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On homotopy types of limits of semi-algebraic sets and additive complexity of polynomials

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Abstract. We prove that the number of homotopy types of limits of one-parameter semi-algebraic families of closed bounded semi-algebraic sets is bounded singly exponentially in the additive complexity of any quantifier-free first-order formula defining the family. As an important consequence, we derive that the number of homotopy types of semi-algebraic subsets of $\mathbb{R}^k$ defined by a quantifier-free first-order formula Φ, where the sum of the additive complexities of the polynomials appearing in Φ is at most a, is bounded by $2^{(k+a)^{O(1)}}$. This proves a conjecture made in [5].

Keywords. Semi-algebraic sets, additive complexity, homotopy types, Hausdorff limit

1. Introduction and statement of the main results

If S is a semi-algebraic subset of $\mathbb{R}^k$ defined by a quantifier-free first-order formula Φ, then various topological invariants of S (such as the Betti numbers) can be bounded in terms of the "format" of the formula Φ (to be defined precisely below). The first results in this direction were proved by Oleĭnik and Petrovskiĭ [19, 20] (also independently by Thom [22] and Milnor [18]) who proved singly exponential bounds on the Betti numbers of real algebraic varieties in $\mathbb{R}^k$ defined by polynomials of degree bounded by d. These results were extended to more general semi-algebraic sets in [1, 12, 13, 14]. As a consequence of more general finiteness results for Pfaffian functions, Khovanskiĭ [17] proved singly exponential bounds on the number of connected components of real algebraic varieties defined by polynomials with a fixed number of monomials. We refer the reader to [3] for a more detailed survey of results on bounding the Betti numbers of semi-algebraic sets.

A second type of quantitative results on the topology of semi-algebraic sets, more directly relevant to the current paper, seeks to obtain tight bounds on the number of different topological types of semi-algebraic sets definable by first-order formulas of bounded format. If the format of a first-order formula is specified by the number and degrees of the polynomials appearing in it (this is often called the "dense format" in the literature), then

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it follows from Hardt's well-known triviality theorem for semi-algebraic sets (see [16, 9]) that this number is finite. However, the quantitative bounds on the number of topological types that follow from the proof of Hardt's theorem are doubly exponential (unlike the singly exponential bounds on the Betti numbers). For some other notions of format, the finiteness of topological types while being true is not an immediate consequence of Hardt's theorem (see below), and tight quantitative bounds on the number of topological types are lacking.

If instead of homeomorphism types, one considers the weaker notion of *homotopy types*, then singly exponential bounds have been obtained on the number of *homotopy types* of semi-algebraic sets defined by different classes of formulas of bounded format [5, 2].

The main motivation behind this paper is to obtain a singly exponential bound on the number of homotopy types of semi-algebraic sets defined by polynomials of bounded "additive complexity" (defined below), answering a question posed in [5].

One notion of format that will play an important role in this paper is that of "additive complexity". Roughly speaking, the additive complexity of a polynomial (see Definition 1.8 below for a precise definition) is bounded from above by the number of additions in any straight line program (allowing divisions) that computes the values of the polynomial at generic points of $\mathbb{R}^n$. This measure of complexity strictly generalizes the more familiar measure of complexity of real polynomials based on counting the number of monomials in the support (as in Khovanskiĭ's theory of "fewnomials" [17]), and is thus of considerable interest in quantitative real algebraic geometry. Additive complexity of real univariate polynomials was first considered in the context of computational complexity theory by Borodin and Cook [10], who proved an effective bound on the number of real zeros of a univariate polynomial in terms of its additive complexity. This result was further improved upon by Grigoriev [15] and Risler [21], who applied Khovanskiĭ's results on fewnomials [17]. A surprising fact conjectured in [7], and proved by Coste [11] and van den Dries [24], is that the number of topological types of real algebraic varieties defined by polynomials of bounded additive complexity is finite.

1.1. Bounding the number of homotopy types of semi-algebraic sets

The problem of obtaining tight quantitative bounds on the number of topological types of semi-algebraic sets defined by formulas of bounded format was considered in [5]. Several results (with different notions of format of formulas) were proved in [5], each giving an explicit singly exponential (in the number of variables and size of the format) bound on the number of homotopy types of semi-algebraic subsets of $\mathbb{R}^k$ defined by formulas having format of bounded size. However, the case of additive complexity was left open in [5], and only a strictly weaker result was proved in the case of *division-free* additive complexity.[1] In order to state this result precisely, we need a few definitions.

[1] Note that what we call "additive complexity" is called "rational additive complexity" in [5], and what we call "division-free additive complexity" is called "additive complexity" there.

Definition 1.1. The *division-free additive complexity* of a polynomial is a non-negative integer, and we say that a polynomial $P \in \mathbb{R}[X_1, \ldots, X_k]$ has *division-free additive complexity at most* a, $a \geq 0$, if there are $Q_1, \ldots, Q_a \in \mathbb{R}[X_1, \ldots, X_k]$ such that

(i)
$$Q_1 = u_1 X_1^{\alpha_{11}} \cdots X_k^{\alpha_{1k}} + v_1 X_1^{\beta_{11}} \cdots X_k^{\beta_{1k}},$$

where $u_1, v_1 \in \mathbb{R}$, and $\alpha_{11}, \ldots, \alpha_{1k}, \beta_{11}, \ldots, \beta_{1k} \in \mathbb{N}$;

(ii)
$$Q_j = u_j X_1^{\alpha_{j1}} \cdots X_k^{\alpha_{jk}} \prod_{1 \leq i \leq j-1} Q_i^{\gamma_{ji}} + v_j X_1^{\beta_{j1}} \cdots X_k^{\beta_{jk}} \prod_{1 \leq i \leq j-1} Q_i^{\delta_{ji}}$$

where $1 < j \leq a$, $u_j, v_j \in \mathbb{R}$, and $\alpha_{j1}, \ldots, \alpha_{jk}, \beta_{j1}, \ldots, \beta_{jk}, \gamma_{ji}, \delta_{ji} \in \mathbb{N}$ for $1 \leq i < j$; and

(iii)
$$P = c X_1^{\zeta_1} \cdots X_k^{\zeta_k} \prod_{1 \leq j \leq a} Q_j^{\eta_j}$$

where $c \in \mathbb{R}$, and $\zeta_1, \ldots, \zeta_k, \eta_1, \ldots, \eta_a \in \mathbb{N}$.

In this case, we say that the above sequence of equations is a *division-free additive representation* of P of length a.

In other words, P has division-free additive complexity at most a if there exists a straight line program which, starting with variables $X_1, \ldots, X_m$ and constants in $\mathbb{R}$ and applying additions and multiplications, computes P and which uses at most a additions (there is no bound on the number of multiplications). Note that the additive complexity of a polynomial (cf. Definition 1.8 below) is clearly at most its division-free additive complexity, but can be much smaller (see Example 1.9 below).

Example 1.2. The polynomial $P := (X + 1)^d \in \mathbb{R}[X]$ with $0 < d \in \mathbb{Z}$ has $d + 1$ monomials when expanded, but division-free additive complexity at most 1.

Notation 1.3. We denote by $\mathcal{A}_{k,a}^{\text{div-free}}$ the family of ordered (finite) lists $\mathcal{P} = (P_1, \ldots, P_s)$ of polynomials $P_i \in \mathbb{R}[X_1, \ldots, X_k]$ with the division-free additive complexity of every P_i not exceeding a_i, and with $a = \sum_{1 \leq i \leq s} a_i$. Note that $\mathcal{A}_{k,a}^{\text{div-free}}$ is allowed to contain lists of different sizes.

Suppose that ϕ is a Boolean formula with atoms $\{p_i, q_i, r_i \mid 1 \leq i \leq s\}$. For an ordered list $\mathcal{P} = (P_1, \ldots, P_s)$ of polynomials $P_i \in \mathbb{R}[X_1, \ldots, X_k]$, we denote by $\phi_{\mathcal{P}}$ the formula obtained from ϕ by replacing for each i, $1 \leq i \leq s$, the atom p_i (respectively, q_i and r_i) by $P_i = 0$ (respectively, by $P_i > 0$ and by $P_i < 0$).

Definition 1.4. We say that two ordered lists $\mathcal{P} = (P_1, \ldots, P_s)$, $\mathcal{Q} = (Q_1, \ldots, Q_s)$ of polynomials $P_i, Q_i \in \mathbb{R}[X_1, \ldots, X_k]$ have the same *homotopy type* if for any Boolean formula ϕ, the semi-algebraic sets defined by $\phi_{\mathcal{P}}$ and $\phi_{\mathcal{Q}}$ are homotopy equivalent. Clearly, in order to be homotopy equivalent two lists should have equal size.

Example 1.5. Consider the lists $\mathcal{P} = (X_1, X_2^2, X_1^2 + X_2^2 + 1)$ and $\mathcal{Q} = (X_1^3, X_2^4, 1)$. It is easy to see that they have the same homotopy type, since in this case for each Boolean formula ϕ with nine atoms, the semi-algebraic sets defined by $\phi_{\mathcal{P}}$ and $\phi_{\mathcal{Q}}$ are identical. A slightly less trivial example is provided by $\mathcal{P} = (X_2 - X_1^2, X_2)$ and $\mathcal{Q} = (X_2, X_2 + X_1^2)$. In this case, for each Boolean formula ϕ with six atoms, the semi-algebraic sets defined by $\phi_{\mathcal{P}}$ and $\phi_{\mathcal{Q}}$ are not identical but homeomorphic. Finally, the singleton sequences $\mathcal{P} = (X_2 X_1 (X_1 - 1))$ and $\mathcal{Q} = (X_2 (X_1^2 - X_2^4))$ are homotopy equivalent. In this case the semi-algebraic sets defined by $\phi_{\mathcal{P}}$ and $\phi_{\mathcal{Q}}$ are homotopy equivalent, but not necessarily homeomorphic. For instance, the algebraic set defined by $X_2 X_1 (X_1 - 1) = 0$ is homotopy equivalent to the algebraic set defined by $X_2 (X_1^2 - X_2^4) = 0$, but they are not homeomorphic to each other.

The following theorem is proved in [5].

Theorem 1.6 ([5]). *The number of homotopy types of ordered lists in $\mathcal{A}_{k,a}^{\text{div-free}}$ does not exceed*

$$2^{O(k+a)^8}. \tag{1.1}$$

In particular, if ϕ is any Boolean formula with $3s$ atoms, then the number of homotopy types of semi-algebraic sets defined by $\phi_{\mathcal{P}}$, where $\mathcal{P} = (P_1, \ldots, P_s) \in \mathcal{A}_{k,a}^{\text{div-free}}$, does not exceed (1.1).

Remark 1.7. The bound in (1.1) is stated in a slightly different form than in the original paper, to take into account the fact that by our definition the division-free additive complexity of a polynomial (for example, that of a monomial) is allowed to be 0. This is not an important issue (see Remark 1.14 below).

The additive complexity of a polynomial is defined as follows [10, 15, 21, 7].

Definition 1.8. A polynomial $P \in \mathbb{R}[X_1, \ldots, X_k]$ is said to have *additive complexity* at most a if there are *rational functions* $Q_1, \ldots, Q_a \in \mathbb{R}(X_1, \ldots, X_k)$ satisfying equations (i), (ii), and (iii) in Definition 1.1 with $\mathbb{N}$ replaced by $\mathbb{Z}$. In this case we say that the corresponding sequence of equations is an *additive representation* of P of length a.

Example 1.9. The polynomial $X^d + \cdots + X + 1 = (X^{d+1} - 1)/(X - 1) \in \mathbb{R}[X]$ with $0 < d \in \mathbb{Z}$ has additive complexity (but not division-free additive complexity) at most 2 (independent of d).

Notation 1.10. We denote by $\mathcal{A}_{k,a}$ the family of ordered (finite) lists $\mathcal{P} = (P_1, \ldots, P_s)$ of polynomials $P_i \in \mathbb{R}[X_1, \ldots, X_k]$ with the additive complexity of every P_i not exceeding a_i, and with $a = \sum_{1 \le i \le s} a_i$.

