



Tonia Ricciardi · Takashi Suzuki

Duality and best constant for a Trudinger–Moser inequality involving probability measures

Received September 2, 2011 and in revised form August 30, 2012

Abstract. We consider the Trudinger–Moser type functional

$$J_\lambda(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 - \lambda \int_I \left(\log \int_\Omega e^{\alpha v} \right) \mathcal{P}(d\alpha),$$

where Ω is a two-dimensional Riemannian surface without boundary, $v \in H^1(\Omega)$, $\int_\Omega v = 0$, $I = [-1, 1]$, \mathcal{P} is a Borel probability measure on I and $\lambda > 0$. The functional J_λ arises in the statistical mechanics description of equilibrium turbulence, under the assumption that the intensity and the orientation of the vortices are determined by \mathcal{P} . We formulate a Toland nonconvex duality principle for J_λ and we compute the optimal value of λ for which J_λ is bounded from below.

Keywords. Trudinger–Moser inequality, mean field equation, logarithmic Hardy–Littlewood–Sobolev inequality

1. Introduction and main results

In the pioneering article [21] (see also [12]), Onsager initiated the equilibrium statistics for a two-dimensional system of point vortices and pointed out the possibility and importance of negative temperatures. Based on this observation, Joyce–Montgomery [16] and Pointin–Lundgren [22] derived the corresponding mean field equation of Liouville type in the high-energy limit. See also [7, 8] for recent developments of the kinetic theory.

Assuming that the distribution of circulations is determined by a general Borel probability measure \mathcal{P} defined on the interval $[-1, 1]$, such methods lead to the following “continuous system”, as derived in [23]:

$$-\Delta v = \lambda \int_I \alpha \left(\frac{e^{\alpha v}}{\int_\Omega e^{\alpha v}} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\alpha), \quad \int_\Omega v = 0. \quad (1)$$

T. Ricciardi: Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, Via Cintia, 80126 Napoli, Italy; e-mail: tonia.ricciardi@unina.it

T. Suzuki: Department of Systems Innovation, Osaka University Graduate School of Engineering Science, 1-3 Machikaneyama, Toyonaka, Osaka 560-8531, Japan; e-mail: suzuki@sigmath.es.osaka-u.ac.jp

Mathematics Subject Classification (2010): Primary 26D15; Secondary 35J20, 35J60

Here, Ω is an orientable compact Riemannian surface without boundary, v is the stream function, $\lambda > 0$ is a constant related to the inverse temperature, $I = [-1, 1]$, and $\mathcal{P} \in \mathcal{M}(I)$ is a Borel probability measure which determines the distribution of the intensity of the vortices. Equation (1) has a variational formulation. Indeed, (1) is the Euler–Lagrange equation for the Trudinger–Moser type functional

$$J_\lambda(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 - \lambda \int_I \left(\log \int_\Omega e^{\alpha v} \right) \mathcal{P}(d\alpha), \tag{2}$$

defined for $v \in H^1(\Omega)$, $\int_\Omega v = 0$.

The special case $\mathcal{P}(d\alpha) = \delta_1$, the Dirac mass concentrated at $\alpha = 1$, corresponds to assuming that all vortex points have the same intensity and the same orientation. In this case, equation (1) reduces to the well known and extensively studied mean field equation

$$-\Delta v = \lambda \left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|} \right), \quad \int_\Omega v = 0. \tag{3}$$

Further applications of (3) include the Nirenberg problem in differential geometry and the description of chemotaxis in biology. In the context of two-dimensional turbulence it was rigorously derived and analyzed in [3, 4, 17]. See [18, 26] for an overview of known results for (3) including uniqueness, topological degree, construction of bubbling solutions, etc. For $\mathcal{P}(d\alpha) = \delta_1$, the functional (2) reduces to

$$J_\lambda(v)|_{\mathcal{P}=\delta_1} = \frac{1}{2} \int_\Omega |\nabla v|^2 - \lambda \log \int_\Omega e^v.$$

In view of the sharp Trudinger–Moser inequality on compact Riemannian surfaces, as established by Fontana [13]:

$$\int_\Omega e^v \leq C \exp \left\{ \frac{1}{16\pi} \int_\Omega |\nabla v|^2 \right\}, \tag{4}$$

the functional $J_\lambda|_{\mathcal{P}=\delta_1}$ is bounded from below if and only if $\lambda \leq 8\pi$. An alternative proof of the sharp constant in the Trudinger–Moser inequality (4) was obtained in [3].

On the other hand, the assumption $\mathcal{P} = \tau\delta_1 + (1 - \tau)\delta_{-1}$ corresponds to assuming that all vortex points have one of two-sided circulations with given rate. In this case, equation (1) takes the form

$$-\Delta v = \lambda\tau \left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|} \right) - \lambda(1 - \tau) \left(\frac{e^{-v}}{\int_\Omega e^{-v}} - \frac{1}{|\Omega|} \right), \tag{5}$$

which received considerable attention in recent years, particularly in view of the possibility of two-sided blow-ups (see [11, 15, 20]). See also [10] for related very recent applications to chemotaxis involving two species. The corresponding Trudinger–Moser functional is given by

$$J_\lambda(v)|_{\mathcal{P}=\tau\delta_1+(1-\tau)\delta_{-1}} = \frac{1}{2} \int_\Omega |\nabla v|^2 - \lambda\tau \log \int_\Omega e^v - \lambda(1 - \tau) \log \int_\Omega e^{-v}.$$

In [20], by blow-up analysis, it is shown that the above functional is bounded below if and only if $\lambda \leq 8\pi/\max\{\tau, 1 - \tau\}$. The arguments in [20] were extended to equation (1) in [19], where in particular a blow-up analysis on the product space $I \times \Omega$ is carried out. As an application of that analysis, it is shown that the functional (2) is bounded below if

$$\lambda \leq \frac{8\pi}{\max\{\int_{[0,1]} \alpha^2 \mathcal{P}(d\alpha), \int_{[-1,0]} \alpha^2 \mathcal{P}(d\alpha)\}}.$$

However, a comparison with the sharp result of Shafrir and Wolansky [24, 25] for an equivalent free energy functional of logarithmic Hardy–Littlewood–Sobolev type with discrete \mathcal{P} indicates that the above estimate may be improved. Indeed, in view of such results, it is conjectured in [19] that J_λ is bounded below if and only if

$$\lambda \leq \bar{\lambda} = \inf \left\{ \frac{8\pi \mathcal{P}(K_\pm)}{[\int_{K_\pm} \alpha \mathcal{P}(d\alpha)]^2} : K_\pm \subset I_\pm \cap \text{supp } \mathcal{P} \right\} \tag{6}$$

where K is a Borel set and $I_+ = [0, 1]$ and $I_- = [-1, 0)$.

Our aim in this article is to confirm the constant appearing in (6). In this direction, our first objective is to identify a duality principle for J_λ . More precisely, we rigorously prove a Toland type nonconvex duality principle for the following Lagrangian from [23, 27]:

$$\mathcal{L}(\oplus \rho_\alpha, v) = \iint_{I \times \Omega} \rho_\alpha (\log \rho_\alpha - 1) \mathcal{P}(d\alpha) + \frac{1}{2} \int_\Omega |\nabla v|^2 - \iint_{I \times \Omega} \alpha \rho_\alpha v \mathcal{P}(d\alpha).$$

Here $(\oplus \rho_\alpha, v) \in \oplus \Gamma_\lambda \times E$, where

$$\begin{aligned} \Gamma_\lambda &= \left\{ \rho \in L \log L(\Omega) : \rho \geq 0, \int_\Omega \rho = \lambda \right\}, \\ \oplus \Gamma_\lambda &= \{ \oplus \rho_\alpha : \rho_\alpha \in \Gamma_\lambda \text{ for } \mathcal{P}\text{-a.e. } \alpha \in I \}, \\ E &= \left\{ v \in H^1(\Omega) : \int_\Omega v = 0 \right\}. \end{aligned}$$

We define the following free-energy functional of logarithmic Hardy–Littlewood–Sobolev type:

$$\Psi(\oplus \rho_\alpha) = \iint_{I \times \Omega} \rho_\alpha (\log \rho_\alpha - 1) - \frac{1}{2} \iint_{I^2} \alpha \beta \int_\Omega \rho_\alpha G \star \rho_\beta \mathcal{P}(d\alpha) \mathcal{P}(d\beta)$$

for $\oplus \rho_\alpha \in \oplus \Gamma_\lambda$. The following duality principle implies that minimization of J_λ is equivalent to minimization of Ψ :

Theorem 1 (Duality principle). *For any $\lambda > 0$ the following relation holds:*

$$\inf_{\oplus \Gamma_\lambda \times E} \mathcal{L} = \inf_E J_\lambda + \lambda(\log \lambda - 1) = \inf_{\oplus \Gamma_\lambda} \Psi. \tag{7}$$

Theorem 1 may be viewed as a Toland nonconvex duality principle for J_λ and Ψ (see [28]). It is stated without proof in [27]. In the special case where $\mathcal{P} = \delta_1$ the duality principle is known, and the corresponding free energy Ψ has been extensively studied by Beckner [1] in arbitrary dimensions. See also [5]. In Section 2 we prove Theorem 1 by a direct minimization argument which requires some care, since on one hand the space $L^1(I, \mathcal{P}; L \log L(\Omega))$ is not reflexive, and on the other hand the logarithmic nonlinearity is not differentiable at 0.

