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Almost sure global well-posedness for the radial nonlinear Schrödinger equation on the unit ball II: the 3d case

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Abstract. We extend the convergence method introduced in our works [8]–[10] for almost sure global well-posedness of Gibbs measure evolutions of the nonlinear Schrödinger (NLS) and nonlinear wave (NLW) equations on the unit ball in $\mathbb{R}^d$ to the case of the three-dimensional NLS. This is the first probabilistic global well-posedness result for NLS with supercritical data on the unit ball in $\mathbb{R}^3$.

The initial data is taken as a Gaussian random process lying in the support of the Gibbs measure associated to the equation, and results are obtained almost surely with respect to this probability measure. The key tools used include a class of probabilistic a priori bounds for finite-dimensional projections of the equation and a delicate trilinear estimate on the nonlinearity, which—when combined with the invariance of the Gibbs measure—enables the a priori bounds to be enhanced to obtain convergence of the sequence of approximate solutions.

Keywords. Gibbs measure, global well-posedness, defocusing nonlinear Schrödinger equation

1. Introduction

We continue our study of Gibbs measure evolution for the nonlinear Schrödinger (NLS) and nonlinear wave (NLW) equations on the unit ball in Euclidean space, initiated in our earlier works [8]–[10]. In particular, the aim of the present article is to extend the almost sure global well-posedness result of [10], which was set on the unit ball in $\mathbb{R}^2$, to the setting of the unit ball in $\mathbb{R}^3$. The techniques involved are a further development of the method introduced in our work [9] for the nonlinear wave equation, combined with a delicate choice of function spaces adapted to the decay properties of the fundamental solution of the Schrödinger equation.

More precisely, we shall consider the initial value problem for the cubic NLS on the unit ball B in $\mathbb{R}^3$,

$$\text{(NLS)}\qquad \begin{cases} iu_t + \Delta u - |u|^2 u = 0, \\ u|_{t=0} = \phi, \end{cases}$$

where $u : I \times B \to \mathbb{C}$, subject to the Dirichlet boundary condition $u|_{I\times\partial B} = 0$, with randomly chosen radial initial data ϕ; the sense in which the randomization is taken will be specified momentarily.

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A fundamental property of (NLS) is that the equation takes the form of an infinite-dimensional Hamiltonian system,

$$iu_t = \frac{\partial H}{\partial \overline{u}},$$

with conserved Hamiltonian

$$H(\phi) = \frac{1}{2}\int_B |\nabla\phi|^2 + \frac{1}{4}\int_B |\phi|^4.$$

In the case that the spatial domain $B \subset \mathbb{R}^3$ is replaced with the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, a robust theory of almost sure global well-posedness for the Cauchy problem was established in the seminal works [3]-[6] for a variety of general classes of nonlinearities including both the attractive and repulsive regimes; see also [7] for a brief survey of these results. The approach pioneered in this line of study was to obtain global control by exploiting the invariant properties of the Gibbs measure inherent in the Hamiltonian structure of the equation.

In preparation for our discussion below, we now outline the main steps of the approach pursued in those works:

(i) The first step is to consider a finite-dimensional projection of the Cauchy problem for (NLS), allowing access to an invariant Gibbs measure which gives global in time estimates for solutions.
(ii) A strong form of the local well-posedness theory driven by a contraction mapping principle then allows one to show convergence of solutions for the finite-dimensional problems to a solution of the original equation. The key point in this step is to obtain estimates which are uniform in the projection parameter.
(iii) The two steps above are then combined to establish almost sure global well-posedness for the original Cauchy problem, (NLS) with no finite-dimensional projection.
(iv) The final step in the analysis is to establish the invariance of the limiting Gibbs measure with respect to the evolution given by the original, non-projected, (NLS) equation.

We remark that the local theory in this approach is a consequence of fixed point arguments in suitable classes of function spaces. Although such results are usually available only for problems in which the initial data is subcritical or critical with respect to the scaling of the equation, nevertheless, the randomization gives additional integrability almost surely in the random variable, and can often enable the application to classes of supercritical data (see for instance [5] as well as [11]).

On the other hand, in the setting of the present paper a more substantial obstruction to implementing the approach described above is posed by the lack of robust Strichartz estimates on domains with boundary, which renders the fixed point technique ineffective for our purposes. Indeed, in the current work our arguments pursue a different path based on the treatment we introduced for the three-dimensional nonlinear wave equation [9] and adapted to the two-dimensional NLS equation [10]. This approach is again based on a procedure of finite-dimensional projection, with the goal of showing global well-posedness by establishing convergence for the sequence of solutions of the projected equations. However, with the fixed-point argument unavailable, the proof of convergence follows from a more delicate analysis of the fine behavior of solutions and their frequency interactions.

More precisely, the strategy in the present paper proceeds in the following steps:

(i′) Construction of a suitable collection of function spaces used to establish convergence for the sequence of solutions to the finite-dimensional projections. Closely related to this is the identification of the relevant embeddings and basic interpolation properties of the spaces.

(ii′) Establishing a priori bounds for solutions of the projected equations which remain uniform in the projection parameter.

(iii′) The formulation of an estimate of the contribution of the nonlinearity. This estimate is the most delicate stage in the process, and serves to provide the decay necessary to establish convergence.

(iv′) The above ingredients are then combined to establish convergence for the sequence of solutions of the projected equations, almost surely in the randomization. The limiting function is a solution of the original equation and is defined for arbitrarily long time intervals.

It is important to note that in our current setting the invariance of the Gibbs measure is an essential ingredient in obtaining the short-time local existence result, whereas in the fixed-point based approach of [3]–[7] the local theory is developed independently of the invariance of the Gibbs measure. This is a major distinction between the two approaches, and our use of the Gibbs measure at this stage of the argument can be seen as the key piece of probabilistic information which allows us to overcome the lack of Strichartz estimates; for a complete discussion of this issue we refer the reader to [9], where the technique was introduced.

We believe that our treatment of the Theorem stated below also applies with minor changes to the cubic defocusing NLS on S^3 as considered in [13], leading to a similar result (in [13], the second Picard iteration with random data was analyzed).

Before giving the precise statement of our main results, we shall now describe the finite-dimensional projections which form the basis of our approach.

1.1. Finite-dimensional model and the Gibbs measure

We shall consider solutions to the *truncated equation*

$$\begin{cases} iu_t + \Delta u - P_N(|u|^2 u) = 0 \\ u|_{t=0} = P_N \phi, \end{cases} \tag{1.1}$$

where the operator P_N is the projection to low frequencies defined by

$$P_N\Big(\sum_{n\in\mathbb{N}} a_n e_n(x)\Big) = \sum_{n\le N} a_n e_n(x)$$

with $(a_n) \in \ell^2$ and (e_n) as the sequence of radial eigenfunctions of $-\Delta$ on B with vanishing Dirichlet boundary conditions.

The initial value problem (1.1) is globally well-posed for every integer $N \geq 1$: indeed, for any initial data $\phi \in L^2_x(B)$, there exists a unique global solution $u_N : \mathbb{R} \times B \to \mathbb{C}$ satisfying the associated *Duhamel formula*,

$$u_N(t) = e^{it\Delta} P_N \phi + i \int_0^t e^{i(t-\tau)\Delta} P_N(|u_N|^2 u_N)(\tau)\, d\tau. \tag{1.2}$$

The *Gibbs measure* $\mu_G^{(N)}$ associated to (1.1) is defined (up to normalization factors) by

$$\mu_G^{(N)}(A) = \int_A \exp(-H_N(\phi)) \prod_{i=1}^{N} d^2\phi = \int_A \exp\Big(-\frac{1}{4}\|P_N\phi\|^4_{L^4_x}\Big)\, d\mu_F^{(N)}(\phi), \qquad A \in \mathcal{M},$$

where

$$H_N(\phi) = \frac{1}{2}\sum_{n \leq N} n^2 |\widehat{\phi}(n)|^2 + \frac{1}{4}\int_B |P_N\phi(x)|^4\, dx.$$

and $\mu_F^{(N)}$ is the free (Wiener) measure induced by the mapping

$$\Omega \ni \omega \mapsto \phi_\omega := \sum_{n \leq N} \frac{g_n(\omega)}{n\pi} e_n,$$

where (g_n) is a sequence of IID normalized complex Gaussian random variables.

As we will see below, basic facts concerning the sequence of eigenfunctions (e_n) ensure that the norms

$$\|\phi\|_{H^s_x(B)},\ s < 1/2, \quad \text{and} \quad \|P_N\phi\|_{L^p_x(B)},\ p < 6,$$

are finite $\mu_F^{(N)}$-almost surely for every $N \geq 1$. These facts dictate the spaces in which we look for solutions, and also serve to ensure that the measure $\mu_G^{(N)}$ is well-defined, nontrivial and normalizable. Finally, we remark that $\mu_G^{(N)}$ is invariant under the evolution of the truncated equation (1.1), that is,

$$\mu_G^{(N)}(\{\phi_\omega : \omega \in \Omega\}) = \mu_G^{(N)}(\{u_N(t) : u_N \text{ solves (1.1) with } \phi = \phi_\omega,\ \omega \in \Omega\})$$

for any $t \in \mathbb{R}$.

We are now ready to state the main result of this paper, which establishes almost sure convergence of the sequence of solutions to the truncated equation (1.1) as the truncation parameter N tends to infinity.

Theorem. *For each $N \in \mathbb{N}$ and $\omega \in \Omega$ let u_N denote the solution to (1.1) with initial data $P_N\phi = P_N\phi^{(\omega)}$. Then, almost surely in ω, for every $s < 1/2$ and $T < \infty$, there exists $u_* \in C_t([0,T); H^s_x(B))$ such that u_N converges to u_* with respect to the norm $C_t([0,T); H^s_x(B))$.*

This function u_* may be formally expressed by a multi-linear Gaussian expansion in the initial data and it follows from our analysis that this expansion is weakly convergent.

The proof of the theorem follows the approach described above, and can be roughly outlined as consisting of the following steps: (1) identification of the Fourier restriction spaces $X^{s,b}$ together with a variant $X_{|||\cdot|||}$ as suitable classes of function spaces, (2) the derivation of a family of a priori bounds which are uniform in the finite-dimensional projection P_N, (3) a trilinear estimate on the nonlinearity which allows us to enhance the a priori bounds into the decay necessary to establish convergence, and (4) a convergence argument for $N \to \infty$ which assembles the above ingredients.

The first step in the analysis is the choice of function spaces. As is by now familiar in the study of nonlinear dispersive equations, the spaces $X^{s,b}$ of [1, **?**] are the natural spaces to carry out perturbation theory from the Duhamel formula (1.2). An additional component in the analysis in the present work is the need to consider short time intervals. To balance this requirement with the degenerating constant in the $X^{s,b}$-localization bound

$$\|\psi f\|_{X^{s,b}} \lesssim \frac{1}{\delta^{b-1/2}} \|f\|_{X^{s,b}}, \qquad b > 1/2,$$

with $\psi(t) = \eta(t/\delta)$, $\delta > 0$, where $\eta : \mathbb{R} \to [0, 1]$ is a smooth function such that $\eta = 1$ on $[-1, 1]$ and $\operatorname{supp} \eta \subset [-2, 2]$ (see, for instance, [7, Lecture 2]), we also introduce the slightly different space $X_{|||\cdot|||}$ for which the degenerating constant does not appear.

With the scale of function spaces identified, we next devote our attention to a priori bounds for solutions of the truncated equations (1.1), uniform in the truncation parameter. We first obtain such bounds in $L_x^p L_t^q$ norms, and subsequently extend the arguments to $X^{s,b}$ norms. To obtain the almost sure global well-posedness result of the theorem, it suffices to establish these bounds up to the exclusion of sets of small measure in the statistical ensemble. In view of this, the key observation is that by exploiting the invariance of the Gibbs measure, it is enough to establish analogous bounds for functions of the form

$$\sum_{n \in \mathbb{N}} \frac{g_n(\omega)}{n\pi} e_n(x).$$

This enables us to combine standard estimates for Gaussian processes and estimates on the eigenfunctions e_n to obtain the desired bounds.

The next step is to obtain a trilinear estimate on the nonlinear term in the Duhamel formula. The argument to establish this bound proceeds by decomposing each of the three linear factors appearing in the nonlinearity $F(u) = |u|^2 u$ into discrete frequencies and estimating the resulting frequency interactions. These estimates are performed using space-time norms, $X^{s,b}$-spaces and further probabilistic considerations based on the Gibbs measure invariance. In fact, we need to distinguish several frequency regions where different arguments apply. Introducing these regions requires certain care.

The final step in establishing the theorem is to assemble the above ingredients to show that the sequence (u_N) of solutions to the truncated equations (1.1) is almost surely a Cauchy sequence in the space $C_t([0, T); H_x^s(B))$. The core step in this argument takes the form of an estimate for the $X_{|||\cdot|||}$ norm of the difference $u_{N_1} - u_{N_0}$ for any integers $N_1 \geq N_0 \geq 1$. This bound is of the form

$$|||u_{N_1} - u_{N_0}||| \lesssim N_0^{-c} \quad \text{for some } c > 0 \tag{1.3}$$

for all $\omega \in \Omega$ outside a singular set having small measure. The measures of these exceptional sets need to be sufficiently small in order to deduce an almost everywhere convergence result. Of course, large deviation estimates for Gaussian processes are essential here. The final stage of the argument consists in revisiting the probabilistic claims in order to justify the required quantitative form.

2. Notation and preliminaries

Throughout our arguments we will frequently make use of a dyadic decomposition in frequency, writing

$$f(x) = \sum_{n} \hat{f}(n)e_n(x) = \sum_{N\geq 1}\sum_{n\sim N} \hat{f}(n)e_n(x),$$

where for each $n \in \mathbb{Z}$, the condition $n \sim N$ is characterized by $N \leq n \leq 2N$.

