p}, \qquad \forall w \in D^1(\mathbb{R}^N).$$

It is known (cf. [32, Theorem 1.42]) that S_* is achieved by translations of the rescaled family

$$W_\lambda(x) := U_\lambda\big(\sqrt{S_*}x\big),$$

where $U_\lambda(x)$ are the ground states of the critical Emden–Fowler equation (R_*), explicitly defined by (2.2). Clearly,

$$\|W_\lambda\|_p = 1, \qquad \|\nabla W_\lambda\|_2^2 = S_*. \tag{4.1}$$

A straightforward computation leads to the explicit expression

$$\|\nabla U_\lambda\|_2^2 = \|U_\lambda\|_p^p = S_*^{N/2}.$$

Note that the family of minimizers W_λ solves the Euler–Lagrange equation

$$-\Delta W = S_* W^{p-1} \quad \text{in } \mathbb{R}^N.$$

4.2. Variational estimates of S_ε

For our purposes it is convenient to consider the dilation invariant Sobolev quotient

$$\mathcal{S}_*(w) := \frac{\int_{\mathbb{R}^N} |\nabla w|^2\,dx}{(\int_{\mathbb{R}^N} |w|^p\,dx)^{(N-2)/N}}, \qquad w \in D^1(\mathbb{R}^N), \quad w \ne 0,$$

so that

$$S_* = \inf_{0 \ne w \in D^1(\mathbb{R}^N)} \mathcal{S}_*(w).$$

Denote

$$\sigma_\varepsilon := S_\varepsilon - S_*.$$

In order to control σ_ε in terms of ε, we shall use Sobolev's minimizers W_λ as a family of test functions for $\mathcal{S}_\varepsilon$. Note that since $W_\lambda \in L^2(\mathbb{R}^N)$ only if $N \ge 5$, we shall consider the higher and lower dimensions separately. Straightforward calculations show that $W_\lambda \in L^s(\mathbb{R}^N)$ for all $s > N/(N-2)$, with

$$\|W_\lambda\|_s^s = \lambda^{N-\frac{2s}{p-2}}\|W_1\|_s^s = \lambda^{-\frac{N-2}{2}(s-p)}\|W_1\|_s^s.$$

In particular, if $N \geq 5$ then $W_\lambda \in L^2(\mathbb{R}^N)$ and

$$\|W_\lambda\|_2^2 = \lambda^2 \|W_1\|_2^2.$$

To consider dimensions $N = 3, 4$, given $R \gg \lambda$, we introduce a cut-off function $\eta_R \in C_c^\infty(\mathbb{R})$ such that $\eta_R(r) = 1$ for $|r| < R$, $0 < \eta(r) < 1$ for $R < |r| < 2R$, $\eta_R(r) = 0$ for $|r| > 2R$ and $|\eta'(r)| \leq 2/R$. We then compute as in, e.g., [28, Chapter III, proof of Theorem 2.1]:[1]

$$\int |\nabla(\eta_R W_\lambda)|^2 = S_* + O((R/\lambda)^{-(N-2)}), \tag{4.2}$$

$$\int |\eta_R W_\lambda|^p = 1 - O((R/\lambda)^{-N}), \tag{4.3}$$

$$\int |\eta_R W_\lambda|^q = \lambda^{-\frac{N-2}{2}(q-p)} \|W_1\|_q^q (1 - O((R/\lambda)^{-(N-2)(q-N/(N-2))})), \tag{4.4}$$

$$\int |\eta_R W_\lambda|^2 = \lambda^2 \|\eta_{R/\lambda}^2 W_1\|_2^2 = \begin{cases} O(\lambda^2 \log(R/\lambda)), & N = 4, \\ O(\lambda R), & N = 3. \end{cases} \tag{4.5}$$

Using the above calculations we obtain an upper estimate of σ_ε which is essential for further considerations.

Lemma 4.1. *We have*

$$0 < \sigma_\varepsilon \lesssim \begin{cases} \varepsilon^{\frac{q-p}{q-2}}, & N \geq 5, \\ \left(\varepsilon \log \frac{1}{\varepsilon}\right)^{\frac{q-4}{q-2}}, & N = 4, \\ \varepsilon^{\frac{q-6}{2q-8}}, & N = 3. \end{cases} \tag{4.6}$$

In particular, $\sigma_\varepsilon \to 0$ as $\varepsilon \to 0$.

Proof. To prove that $\sigma_\varepsilon > 0$ simply note that

$$S_* \leq \mathcal{S}_*(w_\varepsilon) < \mathcal{S}_\varepsilon(w_\varepsilon) = S_\varepsilon.$$

We shall now establish the upper bound on σ_ε, which clearly tends to zero as $\varepsilon \to 0$.

Case $N \geq 5$. Using W_λ as a family of test functions, we obtain $W_\lambda \in \mathcal{M}_\varepsilon$ for sufficiently small ε and sufficiently large λ, and we have

$$\mathcal{S}_\varepsilon(W_\lambda) \leq \frac{S_*}{(1 - \beta_2 \varepsilon \lambda^2 - \beta_q \lambda^{-\frac{N-2}{2}(q-p)})^{(N-2)/N}}, \tag{4.7}$$

[1] Note that if $0 < U \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ solves $-\Delta U = kU^{p-1}$, $x \in \mathbb{R}^N$, for some $k \neq 0$, then

$$\int |\nabla(\eta U)|^2 dx = k \int \eta^2 |U|^p dx + \int |\nabla \eta|^2 U^2 dx \quad \text{for all } \eta \in C_c^\infty(\mathbb{R}^N).$$

See also [28, Chapter III, proof of Theorem 2.1].

where

$$\beta_2 := \frac{p}{2}\|W_1\|_2^2, \qquad \beta_q := \frac{p}{q}\|W_1\|_q^q.$$

To minimize the right hand side of (4.7), we have to minimize the scalar function

$$\psi(\lambda) := \beta_2 \varepsilon \lambda^2 + \beta_q \lambda^{-\frac{N-2}{2}(q-p)}.$$

A direct computation shows that ψ achieves its minimum in scaling at

$$\lambda_\varepsilon = \varepsilon^{-\frac{2}{(N-2)(q-2)}} \tag{4.8}$$

and

$$\min_{\lambda>0} \psi \sim \psi(\lambda_\varepsilon) \sim \varepsilon^{\frac{q-p}{q-2}}.$$

For $N \geq 5$, we conclude that

$$\mathcal{S}_\varepsilon(W_\lambda) \leq \frac{S_*}{(1-\psi(\lambda_\varepsilon))^{(N-2)/N}} = S_*(1 + O(\psi(\lambda_\varepsilon)) = S_* + O(\varepsilon^{\frac{q-p}{q-2}}),$$

and the bound (4.6) is achieved on the function W_{λ_ε}, where λ_ε is given by (4.8).

Case $N = 4$. Assume $R \gg \lambda$. Testing against $\eta_R W_\lambda$ and using the calculations in (4.2)–(4.5) with $p = 4$, we obtain

$$\begin{aligned}\mathcal{S}_\varepsilon(\eta_R W_\lambda) &\leq (S_* + O((R/\lambda)^{-2})) \\ &\quad \times \left([1 - O((R/\lambda)^{-4})] - \varepsilon\lambda^2 O(\log R/\lambda) - \beta_q \lambda^{-(q-4)}[1 - O((R/\lambda)^{-2(q-2)})]\right)^{-1/2} \\ &\leq S_*(1 + O(\psi(\lambda, R))),\end{aligned}$$

where

$$\psi(\lambda, R) = \varepsilon\lambda^2 O(\log R/\lambda) + O((R/\lambda)^{-2}) + \beta_q \lambda^{-(q-4)}[1 - o(1)].$$

Choose

$$\lambda_\varepsilon = \left(\varepsilon \log \frac{1}{\varepsilon}\right)^{-1/(q-2)}, \qquad R_\varepsilon = \varepsilon^{-1/2}. \tag{4.9}$$

A routine calculation shows that as $\varepsilon \to 0$,

$$\log \frac{R_\varepsilon}{\lambda_\varepsilon} \sim \log \frac{1}{\varepsilon},$$

and hence

$$\psi(\lambda_\varepsilon, R_\varepsilon) \sim \left(\varepsilon \log \frac{1}{\varepsilon}\right)^{q-4/q-2}.$$

Thus bound (4.6) is achieved by the test function $\eta_{R_\varepsilon} W_{\lambda_\varepsilon}$, where λ_ε and R_ε are given by (4.9).