With the duality principle at hand, the study of J_λ is reduced to the study of functionals of the form

$$\begin{aligned} \tilde{\Psi}_{\mathcal{P}}(\oplus \rho_\alpha) &= \int_I \int_\Omega \rho_\alpha \log \rho_\alpha \mathcal{P}(d\alpha) \\ &\quad + \iint_{I^2} A(\alpha, \beta) \iint_{\Omega^2} \rho_\alpha(x) \log d(x, y) \rho_\beta(y) \mathcal{P}(d\alpha) \mathcal{P}(d\beta), \end{aligned} \quad (8)$$

where $A \in C(I^2)$ is symmetric and satisfies the sign condition

$$\alpha\beta A(\alpha, \beta) \geq 0 \quad \text{on } I^2, \quad (9)$$

and $\oplus \rho_\alpha \in \oplus \Gamma_{\lambda_\alpha}$, where

$$\oplus \Gamma_{\lambda_\alpha} = \{\oplus \rho_\alpha : \rho_\alpha \in \Gamma_{\lambda_\alpha} \text{ for } \mathcal{P}\text{-a.e. } \alpha \in I\}.$$

In the special case where \mathcal{P} is an atomic measure, namely

$$\mathcal{P} = \sum_{i=1}^n a_i \delta_{\alpha_i},$$

the free energy (8) takes the form

$$\tilde{\Psi}(\rho_1, \dots, \rho_n) = \sum_{i=1}^n a_i \int_\Omega \rho_i \log \rho_i + \sum_{i,j=1}^n a_{ij} \iint_{\Omega^2} \rho_i(x) \log d(x, y) \rho_j(y), \quad (10)$$

with $a_{ij} = A(\alpha_i, \alpha_j) a_i a_j$, $i, j = 1, \dots, n$. Discrete functionals of the form (10) have been extensively investigated by Shafir and Wolansky [24, 25], who derived an optimal condition for boundedness below. In Section 3 we extend the Shafir–Wolansky condition to the case of arbitrary \mathcal{P} in the subcritical case by an approximation argument. We first consider the case of nonnegative $A(\alpha, \beta)$. We prove:

Theorem 2. *Suppose $A \in C(I^2)$ is symmetric and such that $A \geq 0$ on I^2 , $\oplus \lambda_\alpha \in C(I)$ is such that $\inf_{\alpha \in I} \lambda_\alpha > 0$ and $\oplus \rho_\alpha \in C(I, L \log L(\Omega))$. If there exists $\varepsilon > 0$ such that*

$$(2 - \varepsilon) \int_J \lambda_\alpha \mathcal{P}(d\alpha) - \iint_{J^2} A(\alpha, \beta) \lambda_\alpha \lambda_\beta \mathcal{P}(d\alpha) \mathcal{P}(d\beta) \geq 0 \quad (11)$$

for all Borel subsets $J \subset I$, then $\tilde{\Psi}$ is bounded below on $\oplus \Gamma_{\lambda_\alpha}$.

In order to extend Theorem 2 to the case where A satisfies the sign condition (9) we exploit a useful remark concerning collaborating systems with two blocks from [25]. We derive the following result.

Theorem 3. *Suppose $A \in C(I^2)$ is symmetric and satisfies (9), $\oplus\lambda_\alpha \in C(I)$ is such that $\inf_{\alpha \in I} \lambda_\alpha > 0$ and $\oplus\rho_\alpha \in C(I, L \log L(\Omega))$. If there exists $\varepsilon > 0$ such that (11) holds for all Borel subsets $J \subset I_+$ and for all Borel subsets $J \subset I_-$, then $\tilde{\Psi}$ is bounded below on $\oplus\Gamma_{\lambda_\alpha}$.*

At this point, using Theorem 3, we are able to confirm the value (6) in the subcritical case. Namely, we prove the following result.

Theorem 4 (Lower bound for J_λ). *Let $\bar{\lambda}$ be the constant defined in (6). The functional J_λ is bounded below on E if $\lambda < \bar{\lambda}$.*

The constant $\bar{\lambda}$ is sharp in the sense that it cannot be replaced by any larger constant. We expect that the strict inequality is a technical limitation of our approximation argument.

Notation. Throughout this article all integrals over Ω are taken with respect to the Lebesgue measure induced by the underlying metric. All integrals over I are taken with respect to \mathcal{P} . We denote by $G(x, y)$ the Green’s function uniquely defined by $-\Delta_x G(x, y) = \delta_y, \int_\Omega G(x, y) dx = 0$. We denote by C a general constant whose value may vary from line to line.

2. A duality principle

The aim of this section is to prove by an *ad hoc* minimization method, which is perhaps of independent interest, the duality relation between J_λ, Ψ and \mathcal{L} , as stated in Theorem 1.

Our main result in this section is Proposition 1 below, which rigorously establishes for every fixed $v \in E$ the existence of a minimizer \oplus_{ρ_α} for the functional $\mathcal{L}(\cdot, v)$, satisfying the constraint

$$\rho_\alpha = \lambda e^{\alpha v} / \int_\Omega e^{\alpha v} \tag{12}$$

for \mathcal{P} -a.e. $\alpha \in I$.

We note that since the Banach space $L^1(I, \mathcal{P}; L \log L(\Omega))$, on which $\mathcal{L}(\cdot, v)$ is coercive, is not reflexive, the standard minimization argument for convex functionals cannot be applied. Therefore, in Lemma 2 we show that any minimizing sequence in $L^1(I, \mathcal{P}; L \log L(\Omega))$ may be replaced by a minimizing sequence in $L^\infty(I \times \Omega)$. Thus, for every fixed $v \in E$ we obtain \oplus_{ρ_α} such that

$$\mathcal{L}(\oplus_{\rho_\alpha}, v) = \min\{\mathcal{L}(\oplus\rho_\alpha, v) : \rho_\alpha \in \Gamma_\lambda\}.$$

At this point, it may seem natural to derive the asserted constraint by Gâteaux differentiation. However, such an argument is not applicable on the subsets of Ω where $\rho_\alpha = 0$. Therefore, in Lemma 3 we first show that the set $\{(\alpha, x) \in I \times \Omega : \rho_\alpha(x) = 0\}$ has zero measure. Finally, in Lemma 4 we prove that (12) holds for \mathcal{P} -a.e. $\alpha \in I$. The detailed proof is as follows.

We recall that \mathcal{L} is the Lagrangian defined by

$$\mathcal{L}(\oplus\rho_\alpha, v) = \iint_{I \times \Omega} \rho_\alpha (\log \rho_\alpha - 1) \mathcal{P}(d\alpha) + \frac{1}{2} \int_\Omega |\nabla v|^2 - \iint_{I \times \Omega} \alpha \rho_\alpha v \mathcal{P}(d\alpha).$$

The Lagrangian \mathcal{L} is naturally defined for $\oplus\rho_\alpha \geq 0$ belonging to the space

$$X = L^1(I, \mathcal{P}; L \log L(\Omega))$$

and for $v \in E = \{v \in H^1(\Omega) : \int_\Omega v = 0\}$. The space X may be equipped with the norm

$$\|\oplus\rho_\alpha\|_X = \int_I \int_\Omega |\rho_\alpha| \log\left(e + \frac{|\rho_\alpha|}{\|\rho_\alpha\|_1}\right) dx \mathcal{P}(d\alpha).$$

We recall that, in view of results from [14], the functional

$$\int_\Omega |\rho(x)| \log\left(e + \frac{|\rho(x)|}{\|\rho\|_1}\right) dx$$

defines an order-preserving norm on $L \log L(\Omega)$, which is equivalent to the standard Luxemburg and Orlicz norms.

Proposition 1. *For every fixed $v \in E \cap L^\infty(\Omega)$ the functional $\mathcal{L}(\cdot, v)$ admits a minimizer $\oplus_{\underline{\rho}_\alpha} \in \oplus_{\alpha \in I} \Gamma_\lambda$ satisfying (12) for \mathcal{P} -a.e. $\alpha \in I$. Moreover,*

$$\mathcal{L}(\oplus\rho_\alpha, v) \Big|_{\rho_\alpha = \underline{\rho}_\alpha = \lambda e^{\alpha v} / \int_\Omega e^{\alpha v}} = J_\lambda(v) + \lambda(\log \lambda - 1). \tag{13}$$

We divide the proof of Proposition 1 into several steps.

Lemma 1. *For every fixed $v \in H^1(\Omega)$ the functional $\mathcal{L}(\cdot, v)$ is coercive on $\oplus\Gamma_\lambda$.*

Proof. In view of Young’s inequality $st \leq s(\log s - 1) + e^t$ for all $s, t \geq 0$, we estimate

$$\begin{aligned} \left| \iint_{I \times \Omega} \alpha \rho_\alpha v \right| &\leq \iint_{I \times \Omega} [\varepsilon \rho_\alpha (\log(\varepsilon \rho_\alpha) - 1) + e^{|v|/\varepsilon}] \\ &= \varepsilon \iint_{I \times \Omega} \rho_\alpha (\log \rho_\alpha - 1) + \varepsilon \log \varepsilon \cdot \iint_{I \times \Omega} \rho_\alpha + \int_\Omega e^{|v|/\varepsilon}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathcal{L}(\oplus\rho_\alpha, v) &\geq (1 - \varepsilon) \iint_{I \times \Omega} \rho_\alpha (\log \rho_\alpha - 1) + \frac{1}{2} \int_\Omega |\nabla v|^2 \\ &\quad - \varepsilon \log \varepsilon \cdot \iint_{I \times \Omega} \rho_\alpha - \int_\Omega e^{|v|/\varepsilon}. \end{aligned}$$

In view of the elementary inequality $t \leq t(\log t - 1) + e$ for all $t \geq 0$, we derive

$$\begin{aligned} \mathcal{L}(\oplus\rho_\alpha, v) &\geq (1 - \varepsilon - \varepsilon \log \varepsilon) \iint_{I \times \Omega} \rho_\alpha (\log \rho_\alpha - 1) + \frac{1}{2} \int_\Omega |\nabla v|^2 \\ &\quad - e|\Omega|\varepsilon \log \varepsilon - \int_\Omega e^{|v|/\varepsilon}. \end{aligned} \tag{14}$$

Here, we have $C_\lambda > 0$ such that

$$\begin{aligned} \rho \log(e + \rho/\lambda) &\leq \rho \log \rho + C_\lambda, & 0 < \rho \leq 1, \\ \rho \log(e + \rho/\lambda) &\leq \rho(\log \rho + C_\lambda), & \rho \geq 1, \end{aligned}$$

which implies

$$\rho \log(e + \rho/\lambda) \leq \rho(\log \rho + C_\lambda) + C_\lambda, \quad \rho > 0.$$

Hence

$$\int_\Omega \rho(\log \rho - 1) \geq \int_\Omega \rho \log\left(e + \frac{\rho}{\|\rho\|_1}\right) - C'_\lambda$$

with $C'_\lambda > 0$ independent of $\rho \in \Gamma_\lambda$.