For every $n \in \mathbb{N}$, define

$$e_n(x) = \frac{\sin(n\pi|x|)}{|x|} \tag{2.1}$$

and recall that e_n is the nth radial eigenfunction of $-\Delta$ on B, with associated eigenvalue n^2. With this notation, we have the following estimates on the norms of the eigenfunctions:

$$\|e_n\|_{L^p_x} \lesssim 1,\ 1 \leq p < 3, \quad \text{and} \quad \|e_n\|_{L^p_x} \lesssim n^{1-3/p},\ p > 3, \tag{2.2}$$

along with the endpoint-type bound $\|e_n\|_{L^3_x} \lesssim (\log n)^{1/3}$. Moreover, the sequence (e_n) also enjoys the following correlation bound:

$$|c(n, n_1, n_2, n_3)| \lesssim \min\{n, n_1, n_2, n_3\}, \tag{2.3}$$

where we have set

$$c(n, n_1, n_2, n_3) = \int_B e_n(x)e_{n_1}(x)e_{n_2}(x)e_{n_3}(x)\,dx. \tag{2.4}$$

We include a quick proof of (2.3). One needs to bound

$$\int_0^1 \sin(n\pi r)\sin(n_1\pi r)\sin(n_2\pi r)\sin(n_3\pi r)r^{-2}\,dr$$

where we assume $n \leq n_1 \leq n_2 \leq n_3$. The contribution of $r > 1/(10n)$ is obviously bounded by $\int_{r>1/(10n)} r^{-2}dr < O(n)$. For $r \leq \frac{1}{10}n$, write $\frac{\sin(n\pi r)}{r} = n\pi + O(n^2 r)$, which leads to

$$n\pi \int_0^{1/(10n)} (\sin(n_1\pi r)\sin(n_2\pi r)\sin(n_3\pi r))r^{-1}dr + O(n).$$

The integrand in the first term is of the form $\sin(m\pi r)r^{-1}$ with $m = n_1 \pm n_2 \pm n_3$ and hence the integral is bounded.

Another essential tool in our analysis is the following probabilistic estimate for sums of Gaussian random variables:

$$\Big\|\sum_n \alpha_n g_n(\omega)\Big\|_{L^q(d\omega)} \lesssim \sqrt{q}\Big(\sum_n |\alpha_n|^2\Big)^{1/2}, \tag{2.5}$$

where $(\alpha_n) \in \ell^2$, $2 \leq q < \infty$, and (g_n) is a sequence of IID normalized complex Gaussians.

We also have the following multilinear version of the estimate (2.5):

$$\Big\|\sum_n \alpha_n h_n(\omega)\Big\|_{L^q(d\omega)} \lesssim (\sqrt{q})^k \Big\|\sum_n \alpha_n h_n(\omega)\Big\|_{L^2(d\omega)} \tag{2.6}$$

for every $k \geq 1$ and $2 \leq q < \infty$, and each h_n is a product of at most k Gaussians taken from a sequence (g_n) as above.

As a consequence, if (g_n) is a sequence of normalized IID complex Gaussian random variables, the bound

$$\Big\|\sum_n \alpha_n \cdot (|g_n(\omega)|^2 - 1)\Big\|_{L^q(d\omega)} \lesssim q\Big(\sum_n |\alpha_n|^2\Big)^{1/2} \tag{2.7}$$

holds for every $(\alpha_n) \in \ell^2$ and $1 \leq q < \infty$.

In the form (2.6), we note that the inequality remains valid in the vector-valued case, with (α_n) as elements of an arbitrary normed space X. See [12].

2.1. *Description of the function spaces*

Fix a time interval $I = [0, T)$ with $T > 0$ sufficiently small, and let $X^{s,b}(I)$ denote the class of functions $f : I \times B \to \mathbb{C}$ representable as

$$f(x,t) = \sum_{n,m} f_{n,m} e_n(x) e(mt), \qquad (x,t) \in B \times I, \tag{2.8}$$

for which the norm

$$\|f\|_{s,b} := \inf \Big(\sum_{n,m} \langle n\rangle^{2s} \langle n^2 - m\rangle^{2b} |f_{n,m}|^2\Big)^{1/2}$$

is finite, where the infimum is taken over all representations (2.8). We also refer the reader to the works [1]–[2], where these spaces were first introduced.

Moreover, when $f : I \times B \to \mathbb{C}$ has a representation (2.8), we shall define the function $T_{s,b} f$ via

$$(T_{s,b} f)(x,t) = \sum_{n,m} \langle n\rangle^{s} \langle n^2 - m\rangle^{b} f_{n,m} e_n(x) e(mt). \tag{2.9}$$

Our analysis requires considering short time intervals $[0, T]$, where T will depend on the truncation parameters. In order to establish contractive estimates for the nonlinear term, we need a variant of the $\|\cdot\|_{0,1/2}$-norm adapted to the time interval. We denote this norm by $|||\cdot|||_{0,1/2;T}$, and its unit ball is generated by functions of the form

$$\sum_{n,m} \frac{a_{n,m}}{(|n^2-m|+1/T)^{1/2}}\, e_n(x)e(mt) + \sum_{\substack{n,m \\ |n^2-m|>1/T}} \frac{a_n}{|n^2-m|}\, e_n(x)e(mt) \tag{2.10}$$

with

$$\sum_{n,m} |a_{n,m}|^2 \leq 1 \quad \text{and} \quad \sum_n |a_n|^2 \leq 1.$$

Obviously, $\|\cdot\|_{0,b} \lesssim |||\cdot|||$ for $b < 1/2$. One can similarly introduce norms $|||\cdot|||_{s,1/2;T}$ for $s > 0$, but we will not need them.

The next few lemmas put into evidence some basic properties of the norm $|||\cdot|||$.

Lemma 2.1. *Let* $|||f||| \leq 1$. *Then*

$$\frac{1}{T}\int_0^T \|f(t)\|^2_{L^2_x}\, dt < O(1). \tag{2.11}$$

Proof. We first write f as in (2.10). Then

$$\begin{aligned}\|f(t)\|^2_{L^2_x} &= \sum_n \Big|\sum_m \frac{f_{n,m}}{(|n^2-m|+1/T)^{1/2}}\, e(mt)\Big|^2 + \sum_n \Big|\sum_{|n^2-m|>1/T} \frac{e(mt)}{n^2-m}\Big|^2 |f_n|^2 \\ &= (I) + (II).\end{aligned}$$

Taking $0 \leq \varphi \leq 2$ such that $\varphi \geq 1$ on $[0,1]$ and $\operatorname{supp}\hat{\varphi} \subset [-1,1]$, we have

$$\begin{aligned}\int_0^T (I)\, dt &\leq \int (I)\varphi\Big(\frac{t}{T}\Big)\, dt \leq T \sum_n \sum_{\substack{m,m' \\ |m-m'|\leq 1/T}} \frac{|f_{n,m}|\,|f_{n,m'}|}{|n^2-m|+1/T} \\ &\leq T^2 \sum_{|k|\leq 1/T} \sum_{n,m} |f_{n,m}|\,|f_{n,m+k}| \lesssim T\|f\|^2_{L^2_{t,x}} \lesssim T\end{aligned}$$

and similarly

$$\begin{aligned}\int_0^T (II)\, dt &\lesssim T \sum_n \sum_{\substack{m,m',\ |m-m'|\lesssim 1/T \\ |n^2-m|>1/T,\ |n^2-m'|>1/T}} \frac{|f_n|^2}{|n^2-m|\,|n^2-m'|} \\ &\lesssim \sum_{\substack{n,m \\ |n^2-m|>1/T}} \frac{|f_n|^2}{|n^2-m|^2} \lesssim T.\end{aligned}$$

The combination of these two bounds suffices to prove the claim. □

The next statement expresses an important duality property with respect to the Duhamel formula (1.2).

Lemma 2.2. *Assume* $f(x,t)=\sum_{n\in\mathbb{Z}_+,\, m\in\mathbb{Z}} f_{n,m}e_n(x)e(mt)$. *Then*

$$\Big|\Big|\Big|\int_0^t e^{i(t-\tau)\Delta} f(\tau)\,d\tau\Big|\Big|\Big| \lesssim \max_{|||g|||_{0,1/2;T}\le 1}\Big|\sum_{n,m} f_{n,m}g_{n,m}\Big|, \tag{2.12}$$

where $g(x,t)=\sum_{n\in\mathbb{Z}_+,\, m\in\mathbb{Z}} g_{n,m}e_n(x)e(mt)$.

Proof. Write

$$\int_0^t e^{i(t-\tau)\Delta} f(\tau)\,d\tau = \sum_{n,m} f_{n,m}\, e_n \frac{e(mt)-e(n^2t)}{m-n^2}$$

and decompose this as

$$\sum_{|m-n^2|>1/T} \frac{f_{n,m}}{m-n^2}\, e_n\, e(mt) \tag{2.13}$$

$$-\sum_{|m-n^2|>1/T} \frac{f_{n,m}}{m-n^2}\, e_n\, e(n^2t) \tag{2.14}$$

$$+\sum_{|m-n^2|\le 1/T} f_{n,m}e_n \frac{e(mt)-e(n^2t)}{m-n^2}. \tag{2.15}$$

Hence we may write (2.13) as

$$\sum_{|m-n^2|>1/T} \frac{b_{n,m}}{|m-n^2|^{1/2}} e_n\, e(mt) \quad \text{with} \quad b_{n,m}=\frac{\pm f_{n,m}}{|m-n^2|^{1/2}},$$

which satisfies

$$\Big(\sum_{n,m}|b_{n,m}|^2\Big)^{1/2} = \Big(\sum_{|m-n^2|>1/T}\frac{|f_{n,m}|^2}{|m-n^2|}\Big)^{1/2} = \max\Big|\sum_{|m-n^2|>1/T} f_{n,m}\frac{a_{n,m}}{|m-n^2|^{1/2}}\Big|$$

where the maximum is over sequences $(a_{n,m})$ with $(\sum_{n,m}|a_{n,m}|^2)^{1/2}\le 1$, which takes care of the contribution of (2.13) to the left-hand side of (2.12).

Next, let $\varphi(t)=\sum_k \hat\varphi(k)e(kt)$ satisfy $\varphi=1$ on $[0,T]$, $\varphi\ge 0$ together with the condition $|\hat\varphi(k)|\lesssim T/(1+|k|T)^2$.

For $0\le t\le T$, write (2.14) as

$$\Big[\sum_n b_n e_n e(n^2t)\Big]\varphi(t) = \sum_{n,k} b_n e_n e((n^2+k)t)\hat\varphi(k) \tag{2.16}$$

with

$$b_n = \sum_{|m-n^2|>1/T}\frac{f_{n,m}}{m-n^2}.$$

Thus (2.16) becomes

$$\sum \frac{a_{n,m}}{|n^2-m|^{1/2}}\, e_n e(mt)$$

with $a_{n,m} = b_n|n^2 - m|^{1/2}\hat{\varphi}(n^2 - m)$ and

$$\Big(\sum_{n,m} |a_{n,m}|^2\Big)^{1/2} = \Big(\sum_n |b_n|^2 \sum_k |k|\,|\hat{\varphi}(k)|^2\Big)^{1/2} \lesssim \Big(\sum_n |b_n|^2\Big)^{1/2}$$
$$= \max \Big|\sum_{n,m} f_{n,m} \frac{a_n}{|m-n^2|} \chi_{|m-n^2|>1/T}\Big|$$

with maximum taken over (a_n) such that $\sum_n |a_n|^2 \le 1$, which is the desired estimate for the contribution of (2.14).

Finally, for $0 \le t \le T$ and φ as above, write (2.15) as

$$\sum_{|m-n^2|\le 1/T} f_{n,m}\, e_n\, e(n^2t) \frac{e((m-n^2)t) - 1}{m-n^2} \varphi(t)$$

and expand the exponential in a power series

$$\sum_{s\ge 1} \frac{1}{s!} \Big[\sum_n b_n^{(s)} e_n\, e(n^2t)\Big] \Big(\frac{t}{T}\Big)^s \varphi(t) \tag{2.17}$$

with

$$b_n^{(s)} = \sum_{|m-n^2|\le 1/T} f_{n,m}(m-n^2)^{s-1} T^s$$

to obtain

$$\Big(\sum_n |b_n^{(s)}|^2\Big)^{1/2} \le T\Big[\sum_n \Big(\sum_{|m-n^2|\le 1/T} |f_{n,m}|\Big)^2\Big]^{1/2} \le \sqrt{T}\Big(\sum_{|m-n^2|\le 1/T} |f_{n,m}|^2\Big)^{1/2}.$$

For each s, let $\psi_s(t) = \sum_k \hat{\psi}_s e(kt)$ be an extension of $(t/T)^s$, $0 \le t \le T$, such that

$$|\psi_s| \le 2 \quad \text{and} \quad |\psi_s'| \le 10sT^{-1}.$$

Then

$$|\widehat{\varphi\psi_s}(k)| \le \|\varphi\psi_s\|_{L^1_t} \le 2\|\varphi\|_{L^1_t} \lesssim T$$

and

$$\|\varphi\psi_s\|_{H^{1/2}} \lesssim \|\varphi\psi_s\|_{L^2_t}^{1/2} (\|\varphi'\psi_s\|_{L^2_t} + \|\varphi\psi_s'\|_{L^2_t})^{1/2} \lesssim T^{1/4}(T^{-1/2} + sT^{-1/2})^{1/2} \lesssim s^{1/2},$$

which in view of (2.17) gives the desired representation of (2.15). □

The norm $|||\cdot|||$ does not quite control the $L^\infty_{0\le t\le T} L^2_x$ norm. However, the following holds, which will suffice for our purpose.

Lemma 2.3. *Let f and g have expansions as in Lemma 2.2. Then*

$$\Big|\sum_{n,m} f_{n,m} g_{n,m}\Big| \lesssim T|||f||| \cdot |||g|||.$$

Proof. From the representation (2.10), we obtain

$$\sum_{n,m} |f_{n,m}|\,|g_{n,m}| \lesssim \sum_{n,m} \frac{|a_{n,m} b_{n,m}|}{|n^2 - m| + 1/T} + \sum_{|n^2 - m| > 1/T} \frac{|a_n b_{n,m}|}{|n^2 - m|^{3/2}} + \sum_{|n^2 - m| > 1/T} \frac{|a_{n,m} b_n|}{|n^2 - m|^{3/2}} + \sum_{|n^2 - m| > 1/T} \frac{|a_n b_n|}{|n^2 - m|^2}$$

with

$$\sum_{n,m} |a_{n,m}|^2 \le 1, \quad \sum_{n,m} |b_{n,m}|^2 \le 1, \quad \sum_n |a_n|^2 \le 1, \quad \sum_n |b_n|^2 \le 1.$$

By the Cauchy–Schwarz inequality, the first term on the right is bounded by T, while the second term is bounded by

$$\left\{ \sum_n \left(\sum_{\{m:\, |n^2 - m| > 1/T\}} \frac{|b_{n,m}|}{|n^2 - m|^{3/2}} \right)^2 \right\}^{1/2} \lesssim T.$$

The estimate for the third term is similar. Estimating the last term, we obtain the bound $T \sum_n a_n b_n \lesssim T$, which allows us to complete the proof. □

Next, we establish several inequalities bounding suitable $L^p_x L^q_t$ norms in terms of $X^{s,b}$ norms. These will be essential to our analysis.