Case $N = 3$. Assume $R \gg \lambda$. Testing against $\eta_R W_\lambda$ and using the calculations in (4.2)–(4.5) with $p = 6$, we obtain

$$\begin{aligned}\mathcal{S}_\varepsilon(\eta_R W_\lambda) &\le (S_* + O((R/\lambda)^{-1}))\\ &\times \left([1 - O((R/\lambda)^{-3})] - \varepsilon\lambda O(R) - \beta_q \lambda^{-1/2(q-6)}[1 - O((R/\lambda)^{-(q-3)})]\right)^{-1/3}\\ &\le S_*(1 + O(\psi(\lambda, R))),\end{aligned}$$

where

$$\psi(\lambda, R) = \varepsilon\lambda O(R) + O((R/\lambda)^{-1})) + \beta_q \lambda^{-\frac{1}{2}(q-6)}[1 - o(1)].$$

Choosing

$$\lambda_\varepsilon = \varepsilon^{-1/(q-4)}, \qquad R_\varepsilon = \varepsilon^{-1/2}, \tag{4.10}$$

we then find that

$$\psi(\lambda_\varepsilon, R_\varepsilon) \sim \varepsilon^{\frac{q-6}{2q-8}},$$

and the bound (4.6) is achieved on the test function $\eta_{R_\varepsilon} W_{\lambda_\varepsilon}$, where λ_ε and R_ε are given by (4.10). □

4.3. *Pohožaev estimates*

Nehari identity (3.3) combined with Pohožaev's identity (3.4) lead to the following important relations.

Lemma 4.2. *Set* $\kappa := \frac{q(p-2)}{2(q-p)} > 0$. *Then*

$$\|w_\varepsilon\|_q^q = \kappa\varepsilon\|w_\varepsilon\|_2^2,$$
$$\|w_\varepsilon\|_p^p = 1 + (\kappa + 1)\varepsilon\|w_\varepsilon\|_2^2.$$

Proof. Since w_ε is a minimizer of (S_ε), identities (3.3)–(3.5) read

$$1 = \|w_\varepsilon\|_p^p - \|w_\varepsilon\|_q^q - \varepsilon\|w_\varepsilon\|_2^2, \qquad 1 = \|w_\varepsilon\|_p^p - \frac{p}{q}\|w_\varepsilon\|_q^q - \frac{p}{2}\varepsilon\|w_\varepsilon\|_2^2.$$

Then the conclusion follows by a direct algebraic computation. □

Lemma 4.3. $\varepsilon(\kappa + 1)\|w_\varepsilon\|_2^2 \le \frac{N}{N-2} S_*^{-1}\sigma_\varepsilon(1 + o(1))$.

Proof. Since w_ε is a minimizer of (S_ε), with the help of Lemma 4.2 we obtain

$$S_* \le \mathcal{S}_*(w_\varepsilon) = \frac{\|\nabla w_\varepsilon\|_2^2}{\|w_\varepsilon\|_p^2} = \frac{S_\varepsilon}{(1 + (\kappa + 1)\varepsilon\|w_\varepsilon\|_2^2)^{(N-2)/N}},$$

or, equivalently,

$$S_*^{N/(N-2)}(1 + (\kappa + 1)\varepsilon\|w_\varepsilon\|_2^2) \le S_\varepsilon^{N/(N-2)}.$$

Since $\sigma_\varepsilon := S_\varepsilon - S_*$, rearranging and differentiating, for $\varepsilon \to 0$ we obtain

$$S_*^{N/(N-2)}(\kappa+1)\varepsilon\|w_\varepsilon\|_2^2 \le S_\varepsilon^{N/(N-2)} - S_*^{N/(N-2)} = \frac{N}{N-2} S_*^{2/(N-2)}\sigma_\varepsilon + o(\sigma_\varepsilon),$$

so the conclusion follows. □

Combining the results of the three lemmas just proved, we obtain the following result concerning the asymptotic behaviour of different norms associated with the minimizer w_ε of (S_ε).

Corollary 4.4. *As $\varepsilon \to 0$, we have*

$$\|w_\varepsilon\|_p^p \to 1, \qquad \|w_\varepsilon\|_q^q \to 0, \qquad \varepsilon\|w_\varepsilon\|_2^2 \to 0.$$

4.4. Optimal rescaling

Following [19], consider the concentration function

$$Q_\varepsilon(\lambda) = \int_{B_\lambda} |w_\varepsilon|^p \, dx,$$

where B_λ is the ball of radius λ centred at the origin. Clearly, $Q_\varepsilon(\cdot)$ is strictly increasing, with $\lim_{\lambda\to 0} Q_\varepsilon(\lambda) = 0$ and $\lim_{\lambda\to\infty} Q_\varepsilon(\lambda) = \|w_\varepsilon\|_p^p \to 1$ as $\varepsilon \to 0$ in view of Corollary 4.4. Therefore, the equation $Q_\varepsilon(\lambda) = Q_*$ with

$$Q_* := \int_{B_1} |W_1(x)|^p \, dx < 1$$

has a unique solution $\lambda = \lambda_\varepsilon > 0$ whenever $\varepsilon \ll 1$:

$$Q_\varepsilon(\lambda_\varepsilon) = Q_*. \tag{4.11}$$

Similarly, since the function

$$Q_0(\lambda) := \int_{B_{\lambda^{-1}}} |W_1(x)|^p \, dx = \int_{B_1} |W_\lambda(x)|^p \, dx$$

is strictly decreasing, with $\lim_{\lambda\to 0} Q_0(\lambda) = 1$ and $\lim_{\lambda\to\infty} Q_0(\lambda) = 0$, there is a unique solution to the equation $Q_0(\lambda) = Q_*$. In fact, by the definition of Q_* this equation is satisfied if and only if $\lambda = 1$.

Using the value of λ_ε implicitly determined by (4.11), we define the rescaled family

$$v_\varepsilon(x) := \lambda_\varepsilon^{(N-2)/2} w_\varepsilon(\lambda_\varepsilon x). \tag{4.12}$$

Note that

$$\|v_\varepsilon\|_p = \|w_\varepsilon\|_p = 1 + o(1), \qquad \|\nabla v_\varepsilon\|_2^2 = \|\nabla w_\varepsilon\|_2^2 = S_* + o(1), \tag{4.13}$$

i.e. (v_ε) is a minimizing family for S_*. Note also that

$$\int_{B_1} |v_\varepsilon(x)|^p\,dx = Q_*.$$

The next statement is a direct consequence of the Concentration–Compactness Principle of P.-L. Lions (cf. [28, Chapter I, Theorem 4.9]).

