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Bubbling on boundary submanifolds for the Lin–Ni–Takagi problem at higher critical exponents

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Abstract. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We consider the equation $d^2\Delta u - u + u^{\frac{n-k+2}{n-k-2}} = 0$ in Ω, under zero Neumann boundary conditions, where d is a small positive parameter. We assume that there is a k-dimensional closed, embedded minimal submanifold K of $\partial\Omega$ which is nondegenerate, and a certain weighted average of sectional curvatures of $\partial\Omega$ is positive along K. Then we prove the existence of a sequence $d = d_j \to 0$ and a positive solution u_d such that

$$d^2|\nabla u_d|^2 \rightharpoonup S\delta_K \quad \text{as } d \to 0$$

in the sense of measures, where δ_K stands for the Dirac measure supported on K and S is a positive constant.

Keywords. Critical Sobolev exponent, blowing-up solutions, nondegenerate minimal submanifolds

1. Introduction and statement of main results

Let Ω be a bounded, smooth domain in $\mathbb{R}^n$, ν the outer unit normal to $\partial\Omega$ and $q > 1$. The semilinear Neumann elliptic problem

$$d^2\Delta u - u + u^q = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{1.1}$$

has been widely considered in the literature for more than 20 years. In 1988 Lin, Ni and Takagi [27] initiated the study of this problem for small values of d, motivated by the *shadow system* of the Gierer–Meinhardt model of biological pattern formation [20]. In that context, u roughly represents the (steady) concentration of an activating chemical of the process, which is thought to diffuse slowly in the region Ω, leaving patterns of high concentration such as small spots or narrow stripes.

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When $n = 2$ or $q < (n+2)/(n-2)$ the problem is *subcritical*, and a positive least energy solution u_d exists by a standard compactness argument. This solution corresponds to a minimizer for the the Raleigh quotient

$$Q_d(u) = \frac{d^2 \int_\Omega |\nabla u|^2 + \int_\Omega |u|^2}{(\int_\Omega |u|^{q+1})^{2/(q+1)}}. \tag{1.2}$$

In [26, 27, 39, 40] the authors described accurately the asymptotic behavior of u_d as $d \to 0$. This function attains its maximum at exactly one point p_d which lies on $\partial\Omega$. The asymptotic location of p_d gets further characterized as

$$H_{\partial\Omega}(p_d) \to \max_{p \in \partial\Omega} H_{\partial\Omega}(p),$$

where $H_{\partial\Omega}$ denotes the mean curvature of $\partial\Omega$. Moreover, the asymptotic shape of u_d is indeed highly concentrated around p_d:

$$u_d(x) \approx w(|x - p_d|/d), \tag{1.3}$$

where $w(|x|)$ is the unique positive, radially symmetric solution to the problem

$$\Delta w - w + w^p = 0 \quad \text{in } \mathbb{R}^n, \qquad \lim_{|x| \to \infty} w(x) = 0, \tag{1.4}$$

which decays exponentially. See also [13] for a short proof.

Construction of single and multiple *spike-layer patterns* for this problem in the subcritical case has been the object of many studies: see for instance [6, 7, 8, 10, 12, 13, 14, 21, 22, 24, 25, 29, 51] and the surveys [36, 37]. In particular, in [51] it was found that whenever one has a nondegenerate critical point p_0 of the mean curvature $H_{\partial\Omega}(p)$, a solution with a profile of the form (1.3) can be found with $p_d \to p_0$.

It is natural to look for solutions to problem (1.1) that exhibit concentration phenomena as $d \to 0$ not just at points but on higher dimensional sets.

Given a k-dimensional submanifold Γ of $\partial\Omega$ and assuming that either $k \geq n - 2$ or $q < \frac{n-k+2}{n-k-2}$, the question is whether there exists a solution u_d which near Γ looks like

$$u_d(x) \approx w(\mathrm{dist}(x, \Gamma)/d), \tag{1.5}$$

where now $w(|y|)$ denotes the unique positive, radially symmetric solution to the problem

$$\Delta w - w + w^p = 0 \quad \text{in } \mathbb{R}^{n-k}, \qquad \lim_{|y| \to \infty} w(|y|) = 0.$$

In [30, 33, 34, 35], the authors have established the existence of a solution with the profile (1.5) when either $\Gamma = \partial\Omega$ or Γ is an *embedded closed minimal submanifold* of $\partial\Omega$, which is in addition *nondegenerate* in the sense that its Jacobi operator is nonsingular (we recall the exact definitions in the next section). This phenomenon is actually quite subtle compared with concentration at points: existence can only be achieved along a sequence of values $d \to 0$. The parameter d must actually remain suitably away from certain values where resonance occurs, and the topological type of the solution changes:

unlike the point concentration case, the Morse index of these solutions is very large and grows as $d \to 0$.