It was conjectured in [5] that Theorem 1.6 could be strengthened by replacing $\mathcal{A}_{k,a}^{\text{div-free}}$ by $\mathcal{A}_{k,a}$. In this paper we prove this conjecture. More formally, we prove

Theorem 1.11. *The number of homotopy types of ordered lists in $\mathcal{A}_{k,a}$ does not exceed $2^{(k+a)^{O(1)}}$.*

1.2. Additive complexity and limits of semi-algebraic sets

The proof of Theorem 1.6 in [5] proceeds by reducing the problem to the case of bounding the number of homotopy types of semi-algebraic sets defined by polynomials having a bounded number of monomials. The reduction, which was already used by Grigoriev [15] and Risler [21], is as follows. Let $\mathcal{P} \in \mathcal{A}_{k,a}^{\text{div-free}}$ be an ordered list. For each polynomial $P_i \in \mathcal{P}$, $1 \leq i \leq s$, consider the sequence of polynomials $Q_{i1}, \ldots, Q_{ia_i}$ as in Definition 1.1, so that

$$P_i := c_i X_1^{\zeta_{i1}} \cdots X_k^{\zeta_{ik}} \prod_{1 \leq j \leq a_i} Q_{ij}^{\eta_{ij}}.$$

Introduce a_i new variables $Y_{i1}, \ldots, Y_{ia_i}$. Fix a semi-algebraic set $S \subset \mathbb{R}^m$ defined by a formula $\phi_{\mathcal{P}}$. Consider the semi-algebraic set $\widehat{S}$ defined by the conjunction of a 3-nomial equations obtained from the equalities in (i), (ii) of Definition 1.1 by replacing Q_{ij} by Y_{ij} for all $1 \leq i \leq s$, $1 \leq j \leq a_k$, and the formula $\phi_{\mathcal{P}}$ in which every occurrence of an atomic formula of the kind $P_k * 0$, where $* \in \{=, >, <\}$, is replaced by the formula

$$c_i X_1^{\zeta_{i1}} \cdots X_k^{\zeta_{ik}} \prod_{1 \leq j \leq a_i} Y_{ij}^{\eta_{ij}} * 0.$$

Note that $\widehat{S}$ is a semi-algebraic subset of $\mathbb{R}^{k+a}$.

Let $\rho : \mathbb{R}^{k+a} \to \mathbb{R}^k$ be the projection on the subspace spanned by $X_1, \ldots, X_k$. It is clear that the restriction $\rho_{\widehat{S}} : \widehat{S} \to S$ is a homeomorphism, and moreover $\widehat{S}$ is defined by polynomials having at most $k + a$ monomials. Thus, in order to bound the number of homotopy types for S, it suffices to bound the same number for $\widehat{S}$, but since $\widehat{S}$ is defined by at most $2a$ polynomials in $k+a$ variables having at most $k+a$ monomials in total, we have reduced the problem of bounding the number of homotopy types occurring in $\mathcal{A}_{k,a}^{\text{div-free}}$ to that of bounding the number of homotopy types of semi-algebraic sets defined by at most $2a$ polynomials in $k+a$ variables, with the total number of monomials appearing bounded by $k + a$. This allows us to apply a bound proved in the fewnomial case in [5] to obtain a singly exponential bound on the number of homotopy types occurring in $\mathcal{A}_{k,a}^{\text{div-free}}$.

Notice that for the map $\rho_{\widehat{S}}$ to be a homeomorphism it is crucial that the exponents $\eta_{ij}, \gamma_{ij}, \delta_{ij}$ be non-negative, and this restricts the proof to the case of division-free additive complexity. We overcome this difficulty as follows.

Given a polynomial $F \in \mathbb{R}[X_1, \ldots, X_k]$ with additive complexity bounded by a, we prove that F can be expressed as a quotient P/Q, where $P, Q \in \mathbb{R}[X_1, \ldots, X_k]$ with the sum of the *division-free* additive complexities of P and Q bounded by a (see Lemma 3.1 below). We then express the set of real zeros of F in $\mathbb{R}^k$ inside any fixed closed ball as the Hausdorff limit of a one-parameter semi-algebraic family defined using the polynomials P and Q (see Proposition 3.4 and the accompanying Example 3.5 below).

While limits of one-parameter semi-algebraic families defined by polynomials with bounded division-free additive complexities can have complicated descriptions and cannot be described by polynomials of bounded division-free additive complexity, the topological complexity (for example, measured by their Betti numbers) of such limit sets is well controlled. Indeed, the problem of bounding the Betti numbers of Hausdorff limits of one-parameter families of semi-algebraic sets was considered by Zell [27], who proved a singly exponential bound on the Betti numbers of such sets. We prove in this paper (see Theorems 2.1 and 1.16 below) that the number of homotopy types of such limits can indeed be bounded singly exponentially in terms of the format of the formulas defining the one-parameter family. The techniques introduced by Zell [27] (as well certain semi-algebraic constructions described in [6]) play a crucial role in the proof of our bound. These intermediate results may be of independent interest.

Finally, applying Theorem 2.1 to the one-parameter family referred to in the previous paragraph, we obtain a bound on the number of homotopy types of real algebraic varieties defined by polynomials having bounded additive complexity. The semi-algebraic case requires certain additional techniques and is dealt with in Section 3.3.

1.3. Homotopy types of limits of semi-algebraic sets

In order to state our results on bounding the number of homotopy types of limits of one-parameter families of semi-algebraic sets we need to introduce some notation.

Notation 1.12. For any first-order formula Φ with k free variables, if $\mathcal{P} \subset \mathbb{R}[X_1, \dots, X_k]$ consists of the polynomials appearing in Φ, then we call Φ a *$\mathcal{P}$-formula*.

Notation 1.13 (Format of first-order formulas). Suppose Φ is a $\mathcal{P}$-formula defining a semi-algebraic subset of $\mathbb{R}^k$ involving s polynomials of degree at most d. In this case we say that Φ has *dense format* (s, d, k). If $\mathcal{P} \in \mathcal{A}_{k,a}$ then we say that Φ has *additive format bounded by* (a, k). If $\mathcal{P} \in \mathcal{A}_{k,a}^{\text{div-free}}$ then we say that Φ has *division-free additive format bounded by* (a, k).

Remark 1.14. A monomial has additive complexity 0, and every $\mathcal{P}$-formula with $\mathcal{P} \subset \mathbb{R}[X_1, \dots, X_k]$ containing only monomials is equivalent to a $\mathcal{P}'$-formula, where $\mathcal{P}' = \{X_1, \dots, X_k\}$. In particular, if ϕ is a $\mathcal{P}$-formula with (division-free) additive format bounded by (a, k), then ϕ is equivalent to a $\mathcal{P}'$-formula having (division-free) additive format bounded by (a, k) and such that the cardinality of $\mathcal{P}'$ is at most $a + k$.

Notation 1.15. For any $k \geq 1$, and $1 \leq p \leq q \leq k$, we denote by $\pi_{[p,q]} : \mathbb{R}^k = \mathbb{R}^{[1,k]} \to \mathbb{R}^{[p,q]}$ the projection

$$(x_1, \dots, x_k) \mapsto (x_p, \dots, x_q)$$

(omitting the dependence on k which should be clear from context). In case $p = q$ we will denote by π_p the projection $\pi_{[p,p]}$. For any semi-algebraic subset $X \subset \mathbb{R}^{k+1}$, and $\lambda \in \mathbb{R}$, we denote by X_λ the following semi-algebraic subset of $\mathbb{R}^k$:

$$X_\lambda = \pi_{[1,k]}(X \cap \pi_{k+1}^{-1}(\lambda)).$$

We denote by $\mathbb{R}_+$ the set of strictly positive elements of $\mathbb{R}$. If additionally $X \subset \mathbb{R}^k \times \mathbb{R}_+$, then we denote by X_{limit} the following semi-algebraic subset of $\mathbb{R}^k$:

$$X_{\text{limit}} := \pi_{[1,k]}(\overline{X} \cap \pi_{k+1}^{-1}(0)),$$

where $\overline{X}$ denotes the topological closure of X in $\mathbb{R}^{k+1}$.

We have the following theorem which establishes a singly exponential bound on the number of homotopy types of the Hausdorff limit of a one-parameter family of compact semi-algebraic sets defined by a first-order formula of bounded additive format. This result complements the result in [5] giving singly exponential bounds on the number of homotopy types of semi-algebraic sets defined by first-order formulas having bounded division-free additive format on one hand, and the result of Zell [27] bounding the Betti numbers of the Hausdorff limits of one-parameter families of semi-algebraic sets on the other, and could be of independent interest.

Theorem 1.16. *For each $a, k \in \mathbb{N}$, there exists a finite collection $\mathcal{S}_{k,a}$ of semi-algebraic subsets of $\mathbb{R}^N$, $N = (k+2)(k+1) + \binom{k+2}{2}$, with* card $\mathcal{S}_{k,a} = 2^{(k+a)^{O(1)}}$, *which satisfies the following property. If $\mathbb{T} \subset \mathbb{R}^k \times \mathbb{R}_+$ is a bounded semi-algebraic set described by a formula having additive format bounded by $(a, k+1)$ such that $\mathbb{T}_t$ is closed for each $t > 0$, then $\mathbb{T}_{\text{limit}}$ is homotopy equivalent to some $S \in \mathcal{S}_{k,a}$ (cf. Notation 1.15).*

The rest of the paper is devoted to the proofs of Theorems 1.16 and 1.11 and is organized as follows. We first prove a weak version (Theorem 2.1) of Theorem 1.16 in Section 2, in which the term "additive complexity" in the statement of Theorem 1.16 is replaced by "division-free additive complexity". Theorem 2.1 is then used in Section 3 to prove Theorem 1.11 after introducing some additional techniques; the latter theorem is used in turn to prove Theorem 1.16.

2. Proof of a weak version of Theorem 1.16

In this section we prove the following weak version of Theorem 1.16 (using *division-free* additive format rather than additive format) which is needed in the proof of Theorem 1.11.

Theorem 2.1. *For each $a, k \in \mathbb{N}$, there exists a finite collection $\mathcal{S}_{k,a}$ of semi-algebraic subsets of $\mathbb{R}^N$, $N = (k+2)(k+1) + \binom{k+2}{2}$, with* card $\mathcal{S}_{k,a} = 2^{O(k(k^2+a))^8} = 2^{(k+a)^{O(1)}}$, *which satisfies the following property. If $\mathbb{T} \subset \mathbb{R}^k \times \mathbb{R}_+$ is a bounded semi-algebraic set described by a formula having* division-free *additive format bounded by $(a, k+1)$ such that $\mathbb{T}_t$ is closed for each $t > 0$, then $\mathbb{T}_{\text{limit}}$ is homotopy equivalent to some $S \in \mathcal{S}_{k,a}$ (cf. Notation 1.15).*

2.1. Outline of the proof

The main steps in the proof of Theorem 2.1 are as follows. Let $\mathbb{T} \subset \mathbb{R}^k \times \mathbb{R}_+$ be a bounded semi-algebraic set such that $\mathbb{T}_t$ is closed for each $t \in \mathbb{R}$, and let $\mathbb{T}_{\text{limit}}$ be as in Notation 1.15.

We first prove that for all small enough $\lambda > 0$, there exists a semi-algebraic surjection $f_\lambda : \mathbb{T}_\lambda \to \mathbb{T}_{\text{limit}}$ which is metrically close to the identity map $1_{\mathbb{T}_\lambda}$ (see Proposition 2.27 below). Using a semi-algebraic realization of the fibered join described in [6] (see also [13]), we then consider, for any fixed $p \geq 0$, a semi-algebraic set $\mathcal{J}^p_{f_\lambda}(\mathbb{T}_\lambda)$ which is p-equivalent to $\mathbb{T}_{\text{limit}}$ (see Proposition 2.18). The definition of $\mathcal{J}^p_{f_\lambda}(\mathbb{T}_\lambda)$ still involves the map f_λ, whose definition is not simple, and hence we cannot control the topological type of $\mathcal{J}^p_{f_\lambda}(\mathbb{T}_\lambda)$ directly. However, the fact that f_λ is metrically close to the identity map enables us to adapt the main technique in [27] due to Zell. We replace $\mathcal{J}^p_{f_\lambda}(\mathbb{T}_\lambda)$ by another semi-algebraic set, which we denote by $\mathcal{D}^p_\varepsilon(\mathbb{T})$ (for $\varepsilon > 0$ small enough), which is homotopy equivalent to $\mathcal{J}^p_{f_\lambda}(\mathbb{T}_\lambda)$, but whose definition no longer involves the map f_λ (Definition 2.25). We can now bound the format of $\mathcal{D}^p_\varepsilon(\mathbb{T})$ in terms of the format of the formula defining $\mathbb{T}$. This key result is summarized in Proposition 2.3.

We first recall the definition of p-equivalence (see, for example, [23, p. 144]).

Definition 2.2 (p-equivalence). A map $f : A \to B$ between two topological spaces is called a *p-equivalence* if the induced map

$$f_* : \boldsymbol{\pi}_i(A, a) \to \boldsymbol{\pi}_i(B, f(a))$$

is, for each $a \in A$, bijective for $0 \le i < p$, and surjective for $i = p$; we then say that A is *p-equivalent to B*.

