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Counterexample to the convexity of level sets of solutions to the mean curvature equation

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Abstract. The convexity of level sets of solutions to the mean curvature equation is a long standing open problem. In this paper we give a counterexample to it.

Keywords. Mean curvature, convexity, level set

1. Introduction

The convexity of level sets of solutions to elliptic equations is clearly an interesting geometric property, and has been studied by many authors in the last few decades. The convexity of level sets has been proved for a number of elliptic equations, including the Laplacian, the p-Laplacian, and some more general semilinear, quasilinear, and fully nonlinear elliptic equations. We refer the reader to [1, 2, 3, 5, 8, 11, 13, 14] and the references therein.

A long standing open problem is the convexity of level sets of solutions to the constant mean curvature equation, and an explicit conjecture on the problem was stated in [15]. Let Ω be a bounded convex domain in the Euclidean space $\mathbb{R}^n$, $n \geq 2$. Suppose u is a solution to the mean curvature equation

$$\begin{aligned} H[u] &= 1 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where

$$H[u] = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)$$

is the mean curvature of the graph of u. The conjecture in [15] asserts that for all constants h in the range of the solution u, the level sets

$$\mathcal{L}_h = \{x \in \Omega \mid u(x) = h\}$$

are convex.

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The convexity of solutions to the mean curvature equation has been studied in a number of papers [4, 6, 7, 8, 9, 10, 12]. In particular, Sakaguchi [17] proved that for a bounded, convex domain, if the mean curvature is sufficiently small, then the level sets of solutions to (1.1) are convex. See [15, 16] for more discussions.

All the results on the convexity of level sets mentioned above, as well as those in [18, 10, 12] mentioned below, suggest that the level sets of solutions to (1.1) should be convex, so McCuan's conjecture should be true. Surprisingly enough, we found a counterexample to the above conjecture.

Theorem 1.1. *There exists a bounded, smooth, uniformly convex domain $\Omega \subset \mathbb{R}^n$ such that the solution to the Dirichlet problem* (1.1) *has nonconvex level sets.*

A related question is the convexity of level sets of solutions to

$$\begin{aligned} H[u] &= 1 && \text{in } \Omega_1 - \Omega_2, \\ u &= 0 && \text{on } \partial\Omega_1, \\ u &= -1 && \text{on } \partial\Omega_2, \end{aligned} \tag{1.2}$$

where Ω_1 and Ω_2 are bounded convex domains in $\mathbb{R}^n$ and $\Omega_2 \subset\subset \Omega_1$, that is, Ω_2 is strictly contained in Ω_1.

In [18], Shiffman considered a minimal surface S in $\mathbb{R}^3$ with annular topology, bounded by two plane curves Γ_1 and Γ_2 lying in planes parallel to $\{x_3 = 0\}$. He proved that if Γ_1 and Γ_2 are convex curves, then every level line of S is convex. This result was extended to higher dimensions by Korevaar when the minimal surface is a graph [10], and some curvature estimates were given in [12]. In this paper we also give a counterexample to the convexity of level sets of solutions to (1.2).

Theorem 1.2. *There exist bounded, smooth, uniformly convex domains $\Omega_2 \subset\subset \Omega_1 \subset \mathbb{R}^n$ such that the solution to the Dirichlet problem* (1.2) *has nonconvex level sets.*

The proof of Theorem 1.1 is based on an auxiliary function and a rescaling argument, using the comparison principle. Details are given in Sections 2 and 3 respectively. The proof of Theorem 1.1 does not apply to Theorem 1.2 directly, but we show in Section 4 that Theorem 1.2 is a consequence of Theorem 1.1. In Section 5 a brief discussion of the convexity of level sets of solutions to constant Weingarten curvature is given.

2. An auxiliary function

We will prove Theorem 1.1 in dimension two. In higher dimensions, the proof is similar with $t = x_1$ and $r = [x_2^2 + \cdots + x_n^2]^{1/2}$.

We first introduce an auxiliary function. Let

$$\phi(t,r) = \sqrt{1-(1+t)^2} - \sqrt{1-r^2}, \qquad (t,r) \in \Delta, \tag{2.1}$$

where

$$\Delta = \{(t,r) \in \mathbb{R}^2 \mid -1 < t < 0,\ 0 < r < t+1\}.$$

Then $\phi = 0$ on the line $\{t+1 = r\}$ and $\phi < 0$ in the triangle Δ.