Lemma 4.5. *$\|\nabla(v_\varepsilon - W_1)\|_2 \to 0$ and $\|v_\varepsilon - W_1\|_p \to 0$ as $\varepsilon \to 0$.*

Proof. By (4.13), for any sequence $\varepsilon_n \to 0$ there exist a subsequence $(\varepsilon_{n'})$ such that $(v_{\varepsilon_{n'}})$ converges weakly in $D^1(\mathbb{R}^N)$ to some radial function $w_0 \in D^1(\mathbb{R}^N)$. Applying the Concentration–Compactness Principle (cf. [28, Chapter I, Theorem 4.9] or [32, Theorem 1.41]) to $\|v_\varepsilon\|_p^{-1} v_\varepsilon$, we further conclude that in fact $(v_{\varepsilon_{n'}})$ converges to w_0 strongly in $D^1(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$. As a consequence, $\|w_0\|_p = 1$ and hence w_0 is a radial minimizer of (S_*), that is, $w_0 \in \{W_\lambda\}_{\lambda>0}$. Furthermore,

$$\int_{B_1} |w_0(x)|^p\,dx = Q_*.$$

We therefore conclude that $w_0 = W_1$. Finally, by uniqueness of the limit the full sequence (v_n) converges to W_1 strongly in $D^1(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$. □

4.5. *Rescaled equation estimates*

The rescaled minimizer v_ε defined in (4.12) solves the equation

$$-\Delta v_\varepsilon + S_\varepsilon \varepsilon \lambda_\varepsilon^2\, v_\varepsilon = S_\varepsilon\big(|v_\varepsilon|^{p-2} v_\varepsilon - \lambda_\varepsilon^{-\frac{2(q-p)}{p-2}} |v_\varepsilon|^{q-2} v_\varepsilon\big), \qquad (R_\varepsilon^*)$$

obtained from the Euler–Lagrange equation (3.2) for (S_ε). From the definition of v_ε we obtain

$$\|v_\varepsilon\|_q^q = \lambda_\varepsilon^{2(q-p)/p-2}\|w_\varepsilon\|_q^q, \qquad \|v_\varepsilon\|_2^2 = \lambda_\varepsilon^{-2}\|w_\varepsilon\|_2^2.$$

From Lemmas 4.2 and 4.3 we then derive the essential relation

$$\lambda_\varepsilon^{-\frac{2(q-p)}{p-2}} \|v_\varepsilon\|_q^q = \kappa\varepsilon\lambda_\varepsilon^2\|v_\varepsilon\|_2^2 \lesssim \sigma_\varepsilon, \qquad (4.14)$$

which leads to the following two-sided estimate:

Lemma 4.6. $\sigma_\varepsilon^{-\frac{p-2}{2(q-p)}} \lesssim \lambda_\varepsilon \lesssim \varepsilon^{-1/2}\sigma_\varepsilon^{\varepsilon/2}$.

Proof. Follows directly from (4.14) by observing that

$$\liminf_{\varepsilon\to 0} \|v_\varepsilon\|_q > 0, \qquad \liminf_{\varepsilon\to 0} \|v_\varepsilon\|_2 > 0.$$

To prove the latter, we note that by Lemma 4.5 and in view of the embedding $L^q(B_1) \subset L^p(B_1)$ we have

$$c\|v_\varepsilon\chi_{B_1}\|_q \geq \|v_\varepsilon\chi_{B_1}\|_p \geq \|W_1\chi_{B_1}\|_p - \|(W_1 - v_\varepsilon)\chi_{B_1}\|_p = \|W_1\chi_{B_1}\|_p - o(1),$$

where χ_{B_R} is the characteristic function of B_R. Similarly, in view of the embedding $L^p(B_1) \subset L^2(B_1)$ we obtain

$$\|v_\varepsilon \chi_{B_1}\|_2 \geq \|W_1 \chi_{B_1}\|_2 - \|(W_1 - v_\varepsilon)\chi_{B_1}\|_2 = \|W_1 \chi_{B_1}\|_2 - o(1),$$

so the assertion follows. ◻

Using estimate (4.6), we infer from Lemma 4.6 a lower bound

$$\lambda_\varepsilon \gtrsim \sigma_\varepsilon^{-\frac{1}{2}\frac{p-2}{q-p}} \gtrsim \begin{cases} \varepsilon^{-\frac{p-2}{2q-4}}, & N \geq 5, \\ \left(\varepsilon \log \frac{1}{\varepsilon}\right)^{-1/(q-2)}, & N = 4, \\ \varepsilon^{-1/(q-4)}, & N = 3, \end{cases} \tag{4.15}$$

and an upper bound

$$\lambda_\varepsilon \lesssim \begin{cases} \varepsilon^{-\frac{1}{2}\frac{p-2}{q-2}}, & N \geq 5, \\ \varepsilon^{-\frac{1}{q-2}}\left(\log \frac{1}{\varepsilon}\right)^{\frac{q-4}{2q-4}}, & N = 4, \\ \varepsilon^{-\frac{q-2}{4(q-4)}}, & N = 3. \end{cases} \tag{4.16}$$

Note that for $N \geq 5$ the above lower and upper estimates are equivalent, and as a consequence we obtain the following.

Corollary 4.7. *Assume $N \geq 5$. Then $\|v_\varepsilon\|_q$ and $\|v_\varepsilon\|_2$ are bounded.*

Proof. Follows from (4.14)–(4.16). ◻

In the lower dimensions the growth of $\|v_\varepsilon\|_2$ is to be taken into account to obtain matching bounds, so instead of (4.16) we shall use a more explicit upper bound

$$\lambda_\varepsilon \lesssim \frac{\varepsilon^{-1/2}\sigma_\varepsilon^{1/2}}{\|v_\varepsilon\|_2} \lesssim \|v_\varepsilon\|_2^{-1} \begin{cases} \varepsilon^{-\frac{1}{q-2}}\left(\log \frac{1}{\varepsilon}\right)^{\frac{q-4}{2q-4}}, & N = 4, \\ \varepsilon^{-\frac{q-2}{4(q-4)}}, & N = 3, \end{cases} \tag{4.17}$$

which is also a combination of (4.14) and (4.6).

4.6. A lower barrier

To control the norm $\|v_\varepsilon\|_2$, we note that

$$-\Delta v_\varepsilon + S_\varepsilon \varepsilon \lambda_\varepsilon^2 v_\varepsilon = S_\varepsilon\big(v_\varepsilon^{p-1} - \lambda_\varepsilon^{-\frac{2(q-p)}{p-2}} v_\varepsilon^{q-1}\big) \geq -V_\varepsilon(x)\, v_\varepsilon, \qquad x \in \mathbb{R}^N,$$

where

$$V_\varepsilon(x) := S_\varepsilon \lambda_\varepsilon^{-\frac{2(q-p)}{p-2}} v_\varepsilon^{q-2}(x).$$

According to the radial estimate (3.9),

$$u_\varepsilon(x) \leq C_p |x|^{-2/(p-2)} \|u_\varepsilon\|_p.$$

Using (4.13) and the fact that $\lambda_\varepsilon^{-\frac{2(q-p)}{p-2}} \lesssim \sigma_\varepsilon \to 0$ by Lemmas 4.1 and 4.6, for sufficiently small $\varepsilon > 0$ we obtain

$$V_\varepsilon(x) = S_\varepsilon \lambda_\varepsilon^{-\frac{2(q-p)}{p-2}} v_\varepsilon^{q-2}(x) \le S_\varepsilon \lambda_\varepsilon^{-\frac{2(q-p)}{p-2}} C_p^{q-2} \|v_\varepsilon\|_p^{q-2} |x|^{-\frac{2(q-2)}{p-2}} \le C|x|^{-\frac{2(q-2)}{p-2}},$$

where the constant $C > 0$ does not depend on ε or x. Therefore, for small $\varepsilon > 0$ solutions $v_\varepsilon > 0$ satisfy the linear inequality

$$-\Delta v_\varepsilon + V_0(x) v_\varepsilon + S_\varepsilon \varepsilon \lambda_\varepsilon^2 v_\varepsilon \ge 0, \qquad x \in \mathbb{R}^N,$$

where $V_0(x) := C|x|^{-\frac{2(q-2)}{p-2}}$.