Proposition 2.3. *Let $\mathbb{T} \subset \mathbb{R}^k \times \mathbb{R}_+$ be a bounded semi-algebraic set such that $\mathbb{T}_t$ is closed for each $t > 0$, and let $p \ge 0$. Suppose also that $\mathbb{T}$ is described by a formula having (division-free) additive format bounded by $(a, k+1)$ and dense format $(s, d, k+1)$. Then there exists a semi-algebraic set $\mathcal{D}^p \subset \mathbb{R}^N$, $N = (p+1)(k+1) + \binom{p+1}{2}$, such that $\mathcal{D}^p$ is p-equivalent to $\mathbb{T}_{\text{limit}}$ (cf. Notation 1.15) and described by a formula having (division-free) additive format bounded by (M, N) and dense format $(M', d+1, N)$, where $M = (p+1)(k+a+2) + 2k\binom{p+1}{2}$ and $M' = (p+1)(s+2) + 3\binom{p+1}{2} + 3$.*

Finally, Theorem 2.1 is an easy consequence of Proposition 2.3.

2.2. Preliminaries

We need a few facts from the homotopy theory of finite CW-complexes.

We first prove a basic result about p-equivalences (Definition 2.2). It is clear that p-equivalence is not an equivalence relation (e.g., for any $p \ge 0$, the map taking $\mathbf{S}^p$ to a point is a p-equivalence, but no map from a point into $\mathbf{S}^p$ is one). However, we have the following.

Proposition 2.4. *Let A, B, C be finite CW-complexes with $\dim(A), \dim(B) \le k$ and suppose that C is p-equivalent to A and B for some $p > k$. Then A and B are homotopy equivalent.*

The proof of Proposition 2.4 will rely on the following well-known lemmas.

Lemma 2.5 ([26, p. 182, Theorem 7.16]). *Let X, Y be CW-complexes and $f : X \to Y$ a p-equivalence. Then, for each CW-complex M with $\dim(M) \le p$, the induced map*

$$f_* : [M, X] \to [M, Y]$$

is surjective.

Lemma 2.6 ([25, p. 69]). *If A and B are finite CW-complexes with* $\dim(A), \dim(B) \leq p$, *then every p-equivalence from A to B is a homotopy equivalence.*

Proof of Proposition 2.4. Suppose $f : C \to A$ and $g : C \to B$ are two p-equivalences. Applying Lemma 2.5 with $X = C$, $M = Y = A$, we see that the homotopy class of the identity map 1_A has a preimage, $[h]$, under f_*, for some $h \in [A, C]$. Then, for each $a \in A$ and $i \geq 0$,

$$f_* \circ h_* : \boldsymbol{\pi}_i(A, a) \to \boldsymbol{\pi}_i(A, f \circ h(a))$$

is bijective. In particular, since f is a p-equivalence, this implies that $h_* : \boldsymbol{\pi}_i(A, a) \to \boldsymbol{\pi}_i(C, h(a))$ is bijective for $0 \leq i < p$. Composing h with g, and noting that g is also a p-equivalence, we find that $(g \circ h)_* : \boldsymbol{\pi}_i(A, a) \to \boldsymbol{\pi}_i(B, g \circ h(a))$ is bijective for $0 \leq i < p$. Now, Lemma 2.6 shows that $g \circ h$ is a homotopy equivalence. □

We introduce some more notation.

Notation 2.7. For any $R \in \mathbb{R}_+$, we denote by $B_k(0, R) \subset \mathbb{R}^k$ the open ball of radius R centered at the origin.

Notation 2.8. For $P \in \mathbb{R}[X_1, \ldots, X_k]$, we denote by $\mathrm{Zer}(P, \mathbb{R}^k)$ the real algebraic set defined by $P = 0$.

Notation 2.9. For any first-order formula Φ with k free variables, we denote by $\mathrm{Reali}(\Phi)$ the semi-algebraic subset of $\mathbb{R}^k$ defined by Φ.

An important construction that we use later is an efficient semi-algebraic realization (up to homotopy) of the iterated fibered join of a semi-algebraic set over a semi-algebraic map. This construction was introduced in [6].

2.3. *Topological definitions*

We first recall the basic definition of the iterated join of a topological space.

Notation 2.10. For each $p \geq 0$, we denote by

$$\Delta_{[0,p]} = \left\{ \mathbf{t} = (t_0, \ldots, t_p) \;\middle|\; t_i \geq 0,\ 0 \leq i \leq p,\ \sum_{i=0}^{p} t_i = 1 \right\}$$

the standard p-simplex. For each $I = \{i_0, \ldots, i_m\}$, $0 \leq i_0 < \cdots < i_m \leq p$, the set

$$\Delta_I = \{\mathbf{t} = (t_0, \ldots, t_p) \in \Delta_{[0,p]} \mid t_i = 0 \text{ for all } i \notin I\}$$

is a face of $\Delta_{[0,p]}$.

Definition 2.11. For $p \geq 0$, the $(p+1)$*-fold join* $J^p(X)$ of a topological space X is

$$J^p(X) := \underbrace{X \times \cdots \times X}_{p+1 \text{ times}} \times \Delta_{[0,p]}/\sim, \tag{2.1}$$

where

$$(x_0, \ldots, x_p, t_0, \ldots, t_p) \sim (x'_0, \ldots, x'_p, t_0, \ldots, t_p)$$

if $x_i = x'_i$ for each i with $t_i \neq 0$.

In the special situation when X is a semi-algebraic set, the space $J^p(X)$ defined above is not immediately a semi-algebraic set, because of taking quotients. We now define a semi-algebraic set, $\mathcal{J}^p(X)$, that is homotopy equivalent to $J^p(X)$.

Let

$$\Delta'_{[0,p]} = \Big\{\mathbf{t} = (t_0, \ldots, t_p) \in \mathbb{R}^{p+1} \;\Big|\; \sum_{0 \le i \le p} t_i = 1,\ |\mathbf{t}|^2 \le 1\Big\}.$$

For each $I = \{i_0, \ldots, i_m\}$, $0 \le i_0 < \cdots < i_m \le p$, let

$$\Delta'_I = \{\mathbf{t} = (t_0, \ldots, t_p) \in \Delta'_{[0,p]} \mid t_i = 0 \text{ for all } i \notin I\}.$$

It is clear that the standard simplex $\Delta_{[0,p]}$ is a deformation retract of $\Delta'_{[0,p]}$ via a deformation retraction $\rho_p : \Delta'_{[0,p]} \to \Delta_{[0,p]}$ that restricts to a deformation retraction $\rho_p|_{\Delta'_I} : \Delta'_I \to \Delta_I$ for each $I \subset [0, p]$.

We use the lower case bold-face notation $\mathbf{x}$ to denote a point $\mathbf{x} = (x_1, \ldots, x_k)$ of $\mathbb{R}^k$ and upper-case $\mathbf{X} = (X_1, \ldots, X_k)$ to denote a *block of variables*. In the following definition the role of the $\binom{p+1}{2}$ variables $(A_{ij})_{0 \le i < j \le p}$ can be safely ignored, since they are all set to 0. Their significance will be clear later.

Definition 2.12 (The semi-algebraic join [6]). For a semi-algebraic subset $X \subset \mathbb{R}^k$ contained in $B_k(0, R)$, defined by a $\mathcal{P}$-formula Φ, we define

$$\mathcal{J}^p(X) = \{(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}, \mathbf{a}) \in \mathbb{R}^{(p+1)(k+1)+\binom{p+1}{2}} \mid \Omega^R(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}) \wedge \Theta_1(\mathbf{t}, \mathbf{a}) \wedge \Theta_2^{\Phi}(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t})\},$$

where

$$\begin{aligned} \Omega^R &:= \bigwedge_{i=0}^{p} (|\mathbf{X}^i|^2 \le R^2) \wedge |\mathbf{T}|^2 \le 1, \\ \Theta_1 &:= \sum_{i=0}^{p} T_i = 1 \wedge \sum_{0 \le i < j \le p} A_{ij}^2 = 0, \\ \Theta_2^{\Phi} &:= \bigwedge_{i=0}^{p} (T_i = 0 \vee \Phi(\mathbf{X}^i)), \end{aligned} \tag{2.2}$$

We denote the formula $\Omega^R \wedge \Theta_1 \wedge \Theta_2^{\Phi}$ by $\mathcal{J}^p(\Phi)$.

It is checked easily from Definition 2.12 that

$$\mathcal{J}^p(X) \subset \left(\overline{B_k(0, R)}\right)^{p+1} \times \Delta'_{[0,p]} \times \{\mathbf{0}\},$$

and that the deformation retraction $\rho_p : \Delta'_{[0,p]} \to \Delta_{[0,p]}$ extends to a deformation retraction $\tilde{\rho}_p : \mathcal{J}^p(X) \to \tilde{\mathcal{J}}^p(X)$, where

$$\tilde{\mathcal{J}}^p(X) = \Big\{(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}, \mathbf{a}) \in \left(\overline{B_k(0, R)}\right)^{p+1} \times \Delta_{[0,p]} \times \{\mathbf{0}\} \;\Big|\; \Theta_2^{\Phi}(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t})\Big\}.$$

Finally, it is a consequence of the Vietoris–Begle theorem (see [8, Theorem 2]) that $\tilde{\mathcal{J}}^p(X)$ and $J^p(X)$ are homotopy equivalent. We thus have, using the notation introduced above:

Proposition 2.13. *$\mathcal{J}^p(X)$ is homotopy equivalent to $J^p(X)$.*

Remark 2.14. The necessity of defining $\mathcal{J}^p(X)$ instead of just $\tilde{\mathcal{J}}^p(X)$ has to do with removing the inequalities defining the standard simplex from the defining formula $\mathcal{J}^p(\Phi)$; this will simplify certain arguments later.

We now generalize the above constructions and define joins over maps (the topological and semi-algebraic joins defined above are special cases when the map is a constant map to a point).

Notation and Definition 2.15. Let $f : A \to B$ be a map between topological spaces. For each $p \geq 0$, we denote by $W_f^p(A)$ the $(p+1)$-*fold fiber product* of A over f, that is,

$$W_f^p(A) = \{(x_0, \ldots, x_p) \in A^{p+1} \mid f(x_0) = \cdots = f(x_p)\}.$$

Definition 2.16 (Topological join over a map). Let $f : X \to Y$ be a map between topological spaces. For $p \geq 0$, the $(p+1)$-*fold join* $J_f^p(X)$ of X over f is

$$J_f^p(X) := W_f^p(X) \times \Delta^p/\sim, \tag{2.3}$$

where

$$(x_0, \ldots, x_p, t_0, \ldots, t_p) \sim (x_0', \ldots, x_p', t_0, \ldots, t_p)$$

if $x_i = x_i'$ for each i with $t_i \neq 0$.

In the special situation when f is a semi-algebraic continuous map, the space $J_f^p(X)$ defined above is not (as before) immediately a semi-algebraic set, because of taking quotients. Our next goal is to obtain a semi-algebraic set $\mathcal{J}_f^p(X)$ which is homotopy equivalent to $J_f^p(X)$, similar to the case of the ordinary join.

Definition 2.17 (The semi-algebraic fibered join [6]). For a semi-algebraic subset $X \subset \mathbb{R}^k$ contained in $B_k(0, R)$, defined by a $\mathcal{P}$-formula Φ, and $f : X \to Y$ a semi-algebraic map, we define

$$\begin{aligned}\mathcal{J}_f^p(X) = \{(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}, \mathbf{a}) \in \mathbb{R}^{(p+1)(k+1)+\binom{p+1}{2}} \mid \\ \Omega^R(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}) \wedge \Theta_1(\mathbf{t}, \mathbf{a}) \wedge \Theta_2^\Phi(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}) \wedge \Theta_3^f(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}, \mathbf{a})\},\end{aligned}$$

where Ω^R, Θ_1, Θ_2^Φ have been defined previously, and

$$\Theta_3^f := \bigwedge_{0 \leq i < j \leq p} (T_i = 0 \vee T_j = 0 \vee |f(\mathbf{X}^i) - f(\mathbf{X}^j)|^2 = A_{ij}). \tag{2.4}$$

We denote the formula $\Omega^R \wedge \Theta_1 \wedge \Theta_2^\Phi \wedge \Theta_3^f$ by $\mathcal{J}_f^p(\Phi)$.

Observe that there exists a natural map $J^p(f) : \mathcal{J}^p_f(X) \to Y$ which maps a point $(\mathbf{x}^0, \dots, \mathbf{x}^p, \mathbf{t}, \mathbf{0}) \in \mathcal{J}^p_f(X)$ to $f(\mathbf{x}^i)$ (where i is such that $t_i \neq 0$). It is easy to see that for each $\mathbf{y} \in Y$, $J^p(f)^{-1}(\mathbf{y}) = \mathcal{J}^p(f^{-1}(\mathbf{y}))$.

The following proposition follows from the above observation and the generalized Vietoris–Begle theorem (see [8, Theorem 2]) and is important in the proof of Proposition 2.3; it relates up to p-equivalence the semi-algebraic set $\mathcal{J}^p_f(X)$ to the image of a closed continuous semi-algebraic surjection $f : X \to Y$. Its proof is similar to the proof of Theorem 2.12 proved in [6] and is omitted.