It is natural to analyze the critical case $q = \frac{n+2}{n-2}$, namely the problem

$$d^2 \Delta u - u + u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{1.6}$$

The lack of compactness of Sobolev's embedding makes it harder to apply variational arguments. On the other hand, in [1, 48] it was proven that a nonconstant least energy solution u_d of (1.6), i.e. a minimizer of (1.2), exists, provided that d is sufficiently small. The behavior of u_d as $d \to 0$ has been clarified in the subsequent works [4, 38, 43]: as in the subcritical case, u_d concentrates, having a unique maximum point p_d which lies on $\partial\Omega$ with

$$H_{\partial\Omega}(p_d) \to \max_{p \in \partial\Omega} H_{\partial\Omega}(p).$$

Pohožaev's identity [41] yields nonexistence of positive solutions to problem (1.4) when $q = \frac{n+2}{n-2}$, and thus the concentration phenomenon must necessarily be different. Unlike the subcritical case, $u_d(p_d) \to \infty$ and the profile of u_d near p_d is given, for suitable $\mu_d \to 0$, by

$$u_d(x) \approx d^{(n-2)/2} w_{\mu_d}(|x - p_d|), \tag{1.7}$$

where $w_\mu(|x|)$ corresponds to a family of radial positive solutions of

$$\Delta w + w^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n, \tag{1.8}$$

namely

$$w_\mu(|x|) = \alpha_n \left(\frac{\mu}{\mu^2 + |x|^2} \right)^{(n-2)/2}, \qquad \alpha_n = (n(n-2))^{(n-2)/4}, \tag{1.9}$$

which, up to translations, correspond to all positive solutions of (1.8) (see [9]). The precise concentration rates μ_d are dimension dependent, and found in [5, 23, 43]. In particular $\mu_d \sim d^2$ for $n \geq 5$, so that $u_d(p_d) \sim d^{-(n-2)/2}$.

As in the subcritical case, construction and estimates for bubbling solutions to problem (1.6) have been broadly treated. In addition to the above references we refer the reader to [2, 3, 17, 18, 19, 28, 32, 42, 44, 45, 46, 49, 50, 52].

In particular, in [3] it was found that for $n \geq 6$ and a nondegenerate critical point p_0 of the mean curvature with $H_{\partial\Omega}(p_0) > 0$, there exists a solution whose profile near p_0 is given by

$$u_d(x) \approx d^{(n-2)/2} w_{\mu_d}(|x - p_0|), \qquad \mu_d = a_n H_{\partial\Omega}(p_0)^{1/(n-2)} d^2, \tag{1.10}$$

for a certain explicit constant $a_n > 0$. See also [42, 43] for the lower dimensional case. The condition of critical point for $H_{\partial\Omega}$ with $H_{\partial\Omega}(p_0) > 0$ turns out to be necessary for the boundary bubbling phenomenon to take place (see [5, 23]).

The concentration phenomenon in the critical scenario is more degenerate than in the subcritical case, and its features are harder to detect because of the rather subtle role of the scaling parameter μ. The purpose of this paper is to unveil the corresponding analog of a solution like (1.10) for the k-dimensional concentration question, in the so far open critical case of the *k-th critical exponent* $q = \frac{n-k+2}{n-k-2}$, that is, for the problem

$$d^2\Delta u - u + u^{\frac{n-k+2}{n-k-2}} = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{1.11}$$

Notice that for the Dirichlet problem, solutions concentrating along boundary geodesics near the second critical exponent have been considered by del Pino, Musso and Pacard [16].

Let K be k-dimensional embedded submanifold of $\partial\Omega$. Under suitable assumptions we shall find a solution $u_d(x)$ which, for points $x \in \mathbb{R}^n$ near K,

$$x = p + z, \qquad p \in K, \qquad |z| = \operatorname{dist}(x, K),$$

can be described as

$$u_d(x) \approx d^{(n-k-2)/2} w_{\mu_d}(|z|), \qquad \mu_d(p) = a_{n-k}\bar{H}(p)^{1/(n-k-2)} d^2, \tag{1.12}$$

where now

$$w_\mu(|z|) = \alpha_{n-k}\left(\frac{\mu}{\mu^2 + |z|^2}\right)^{(n-k-2)/2}.$$

The form of the quantity $\bar{H}(p)$ is of course not obvious. It turns out to correspond to a weighted average of sectional curvatures of $\partial\Omega$ along K, which we shall need to assume positive. To explain what it is we need some notation.