Let $h \in (0, 1)$ be a constant. Consider the level set $\mathcal{L}_h = \{\phi = -h\}$, which is given by

$$\sqrt{1-r^2} = h + \sqrt{1-s^2},$$

where $s = t + 1$. We have

$$r = [s^2 - 2h(1-s^2)^{1/2} - h^2]^{1/2}, \tag{2.2}$$

where $\sqrt{2h - h^2} < s < 1$.

We calculate the first and the second derivatives of the function r:

$$r' = s\left(1 + \frac{h}{\sqrt{1-s^2}}\right)[s^2 - 2h(1-s^2)^{1/2} - h^2]^{-1/2} = s\left(1 + \frac{h}{\sqrt{1-s^2}}\right)r^{-1}$$

and

$$\begin{aligned} r'' &= \left[1 + \frac{h}{\sqrt{1-s^2}} + \frac{hs^2}{(\sqrt{1-s^2})^3}\right]r^{-1} - \left[s\left(1 + \frac{h}{\sqrt{1-s^2}}\right)\right]^2 r^{-3} \\ &= \left[1 + \frac{h}{(\sqrt{1-s^2})^3}\right]r^{-1} - \left[s\left(1 + \frac{h}{\sqrt{1-s^2}}\right)\right]^2 r^{-3}. \end{aligned}$$

From the above formula we see that $r'' < 0$ when $r > 0$ is small, and $r'' > 0$ when s is close to 1. To find the point where r'' vanishes, we write $r'' = 0$ in the form

$$[s^2 - 2h(1-s^2)^{1/2} - h^2]\left[1 + \frac{h}{(\sqrt{1-s^2})^3}\right] = s^2\left(1 + \frac{h}{\sqrt{1-s^2}}\right)^2.$$

Denote $\theta = \sqrt{1-s^2}$. Then $s^2 = 1 - \theta^2$ and we obtain

$$[1 - (h+\theta)^2]\left(1 + \frac{h}{\theta^3}\right) = (1-\theta^2)\left(1 + \frac{h}{\theta}\right)^2.$$

Therefore,

$$1 + \frac{h}{\theta^3} = \frac{h}{\theta^3}(h+\theta)^2 + \frac{1}{\theta^2}(h+\theta)^2.$$

Hence $\theta^3 + h = (\theta + h)^3$, i.e.,

$$\theta^2 + \theta h + \frac{1}{3}(h^2 - 1) = 0.$$

From this we obtain the unique positive solution

$$\theta = \frac{1}{2}\left[-h + \left(\frac{4}{3} - \frac{h^2}{3}\right)^{1/2}\right] =: \theta(h).$$

Computing the derivative we see that $\theta(h)$ is decreasing in h. Hence

$$s = \sqrt{1-\theta^2} = \left[\frac{1}{2}\sqrt{\frac{4}{3} - \frac{h^2}{3}}\left(\sqrt{\frac{4}{3} - \frac{h^2}{3}} + h\right)\right]^{1/2} =: s(h)$$

is increasing in h, and

$$\begin{aligned} r'' &< 0 \quad \text{if } s < s(h), \\ r'' &> 0 \quad \text{if } s > s(h). \end{aligned}$$

In the following we will consider the level set at height $h = 1/4$. We have $s(1/4) \approx 0.9$.

We extend the function ϕ to the square

$$S = \{(t, r) \in \mathbb{R}^2 \mid |r| + |t| < 1\} \tag{2.3}$$

so that

$$\phi(t, r) = \sqrt{1 - (1 - |t|)^2} - \sqrt{1 - r^2}, \qquad (t, r) \in S. \tag{2.4}$$