Proposition 2.18 ([6]). *Let $f : X \to Y$ a closed continuous semi-algebraic surjection with $X \subset B_k(0, R)$ a closed semi-algebraic set. Then, for every $p \geq 0$, the map $J^p(f) : \mathcal{J}^p_f(X) \to Y$ is a p-equivalence.*

We now define a thickened version of the semi-algebraic set $\mathcal{J}^p_f(X)$ defined above and prove that it is homotopy equivalent to $\mathcal{J}^p_f(X)$. The variables A_{ij}, $0 \leq i < j \leq p$, play an important role in the thickening process.

Definition 2.19 (The thickened semi-algebraic fibered join). For $X \subset \mathbb{R}^k$ a semi-algebraic set contained in $B_k(0, R)$, defined by a $\mathcal{P}$-formula Φ, $p \geq 1$, and $\varepsilon > 0$, define

$$\mathcal{J}^p_{f,\varepsilon}(X) = \{(\mathbf{x}^0, \dots, \mathbf{x}^p, \mathbf{t}, \mathbf{a}) \in \mathbb{R}^{(p+1)(k+1)+\binom{p+1}{2}} \mid \Omega^R(\mathbf{x}^0, \dots, \mathbf{x}^p, \mathbf{t}) \wedge \Theta^\varepsilon_1(\mathbf{t}, \mathbf{a}) \wedge \Theta^\Phi_2(\mathbf{x}^0, \dots, \mathbf{x}^p, \mathbf{t}) \wedge \Theta_3(\mathbf{x}^0, \dots, \mathbf{x}^p, \mathbf{t}, \mathbf{a})\},$$

where

$$\begin{aligned}
\Omega^R &:= \bigwedge_{i=0}^{p} (|\mathbf{X}^i|^2 \leq R^2) \wedge |\mathbf{T}|^2 \leq 1, \\
\Theta^\varepsilon_1 &:= \sum_{i=0}^{p} T_i = 1 \wedge \sum_{1 \leq i < j \leq p} A^2_{ij} \leq \varepsilon, \\
\Theta^\Phi_2 &:= \bigwedge_{i=0}^{p} (T_i = 0 \vee \Phi(\mathbf{X}^i)), \\
\Theta^f_3 &:= \bigwedge_{0 \leq i < j \leq p} (T_i = 0 \vee T_j = 0 \vee |f(\mathbf{X}^i) - f(\mathbf{X}^j)|^2 = A_{ij}).
\end{aligned} \tag{2.5}$$

Note that if X is closed (and bounded), then $\mathcal{J}^p_{f,\varepsilon}(X)$ is again closed (and bounded).

The relation between $\mathcal{J}^p_f(X)$ and $\mathcal{J}^p_{f,\varepsilon}(X)$ is described in the following proposition.

Proposition 2.20. *For $p \in \mathbb{N}$ and $f : X \to Y$ semi-algebraic there exists $\varepsilon_0 > 0$ such that $\mathcal{J}^p_f(X)$ is homotopy equivalent to $\mathcal{J}^p_{f,\varepsilon}(X)$ for all $0 < \varepsilon \leq \varepsilon_0$.*

Proposition 2.20 follows from the next two lemmas.

Lemma 2.21. *For $p \in \mathbb{N}$ and $f : X \to Y$ semi-algebraic we have*

$$\mathcal{J}_f^p(X) = \bigcap_{t>0} \mathcal{J}_{f,t}^p(X).$$

Proof. Obvious from Definitions 2.17 and 2.19. □

Lemma 2.22. *Let $\mathbb{T} \subset \mathbb{R}^k \times \mathbb{R}_+$ be such that each $\mathbb{T}_t$ is closed and contained in $B_k(0, R)$. Suppose further that $\mathbb{T}_t \subseteq \mathbb{T}_{t'}$ for all $0 < t \leq t'$. Then*

$$\bigcap_{t>0} \mathbb{T}_t = \pi_{[1,k]}(\overline{\mathbb{T}} \cap \pi_{k+1}^{-1}(0)).$$

Furthermore, there exists $\varepsilon_0 > 0$ such that for all ε with $0 < \varepsilon \leq \varepsilon_0$ the set $\mathbb{T}_\varepsilon$ is semi-algebraically homotopy equivalent to $\mathbb{T}_{\text{limit}}$ (cf. Notation 1.15).

Proof. The first part of the proposition is straightforward. The second part follows easily from [4, Lemma 16.16]. □

Proof of Proposition 2.20. The set $\mathbb{T} = \{(\mathbf{x}, t) \in \mathbb{R}^{k+1} \mid t > 0 \wedge \mathbf{x} \in \mathcal{J}_{f,t}^p(X)\}$ satisfies the conditions of Lemma 2.22. The proposition now follows from Lemmas 2.22 and 2.21. □

Proposition 2.23. *For $p \in \mathbb{N}$, $f : X \to Y$ semi-algebraic, and $0 < t \leq t'$,*

$$\mathcal{J}_{f,t}^p(X) \subseteq \mathcal{J}_{f,t'}^p(X).$$

Moreover, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon' < \varepsilon_0$ the above inclusion with $t = \varepsilon$, $t' = \varepsilon'$ is a semi-algebraic homotopy equivalence.

The first part of Proposition 2.23 is obvious from the definition of $\mathcal{J}_{f,\varepsilon}^p(X)$. The second part follows from Lemma 2.24 below.

The following lemma is probably well known and easy. However, since we were unable to locate an exact statement to this effect in the literature, we include a proof.

Lemma 2.24. *Let $\mathbb{T} \subset \mathbb{R}^k \times \mathbb{R}_+$ be a semi-algebraic set, and suppose that $\mathbb{T}_t \subset \mathbb{T}_{t'}$ for all $0 < t < t'$. Then there exists ε_0 such that for each $0 < \varepsilon < \varepsilon' \leq \varepsilon_0$ the inclusion map $\mathbb{T}_\varepsilon \overset{i_{\varepsilon'}}{\hookrightarrow} \mathbb{T}_{\varepsilon'}$ is a semi-algebraic homotopy equivalence.*

Proof. We prove that there exists $\phi_{\varepsilon'} : \mathbb{T}_{\varepsilon'} \to \mathbb{T}_\varepsilon$ such that

$$\begin{aligned} \phi_{\varepsilon'} \circ i_{\varepsilon'} &: \mathbb{T}_\varepsilon \to \mathbb{T}_\varepsilon, & \phi_{\varepsilon'} \circ i_{\varepsilon'} &\simeq \mathrm{Id}_{\mathbb{T}_\varepsilon}, \\ i_{\varepsilon'} \circ \phi_{\varepsilon'} &: \mathbb{T}_{\varepsilon'} \to \mathbb{T}_{\varepsilon'}, & i_{\varepsilon'} \circ \phi_{\varepsilon'} &\simeq \mathrm{Id}_{\mathbb{T}_{\varepsilon'}}. \end{aligned}$$

We first define $i_t : \mathbb{T}_\varepsilon \hookrightarrow \mathbb{T}_t$ and $\widehat{i_t} : \mathbb{T}_t \hookrightarrow \mathbb{T}_{\varepsilon'}$, and note that trivially $i_\varepsilon = \mathrm{Id}_{\mathbb{T}_\varepsilon}$, $\widehat{i_{\varepsilon'}} = \mathrm{Id}_{\mathbb{T}_{\varepsilon'}}$, and $i_{\varepsilon'} = \widehat{i_\varepsilon}$. Now, by Hardt triviality there exists $\varepsilon_0 > 0$ such that there is a definably trivial homeomorphism h which commutes with the projection π_{k+1}, i.e., the following diagram commutes:

$$\begin{array}{ccc} \mathbb{T}_{\varepsilon_0} \times (0, \varepsilon_0] & \xrightarrow{h} & \mathbb{T} \cap \{(\mathbf{x}, t) \mid 0 < t \leq \varepsilon_0\} \\ \downarrow \pi_{k+1} & \swarrow \pi_{k+1} & \\ (0, \varepsilon_0] & & \end{array}$$

Define $F(\mathbf{x}, t, s) = h(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, t), s)$. Note that $F(\mathbf{x}, t, t) = h(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, t), t) = h(h^{-1}(\mathbf{x}, t)) = (\mathbf{x}, t)$. We set

$$\begin{aligned} \phi_t &: \mathbb{T}_t \to \mathbb{T}_\varepsilon, & \phi_t(\mathbf{x}) &= \pi_{[1,k]} \circ F(\mathbf{x}, t, \varepsilon), \\ \widehat{\phi}_t &: \mathbb{T}_{\varepsilon'} \to \mathbb{T}_t, & \widehat{\phi}_t(\mathbf{x}) &= \pi_{[1,k]} \circ F(\mathbf{x}, \varepsilon', t), \end{aligned}$$

and note that $\phi_{\varepsilon'} = \widehat{\phi}_\varepsilon$.

Finally, define

$$\begin{aligned} H_1(\cdot, t) &= \phi_t \circ i_t : \mathbb{T}_\varepsilon \to \mathbb{T}_\varepsilon, & H_2(\cdot, t) &= \widehat{i}_t \circ \widehat{\phi}_t : \mathbb{T}_{\varepsilon'} \to \mathbb{T}_{\varepsilon'}, \\ H_1(\cdot, \varepsilon) &= \phi_\varepsilon \circ i_\varepsilon = \mathrm{Id}_{\mathbb{T}_\varepsilon}, & H_2(\cdot, \varepsilon) &= \widehat{i}_\varepsilon \circ \widehat{\phi}_\varepsilon = i_{\varepsilon'} \circ \phi_{\varepsilon'}, \\ H_1(\cdot, \varepsilon') &= \phi_{\varepsilon'} \circ i_{\varepsilon'}, & H_2(\cdot, \varepsilon') &= \widehat{i}_{\varepsilon'} \circ \widehat{\phi}_{\varepsilon'} = \mathrm{Id}_{\mathbb{T}_{\varepsilon'}}. \end{aligned}$$

The semi-algebraic continuous maps H_1 and H_2 defined above give semi-algebraic homotopies $\phi_{\varepsilon'} \circ i_{\varepsilon'} \simeq \mathrm{Id}_{\mathbb{T}_\varepsilon}$ and $i_{\varepsilon'} \circ \phi_{\varepsilon'} \simeq \mathrm{Id}_{\mathbb{T}_{\varepsilon'}}$ proving the required semi-algebraic homotopy equivalence. □

As mentioned before, we would like to replace $\mathcal{J}^p_{f,\varepsilon}(X)$ by another semi-algebraic set, which we denote by $\mathcal{D}^p_\varepsilon(X)$, which is homotopy equivalent to $\mathcal{J}^p_{f,\varepsilon}(X)$, under certain assumptions on f and ε, and whose definition no longer involves the map f. This is what we do next.

Definition 2.25 (The thickened diagonal). For a semi-algebraic set $X \subset \mathbb{R}^k$ contained in $B_k(0, R)$ defined by a $\mathcal{P}$-formula Φ, $p \geq 1$, and $\varepsilon > 0$, define

$$\begin{aligned} \mathcal{D}^p_\varepsilon(X) = \{(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}, \mathbf{a}) \in \mathbb{R}^{(p+1)(k+1)+\binom{p+1}{2}} \mid \\ \Omega^R(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}) \wedge \Theta_1(\mathbf{t}, \mathbf{a}) \wedge \Theta_2^\Phi(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}) \wedge \Upsilon(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}, \mathbf{a})\}, \end{aligned}$$

where Ω^R, Θ_1^ε, Θ_2^Φ are defined as in (2.5), and

$$\Upsilon := \bigwedge_{0 \leq i < j \leq p} (T_i = 0 \vee T_j = 0 \vee |\mathbf{X}^i - \mathbf{X}^j|^2 = A_{ij}).$$

Notice that the formula defining the thickened diagonal $\mathcal{D}^p_\varepsilon(X)$ in Definition 2.25 is identical to that defining the thickened semi-algebraic fibered join $\mathcal{J}^p_{f,\varepsilon}(X)$ in Definition 2.19, except that Θ_3^f is replaced by Υ, and Υ does not depend on the map f or on the set X.