Lemma 2.4. *The spaces $X^{s,b}$ obey the following embedding relations:*

(i) *For $2 < p < 3$ and $b_1 > 1/4$,*

$$\|f\|_{L^p_x L^2_t} \lesssim \|f\|_{0,b_1}.$$

(ii) *For $3 < p < 6$, $s > 1 - 3/p$ and $b_2 > 1/2$,*

$$\|f\|_{L^p_x L^4_t} \lesssim \|f\|_{s,b_2}.$$

(iii) *For $1/4 < b_3 < 1/2$ and $\epsilon > 0$,*

$$\|f\|_{L^3_x L_t^{\frac{4}{3-4b_3}}} \lesssim \|f\|_{\epsilon,b_3}.$$

(iv) *For $b_4 > 1/2$ and $s > 1/2$,*

$$\|f\|_{L^3_x L^\infty_t} \lesssim \|f\|_{s,b_4}.$$

(v) *For $3 \le p \le 6$, $4 \le q \le \infty$, $s > 3/2 - 3/p - 2/q$ and $b_5 > 1/2$,*

$$\|f\|_{L^p_x L^q_t} \lesssim \|f\|_{s,b_5}.$$

(vi) *For* $1/4 < b_6 < 1/2$, $3 < p < 6/(3-4b_6)$, $4/(3-4b_6) < q < \infty$, *and* $s > 5/2 - 3/p - 2/q - 2b_6$,

$$\|f\|_{L^p_x L^q_t} \lesssim \|f\|_{s,b_6}.$$

(vii) *For* $2 \le p < 8/3$ *and* $b_7 > 1/2$,

$$\|f\|_{L^p_x L^p_t} \lesssim \|f\|_{0,b_7}.$$

(viii) *For* $1/4 < b_8 < 1/2$, $p < 24/(4b_8+7)$, *and* $q < 8/(5-4b_8)$,

$$\|f\|_{L^p_x L^q_t} \lesssim \|f\|_{0,b_8}.$$

Proof. We begin with (i). Let $2 < p < 3$ be given. Then for every f as in (2.8), applying the Plancherel identity in time followed by the Minkowski inequality, the eigenfunction estimate (2.4) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\|f\|_{L^p_x L^2_t} &\lesssim \Big(\sum_m \Big\|\sum_n f_{m,n} e_n(x)\Big\|^2_{L^p_x}\Big)^{1/2} \lesssim \Big(\sum_m \Big(\sum_n |f_{m,n}|\Big)^2\Big)^{1/2} \\ &\lesssim \Big(\sum_m \Big(\sum_n \langle m-n^2\rangle^{2b} |f_{m,n}|^2\Big)\Big(\sum_n \frac{1}{\langle m-n^2\rangle^{2b}}\Big)\Big)^{1/2}.\end{aligned}$$

Observing that $b > 1/4$ implies

$$\sup_m \sum_n \frac{1}{\langle m-n^2\rangle^{2b}} < \infty$$

then establishes (i) as desired.

We now turn to (ii), for which we argue as in the proof of [10, Lemma 2.3]. Let $3 < p < 6$ be given. Then, writing (2.8) in the form

$$f(t,x) = \sum_m \Big(\sum_n f_{m+n^2,n} e_n(x) e(n^2 t)\Big) e(mt),$$

we perform a dyadic decomposition into intervals $m \sim M$, $n \sim N$, expand the square inside the norm $\| |\cdot|^2 \|^{1/2}_{L^{p/2}_x L^2_t}$, and use the Plancherel identity in the t variable to obtain

$$\begin{aligned}\|f\|_{L^p_x L^4_t} &\lesssim \sum_{M,N} \sum_{m\sim M} \Big\|\sum_{n\sim N} f_{m+n^2,n} e_n(x) e(n^2 t)\Big\|_{L^p_x L^4_t} \\ &\lesssim \sum_{M,N} \sum_{m\sim M} \Big\|\Big(\sum_\ell \Big|\sum_{\substack{n,n'\sim N\\ n^2+(n')^2=\ell}} f_{m+n^2,n} f_{m+(n')^2,n'} e_n(x) e_{n'}(x)\Big|^2\Big)^{1/2}\Big\|^{1/2}_{L^{p/2}_x} \\ &\lesssim \sum_{M,N} \sum_{m\sim M} \Big(\sup_\ell \sum_{\substack{n,n'\sim N\\ n^2+(n')^2=\ell}} 1\Big)^{1/4} \Big\|\Big(\sum_{n\sim N} |f_{m+n^2,n}|^2 e_n(x)^2\Big)\Big\|^{1/2}_{L^{p/2}_x}, \qquad (2.18)\end{aligned}$$

where we have used the Cauchy–Schwarz inequality to obtain the last bound.

Note that arithmetic considerations associated with lattice points on circles (see, for instance [10, Lemma 2.1] and the comments in the proof of [10, Lemma 2.3]) entail the bound

$$\sup_{\ell \geq 0} |\{(n, n') \in [0, N]^2 : n^2 + (n')^2 = \ell\}| \lesssim N^{\epsilon} \tag{2.19}$$

for any $\epsilon > 0$ (where the implicit constant may depend on ϵ).

Set $\epsilon = s - (1 - 3/p)$. Then, using (2.19) followed by the Minkowski inequality, the eigenfunction estimates (2.19), and the Cauchy–Schwarz inequality in the summation over $m \sim M$, we obtain

$$\begin{aligned}
(2.18) &\lesssim \sum_{M,N} \sum_{m \sim M} N^{\epsilon/4} \Big(\sum_{n \sim N} |f_{m+n^2,n}|^2 \|e_n(x)\|^2_{L^p_x} \Big)^{1/2} \\
&\lesssim \sum_{M,N} \sum_{m \sim M} N^{\epsilon/4} \Big(\sum_{n \sim N} n^{2-6/p} |f_{m+n^2,n}|^2 \Big)^{1/2} \\
&\lesssim \sum_{M,N} N^{\epsilon/4} M^{1/2} \Big(\sum_{m \sim M} \sum_{n \sim N} n^{2-6/p} |f_{m+n^2,n}|^2 \Big)^{1/2} \\
&\lesssim \sum_{M,N} N^{-3\epsilon/4} M^{1/2-b} \Big(\sum_{\substack{n \sim N \\ m-n^2 \sim M}} \langle n \rangle^{2s} \langle m - n^2 \rangle^{2b} |f_{m,n}|^2 \Big)^{1/2} \lesssim \|f\|_{s,b}
\end{aligned}$$

since $\sum_{M,N} N^{-3\epsilon/2} M^{1-2b} < \infty$. This completes the proof of part (ii) of the lemma.

The inequality stated in part (iii) now follows from (i) and (ii) by standard interpolation arguments.

Next, we prove (iv). Since $b_4 > 1/2$, it suffices to consider f of the form

$$f(x,t) = \sum_n a_n e_n(x) e(n^2 t) \quad \text{with} \quad \sum_n n^{2s} |a_n|^2 \leq 1.$$

It follows from the Cauchy–Schwarz inequality that for any $\epsilon > 0$,

$$|f(x,t)| \lesssim \Big[\sum_n |a_n|^2 n^{1+\epsilon} |e_n(x)|^2 \Big]^{1/2}$$

and hence

$$\|f\|_{L^3_x L^\infty_t} \lesssim \Big[\sum_n |a_n|^2 n^{1+\epsilon} \|e_n\|^2_{L^3_x} \Big]^{1/2} < O(1).$$

Inequality (v) then follows by interpolation between (ii) and (iv), while (vi) is obtained by interpolating between (i) and (v).

We prove (vii), taking f of the form

$$f(x,t) = \sum_n a_n e_n(x) e(n^2 t) = \sum_n a_n \frac{\sin(\pi n r)}{r} e(n^2 t)$$

with $r = |x|$ and $\sum_n |a_n|^2 \leq 1$.

Fix $0 < \rho \le 1$ and consider values of x in the annulus $\rho/2 \le r \le \rho$. We make two estimates. We first note that

$$\begin{aligned}\|f\|_{L^4_{|x|\sim\rho}L^4_t} &\le \frac{1}{\sqrt{\rho}}\Big\|\sum_n a_n \sin(\pi n r)e(n^2 t)\Big\|_{L^4_{r\le 1}L^4_{|t|\le 1}}\\ &\le \frac{1}{\sqrt{\rho}}\Big[\max_{k,\ell}\big|\{(n,n')\in\mathbb{Z}^2 : n\pm n' = k, n^2+(n')^2=\ell\}\big|\Big]^{1/4}\\ &\lesssim \frac{1}{\sqrt{\rho}}. \end{aligned} \tag{2.20}$$

On the other hand, one has

$$\|f\|_{L^2_{|x|\sim\rho}L^2_t} \le \Big\|\sum_n a_n \sin(\pi n r)e(n^2 t)\Big\|_{L^2_{r\sim\rho}L^2_{|t|<1}} \lesssim \sqrt{\rho}. \tag{2.21}$$

Hence (vii) follows by interpolation between (2.20), (2.21) and summation over dyadic $\rho = 2^{-j}$.

Finally, (viii) is obtained by interpolation between (i) and (vii). This completes the proof of Lemma 2.4. □

3. A priori uniform bounds

In this section, we establish $X^{s,b}$ bounds on solutions of the truncated equation (1.1) which are uniform in the truncation parameter N. For this purpose, we will first obtain a preliminary uniform estimate on the norms $L^p_x L^q_t$ for suitable values of p and q. In particular, we have the following:

Lemma 3.1. *For every $0 \le s < 1/2$, $1 \le p < 6/(1+2s)$, $1 \le q < \infty$, there exists a constant $C > 0$ such that for every $N > 0$ one has the bound*

$$\mu_F^{(N)}(\{\phi : \|(\sqrt{-\Delta})^s u\|_{L^p_x L^q_t} > \lambda\}) \lesssim \exp(-c\lambda^c), \tag{3.1}$$

where $u = u_N$ is a solution to the truncated equation (1.1) associated to initial data ϕ (truncated as $P_N\phi$).

Proof. Without loss of generality we may assume $p > 3$. It suffices to show that (3.1) holds with $\mu_F^{(N)}$ replaced by the Gibbs measure μ_G. Indeed, suppose that

$$\mu_G(A_\lambda) \le C\exp(-c\lambda^c) \tag{3.2}$$

with

$$A_\lambda := \{\phi : \|(\sqrt{-\Delta})^s u\|_{L^p_x L^q_t} > \lambda\}, \qquad \lambda > 0.$$

Then, fixing $\lambda_1 > 0$, we have

$$\begin{aligned}\mu_F^{(N)}(A_\lambda) &= \mu_F^{(N)}(A_\lambda \cap \{\phi : \|\phi\|_{L_x^4} > \lambda_1\}) + \mu_F^{(N)}(A_\lambda \cap \{\phi : \|\phi\|_{L_x^4} \le \lambda_1\}) \\ &\lesssim \mu_F^{(N)}(\{\phi : \|\phi\|_{L_x^4} > \lambda_1\}) + \exp\left(\frac{1}{4}\lambda_1^4\right)\mu_G(A_\lambda) \\ &\lesssim \mu_F^{(N)}(\{\phi : \|\phi\|_{L_x^4} > \lambda_1\}) + \exp\left(\frac{1}{4}\lambda_1^4\right)\exp(-c\lambda^c). \end{aligned} \tag{3.3}$$

To estimate the first term in (3.3), we fix $q_1 \ge 4$ and appeal to the Chebyshev and Minkowski inequalities followed by the estimate (2.5) on sums of Gaussian random variables. This gives

$$\begin{aligned}\mu_F^{(N)}(\{\phi : \|\phi\|_{L_x^4} > \lambda_1\}) &\lesssim \frac{1}{\lambda_1^{q_1}}\left[\mathbb{E}_{\mu_F^{(N)}}\|\phi\|_{L_x^4}^{q_1}\right] \\ &\le \frac{1}{\lambda_1^{q_1}}\left\|\left(\mathbb{E}_{\mu_F^{(N)}}\left[\left(\sum_n \frac{g_n(\omega)}{n}e_n(x)\right)^{q_1}\right]\right)^{1/q_1}\right\|_{L_x^4}^{q_1} \\ &\lesssim \left(\frac{\sqrt{q_1}}{\lambda_1}\right)^{q_1}\left\|\left(\sum_n \frac{|e_n(x)|^2}{n^2}\right)^{1/2}\right\|_{L_x^4}^{q_1} \\ &\lesssim \left(\frac{\sqrt{q_1}}{\lambda_1}\right)^{q_1}\left(\sum_n \frac{\|e_n\|_{L_x^4}^2}{n^2}\right)^{q_1/2}. \end{aligned} \tag{3.4}$$

where in obtaining the last inequality we have used the Minkowski inequality.

Invoking now the eigenfunction estimate (2.2) yields

$$(3.4) \lesssim \left(\frac{\sqrt{q_1}}{\lambda_1}\right)^{q_1}\left(\sum_n n^{-3/2}\right)^{q_1/2} \lesssim \left(\frac{\sqrt{q_1}}{\lambda_1}\right)^{q_1}.$$

We therefore obtain

$$\mu_F^{(N)}(A_\lambda) \lesssim \left(\frac{\sqrt{q_1}}{\lambda_1}\right)^{q_1} + \exp\left(\frac{1}{4}\lambda_1^4\right)\mu_G(A),$$

so that optimizing in the choice of q_1 gives $\mu_F^{(N)}(A_\lambda) \lesssim \exp(-c\lambda_1^c)$ as desired.