We denote as usual by $T_p\partial\Omega$ the tangent space to $\partial\Omega$ at the point p. We consider the *shape operator* $\mathtt{L} : T_p\partial\Omega \to T_p\partial\Omega$ defined as

$$\mathtt{L}[e] := -\nabla_e \nu(p),$$

where $\nabla_e \nu(p)$ is the directional derivative of the vector field ν in the direction e. Let us consider the orthogonal decomposition

$$T_p\partial\Omega = T_pK \oplus N_pK,$$

where N_pK stands for the normal bundle of K. We choose orthonormal bases $(e_a)_{a=1}^k$ of T_pK and $(e_i)_{i=k+1}^{n-1}$ of N_pK.

Let us consider the $(n-1)\times(n-1)$ matrix $H(p)$ representing $\mathtt{L}$ in these bases, i.e.

$$H_{\alpha\beta}(p) = e_\alpha \cdot \mathtt{L}[e_\beta].$$

This matrix also represents the second fundamental form of $\partial\Omega$ at p in this basis. $H_{\alpha\alpha}(p)$ corresponds to the curvature of $\partial\Omega$ in the direction e_α. By definition, the mean curvature of $\partial\Omega$ at p is given by the trace of this matrix,

$$H_{\partial\Omega}(p) = \sum_{j=1}^{n-1} H_{jj}(p).$$

In order to state our result we need to consider the mean of the curvatures in the directions of T_pK and N_pK, i.e. the numbers $\sum_{i=1}^{k} H_{ii}(p)$ and $\sum_{j=k+1}^{n-1} H_{jj}(p)$.

Theorem 1. *Assume that $\partial\Omega$ contains a closed, embedded, nondegenerate minimal submanifold K of dimension $k \geq 1$ with $n-k \geq 7$ such that*

$$\bar{H}(p) := 2\sum_{a=1}^{k} H_{aa}(p) + \sum_{j=k+1}^{n-1} H_{jj}(p) > 0 \quad \textit{for all } p \in K. \tag{1.13}$$

Then, for a sequence $d = d_j \to 0$, problem (1.11) has a positive solution u_d concentrating along K in the sense that expansion (1.12) holds as $d \to 0$ and moreover

$$d^2|\nabla u_d|^2 \rightharpoonup S_{n-k}\delta_K \quad \textit{as } d \to 0,$$

where δ_K stands for the Dirac measure supported on K, and S_{n-k} is an explicit positive constant.

Condition (1.13) is new and unexpected. It is worth noticing that it can be rewritten as

$$2H_{\partial\Omega}(p) - \sum_{j=k+1}^{n-1} H_{jj}(p) > 0 \quad \text{for all } p \in K.$$

Formally in the case of point concentration, i.e. $k = 0$, this reduces precisely to $H_{\partial\Omega}(p) > 0$, which is exactly the condition known to be necessary for point concentration. We suspect that this condition is essential for the phenomenon to take place. On the other hand, while the high codimension assumption $n-k \geq 7$ is important in our proof, we expect that a similar phenomenon holds provided just $n-k \geq 5$, and with a suitable change in the bubbling scales for $n-k \geq 3$ (the difference of rates is formally due to the fact that $\int_{\mathbb{R}^n} w_\mu^2$ is finite if and only if $n \geq 5$).

It will be convenient to rewrite problem (1.11) in an equivalent form: Set $N = n-k$ and $d^2 = \varepsilon$. Define

$$u(x) = \varepsilon^{-(N-2)/4} v(\varepsilon^{-1}x).$$

Then, setting $\Omega_\varepsilon := \varepsilon^{-1}\Omega$, problem (1.11) becomes

$$\begin{cases} \Delta v - \varepsilon v + v^{\frac{N+2}{N-2}} = 0 & \text{in } \Omega_\varepsilon, \\ \partial v/\partial \nu = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{1.14}$$

The proof of the theorem has as a main ingredient the construction of an approximate solution with arbitrary degree of accuracy in powers of ε, in a neighborhood of the manifold $K_\varepsilon = \varepsilon^{-1}K$. Later we build the desired solution by linearizing equation (1.14) around this approximation. The associated linear operator turns out to be invertible with inverse controlled in a suitable norm by a certain large negative power of ε, provided that ε remains away from certain critical values where resonance occurs. The interplay of the size of the error and the size of the inverse of the linearization then makes a fixed point scheme possible.