Proposition 2.26. *Let $X \subset \mathbb{R}^k$ be a semi-algebraic set defined by a quantifier-free formula Φ having (division-free) additive format bounded by (a, k) and dense format bounded by (s, d, k). Then $\mathcal{D}^p_\varepsilon(X)$ is a semi-algebraic subset of $\mathbb{R}^N$, defined by a formula with (division-free) additive format bounded by (M, N) and dense format bounded by $(M', d+1, N)$, where $M = (p+1)(k+a+2)+2k\binom{p+1}{2}$, $M' = (p+1)(s+2)+3\binom{p+1}{2}+3$, and $N = (p+1)(k+1)+\binom{p+1}{2}$.*

Proof. It is a straightforward computation to bound the division-free additive format and give the dense format of the formulas Ω^R, Θ^ε_1, Υ as well as the (division-free) additive format and dense format of the formula Θ^Φ_2. More precisely, let

$$\begin{aligned} M_{\Omega^R} &= (p+1)k + (p+1), & M'_{\Omega^R} &= (p+1)+1,\\ M_{\Theta^\varepsilon_1} &= (p+1) + \tbinom{p+1}{2}, & M'_{\Theta^\varepsilon_1} &= 2,\\ M_{\Theta^\Phi_2} &= (p+1)a, & M'_{\Theta^\Phi_2} &= (p+1)(s+1),\\ M_\Upsilon &= 2k\tbinom{p+1}{2}, & M'_\Upsilon &= 3\tbinom{p+1}{2}. \end{aligned}$$

It is clear from Definition 2.25 that the division-free additive format (resp. dense format) of Ω^R is bounded by (M_{Ω^R}, N), $N = (p+1)(k+1) + \binom{p+1}{2}$ (resp. $(M'_{\Omega^R}, 2, N)$). Similarly, the division-free additive format (resp. dense format) of Θ^ε_1, Υ is bounded by $(M_{\Theta^\varepsilon_1}, N)$, (M_Υ, N) (resp. $(M'_{\Theta^\varepsilon_1}, 2, N)$, $(M'_\Upsilon, 2, N)$). Finally, the (division-free) additive format of Θ^Φ_2 is bounded by $(M_{\Theta^\Phi_2}, N)$ and its dense format is $(M'_{\Theta^\Phi_2}, d+1, N)$. The (division-free) additive format (resp. dense format) of the formula defining $\mathcal{D}^p_\varepsilon(X)$ is thus bounded by

$$(M_{\Omega^R} + M_{\Theta^\varepsilon_1} + M_{\Theta^\Phi_2} + M_\Upsilon, N) \quad (\text{resp. } (M'_{\Omega^R} + M'_{\Theta^\varepsilon_1} + M'_{\Theta^\Phi_2} + M'_\Upsilon, d+1, N)).$$

□

We now relate the thickened semi-algebraic fibered-join and the thickened diagonal using a sandwiching argument similar in spirit to that used in [27].

2.3.1. Limits of one-parameter families. In this section, we fix a bounded semi-algebraic set $\mathbb{T} \subset \mathbb{R}^k \times \mathbb{R}_+$ such that $\mathbb{T}_t$ is closed and $\mathbb{T}_t \subseteq B_k(0, R)$ for some $R \in \mathbb{R}_+$ and all $t > 0$. Let $\mathbb{T}_{\mathrm{limit}}$ be as in Notation 1.15.

We need the following proposition proved in [27].

Proposition 2.27 ([27, Proposition 8]). *There exists $\lambda_0 > 0$ and a family $\{f_\lambda\}_{0<\lambda\leq\lambda_0}$ of continuous semi-algebraic surjections $f_\lambda : \mathbb{T}_\lambda \to \mathbb{T}_{\mathrm{limit}}$ such that*

(A) $\lim_{\lambda\to 0} \max_{\mathbf{x}\in\mathbb{T}_\lambda} |\mathbf{x} - f_\lambda(\mathbf{x})| = 0$,
(B) *for each $\lambda, \lambda' \in (0, \lambda_0)$, $f_\lambda = f_{\lambda'} \circ g$ for some semi-algebraic homeomorphism $g : \mathbb{T}_\lambda \to \mathbb{T}_{\lambda'}$.*

Proposition 2.28. *There exist λ_1 satisfying $0 < \lambda_1 \leq \lambda_0$ and semi-algebraic functions $\delta_0, \delta_1 : (0, \lambda_1) \to \mathbb{R}$ such that*

(A) $0 < \delta_0(\lambda) < \delta_1(\lambda)$ *for* $\lambda \in (0, \lambda_1)$,
(B) $\lim_{\lambda \to 0} \delta_0(\lambda) = 0$ *and* $\lim_{\lambda \to 0} \delta_1(\lambda) \neq 0$,
(C) *for each* $\lambda \in (0, \lambda_1)$ *and* δ, δ' *satisfying* $0 < \delta_0(\lambda) < \delta < \delta' < \delta_1(\lambda)$, *the inclusion* $\mathcal{D}^p_{\delta'}(\mathbb{T}_\lambda) \hookrightarrow \mathcal{D}^p_\delta(\mathbb{T}_\lambda)$ *is a semi-algebraic homotopy equivalence.*

Proposition 2.28 is adapted from [27, Proposition 20] and the proof is identical after replacing $D^p_\lambda(\delta)$ (defined in [27]) with the semi-algebraic set $\mathcal{D}^p_\delta(\mathbb{T}_\lambda)$ defined above (Definition 2.25).

Let f_λ, $\lambda \in (0, \lambda_0]$, satisfy the conclusion of Proposition 2.27. As in [27], define, for $p \in \mathbb{N}$,

$$\eta_p(\lambda) = p(p+1)\Big(4R \max_{\mathbf{x}\in\mathbb{T}_\lambda} |\mathbf{x} - f_\lambda(\mathbf{x})| + 2\Big(\max_{\mathbf{x}\in\mathbb{T}_\lambda} |\mathbf{x} - f_\lambda(\mathbf{x})|\Big)^2\Big). \tag{2.6}$$

Note that, for every $\lambda \in (0, \lambda_0]$ and every $q \leq p$, we have $\eta_q(\lambda) \leq \eta_p(\lambda)$. Additionally, for each $p \in \mathbb{N}$, $\lim_{\lambda \to 0} \eta_p(\lambda) = 0$ by Proposition 2.27(A).

For $\overline{\mathbf{x}} = (\mathbf{x}^0, \ldots, \mathbf{x}^p) \in \mathbb{R}^{(p+1)k}$ define

$$\rho_p(\mathbf{x}^0, \ldots, \mathbf{x}^p) = \sum_{1 \leq i < j \leq p} |\mathbf{x}^i - \mathbf{x}^j|^2.$$

A special case of this sum corresponding to all $t_i \neq 0$ appears in the formula Υ^ε_1 of Definition 2.25 after making the replacement $a_{ij} = |\mathbf{x}^i - \mathbf{x}^j|$. The next lemma is taken from [27], to which we refer the reader for the proof.

Lemma 2.29 ([27, Lemma 21]). *Given* $\eta_p(\lambda)$ *and* $f_\lambda : \mathbb{T}_\lambda \to \mathbb{T}_{\mathrm{limit}}$ *as above, we have*

$$\Big|\sum_{i<j} |f_\lambda(\mathbf{x}^i) - f_\lambda(\mathbf{x}^j)|^2 - \sum_{i<j} |\mathbf{x}^i - \mathbf{x}^j|^2\Big| \leq \eta_p(\lambda),$$

and in particular

$$\rho_p(\mathbf{x}^0, \ldots, \mathbf{x}^p) \leq \rho_p(f_\lambda(\mathbf{x}^0), \ldots, f_\lambda(\mathbf{x}^p)) + \eta_p(\lambda) \leq \rho_p(\mathbf{x}^0, \ldots, \mathbf{x}^p) + 2\eta_p(\lambda).$$

The next proposition follows immediately from Lemma 2.29, Definition 2.19, and Definition 2.25.

Proposition 2.30. *For every* $\lambda \in (0, \lambda_0)$ *and* $\varepsilon > 0$, *we have*

$$\mathcal{J}^p_{f_\lambda,\varepsilon}(\mathbb{T}_\lambda) \subseteq \mathcal{D}^p_{\varepsilon+\eta_p(\lambda)}(\mathbb{T}_\lambda) \subseteq \mathcal{J}^p_{f_\lambda,\varepsilon+2\eta_p(\lambda)}(\mathbb{T}_\lambda).$$

Let $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_+$ satisfy the conclusions of Propositions 2.20 and 2.23, respectively. Set $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$.

Proposition 2.31. *For any* $p \in \mathbb{N}$, *there exist* $\lambda, \varepsilon, \delta \in \mathbb{R}_+$ *such that* $\varepsilon \in (0, \varepsilon_0)$, $\lambda \in (0, \lambda_0)$, *and*

$$\mathcal{D}^p_\delta(\mathbb{T}_\lambda) \simeq \mathcal{J}^p_{f_\lambda,\varepsilon}(\mathbb{T}_\lambda).$$

Proof. We first describe how to choose $\varepsilon, \varepsilon' \in (0, \varepsilon_0)$, $\lambda \in (0, \lambda_0)$ and $\delta, \delta' \in (\delta_0(\lambda), \delta_1(\lambda))$ (cf. Proposition 2.28) so that

$$\mathcal{D}^p_{\delta'}(\mathbb{T}_\lambda) \subseteq \mathcal{J}^p_{f_\lambda,\varepsilon}(\mathbb{T}_\lambda) \overset{*}{\subseteq} \mathcal{D}^p_{\delta}(\mathbb{T}_\lambda) \subseteq \mathcal{J}^p_{f_\lambda,\varepsilon'}(\mathbb{T}_\lambda),$$

and secondly we show that, with these choices, the inclusion $(*)$ is a homotopy equivalence.

Since the limit of $\delta_1(\lambda) - \delta_0(\lambda)$ is not zero for $0 < \lambda < \lambda_1 \le \lambda_0$ and λ tending to zero, while the limits of $\eta_p(\lambda)$ and $\delta_0(\lambda)$ are zero (by Propositions 2.28 and 2.27(A)), we can choose $0 < \lambda < \lambda_0$ which simultaneously satisfies

$$2\eta_p(\lambda) < \frac{\delta_1(\lambda) - \delta_0(\lambda)}{2} \quad \text{and} \quad \delta_0(\lambda) + 4\eta_p(\lambda) < \varepsilon_0.$$

Set $\delta' = \delta_0 + \eta_p(\lambda)$, $\varepsilon = \delta_0 + 2\eta_p(\lambda)$, $\delta = \delta_0 + 3\eta_p(\lambda)$, and $\varepsilon' = \delta_0 + 4\eta_p(\lambda)$. From Proposition 2.30 we have the inclusions

$$\mathcal{D}^p_{\delta'}(\mathbb{T}_\lambda) \overset{i}{\hookrightarrow} \mathcal{J}^p_{f_\lambda,\varepsilon}(\mathbb{T}_\lambda) \overset{j}{\hookrightarrow} \mathcal{D}^p_{\delta}(\mathbb{T}_\lambda) \overset{k}{\hookrightarrow} \mathcal{J}^p_{f_\lambda,\varepsilon'}(\mathbb{T}_\lambda).$$

Furthermore, it is easy to see that $\delta, \delta' \in (\delta_0(\lambda), \delta_1(\lambda))$ and $\varepsilon, \varepsilon' \in (0, \varepsilon_0)$, and so both $j \circ i$ and $k \circ j$ are semi-algebraic homotopy equivalences (Propositions 2.28 and 2.23 resp.).

For each $\mathbf{z} \in \mathcal{D}^p_{\delta'}(\mathbb{T}_\lambda)$ we have the following diagram of homotopy groups:

$$\begin{array}{ccccc}
\boldsymbol{\pi}_*(\mathcal{D}^p_{\delta'}(\mathbb{T}_\lambda), \mathbf{z}) & \xrightarrow[\cong]{(j\circ i)_*} & \boldsymbol{\pi}_*(\mathcal{D}^p_{\delta}(\mathbb{T}_\lambda), \mathbf{z}) & & \\
& \searrow^{i_*} & \nearrow_{j_*} \quad \searrow^{k_*} & & \\
& & \boldsymbol{\pi}_*(\mathcal{J}^p_{f_\lambda,\varepsilon}(\mathbb{T}_\lambda), \mathbf{z}) & \xrightarrow[\cong]{(k\circ j)_*} & \boldsymbol{\pi}_*(\mathcal{J}^p_{f_\lambda,\varepsilon'}(\mathbb{T}_\lambda), \mathbf{z})
\end{array}$$

where we have identified $\mathbf{z}$ with its images under various inclusion maps.

Since $(j \circ i)_* = j_* \circ i_*$, the surjectivity of $(j \circ i)_*$ implies that j_* is surjective, and similarly $(k \circ j)_*$ injective ensures that j_* is injective. Hence, j_* is an isomorphism as required.

This implies that the inclusion map $\mathcal{J}^p_{f_\lambda,\varepsilon}(\mathbb{T}_\lambda) \overset{j}{\hookrightarrow} \mathcal{D}^p_{\delta}(\mathbb{T}_\lambda)$ is a weak homotopy equivalence (see [26, p. 181]). Since both spaces have the structure of a finite CW-complex, every weak equivalence is in fact a homotopy equivalence ([26, Theorem 3.5, p. 220]). □

We now prove Proposition 2.3.