Finally, by choosing ε small in (14) and in view of the Trudinger–Moser inequality, we conclude that for every fixed $v \in H^1(\Omega)$ there exists a constant $C(v) > 0$ such that

$$\mathcal{L}(\oplus \rho_\alpha, v) \geq \frac{1}{2} \|\oplus \rho_\alpha\|_X - C(v)$$

for any $\oplus \rho_\alpha \in \oplus \Gamma_\lambda$. □

Since $\mathcal{L}(\cdot, v)$ is bounded below and coercive, there exists a minimizing sequence which is bounded in X . However, since X is not reflexive, we cannot directly deduce the existence of a minimizer. Therefore, we first show that there exists a minimizing sequence which is bounded in $L^\infty(I \times \Omega)$ (see Lemma 2 below). In what follows it will be convenient to denote

$$\Phi(t) = t(\log t - 1).$$

We consider, for $w \in L^\infty(\Omega)$ and $\rho \in \Gamma_\lambda$, the functional

$$F_w(\rho) = \int_\Omega \Phi(\rho) - \int_\Omega \rho w.$$

Lemma 2. *There exists $M > 0$, depending on $|\Omega|$, λ and $\|w\|_\infty$ only, such that for any $\rho \in \Gamma_\lambda$ there exists $\tilde{\rho} \in \Gamma_\lambda$ with $\tilde{\rho} \leq M$ such that*

$$F_w(\tilde{\rho}) < F_w(\rho).$$

Proof. For $M > 0$ large, we define

$$A := \{\rho \geq M\}, \quad E := \{\rho < 2\lambda/|\Omega|\}, \quad k^M := \int_A (\rho - M).$$

We note that $k^M \rightarrow 0$ as $M \rightarrow +\infty$ and $k^M \leq \lambda$. Moreover,

$$\lambda = \int_\Omega \rho = \int_E \rho + \int_{\Omega \setminus E} \rho \geq \frac{2\lambda}{|\Omega|} (|\Omega| - |E|) = 2\lambda - 2\lambda \frac{|E|}{|\Omega|}$$

and therefore

$$|E| \geq |\Omega|/2. \tag{15}$$

Hence, we may define

$$\tilde{\rho} = M\chi_A + \rho\chi_{(\Omega\setminus A)\setminus E} + (\rho + k^M/|E|)\chi_E.$$

We note that

$$\int_{\Omega} \tilde{\rho} = M|A| + \int_{(\Omega\setminus A)\setminus E} \rho + \int_E \rho + k^M = \int_{\Omega} \rho - \int_A (\rho - M) + k^M = \lambda,$$

so that $\tilde{\rho} \in \Gamma_{\lambda}$. In view of (15) we also have

$$k^M/|E| \leq 2\lambda/|\Omega|. \tag{16}$$

We have

$$F_w(\tilde{\rho}) - F_w(\rho) = \int_A [\Phi(M) - \Phi(\rho)] + \int_E \left[\Phi\left(\rho + \frac{k^M}{|E|}\right) - \Phi(\rho) \right] - \int_{\Omega} (\tilde{\rho} - \rho)w.$$

By the Mean Value Theorem, we estimate

$$\begin{aligned} \int_A [\Phi(\rho) - \Phi(M)] &= \int_A \Phi'(M + \theta(x)(\rho - M))(\rho - M) \\ &\geq \log M \cdot \int_A (\rho - M) = k^M \log M \end{aligned} \tag{17}$$

with $0 < \theta(x) < 1$. On the other hand, by the same argument and (16), we have

$$\begin{aligned} &\left| \int_{E \cap \{1/2 \leq \rho \leq 2\lambda/|\Omega|\}} \left[\Phi\left(\rho + \frac{k^M}{|E|}\right) - \Phi(\rho) \right] \right| \\ &= \left| \int_{E \cap \{1/2 \leq \rho \leq 2\lambda/|\Omega|\}} \left[\Phi'\left(\rho + \theta(x)\frac{k^M}{|E|}\right)\frac{k^M}{|E|} \right] \right| \leq k^M \max_{1/2 \leq s \leq 4\lambda/|\Omega|} |\Phi'(s)|. \end{aligned}$$

We claim that

$$\int_{E \cap \{0 \leq \rho \leq 1/2\}} \left[\Phi(\rho) - \Phi\left(\rho + \frac{k^M}{|E|}\right) \right] \geq -k^M \max_{1/2 \leq s \leq 1/2 + 2\lambda/|\Omega|} |\Phi'(s)|. \tag{18}$$

Indeed, since Φ is decreasing on $[0, 1]$, if $k^M/|E| \leq 1/2$, we readily have

$$\int_{E \cap \{0 \leq \rho \leq 1/2\}} \left[\Phi(\rho) - \Phi\left(\rho + \frac{k^M}{|E|}\right) \right] \geq 0.$$

If $k^M/|E| \geq 1/2$, then $0 \leq \rho + k^M/|E| - 1/2 \leq k^M/|E|$ and therefore

$$\begin{aligned} \int_{E \cap \{0 \leq \rho \leq 1/2\}} \left[\Phi(\rho) - \Phi\left(\rho + \frac{k^M}{|E|}\right) \right] &\geq \int_{E \cap \{0 \leq \rho \leq 1/2\}} \left[\Phi\left(\frac{1}{2}\right) - \Phi\left(\rho + \frac{k^M}{|E|}\right) \right] \\ &= \int_{E \cap \{0 \leq \rho \leq 1/2\}} \Phi'\left(\frac{1}{2} + \theta(x)\left(\rho + \frac{k^M}{|E|} - \frac{1}{2}\right)\right)\left(\rho + \frac{k^M}{|E|} - \frac{1}{2}\right) \\ &\geq -k^M \max_{1/2 \leq s \leq 1/2 + 2\lambda/|\Omega|} |\Phi'(s)|. \end{aligned}$$

Hence, (18) is established. It follows that

$$\int_E \left[\Phi \left(\rho + \frac{k^M}{|E|} \right) - \Phi(\rho) \right] \leq k^M \left(\max_{1/2 \leq s \leq 4\lambda/|\Omega|} |\Phi'(s)| + \max_{1/2 \leq s \leq 1/2+2\lambda/|\Omega|} |\Phi'(s)| \right). \tag{19}$$

Furthermore, we have

$$\int_{\Omega} (\tilde{\rho} - \rho)w = - \int_A (\rho - M)w + \int_E \frac{k^M}{|E|} w$$

and therefore

$$\left| \int_{\Omega} (\tilde{\rho} - \rho)w \right| \leq 2k^M \|w\|_{\infty}. \tag{20}$$

From (17), (19) and (20) we conclude that

$$\begin{aligned} F_w(\tilde{\rho}) - F_w(\rho) &\leq \left(-\log M + \max_{1/2 \leq s \leq 4\lambda/|\Omega|} |\Phi'(s)| + \max_{1/2 \leq s \leq 1/2+2\lambda/|\Omega|} |\Phi'(s)| + 2\|w\|_{\infty} \right) k^M. \end{aligned}$$

In particular, for large M we have $F_w(\tilde{\rho}) - F_w(\rho) < 0$, as asserted. □

In view of Lemma 2, we can prove the existence part of Proposition 1:

Proof of Proposition 1 (existence of a minimizer). Fix $v \in E \cap L^{\infty}(\Omega)$ and consider the functional

$$\mathcal{F}_v(\oplus \rho_{\alpha}) = \int_I F_{\alpha v}(\rho_{\alpha}) \mathcal{P}(d\alpha).$$

In view of Lemma 1, \mathcal{F} is convex and coercive on the closed convex subset $X_+ = \{\oplus \rho_{\alpha} \in X : \rho_{\alpha} \geq 0, \int_{\Omega} \rho_{\alpha} = \lambda \text{ a.e. } \Omega \text{ for } \mathcal{P}\text{-a.e. } \alpha \in I\}$. In view of Lemma 2 and $\|\alpha v\|_{\infty} \leq \|v\|_{\infty}$, there exists a minimizing sequence $\oplus \rho_{\alpha}^n$ which is bounded in $L^{\infty}(I \times \Omega)$. In particular, $\oplus \rho_{\alpha}^n$ is bounded in $L^p(I \times \Omega)$ for all $p > 1$. It follows that there exists $\oplus \rho_{\alpha} \in L^p(I \times \Omega)$ such that $\oplus \rho_{\alpha}^n \rightharpoonup \oplus \rho_{\alpha}$ weakly in $L^p(I \times \Omega)$. By weak $L^p(I \times \Omega)$ -lower semicontinuity of the convex functional \mathcal{F}_v , we conclude that

$$\mathcal{F}_v(\oplus \rho_{\alpha}) \leq \liminf_n \mathcal{F}_v(\oplus \rho_{\alpha}^n).$$

Since $\mathcal{L}(\cdot, v) = \mathcal{F}_v(\cdot) + \|\nabla v\|_2^2/2$, we conclude that $\oplus \rho_{\alpha}$ is the desired minimizer. □

Lemma 3. *Let $\oplus \rho_{\alpha} \in \oplus \Gamma_{\lambda}$ and for every $\alpha \in I$ let*

$$E_{\alpha} = \{x \in \Omega : \rho_{\alpha} = 0\}.$$

If there exists $J \subset I$ such that $\mathcal{P}(J) > 0$ and $|E_{\alpha}| > 0$ for all $\alpha \in J$, then there exists $\oplus \tilde{\rho}_{\alpha} \in \oplus \Gamma_{\lambda}$ such that $\tilde{\rho}_{\alpha} > 0$ for all $\alpha \in J$ and $\mathcal{L}(\oplus \tilde{\rho}_{\alpha}, v) < \mathcal{L}(\oplus \rho_{\alpha}, v)$.