The accurate approximate solution to (1.14) is built by using an iterative scheme of Picard's type which we describe in general next.

Observe that the desired asymptotic behavior (1.12) translates in terms of v as

$$v(x) \approx \mu_0(\varepsilon z)^{-(N-2)/2} w_0(\mu_0^{-1}(\varepsilon z)|\zeta|), \quad x = z + \zeta, \quad z \in K_\varepsilon, \quad |\zeta| = \operatorname{dist}(x, K_\varepsilon), \tag{1.15}$$

where

$$\mu_0(y) = a_N \bar{H}(y)^{1/(N-2)}, \qquad y = \varepsilon z \in K.$$

Here and in what follows, w_0 designates the standard bubble,

$$w_0(\xi) = w_0(|\xi|) = \alpha_N \left(\frac{1}{1 + |\xi|^2} \right)^{(N-2)/2}, \qquad \alpha_N = (N(N-2))^{(N-2)/4}. \tag{1.16}$$

We introduce the so-called Fermi coordinates on a neighborhood of $K_\varepsilon := \varepsilon^{-1} K$, as a suitable tool to describe the approximation (1.15). They are defined as follows (we refer to Subsection 2.2 for further details): we parameterize a neighborhood of K_ε using the exponential map in $\partial\Omega_\varepsilon$,

$$\begin{aligned} K_\varepsilon \times \mathbb{R}^{N-1} \times \mathbb{R}_+ \ni (z, \bar{X}, X_N) &\mapsto \Upsilon(z, \overline{X}, X_N) \\ &:= \exp_z^{\partial\Omega_\varepsilon}\Big(\sum_{i=1}^{N-1} X_i E_i \Big) - X_N \nu \Big(\exp_z^{\partial\Omega_\varepsilon}\Big(\sum_{i=1}^{N-1} X_i E_i \Big) \Big). \end{aligned}$$

Here the vector fields $E_i(z)$ represent an orthonormal basis of $N_z K_\varepsilon$. Thus, (1.15) corresponds to the statement that after expressing v in these coordinates we get

$$v(z, \bar{X}, X_N) \approx \mu_0(\varepsilon z)^{-(N-2)/2} w_0(\mu_0^{-1}(\varepsilon z)(\bar{X}, X_N)),$$

where suitable corrections need to be introduced if we want further accuracy on the induced error: we consider a positive smooth function $\mu_\varepsilon = \mu_\varepsilon(p)$ defined on K, a smooth function $\Phi_\varepsilon : K \to \mathbb{R}^{N-1}$, and the change of variables (with some abuse of notation)

$$v(z, \overline{X}, X_N) = \mu_\varepsilon^{-(N-2)/2}(\varepsilon z) W\big(\varepsilon^{-1} y, \mu_\varepsilon^{-1}(\varepsilon z)(\overline{X} - \Phi_\varepsilon(\varepsilon z)), \mu_\varepsilon^{-1}(\varepsilon z) X_N\big), \tag{1.17}$$

with the new W being a function

$$W(z, \xi), \qquad z = \frac{y}{\varepsilon}, \qquad \overline{\xi} = \frac{\overline{X} - \Phi_\varepsilon}{\mu_\varepsilon}, \qquad \xi_N = \frac{X_N}{\mu_\varepsilon}.$$

We will formally expand $W(z, \xi)$ in powers of ε starting with $w_0(\xi)$, with the functions $\Phi_\varepsilon(y)$ and $\mu_\varepsilon(y)$ correspondingly expanded.

Substituting into the equation, we will arrive formally at linear equations satisfied by the successive remainders of $w_0(\xi)$ (as functions of ξ). These linear equations involve the basic linearized operator $\mathcal{L} := -\Delta - p w_0^{p-1}$.