It therefore suffices to show (3.2), which we recall was the desired inequality with the measure $\mu_F^{(N)}$ replaced by the (invariant) Gibbs measure $\mu_G = \mu_G^{(N)}$. We argue as above: fixing $q_2 \ge \max\{p, q\}$ and invoking the Chebyshev and Minkowski inequalities, one has

$$\begin{aligned}\mu_G(A_\lambda) &\le \lambda^{-q_2}\mathbb{E}_{\mu_G}\left[\|(\sqrt{-\Delta})^s u\|_{L_x^p L_t^q}^{q_2}\right] \\ &\lesssim \lambda^{-q_2}\|(\mathbb{E}_{\mu_G}[((\sqrt{-\Delta})^s u)^{q_2}])^{1/q_2}\|_{L_x^p L_t^q}^{q_2}. \end{aligned} \tag{3.5}$$

Now, using the invariance of the Gibbs measure $\mu_G = \mu_G^{(N)}$ with respect to the truncated evolution (with $u = u_N$ being a solution of the truncated equation) followed by the estimate for sums of Gaussian random variables given by (2.5), we obtain

$$(3.5) \lesssim \lambda^{-q_2} \left\| \left(\mathbb{E}_{\mu_G} \left[\left(\sum_n \frac{g_n(\omega)}{n^{1-s}} e_n \right)^{q_2} \right] \right)^{1/q_2} \right\|_{L_x^p L_t^q}^{q_2} \lesssim \left(\frac{\sqrt{q_2}}{\lambda} \right)^{q_2} \left\| \sum_n \frac{|e_n(x)|^2}{n^{2(1-s)}} \right\|_{L_x^{p/2}}^{q_2/2}.$$

To conclude, we use the eigenfunction estimate (2.2) together with the condition $p < 6/(1+2s)$ to get the bound

$$\left\| \sum_n \frac{|e_n(x)|^2}{n^{2(1-s)}} \right\|_{L_x^{p/2}}^{q_2/2} \lesssim \left(\sum_n \frac{1}{n^{2(1-s)}} \|e_n\|_{L_x^p}^2 \right)^{q_2/2} \lesssim \left(\sum_n n^{2(s-3/p)} \right)^{q_2/2} \lesssim 1.$$

Hence

$$\mu_G(A_\lambda) \lesssim (\sqrt{q_2}/\lambda)^{q_2}. \tag{3.6}$$

Optimizing the choice of q_2 in (3.6) as for q_1 above gives

$$\mu_F^{(N)}(\{\phi : \|(\sqrt{-\Delta})^s u\|_{L_x^p L_t^q} > \lambda\}) \lesssim \exp(-c\lambda^2)$$

as desired. □

We are now ready to establish uniform $X^{s,b}$ bounds.

Proposition 3.2. *Fix $0 \le s < 1/2$ and $1/2 < b < 3/4$. Then there exists $C > 0$ such that for all $N > 0$, if $u = u_N$ is a solution to the truncated equation (1.1), then*

$$\mu_F^{(N)}(\{\phi : \|u\|_{s,b} > \lambda\}) \lesssim \exp(-c_1\lambda^{c_2}).$$

Proof. Let $s \in [0, 1/2)$ and $b \in (1/2, 3/4)$ be given. Fix $N \ge 1$ and write the Duhamel formula

$$u(t) = e^{it\Delta}\phi + \int_0^t e^{i(t-\tau)\Delta} |u(\tau)|^2 u(\tau)\, d\tau. \tag{3.7}$$

We estimate both the linear and nonlinear terms in (3.7) individually. We begin with the linear term. Let $T_{s,b}$ be the operator defined in (2.9). Then, fixing $q \ge 2$ and invoking the Chebyshev and Minkowski inequalities, one has

$$\begin{aligned} \mu_F^{(N)}(\{\phi : \|e^{it\Delta}\phi\|_{s,b} > \lambda\}) &\le \lambda^{-q} \mathbb{E}_\omega\big[\|T_{s,b} e^{it\Delta}\phi\|_{L_{t,x}^2}^q\big] \lesssim \lambda^{-q} \|\mathbb{E}_\omega[(T_{s,b} e^{it\Delta}\phi)^q]^{1/q}\|_{L_{t,x}^2}^q \\ &\lesssim \lambda^{-q} q^{q/2} \left\| \sum_n \frac{|e_n(x)|^2}{n^{2(1-s)}} \right\|_{L_{t,x}^1}^{q/2} \lesssim \lambda^{-q} q^{q/2}. \end{aligned}$$

An appropriate choice of q gives

$$\mu_F^{(N)}(\{\phi : \|e^{it\Delta}\phi\|_{s,b} > \lambda\}) \lesssim \exp(-c\lambda^2). \tag{3.8}$$

Turning to the integral term, we set $f = |u(\tau)|^2 u(\tau)$ and observe that the expansion $f(x,\tau) = \sum_{m,n} f_{n,m} e_n(x) e(m\tau)$ leads to

$$\int_0^t e^{i(t-\tau)\Delta} f(\tau)\, d\tau = \int_0^t \Big(\sum_{m,n} f_{n,m} e_n(x) e((t-\tau)n^2 + m\tau) \Big) d\tau$$
$$= \sum_{m,n} \frac{i f_{n,m}}{(n^2 - m)} e_n(x)(e(tn^2) - e(tm)).$$

Applying Hölder's inequality and recalling $b > 1/2$, we obtain

$$\Big\| \int_0^t e^{i(t-\tau)\Delta} f(\tau)\, d\tau \Big\|_{s,b} \lesssim \Big(\sum_{n,m} \frac{\langle n\rangle^{2s} |f_{n,m}|^2}{\langle n^2 - m\rangle^{2(1-b)}} \Big)^{1/2}$$
$$= \sup_{\substack{v \in X^{0,1-b} \\ \|v\|_{0,1-b} \le 1}} \Big| \int_0^1 \int_B v(t,x) (\sqrt{-\Delta})^s f(t,x)\, dx\, dt \Big|$$
$$\lesssim \sup_{\substack{v \in X^{0,1-b} \\ \|v\|_{0,1-b} \le 1}} \|v\|_{L_x^{3-\epsilon} L_t^2} \|(\sqrt{-\Delta})^s u\|_{L_x^{\frac{3-\epsilon}{1-\epsilon}} L_t^6} \|u\|^2_{L_x^{6-2\epsilon} L_t^6}.$$

Now, invoking Lemma 2.4(i) in the form

$$\|v\|_{L_x^{3-\epsilon} L_t^2} \lesssim \|v\|_{0,1-b}$$

and using Lemma 3.1 to estimate the norms of u yields

$$\Big\| \int_0^t e^{i(t-\tau)\Delta} f(\tau)\, d\tau \Big\|_{s,b} \lesssim \lambda^3$$

for each $\lambda > 0$ and all $\omega \in \Omega$ outside a set of measure $O(\exp(-c\lambda^c))$.

We therefore have (adjusting the value of the constant c as well as the implicit constant)

$$\mu_F^{(N)} \Big(\Big\{ \phi : \Big\| \int_0^t e^{i(t-\tau)\Delta} f(\tau)\, d\tau \Big\|_{s,b} > \lambda \Big\} \Big) \lesssim \exp(-c\lambda^c). \tag{3.9}$$

To conclude, collecting (3.8) and (3.9), we have

$$\mu_F^{(N)}(\{\phi : \|u\|_{s,b} > \lambda\}) \lesssim \exp(-c\lambda^c), \tag{3.10}$$

which gives the desired inequality. □

4. The nonlinear term

The main issue is an estimate on the $|||\cdot|||$ norm of trilinear expressions of the form

$$\int_0^t e^{i(t-\tau)\Delta} P_N[P_{N_1} U^1 \, \overline{P_{N_2} u^{(2)}} \, P_{N_3} u^{(3)}](\tau)\, d\tau \tag{4.1}$$

with $t < T$, where U^1 belongs to $X_{|||\cdot|||}$ and $u^{(2)}, u^{(3)} : \mathbb{R} \times B \to \mathbb{C}$ are solutions to truncated equations (1.1) for possibly different truncations $N^{(2)} \geq N_2$, $N^{(3)} \geq N_3$ and initial data

$$u^{(i)}\big|_{t=0} = P_{N^{(i)}}(\phi), \qquad i = 2, 3.$$

In order to establish a contractive estimate on (4.1), T will have to be chosen sufficiently small; more specifically, we shall require

$$T \sim 1/\log N_* \quad \text{ with } N_* = \max(N_1, N_2, N_3). \tag{4.2}$$

As will be clear later on, this choice of T is essential in our argument due to the presence of a certain logarithmic divergence.

Our analysis is based on $L^p_x L^q_t$ norms as well as the norms $\|\cdot\|_{s,b}$ and $|||\cdot|||$. Various contributions are considered, requiring different arguments. While the norms $\|\cdot\|_{s,b}$ and $|||\cdot|||$ allow in particular for Fourier restrictions of the form $\chi_{n^2-m\lesssim K}$, these operations are in general not allowed for $L^p_x L^q_t$ norms. For this reason, certain care is required in organizing the argument.

We denote by N, N_i, $i = 1, 2, 3$, integers of the form 2^j, and $n \sim N_i$ means $N_i \leq n < 2N_i$. Denote $u_2 = P_{N_2}u^{(2)}$ and $u_3 = P_{N_3}u^{(3)}$.

We start by applying Lemma 2.2, and estimate $|||(4.1)|||$ by

$$\int_0^1 \int_B \bar{v}(P_{N_1}U^1)\bar{u}_2 u_3\, dx\, dt \tag{4.3}$$

with $|||v||| \leq 1$. By Cauchy–Schwarz,

$$(4.3) \leq \left[\iint |v|^2|u_2|^2\, dx\, dt\right]^{1/2}\left[\iint |P_{N_1}U^1|^2|u_3|^2\, dx\, dt\right]^{1/2}. \tag{4.4}$$

In each factor on the right-hand side of (4.4), $u^{(2)}$ and $u^{(3)}$ are obtained from the same truncated equation. This is essential for our analysis.

We have therefore reduced the estimate of (4.3) to estimating

$$\iint \overline{P_N v}\, P_{N_1} v_1\, \overline{P_{N_2} u}\, P_{N_3} u \tag{4.5}$$

with u obtained from some truncated equation (1.1) and $|||v|||, |||v_1||| \leq 1$.

Write (4.5) as

$$\sum_{\substack{n\leq N, n_i\leq N_i\\ m-m_1+m_2-m_3=0}} \overline{\hat{v}(n,m)}\, v_1(n_1,m_1)\, \overline{\hat{u}(n_2,m_2)}\, \hat{u}(n_3,m_3)\, c(n,\bar{n}) \tag{4.6}$$

with $c(n, \bar{n}) = c(n, n_1, n_2, n_3)$. Subdividing $[0, N_i]$ into dyadic intervals $[N_i', 2N_i']$, we estimate

$$(4.6) \leq \sum_{N_2', N_3'} \Bigg| \sum_{\substack{n\leq N,\, n_1\leq N_1,\, n_2\sim N_2',\, n_3\sim N_3'\\ m-m_1+m_2-m_3=0}} c(n,\bar{n}) A_{n,m,\bar{n},\bar{m}} \Bigg| \tag{4.7}$$

with

$$A_{n,m,\bar{n},\bar{m}} = \overline{\hat{v}(n,m)}\,\hat{v}_1(n_1,m_1)\,\overline{\hat{u}(n_2,m_2)}\,\hat{u}(n_3,m_3).$$

Fix N_2', N_3' and assume $N_2' \geq N_3'$. Set

$$K = (N_2')^{10^{-3}}$$

and define

$$c_K(n,\bar{n}) = \begin{cases} c(n,\bar{n}) & \text{if } |n^2 - n_1^2 + n_2^2 - n_3^2| < 10K, \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

We now estimate

$$\Big| \sum_{\substack{n \leq N,\, n_1 \leq N_1,\, n_2 \sim N_2',\, n_3 \sim N_3' \\ m - m_1 + m_2 - m_3 = 0}} c(n,\bar{n}) A_{n,m,\bar{n},\bar{m}} \Big|$$

$$\leq \sum_{N', N_1'} \Big| \sum_{\substack{n \sim N',\, n_1 \sim N_1',\, n_2 \sim N_2',\, n_3 \sim N_3' \\ m - m_1 + m_2 - m_3 = 0, |m - n^2| \geq K}} c(n,\bar{n}) A_{n,m,\bar{n},\bar{m}} \Big| \tag{4.9}$$

$$+ \sum_{N', N_1'} \Big| \sum_{\substack{n \sim N',\, n_1 \sim N_1',\, n_2 \sim N_2',\, n_3 \sim N_3' \\ m - m_1 + m_2 - m_3 = 0, |m - n^2| < K \\ |m_1 - n_1^2| \geq K}} c(n,\bar{n}) A_{n,m,\bar{n},\bar{m}} \Big| \tag{4.10}$$

$$+ \Big| \sum_{\substack{n \leq N,\, n_1 \leq N_1,\, n_2 \sim N_2',\, n_3 \sim N_3' \\ m - m_1 + m_2 - m_3 = 0,\, |m - n^2| < K,\, |m_1 - n_1^2| < K}} c(n,\bar{n}) A_{n,m,\bar{n},\bar{m}} \Big|. \tag{4.11}$$

Making a further decomposition according to whether $|n^2 - n_1^2 + n_2^2 - n_3^2| > 10K$ or $|n^2 - n_1^2 + n_2^2 - n_3^2| \lesssim 10K$ in (4.11), the contribution of (4.11) may be evaluated by bounding

$$\sum_{N', N_1'} \Big| \sum_{\substack{n \sim N',\, n_i \sim N_i',\, |n^2 - n_1^2 + n_2^2 - n_3^2| > 10K \\ m - m_1 + m_2 - m_3 = 0,\, |m - n^2| < K,\, |m_1 - n_1^2| < K}} c(n,\bar{n}) A_{n,m,\bar{n},\bar{m}} \Big| \tag{4.12}$$

$$+ \Big| \sum_{\substack{n \leq N,\, n_1 \leq N_1,\, n_2 \sim N_2',\, n_3 \sim N_3' \\ m - m_1 + m_2 - m_3 = 0}} c_K(n,\bar{n}) A_{n,m,\bar{n},\bar{m}} \Big|, \tag{4.13}$$

where in (4.13) we replaced $\hat{v}$ by $\hat{v}\chi_{|m-n^2| \leq K}$ and $\hat{v}_1$ by $\hat{v}_1\chi_{|m_1 - n_1^2| < K}$ (noting that the norm $|||\cdot|||$ is unconditional).