Then ϕ is symmetric in both axes. But ϕ is not $C^{1,1}$ smooth at $t = 0$. To apply the comparison principle in the next section, it is convenient to smoothen it near $t = 0$. Let $a_0 = 10^{-3}$ be a small constant, such that the level set $\{(t, r) \in \Delta \mid \phi(t, r) = -1/4, t < -a_0\}$ is not convex. Let

$$\gamma(t) = \begin{cases} \dfrac{1}{8a_0^3} t^4 - \dfrac{3}{4a_0} t^2 + 1 - \dfrac{3}{8} a_0 & \text{if } |t| < a_0, \\ t + 1 & \text{if } -1 < t < -a_0, \\ 1 - t & \text{if } a_0 < t < 1, \end{cases} \tag{2.5}$$

which is $C^{2,1}$ and piecewise C^∞ smooth. Let

$$\Omega = \{(t, r) \in \mathbb{R}^2 \mid -1 < t < 1, \ -\gamma(t) < r < \gamma(t)\}, \tag{2.6}$$

and

$$w(t, r) = \sqrt{1 - \gamma^2(t)} - \sqrt{1 - r^2}. \tag{2.7}$$

Then

$$\begin{aligned} w &= 0 \quad \text{on } \partial\Omega, \\ w &< 0 \quad \text{in } \Omega, \end{aligned}$$

and $w = \phi$ when $|t| \geq a_0$. Noting that $|\gamma(t)| \leq 1 - 3a_0/8$, we see that the gradient and the second derivatives of w are uniformly bounded. Denote

$$\mathcal{L} = \{(t, r) \in \Omega \mid w(t, r) = -1/4\}.$$

As $w = \phi$ in $|t| > a_0$, the level set $\mathcal{L}$ is not convex. Let $\mathcal{C}$ be the convex envelope of $\mathcal{L}$, that is, the boundary of the smallest convex domain containing $\mathcal{L}$. Denote

$$d(\mathcal{L}, \mathcal{C}) = \sup\{r_2 - r_1 \mid (t, r_1) \in \mathcal{L}, \ (t, r_2) \in \mathcal{C}, \ t \in (-1, 1), \ r_2 \geq r_1 > 0\}, \tag{2.8}$$

which is the (vertical) distance from the level set $\mathcal{L}$ to its convex envelope.

Next we introduce a family of domains and functions. For $\epsilon > 0$ small, let

$$\Omega_\epsilon = \{(t, r) \in \mathbb{R}^2 \mid (\epsilon t, r) \in \Omega\}, \tag{2.9}$$

$$w_\epsilon(t, r) = w(\epsilon t, r), \qquad (t, r) \in \Omega_\epsilon. \tag{2.10}$$

The domain Ω_ϵ and the function w_ϵ are obtained from Ω and w by the linear coordinate transformation

$$T_\epsilon : (t, r) \mapsto (t_\epsilon, r_\epsilon) \quad \text{with} \quad \begin{cases} t_\epsilon = t/\epsilon, \\ r_\epsilon = r. \end{cases} \tag{2.11}$$

Therefore the level set $\mathcal{L}_\epsilon = \{w_\epsilon = -1/4\}$ is not convex. As above we denote by $\mathcal{C}_\epsilon$ the convex envelope of $\mathcal{L}_\epsilon$ and by $d(\mathcal{L}_\epsilon, \mathcal{C}_\epsilon)$ the vertical distance between $\mathcal{L}_\epsilon$ and $\mathcal{C}_\epsilon$. Apparently we have $\mathcal{L}_\epsilon = T_\epsilon(\mathcal{L})$, $\mathcal{C}_\epsilon = T_\epsilon(\mathcal{C})$, and

$$d(\mathcal{L}_\epsilon, \mathcal{C}_\epsilon) = d(\mathcal{L}, \mathcal{C}) > 0, \tag{2.12}$$

which is independent of $\epsilon > 0$.