Lemma 4.8. *There exists $R > 0$ and $c > 0$ such that for all small $\varepsilon > 0$,*

$$v_\varepsilon(x) \ge c|x|^{-(N-2)} e^{-\sqrt{\varepsilon S_\varepsilon}\,\lambda_\varepsilon |x|} \qquad (|x| > R).$$

Proof. Define the barrier

$$h_\varepsilon(x) := (|x|^{-(N-2)} + |x|^\beta) e^{-\sqrt{\varepsilon S_\varepsilon}\,\lambda_\varepsilon |x|},$$

where $\beta < 0$ is fixed in such a way that

$$-(N-2) - \frac{2(q-p)}{p-2} < \beta < -(N-2),$$

and the value of $c > 0$ will be specified later. A direct computation then shows that for some $R \gg 1$ one gets

$$\begin{aligned}
&-\Delta h_\varepsilon + V_0(x) h_\varepsilon + \varepsilon \lambda_\varepsilon^2 h_\varepsilon \\
&\quad = \{-\beta(\beta + N - 2)|x|^{\beta-2} + C(|x|^{-(N-2)} + |x|^\beta)|x|^{-2\frac{q-2}{p-2}} \\
&\qquad + \sqrt{\varepsilon S_\varepsilon}\,\lambda_\varepsilon((2\beta + N - 1)|x|^{\beta-1} + (3 - N)|x|^{-(N-1)})\}\, e^{-\sqrt{\varepsilon S_\varepsilon}\,\lambda_\varepsilon |x|} \\
&\quad \le \{-\beta(\beta + N - 2)|x|^{\beta-2} + 2C|x|^{-(N-2)-2\frac{q-2}{p-2}}\}\, e^{-\sqrt{\varepsilon S_\varepsilon}\lambda_\varepsilon |x|} \le 0,
\end{aligned}$$

for all $|x| > R$, where $R \gg 1$ can be chosen independent of $\varepsilon > 0$.

Note that Lemmas 4.5 and 3.1 imply

$$\|(v_\varepsilon - W_1)\chi_{B_{2R} \setminus B_{R/2}}\|_\infty \to 0,$$

and hence

$$v_\varepsilon(R) \ge \tfrac{1}{2} W_1(R),$$

for all sufficiently small $\varepsilon > 0$. Choose $c > 0$ so that

$$c(R^{-(N-2)} + R^\beta) \le \tfrac{1}{2} W_1(R).$$

Then

$$v_\varepsilon \ge c h_\varepsilon \quad \text{for } |x| > R,$$

by the comparison principle for the operator $-\Delta + V_0 + \varepsilon\lambda_\varepsilon^2$ (see, e.g., [2, Theorem 2.7]). $\square$

4.7. Cases $N = 3$ and $N = 4$ completed

We shall apply Lemma 4.8 to obtain matching estimates on the blow-up of $\|v_\varepsilon\|_2$ in low dimensions.

Lemma 4.9. *If $N = 3$ then $\|v_\varepsilon\|_2^2 \gtrsim \frac{1}{\sqrt{\varepsilon}\,\lambda_\varepsilon}$.*

Proof. Assuming $N = 3$ we directly calculate from Lemma 4.8:

$$\|v_\varepsilon\|_2^2 \geq \int_{\mathbb{R}^3 \setminus B_R} |v_\varepsilon|^2 \, dx \geq \int_R^\infty c^2 e^{-2\sqrt{\varepsilon S_\varepsilon}\,\lambda_\varepsilon r} \, dr \geq \frac{C}{\sqrt{\varepsilon}\,\lambda_\varepsilon},$$

as required. □

As an immediate corollary, using (4.17), we obtain an upper estimate of λ_ε which matches the lower bound of (4.15) in the case $N = 3$.

Corollary 4.10. *If $N = 3$ then $\lambda_\varepsilon \lesssim \varepsilon^{-1/(q-4)}$.*

Next we consider the case $N = 4$.

Lemma 4.11. *If $N = 4$ then $\|v_\varepsilon\|_2^2 \gtrsim \log \frac{1}{\sqrt{\varepsilon}\,\lambda_\varepsilon}$.*

Proof. Assuming $N = 4$ we directly calculate using Lemma 4.8:

$$\|v_\varepsilon\|_2^2 \geq \int_{\mathbb{R}^4 \setminus B_R} |v_\varepsilon|^2 \, dx \geq \int_R^\infty c^2 r^{-1} e^{-2\sqrt{\varepsilon S_\varepsilon}\,\lambda_\varepsilon r} \, dr = c^2 \Gamma(0, 2\sqrt{\varepsilon S_\varepsilon}\,\lambda_\varepsilon R),$$

where

$$\Gamma(0, t) = -\log(t) - \gamma + O(t), \qquad t \searrow 0,$$

is the incomplete Gamma function and $\gamma \approx 0.5772$ is the Euler constant [1]. Hence we obtain, for sufficiently small ε,

$$\|v_\varepsilon\|_2^2 \geq c^2(-\log(2\sqrt{\varepsilon S_\varepsilon}\,\lambda_\varepsilon R) - \gamma) \geq C \log\left(\frac{1}{\sqrt{\varepsilon}\,\lambda_\varepsilon}\right),$$

as required. □

Corollary 4.12. *If $N = 4$ then $\lambda_\varepsilon \lesssim \left(\varepsilon \log \frac{1}{\varepsilon}\right)^{-1/(q-2)}$.*

Proof. An immediate corollary of (4.14) and (4.6) is the relation

$$C\varepsilon\lambda_\varepsilon^2 \log \frac{1}{\sqrt{\varepsilon}\,\lambda_\varepsilon} \leq \left(\varepsilon \log \frac{1}{\varepsilon}\right)^{\frac{q-4}{q-2}}.$$

Note that $\varepsilon^{\delta_1} \leq \sqrt{\varepsilon}\,\lambda_\varepsilon \leq \varepsilon^{\delta_2}$ for some $\delta_{1,2} \geq 0$ and ε small enough, which is a consequence of (4.16) and (4.15). Therefore,

$$\log \frac{1}{\sqrt{\varepsilon}\,\lambda_\varepsilon} \sim \log \frac{1}{\varepsilon},$$

and the conclusion follows. □

4.8. *Further estimates*

The results in the previous section could be used in a standard way to improve upon some earlier estimates.

An immediate consequence of the sharp upper estimates of λ_ε is the following.

Corollary 4.13. $\|v_\varepsilon\|_q = O(1)$.

The boundedness of the L^q norm also allows one to reverse estimates of $\|v_\varepsilon\|_2$ via (4.14).

Corollary 4.14.

$$\|v_\varepsilon\|_2^2 = \begin{cases} O(1), & N \geq 5, \\ O\big(\log \frac{1}{\varepsilon}\big), & N = 4, \\ O(\varepsilon^{-\frac{1}{2}\frac{q-6}{q-4}}), & N = 3. \end{cases}$$

We now prove that the L^q bound also implies an L^∞ bound.

Lemma 4.15. $\|v_\varepsilon\|_\infty = O(1)$.

Proof. Note that by (R_ε^*) the function v_ε is a positive solution of the linear inequality

$$-\Delta v_\varepsilon - V_\varepsilon(x) v_\varepsilon \leq 0, \qquad x \in \mathbb{R}^N,$$

where

$$V_\varepsilon(x) := S_\varepsilon v_\varepsilon^{p-2}(x).$$

From the radial estimate (3.9) we obtain

$$v_\varepsilon(x) \leq C_q \|v_\varepsilon\|_q |x|^{-N/q}. \tag{4.18}$$

Hence, using Corollary 4.13 we obtain

$$V_\varepsilon(x) \leq S_\varepsilon C_q^{p-2} \|v_\varepsilon\|_q^{p-2} |x|^{-N(p-2)/q} \leq C_* |x|^{-2p/q},$$

for some constant $C_* > 0$ which does not depend on ε or x. As a consequence, v_ε is a positive solution of the linear inequality

$$-\Delta v_\varepsilon - V_*(x) v_\varepsilon \leq 0, \qquad x \in \mathbb{R}^N, \tag{4.19}$$

where $V_*(x) = C_* |x|^{-2p/q} \in L^s_{\text{loc}}(\mathbb{R}^N)$ for some $s > N/2$. The result can then be deduced from the weak Harnack inequality for subsolutions of (4.19) (cf. [26, Remark 5.1 on p. 226]). Here we give an elementary proof that also works in the present context. Integrating the inequality (4.19) over a ball and applying the divergence theorem, by monotonic decrease of $v_\varepsilon(x)$ in $|x|$ we have

$$|\nabla v_\varepsilon(x)| \leq \frac{C}{|x|^{N-1}} \int_{B_{|x|}(0)} V_*(y) v_\varepsilon(y)\, dy \leq C' v_\varepsilon(0) |x|^{1-2p/q},$$

for some $C, C' > 0$ independent of ε and x. Integrating again along the straight line from 0 to x_0, we obtain

$$v_\varepsilon(0) \le v_\varepsilon(x_0) + C'' v_\varepsilon(0)|x_0|^{2(q-p)/q},$$

for some $C'' > 0$ independent of ε and x. We then conclude by choosing $|x_0|$ sufficiently small independently of ε, using (4.18) and Corollary 4.13. □

A standard consequence of the L^∞ bound and elliptic regularity theory is the following convergence statement.