Note that if $n_2 = n_3$ and $|n^2 - n_1^2 + n_2^2 - n_3^2| \leq 10K$, then either $n = n_1$ or $N' + N_1' \lesssim (N_2')^{10^{-3}}$. Hence, (4.13) is bounded by

$$\sum_{N', N_1'} \Big| \sum_{\substack{n \sim N',\, n_i \sim N_i',\, n_2 \neq n_3 \\ m - m_1 + m_2 - m_3 = 0}} c_K(n, \bar{n}) A_{n,m,\bar{n},\bar{m}} \Big| \tag{4.14}$$

$$+ \sum_{N',\, N_1' \lesssim (N_2')^{10^{-3}}} \Big| \sum_{\substack{n \sim N',\, n_1 \sim N_1',\, n_2 \sim N_2' \\ m - m_1 + m_2 - m_3 = 0}} c_K(n, n_1, n_2, n_2) A_{n,m,(n_1,n_2,n_2),\bar{m}} \Big| \tag{4.15}$$

$$+ \Big| \sum_{\substack{n \leq N,\, n_2 \sim N_2' \\ m - m_1 + m_2 - m_3 = 0}} c(n, n, n_2, n_2) A_{n,m,(n,n_2,n_2),\bar{m}} \Big|. \tag{4.16}$$

Let

$$\sigma_{n,N_2'} = \sum_{n_2 \sim N_2'} \frac{1}{n_2^2} c(n, n, n_2, n_2) = O(1)$$

and estimate (4.16) by

$$\sum_{N'} \Big| \sum_{n \sim N'} \int_0^1 \overline{\hat{v}(n)(\tau)}\, \hat{v}_1(n)(\tau) \Big[\sum_{n_2 \sim N_2'} c(n, n, n_2, n_2) |\hat{u}(n)(\tau)|^2 - \sigma_{n,N_2'} \Big] d\tau \Big| \tag{4.17}$$

$$+ \Big| \sum_{n \leq N} \Big(\int_0^1 \overline{\hat{v}(n)(\tau)}\, \hat{v}_1(n)(\tau)\, d\tau \Big) \sigma_{n,N_2'} \Big|. \tag{4.18}$$

In view of the above observations, our estimate of |||(4.1)||| reduces to establishing bounds on (4.9), (4.10), (4.12), (4.14), (4.15), (4.17) and (4.18); this will be the topic of the following two sections.

The choice of T is dictated by (4.18), and we treat this term first. Indeed, taking T sufficiently small, Lemma 2.3 gives

$$\Big| \sum_{n \leq N, m} \overline{\hat{v}(n, m)}\, \hat{v}_1(n, m) \sigma_{n,N_2'} \Big| \lesssim \sum_{n \leq N, m} |\hat{v}(n, m)|\, |\hat{v}_1(n, m)| \lesssim T$$
$$= o\Big(\frac{1}{\log N_*} \Big). \tag{4.19}$$

Evaluating the summation over dyadic $N_2' \leq N_*$ then allows us to conclude that the contribution of (4.18) can be estimated by $o(1)$.

5. Multilinear estimates (I)

In this section, we obtain bounds on the terms (4.9), (4.10) and (4.12). The remaining terms will be treated in the next section.

We begin with the contribution of (4.9). Fix the values N', N_1' and rewrite the inner sum in (4.9) as

$$\int_B \int_0^1 \overline{P_{n\sim N'} v}\, P_{n_1\sim N_1'} v_1\, \overline{P_{n_2\sim N_2'} u}\, P_{n_3\sim N_3'} u\, dt\, dx, \tag{5.1}$$

where $\hat{v}(n,m) = 0$ for $|m-n^2| < K$.

It follows from the definition of the $|||\cdot|||$ norm that

$$\|P_{n\sim N'} v\|_{0,1/3} < K^{-1/7} |||P_{n\sim N'} v|||.$$

Moreover, by Lemma 2.4(viii), applied with $b = 1/4 + 3\epsilon/2$, $\epsilon = 10^{-6}$, we obtain

$$\|P_{n\sim N'} v\|_{L_x^{\frac{3}{1+\epsilon}} L_t^{\frac{2}{1-\epsilon}}} < K^{-1/7} |||P_{n\sim N'} v|||. \tag{5.2}$$

Also by Lemma 2.4(viii),

$$\|P_{n_1\sim N_1'} v_1\|_{L_x^{\frac{3}{1+\epsilon}} L_t^{\frac{2}{1-\epsilon}}} \lesssim |||P_{n\sim N_1'} v_1|||. \tag{5.3}$$

To estimate the contributions of u_2 and u_3 to (5.1), we use the a priori bound given by Lemma 3.1 with $q = 2/\epsilon$, where $\epsilon = 10^{-6}$ as before. In particular, we may ensure that

$$\max\{\|P_{n_2\sim N_2'} u\|_{L_x^{6-\epsilon} L_t^q}, \|P_{n_3\sim N_3'} u\|_{L_x^{6-\epsilon} L_t^q}\} < (N_2')^{10^{-6}} \tag{5.4}$$

outside an exceptional set of measure at most $\exp(-c(N_2')^{c10^{-6}})$ in the initial datum ϕ. Taking $p = 6/(1-2\epsilon)$, we then obtain

$$\|P_{n_2\sim N_2'} u\|_{L_x^p L_t^q} \lesssim (N_2')^{10^{-6} + \frac{3}{6-\epsilon} - 1/2 + \epsilon}. \tag{5.5}$$

From (5.2)–(5.5) and recalling that $K = (N_2')^{10^{-3}}$ and $\epsilon = 10^{-6}$, it follows that

$$\begin{aligned}(5.1) &< K^{-1/7} (N_2')^{10^{-6} + \frac{3}{6-\epsilon} - 1/2 + \epsilon} |||P_{n\sim N'} v|||\, |||P_{n\sim N_1'} v_1|||\\ &< (N_2')^{-1/2 10^{-4}} |||P_{n\sim N'} v|||\, |||P_{n\sim N_1'} v_1|||.\end{aligned} \tag{5.6}$$

To complete the estimate of the contribution of (4.9), it remains to perform dyadic summation over N', N_1', N_2' and N_3', with $N_2' \geq N_3'$. Note that from the definition of the $|||\cdot|||$ norm, one has

$$|||v|||^2 \sim \sum_{N'} |||P_{n\sim N'} v|||^2. \tag{5.7}$$

In view of (5.6), there is of course no problem with the summation over values of N_2' and N_3', and we may also assume $\max\{N', N_1'\} > \exp((N_2')^{10^{-5}})$. Consider the case $N' \geq N_1'$. If $N' \sim N_1'$, the estimate follows by using Cauchy–Schwarz and (5.7) for v and v_1. Assume now that $N' > 4N_1'$. We estimate the contribution of such terms to (4.9) by

$$\gamma \int \Big[\sum_{n\sim N'} |\hat{v}(n)(t)|\Big]\Big[\sum_{n_1\sim N_1'} |\hat{v}_1(n_1)(t)|\Big]\Big[\sum_{n_2\sim N_2'} |\hat{u}(n_2)(t)|\Big]\Big[\sum_{n_3\sim N_3'} |\hat{u}(n_3)(t)|\Big] dt \tag{5.8}$$

with

$$\gamma = \max_{n\sim N',\, n_i\sim N_i'} |c(n,n_1,n_2,n_3)|.$$

We then have the bound

$$\begin{aligned}(5.8) &\leq \gamma \cdot (N'N_1'N_2'N_3')^{1/2}\|v\|_{L_t^2L_x^2}\|v_1\|_{L_t^2L_x^2}\|u\|^2_{L_t^\infty L_x^2}\\ &\leq \gamma \cdot (N'N_1'N_2'N_3')^{1/2}|||v|||\,|||v_1|||\,\|u\|^2_{L_t^\infty L_x^2}.\end{aligned} \tag{5.9}$$

To evaluate γ, we write

$$\int_B e_n e_{n_1} e_{n_2} e_{n_3}\,dx = \int_0^1 \sin(n\pi r)\sin(n_1\pi r)\varphi(r)\,dr \tag{5.10}$$

with $\varphi(r) = \frac{\sin(\pi n_2 r)}{r}\cdot\frac{\sin(\pi n_3 r)}{r}$, and note that integration by parts gives

$$\begin{aligned}\int_0^1 \cos((n\pm n_1)\pi r)\,\varphi(r)\,dr &= -\frac{1}{\pi(n\pm n_1)}\int_0^1 \varphi'(r)\sin((n\pm n_1)\pi r)\,dr\\ &< O\Big(\frac{\|\varphi'\|_{L^\infty}}{(n\pm n_1)^2}\Big) < O\Big(\frac{(N_2')^2N_3'}{(N')^2}\Big),\end{aligned}$$

where the last inequality follows from $N' > 4N_1'$. Hence

$$(5.9) \lesssim \frac{(N_2')^4}{N'}.$$

Summing (5.9) over dyadic N', N_1', N_2' and N_3' with $N' > \max\{\exp((N_2')^{10^{-5}}), 4N_1'\}$ and $N_3' \leq N_2'$ shows that the contribution of (5.8) is bounded by

$$\sum_{N',N_2'} \frac{(N_2')^4(\log N_2')(\log N')}{N'} < \frac{1}{N_2'},$$

which completes the estimate of the contribution of (4.9).

Since v and v_1 play the same role, the same argument also takes care of the contribution of (4.10).

We now address the contribution of (4.12). Since the estimate relies only on $X_{s,b}$ norms, Fourier restrictions are not an issue. Note that since $|m-n^2| < K$, $|m_1 - n_1^2| < K$ and $|n^2 - n_1^2 + n_2^2 - n_3^2| > 10K$, at least one of the conditions

$$|m_2 - n_2^2| > K \quad \text{or} \quad |m_3 - n_3^2| > K$$

holds.

Assume

$$|m_2 - n_2^2| \gtrsim |m_3 - n_3^2| > K. \tag{5.11}$$

We distinguish several cases.

Case 1: $N' + N_1' < (N_2')^3$. Consider the expression

$$\sum_{\substack{n\sim N',\, n_i\sim N_i',\, |n^2-n_1^2+n_2^2-n_3^2|\gtrsim K \\ m-m_1+m_2-m_3=0 \\ |m_2-n_2^2|\gtrsim|m_3-n_3^2|>K}} c(n,\bar{n}) A_{n,m,\bar{n},\bar{m}} \tag{5.12}$$

where we assume $|||v|||, |||v_1||| \le 1$ and, according to Proposition 3.2, that $\|u\|_{\frac{1}{2}-,\frac{3}{4}-} < O(1)$.

The restriction $|n^2-n_1^2+n_2^2-n_3^2| \gtrsim K$ in (5.12) may be removed arguing as follows: Let $0 \le \psi \le 1$ be a parameter, and replace $\hat{v}(n,m)$ by

$$e(n^2\psi)\hat{v}(n,m),$$

and $\hat{v}_1(n_1,m_1)$, $\hat{u}(n_2,m_2)$ and $\hat{u}(n_3,m_3)$ by

$$e(n_1^2\psi)\hat{v}_1(n_1,m_1), \qquad e(n_2^2\psi)\hat{u}(n_2,m_2) \qquad \text{and} \qquad e(n_3^2\psi)\hat{u}(n_3,m_3),$$

respectively. The restriction $|n^2-n_1^2+n_2^2-n_3^2| \lesssim K$ may then be achieved by taking a suitable average over ψ.

It thus suffices to bound the expression

$$\sum_{\substack{n\sim N',\, n_i\sim N_i' \\ m-m_1+m_2-m_3=0 \\ |m_2-n_2^2|\gtrsim|m_3-n_3^2|>K}} c(n,\bar{n})\, \overline{\hat{v}(n,m)}\, \hat{v}_1(n_1,m_1)\, \overline{\hat{u}_2(n_2,m_2)}\, \hat{u}_3(n_3,m_3) \tag{5.13}$$

with $|||v|||, |||v_1||| \le 1$, $\|u_2\|_{\frac{1}{2}-,\frac{3}{4}-} < O(1)$ and $\|u_3\|_{\frac{1}{2}-,\frac{3}{4}-} < O(1)$.

To bound this quantity, we re-express (5.13) as

$$\int_B \int_0^1 \overline{K^{-\epsilon}(P_{n\sim N'}v)}\, K^{-\epsilon}(P_{n_1\sim N_1'}v_1)\, \overline{K^{2\epsilon}(P_{n_2\sim N_2'}u_2)}\, P_{n_3\sim N_3'}u_3\, dt\, dx \tag{5.14}$$

with $\epsilon = 10^{-6}$, where to simplify notation we have suppressed an additional Fourier restriction on the u_2 and u_3 factors.

Since the norm $|||\cdot|||$ indeed controls the norm $\|\cdot\|_{0,\frac{1}{2}-}$, and the condition $K^\epsilon > (N')^{\frac{1}{3}10^{-9}}$ holds by assumption, we may apply Lemma 2.4(iii) to obtain

$$\|K^{-\epsilon}P_{n\sim N'}v\|_{L_x^3L_t^{4-}} < O(1) \tag{5.15}$$

and, similarly,

$$\|K^{-\epsilon}P_{n_1\sim N_1'}v_1\|_{L_x^3L_t^{4-}} < O(1). \tag{5.16}$$

On the other hand, using the Fourier restriction due to (5.11), we have

$$\|P_{n_2\sim N_2'}u_2\|_{\frac{1}{2}-,\frac{5}{8}} < K^{-1/16}\|u_2\|_{\frac{1}{2}-,\frac{3}{4}-},$$
$$\|P_{n_2\sim N_2'}u_2\|_{\frac{1}{2}+\epsilon,\frac{5}{8}} < K^{-1/16}(N_2')^\epsilon\|u_2\|_{\frac{1}{2}-,\frac{3}{4}-}.$$

Applying Lemma 2.4(v), it follows that

$$\|P_{n_2\sim N_2'}u_2\|_{L_x^pL_t^q} \lesssim K^{-1/16}(N_2')^\epsilon\|u_2\|_{\frac{1}{2}-,\frac{3}{4}-} \tag{5.17}$$

with
$$p = \frac{6}{1-\epsilon/2}, \qquad q = \frac{4}{1-\epsilon/2}. \tag{5.18}$$

In addition, Lemma 2.4(v) gives
$$\|P_{n_3 \sim N_3'} u_3\|_{L_x^{6-} L_t^{4-}} < O(1). \tag{5.19}$$

Combining (5.15)–(5.19), we obtain
$$|(5.14)| \lesssim (N_2')^{-10^{-3}/16+10^{-6}+2\cdot 10^{-9}} \lesssim (N_2')^{-10^{-5}}.$$

Summing in N' and N_1' now gives the bound
$$(N_2')^{-10^{-5}}[\log(N_2')]^2 \lesssim (N_2')^{-10^{-5}/2}$$
for the contribution of these terms to (4.12).

Case 2: $N' + N_1' > (N_2')^3$ and $n \neq n_1$. In this case, we have
$$N' + N_1' - (N_2')^2 < |n^2 - n_1^2 + n_2^2 - n_3^2| < |n_2^2 - m_2| + |n_3^2 - m_3| + 2K$$
and hence $|n_2^2 - m_2| > \frac{1}{3}(N' + N_1')$. This clearly allows us to repeat the analysis of Case 1 with $\frac{1}{3}(N' + N_1')$ in place of K, giving again the bound $(N_2')^{-10^{-5}}$.