3. Proof of Theorem 1.1

First we compute

$$\begin{aligned} H[w_\epsilon] &= \operatorname{div}\left(\frac{Dw_\epsilon}{\sqrt{1 + |Dw_\epsilon|^2}}\right) \\ &= \frac{1}{\sqrt{1 + |Dw_\epsilon|^2}}\left[\frac{1 + (w_\epsilon)_r^2}{1 + |Dw_\epsilon|^2}(w_\epsilon)_{tt} + \frac{1 + (w_\epsilon)_t^2}{1 + |Dw_\epsilon|^2}(w_\epsilon)_{rr}\right] \\ &= \frac{1}{\sqrt{1 + w_r^2 + \epsilon^2 w_t^2}}\left[\frac{(1 + w_r^2)\epsilon^2 w_{tt}}{1 + w_r^2 + \epsilon^2 w_t^2} + \frac{(1 + \epsilon^2 w_t^2) w_{rr}}{1 + w_r^2 + \epsilon^2 w_t^2}\right] \\ &= \frac{w_{rr}}{(1 + w_r^2)^{3/2}} + O(\epsilon^2) = 1 + O(\epsilon^2). \end{aligned} \tag{3.1}$$

Heuristically the above formula can be explained as follows. Note that even after the modication of ϕ involving γ, the "sectional graphs" parameterized by $r \mapsto w(t, r)$ are still portions of circles of radius 1. As a consequence, the scaling which is used to construct w_ϵ leads to uniform convergence of the first and second derivatives to those of the lower half $z = -\sqrt{1 - r^2}$ of a cylinder of radius 1.

It can also be shown, incidentally, that the convergence is uniform in C^2 as $\epsilon \searrow 0$ on each fixed compactly contained subsets in Ω_ϵ to the specific cylindrical graph $z = -\sqrt{1 - r^2}$. This makes sense because the domains Ω_ϵ are nested. This second observation is neither necessary nor sufficient for the calculation above, but is only intended to give a heuristic explanation of the fact that the graphs are getting close to the bottom half of a cylinder of radius 1.

From (3.1), one sees that w_ϵ is a subsolution to

$$\begin{aligned} H[u] &= 1-\epsilon \quad \text{in } \Omega_\epsilon, \\ u &= 0 \quad \text{on } \partial\Omega_\epsilon. \end{aligned} \tag{3.2}$$

By the Perron method, there is a solution u_ϵ to the above Dirichlet problem.

Next we construct a supersolution for u_ϵ. Let

$$v(t,r) = \sqrt{1+\delta-\gamma^2(t)} - \sqrt{1+\delta-r^2},$$

where $\delta = \epsilon^{1/2}$. Apparently $v > w$ in Ω and $v = w = 0$ on $\partial\Omega$. Let

$$v_\epsilon(t,r) = v(\epsilon t, r), \qquad (t,r) \in \Omega_\epsilon. \tag{3.3}$$

By the same computation as above, we have

$$\begin{aligned} H[v_\epsilon] &= \operatorname{div}\left(\frac{Dv_\epsilon}{\sqrt{1+|Dv_\epsilon|^2}}\right) = \frac{1}{\sqrt{1+|Dv_\epsilon|^2}}\left[\frac{1+(v_\epsilon)_r^2}{1+|Dv_\epsilon|^2}(v_\epsilon)_{tt} + \frac{1+(v_\epsilon)_t^2}{1+|Dv_\epsilon|^2}(v_\epsilon)_{rr}\right] \\ &= \frac{1}{\sqrt{1+v_r^2+\epsilon^2 v_t^2}}\left[\frac{(1+v_r^2)\epsilon^2 v_{tt}}{1+v_r^2+\epsilon^2 v_t^2} + \frac{(1+\epsilon^2 v_t^2)v_{rr}}{1+v_r^2+\epsilon^2 v_t^2}\right] \\ &= \frac{v_{rr}}{(1+v_r^2)^{3/2}} + O(\epsilon^2) = (1+\delta)^{-1/2} + O(\epsilon^2). \end{aligned}$$

Hence

$$H[v_\epsilon] < 1-\epsilon = H[u_\epsilon] \tag{3.4}$$

if $\delta = \epsilon^{1/2}$ and ϵ is sufficiently small. Note that $v_\epsilon = u_\epsilon = 0$ on $\partial\Omega_\epsilon$. By the comparison principle, we obtain

$$v_\epsilon \geq u_\epsilon \geq w_\epsilon. \tag{3.5}$$

Denote

$$\begin{aligned} \mathcal{L}_\epsilon &= \{(t,r) \in \Omega_\epsilon \mid w_\epsilon(t,r) = -1/4\}, \\ \mathcal{L}'_\epsilon &= \{(t,r) \in \Omega_\epsilon \mid u_\epsilon(t,r) = -1/4\}, \\ \mathcal{L}''_\epsilon &= \{(t,r) \in \Omega_\epsilon \mid v_\epsilon(t,r) = -1/4\}. \end{aligned}$$