Corollary 4.16. *$v_\varepsilon \to U_1$ in $C^2(\mathbb{R}^N)$ and $L^s(\mathbb{R}^N)$ for any $s \ge p$. In particular,*

$$v_\varepsilon(0) \ \simeq \ W_1(0). \tag{4.20}$$

Proof. Indeed, a consequence of the L^∞ bound of Lemma 4.15 and convergence in $D^1(\mathbb{R}^N)$ via the compactness result for monotone radial functions in Lemma 3.1 is convergence in $L^s(\mathbb{R}^N)$ for any $s \ge p$. Then a Calderón–Zygmund estimate [15, Theorem 9.11] implies convergence in $W^{2,s}_{\mathrm{loc}}(\mathbb{R}^N)$ and, hence, by Sobolev embedding, also in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. Since the nonlinearity in (R^*_ε) is smooth, using Schauder's estimates [15, Theorem 6.2, 6.6] we deduce convergence in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$. Finally, taking into account that the constants in Schauder estimates are uniform with respect to translations, we conclude convergence in $C^2(\mathbb{R}^N)$. □

Taking into account that

$$u_\varepsilon(0) \ \sim \ \lambda_\varepsilon^{-2/(p-2)} v_\varepsilon(0),$$

we can use (4.20) to estimate the amplitude of $u_\varepsilon(0)$ to derive (2.4), which completes the proof of Theorem 2.5.

5. Supercritical case $p > 2^*$

5.1. *The limit equation*

For $p > 2^*$ the limit equation

$$-\Delta u - |u|^{p-2}u + |u|^{q-2}u = 0 \quad \text{in } \mathbb{R}^N \tag{P_0}$$

admits a unique positive radial ground state solution $u_0 \in D^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$. Further, it is known that $u_0 \in C^2(\mathbb{R}^N)$, $u_0(x)$ is a decreasing function of $|x|$, and there exists $C_0 > 0$ such that

$$\lim_{|x|\to\infty} |x|^{N-2} u_0(x) = C_0 > 0 \tag{5.1}$$

(see [5, Theorem 4] for existence, or [20, 21] for existence and asymptotic decay, and [21, 18] for the uniqueness proofs).

Similarly to (3.6), the ground state u_0 admits a variational characterization in the Sobolev space $D^1(\mathbb{R}^N)$ via the rescaling

$$u_0(x) := w_0(x/\sqrt{S_0}), \tag{5.2}$$

where w_0 is the positive radial (i.e., depending only on $|x|$) minimizer of the constrained minimization problem

$$S_0 := \inf\left\{\int_{\mathbb{R}^N} |\nabla w|^2\,dx \;\middle|\; w \in D^1(\mathbb{R}^N),\ 2^*\int_{\mathbb{R}^N} F_0(w)\,dx = 1\right\}, \tag{S_0}$$

where F_0 is the truncated nonlinearity defined by (3.1) (see [5, Section 5]), so that the minimization problem (S_0) is well defined on $D^1(\mathbb{R}^N)$. Similarly to (3.2)–(3.5), one concludes that the minimizer w_0 solves the Euler–Lagrange equation

$$-\Delta w_0 = S_0(|w_0|^{p-2}w_0 - |w_0|^{q-2}w_0) \quad \text{in } \mathbb{R}^N. \tag{5.3}$$

Further, w_0 satisfies Nehari's identity

$$\int_{\mathbb{R}^N} |\nabla w_0|^2\,dx = S_0\int (|w_0|^p - |w_0|^q)\,dx,$$

and Pohožaev's identity (see e.g. [5, Proposition 1])

$$\int_{\mathbb{R}^N} |\nabla w_0|^2\,dx = S_0 2^*\int\left(\frac{|w_0|^p}{p} - \frac{|w_0|^q}{q}\right)dx.$$

Taking into account that $\|\nabla w_0\|_2^2 = S_0$, we then derive from Nehari and Pohožaev's identities the relation

$$\|w_0\|_p^p - \|w_0\|_q^q = \frac{2^*}{p}\|w_0\|_p^p - \frac{2^*}{q}\|w_0\|_q^q = 1, \tag{5.4}$$

which leads to the explicit expressions

$$\|w_0\|_p^p = \frac{(q-2^*)p}{(q-p)2^*}, \qquad \|w_0\|_q^q = \frac{(p-2^*)q}{(q-p)2^*}. \tag{5.5}$$

Remark 5.1. Note that the arguments leading to (5.5) also give nonexistence of nontrivial weak solutions $u \in D^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ of problem (P_0) in the case $2 < p \le 2^*$ and $q > p$.

5.2. Energy and norm estimates

To control the relations between S_ε and S_0 it is convenient to consider the (equivalent to (S_0)) scaling invariant quotient

$$\mathcal{S}_0(w) := \frac{\int_{\mathbb{R}^N} |\nabla w|^2\,dx}{(2^*\int_{\mathbb{R}^N} F_0(w)\,dx)^{(N-2)/N}}, \qquad w \in D^1(\mathbb{R}^N), \qquad \int_{\mathbb{R}^N} F_0(w)\,dx > 0. \tag{5.6}$$

Then

$$S_0 = \inf_{\substack{w\in D^1(\mathbb{R}^N)\\ F_0(w)>0}} \mathcal{S}_0(w). \tag{5.7}$$

Lemma 5.2. $0 < S_\varepsilon - S_0 \to 0$ *as* $\varepsilon \to 0$.

Proof. To show that $S_0 < S_\varepsilon$ simply note that

$$S_0 \le \mathcal{S}_0(w_\varepsilon) < \mathcal{S}_\varepsilon(w_\varepsilon) = S_\varepsilon. \tag{5.8}$$

To control S_ε from above we will use the minimizer w_0 as a test function for (S_ε). In view of (5.1), we have $w_0 \in L^2(\mathbb{R}^N)$ if and only if $N \ge 5$. Therefore we shall consider the higher and lower dimensions separately.

Case $N \ge 5$. Testing (S_ε) against w_0, we obtain

$$S_\varepsilon \le \mathcal{S}_\varepsilon(w_0) \le \frac{S_0}{(1 - \varepsilon\|w_0\|^2_{L^2(\mathbb{R}^N)})^{(N-2)/N}} \le S_0 + O(\varepsilon), \tag{5.9}$$

which proves the claim for $N \ge 5$.

To consider the lower dimensions, given $R > 1$ we introduce a cut-off function $\eta_R \in C_c^\infty(\mathbb{R})$ such that $\eta_R(r) = 1$ for $|r| < R$, $0 < \eta(r) < 1$ for $R < |r| < 2R$, $\eta_R(r) = 0$ for $|r| > 2R$ and $|\eta'(r)| \le 2/R$. Then taking into account (5.1), for $s > N/(N-2)$ we compute

$$\int_{\mathbb{R}^N} |\nabla(\eta_R w_0)|^2 = S_0 + O(R^{-(N-2)}),$$
$$\int_{\mathbb{R}^N} |\eta_R w_0|^s\,dx = \|w_0\|^s_{L^s(\mathbb{R}^N)}(1 - O(R^{N-s(N-2)})),$$
$$\int_{\mathbb{R}^N} |\eta_R w_0|^2 = \begin{cases} O(\log(R)), & N = 4,\\ O(R), & N = 3.\end{cases}$$

Case $N = 4$. Let $R = \varepsilon^{-1}$. Testing (S_ε) against $\eta_R w_0$ and using the fact that $p > 4$, we obtain

$$\begin{aligned} S_\varepsilon \le \mathcal{S}_\varepsilon(w_0) &\le \frac{S_0 + O(R^{-2})}{(1 - O(R^{-4}) - \varepsilon O(\log R))^{1/2}}\\ &\le \frac{S_0 + O(\varepsilon^2)}{\big(1 - O(\varepsilon^4) - O\big(\varepsilon\log\frac{1}{\varepsilon}\big)\big)^{1/2}} \le S_0 + O\left(\varepsilon\log\frac{1}{\varepsilon}\right),\end{aligned}$$

which proves the claim.