Proof. Let $A_\alpha = \{\rho_\alpha \geq \lambda/(2|\Omega|)\}$. We claim that

$$|A_\alpha| > 0. \tag{21}$$

Indeed, if this is not the case we have $\rho_\alpha < \lambda/(2|\Omega|)$ a.e. in Ω and hence

$$\lambda = \int_\Omega \rho_\alpha \leq |\Omega| \cdot \frac{\lambda}{2|\Omega|} = \frac{\lambda}{2},$$

a contradiction. In view of our assumption on E_α and (21), for all $\alpha \in J$ we define

$$\varphi_\alpha = \frac{1}{|E_\alpha|} \chi_{E_\alpha} - \frac{1}{|A_\alpha|} \chi_{A_\alpha}$$

and we set $\varphi_\alpha \equiv 0$ for $\alpha \notin J$. We note that $\int_\Omega \varphi_\alpha = 0$. We have, recalling the definition of φ_α ,

$$\begin{aligned} & \mathcal{L}(\oplus(\rho_\alpha + t_\alpha \varphi_\alpha), v) - \mathcal{L}(\oplus \rho_\alpha, v) \\ &= \iint_{I \times \Omega} \rho_\alpha (\log(\rho_\alpha + t_\alpha \varphi_\alpha) - \log \rho_\alpha) \\ & \quad + \iint_{I \times \Omega} t_\alpha \varphi_\alpha (\log(\rho_\alpha + t_\alpha \varphi_\alpha) - 1) - \iint_{I \times \Omega} \alpha t_\alpha \varphi_\alpha v \\ &= \iint_{J \times A_\alpha} \rho_\alpha \left(\log\left(\rho_\alpha - \frac{t_\alpha}{|A_\alpha|}\right) - \log \rho_\alpha \right) + \iint_{J \times E_\alpha} \frac{t_\alpha}{|E_\alpha|} \left(\log \frac{t_\alpha}{|E_\alpha|} - 1 \right) \\ & \quad - \iint_{J \times A_\alpha} \frac{t_\alpha}{|A_\alpha|} \left(\log\left(\rho_\alpha - \frac{t_\alpha}{|A_\alpha|}\right) - 1 \right) - \iint_{J \times E_\alpha} \alpha \frac{t_\alpha}{|E_\alpha|} v \\ & \quad + \iint_{J \times A_\alpha} \alpha \frac{t_\alpha}{|A_\alpha|} v \\ &= \int_J t_\alpha \left\{ \left(\log \frac{t_\alpha}{|E_\alpha|} - 1 \right) + \int_{A_\alpha} \frac{\rho_\alpha}{t_\alpha} \log\left(1 - \frac{t_\alpha}{|A_\alpha| \rho_\alpha}\right) \right. \\ & \quad \left. - \frac{1}{|A_\alpha|} \int_{A_\alpha} \left(\log\left(\rho_\alpha - \frac{t_\alpha}{|A_\alpha|}\right) - 1 \right) - \frac{\alpha}{|E_\alpha|} \int_{E_\alpha} v + \frac{\alpha}{|A_\alpha|} \int_{A_\alpha} v \right\}. \end{aligned}$$

In view of the elementary expansion $x^{-1} \log(1 + x) = 1 + O(x)$ as $x \downarrow 0$, we have

$$\int_{A_\alpha} \frac{\rho_\alpha}{t_\alpha} \log\left(1 - \frac{t_\alpha}{|A_\alpha| \rho_\alpha}\right) = \frac{1}{|A_\alpha|} \int_{A_\alpha} \left[1 + O\left(\frac{t_\alpha}{|A_\alpha| \rho_\alpha}\right) \right].$$

We note that since $\rho_\alpha \geq \lambda/(2|\Omega|)$ on A_α , by choosing t_α suitably small we may assume that $t_\alpha/(|A_\alpha| \rho_\alpha)$ is small. We conclude that

$$b_{1,\alpha}(t_\alpha) = \int_{A_\alpha} \frac{\rho_\alpha}{t_\alpha} \log\left(1 - \frac{t_\alpha}{|A_\alpha| \rho_\alpha}\right)$$

is bounded with respect to t_α . On the other hand, we have

$$\begin{aligned} \frac{1}{|A_\alpha|} \int_{A_\alpha} \left(\log \left(\frac{\lambda}{2|\Omega|} - \frac{t_\alpha}{|A_\alpha|} \right) - 1 \right) &\leq \frac{1}{|A_\alpha|} \int_{A_\alpha} \left(\log \left(\rho_\alpha - \frac{t_\alpha}{|A_\alpha|} \right) - 1 \right) \\ &\leq \frac{1}{|A_\alpha|} \int_{A_\alpha} \log \rho_\alpha \leq \frac{1}{|A_\alpha|} \int_{A_\alpha} \rho_\alpha \leq \frac{\lambda}{|A_\alpha|}. \end{aligned}$$

We conclude that there exists $c_\alpha = c_\alpha(\lambda, v)$ such that

$$\left| \int_{A_\alpha} \frac{\rho_\alpha}{t_\alpha} \log \left(1 - \frac{t_\alpha}{|A_\alpha| \rho_\alpha} \right) - \frac{1}{|A_\alpha|} \int_{A_\alpha} \left(\log \left(\rho_\alpha - \frac{t_\alpha}{|A_\alpha|} \right) - 1 \right) - \frac{\alpha}{|E_\alpha|} \int_{E_\alpha} v + \frac{\alpha}{|A_\alpha|} \int_{A_\alpha} v \right| \leq c_\alpha.$$

For every $\alpha \in J$, choosing $0 < t_\alpha < |E_\alpha| \exp\{1 - c_\alpha\}$ we obtain $\log(t_\alpha/|E_\alpha|) - 1 + c_\alpha < 0$ and therefore

$$\mathcal{L}(\oplus(\rho_\alpha + t_\alpha \varphi_\alpha), v) - \mathcal{L}(\oplus \rho_\alpha, v) < 0,$$

as asserted. □

We denote by $\oplus \rho_\alpha$ the minimizer of $\mathcal{L}(\cdot, v)$ on Γ_λ . In view of Lemma 3 we may assume that $\rho_\alpha > 0$ a.e. in Ω for \mathcal{P} -a.e. $\alpha \in I$. However, this condition is not sufficient to differentiate \mathcal{L} at ρ_α . Therefore, we derive the constraint (12) by the following direct arguments.

Lemma 4. *The minimizer $\oplus \rho_\alpha$ satisfies the constraint (12).*

Proof. Throughout this proof, for the sake of simplicity, we omit the underlining of ρ_α . We define

$$F_{\alpha,n} = \{\rho_\alpha \geq 1/n\}.$$

Then, for all $\oplus \varphi_\alpha \in L^\infty(I \times \Omega)$ such that $\text{supp } \varphi_\alpha \subset F_{\alpha,n}$, $\int_\Omega \varphi_\alpha = 0$, $\|\varphi_\alpha\|_\infty \leq 1/n$, and for all $0 < t < 1$, we have $\rho_\alpha + t\varphi_\alpha \in \Gamma_\lambda$, and consequently, by minimality of $\oplus \rho_\alpha$,

$$\mathcal{L}(\oplus(\rho_\alpha + t\varphi_\alpha), v) \geq \mathcal{L}(\oplus \rho_\alpha, v).$$

We derive

$$\begin{aligned} 0 &\leq \frac{1}{t} [\mathcal{L}(\oplus(\rho_\alpha + t\varphi_\alpha), v) - \mathcal{L}(\oplus \rho_\alpha, v)] \\ &= \frac{1}{t} \left[\iint_{I \times F_{\alpha,n}} \{(\rho_\alpha + t\varphi_\alpha)[\log(\rho_\alpha + t\varphi_\alpha) - 1] - \rho_\alpha(\log \rho_\alpha - 1)\} - \iint_{I \times F_{\alpha,n}} t\alpha\varphi_\alpha v \right]. \end{aligned}$$

By the Mean Value Theorem applied to $f(s) = s(\log s - 1)$ we have

$$0 \leq \iint_{I \times F_{\alpha,n}} \log(\rho_\alpha + \theta(t, x, \alpha)t\varphi_\alpha)\varphi_\alpha - \iint_{I \times F_{\alpha,n}} \alpha\varphi_\alpha v$$

with $0 < \theta(t, x, \alpha) < 1$. Taking limits as $t \rightarrow 0$, we derive

$$0 \leq \iint_{I \times F_{\alpha,n}} [\log \rho_\alpha - \alpha v] \varphi_\alpha$$

for all $\varphi_\alpha \in L^\infty(\Omega)$ with $\text{supp } \varphi_\alpha \subset F_{\alpha,n}$, $\int_\Omega \varphi_\alpha = 0$, and $\|\varphi_\alpha\|_\infty \leq 1/n$. By considering $-\varphi_\alpha$, we conclude that

$$\iint_{I \times F_{\alpha,n}} [\log \rho_\alpha - \alpha v] \varphi_\alpha = 0$$

for all φ_α satisfying the above assumptions. It follows that for any $\psi_\alpha \in L^\infty(\Omega)$ such that $\text{supp } \psi_\alpha \subset F_{\alpha,n}$,

$$\iint_{I \times F_{\alpha,n}} [\log \rho_\alpha - \alpha v] \left(\psi_\alpha - \frac{1}{|\Omega|} \int_\Omega \psi_\alpha \right) = 0.$$