The bounded solvability of the linear equations at each step of the iteration is guaranteed by imposing orthogonality conditions on their right-hand sides, with respect to ker $\mathcal{L}$ in $L^\infty(\mathbb{R}^N)$. These orthogonality conditions amount to choices of the coefficients of an expansions of μ_ε and Φ_ε: for the latter, the equations involve the Jacobi operator of K and it is where the nondegeneracy assumption is used. The coefficients for the expansion of $\mu_\varepsilon(\varepsilon z)$ come from algebraic relations, in particular an orthogonality condition in the first iteration yields

$$\mu_0(y) := a_N\Big[2\sum_{j=1}^{k} H_{jj}(y) + \sum_{i=k+1}^{N+k-1} H_{ii}(y)\Big], \qquad y \in K.$$

This is exactly where the sign condition (1.13) in the theorem appears.

The rest of the paper is organized as follows. We first introduce some notation and conventions. Next, we collect some notions in differential geometry, like the Fermi coordinates (geodesic normal coordinates) near a minimal submanifold, and we expand the coefficients of the metric near these Fermi coordinates. In Section 3 we expand the Laplace–Beltrami operator. Section 4 will be mainly devoted to the construction of an approximate solution to our problem using the local coordinates around the submanifold K introduced before. In Section 5 we define the approximation globally and we write the solution to our problem as the sum of the global approximation plus a remainder term. Thus we express our original problem as a nonlinear problem for the remainder term. To solve the latter, we need to understand the invertibility properties of a linear operator. To do so we start by expanding a quadratic functional associated to the linear problem. In Section 6 we develop a linear theory to study our problem. Then we turn to the proof of our main theorem in Section 7. Sections 8 and 9 are appendices, to which we postpone the proofs of some technical facts to facilitate the reading of the paper.

2. Geometric setting

In this section we first introduce Fermi coordinates near a k-dimensional submanifold of $\partial\Omega \subset \mathbb{R}^n$ (with $n = N + k$) and we expand the coefficients of the metric in these coordinates. Then, we recall some basic notions about minimal and nondegenerate submanifolds.

2.1. *Notation and conventions*

Dealing with coordinates, Greek letters like $\alpha, \beta, \dots,$ will denote indices varying between 1 and $n-1$, while capital letters like $A, B, \dots$ will vary between 1 and n; Roman letters like a or b will run from 1 to k, while indices like $i, j, \dots$ will run between 1 and $N-1 := n-k-1$.

$\xi_1, \dots, \xi_{N-1}, \xi_N$ will denote coordinates in $\mathbb{R}^N = \mathbb{R}^{n-k}$, and they will also be written as $\bar{\xi} = (\xi_1, \dots, \xi_{N-1})$, $\xi = (\bar{\xi}, \xi_N)$.

The manifold K will be parameterized with coordinates $y = (y_1, \dots, y_k)$. Its dilation $K_\varepsilon := (1/\varepsilon)K$ will be parameterized by coordinates $z = (z_1, \dots, z_k)$ related to the y's simply by $y = \varepsilon z$.

Derivatives with respect to the variables y, z or ξ will be denoted by ∂_y, ∂_z, ∂_ξ, and for brevity we might sometimes use the symbols ∂_a, $\partial_{\overline{a}}$ and ∂_i for ∂_{y_a}, ∂_{z_a} and ∂_{ξ_i} respectively. In a local system of coordinates, $(\overline{g}_{\alpha\beta})_{\alpha\beta}$ are the components of the metric on $\partial\Omega$ naturally induced by $\mathbb{R}^n$. Similarly, $(\overline{g}_{AB})_{AB}$ are the entries of the metric on Ω in a neighborhood of the boundary. $(H_{\alpha\beta})_{\alpha\beta}$ will denote the components of the mean curvature operator of $\partial\Omega$ into $\mathbb{R}^n$.

2.2. Fermi coordinates on $\partial\Omega$ near K and expansion of the metric

Let K be a k-dimensional submanifold of $(\partial\Omega, \overline{g})$ $(1 \le k \le N-1)$. We choose along K a local orthonormal frame field $((E_a)_{a=1}^k, (E_i)_{i=1}^{N-1})$ which is oriented. At points of K, we have the natural splitting

$$T\partial\Omega = TK \oplus NK,$$

where TK is the tangent space to K and NK represents the normal bundle; they are spanned respectively by $(E_a)_a$ and $(E_j)_j$.