Case $N = 3$. Let $R = \varepsilon^{-1/2}$. Testing (S_ε) against $\eta_R w_0$ and using the fact that $p > 6$, we obtain

$$\begin{aligned} S_\varepsilon \le \mathcal{S}_\varepsilon(w_0) &\le \frac{S_0 + O(R^{-1})}{(1 - O(R^{-6}) - \varepsilon O(R))^{1/3}}\\ &\le \frac{S_0 + O(\varepsilon^{1/2})}{(1 - O(\varepsilon^{3/2}) - O(\varepsilon^{1/2}))^{1/3}} \le S_0 + O(\varepsilon^{1/2}),\end{aligned}$$

which completes the proof. □

Lemma 5.3. $\|w_\varepsilon\|_\infty \le 1$ *and* $\|w_\varepsilon\|_s \lesssim 1$ *for all* $s > 2^*$.

Proof. In view of (1.2) and (3.6) we have

$$\|w_\varepsilon\|_\infty = \|u_\varepsilon\|_\infty \le 1.$$

Using Sobolev's inequality and Lemma 5.2 we also obtain

$$\|w_\varepsilon\|_{2^*}^2 \le S_*^{-1}\|\nabla w_\varepsilon\|_2^2 = S_*^{-1} S_\varepsilon = S_*^{-1} S_0(1+o(1)).$$

Then for every $s > 2^*$,

$$\|w_\varepsilon\|_s^s \le \|w_\varepsilon\|_{2^*}^{2^*},$$

so the assertion follows. □

Lemma 5.4. $\varepsilon\|w_\varepsilon\|_2^2 \to 0$.

Proof. Since w_ε is a minimizer of (S_ε), we have

$$1 = 2^* \int_{\mathbb{R}^N} F_\varepsilon(w_\varepsilon)\,dx = 2^* \int_{\mathbb{R}^N} F_0(w_\varepsilon)\,dx - 2^* \frac{\varepsilon}{2}\|w_\varepsilon\|_2^2. \tag{5.10}$$

Therefore

$$\mathcal{S}_0(w_\varepsilon) = \frac{\|\nabla w_\varepsilon\|_2^2}{\left(2^* \int_{\mathbb{R}^N} F_0(w)\,dx\right)^{(N-2)/N}} = \frac{S_\varepsilon}{\left(1 + \frac{2^*}{2}\varepsilon\|w_\varepsilon\|_2^2\right)^{(N-2)/N}}.$$

Assume contrary to the statement of the lemma that $\limsup_{\varepsilon\to 0} \varepsilon\|w_\varepsilon\|_2^2 = m > 0$. Then by Lemma 5.2 for any sequence $\varepsilon_n \to 0$ we obtain

$$S_0 \le \mathcal{S}_0(w_{\varepsilon_n}) = \frac{S_{\varepsilon_n}}{\left(1 + \frac{2^*}{2}\varepsilon_n\|w_{\varepsilon_n}\|_2^2\right)^{(N-2)/N}} \le \frac{S_0(1+o(1))}{1 + \frac{2^*}{2}m} < S_0,$$

a contradiction. □

5.3. *Proof of Theorem 2.3*

Consider a sequence of $\varepsilon_n \to 0$. Since $\|\nabla w_{\varepsilon_n}\|_2^2 = S_{\varepsilon_n} \to S_0$, the sequence (ε_n) contains a subsequence, still denoted (ε_n), such that

$$w_{\varepsilon_n} \rightharpoonup \bar{w} \quad \text{in } D^1(\mathbb{R}^N) \quad \text{and} \quad w_{\varepsilon_n} \to \bar{w} \quad \text{a.e. in } \mathbb{R}^N,$$

where $\bar{w} \in D^1(\mathbb{R}^N)$ is a radial function. By Lemma 5.3, the sequence (w_{ε_n}) is bounded in $L^{2^*}(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$. Using Lemma 3.1 and Sobolev's inequality, we also obtain a uniform bound

$$w_\varepsilon(x) \le C|x|^{-(N-2)/2}\|\nabla w_\varepsilon\|_2 \le 2C|x|^{-(N-2)/2} S_0,$$

for ε sufficiently small. Using Lemma 3.1 we conclude that

$$w_{\varepsilon_n} \to \bar{w} \quad \text{in } L^s(\mathbb{R}^N) \quad \text{for any } s \in (2^*, \infty).$$

Taking into account Lemma 5.4 and (5.10) we also obtain

$$\int_{\mathbb{R}^N} F_0(\bar{w})\,dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} F_0(w_{\varepsilon_n})\,dx = \lim_{n\to\infty}\Big(1 + 2^*\frac{\varepsilon_n}{2}\|w_{\varepsilon_n}\|_2^2\Big) = 1.$$

By weak lower semicontinuity we also conclude that

$$\|\nabla \bar{w}\|_2^2 \le \liminf_{n\to\infty} \|\nabla w_{\varepsilon_n}\|_2^2 = S_0,$$

that is, $\bar{w}$ is a minimizer for (S_0). By the uniqueness of the radial minimizer of (S_0) we conclude that $\bar{w} = w_0$.

We now claim that (w_{ε_n}) converges strongly to w_0 in $D^1(\mathbb{R}^N)$. Indeed, we have

$$\begin{aligned}\|\nabla(w_{\varepsilon_n} - w_0)\|_2^2 &= \|\nabla w_{\varepsilon_n}\|_2^2 + \|\nabla w_0\|_2^2 - 2\int_{\mathbb{R}^N} \nabla w_{\varepsilon_n}\cdot\nabla w_0\,dx \\ &= S_{\varepsilon_n} + S_0 - 2\int_{\mathbb{R}^N} \nabla w_{\varepsilon_n}\cdot\nabla w_0\,dx.\end{aligned}$$

Estimating the last term and taking into account (5.3), (5.4) and the fact that by the Hölder inequality

$$\begin{aligned}\left|\int_{\mathbb{R}^N} f_0(w_0)(w_\varepsilon - w_0)\,dx\right| &\le \|f_0(w_0)\|_{p/(p-1)}\|w_\varepsilon - w_0\|_p \\ &\le C\|w_0\|_p^{p-1}\|w_\varepsilon - w_0\|_p \to 0,\end{aligned}$$

we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla w_{\varepsilon_n}\cdot\nabla w_0\,dx &= S_0\int_{\mathbb{R}^N} f_0(w_0)w_{\varepsilon_n}\,dx \\ &= S_0\int_{\mathbb{R}^N} f_0(w_0)w_0\,dx + S_0\int_{\mathbb{R}^N} f_0(w_0)(w_{\varepsilon_n} - w_0)\,dx \\ &= S_0(1 + o(1)),\end{aligned}$$

which proves the claim.

Since (w_{ε_n}) converges to w_0 in $D^1(\mathbb{R}^N)$ and in $L^s(\mathbb{R}^N)$ for any $s \ge 2^*$, similarly to the proof of Corollary 4.16 by standard elliptic regularity we conclude that (w_{ε_n}) converges to w_0 in $C^2(\mathbb{R}^N)$. The proof of of Theorem 2.3 is then completed by taking into account the uniqueness of w_0.

5.4. Remarks on a slightly supercritical limit problem

Here we discuss the asymptotic behaviour as $p \downarrow 2^*$ of the minimizer w_0 of the limit variational problem (S_0). For convenience, set $\delta := p - 2^* > 0$. To highlight the dependence on δ, in this section we denote the ground state energy in (5.7) by S_0^δ, while w_0^δ will be used to denote the corresponding minimizer. Also, in this section the asymptotic notation such as $\lesssim$ etc. is in terms of $\delta \to 0$.

The following summarizes our results regarding the asymptotic behaviour of w_0^δ as $\delta \downarrow 0$.