Then by (3.5), $\mathcal{L}''_\epsilon$ stays inside of $\mathcal{L}'_\epsilon$, and $\mathcal{L}'_\epsilon$ stays inside of $\mathcal{L}_\epsilon$. By our construction, we have

$$d(\mathcal{L}''_\epsilon, \mathcal{L}_\epsilon) \to 0 \quad \text{as } \epsilon \to 0.$$

Hence

$$d(\mathcal{L}'_\epsilon, \mathcal{L}_\epsilon) \to 0 \quad \text{as } \epsilon \to 0.$$

Here $d(\mathcal{L}_\epsilon, \mathcal{L}'_\epsilon)$ denotes the vertical distance between $\mathcal{L}_\epsilon$ and $\mathcal{L}'_\epsilon$, as defined in (2.8). But since $\mathcal{L}_\epsilon$ is not convex, $\mathcal{L}'_\epsilon$ is not convex either, and we have

$$d(\mathcal{L}'_\epsilon, \mathcal{C}'_\epsilon) \geq \tfrac{1}{2} d(\mathcal{L}, \mathcal{C}) > 0$$

when $\epsilon > 0$ is sufficiently small, where $\mathcal{C}'_\epsilon$ is the convex envelope of $\mathcal{L}'_\epsilon$ and $d(\mathcal{L}, \mathcal{C})$ is the constant in (2.12). Therefore the level set $\mathcal{L}'_\epsilon$ is not convex.

We have proved that for any given $\epsilon > 0$, the solution to the Dirichlet problem (3.2) in the domain Ω_ϵ has nonconvex level sets provided ϵ is sufficiently small. Note that the solution of (3.2) is a surface of mean curvature $1-\epsilon$. But by a dilation of the coordinates, we can make the mean curvature 1.

To finish the proof of Theorem 1.1, we first fix a small $\epsilon > 0$ such that the solution to the Dirichlet problem (3.2) in the domain Ω_ϵ has nonconvex level sets. By a dilation, we see that the solution to the Dirichlet problem (1.1) in the domain $\Omega := \Omega_\epsilon$ has nonconvex level sets. We then choose a sequence of bounded, uniformly convex domain Ω_k such that the boundary $\partial\Omega_k$ converges to $\partial\Omega$ smoothly, namely $\partial\Omega_k$ is in the δ_k-neighborhood of $\partial\Omega$ with $\delta_k \to 0$ as $k \to \infty$. Recall that the a priori estimates and existence for solutions to the Dirichlet problem (1.1) depend continuously on the domain and the inhomogeneous term, so one sees that for any domain Ω' sufficiently close to Ω in the $C^{2+\alpha}$ norm, there is also a solution to (1.1) in the domain Ω'. (Alternatively, let T be a diffeomorphism from Ω to Ω' such that $\|T - \mathrm{Id}\|_{C^{2+\alpha}}$ is sufficiently small. Then $u(T(x))$ is a subsolution to (1.1) in Ω' with the right hand side $1 - \delta$ for small $\delta > 0$.) In particular there is a solution u_k to (1.1) in the domain Ω_k, provided k is sufficiently large. As Ω is fixed, we have $u_k \to u$ uniformly as $k \to \infty$. Hence when k is sufficiently large, the level sets of u_k cannot all be convex. This finishes the proof of Theorem 1.1.

Remark 3.1. By approximation and the uniqueness of solutions to the Dirichlet problem, Theorem 1.1 also applies to the case when the mean curvature is sufficiently close to a constant, as is the case for solutions to the problem

$$\begin{aligned} H[u] &= \kappa u + 1 \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.6}$$

where κ is a small constant.

4. Proof of Theorem 1.2

The proof of Theorem 1.1 does not work directly for Theorem 1.2, as the prescribed mean curvature equation is very special. For example, the existence of a subsolution to the Dirichlet problem does not mean the existence of a solution. However, we will show that Theorem 1.2 follows readily from Theorem 1.1.