We denote by ∇ the connection induced by the metric $\overline{g}$ and by ∇^N the corresponding normal connection on the normal bundle. Given $p \in K$, we use some geodesic coordinates y centered at p. We also assume that the normal vectors $(E_i)_{i=1}^n$ are transported parallel (with respect to ∇^N) along geodesics from p, so in particular

$$\overline{g}(\nabla_{E_a} E_j, E_i) = 0 \quad \text{at } p, \quad i, j = 1, \dots, n, \ a = 1, \dots, k. \tag{2.1}$$

In a neighborhood of p in K, we consider normal geodesic coordinates

$$f(y) := \exp_p^K(y_a E_a), \qquad y := (y_1, \dots, y_k),$$

where $\exp^K$ is the exponential map on K and summation over repeated indices is understood. This yields the coordinate vector fields $X_a := f_*(\partial_{y_a})$. We extend the E_i along each $\gamma_E(s)$ so that they are parallel with respect to the induced connection on the normal bundle NK. This yields an orthonormal frame field X_i for NK in a neighborhood of p in K which satisfies

$$\nabla_{X_a} X_i|_p \in T_p K.$$

A coordinate system in a neighborhood of p in $\partial\Omega$ is now defined by

$$F(y, \bar{x}) := \exp_{f(y)}^{\partial\Omega}(x_i X_i), \qquad (y, \bar{x}) := (y_1, \dots, y_k, x_1, \dots, x_{N-1}), \tag{2.2}$$

with corresponding coordinate vector fields

$$X_i := F_*(\partial_{x_i}) \quad \text{and} \quad X_a := F_*(\partial_{y_a}).$$

By our choice of coordinates, on K the metric $\overline{g}$ splits as

$$\overline{g}(q) = \overline{g}_{ab}(q) dy_a \otimes dy_b + \overline{g}_{ij}(q) dx_i \otimes dx_j, \qquad q \in K. \tag{2.3}$$

We denote by $\Gamma_a^b(\cdot)$ the 1-forms defined on the normal bundle NK by

$$\overline{g}_{bc}\Gamma_{ai}^c := \overline{g}_{bc}\Gamma_a^c(X_i) = \overline{g}(\nabla_{X_a} X_b, X_i) \quad \text{at } q = f(y). \tag{2.4}$$

Notice that

$$K \text{ is minimal } \Leftrightarrow \sum_{a=1}^{k} \Gamma_a^a(E_i) = 0 \text{ for any } i = 1, \ldots, N-1. \tag{2.5}$$

Define $q = f(y) = F(y, 0) \in K$ and let $(\tilde{g}_{ab}(y))$ be the induced metric on K.

When we consider the metric coefficients in a neighborhood of K, we obtain a deviation from formula (2.3), which is expressed by the next lemma. The proof follows the same ideas as in [31, Proposition 2.1] but we give it here for completeness. See also the book [47]. We will denote by $R_{\alpha\beta\gamma\delta}$ the components of the curvature tensor with lowered indices, which are obtained from the usual ones $R^{\sigma}_{\beta\gamma\delta}$ by means of

$$R_{\alpha\beta\gamma\delta} = \overline{g}_{\alpha\sigma} R^{\sigma}_{\beta\gamma\delta}. \tag{2.6}$$

Lemma 2.1. *At the point $F(y, \bar{x})$, for any $a = 1, \ldots, k$ and any $i, j = 1, \ldots, N-1$, we have*

$$\begin{aligned}
\overline{g}_{ij} &= \delta_{ij} + \tfrac{1}{3} R_{istj} x_s x_t + O(|x|^3);\\
\overline{g}_{aj} &= O(|x|^2);\\
\overline{g}_{ab} &= \tilde{g}_{ab} - \{\tilde{g}_{ac}\Gamma^c_{bi} + \tilde{g}_{bc}\Gamma^c_{ai}\} x_i + [R_{sabl} + \tilde{g}_{cd}\Gamma^c_{as}\Gamma^d_{bl}] x_s x_l + O(|x|^3).
\end{aligned}$$

Here R_{istj} (see (2.6)) are computed at the point of K parameterized by $(y, 0)$.