Proposition 5.5. $0 < S_0^\delta - S_* \to 0$ *for* $\delta \downarrow 0$. *In addition,*

$$\delta^{1/(q-2^*)} \lesssim w_0^\delta(0) \lesssim \delta^{1/(q+N)},$$

and, provided that $q > \frac{N(N+2)}{2(N-2)}$,

$$w_0^\delta(0) \sim \delta^{1/(q-2^*)}. \tag{5.11}$$

Let us note, however, that the asymptotic of $w_0^\delta(0)$ for general values of q is open, and numerical evidence suggests that the conclusion of (5.11) is *false* for q sufficiently close to 2^*.

To prove Proposition 5.5, we first establish a few basic estimates for the behaviour of the minimizer of the quotient in (5.6) as $\delta \to 0$.

Lemma 5.6. $\|w_0^\delta\|_\infty \le 1$, $\|\nabla w_0^\delta\|_2 \lesssim 1$, $\|w_0^\delta\|_p \lesssim 1$ *and* $\|w_0^\delta\|_q \lesssim \delta$.

Proof. The first inequality is an immediate consequence of $\|u_0\|_\infty \le 1$ and (5.2). To prove the second estimate, consider a suitable fixed test function $w \in C_0^\infty(\mathbb{R}^N)$ satisfying $0 \le w \le 1$. Then from (5.6) and (5.7) we conclude that $S_0^\delta \lesssim 1$ as $\delta \to 0$, implying the result. The last two inequalities are immediate consequences of (5.5). □

We now establish a rough upper bound on the amplitude of w_0^δ.

Lemma 5.7. $\|w_0^\delta\|_\infty \lesssim \delta^{1/(q+N)}$.

Proof. In view of the gradient estimate of Lemma 5.6, by the Calderón–Zygmund inequality [15, Theorem 9.11] applied to w_0^δ solving (5.3) we conclude that $\|w_0^\delta\|_{W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)}$ is uniformly bounded, and hence $\|\nabla w_0^\delta\|_\infty \le C$ for some $C > 0$ independent of δ for sufficiently small $\delta > 0$. This yields the following estimate for some $c > 0$ independent of δ:

$$c\|w_0^\delta\|_\infty^{q+N} \le \frac{1}{2^q}\|w_0^\delta\|_\infty^q |B_R(0)| \le \int_{B_R(0)} |w_0^\delta|^q dx \le \|w_0^\delta\|_q^q,$$

where $R = \|w_0^\delta\|_\infty/(2C)$, and we used monotonicity of $w_0^\delta(x)$ in $|x|$. The result then follows from the fact that $\|w_0^\delta\|_q^q \sim \delta$ by (5.5). □

The relations in (5.5) immediately lead to the following lower bound on $w_0^\delta(0)$, which was established in [18, Theorem A (1.7)] (see also [20, Proposition B]). We present a proof for completeness.

Lemma 5.8. $\|w_0^\delta\|_\infty \gtrsim \delta^{1/(q-2^*)}$.

Proof. Indeed, by (5.5) we have

$$\delta\|w_0^\delta\|_p^p \le \frac{p}{q}(q-2^*)\|w_0^\delta\|_\infty^{q-2^*-\delta}\|w_0^\delta\|_p^p,$$

and the result follows from $\|w_0^\delta\|_p^p > 0$ and smallness of δ. □

Importantly, for sufficiently large q we can prove a matching upper bound, yielding the precise asymptotic behaviour of the minimizer's amplitude as $\delta \to 0$.

Lemma 5.9. *If* $q > \frac{N(N+2)}{2(N-2)}$ *then* $\|w_0^\delta\|_\infty \lesssim \delta^{1/(q-2^*)}$.

Proof. In view of Lemmas 3.1 and 5.6 and the Sobolev inequality, we have

$$\begin{aligned} w_0^\delta(x) &\le \min\{C_{2^*}|x|^{-N/2^*}\|w_0^\delta\|_{2^*}, C_q|x|^{-N/q}\|w_0^\delta\|_q\}, \\ &\lesssim \min\{|x|^{-(N-2)/2}, \delta^{1/q}|x|^{-N/q}\}. \end{aligned}$$

In view of (5.1), (5.2) and Lemma 5.3, we can apply Newtonian kernel to (P_0^δ). We obtain

$$w_0^\delta(x) = S_0^\delta A_N \int_{\mathbb{R}^N} \frac{(w_0^\delta(y))^{p-1} - (w_0^\delta(y))^{q-1}}{|x-y|^{N-2}}\, dy,$$

where $A_N = \frac{\Gamma((N-2)/2)}{4\pi^{N/2}}$. In particular, for $q > \frac{N(N+2)}{2(N-2)}$ we have (with a slight abuse of notation)

$$\begin{aligned} w_0^\delta(0) &= \frac{S_0^\delta}{N-2}\int_0^\infty \big((w_0^\delta(r))^{p-1} - (w_0^\delta(r))^{q-1}\big) r\, dr \\ &\le S_*(1+o(1))\int_0^\infty (w_0^\delta(r))^{2^*-1} r\, dr \lesssim \int_0^\infty \min\{r^{-(N-2)/2}, \delta^{1/q} r^{-N/q}\}^{\frac{N+2}{N-2}} r\, dr, \\ &\lesssim \int_R^\infty r^{-N/2}\, dr + \delta^{\frac{N+2}{q(N-2)}} \int_0^R r^{1-\frac{N(N+2)}{q(N-2)}}\, dr \lesssim R^{-(N-2)/2} + \delta^{\frac{N+2}{q(N-2)}} R^{2-\frac{N(N+2)}{q(N-2)}}. \end{aligned}$$

Minimizing for $q > \frac{N(N+2)}{2(N-2)}$ the function

$$\psi_\delta(R) := R^{-(N-2)/2} + \delta^{\frac{N+2}{q(N-2)}} R^{2-\frac{N(N+2)}{q(N-2)}}$$

we obtain

$$\min_{R>0} \psi_\delta(R) = \psi_\delta(R_*) \sim \delta^{1/(q-2^*)},$$

where $R_* \sim \delta^{-\frac{2}{(N-2)(q-2^*)}}$. □

We also establish the energy convergence estimate.

Lemma 5.10. $0 < S_0^\delta - S_* \to 0$ *as* $\delta \to 0$.

Proof. Taking into account (5.5) we obtain

$$S_* \le \mathcal{S}_*(w_0^\delta) = \frac{\|\nabla w_0^\delta\|_2^2}{\|w_0^\delta\|_p^2} = \left(\frac{2^*(q-p)}{p(q-2^*)}\right)^{2/p} S_0^\delta < S_0^\delta.$$

To control S_0^δ from above we will use the Sobolev minimizers $(W_\lambda)_{\lambda>0}$ as a family of test functions for (S_0^δ). Using (4.1) we obtain

$$\mathcal{S}_0^\delta(W_\lambda) = \frac{S_*}{\left(\frac{2^*}{p}\lambda^{-\frac{N-2}{2}\delta}\|W_1\|_p^p - \frac{2^*}{q}\lambda^{-\frac{N-2}{2}(q-2^*)}\|W_1\|_q^q\right)^{(N-2)/N}}. \tag{5.12}$$

To minimize the right hand side of (5.12), we need to maximize for $\lambda > 0$ the scalar function

$$\psi(\lambda) := \frac{2^*}{p}\|W_1\|_p^p \lambda^{-\frac{1}{2}(N-2)(p-2^*)} - \frac{2^*}{q}\|W_1\|_q^q \lambda^{-\frac{1}{2}(N-2)(q-2^*)}.$$

A direct calculation shows that ψ achieves its maximum at

$$\lambda_* := \left(\frac{p(q-2^*)\|W_1\|_q^q}{q(p-2^*)\|W_1\|_p^p}\right)^{\frac{2}{N-2}\frac{1}{q-p}},$$

and

$$\psi(\lambda_*) = A(p, 2^*, q)\|W_1\|_p^{\frac{p(q-2^*)}{q-p}}\|W_1\|_q^{-\frac{q(p-2^*)}{q-p}},$$

where

$$A(p, 2^*, q) := 2^* p^{-\frac{q-2^*}{q-p}} q^{\frac{p-2^*}{q-p}} \left\{\left(\frac{q-2^*}{p-2^*}\right)^{-\frac{p-2^*}{q-p}} - \left(\frac{q-2^*}{p-2^*}\right)^{-\frac{q-2^*}{q-p}}\right\} > 0.$$