We derive that

$$\begin{aligned} \iint_{I \times F_{\alpha,n}} [\log \rho_\alpha - \alpha v] \psi_\alpha &= \frac{1}{|\Omega|} \int_\Omega \psi_\alpha \iint_{I \times F_{\alpha,n}} [\log \rho_\alpha - \alpha v] \\ &= \frac{1}{|\Omega|} \iint_{I \times F_{\alpha,n}} \left(\int_{F_{\alpha,n}} \log \rho_\alpha - \alpha \int_{F_{\alpha,n}} v \right) \psi_\alpha \end{aligned}$$

and, in turn, that

$$\iint_{I \times F_{\alpha,n}} \left[\log \rho_\alpha - \alpha v - \frac{1}{|\Omega|} \int_{F_{\alpha,n}} \log \rho_\alpha + \frac{\alpha}{|\Omega|} \int_{F_{\alpha,n}} v \right] \psi_\alpha = 0$$

for all $\psi_\alpha \in L^\infty(\Omega)$. It follows that

$$\log \rho_\alpha - \alpha v = \frac{1}{|\Omega|} \int_{F_{\alpha,n}} \log \rho_\alpha - \frac{\alpha}{|\Omega|} \int_{F_{\alpha,n}} v =: c_{\alpha,n}$$

on $F_{\alpha,n}$. Now, we observe that $F_{\alpha,n} \subset F_{\alpha,n+1}$, $\bigcup_n F_{\alpha,n} = \Omega \setminus E_\alpha$ and in view of Lemma 3 we have $|E_\alpha| = 0$. We conclude that $c_{\alpha,n}$ does not depend on n and

$$\log \rho_\alpha - \alpha v = \frac{1}{|\Omega|} \int_\Omega \log \rho_\alpha - \frac{\alpha}{|\Omega|} \int_\Omega v = \frac{1}{|\Omega|} \int_\Omega \log \rho_\alpha.$$

This, in turn, implies (12). □

Proof of Proposition 1 completed. We have already established (12). On the other hand, if $\rho_\alpha = \lambda e^{\alpha v} / \int_\Omega e^{\alpha v}$, we have

$$\alpha v = \log \rho_\alpha - \log \lambda + \log \int_\Omega e^{\alpha v},$$

and therefore, recalling that $\int_\Omega \rho_\alpha = \lambda$,

$$\iint_{I \times \Omega} \alpha \rho_\alpha v = \iint_{I \times \Omega} \rho_\alpha (\log \rho_\alpha - 1) + \lambda \int_I \log \int_\Omega e^{\alpha v} - \lambda (\log \lambda - 1).$$

Inserting the above identity into \mathcal{L} , we obtain (13). □

Proof of Theorem 1. By density, we have

$$\inf_{\oplus \Gamma_\lambda \times E} \mathcal{L} = \inf_{\oplus \Gamma_\lambda \times (E \cap L^\infty(\Omega))} \mathcal{L}.$$

The first equality in (7) is a direct consequence of (13). Hence, we are left to verify the second equality in (7), that is,

$$\inf_{\oplus \Gamma_\lambda \times E} \mathcal{L} = \inf_{\oplus \Gamma_\lambda} \Psi.$$

By density, we have

$$\inf_{\oplus \Gamma_\lambda \times E} \mathcal{L} = \inf_{(\oplus \Gamma_\lambda \cap L^\infty(I \times \Omega)) \times E} \mathcal{L}, \quad \inf_{\oplus \Gamma_\lambda} \Psi = \inf_{\oplus \Gamma_\lambda \cap L^\infty(I \times \Omega)} \Psi.$$

Let $\oplus \rho_\alpha \in L^\infty(I \times \Omega)$. In view of the estimate

$$\left| \iint_{I \times \Omega} \alpha \rho_\alpha v \right| \leq \frac{1}{2\varepsilon} \|\oplus \rho_\alpha\|_{L^\infty(I \times \Omega)}^2 + \frac{\varepsilon}{2} \int_\Omega |v|^2,$$

it is readily seen that $\mathcal{L}(\oplus \rho_\alpha, \cdot)$ is coercive on E for any λ . Moreover, the minimum is attained at $v = \underline{v}$, where

$$\underline{v} = G \star \int_I \alpha \rho_\alpha \mathcal{P}(d\alpha).$$

Integration by parts and Fubini’s Theorem yield

$$\int_\Omega |\nabla v|^2 = \int_\Omega \nabla G \star \int_I \alpha \rho_\alpha \mathcal{P}(d\alpha) \cdot \nabla G \star \int_I \beta \rho_\beta \mathcal{P}(d\beta) = \iint_{I^2} \alpha \beta \int_\Omega \rho_\alpha G \star \rho_\beta.$$

On the other hand, we also have

$$\int_I \int_\Omega \alpha \rho_\alpha \underline{v} = \iint_{I \times \Omega} \alpha \rho_\alpha G \star \int_I \beta \rho_\beta \mathcal{P}(d\beta) = \iint_{I^2} \alpha \beta \int_\Omega \rho_\alpha G \star \rho_\beta.$$

We conclude that $\mathcal{L}(\oplus \rho_\alpha, \underline{v}) = \Psi(\oplus \rho_\alpha)$ and the second inequality in (7) follows.

Now, Theorem 1 is completely established. □

3. An approximation argument

This section is devoted to obtaining a sufficient condition for boundedness from below of the functional

$$\begin{aligned} \Psi(\oplus \rho_\alpha) &= \iint_{I \times \Omega} \rho_\alpha \log \rho_\alpha \mathcal{P}(d\alpha) \\ &\quad + \iint_{I^2} A(\alpha, \beta) \iint_{\Omega^2} \rho_\alpha(x) \log d(x, y) \rho_\beta(y) \mathcal{P}(d\alpha) \mathcal{P}(d\beta), \end{aligned}$$

as stated in Theorem 2.

Our aim is to approximate \mathcal{P} by atomic measures. More precisely, for a fixed interval $J \subset I$ we define a sequence of atomic measures \mathcal{P}_n as follows. For every $n \in \mathbb{N}$, let $J_{i,n}$,

$i = 1, \dots, n$, be disjoint semi-closed intervals such that $\sup_{i=1, \dots, n} |J_{i,n}| \rightarrow 0$ as $n \rightarrow \infty$ and $J = \bigcup_{i=1}^n J_{i,n}$. Let $\alpha_{i,n} \in J_{i,n}$, $i = 1, \dots, n$. We define

$$\mathcal{P}_n = \sum_{i=1}^n \mathcal{P}(J_{i,n}) \delta_{\alpha_{i,n}}, \tag{22}$$

where δ_α denotes the Dirac measure concentrated at $\alpha \in J$. The following result holds.

Lemma 5. *Let \mathcal{P} be a Borel probability measure on $I = [-1, 1]$, let $J \subset I$ be a fixed interval and let \mathcal{P}_n be defined by (22). Then:*

(i) *For every $f \in C(\bar{J})$ we have*

$$\int_J f d\mathcal{P}_n \rightarrow \int_J f d\mathcal{P} \quad \text{as } n \rightarrow \infty.$$

(ii) *For every $g \in C(\bar{J}^2)$ we have*

$$\iint_{J^2} g(\alpha, \beta) \mathcal{P}_n(d\alpha) \mathcal{P}_n(d\beta) \rightarrow \iint_{J^2} g(\alpha, \beta) \mathcal{P}(d\alpha) \mathcal{P}(d\beta).$$

Proof. (i) We have

$$\int_J f d\mathcal{P}_n = \sum_{i=1}^n f(\alpha_{i,n}) \mathcal{P}(J_{i,n}) = \int_J \sum_{i=1}^n f(\alpha_{i,n}) \chi_{J_{i,n}} d\mathcal{P}.$$

Setting $f_n = \sum_{i=1}^n f(\alpha_{i,n}) \chi_{J_{i,n}}$ we have $\|f_n\|_\infty \leq \|f\|_\infty$. Moreover, for any $\alpha \in J$ and any n let $\alpha \in J_{j,n}$. By uniform continuity of f on \bar{J} we have

$$|f(\alpha) - f_n(\alpha)| \leq \text{osc}_{J_{j,n}} f \rightarrow 0$$

as $n \rightarrow \infty$. Here, $\text{osc}_J f = \sup_{x,y \in J} |f(x) - f(y)|$. Now, the claim follows by standard Riemann integration arguments.

The proof of (ii) is similar. □

Now, we strengthen the previous lemma as follows.

Lemma 6. *Let $J \subset I$ be a fixed interval and let $f \in C(\bar{J})$ and $g \in C(\bar{J}^2)$. Then for every $\eta > 0$ there exists an $n_\eta \in \mathbb{N}$ such that for all $n \geq n_\eta$ and for all $\omega_n \subset \{1, \dots, n\}$ we have*

$$\left| \sum_{i \in \omega_n} \mathcal{P}(J_{i,n}) f(\alpha_{i,n}) - \int_{J_{\omega_n}} f d\mathcal{P} \right| < \eta, \tag{23}$$

and

$$\left| \sum_{i,j \in \omega_n} \mathcal{P}(J_{i,n}) \mathcal{P}(J_{j,n}) g(\alpha_{i,n}, \alpha_{j,n}) - \iint_{J_{\omega_n}^2} g(\alpha, \beta) \mathcal{P}(d\alpha) \mathcal{P}(d\beta) \right| < \eta, \tag{24}$$

where $J_{\omega_n} = \bigcup_{i \in \omega_n} J_{i,n}$.