By Theorem 1.1, there exists a smooth, bounded, uniformly convex domain $\Omega \subset \mathbb{R}^n$ such that the Dirichlet problem (1.1) has a solution $u \in C^4(\overline{\Omega})$ which has nonconvex level sets. By the maximum principle, $u < 0$ in Ω and u attains its minimum at an interior point of Ω.

Since Ω is uniformly convex, there exists $\delta_0 > 0$ such that for any hyperplane L : $x_{n+1} = \ell(x_1, \ldots, x_n)$ with the properties

$$|D\ell| \le \delta_0 \quad \text{and} \quad \sup_\Omega \ell(x) = 0, \tag{4.1}$$

the intersection $L \cap G_u$ is a closed convex hypersurface, where G_u denotes the graph of u.

Let

$$f(x) = \delta|x|^2 + c$$

be a uniformly convex function. When $c < 0$ is sufficiently large in absolute value, we have $f < u$ in $\overline{\Omega}$. We increase the constant c until $c = c_0$ such that the graph of f touches that of u, that is, we choose c_0 such that

$$f(x) \le u(x) \quad \text{in } \Omega, \tag{4.2}$$

$$f(x_0) = u(x_0) \tag{4.3}$$

for some $x_0 \in \Omega$. The point x_0 is an interior point of Ω if we choose $\delta > 0$ sufficiently small.

Let

$$L_a : \; p(x) = a + \sum a_i x_i \tag{4.4}$$

be the tangent plane of u at x_0. By (4.2) and (4.3), u is uniformly convex at x_0. Hence the set $\{x \in \Omega \mid u(x) < \epsilon + p(x)\}$ is uniformly convex if ϵ is sufficiently small. Thus the intersection $G_u \cap L_{a+\epsilon}$ is uniformly convex, where $L_{a+\epsilon}$ denotes the plane $x_{n+1} = a + \epsilon + \sum a_i x_i$.

On the other hand, choose a constant $h > 0$ such that $p(x) + h \le u(x) \le 0$ in Ω and $\sup_\Omega (p(x) + h) = 0$. When $\delta \gg \delta_0$, from (4.1) we see that the intersection of the plane $p(x) + h$ with G_u is uniformly convex. Thus the intersection $G_u \cap L_{a+h}$ is uniformly convex.

We restrict the graph G_u to the part between the parallel planes $L_{a+\epsilon}$ and L_{a+h}, that is,

$$\mathcal{M} = G_u \cap \{\epsilon + p(x) < x_{n+1} < h + p(x)\}.$$

Choose new coordinates $(y_1, \dots, y_n, y_{n+1})$ so that the y_{n+1}-axis is perpendicular to the plane L_a. Then when δ is sufficiently small, the hypersurface $\mathcal{M}$ is a graph in the new coordinates, with constant boundary values on two parallel planes. Theorem 1.2 is proved.

We remark that when $n = 2$, we may simply choose the point x_0 in (4.3) as the minimum point of u, as u is convex near x_0 (see [16, Theorem 9(ii)]).

5. Remark on the Weingarten curvature

Our proof of Theorem 1.1 also applies to some other constant Weingarten curvature equations. Indeed, the proof of Theorem 1.1 relies on two observations. The first is that the level set of the function ϕ in (2.1) is not convex. The second is that if we write $w_\epsilon(t, r) = w_\epsilon(a, r) + [w_\epsilon(t, r) - w_\epsilon(a, r)]$, then $w_\epsilon(a, r)$, as a function of $r = [x_2^2 + \cdots + x_n^2]^{1/2}$, has positive constant (Weingarten) curvature, and the function $w_\epsilon(t, r) - w_\epsilon(a, r)$ only gives a small perturbation of the curvature.

In particular Theorem 1.1 also holds for the k-curvature equation

$$\begin{aligned} \sigma_k(\kappa) = 1 \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded, uniformly convex domain in $\mathbb{R}^n$, $n \geq 2$, $1 \leq k \leq n-1$, and $\sigma_k(\kappa)$ is the k^{th} elementary symmetric polynomial of the principal curvatures $\kappa = (\kappa_1, \dots, \kappa_n)$. One can also extend Theorem 1.2 to the k-curvature equation similarly.

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