Case 3: $N' = N_1' > (N_2')^3$, $n = n_1$. Proceeding as in Case 1 above, we obtain
$$\sum_{n\sim N',\, n_i \sim N_i',\, m-m_1+m_2-m_3=0} c(n, n, n_2, n_3) A_{n,m,(n,n_2,n_3),\bar{m}} \tag{5.20}$$
with $|m_2 - n_2^2| > K$. Rewrite (5.20) as
$$\int_B \int_0^1 \Big[\sum_{n\sim N'} \overline{\hat{v}(n)}\, \hat{v}_1(n) e_n^2\Big] \overline{P_{n_2\sim N_2'} u_2}\, P_{n_3 \sim N_3'} u_3 \, dt\, dx. \tag{5.21}$$

Now, observe that it follows from (5.17) and (5.18) that
$$\|P_{n_2\sim N_2'} u_2\|_{L_x^p L_t^q} < (N_2')^{-10^{-5}}, \tag{5.22}$$
while (5.19) gives
$$\|P_{n_3 \sim N_3'} u_3\|_{L_x^{6-} L_t^{4-}} < O(1). \tag{5.23}$$

On the other hand, since $e_n^2(x) \leq 1/|x|^2$, the first factor in the integrand of (5.21) is bounded by
$$\frac{1}{|x|^2}\Big(\sum_{n\sim N'} |\hat{v}(n)|^2\Big)^{1/2}\Big(\sum_{n\sim N'} |\hat{v}_1(n)|^2\Big)^{1/2} \tag{5.24}$$
where for any $q_1 < \infty$ one has
$$\Big\|\Big(\sum_n |\hat{v}(n)|^2\Big)^{1/2}\Big\|_{L_t^{q_1}} = \|v\|_{L_t^{q_1} L_x^2} \lesssim \|v\|_{0,1/2-} \lesssim |||v|||,$$
with the analogous bound for v_1.

It then follows that

$$\|(5.24)\|_{L_x^{\frac{3}{2}-} L_t^{q_1}} < O(1). \tag{5.25}$$

Combining (5.25) with (5.22) along with (5.23) and summing in N' now shows that the contribution of (5.20) is bounded by $(N_2')^{-10^{-5}}$, completing the bound in this case.

6. Multilinear estimates (II)

In this section, we estimate the remaining contributions, those of (4.14), (4.15) and (4.17). This will involve a different type of analysis than that used in the previous section; in particular we will make essential use of several further probabilistic considerations related to the solution map.

We begin with (4.14). Rewrite this quantity as a sum over N', N_1' of

$$\left|\int_0^1 \Big[\sum_{n\sim N',\, n_i\sim N_i',\, n_2\neq n_3} c_K(n,\bar{n})\,\overline{\hat{v}(n)}\,\hat{v}_1(n_1)\,\overline{\hat{u}(n_2)}\,\hat{u}(n_3)\Big]\,dt\right|. \tag{6.1}$$

Note that in the sum we necessarily have $n \neq n_1$, since otherwise

$$N_2' + N_3' \leq |n_2^2 - n_3^2| \leq 10K = 10(N_2')^{10^{-3}},$$

giving a contradiction.

Hence, it follows that

$$N' + N_1' \leq |n^2 - n_1^2| \leq K + 8(N_2')^2 < 9(N_2')^2.$$

We first examine the contribution from $n \neq n_3$. Denote N', N_i' by N, N_i for simplicity. Since $\|v\|_{L^2_{t,x}} \lesssim 1$, it follows from Cauchy–Schwarz that (6.1) is bounded by the L_t^2 norm of

$$\Big[\sum_n \Big|\sum_{n_1,n_2,n_3} \hat{v}_1(n_1)\,\overline{\hat{u}(n_2)}\,\hat{u}(n_3)c_K(n,n_1,n_2,n_3)\Big|^2\Big]^{1/2}$$
$$\leq \Big[\sum_{n_1,n_1'} |\hat{v}_1(n_1)|\,|\hat{v}_1(n_1')|\Big|\sum_{\substack{n,n_2,n_2'\\ n_3,n_3'}} B_{n,\bar{n},\bar{n}'}\Big|\Big]^{1/2}$$

where

$$B_{n,\bar{n},\bar{n}'} = \overline{\hat{u}(n_2)}\,\hat{u}(n_3)\hat{u}(n_2')\,\overline{\hat{u}(n_3')}\,c_K(n,n_1,n_2,n_3)c_K(n,n_1',n_2',n_3'),$$

and again by Cauchy–Schwarz,

$$\Big[\sum_{n_1} |\hat{v}_1(n_1)|^2\Big]^{1/2}\Big[\sum_{n_1\neq n_1'} |B_{n,\bar{n},\bar{n}'}|^2\Big]^{1/4} + \Big[\sum_{n_1} |\hat{v}_1(n_1)|^2\Big]^{1/2}\Big[\max_{n_1=n_1'} |B_{n,\bar{n},\bar{n}'}|\Big]^{1/2}$$
$$\leq \|v_1(t)\|_{L_x^2}\Big\{\Big[\sum_{n_1\neq n_1'} |B_{n,\bar{n},\bar{n}'}|^2\Big]^{1/4} + \max_{n_1=n_1'} |B_{n,\bar{n},\bar{n}'}|^{1/2}\Big\}. \tag{6.2}$$

Since $\|v_1\|_{L^q_t L^2_x} \lesssim |||v_1||| < O(1)$ for all $q < \infty$, it suffices to bound

$$\|\{\cdots\}\|_{L^4_t}, \tag{6.3}$$

where $\{\cdots\}$ is the quantity appearing in (6.2).

Note that (6.3) involves only the truncated solution u with initial data $\phi = \phi_\omega$, and we view u as a random variable of ω. For fixed t, the distribution of $u_\omega(t)$ is given by the Gibbs measure. Since the Gibbs measure is absolutely continuous with respect to the free measure with bounded density, we may replace $u_\omega(t)$ by a Gaussian Fourier series

$$\sum_n \frac{g_n(\omega)}{n} e_n$$

with $\{g_n\}$ as a sequence of IID normalized complex Gaussians. This fact is essential to our analysis in this section.

For sufficiently large q, we may estimate

$$\big(\mathbb{E}_\omega\big[\|\{\cdots\}\|^q_{L^4_t}\big]\big)^{1/q} \le \big\| \, \|\{\cdots\}\|_{L^q_\omega} \big\|_{L^4_t} \le \max_{0\le t\le 1} \|\{\cdots\}\|_{L^q_\omega}$$

and, fixing t, we accordingly write

$$\|\{\cdots\}\|_{L^q_\omega}$$
$$\le \Bigg\{ \sum_{n_1 \neq n_1'} \Bigg\| \sum_{n,n_2,n_3,n_2',n_3'} \frac{\overline{g_{n_2}}}{n_2} \frac{g_{n_3}}{n_3} \frac{g_{n_2'}}{n_2'} \frac{\overline{g_{n_3'}}}{n_3'} c_K(n,\bar{n}) c_K(n,\bar{n}') \Bigg\|^2_{L^{q/2}_\omega} \Bigg\}^{1/4} \tag{6.4}$$
$$+ \Bigg\| \max_{n_1} \Bigg| \sum_{n,n_2,n_3,n_2',n_3'} \frac{\overline{g_{n_2}}}{n_2} \frac{g_{n_3}}{n_3} \frac{g_{n_2'}}{n_2'} \frac{\overline{g_{n_3'}}}{n_3'} c_K(n,\bar{n}) c_K(n,n_1,n_2',n_3') \Bigg| \Bigg\|^{1/2}_{L^{q/2}_\omega}. \tag{6.5}$$

We first analyze (6.4) by considering several cases, recalling that $n_2 \neq n_3$ and $n_2' \neq n_3'$.

Case 1: $n_2 \neq n_2'$, $n_3 \neq n_3'$. In this case, we note that the bound

$$c_K(n,n_1,n_2,n_3) \lesssim N_3 \chi_{|n^2-n_1^2+n_2^2-n_3^2|<K}$$

gives the estimate

$$\mathbb{E}_\omega\Bigg[\Bigg| \sum_{n,n_2,n_3,n_2',n_3'} \frac{\overline{g_{n_2}}}{n_2} \frac{g_{n_3}}{n_3} \frac{g_{n_2'}}{n_2'} \frac{\overline{g_{n_3'}}}{n_3'} c_K(n,\bar{n}) c_K(n,\bar{n}') \Bigg|^2\Bigg]$$
$$\lesssim \frac{1}{N_2^4} \sum_{n_2,n_2',n_3,n_3'} \Big(\sum_n \chi_{|n^2-n_1^2+n_2^2-n_3^2|<K} \cdot \chi_{|n^2-(n_1')^2+(n_2')^2-(n_3')^2|<K} \Big)^2$$
$$\lesssim \frac{\sqrt{K}}{N_2^4} \sum_{n,n_2,n_2',n_3,n_3'} \chi_{|n^2-n_1^2+n_2^2-n_3^2|<K} \cdot \chi_{|n^2-(n_1')^2+(n_2')^2-(n_3')^2|<K}.$$

For the summation over n_1 and n_1' in (6.4), this gives the bound

$$\frac{\sqrt{K}}{N_2^4}\big|\big\{(n, n_1, n_1', n_2, n_2', n_3, n_3') : n_i, n_i' \sim N_i,\ n \neq n_1, n_1',\ \text{and}\\ |n^2 - n_1^2 + n_2^2 - n_3^2| < K,\ |n^2 - (n_1')^2 + (n_2')^2 - (n_3')^2| < K\big\}\big|. \tag{6.6}$$

Fix values of k, k' with $|k|, |k'| < K$, and evaluate the number of solutions of the equations

$$\begin{cases} n^2 - n_1^2 + n_2^2 - n_3^2 = k, \\ n^2 - (n_1')^2 + (n_2')^2 - (n_3')^2 = k', \end{cases} \tag{6.7}$$

in the variables $n, n_1, n_1', n_2, n_2', n_3$ and n_3'.

For this purpose, further fix n_2, n_2', n_3. Since $n \pm n_1$ are divisors of $k - n_2^2 + n_3^2 \neq 0$, this specifies n, n_1 up to N_2^{0+} possibilities. Next, if we write

$$(n_1')^2 + (n_3')^2 = n^2 + (n_2')^2 - k', \tag{6.8}$$

the usual bounds for the number of $\mathbb{Z}^2$-points on circles (and circle arcs) imply that (6.8) has at most N_3^{0+} solutions in (n_1', n_3').

Summarizing, this proves that

$$(6.6) < \sqrt{K}\, N_2^{-4} K^2 N_2^{2+} N_3 < N_2^{-1/2}. \tag{6.9}$$

Case 2: $n_2 = n_2'$, $n_3 \neq n_3'$. We obtain

$$\mathbb{E}_\omega\Big[\Big|\sum_{n,n_2,n_3,n_3'} \frac{|g_{n_2}|^2}{(n_2)^2}\frac{g_{n_3}}{n_3}\frac{\overline{g_{n_3'}}}{n_3'} c_K(n,\bar{n}) c_K(n, n_1', n_2, n_3')\Big|^2\Big]\\ \lesssim \frac{1}{N_2^4}\sum_{n_3,n_3'}\Big(\sum_{n,n_2}\chi_{|n^2-n_1^2+n_2^2-n_3^2|<K}\cdot\chi_{|n^2-(n_1')^2+n_2^2-(n_3')^2|<K}\Big)^2, \tag{6.10}$$

and since the number of (n, n_2)-terms in the inner sum is at most $K N_2^{0+}$ (for given n_1, n_1', n_3, n_3'), we obtain

$$(6.10) \ll N_2^{-4+} K \sum_{n,n_2,n_3,n_3'} \chi_{|n^2-n_1^2+n_2^2-n_3^2|<K}\cdot\chi_{|n^2-(n_1')^2+n_2^2-(n_3')^2|<K}. \tag{6.11}$$

Summing (6.11) over n_1 and n_1' then gives the bound

$$N_2^{-4+} K^3 N_2 N_3 < N_2^{-1}$$

for the contribution to the sum in (6.4).

Case 3: $n_2 \neq n_2'$, $n_3 = n_3'$. In place of (6.10), we get

$$\frac{1}{N_2^4}\sum_{n_2,n_2'}\Big(\sum_{n,n_3}\chi_{|n^2-n_1^2+n_2^2-n_3^2|<K}\cdot\chi_{|n^2-(n_1')^2+(n_2')^2-n_3^2|<K}\Big)^2. \tag{6.12}$$

Writing $n^2 - n_1^2 + n_2^2 - n_3^2 = k$, $|k| < K$, in the inner sum, it follows that $n \pm n_3$ divides $k + n_1^2 - n_2^2 \neq 0$, since $n \neq n_3$. Thus, there are at most $K N_2^{0+}$ terms in the inner sum and we obtain the bound

$$N_2^{-4+} K^3 N_2^2 N_3 < N_2^{-1/2}$$

for the contribution of (6.12) to the sum in (6.4).

Case 4: $n_2 = n_2'$, $n_3 = n_3'$. In this case, the inner sum in (6.4) becomes

$$N_2^2 N_3^{-2} \sum_{n,n_2,n_3} c_K(n, n_1, n_2, n_3) c_K(n, n_1', n_2, n_3). \tag{6.13}$$

It follows from the definition of c_K that the quantity (6.13) vanishes unless

$$|n_1^2 - (n_1')^2| < 2K;$$

note that this implies $N_1 = O(K)$, since $n_1 \neq n_1'$. Thus

$$\begin{aligned}(6.13) &< N_2^{-2} \big|\{(n, n_2, n_3) : n_2 \sim N_2,\ n_3 \sim N_3 \text{ and } |n^2 + n_2^2 - n_3^2| \lesssim K^2\}\big| \\ &\lesssim N_2^{-2} K^2 N_3 N_2^{0+} < N_2^{-3/4},\end{aligned}$$

and the corresponding contribution to (6.4) is bounded by $N_2^{-1/4}$.

The considerations in Cases 1–4 take care of the estimate of (6.4).

We next consider the estimate of (6.5). Note that the analogues of Cases 1, 2 and 3 in this setting are captured by the previous analysis, since we did not use the condition $n_1 \neq n_1'$.