Proof. We first prove (23). We have

$$\sum_{i \in \omega_n} \mathcal{P}(J_{i,n}) f(\alpha_{i,n}) = \sum_{i \in \omega_n} f(\alpha_{i,n}) \int_{J_{i,n}} d\mathcal{P}$$

and therefore

$$\left| \sum_{i \in \omega_n} \mathcal{P}(J_{i,n}) f(\alpha_{i,n}) - \int_{J_{\omega_n}} f d\mathcal{P} \right| \leq \sum_{i \in \omega_n} \int_{J_{i,n}} |f - f(\alpha_{i,n})| d\mathcal{P}.$$

Choosing n_η sufficiently large so that $\sup_{i \in \{1, \dots, n\}} \text{osc}_{J_{i,n}} f < \eta$ for all $n \geq n_\eta$, and recalling that $\mathcal{P}(J) \leq 1$, we conclude that

$$\sum_{i \in \omega_n} \int_{J_{i,n}} |f - f(\alpha_{i,n})| d\mathcal{P} \leq \eta,$$

and the claim follows.

We now prove (24). As before, we have

$$\sum_{i,j \in \omega_n} g(\alpha_{i,n}, \alpha_{j,n}) \mathcal{P}(J_{i,n}) \mathcal{P}(J_{j,n}) = \iint_{J_{\omega_n}^2} g(\alpha, \beta) \mathcal{P}_n(d\alpha) \mathcal{P}_n(d\beta)$$

and therefore

$$\begin{aligned} & \left| \sum_{i,j \in \omega_n} \mathcal{P}(J_{i,n}) \mathcal{P}(J_{j,n}) g(\alpha_{i,n}, \alpha_{j,n}) - \iint_{J_{\omega_n}^2} g(\alpha, \beta) \mathcal{P}(d\alpha) \mathcal{P}(d\beta) \right| \\ & \leq \sum_{i,j \in \omega_n} \iint_{J_{i,n} \times J_{j,n}} |g(\alpha_{i,n}, \alpha_{j,n}) - g(\alpha, \beta)| \mathcal{P}(d\alpha) \mathcal{P}(d\beta). \end{aligned}$$

Choosing n_η sufficiently large so that $\sup_{i,j \in \{1, \dots, n\}} \text{osc}_{J_{i,n} \times J_{j,n}} g < \eta$ for all $n \geq n_\eta$, we conclude the proof. \square

Now we consider the functional

$$\Psi(\rho_1, \dots, \rho_n) = \sum_{i=1}^n b_i \int_{\Omega} \rho_i \log \rho_i + \sum_{i,j=1}^n a_{ij} \iint_{\Omega^2} \rho_i(x) \log d(x, y) \rho_j(y) \quad (25)$$

where $(\rho_1, \dots, \rho_n) \in \prod_{i=1}^n \Gamma_{\lambda_i}$, (a_{ij}) is a symmetric matrix with nonnegative entries and $b_i > 0$ for every $i = 1, \dots, n$. The next result, which is essentially contained in [24], provides the optimal condition for boundedness below of Ψ in the subcritical case. Since we are interested in keeping track of the explicit values of the constants with respect to n , we provide the proof.

Proposition 2. *Suppose that there exists $\varepsilon > 0$ such that*

$$(2 - \varepsilon) \sum_{i \in J} b_i \lambda_i - \sum_{i, j \in J} a_{ij} \lambda_i \lambda_j \geq 0 \tag{26}$$

for all $J \subset \{1, \dots, n\}$. Then Ψ is bounded below on $\prod_{i=1}^n \Gamma_{\lambda_i}$. Moreover, there exist $\tau_{ij} \geq 0$ such that $\tau_{ij} + \tau_{ji} = 1$ and

$$\Psi(\rho_1, \dots, \rho_n) \geq -\frac{|\Omega|}{e} \sum_{i=1}^n b_i - \sum_{i, j=1}^n \frac{a_{ij} \lambda_i \lambda_j}{2 - \varepsilon} [C_{\Omega}(\varepsilon) - \tau_{ij} \log \lambda_i - \tau_{ji} \log \lambda_j]$$

for all $\rho_1, \dots, \rho_n \in \prod_{i=1}^n \Gamma_{\lambda_i}$, where $C_{\Omega}(\varepsilon) > 0$ is a constant depending on ε and Ω only.

The constant $C_{\Omega}(\varepsilon)$ is defined in the following lemma.

Lemma 7 ([24]). *For every $\varepsilon > 0$ there exists $C_{\Omega}(\varepsilon) > 0$ such that for every $\tau \in [0, 1]$,*

$$(2 - \varepsilon) \iint_{\Omega^2} F(x) \log \frac{1}{d(x, y)} G(y) \leq (1 - \tau) \int_{\Omega} F \log F + \tau \int_{\Omega} G \log G + C_{\Omega}(\varepsilon)$$

for all $F, G \in \Gamma_{\lambda=1}$.

Proof of Proposition 2. We use Lemma 7 with $F = \rho_i/\lambda_i$, $G = \rho_j/\lambda_j$, $\tau = 1 - \tau_{ij}$ in order to estimate the cross-term in the functional Ψ defined in (25). We have

$$\begin{aligned} \iint_{\Omega^2} \frac{\rho_i(x)}{\lambda_i} \log \frac{1}{d(x, y)} \frac{\rho_j(y)}{\lambda_j} \\ \leq \frac{1}{2 - \varepsilon} \left[\tau_{ij} \int_{\Omega} \frac{\rho_i}{\lambda_i} \log \frac{\rho_i}{\lambda_i} + \tau_{ji} \int_{\Omega} \frac{\rho_j}{\lambda_j} \log \frac{\rho_j}{\lambda_j} \right] + \frac{C_{\Omega}(\varepsilon)}{2 - \varepsilon}. \end{aligned}$$

Multiplying by $\lambda_i \lambda_j$ and recalling that $\int_{\Omega} \rho_i = \lambda_i$, $\int_{\Omega} \rho_j = \lambda_j$, we derive

$$\begin{aligned} \iint_{\Omega^2} \rho_i(x) \log \frac{1}{d(x, y)} \rho_j(y) \\ \leq \frac{1}{2 - \varepsilon} \left[\tau_{ij} \lambda_j \int_{\Omega} \rho_i \log \frac{\rho_i}{\lambda_i} + \tau_{ji} \lambda_i \int_{\Omega} \rho_j \log \frac{\rho_j}{\lambda_j} \right] + \frac{\lambda_i \lambda_j C_{\Omega}(\varepsilon)}{2 - \varepsilon} \\ = \frac{1}{2 - \varepsilon} \left(\tau_{ij} \lambda_j \int_{\Omega} \rho_i \log \rho_i + \tau_{ji} \lambda_i \int_{\Omega} \rho_j \log \rho_j \right) + C_{ij}(\varepsilon), \end{aligned}$$

where

$$C_{ij}(\varepsilon) = \frac{\lambda_i \lambda_j}{2 - \varepsilon} [C_{\Omega}(\varepsilon) - \tau_{ij} \log \lambda_i - \tau_{ji} \log \lambda_j].$$

In view of symmetry of a_{ij} and relabelling, we have

$$\begin{aligned} & \sum_{i,j=1}^n a_{ij} \iint_{\Omega^2} \rho_i(x) \log \frac{1}{d(x,y)} \rho_j(y) \\ & \leq \frac{1}{2-\varepsilon} \sum_{i,j=1}^n a_{ij} \left(\tau_{ij} \lambda_j \int_{\Omega} \rho_i \log \rho_i + \tau_{ji} \lambda_i \int_{\Omega} \rho_j \log \rho_j \right) + \sum_{i,j=1}^n a_{ij} C_{ij}(\varepsilon) \\ & = \frac{2}{2-\varepsilon} \sum_{i,j=1}^n a_{ij} \tau_{ij} \lambda_j \int_{\Omega} \rho_i \log \rho_i + \sum_{i,j=1}^n a_{ij} C_{ij}(\varepsilon). \end{aligned}$$

Note that

$$\sum_{i,j=1}^n a_{ij} C_{ij}(\varepsilon) = \frac{1}{2-\varepsilon} \sum_{i,j=1}^n a_{ij} \lambda_i \lambda_j [C_{\Omega}(\varepsilon) - \tau_{ij} \log \lambda_i - \tau_{ji} \log \lambda_j].$$

Plugging this into Ψ we estimate

$$\Psi(\rho_1, \dots, \rho_n) \geq \sum_{i=1}^n \left(b_i - \frac{2}{2-\varepsilon} \sum_{j=1}^n a_{ij} \tau_{ij} \lambda_j \right) \int_{\Omega} \rho_i \log \rho_i - \sum_{i,j=1}^n a_{ij} C_{ij}(\varepsilon).$$

We note that $\int_{\Omega} \rho_i \log \rho_i \geq -|\Omega|/e$. Therefore, Ψ is bounded below on $\prod_{i=1}^n \Gamma_{\lambda_i}$ if there exist $(\tau_{ij})_{i,j}$ such that

$$b_i - \frac{2}{2-\varepsilon} \sum_{j=1}^n a_{ij} \tau_{ij} \lambda_j \geq 0 \quad \forall i = 1, \dots, n. \tag{27}$$

If (27) holds, then we have the lower bound

$$\Psi(\rho_1, \dots, \rho_n) \geq -\frac{|\Omega|}{e} \sum_{i=1}^n b_i - \sum_{i,j=1}^n a_{ij} C_{ij}(\varepsilon).$$

Finding $(\tau_{ij})_{i,j}$ such that condition (27) holds is equivalent to the following problem:

$$\begin{cases} \tau_{ij} \geq 0, \\ \tau_{ij} + \tau_{ji} = 1, \\ 2 \sum_{j=1}^n a_{ij} \tau_{ij} \lambda_i \lambda_j \leq (2-\varepsilon) b_i \lambda_i. \end{cases} \tag{28}$$

Similarly to [24], we introduce new unknowns $x_{ij} = 2a_{ij} \tau_{ij} \lambda_i \lambda_j$ and new coefficients $d_{ij} = a_{ij} \lambda_i \lambda_j, c_i = (2-\varepsilon) b_i \lambda_i$. Then a solution (τ_{ij}) to (28) exists if and only if there

exists a solution (x_{ij}) to the problem

$$\begin{cases} x_{ij} \geq 0, \\ x_{ij} + x_{ji} = 2d_{ij}, \\ \sum_{j=1}^n x_{ij} \leq c_i. \end{cases} \tag{29}$$

This is exactly Problem (P) in the Appendix of [24]. Hence, we may use the linear programming argument in [24, Proposition A.1] (see also [27]) in order to conclude that a solution to system (29) exists if and only if

$$\sum_{i,j \in J} d_{ij} \leq \sum_{i \in J} c_i \quad \forall J \subset \{1, \dots, n\}.$$

By definition of d_{ij} and c_i , the above is equivalent to (26). □

We shall also need the following estimate concerning functions in the logarithmic space $L \log L(\Omega)$. We note that it may be alternatively derived by duality from Lemma 7. The proof below uses standard rearrangement arguments.