Proof of Proposition 2.3. Let $\mathbb{T} \subset \mathbb{R}^k \times \mathbb{R}_+$ be such that $\mathbb{T}_\lambda$ is closed and $\mathbb{T}_\lambda \subset B_k(0, R)$ for some $R \in \mathbb{R}$ and all $\lambda \in \mathbb{R}_+$. Proposition 2.31 shows that there exist $\lambda \in (0, \lambda_0)$ and $\varepsilon \in (0, \varepsilon_0)$ such that the sets $\mathcal{D}^p_{\delta}(\mathbb{T}_\lambda)$ and $\mathcal{J}^p_{f_\lambda,\varepsilon}(\mathbb{T}_\lambda)$ are semi-algebraically homotopy equivalent. Also, by Proposition 2.20 the sets $\mathcal{J}^p_{f_\lambda,\varepsilon}(\mathbb{T}_\lambda)$ and

$\mathcal{J}^p_{f_\lambda}(\mathbb{T}_\lambda)$ are semi-algebraically homotopy equivalent. By Propositions 2.18 and 2.27 the map $J(f_\lambda) : \mathcal{J}^p_{f_\lambda}(\mathbb{T}_\lambda) \twoheadrightarrow \mathbb{T}_0$ is a p-equivalence.

Thus we have the following sequence of homotopy equivalences and p-equivalence:

$$\mathcal{D}^p_\delta(\mathbb{T}_\lambda) \simeq \mathcal{J}^p_{f_\lambda,\varepsilon}(\mathbb{T}_\lambda) \simeq \mathcal{J}^p_{f_\lambda}(\mathbb{T}_\lambda) \xrightarrow{\sim}_p \mathbb{T}_{\text{limit}}. \tag{2.7}$$

The first homotopy equivalence follows from Proposition 2.31, the second from Proposition 2.20, and the last p-equivalence is a consequence of Propositions 2.18 and 2.27. The bound on the format of the formula defining $\mathcal{D}^p := \mathcal{D}^p_\delta(\mathbb{T}_\lambda)$ follows from Proposition 2.26. This finishes the proof. □

Proof of Theorem 2.1. The theorem follows directly from Proposition 2.3, Theorem 1.6 and Proposition 2.4 after choosing $p = k + 1$. □

3. Proofs of Theorems 1.11 and 1.16

3.1. Algebraic preliminaries

We start with a lemma that provides a slightly different characterization of additive complexity from that given in Definition 1.8. Roughly speaking, the lemma states that any given additive representation of a given polynomial P can be modified without changing its length to another additive representation of P in which any negative exponents occur only in the very last step. This simplification will be very useful in what follows.

Lemma 3.1 ([24, p. 152]). *For any $P \in \mathbb{R}[X_1, \dots, X_k]$ and $a \in \mathbb{N}$, the polynomial P has additive complexity at most a if and only if there exists a sequence of equations*

(i) $$Q_1 = u_1 X_1^{\alpha_{11}} \cdots X_k^{\alpha_{1k}} + v_1 X_1^{\beta_{11}} \cdots X_k^{\beta_{1k}},$$

where $u_1, v_1 \in \mathbb{R}$, and $\alpha_{11}, \dots, \alpha_{1k}, \beta_{11}, \dots, \beta_{1k} \in \mathbb{N}$;

(ii) $$Q_j = u_j X_1^{\alpha_{j1}} \cdots X_k^{\alpha_{jk}} \prod_{1 \le i \le j-1} Q_i^{\gamma_{ji}} + v_j X_1^{\beta_{j1}} \cdots X_k^{\beta_{jk}} \prod_{1 \le i \le j-1} Q_i^{\delta_{ji}},$$

where $1 < j \le a$, $u_j, v_j \in \mathbb{R}$, and $\alpha_{j1}, \dots, \alpha_{jk}, \beta_{j1}, \dots, \beta_{jk}, \gamma_{ji}, \delta_{ji} \in \mathbb{N}$ for $1 \le i < j$; and

(iii) $$P = c X_1^{\zeta_1} \cdots X_k^{\zeta_k} \prod_{1 \le j \le a} Q_j^{\eta_j},$$

where $c \in \mathbb{R}$, and $\zeta_1, \dots, \zeta_k, \eta_1, \dots, \eta_a \in \mathbb{Z}$.

Remark 3.2. Observe that in Lemma 3.1 all exponents other than those in (iii) are in $\mathbb{N}$ rather than in $\mathbb{Z}$ (cf. Definition 1.8). Observe also that if a polynomial P satisfies the conditions of the lemma, then it has additive complexity at most a.

3.2. *The algebraic case*

Before proving Theorem 1.11 it is useful to first consider the algebraic case separately, since the main technical ingredients used in the proof of Theorem 1.11 are more clearly visible in this case. With this in mind, in this section we consider the algebraic case and prove the following theorem, deferring the proof in the general semi-algebraic case until the next section.

Theorem 3.3. *The number of homotopy types of* $\mathrm{Zer}(F, \mathbb{R}^k)$ *amongst all polynomials* $F \in \mathbb{R}[X_1, \ldots, X_k]$ *having additive complexity at most a does not exceed*

$$2^{O(k(k^2+a))^8} = 2^{(k+a)^{O(1)}}.$$

Before proving Theorem 3.3 we need a few preliminary results.

Proposition 3.4. *Let* $F, P, Q \in \mathbb{R}[\mathbf{X}]$ *be such that* $FQ = P$, $R \in \mathbb{R}_+$, *and define*

$$\mathbb{T} := \{(\mathbf{x}, t) \in \mathbb{R}^k \times \mathbb{R}_+ |\ P^2(\mathbf{x}) \leq t(Q^2(\mathbf{x}) - t^N) \wedge |\mathbf{x}|^2 \leq R^2\}, \tag{3.1}$$

where $N = 2\deg(Q) + 1$. *Then, using Notation 1.15,*

$$\mathbb{T}_{\mathrm{limit}} = \mathrm{Zer}(F, \mathbb{R}^k) \cap \overline{B_k(0, R)}.$$

Before proving Proposition 3.4 we first discuss an illustrative example.

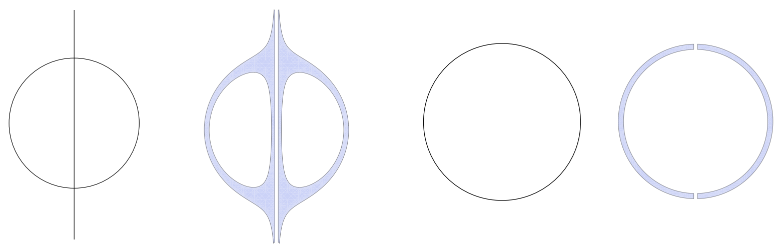
Example 3.5. Let

$$\begin{aligned} F_1 &= X(X^2 + Y^2 - 1), & F_2 &= X^2 + Y^2 - 1. \\ P_1 &= X^2(X^2 + Y^2 - 1), & P_2 &= X(X^2 + Y^2 - 1), \\ Q_1 &= Q_2 = X. \end{aligned}$$

For $i = 1, 2$ and $R > 0$, let

$$\mathbb{T}^i = \{(\mathbf{x}, t) \in \mathbb{R}^k \times \mathbb{R}_+ \mid P_i^2(\mathbf{x}) \leq t(Q_i^2(\mathbf{x}) - t^N) \wedge |\mathbf{x}|^2 \leq R^2\}$$

as in Proposition 3.4.



(a) (b) (c) (d)

Fig. 1: Two examples.

In Figure 1, we display, from left to right, $\mathrm{Zer}(F_1, \mathbb{R}^2)$, $\mathbb{T}^1_\varepsilon$, $\mathrm{Zer}(F_2, \mathbb{R}^2)$ and $\mathbb{T}^2_\varepsilon$ (where $\varepsilon = .005$ and $N = 3$). Notice that, for $i = 1, 2$ and any fixed $R > 0$, the semi-algebraic set $\mathbb{T}^i_\varepsilon$ approaches (in the sense of Hausdorff distance) the set $\mathrm{Z}(F_i, \mathbb{R}^2) \cap \overline{B_2(0, R)}$ as $\varepsilon \to 0$.

We now prove Proposition 3.4.

Proof of Proposition 3.4. We show both inclusions. First let $\mathbf{x} \in \mathbb{T}_{\mathrm{limit}}$; we will show that $F(\mathbf{x}) = 0$. In particular, we will prove that $0 \le F^2(\mathbf{x}) < \varepsilon$ for every $\varepsilon > 0$.

Let $\varepsilon > 0$. Since F^2 is continuous, there exists $\delta > 0$ such that

$$|\mathbf{x} - \mathbf{y}|^2 < \delta \;\Rightarrow\; |F^2(\mathbf{x}) - F^2(\mathbf{y})| < \varepsilon/2. \tag{3.2}$$

After possibly making δ smaller we can suppose that $\delta < \varepsilon^2/4$.

From the definition of $\mathbb{T}_{\mathrm{limit}}$ (cf. Notation 1.15), we have

$$\mathbb{T}_{\mathrm{limit}} = \{\mathbf{x} \mid (\forall\delta)(\delta > 0 \Rightarrow (\exists t)(\exists \mathbf{y})(\mathbf{y} \in \mathbb{T}_t \wedge |\mathbf{x} - \mathbf{y}|^2 + t^2 < \delta))\}. \tag{3.3}$$

Since $\mathbf{x} \in \mathbb{T}_{\mathrm{limit}}$, there exist $t \in \mathbb{R}_+$ and $\mathbf{y} \in \mathbb{T}_t$ such that $|\mathbf{x} - \mathbf{y}|^2 + t^2 < \delta$, and in particular both $|\mathbf{x} - \mathbf{y}|^2 < \delta$ and $t^2 < \delta < \varepsilon^2/4$. The former inequality implies that $|F^2(\mathbf{x}) - F^2(\mathbf{y})| < \varepsilon/2$. The latter inequality implies $t < \varepsilon/2$, and this together with $\mathbf{y} \in \mathbb{T}_t$ implies

$$\begin{aligned}
& P^2(\mathbf{y}) \le t(Q^2(\mathbf{y}) - t^N), \\
\text{so} \quad & F^2(\mathbf{y})Q^2(\mathbf{y}) \le t(Q^2(\mathbf{y}) - t^N), \\
\text{so} \quad & 0 \le F^2(\mathbf{y}) \le t - \frac{t^{N+1}}{Q^2(\mathbf{y})} < t, \\
\text{so} \quad & 0 \le F^2(\mathbf{y}) < \varepsilon/2.
\end{aligned}$$

Finally, note that $|F^2(\mathbf{x})| \le |F^2(\mathbf{x}) - F^2(\mathbf{y})| + |F^2(\mathbf{y})| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

We next prove the other inclusion, namely $\mathrm{Zer}(F, \mathbb{R}^k) \cap \overline{B_k(0, R)} \subseteq \mathbb{T}_{\mathrm{limit}}$. Let $\mathbf{x} \in \mathrm{Zer}(F, \mathbb{R}^k) \cap \overline{B_k(0, R)}$. We fix $\delta > 0$ and show that there exist $t \in \mathbb{R}_+$ and $\mathbf{y} \in \mathbb{T}_t$ such that $|\mathbf{x} - \mathbf{y}|^2 + t^2 < \delta$ (cf. (3.3)).

There are two cases to consider.

Case 1: $Q(\mathbf{x}) \neq 0$. Then there exists $t > 0$ such that $Q^2(\mathbf{x}) \ge t^N$ and $t^2 < \delta$. Now, $\mathbf{x} \in \mathbb{T}_t$ and

$$|\mathbf{x} - \mathbf{x}|^2 + t^2 = t^2 < \delta,$$

so setting $\mathbf{y} = \mathbf{x}$ we see that $\mathbf{y} \in \mathbb{T}_t$ and $|\mathbf{x} - \mathbf{y}| + t^2 < \delta$. Thus, $\mathbf{x} \in \mathbb{T}_{\mathrm{limit}}$ as desired.

Case 2: $Q(\mathbf{x}) = 0$. Let $\mathbf{v} \in \mathbb{R}^k$ be generic, and denote $\widehat{P}(U) = P(\mathbf{x} + U\mathbf{v})$, $\widehat{Q}(U) = Q(\mathbf{x} + U\mathbf{v})$, and $\widehat{F}(U) = F(\mathbf{x} + U\mathbf{v})$. Note that

$$\widehat{P} = \widehat{F}\widehat{Q}, \qquad \widehat{P}(0) = \widehat{Q}(0) = \widehat{F}(0) = 0. \tag{3.4}$$

If F is not the zero polynomial, then neither is $\widehat{P}$. Indeed, assume F is not identically zero, and hence P is not identically zero. In order to prove that $\widehat{P}$ is not identically zero for a generic choice of $\mathbf{v}$, write $P = \sum_{0 \le i \le d} P_i$ where P_i is the homogeneous part of P of degree i, and P_d is not identically zero. Then it is easy to see that $\widehat{P}(U) = P_d(\mathbf{v})U^d +$ (lower degree terms). Since $\mathbb{R}$ is an infinite field, a generic choice of $\mathbf{v}$ will avoid the set of zeros of P_d, and thus $\widehat{P}$ is not identically zero.