In particular, when $\delta = p - 2^* \to 0$ we have $A(p, 2^*, q) \simeq 1$ and $\psi(\lambda_*) \simeq 1$, so

$$S_0^\delta \le \mathcal{S}_0^\delta(W_{\lambda_*}) = (\psi(\lambda_*))^{-(N-2)/N} S_* = S_*(1 + o(1)),$$

which completes the proof. □

Remark 5.11. Instead of W_λ we can use rescalings of an arbitrary functions $w \in D^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ as a family of test function in (5.12). Then, taking into account that by Sobolev imbedding $w \in L^{2^*}(\mathbb{R}^N)$, and hence by interpolation we have $w \in L^p(\mathbb{R}^N)$ as well, the above argument with generic $q > p > 2^*$ leads to

$$S_0(p, 2^*, q) A^{2/2^*}(p, 2^*, q)\|w\|_p^{\frac{2p(q-2^*)}{2^*(q-p)}} \le \|\nabla w\|_2^2 \|w\|_q^{\frac{2q(p-2^*)}{2^*(q-p)}},$$

which could be interpreted as a *supercritical* Gagliardo–Nirenberg type inequality. Similar ideas were used in [31, 10] to characterize sharp constants in the Gagliardo–Nirenberg inequality for $1 < q < p < 2^*$.

6. Subcritical case $2 < p < 2^*$ revisited: Proof of Theorem 2.1

In the subcritical case Pohožaev's identity implies that the limit equation (P_0) has no positive finite energy solutions. As discussed in the Introduction, to understand the asymptotic behaviour of the ground states u_ε we consider the rescaling in (1.3), which transforms (P_ε) into (R_ε), with the associated limit problem as $\varepsilon \to 0$ given by (R_0) (see Sec. 1).

Let $G_\varepsilon : \mathbb{R} \to \mathbb{R}$ be a bounded C^2 function such that

$$G_\varepsilon(w) := \frac{1}{p}|w|^p - \frac{1}{2}|w|^2 - \frac{\varepsilon^{\frac{q-p}{p-2}}}{q}|w|^q$$

for $0 \le w \le \varepsilon^{-1/(p-2)}$, $G_\varepsilon(w) \le 0$ for $w > \varepsilon^{-1/(p-2)}$, and $G_\varepsilon(w) = 0$ for $w \le 0$. For $\varepsilon \in [0, \varepsilon^*)$, consider the family of constrained minimization problems

$$S'_\varepsilon := \inf\left\{\int_{\mathbb{R}^N} |\nabla w|^2\, dx \;\middle|\; w \in H^1(\mathbb{R}^N),\ 2^* \int_{\mathbb{R}^N} G_\varepsilon(w)\, dx = 1\right\}. \qquad (S'_\varepsilon)$$

Note that all the problems (S'_ε), including the limit problem (S'_0), are well-posed in the same energy space $H^1(\mathbb{R}^N)$. According to [5, Theorem 2], (S_ε) admits a radial positive minimizer w_ε for every $\varepsilon \in [0, \varepsilon^*)$. In view of its uniqueness [17], the rescaled function

$$v_\varepsilon(x) := w_\varepsilon(x/\sqrt{S'_\varepsilon}),$$

coincides with the radial ground state of (R_ε).

In order to estimate S'_ε, consider the associated dilation invariant representation

$$\mathcal{S}'_\varepsilon(w) := \frac{\int_{\mathbb{R}^N} |\nabla w|^2\, dx}{(2^* \int_{\mathbb{R}^N} G_\varepsilon(w)\, dx)^{(N-2)/N}}, \qquad w \in \mathcal{M}'_\varepsilon,$$

where $\mathcal{M}'_\varepsilon := \{0 \le u \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} G_\varepsilon(w)\, dx > 0\}$. Clearly

$$S'_\varepsilon = \inf_{w \in \mathcal{M}'_\varepsilon} \mathcal{S}'_\varepsilon(w)$$

and for sufficiently small ε we have

$$S'_0 \le \mathcal{S}'_0(w_\varepsilon) < \mathcal{S}'_\varepsilon(w_\varepsilon) = S'_\varepsilon. \qquad (6.1)$$

Indeed, since by definition $2^* \int_{\mathbb{R}^N} G_\varepsilon(w_\varepsilon)\, dx = 1$ and $G_\varepsilon(s)$ is a decreasing function of ε for each $s > 0$, we have $w_\varepsilon \in \mathcal{M}'_0$, and the second inequality again follows by monotonicity of $G_\varepsilon(s)$ in ε. At the same time, by continuity $w_0 \in \mathcal{M}'_\varepsilon$ for sufficiently small ε. Therefore, using w_0 as a test function for (S'_ε), we obtain, for sufficiently small ε,

$$S'_\varepsilon \le \mathcal{S}'_\varepsilon(w_0) \;=\; \frac{S'_0}{\left(1 - \frac{2^*}{q}\varepsilon^{\frac{q-p}{p-2}} \|w_0\|_q^q\right)^{(N-2)/N}} \le S'_0 + O\big(\varepsilon^{\frac{q-p}{p-2}}\big).$$

Therefore, $S'_\varepsilon \to S'_0$.

Arguing as in the proof of Lemma 4.2, we may conclude that

$$\|w_\varepsilon\|_p^p = \frac{(q-2^*)p}{(q-p)2^*} + \frac{p(q-2)}{2(q-p)} \|w_\varepsilon\|_2^2.$$

Then, using this identity to compute $\mathcal{S}'_0(w_\varepsilon)$ and the convergence of S'_ε to S'_0, after some tedious algebra we obtain

$$\lim_{\varepsilon \to 0} \|w_\varepsilon\|_2^2 = \frac{2(2^*-p)}{2^*(p-2)}, \qquad \lim_{\varepsilon \to 0} \|w_\varepsilon\|_p^p = \frac{(2^*-2)p}{(p-2)2^*}.$$

In particular, this implies that $2^* \int_{\mathbb{R}^N} G_0(w_\varepsilon)\, dx \to 1$ as $\varepsilon \to 0$. Hence, there exists a rescaling $\lambda_\varepsilon \to 1$ such that $2^* \int_{\mathbb{R}^N} G_0(\tilde{w}_\varepsilon)\, dx = 1$ and $\mathcal{S}'_\varepsilon(\tilde{w}_\varepsilon) \to S'_0$ for $\tilde{w}_\varepsilon(x) :=$

$w_\varepsilon(\lambda_\varepsilon x)$. This implies that $(\tilde{w}_\varepsilon)$ is a minimizing family for (S_0') that satisfies the constraint used in the analysis of [5]. Then, applying [5, Theorem 2] we conclude that for a sequence $\varepsilon_n \to 0$ we have $\tilde{w}_{\varepsilon_n} \to \bar{w}$ strongly in $H^1(\mathbb{R}^N)$, and in view of the convergence of (λ_ε) we have $w_{\varepsilon_n} \to \bar{w}$ as well, where $\bar{w}$ is the minimizer of (S_0') satisfying the constraint. Therefore, by uniqueness of minimizers of (R_0) [17], we have $\bar{w} = w_0$ and the limit is a full limit.

Finally, arguing as in the proof of Lemma 4.15, using $\|w_\varepsilon\|_{2^*}$ instead of the L^q norm to control the growth of w_ε at the origin, we also conclude that $\|w_\varepsilon\|_\infty \lesssim 1$ as $\varepsilon \to 0$. Then by standard elliptic regularity, similarly to the proof of Corollary 4.16, we conclude that w_ε converges to w_0 in $L^s(\mathbb{R}^N)$ for any $s \geq 2$ and in $C^2(\mathbb{R}^N)$, which completes the proof of Theorem 2.1.

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