Lemma 8. *There exists a constant $C > 0$ such that*

$$\left\| \int_{\Omega} |\log d(\cdot, y)| f(y) dy \right\|_{\infty} \leq C(1 + \|f\|_{L \log L(\Omega)}) \tag{30}$$

for all $f \in L \log L(\Omega)$.

Proof. For a fixed $x \in \Omega$ we take a normal coordinate system on a geodesic disc $B_r(x)$. Moreover, for every measurable function g defined on $E \subset \Omega$ we denote by g^* the decreasing rearrangement of g defined on $(0, |E|)$. See, e.g., [2]. We note that the decreasing rearrangement of the function $g(y) = |\log d(\cdot, y)|$ on $E = B_r(x)$ is given by $g^*(t) = \log \sqrt{\pi/t}$, $t \in (0, \pi r^2)$. Hence, by standard rearrangement properties,

$$\left| \int_{B_r(x)} |\log d(x, y)| f(y) dy \right| = \left| \int_{B_r(0)} |\log r| f \right| \leq \int_0^{\pi r^2} \log \sqrt{\frac{\pi}{t}} f^*(t) dt.$$

In view of Lemma 6.2, p. 243 in [2] we have

$$\int_0^{\pi r^2} \log \sqrt{\frac{\pi}{t}} f^*(t) dt \leq C(1 + \|f\|_{L \log L(\Omega)}).$$

On the other hand,

$$\int_{\Omega \setminus B_r(x)} |\log d(x, y) f(y)| dy \leq C \|f\|_1 \leq C \|f\|_{L \log L(\Omega)}.$$

Hence, (30) is established. □

Finally, we provide the proof of Theorem 2.

Proof of Theorem 2. For every $\oplus\rho_\alpha \in \oplus\Gamma_{\lambda_\alpha}$, using the discrete probability measure \mathcal{P}_n defined in (22) with $J = I$, we consider the discretized functional $\Psi_n = \Psi_{\mathcal{P}=\mathcal{P}_n}$ defined by

$$\begin{aligned} \Psi_n(\oplus\rho_\alpha) &= \sum_{i=1}^n \mathcal{P}(J_{i,n}) \int_{\Omega} \rho_{\alpha_{i,n}} \log \rho_{\alpha_{i,n}} \\ &\quad + \sum_{i,j=1}^n A(\alpha_{i,n}, \alpha_{j,n}) \mathcal{P}(J_{i,n}) \mathcal{P}(J_{j,n}) \iint_{\Omega^2} \rho_{\alpha_{i,n}}(x) \log d(x, y) \rho_{\alpha_{j,n}}(y). \end{aligned} \tag{31}$$

Note that $(\rho_{\alpha_{i,n}})_{i=1}^n \in \prod_{i=1}^n \Gamma(\lambda_{\alpha_{i,n}})$. Moreover, Ψ_n has the form (25) with $b_i = \mathcal{P}(J_{i,n})$, $a_{ij} = A(\alpha_{i,n}, \alpha_{j,n}) \mathcal{P}(J_{i,n}) \mathcal{P}(J_{j,n})$. In view of assumption (11) and Lemma 6 we take $\eta > 0$ with

$$\frac{\varepsilon}{2} \int_I \lambda_\alpha \mathcal{P}(d\alpha) > \eta.$$

We conclude that there exist $\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$ such that

$$(2 - \varepsilon/2) \sum_{i \in J} \mathcal{P}(J_{i,n}) \lambda_{\alpha_{i,n}} - \sum_{i,j \in J} A(\alpha_{i,n}, \alpha_{j,n}) \mathcal{P}(J_{i,n}) \mathcal{P}(J_{j,n}) \lambda_{\alpha_{i,n}} \lambda_{\alpha_{j,n}} \geq 0$$

for all $J \subset \{1, \dots, n\}$ with $n \geq n_\varepsilon$. Throughout the rest of this proof, we assume $n \geq n_\varepsilon$. In view of Proposition 2 and recalling that $\sum_{i=1}^n \mathcal{P}(J_{i,n}) \leq 1$, we conclude that $\Psi_n(\oplus\rho_\alpha) \geq -|\Omega|/e - C_n(\varepsilon)$, where

$$\begin{aligned} C_n(\varepsilon) &= \frac{1}{2 - \varepsilon} \sum_{i,j=1}^n A(\alpha_{i,n}, \alpha_{j,n}) \mathcal{P}(J_{i,n}) \mathcal{P}(J_{j,n}) \lambda_{\alpha_{i,n}} \lambda_{\alpha_{j,n}} \\ &\quad \times [C_\Omega(\varepsilon) - \tau_{ij} \log \lambda_{\alpha_{i,n}} - \tau_{ji} \log \lambda_{\alpha_{j,n}}] \end{aligned}$$

for some $0 \leq \tau_{ij} \leq 1$. We note that in view of Lemma 5 we have

$$\sum_{i,j=1}^n A(\alpha_{i,n}, \alpha_{j,n}) \mathcal{P}(J_{i,n}) \mathcal{P}(J_{j,n}) \rightarrow \iint_{I^2} A(\alpha, \beta) \mathcal{P}(d\alpha) \mathcal{P}(d\beta),$$

and therefore the constant $C_n(\varepsilon)$ is uniformly bounded with respect to n . Hence, we may take limits. We recall that by assumption $\oplus\rho_\alpha \in C(I, L \log L(\Omega))$ and therefore $f(\alpha) = \int_{\Omega} \rho_\alpha \log \rho_\alpha$ is continuous on I . Letting $n \rightarrow \infty$ and using Lemma 5(i) we conclude that

$$\sum_{i=1}^n \mathcal{P}(J_{i,n}) \int_{\Omega} \rho_{\alpha_{i,n}} \log \rho_{\alpha_{i,n}} = \int_I \int_{\Omega} \rho_\alpha \log \rho_\alpha \mathcal{P}_n(d\alpha) \rightarrow \int_I \int_{\Omega} \rho_\alpha \log \rho_\alpha \mathcal{P}(d\alpha).$$

On the other hand, in view of the assumption $\oplus\rho_\alpha \in C(I, L \log L)$ and Lemma 8 we derive that the function

$$g(\alpha, \beta) = \iint_{\Omega^2} \rho_\alpha(x) \log d(x, y) \rho_\beta(y) dx dy$$

is continuous on I^2 . Letting $n \rightarrow \infty$ and using Lemma 5(ii) we conclude that

$$\begin{aligned} \sum_{i,j \in J} A(\alpha_{i,n}, \alpha_{j,n}) \mathcal{P}(J_{i,n}) \mathcal{P}(J_{j,n}) \iint_{\Omega^2} \rho_{\alpha_{i,n}}(x) \log d(x, y) \rho_{\alpha_{j,n}}(y) \\ = \iint_{I^2} A(\alpha, \beta) \iint_{\Omega^2} \rho_{\alpha}(x) \log d(x, y) \rho_{\beta}(y) \mathcal{P}_n(d\alpha) \mathcal{P}_n(d\beta) \\ \rightarrow \iint_{I^2} A(\alpha, \beta) \iint_{\Omega^2} \rho_{\alpha}(x) \log d(x, y) \rho_{\beta}(y) \mathcal{P}(d\alpha) \mathcal{P}(d\beta). \end{aligned}$$

We conclude that $\tilde{\Psi}$ is bounded below on $\oplus \Gamma_{\lambda_{\alpha}}$ if condition (11) holds for all $J \subset I$, J a finite union of intervals. Since subintervals of I form a base for the standard topology on I , we conclude that it is equivalent to assume that (11) holds for all Borel subsets $J \subset I$.

The proof of Theorem 2 is now complete. □

The proof of Theorem 3 relies on the observation that a kernel $A(\alpha, \beta)$ satisfying the sign condition (9) may be viewed as a ‘‘collaborating system with two blocks’’, as defined in [25, Section 5.2].

Proof of Theorem 3. We rewrite

$$\begin{aligned} \tilde{\Psi}(\oplus \rho_{\alpha}) &= \tilde{\Psi}(\oplus_{\alpha \in I_+} \rho_{\alpha}) + \tilde{\Psi}(\oplus_{\alpha \in I_-} \rho_{\alpha}) \\ &\quad + 2 \iint_{I_+ \times I_-} A(\alpha, \beta) \iint_{\Omega^2} \rho_{\alpha}(x) \log d(x, y) \rho_{\beta}(y) \mathcal{P}(d\alpha) \mathcal{P}(d\beta). \end{aligned}$$

In view of the sign assumption (9), we have $A(\alpha, \beta) \leq 0$ on $I_+ \times I_-$. Therefore, we estimate

$$\begin{aligned} \iint_{I_+ \times I_-} A(\alpha, \beta) \iint_{\Omega^2} \rho_{\alpha}(x) \log d(x, y) \rho_{\beta}(y) \mathcal{P}(d\alpha) \mathcal{P}(d\beta) \\ = \iint_{I_+ \times I_-} |A(\alpha, \beta)| \iint_{\Omega^2} \rho_{\alpha}(x) \log \frac{1}{d(x, y)} \rho_{\beta}(y) \mathcal{P}(d\alpha) \mathcal{P}(d\beta) \\ \geq \log \frac{1}{\text{diam } \Omega} \iint_{I_+ \times I_-} |A(\alpha, \beta)| \lambda_{\alpha} \lambda_{\beta} \mathcal{P}(d\alpha) \mathcal{P}(d\beta). \end{aligned}$$

It follows that $\tilde{\Psi}(\oplus \rho_{\alpha})$ is bounded below on $\oplus \Gamma_{\lambda_{\alpha}}$ if and only if $\tilde{\Psi}(\oplus_{\alpha \in I_+} \rho_{\alpha})$ is bounded below on $\oplus_{\alpha \in I_+} \Gamma_{\lambda_{\alpha}}$ and $\tilde{\Psi}(\oplus_{\alpha \in I_-} \rho_{\alpha})$ is bounded below on $\oplus_{\alpha \in I_-} \Gamma_{\lambda_{\alpha}}$. In view of Theorem 2 with $J = I_+$ and $J = I_-$, we derive the conclusion of Theorem 3. □

Finally, we provide the proof of Theorem 4.