Proof. The Fermi coordinates above are defined so that the metric coefficient

$$g_{\alpha\beta} = g(X_\alpha, X_\beta)$$

is equal to $\delta_{\alpha\beta}$ at $p = F(0, 0)$, and $g_{ab} = \tilde{g}_{ab}(y)$ at $q = F(y, 0)$; furthermore, $g(X_a, X_i) = 0$ in some neighborhood of q in K. Taylor expansion of the metric $\overline{g}_{\alpha\beta}(\overline{x}, y)$ at q gives

$$\begin{aligned}
\overline{g}_{\alpha\beta} &= \overline{g}(X_\alpha, X_\beta)|_q + X_j \overline{g}(X_\alpha, X_\beta)|_q x_j + O(|x|^2)\\
&= \overline{g}(X_\alpha, X_\beta)|_q + \overline{g}(\nabla_{X_j} X_\alpha, X_\beta)|_q x_j + \overline{g}(\nabla_{X_j} X_\beta, X_\alpha)|_q x_j + O(|x|^2).
\end{aligned}$$

Since $\overline{g}(X_b, X_i) = 0$ in a neighborhood of q, we have

$$0 = X_b \overline{g}(X_i, X_a) = \overline{g}(\nabla_{X_b} X_i, X_a) + \overline{g}(X_i, \nabla_{X_b} X_a) = \overline{g}(\nabla_{X_i} X_b, X_a) + \overline{g}(X_i, \nabla_{X_b} X_a).$$

This implies in particular that

$$\overline{g}(\nabla_{X_i} X_b, X_a) = -\overline{g}(X_i, \nabla_{X_b} X_a) = -\Gamma^c_{ai}\tilde{g}_{cb}.$$

Then at first order we have

$$\begin{aligned}
\overline{g}_{ab} &= \overline{g}(X_a, X_b)|_q + \overline{g}(\nabla_{X_j} X_a, X_b)|_q x_j + \overline{g}(\nabla_{X_j} X_b, X_a)|_q x_j + O(|x|^2)\\
&= \tilde{g}_{ab} - (\Gamma^c_{ai}\tilde{g}_{cb} + \Gamma^c_{bi}\tilde{g}_{ca}) x_i + O(|x|^2).
\end{aligned}$$

Similarly using (2.1) we get

$$\overline{g}_{ai} = \overline{g}(X_a, X_i)|_q + \overline{g}(\nabla_{X_j} X_a, X_i)|_q x_j + \overline{g}(\nabla_{X_j} X_i, X_a)|_q x_j + O(|x|^2) = O(|x|^2).$$

On the other hand, since every vector field $X \in N_q K$ is tangent to the geodesic $s \mapsto \exp_q^{\partial\Omega}(sX)$, we have

$$\nabla_{X_\ell + X_j}(X_\ell + X_j)|_q = 0.$$

This clearly implies that

$$(\nabla_{X_\ell} X_j + \nabla_{X_j} X_\ell)|_q = 0.$$

Then the following expansion holds:

$$\overline{g}_{ij} = \overline{g}(X_i, X_j)|_q + \overline{g}(\nabla_{X_l} X_i, X_j)|_q x_l + \overline{g}(\nabla_{X_l} X_j, X_i)|_q x_l + O(|x|^2) = \delta_{ij} + O(|x|^2).$$

To compute the terms of order two in the Taylor expansion it suffices to compute $X_k X_k \overline{g}_{\alpha\beta}$ at q and polarize (i.e. replace X_k by $X_i + X_j$). We have

$$X_k X_k \overline{g}_{\alpha\beta} = \overline{g}(\nabla^2_{X_k} X_\alpha, X_\beta) + \overline{g}(X_\alpha, \nabla^2_{X_k} X_\beta) + 2\overline{g}(\nabla_{X_k} X_\alpha, \nabla_{X_k} X_\beta). \tag{2.7}$$

Now, using the fact every normal vector $X \in N_q K$ is tangent to the geodesic $s \mapsto \exp_q^{\partial\Omega}(sX)$, we see that

$$\nabla_X X|_q = \nabla^2_X X|_q = 0$$

for every $X \in N_q K$. In particular, choosing $X = X_k + \varepsilon X_j$, we obtain

$$0 = \nabla_{X_k + \varepsilon X_j} \nabla_{X_k + \varepsilon X_j}(X_k + \varepsilon X_j)||_q$$

for every ε, which implies $\nabla_{X_j} \nabla_{X_k} X_k|_p = -2\nabla_{X_k} \nabla_{X_k} X_j|_p$, and hence

$$3\nabla^2_{X_k} X_j|_q = R(X_k, X_j) X_k|_q.$$

We then deduce from (2.7) that

$$X_k X_k \overline{g}_{ij}|_q = \tfrac{2}{3}\overline{g}(R(X_k, X_i) X_k, X_j)|_q.$$

On the other hand,

$$\nabla^2_{X_k} X_\gamma = \nabla_{X_k} \nabla_{X_\gamma} X_k = \nabla_{X_\gamma} \nabla_{X_k} X_k + R(X_k, X_\gamma) X_k.$$