To treat the estimate in the analogue of Case 4, we bound the contribution to (6.5) by

$$\begin{aligned}(\log N_1)\Big[\max_{n_1 \sim N_1} N_2^{-2} N_3^{-2} \sum_{n,n_2,n_3} c_K(n, n_1, n_2, n_3)^2\Big]^{1/2} \\ \lesssim (\log N_1) N_2^{-1} \Big[\max_{n_1 \sim N_1} \sum_{n,n_2,n_3} \chi_{|n^2 - n_1^2 + n_2^2 - n_3^2| < K}\Big]^{1/2} \\ \lesssim (\log N_1) N_2^{-1} (K N_3 N_2^{0+})^{1/2} \lesssim N_2^{-1/3}.\end{aligned}$$

This completes the treatment of Case 4 for the estimate of (6.5). Combining the estimates of (6.4) and (6.5) then completes the analysis of the contribution of the terms with $n \neq n_3$.

We now consider the terms for which $n_3 = n$. Note that since under this condition we have

$$|n_1^2 - n_2^2| \lesssim K = (N_2)^{10^{-3}},$$

it also follows that $n_1 = n_2$ in this setting. We then estimate the contribution to (6.1) by

$$\min(N, N_1) \int \Big[\sum_{n \sim N,\, n_1 \sim N_1} |\hat{v}(n)|\, |\hat{v}_1(n_1)|\, |\hat{u}(n)|\, |\hat{u}(n_1)|\Big] dt,$$

which, after using Cauchy–Schwarz, is in turn estimated by

$$\min(N, N_1) \int \|P_{n\sim N} v\|_{L^2_x} \|P_{n\sim N} u\|_{L^2_x} \|P_{n_1\sim N_1} v_1\|_{L^2_x} \|P_{n_1\sim N_1} u\|_{L^2_x}\, dt.$$

Using Hölder and summing over N and N_1, we obtain the bound

$$\sum_{N,N_1} \min\{N, N_1\} \|P_{n\sim N} v\|_{L^2_{t,x}} \|P_{n\sim N} u\|_{L^6_t L^2_x} \|P_{n_1\sim N_1} v_1\|_{L^6_t L^2_x} \|P_{n_1\sim N_1} u\|_{L^6_t L^2_x}. \tag{6.14}$$

Moreover, since $\|\cdot\|_{L^q_t L^2_x} \lesssim |||\cdot|||$ for all q, and, by Lemma 2.3,

$$\Big(\sum_N \|P_{n\sim N} v\|^2_{L^2_{x,t}}\Big)^{1/2} = \|v\|_{L^2_{x,t}} \lesssim \sqrt{T}\, |||v|||,$$

it follows from Cauchy–Schwarz that

$$(6.14) \lesssim \sqrt{T} \Big\{ \sum_N \|P_{n\sim N} u\|^2_{L^6_t L^2_x} \Big(\sum_{N_1} \min\{N, N_1\} |||P_{n_1\sim N_1} v_1|||\, \|P_{n_1\sim N_1} u\|_{L^6_t L^2_x} \Big)^2 \Big\}^{1/2}. \tag{6.15}$$

To control the norms of projections of u appearing in (6.15) we require the following probabilistic estimate.

Lemma 6.1. *Let $1 \le q < \infty$ be given. Then there exists $c > 0$ such that for every $\lambda \ge 1$,*

$$\mu_F^{(N)}\Big(\Big\{\phi : \max_N N^{1/2} \|P_{n\sim N} u_\phi\|_{L^q_t L^2_x} > \lambda\Big\}\Big) \le \exp(-c\lambda^c), \tag{6.16}$$

where the maximum is taken over dyadic integers N.

Assuming that Lemma 6.1 holds, we use it to estimate (6.15) by

$$\sqrt{T} \Big\{ \sum_N \Big(\sum_{N_1} \frac{\min\{N, N_1\}}{\sqrt{N N_1}} |||P_{n_1\sim N_1} v_1||| \Big)^2 \Big\}^{1/2} \lesssim \sqrt{T}\, |||v_1|||.$$

This leads to the bound $O(\sqrt{T})$ on (6.14).

This completes the analysis of the contribution of (4.14) except for the proof of Lemma 6.1, which we address presently.

Proof of Lemma 6.1. We begin by noting that it suffices to establish

$$\mu_G(A_\lambda) \le \exp(-c\lambda^c) \tag{6.17}$$

for $A_\lambda := \{\phi : \max_N N^{1/2} \|P_{n\sim N} u_\phi\|_{L^q_t([0,T_*); L^2_x(B))} > \lambda\}$. Indeed, if we argue as in the proof of Lemma 3.1, then (6.17) implies an inequality of the type (6.16).

It therefore remains to establish (6.17). Toward this end, fixing $q_1 > q$ and applying the Chebyshev inequality and the Plancherel identity followed by the Minkowski inequality, one obtains

$$\begin{aligned}\mu_G(A_\lambda) &\le \lambda^{-q_1} \Big\| \max_N N^{1/2} \|P_{n\sim N} u_\phi\|_{L_t^q L_x^2} \Big\|_{L^{q_1}(d\mu_G)}^{q_1} \\ &\lesssim \lambda^{-q_1} \Big\| \Big\| \max_N N^{1/2} \Big(\sum_{n\sim N} |\widehat{u}_\phi(n)|^2 \Big)^{1/2} \Big\|_{L^{q_1}(d\mu_G)} \Big\|_{L_t^q}^{q_1} \\ &\lesssim \lambda^{-q_1} \Big\| \max_N N^{1/2} \Big(\sum_{n\sim N} |\widehat{\phi}(n)|^2 \Big)^{1/2} \Big\|_{L^{q_1}(d\mu_G)}^{q_1} \\ &\lesssim \lambda^{-q_1} \Big\| \max_N N^{1/2} \Big(\sum_{n\sim N} \frac{|g_n(\omega)|^2}{n^2} \Big)^{1/2} \Big\|_{L_\omega^{q_1}}^{q_1}, \end{aligned} \tag{6.18}$$

where we have used the invariance of the Gibbs measure to obtain the third inequality.

We therefore have

$$\begin{aligned}(6.18) &\lesssim \lambda^{-q_1} \Big\{ 1 + \Big(\sum_N \Big\| \sum_{n\sim N} \frac{N}{n^2} (|g_n(\omega)|^2 - 1) \Big\|_{L_\omega^{q_1}} \Big)^{1/2} \Big\}^{q_1} \\ &\lesssim \lambda^{-q_1} \Big\{ 1 + \Big(\sum_N \frac{q_1}{\sqrt{N}} \Big)^{1/2} \Big\}^{q_1} \lesssim \Big(\frac{\sqrt{q_1}}{\lambda} \Big)^{q_1}, \end{aligned}$$

where we have used the estimate

$$\Big\| \sum_{n\sim N} \frac{N}{n^2} (|g_n(\omega)|^2 - 1) \Big\|_{L_\omega^{q_1}} \lesssim N q_1 \Big(\sum_{n\sim N} \frac{1}{n^4} \Big)^{1/2} \lesssim \frac{q_1}{\sqrt{N}},$$

which follows from (2.7). Optimizing the choice of q_1 (by essentially taking $q_1 = \lambda^2/2$; see, for instance, the proof of Lemma 3.1) now yields the desired claim.

This completes the proof of Lemma 6.1. □

It remains to bound the contributions of (4.15) and (4.17). We begin with (4.15), for which we argue by rewriting the inner sum in this expression as

$$\sum_{n\sim N, n_1\sim N_1, n_2\sim N_2} c_K(n, n_1, n_2, n_2) \int_0^1 \overline{\hat{v}(n)}\, \hat{v}_1(n_1) |\hat{u}(n_2)|^2 \, dt.$$

In view of Lemma 6.1, this is in turn bounded by

$$\begin{aligned} N &\int_0^1 \Big(\sum_{n\sim N} |\hat{v}(n)| \Big) \Big(\sum_{n_1\sim N_1} |\hat{v}_1(n_1)| \Big) \Big(\sum_{n_2\sim N_2} |\hat{u}(n_2)|^2 \Big) dt \\ &\le N^{3/2} N_1^{1/2} \int_0^1 \|P_{n\sim N} v\|_{L_x^2} \|P_{n_1\sim N_1} v_1\|_{L_x^2} \|P_{n_2\sim N_2} u\|_{L_x^2}^2 \, dt \\ &\lesssim (N_2)^{2\cdot 10^{-3}} \|P_{n\sim N} v\|_{L_t^4 L_x^2} \|P_{n_1\sim N_1} v_1\|_{L_t^4 L_x^2} \|P_{n_2\sim N_2} u\|_{L_t^4 L_x^2}^2 \lesssim (N_2)^{2\cdot 10^{-3} - 1}. \end{aligned}$$

We next consider (4.17). We use the Cauchy–Schwarz inequality to bound this expression by

$$\sum_N \|P_{n\sim N} v\|_{L^2_{t,x}} \|P_{n\sim N} v_1\|_{L^4_t L^2_x} \Big\| \max_{n\sim N} \Big| \sum_{n_2\sim N_2} c(n,n,n_2,n_2)|\hat{u}(n_2)|^2 - \sigma_{n,N_2}\Big| \Big\|_{L^4_t}$$
$$\lesssim T^{1/2} \sup_N \Big\| \max_{n\sim N} \Big| \sum_{n_2\sim N_2} c(n,n,n_2,n_2)|\hat{u}(n_2)|^2 - \sigma_{n,N_2}\Big| \Big\|_{L^4_t}. \tag{6.19}$$

Recall that

$$\sigma_n = \sigma_{n,N_2} = \mathbb{E}_\phi\Big[\sum_{n_2\sim N_2} c(n,n,n_2,n_2)|\hat{\phi}(n_2)|^2\Big].$$

The bound on the second factor in (6.19) again follows from probabilistic considerations. We have the following:

Lemma 6.2. *For $\lambda \gg 1$, we have, for some constant $c > 0$,*

$$\mu_F\Big[\phi : \Big\|\max_n \Big| \sum_{n_2\sim N_2} |\widehat{u_\phi(t)}(n_2)|^2 c(n,n,n_2,n_2) - \sigma_n \Big|\Big\|_{L^4_t} > \lambda\Big] \lesssim e^{-c\lambda^c N_2^c}. \tag{6.20}$$

Proof. It suffices again to prove (6.20) with μ_F replaced by the Gibbs measure μ_G. Proceeding as in Lemma 6.1, take $q_1 = q_1(\lambda)$ and write

$$\Big\| \Big\|\max_n \Big| \sum_{n_2\sim N_2} |\widehat{u_\phi(t)}(n_2)|^2 c(n,n,n_2,n_2) - \sigma_n \Big|\Big\|_{L^4_t}\Big\|_{L^{q_1}(\mu_G(d\phi))}$$
$$\le \Big\| \Big\|\max_n \Big| \sum_{n_2\sim N_2} |\widehat{u_\phi(t)}(n_2)|^2 c(n,n,n_2,n_2) - \sigma_n \Big|\Big\|_{L^{q_1}(\mu_G(d\phi))}\Big\|_{L^4_t}.$$

In view of the Gibbs measure invariance under the flow, the above is bounded by

$$\Big\|\max_n \Big| \sum_{n_2\sim N_2} |\hat{\phi}(n_2)|^2 c(n,n,n_2,n_2) - \sigma_n \Big|\Big\|_{L^{q_1}(\mu_G(d\phi))}$$
$$\le \Big\|\max_n \Big| \sum_{n_2\sim N_2} \frac{c(n,n,n_2,n_2)}{n_2^2}(|g_{n_2}(\omega)|^2 - 1)\Big|\Big\|_{L^{q_1}(d\omega)}. \tag{6.21}$$

Note that

$$c(n,n,n_2,n_2) = \int_0^1 \sin^2(\pi n r)\frac{\sin^2(\pi n_2 r)}{r^2}\,dr$$
$$= \frac{1}{2}\int_0^1 \frac{\sin^2(\pi n_2 r)}{r^2}\,dr - \frac{1}{2}\int_0^1 \cos(2\pi n r)\frac{\sin^2(\pi n_2 r)}{r^2}\,dr. \tag{6.22}$$

The second term in (6.22) is bounded by $O(N_2^4/N^2)$ for $n > N$, and therefore its contribution to (6.21) is at most

$$O(N_2^2/N^2)\Big\|\sum_{n_2\sim N_2}(|g_{n_2}(\omega)|^2 + 1)\Big\|_{L^{q_1}(d\omega)} < O(q_1 N_2^3/N^2) < O(q_1 N_2^{-1})$$

for $N > N_2^2$.

Hence, we may restrict n in (6.21) to the range $n \le N_2^2$ and get the bound

$$O(\log N_2) \max_{n<N_2^2} \Big\| \sum_{n_2\sim N_2} \frac{c(n,n,n_2,n_2)}{n_2^2}(|g_{n_2}(\omega)|^2-1)\Big\|_{L^{q_1}(d\omega)} < O(\log N_2) q_1 N_2^{-1/2}.$$

Taking $q_1 \sim \lambda N_2^{1/3}$ and applying Chebyshev's inequality yields (6.20). □

Having estimated the contributions of (4.14), (4.15) and (4.17), we complete our analysis of the nonlinear term (4.5).

7. Further probabilistic considerations

Returning to the nonlinear term (4.1), an inspection of the estimates in Sections 5 and 6—including Lemmas 6.1 and 6.2—as well as the non-probabilistic inequality (4.19) which determines the size of T, gives the following statement.

Proposition 7.1. *Let T be as in (4.2) and take $M_i \le N_i$ for $i = 2, 3$, $M = M_2 + M_3$. Moreover, let $u = u_\phi$ denote the solution of some truncated equation (1.1). Then*

$$\Big|\Big|\Big| \int_0^t e^{i(t-\tau)\Delta} P_N[P_{N_1}U^1 \overline{P_{M_2\le n\le N_2}u}\, P_{M_3\le n\le N_3}u](\tau)\, d\tau \Big|\Big|\Big| \le 10^{-3}|||U^1||| \tag{7.1}$$

for all U^1 for which the right side is finite, assuming that ϕ is restricted to the complement of an exceptional set of measure at most $\exp(-M^c)$ *(with $c > 0$ some constant).*

Note that for M small, we have the bound (cf. (5.1))

$$\begin{aligned}\sup_{|||v|||\le 1}\Big(\int_B\int_0^1 |P_N v|\,|P_{N_1}U^1|\,|P_{M_2}u|\,|P_{M_3}u|\,dx\,dt\Big) \\ \le \sup_{|||v|||\le 1}\|v\|_{L^2_{t,x}}\|U^1\|_{L^2_{t,x}}\|P_M u\|^2_{L^\infty_t L^\infty_x} \\ \lesssim T|||U^1|||M^3\|u\|^2_{L^\infty_t L^2_x} \le TM^3\|\phi\|^2_{L^2_x}|||U^1|||,\end{aligned} \tag{7.2}$$

where the second inequality follows from Lemma 2.3 and the third is a consequence of the conservation of the L^2_x norm under the flow.