Proof of Theorem 4. By Theorem 1, it is equivalent to obtain boundedness below of the functional

$$\Psi(\oplus \rho_{\alpha}) = \iint_{I \times \Omega} \rho_{\alpha} (\log \rho_{\alpha} - 1) - \frac{1}{2} \iint_{I^2} \alpha \beta \int_{\Omega} \rho_{\alpha} G \star \rho_{\beta} \mathcal{P}(d\alpha) \mathcal{P}(d\beta)$$

on $\oplus\Gamma_\lambda$. In view of standard properties of the Green’s function on compact manifolds, we equivalently consider the functional

$$\begin{aligned}\tilde{\Psi}(\oplus\rho_\alpha) &= \iint_{I \times \Omega} \rho_\alpha \log \rho_\alpha \\ &+ \frac{1}{4\pi} \iint_{J^2} \alpha\beta \iint_{\Omega} \rho_\alpha(x) \log d(x, y) \rho_\beta(y) dx dy \mathcal{P}(d\alpha) \mathcal{P}(d\beta).\end{aligned}$$

Using Theorem 3 with $A(\alpha, \beta) = (4\pi)^{-1}\alpha\beta$ and $\lambda_\alpha = \lambda$ for all $\alpha \in I$, we derive that Ψ is bounded below if there exists $\varepsilon > 0$ such that

$$(2 - \varepsilon)\lambda\mathcal{P}(J) - \lambda^2 \iint_{J^2} \frac{\alpha\beta}{4\pi} \mathcal{P}(d\alpha) \mathcal{P}(d\beta) \geq 0$$

for all Borel sets $J \subset I_+$ and $J \subset I_-$. At this point, it suffices to observe that

$$\iint_{J^2} \alpha\beta \mathcal{P}(d\alpha) \mathcal{P}(d\beta) = \left(\int_J \alpha \mathcal{P}(d\alpha) \right)^2. \quad \square$$

Acknowledgments. T.R. thanks Osaka University for kind hospitality and for support. T.S. acknowledges support from the Graduate School in Computation and Information Sciences of Naples Federico II University, on invitation of the late Professor Luigi M. Ricciardi. The authors also thank the referees for careful reading and useful comments. This research is partially supported by the FP7 Marie Curie IRSES-2009-247486 “MaNEqui” International Research Staff Exchange Scheme.

References

- [1] Beckner, W.: Sharp Sobolev inequalities on the sphere and the Moser–Trudinger inequality. *Ann. of Math. (2)* **138**, 213–242 (1993) [Zbl 0826.58042](#) [MR 1230930](#)
- [2] Bennett, C., Sharpley, R.: *Interpolation of Operators*. Academic Press, London (1988) [Zbl 0647.46057](#) [MR 0928802](#)
- [3] Caglioti, E., Lions, P.-L., Marchioro, C., Pulvirenti, M.: A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description. *Comm. Math. Phys.* **143**, 501–525 (1992) [Zbl 0745.76001](#) [MR 1145596](#)
- [4] Caglioti, E., Lions, P.-L., Marchioro, C., Pulvirenti, M.: A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description, Part II. *Comm. Math. Phys.* **174**, 229–260 (1995) [Zbl 0840.76002](#) [MR 1362165](#)
- [5] Carlen, E., Loss, M.: Competing symmetries, the logarithmic HLS inequality and Onofri’s inequality on S^n . *Geom. Funct. Anal.* **2**, 90–104 (1992) [Zbl 0754.47041](#) [MR 1143664](#)
- [6] Chanillo, S., Kiessling, M.: Rotational symmetry of solutions of some nonlinear problems in statistical mechanics and in geometry. *Comm. Math. Phys.* **160**, 217–238 (1994) [Zbl 0821.35044](#) [MR 1262195](#)
- [7] Chavanis, P. H.: Kinetic theory of point vortices in two dimensions: analytical results and numerical simulations. *Eur. Phys. J. B* **59**, 217–247 (2007) [Zbl 1195.85007](#) [MR 2350178](#)
- [8] Chavanis, P. H.: Statistical mechanics of two-dimensional vortices and stellar systems. In: *Dynamics and Thermodynamics of Systems with Long-Range Interactions*, Lecture Notes in Phys. 602, Springer, 208–289 (2002) [MR 2008183](#)

- [9] Chipot, M., Shafrir, I., Wolansky, G.: On the solutions of Liouville systems. *J. Differential Equations* **140**, 59–105 (1997) [Zbl 0902.35039](#) [MR 1473855](#)
- [10] Espejo, E. E., Stevens, A., Suzuki, T.: Simultaneous blowup and mass separation during collapse in an interacting system of chemotactic species. *Differential Integral Equations* **25**, 251–288 (2012) [Zbl 1265.35135](#) [MR 2917884](#)
- [11] Esposito, P., Wei, J.: Non-simple blow-up solutions for the Neumann two-dimensional sinh-Gordon equation. *Calc. Var. Partial Differential Equations* **34**, 341–375 (2009) [Zbl 1169.35049](#) [MR 2471141](#)
- [12] Eyink, G. L., Sreenivasan, K. R.: Onsager and the theory of hydrodynamic turbulence. *Rev. Modern Phys.* **78**, 87–135 (2006) [Zbl 1205.01032](#) [MR 2214822](#)
- [13] Fontana, L.: Sharp borderline Sobolev inequalities on compact Riemannian manifolds. *Comment. Math. Helv.* **68**, 415–375 (1993) [Zbl 0844.58082](#) [MR 1236762](#)
- [14] Iwaniec, T., Verde, A.: On the operator $\mathcal{L}(f) = f \log |f|$. *J. Funct. Anal.* **169**, 391–420 (1999) [Zbl 0961.47020](#) [MR 1730563](#)
- [15] Jost, J., Wang, G., Ye, D., Zhou, C.: The blow up analysis of solutions of the elliptic sinh-Gordon equation. *Calc. Var. Partial Differential Equations* **31**, 263–276 (2008) [Zbl 1137.35061](#) [MR 2353873](#)
- [16] Joyce, G., Montgomery, D.: Negative temperature states for two-dimensional guiding-centre plasma. *J. Plasma Phys.* **10**, 107–121 (1973)
- [17] Kiessling, M. K. H.: Statistical mechanics of classical particles with logarithmic interaction. *Comm. Pure Appl. Math.* **46**, 27–56 (1993) [Zbl 0811.76002](#) [MR 1193342](#)
- [18] Lin, C. S.: An expository survey on the recent development of mean field equations. *Discrete Contin. Dynam. Systems* **19**, 387–410 (2007) [Zbl 1159.35025](#) [MR 2335753](#)
- [19] Ohtsuka, H., Ricciardi, T., Suzuki, T.: Blow-up analysis for an elliptic equation describing stationary vortex flows with variable intensities in 2D-turbulence. *J. Differential Equations* **249**, 1436–1465 (2010) [Zbl 1201.35163](#) [MR 2677801](#)
- [20] Ohtsuka, H., Suzuki, T.: Mean field equation for the equilibrium turbulence and a related functional inequality. *Adv. Differential Equations* **11**, 281–304 (2006) [Zbl 1109.26014](#) [MR 2221484](#)
- [21] Onsager, L.: Statistical hydrodynamics. *Suppl. Nuovo Cimento* **6**, 279–287 (1949) [MR 0036116](#)
- [22] Pointin, Y. B., Lundgren, T. S.: Statistical mechanics of two-dimensional vortices in a bounded container. *Phys. Fluids* **19**, 1459–1470 (1976) [Zbl 0339.76013](#)
- [23] Sawada, K., Suzuki, T.: Derivation of the equilibrium mean field equations of point vortex and vortex filament system. *Theoret. Appl. Mech. Japan* **56**, 285–290 (2008)
- [24] Shafrir, I., Wolansky, G.: The logarithmic HLS inequality for systems on compact manifolds. *J. Funct. Anal.* **227**, 200–226 (2005) [Zbl 1082.58034](#) [MR 2165092](#)
- [25] Shafrir, I., Wolansky, G.: Moser–Trudinger and logarithmic HLS inequalities for systems. *J. Eur. Math. Soc.* **7**, 413–448 (2005) [Zbl 1129.26304](#) [MR 2159222](#)
- [26] Suzuki, T.: *Free Energy and Self-Interacting Particles*. Birkhäuser, Boston (2005) [Zbl 1082.35006](#) [MR 2135150](#)
- [27] Suzuki, T.: *Mean Field Theories and Dual Variation*. Atlantis Press, Amsterdam–Paris (2008) [Zbl 1247.35001](#) [MR 2510744](#)
- [28] Toland, J.: A duality principle for non-convex optimization and the calculus of variations. *Arch. Ration. Mech. Anal.* **71**, 41–61 (1979) [Zbl 0411.49012](#) [MR 0522706](#)