We further require that $\mathbf{x} + t\mathbf{v} \in B_k(0, R)$ for $t > 0$ sufficiently small. For generic $\mathbf{v}$, this is true for either $\mathbf{v}$ or $-\mathbf{v}$, and so after possibly replacing $\mathbf{v}$ by $-\mathbf{v}$ (and noticing that since P_d is homogeneous we have $P_d(\mathbf{v}) = (-1)^d P_d(-\mathbf{v})$) we may assume $\mathbf{x} + t\mathbf{v} \in B_k(0, R)$ for $t > 0$ sufficiently small, say for $0 < t < t_0$.

Denoting $\nu = \mathrm{mult}_0(\widehat{P})$ and $\mu = \mathrm{mult}_0(\widehat{Q})$, we see from (3.4) that $\nu > \mu$. Let

$$\widehat{P}(U) = \sum_{i=\nu}^{\deg_U \widehat{P}} c_i U^i = U^\nu \cdot \sum_{i=0}^{\deg_U \widehat{P} - \nu} c_{\nu+i} U^i = c_\nu U^\nu + \text{(higher order terms)},$$
$$\widehat{Q}(U) = \sum_{i=\mu}^{\deg_U \widehat{Q}} d_i U^i = U^\mu \cdot \sum_{i=0}^{\deg_U \widehat{Q} - \mu} d_{\mu+i} U^i = d_\mu U^\mu + \text{(higher order terms)},$$

where $c_\nu, d_\mu \neq 0$. Then

$$\begin{aligned}
\widehat{P}^2(U) &= c_\nu^2 U^{2\nu} + \text{(higher order terms)},\\
\widehat{Q}^2(U) &= d_\mu^2 U^{2\mu} + \text{(higher order terms)},\\
D(U) := U(\widehat{Q}^2(U) - U^N) &= U(d_\mu^2 U^{2\mu} + \text{(higher order terms)} - U^N),\\
D(U) - \widehat{P}^2(U) &= d_\mu^2 U^{2\mu+1} + \text{(higher order terms)} - U^{N+1}.
\end{aligned}$$

Since $\mu \le \deg(Q)$ and $N = 2\deg(Q) + 1$, we have $2\mu + 1 < N + 1$. Hence, there exists $t_1 \in \mathbb{R}_+$ such that for each t with $0 < t < t_1$, we have $D(t) - \widehat{P}^2(t) \ge 0$. Thus, $\mathbf{x} + t\mathbf{v} \in \mathbb{T}_t$ for each t with $0 < t < \min\{t_0, t_1\}$. Let $t_2 = \left(\frac{\delta}{|\mathbf{v}|^2+1}\right)^{1/2}$ and note that for all t with $0 < t < t_2$, we have $(|\mathbf{v}|^2 + 1)t^2 < \delta$. Finally, if $0 < t < \min\{t_0, t_1, t_2\}$ then $\mathbf{x} + t\mathbf{v} \in \mathbb{T}_t$, and

$$|\mathbf{x} - (\mathbf{x} + t\mathbf{v})|^2 + t^2 = (|\mathbf{v}|^2 + 1)t^2 < \delta.$$

Hence, setting $\mathbf{y} = \mathbf{x} + t\mathbf{v}$ (cf. (3.3)) we have shown that $\mathbf{x} \in \mathbb{T}_{\mathrm{limit}}$ as desired.

The case where F is the zero polynomial is straightforward. □

Proof of Theorem 3.3. For each $F \in \mathbb{R}[X_1, \ldots, X_k]$, by the conical structure at infinity of semi-algebraic sets (see for instance [4, p. 188]), there exists $R_F \in \mathbb{R}_+$ such that, for every $R > R_F$, the semi-algebraic sets $\mathrm{Zer}(F, \mathbb{R}^k) \cap \overline{B_k(0, R)}$ and $\mathrm{Zer}(F, \mathbb{R}^k)$ are semi-algebraically homeomorphic.

Let $\ell \in \mathbb{N}$ and let $F_1, \ldots, F_\ell \in \mathbb{R}[X_1, \ldots, X_k]$ be such that each F_i has additive complexity at most a and, for every F having additive complexity at most a, the algebraic sets $\mathrm{Zer}(F, \mathbb{R}^k)$, $\mathrm{Zer}(F_i, \mathbb{R}^k)$ are semi-algebraically homeomorphic for some i, $1 \le i \le \ell$ (see, for example [24, Theorem 3.5]). Let $R = \max_{1 \le i \le \ell}\{R_{F_i}\}$.

Let $F \in \{F_i\}_{1 \le i \le \ell}$. By Lemma 3.1 there exist $P, Q \in \mathbb{R}[X_1, \dots, X_k]$ such that $FQ = P$ and $P^2 - T(Q^2 - T^N) \in \mathbb{R}[X_1, \dots, X_k, T]$ has division-free additive complexity bounded by $a + 2$. Let

$$\mathbb{T} = \{(\mathbf{x}, t) \in \mathbb{R}^k \times \mathbb{R}_+ |\ P^2(\mathbf{x}) \le t(Q^2(\mathbf{x}) - t^N) \wedge |\mathbf{x}|^2 \le R^2\}.$$

By Proposition 3.4 we have $\mathbb{T}_{\text{limit}} = \text{Zer}(F, \mathbb{R}^k) \cap \overline{B_k(0, R)}$. Note that the one-parameter semi-algebraic family $\mathbb{T}$ (where the last coordinate is the parameter) is described by a formula having division-free additive format $(a + k + 2, k + 1)$.

By Theorem 2.1 we obtain a collection of semi-algebraic sets $\mathcal{S}_{k,a+k+2}$ such that $\mathbb{T}_{\text{limit}}$, and hence $\text{Zer}(F, \mathbb{R}^k)$, is homotopy equivalent to some $S \in \mathcal{S}_{k,a+k+2}$ and $\#\mathcal{S}_{k,a+k+2} = 2^{O(k(k^2+a))^8}$, which proves the theorem. □

3.3. The semi-algebraic case

We first prove a generalization of Proposition 3.4.

Notation 3.6. Let $\mathbf{X} = (X_1, \dots, X_k)$ be a block of variables and $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$ with $\sum_{j=1}^n k_j = k$. Let $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{R}^n$ with $r_j > 0$, $j = 1, \dots, n$. Let $B_{\mathbf{k}}(0, \mathbf{r})$ denote the product

$$B_{\mathbf{k}}(0, \mathbf{r}) := B_{k_1}(0, r_1) \times \cdots \times B_{k_n}(0, r_n).$$

Proposition 3.7. *Let $F_1, \dots, F_s, P_1, \dots, P_s, Q_1, \dots, Q_s \in \mathbb{R}[\mathbf{X}^1, \dots, \mathbf{X}^n]$, $\mathcal{P} = \{F_1, \dots, F_s\}$ such that $F_i Q_i = P_i$ for all $i = 1, \dots, s$. Suppose $\mathbf{X}^j = (X_1^j, \dots, X_{k_i}^j)$ and let $\mathbf{k} = (k_1, \dots, k_n)$. Suppose ϕ is a $\mathcal{P}$-formula containing no negations and no inequalities. Let*

$$\bar{P}_i := P_i \prod_{i' \ne i} Q_{i'}, \qquad \bar{Q} := \prod_i Q_i,$$

and let $\bar{\phi}$ denote the formula obtained from ϕ by replacing each $F_i = 0$ with

$$\bar{P}_i^2 - U(\bar{Q}^2 - U^N) \le 0,$$

where U is the last variable of $\bar{\phi}$, $N = 2\deg(\bar{Q}) + 1$. Then, for every $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{R}_+^n$, we have (cf. Notation 2.9 and Notation 1.15)

$$\text{Reali}\Big(\bigwedge_{j=1}^n (|\mathbf{X}^j|^2 \le r_j^2) \wedge \bar{\phi} \wedge U > 0\Big)_{\text{limit}} = \text{Reali}(\phi) \cap \overline{B_{\mathbf{k}}(0, \mathbf{r})}. \tag{3.5}$$

Proof. We follow the proof of Proposition 3.4. The only case which is not immediate is when $\mathbf{x} \in \text{Reali}(\phi) \cap \overline{B_{\mathbf{k}}(0, \mathbf{r})}$ and $\bar{Q}(\mathbf{x}) = 0$, so suppose this holds. Since ϕ is a formula containing no negations and no inequalities, it consists of conjunctions and disjunctions of equalities. Without loss of generality we can assume that ϕ is written as a disjunction of conjunctions, and still without negations. Let

$$\phi = \bigvee_\alpha \phi_\alpha$$

where ϕ_α is a conjunction of equations. As above let $\bar{\phi}_\alpha$ be the formula obtained from ϕ_α after replacing each $F_i = 0$ in ϕ_α with

$$\bar{P}_i^2 \le U(\bar{Q}^2 - U^N), \qquad N = 2\deg(\bar{Q}) + 1.$$

We have

$$\begin{aligned}
\mathrm{Reali}\Big(\bigwedge_{j=1}^n (|\mathbf{X}^j|^2 \le r_j^2) \wedge \bar{\phi} \wedge U > 0\Big)_{\mathrm{limit}}
&= \mathrm{Reali}\Big(\bigwedge_{j=1}^n (|\mathbf{X}^j|^2 \le r_j^2) \wedge \Big(\bigvee_\alpha \bar{\phi}_\alpha\Big) \wedge U > 0\Big)_{\mathrm{limit}}\\
&= \mathrm{Reali}\Big(\bigvee_\alpha \bigwedge_{j=1}^n (|\mathbf{X}^j|^2 \le r_j^2) \wedge \bar{\phi}_\alpha \wedge U > 0\Big)_{\mathrm{limit}}\\
&= \bigcup_\alpha \mathrm{Reali}\Big(\bigwedge_{j=1}^n (|\mathbf{X}^j|^2 \le r_j^2) \wedge \bar{\phi}_\alpha \wedge U > 0\Big)_{\mathrm{limit}}.
\end{aligned}$$

In order to show that $\mathbf{x} \in \mathrm{Reali}(\bigwedge_{j=1}^n (|\mathbf{X}^j|^2 \le r_j^2) \wedge \bar{\phi} \wedge U > 0)_{\mathrm{limit}}$ it now suffices to show that if $\mathbf{x} \in \mathrm{Reali}(\phi_\alpha) \cap \overline{B_{\mathbf{k}}(0, \mathbf{r})}$ and $\bar{Q}(\mathbf{x}) = 0$, then $\mathbf{x}$ belongs to $\mathrm{Reali}(\bigwedge_{j=1}^n (|\mathbf{X}^j|^2 \le r_j^2) \wedge \bar{\phi}_\alpha \wedge U > 0)_{\mathrm{limit}}$.

Let $\mathbf{x} \in \mathrm{Reali}(\phi_\alpha) \cap \overline{B_{\mathbf{k}}(0, \mathbf{r})}$ and suppose $\bar{Q}(\mathbf{x}) = 0$. Let $\mathcal{Q} \subseteq \mathcal{P}$ consist of the polynomials of $\mathcal{P}$ appearing in ϕ_α. Let $\mathbf{v} \in \mathbb{R}^k$ be generic, and set $\widehat{P}_i(U) = \bar{P}_i(\mathbf{x} + U\mathbf{v})$, $\widehat{Q}(U) = \bar{Q}(\mathbf{x} + U\mathbf{v})$, and $\widehat{F}_i(U) = \bar{F}(\mathbf{x} + U\mathbf{v})$. Note that

$$\widehat{P}_i = \widehat{F}_i \widehat{Q}, \qquad \widehat{P}_i(0) = \widehat{Q}(0) = \widehat{F}_i(0) = 0. \tag{3.6}$$

As in the proof of Proposition 3.4, if $F_i \in \mathcal{Q}$ is not the zero polynomial then $\widehat{P}_i$ is not identically zero. Since ϕ_α consists of a conjunction of equalities and

$$\bigwedge_{\substack{F \in \mathcal{Q}\\ F \not\equiv 0}} F = 0 \Rightarrow \bigwedge_{F \in \mathcal{Q}} F = 0,$$

we may assume that $\mathcal{Q}$ does not contain the zero polynomial. Under this assumption, for every $F_i \in \mathcal{Q}$ the univariate polynomial $\widehat{P}_i$ is not identically zero. As in the proof of Proposition 3.4, there exists $t_0 \in \mathbb{R}_+$ such that for all t with $0 < t < t_0$, we have $\mathbf{x} + t\mathbf{v} \in B_{\mathbf{k}}(0, \mathbf{r})$. Denoting $\nu_i = \mathrm{mult}_0(\widehat{P}_i)$ and $\mu = \mathrm{mult}_0(\widehat{Q})$, we infer from (3.6) that $\nu_i > \mu$ for all $i = 1, \ldots, s$.