Recalling also the discussion in Section 4 on how to treat (4.1) with solutions $u^{(2)}$ and $u^{(3)}$ obtained from different truncations, we obtain

Proposition 7.2. *Let T be given by (4.2). Then*

$$\Big|\Big|\Big| \int_0^t e^{i(t-\tau)\Delta} P_N[P_{N_1}U^1 \overline{P_{N_2}u^{(2)}}\, P_{N_3}u^{(3)}](\tau)\, d\tau \Big|\Big|\Big| \le 10^{-3}|||U^1||| \tag{7.3}$$

for all U^1 for which the right side is finite. Here $u^{(i)}|_{t=0} = P_{N^{(i)}}\phi$ satisfies the $N^{(i)}$-truncated equation ($i = 2, 3$) and we assume ϕ is outside an exceptional set of measure at most $O(\exp(-T^{-c}))$ (independent of U^1).

As we will see in the next section, Proposition 7.2 suffices to establish almost sure convergence of the sequence $\{u^N\}$ of truncated solutions of (1.1), with N running over the integers 2^j (or any sufficiently rapidly increasing sequence). However, the measure estimates do not quite suffice to deduce immediately the a.s. convergence of the full sequence, and an additional consideration is needed. The idea is basically the following: in view of Proposition 7.1, we obtain the desired measure estimates for the factors $P_{n\geq M_2}u^{(2)}$ and $P_{n\geq M_3}u^{(3)}$ provided that for instance M satisfies

$$M = M_2 + M_3 > (\log(N^{(2)} + N^{(3)}))^C$$

with C an appropriate constant.

It then remains to consider

$$\int_0^t e^{i(t-T)\Delta} P_N[P_{N_1}U^1\, \overline{P_M u^{(2)}}\, P_M u^{(3)}](\tau)\, d\tau. \tag{7.4}$$

Fix some truncation $M < N^{(0)} < N^{(2)}, N^{(3)}$ and let $u^{(0)} = P_{N^{(0)}}u^{(0)}$ be the corresponding solution of (1.1) with initial data $u^{(0)}|_{t=0} = P_{N^{(0)}}\phi$.

We compare (7.4) with

$$\int_0^t e^{i(t-\tau)\Delta} P_N[P_{N_1}U^1\, \overline{P_M u^{(0)}}\, P_M u^{(0)}](\tau)\, d\tau. \tag{7.5}$$

The difference between (7.4) and (7.5) may then be bounded by

$$\begin{aligned}
&\|P_{N_1}U^1\|_{L^4_t L^2_x}\big[\|P_M u^{(0)} - P_M u^{(2)}\|_{L^4_t L^\infty_x} + \|P_M u^{(0)} - P_M u^{(3)}\|_{L^4_t L^\infty_x}\big]\|P_M u^{(0)}\|_{L^4_t L^\infty_x}\\
&\qquad + \|P_{N_1}U^1\|_{L^4_t L^2_x}\|P_M u^{(0)} - P_M u^{(2)}\|_{L^4_t L^\infty_x}\|P_M u^{(0)} - P_M u^{(3)}\|_{L^4_t L^\infty_x}\\
&\qquad \lesssim |||P_{N_1}U^1|||M^3\|P_M\phi\|_{L^2_x}\big[|||P_M u^{(0)} - P_M u^{(2)}||| + |||P_M u^{(0)} - P_M u^{(3)}|||\big]\\
&\qquad + |||P_{N_1}U^1|||M^3|||P_M u^{(0)} - P_M u^{(2)}|||\ |||P_M u^{(0)} - P_M u^{(3)}|||.
\end{aligned} \tag{7.6}$$

The interest of this construction is that in order to bound (7.5), only exceptional sets related to $u^{(0)}_\phi$ have to be removed, while the prefactor M^3 in (7.6) is harmless in view of the smallness of $|||P_M u^{(0)} - P_M u^{(i)}|||$, $i = 2, 3$. This will be made more precise in the next section.

8. Proof of the theorem

In this section, we complete the proof of our main theorem. Toward this end, let $1 \ll N_0 < N$ be given. Our goal is to compare the solutions u^{N_0} and u^N of

$$\begin{cases} iu^{N_0}_t + \Delta u^{N_0} - P_{N_0}(u^{N_0}|u^{N_0}|^2) = 0,\\ u^{N_0}(0) = P_{N_0}\phi, \end{cases} \tag{8.1}$$

and

$$\begin{cases} iu_t^N + \Delta u^N - P_N(u^N|u^N|^2) = 0, \\ u^N(0) = P_N\phi, \end{cases} \tag{8.2}$$

on a time interval $I = [0, \eta]$ with $\eta > 0$ a sufficiently small constant.

Let $1 \ll M \le N_0$ and set

$$T = c/\log M \tag{8.3}$$

with $c > 0$ taken as in Proposition 7.2 with $N_i \le M$ for $i = 2, 3$.

The argument consists in dividing $[0, \eta]$ into time intervals of size T and applying Duhamel's formula on each of these subintervals in order to obtain recursive inequalities.

Taking $0 \le t \le T$, we have

$$u^N(t) = e^{it\Delta}(P_N\phi) + i\int_0^t e^{i(t-\tau)\Delta} P_N(u^N|u^N|^2)(\tau)\, d\tau$$

and

$$P_M(u^N - u^{N_0})(t) = i\int_0^t e^{i(t-\tau)\Delta}[P_M(u^N|u^N|^2) - P_M(u^{N_0}|u^{N_0}|^2)](\tau)\, d\tau. \tag{8.4}$$

We will estimate the $|||\cdot|||$ norm of this quantity.

We first replace u^N and u^{N_0} in (8.4) by $P_M u^N$ and $P_M u^{N_0}$, respectively. The $|||\cdot|||$ norm of the difference may then be estimated by

$$\begin{aligned} &\big[\|u^{N_0} - P_M u^{N_0}\|_{L_x^{3+}L_t^6} + \|u^N - P_M u^N\|_{L_x^{3+}L_t^6}\big] \\ &\qquad \cdot \big[\|u^{N_0}\|^2_{L_x^{6-}L_t^6} + \|u^N\|^2_{L_x^{6-}L_t^6}\big] < M^{-1/4}, \end{aligned} \tag{8.5}$$

where we have used the a priori bound given by Lemma 3.1; again, (8.5) holds outside an exceptional set of measure at most $O(e^{-M^c})$.

We then obtain

$$|||P_M(u^N - u^{N_0})||| < M^{-1/4}$$
$$+ \left|\left|\left|\int_0^t e^{i(t-\tau)\Delta}[P_M(u^N - u^{N_0})|P_M u^N|^2](\tau)\, d\tau\right|\right|\right| \tag{8.6}$$
$$+ \left|\left|\left|\int_0^t e^{i(t-\tau)\Delta}[P_M u^{N_0}\, \overline{P_M(u^N - u^{N_0})}\, P_M u^N](\tau)\, d\tau\right|\right|\right| \tag{8.7}$$
$$+ \left|\left|\left|\int_0^t e^{i(t-\tau)\Delta}[|P_M u^{N_0}|^2 P_M(u^N - u^{N_0})](\tau)\, d\tau\right|\right|\right|. \tag{8.8}$$

In view of Proposition 7.2, each of the terms (8.6)–(8.8) may be bounded by $10^{-3}|||P_M(u^N - u^{N_0})|||$, provided that ϕ is taken outside an exceptional set of measure at most $\exp(-T^{-c})$. Note that this set depends on N_0 and N. The preceding discussion then implies that

$$|||P_M(u^N - u^{N_0})|||_{0,1/2;T} < 2M^{-1/4} \tag{8.9}$$

and an application of Lemma 2.1 gives the existence of some $t_1 \in [T/2, T]$ such that

$$\|P_M(u^N - u^{N_0})(t_1)\|_{L^2_x} < 2C_1 M^{-1/4}. \tag{8.10}$$

Consider now the next time interval $[t_1, t_1 + T]$ and write, for each $t \in [0, T]$,

$$u^N(t_1 + t) = e^{it\Delta}(u^N(t_1)) + i \int_0^t e^{i(t-\tau)\Delta} P_N(u^N |u^N|^2)(t_1 + \tau)\, d\tau. \tag{8.11}$$

Repeating the above argument, we obtain

$$|||P_M(u^N - u^{N_0})(t_1 + \cdot)||| \le C_0 \|P_M(u^N - u^{N_0})(t_1)\|_{L^2_x} + M^{-1/4} + \tfrac{3}{10^3} |||P_M(u^N - u^{N_0})(t_1 + \cdot)|||$$

and thus

$$|||P_M(u^N - u^{N_0})(t_1 + \cdot)||| < 2(2C_0C_1 + 1)M^{-1/4} \tag{8.12}$$

for ϕ outside a set of measure at most $\exp(-T^{-c})$.

Note that the value of t_1 in (8.10) depends on ϕ, but this does not create problems with the estimates of the nonlinear terms.

Again by Lemma 2.1, (8.12) gives a $t_2 \in [t_1 + T/2, t_1 + T]$ with

$$\|P_M(u^N - u^{N_0})(t_2)\|_{L^2_x} < 2C_1(C_0C_1 + 1)M^{-1/4}. \tag{8.13}$$

Repeating this argument recursively, we obtain times $t_{j+1} \in [t_j + T/2, t_j + T]$ for each $j \ge 1$ with

$$\|P_M(u^N - u^{N_0})(t_{j+1})\|_{L^2_x} \le 2C_1[C_0 \|P_M(u^N - u_{N_0})(t_j)\|_2 + M^{-1/4}]. \tag{8.14}$$

Iterating the resulting bounds gives

$$\|P_M(u^N - u^{N_0})(t_j)\|_{L^2_x} < (4C_1C_0)^j M^{-1/4} < M^{-1/8}, \tag{8.15}$$

since $j \le T^{-1}\eta = c^{-1}\eta \log M$ by (8.3), and provided that η is chosen sufficiently small.

Since $|||P_M(u^N - u^{N_0})(t_j + \cdot)||| < M^{-1/8}$ for each j, it follows from Lemma 2.1 that

$$\frac{1}{T} \int_I \|P_M(u^N - u^{N_0})\|^2_{L^2_x}\, dt \lesssim M^{-1/4}$$

for each subinterval $I \subset [0, \eta]$ of size T. We therefore obtain

$$\|P_M(u^N - u^{N_0})\|_{L^2_{t<\eta} L^2_x} \lesssim M^{-1/8} \tag{8.16}$$

for ϕ outside an exceptional set of measure at most $T^{-1}\exp(-T^{-c}) < \exp(-T^{-c'})$ (depending on N_0 and N).

In view of the a priori bounds of Proposition 3.2 on the quantities $\|u^{N_0}\|_{X^{s,b}}$ and $\|u^N\|_{X^{s,b}}$ for $s < 1/2$ and $b < 3/4$) and interpolation arguments, the bound (8.16) also implies

$$\|u^N - u^{N_0}\|_{X^{s,b}[0,\eta]} < M^{-c(s,b)} \tag{8.17}$$

for $s < 1/2$ and $b < 3/4$.

To consider the interval $[\eta, 2\eta]$, we repeat the previous reasoning with M replaced by $M_1 = M^c$ and T by $T_1 = c/\log M_1$. This gives

$$\|u^N - u^{N_0}\|_{X^{s,b}[\eta,2\eta]} < M_1^{-c(s,b)} \tag{8.18}$$

and so on.

Starting from $M = N_0$, the above argument shows that for any given time interval $[0, T] = I$ with $T < \infty$, the estimate

$$\|u^N - u^{N_0}\|_{X^{s,b}(I)} < N_0^{-c(s,b,T)} \tag{8.19}$$

holds for $s < 1/2$, $b < 3/4$ and all ϕ outside a set of measure at most $e^{-(\log N_0)^c}$, depending on N and N_0. This statement clearly implies convergence of the sequence $\{u^N\}$, $N = 2^j$, in

$$\bigcap_{s<1/2,\, b<3/4} X^{s,b}(I)$$

almost surely in ϕ.

Since the series $\sum_{N\in\mathbb{Z}_+} e^{-(\log N)^c}$ diverges, this does not immediately imply the convergence of the full sequence. In order to achieve this improvement of the convergence properties, we use the procedure discussed at the end of Section 7.

Toward this end, fix $N_0 \gg 1$ and let N range between N_0 and $2N_0$. In (7.5), let $u^{(0)} = u^{N_0}$, and take M as the truncation

$$K = (\log N_0)^C$$

with C a sufficiently large constant.

On the other hand, in (8.6)–(8.8) above, $\log M \sim \log N_0$. Recalling (7.6), the estimation of (8.6)–(8.8) gives some additional terms:

$$\begin{aligned}
|||P_M(u^N - u^{N_0})||| &< M^{-1/4} + 10^{-3}|||P_M(u^N - u^{N_0})||| \\
&\quad + K^3\|P_K\phi\|_{L^2_x}|||P_K(u^N - u^{N_0})|||\ |||P_M(u^N - u^{N_0})||| \\
&\quad + K^3|||P_K(u^N - u^{N_0})|||^2|||P_M(u^N - u^{N_0})||| \\
&< M^{-1/4} + \big[10^{-3} + K^3\|\phi\|_{L^2_x}|||P_M(u^N - u^{N_0})||| \\
&\quad + K^3|||P_M(u^N - u^{N_0})|||^2\big]\cdot|||P_M(u^N - u^{N_0})|||.
\end{aligned} \tag{8.20}$$

The inequality (8.20) holds for ϕ outside an exceptional set which is the union of a set of measure at most $e^{-(\log N_0)^c}$ depending on N_0 and an exceptional set of measure at most $e^{-K^c} < N_0^{-2}$ depending on N.

Taking $\|\phi\|_{L^2_x} < K$ in (8.20) and recalling that $\log M \sim \log N_0$, we may again deduce (8.9), which is now valid for all $N_0 \le N \le 2N_0$ and ϕ outside an exceptional set of measure at most $e^{-(\log N_0)^c}$.

This completes the proof of the main theorem.

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