Let

$$\widehat{P}_i(U) = \sum_{j=\nu_i}^{\deg_U \widehat{P}_i} c_j U^j = U^{\nu_i} \cdot \sum_{j=0}^{\deg_U \widehat{P}_i - \nu_i} c_{\nu_i + j} U^j = c_{\nu_i} U^{\nu_i} + (\text{higher order terms}),$$

$$\widehat{Q}(U) = \sum_{j=\mu}^{\deg_U \widehat{Q}} d_j^j = U^\mu \cdot \sum_{j=0}^{\deg_U \widehat{Q} - \mu} d_{\mu + j} U^j = d_\mu U^\mu + (\text{higher order terms}),$$

where $d_\mu \neq 0$ and $c_{\nu_i} \neq 0$. Then

$$\begin{aligned}\widehat{P}_i^2(U) &= c_{\nu_i}^2 U^{2\nu_i} + \text{(higher order terms)},\\ \widehat{Q}^2(U) &= d_\mu^2 U^{2\mu} + \text{(higher order terms)},\\ D(t) := U(\widehat{Q}^2(U) - U^N) &= U(d_\mu^2 U^{2\mu} + \text{(higher order terms)} - U^N),\\ D(U) - \widehat{P}_i^2(U) &= d_\mu^2 U^{2\mu+1} + \text{(higher order terms)} - U^{N+1}.\end{aligned}$$

Since $\mu \leq \deg(\bar{Q})$ and $N = 2\deg(\bar{Q}) + 1$, we have $2\mu + 1 < N + 1$. Hence, there exists $t_{1,i} \in \mathbb{R}_+$ such that for all t with $0 < t < t_{1,i}$, we have $D(t) - \widehat{P}_i^2(t) \geq 0$, and thus $\mathbf{x} + t\mathbf{v}$ satisfies

$$\bar{P}_i^2(\mathbf{x} + t\mathbf{v}) \leq t(\bar{Q}^2(\mathbf{x} + t\mathbf{v}) - t^N).$$

Let $t_1 = \min\{t_{1,1}, \ldots, t_{1,s}\}$. Let $t_2 = \left(\frac{\delta}{|\mathbf{v}|^2+1}\right)^{1/2}$ and note that for all $t \in \mathbb{R}$ with $0 < t < t_2$, we have $(|\mathbf{v}|^2 + 1)t^2 < \delta$. Finally, if $0 < t < \min\{t_0, t_1, t_2\}$ then

$$(\mathbf{x} + t\mathbf{v}, t) \in \operatorname{Reali}\Big(\bigwedge_{j=1}^{n} (|\mathbf{X}^j|^2 \leq r_j^2) \wedge \bar{\phi}_\alpha \wedge U > 0\Big)$$

and

$$|\mathbf{x} - (\mathbf{x} + t\mathbf{v})|^2 + t^2 \;=\; (|\mathbf{v}|^2)t^2 < \delta,$$

and so we have shown that

$$\mathbf{x} \in \operatorname{Reali}\Big(\bigwedge_{j=1}^{n} (|\mathbf{X}^j|^2 \leq r_j^2) \wedge \bar{\phi}_\alpha \wedge U > 0\Big)_{\text{limit}}. \qquad \square$$

Using the same notation as in Proposition 3.7 above, we have

Corollary 3.8. *Let ϕ be a $\mathcal{P}$-formula containing no negations and no inequalities, with $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k]$ with $\mathcal{P} \in \mathcal{A}_{k,a}$. Then there exists a family of polynomials $\mathcal{P}' \subset \mathbb{R}[X_1, \ldots, X_k, U]$ and a $\mathcal{P}'$-formula $\bar{\phi}$ satisfying (3.5) and such that $\mathcal{P}' \in \mathcal{A}^{\text{div-free}}_{k+1,(k+a)(a+2)}$.*

Proof. The proof is immediate from Lemma 3.1, Remark 1.14, and the definition of $\bar{\phi}$. □

Definition 3.9. Let Φ be a $\mathcal{P}$-formula with $\mathcal{P} \subseteq \mathbb{R}[\mathbf{X}_1, \ldots, \mathbf{X}_k]$. We say that Φ is a *$\mathcal{P}$-closed* formula if it contains no negations and all the inequalities in atoms of Φ are weak inequalities.

Let $\mathcal{P} = \{F_1, \ldots, F_s\} \subset \mathbb{R}[X_1, \ldots, X_k]$, and let Φ be a $\mathcal{P}$-closed formula.
For $R \in \mathbb{R}_+$, let Φ_R denote the formula $\Phi \wedge (|\mathbf{X}|^2 - R^2 \leq 0)$.

Let $\Phi^\dagger$ be the formula obtained from Φ by replacing each occurrence of the atom $F_i * 0, * \in \{=, \leq, \geq\}, i = 1, \ldots, s$, with

$$\begin{aligned} F_i - V_i^2 &= 0 \quad \text{if } * \in \{\leq\}, \\ -F_i - V_i^2 &= 0 \quad \text{if } * \in \{\geq\}, \\ F_i &= 0 \quad \text{if } * \in \{=\}, \end{aligned}$$

and for $R, R' \in \mathbb{R}_+$, let $\Phi^\dagger_{R,R'}$ denote the formula

$$\Phi^\dagger \wedge (U_1^2 + |\mathbf{X}|^2 - R^2 = 0) \wedge (U_2^2 + |\mathbf{V}|^2 - R'^2 = 0).$$

We have

Proposition 3.10.

$$\mathrm{Reali}(\Phi) = \pi_{[1,k]}(\mathrm{Reali}(\Phi^\dagger)),$$

and for all $0 < R \ll R'$,

$$\mathrm{Reali}(\Phi_R) = \pi_{[1,k]}(\mathrm{Reali}(\Phi^\dagger_{R,R'})).$$

Proof. Obvious. □

Note that, for $0 < R \ll R'$, $\pi_{[1,k]}|_{\mathrm{Reali}(\Phi^\dagger_{R,R'})}$ is a continuous semi-algebraic surjection onto $\mathrm{Reali}(\Phi_R)$. Let $\pi_{R,R'}$ denote the map $\pi_{[1,k]}|_{\mathrm{Reali}(\Phi^\dagger_{R,R'})}$.

Proposition 3.11. $\mathcal{J}^p_{\pi_{R,R'}}(\mathrm{Reali}(\Phi^\dagger_{R,R'}))$ *is* p*-equivalent to* $\pi_{[1,k]}(\mathrm{Reali}(\Phi^\dagger_{R,R'}))$*. Moreover, for any two formulas* Φ, Ψ*, the realizations* $\mathrm{Reali}(\Phi)$ *and* $\mathrm{Reali}(\Psi)$ *are homotopy equivalent if, for all* $1 \ll R \ll R'$,

$$\mathrm{Reali}(\mathcal{J}^p_{\pi_{R,R'}}(\Phi^\dagger_{R,R'})) \simeq \mathrm{Reali}(\mathcal{J}^p_{\pi_{R,R'}}(\Psi^\dagger_{R,R'}))$$

are homotopy equivalent for some $p > k$.

Proof. Immediate from Propositions 2.18, 2.4 and 3.10. □

Suppose that Φ has additive format bounded by (a, k), and suppose that the number of polynomials appearing Φ is s; without loss of generality we can assume that $s \leq k+a$ (see Remark 1.14). Then the sum of the additive complexities of the polynomials appearing in $\Phi^\dagger_{R,R'}$ is bounded by $3a+3s+2 \leq 3a+3(a+k)+2 \leq 6(k+a)$, and $\Phi^\dagger_{R,R'}$ has additive format bounded by $(6(k+a), 2k+a+2)$.

Consequently, the additive format of the formula

$$\Theta_1 \wedge \Theta_2^{\Phi^\dagger_{R,R'}} \wedge \Theta_3^{\pi_{R,R'}}$$

is bounded by (M, N), where

$$\begin{aligned} M &= (p+1)(6k+6a+1) + \binom{p+1}{2}(4k+2a+3), \\ N &= (p+1)(2k+a+3) + \binom{p+1}{2}. \end{aligned}$$

In the above, the estimates of Proposition 2.26 suffice, with (a,k) replaced by $(6(k+a), 2k+a+2)$. Now, Corollary 3.8 shows that there exists a $\mathcal{P}'$-formula

$$\overline{(\Theta_1 \wedge \Theta_2^{\Phi^\dagger_{R,R'}} \wedge \Theta_3^{\pi_{R,R'}})}$$

which satisfies (3.5) and whose *division-free* additive format is bounded by $((N+M)(M+2), N+1)$. Finally, let $\mathcal{J}^p_{\pi_{R,R'}}(\Phi^\dagger_{R,R'})^\star$ denote the formula, with last variable U,

$$\Omega^R \wedge \overline{(\Theta_1 \wedge \Theta_2^{\Phi^\dagger_{R,R'}} \wedge \Theta_3^{\pi_{R,R'}})} \wedge U > 0; \tag{3.7}$$

then the *division-free* additive format of $\mathcal{J}^p_{\pi_{R,R'}}(\Phi^\dagger_{R,R'})^\star$ is bounded by $(M', N+1)$, where

$$M' = (p+1)(2k+a+3) + (N+M)(M+2).$$

Note that $M' \leq 5M^2$.

We have shown the following:

Proposition 3.12. *Suppose that the sum of the additive complexities of F_i, $1 \leq i \leq s$, is bounded by a. Then the semi-algebraic sets* $\mathrm{Reali}(\mathcal{J}^p_{\pi_{R,R'}}(\Phi^\dagger_{R,R'})^\star)$ *can be defined by a $\mathcal{P}'$-formula with $\mathcal{P}' \in \mathcal{A}^{\text{div-free}}_{5M^2,N+1}$, where*

$$M = (p+1)(6k+6a+1) + 2\binom{p+1}{2}(4k+2a+3),$$
$$N = (p+1)(2k+a+3) + \binom{p+1}{2}.$$

Finally, we obtain

Proposition 3.13. *The number of homotopy types of semi-algebraic subsets of $\mathbb{R}^k$ defined by $\mathcal{P}$-closed formulas with $\mathcal{P} \in \mathcal{A}_{a,k}$ is bounded by $2^{(k(k+a))^{O(1)}}$.*

Proof. Let $\mathcal{P} \in \mathcal{A}_{a,k}$. By the conical structure at infinity of semi-algebraic sets (see, for instance [4, p. 188]) there exists $R_\mathcal{P} > 0$ such that, for all $R > R_\mathcal{P}$ and every $\mathcal{P}$-closed formula Φ, the semi-algebraic sets $\mathrm{Reali}(\Phi_R)$, $\mathrm{Reali}(\Phi)$ are semi-algebraically homeomorphic.

For each $a, k \in \mathbb{N}$, there are only finitely many semi-algebraic homeomorphism types of semi-algebraic sets described by a $\mathcal{P}$-formula having additive complexity at most (a,k) [24, Theorem 3.5]. Let $\ell \in \mathbb{N}$, $\mathcal{P}_i \in \mathcal{A}_{a,k}$, and Φ_i a $\mathcal{P}_i$-formula, $1 \leq i \leq \ell$, such that every semi-algebraic set described by a formula of additive complexity at most (a,k) is semi-algebraically homeomorphic to $\mathrm{Reali}(\Phi_i)$ for some i, $1 \leq i \leq \ell$. Let $R = \max_{1\leq i\leq \ell}\{R_{\mathcal{P}_i}\}$ and $R' \gg R$.

Let $\Phi \in \{\Phi_i\}_{1\leq i\leq \ell}$. By Proposition 3.11 it suffices to bound the number of homotopy types of the semi-algebraic sets $\mathrm{Reali}(\mathcal{J}^{k+1}_{\pi_{R,R'}}(\Phi^\dagger_{R,R'}))$. By Proposition 3.7,

$$\mathrm{Reali}(\mathcal{J}^{k+1}_{\pi_{R,R'}}(\Phi^\dagger_{R,R'})^\star)_{\text{limit}} = \mathrm{Reali}(\mathcal{J}^{k+1}_{\pi_{R,R'}}(\Phi^\dagger_{R,R'})).$$

By Proposition 3.12, the division-free additive format of the formula $\mathcal{J}^{k+1}_{\pi_{R,R'}}(\Phi^{\dagger}_{R,R'})^{\star}$ is bounded by $(2M, N)$, where $p = k + 1$. The proposition now follows immediately from Theorem 2.1. □

Proof of Theorem 1.11. Using the construction of Gabrielov and Vorobjov [14] one can reduce the case of arbitrary semi-algebraic sets to that of a closed bounded one, defined by a $\mathcal{P}$-closed formula, without changing asymptotically the complexity estimates (see for example [5]). The theorem then follows directly from Proposition 3.13 above. □

Proof of Theorem 1.16. The proof is identical to that of Theorem 2.1, except that we use Theorem 1.11 instead of Theorem 1.6. □

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