Hence

$$\begin{aligned} X_k X_k g_{ab} &= 2\overline{g}(R(X_k, X_a) X_k, X_b) + 2\overline{g}(\nabla_{X_k} X_a, \nabla_{X_k} X_b) \\ &\quad + \overline{g}(\nabla_{X_a} \nabla_{X_k} X_k, X_b) + \overline{g}(X_a, \nabla_{X_b} \nabla_{X_k} X_k). \end{aligned}$$

Now using the fact that $\nabla_X X = 0|_q$ at $q \in K$ for every $X \in N_q K$, the definition of Γ^c_{ak} in (2.4) and the formula

$$R(X_k, X_a) X_l = R^\gamma_{kal} X_\gamma,$$

we deduce that at the point q,

$$\begin{aligned} X_k X_k \overline{g}_{ab}|_q &= 2\overline{g}(R(X_k, X_a)X_k, X_b) + 2\tilde{g}_{cd}\Gamma^c_{ak}\Gamma^d_{bk} = 2R^c_{kak}\overline{g}(X_c, X_b) + 2\tilde{g}_{cd}\Gamma^c_{ak}\Gamma^d_{bk} \\ &= 2R^c_{kak}\,\tilde{g}_{cb} + 2\tilde{g}_{cd}\Gamma^c_{ak}\Gamma^d_{bk} = 2R_{kabk} + 2\tilde{g}_{cd}\Gamma^c_{ak}\Gamma^d_{bk}. \end{aligned}$$

This proves the lemma. □

Next we introduce a parameterization of a neighborhood in Ω of $q \in \partial\Omega$ via the map Υ given by

$$\Upsilon(y, x) = F(y, \bar{x}) + x_N \nu(y, \bar{x}), \qquad x = (\bar{x}, x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}, \tag{2.8}$$

where F is the parameterization introduced in (2.2) and $\nu(y, \bar{x})$ is the inner unit normal to $\partial\Omega$ at $F(y, \bar{x})$. We have

$$\frac{\partial \Upsilon}{\partial y_a} = \frac{\partial F}{\partial y_a}(y, \bar{x}) + x_N \frac{\partial \nu}{\partial y_a}(y, \bar{x}), \qquad \frac{\partial \Upsilon}{\partial x_i} = \frac{\partial F}{\partial x_i}(y, \bar{x}) + x_N \frac{\partial \nu}{\partial x_i}(y, \bar{x}).$$

Let us define the tensor matrix H by

$$d\nu_x[v] = -H(x)[v]. \tag{2.9}$$

We thus find

$$\frac{\partial \Upsilon}{\partial y_a} = [\mathrm{Id} - x_N H(y, \bar{x})]\frac{\partial F}{\partial y_a}(y, \bar{x}), \tag{2.10}$$

$$\frac{\partial \Upsilon}{\partial x_i} = [\mathrm{Id} - x_N H(y, \bar{x})]\frac{\partial F}{\partial x_i}(y, \bar{x}). \tag{2.11}$$

Differentiating Υ with respect to x_N we also get

$$\frac{\partial \Upsilon}{\partial x_N} = \nu(y, \bar{x}). \tag{2.12}$$

Hence, letting $g_{\alpha\beta}$ be the coefficients of the flat metric g of $\mathbb{R}^{N+k}$ in the coordinates $(y, \bar{x}, x_N)$, by easy computations we deduce for $\tilde{y} = (y, \bar{x})$ that

$$g_{\alpha\beta}(\tilde{y}, x_N) = \overline{g}_{\alpha\beta}(\tilde{y}) - x_N(H_{\alpha\delta}\overline{g}_{\delta\beta} + H_{\beta\delta}\overline{g}_{\delta\alpha})(\tilde{y}) + x_N^2 H_{\alpha\delta}H_{\sigma\beta}\overline{g}_{\delta\sigma}(\tilde{y}); \tag{2.13}$$

$$g_{\alpha N} \equiv 0; \qquad g_{NN} \equiv 1. \tag{2.14}$$

In the above expressions, α and β denote any index of the form $a = 1, \ldots, k$ or $i = 1, \ldots, N-1$.

We first provide a Taylor expansion of the coefficients of the metric g. From Lemma 2.1 and formula (2.13) we immediately obtain the following result.

Lemma 2.2. *For the (Euclidean) metric g in the above coordinates we have the expansions*

$$\begin{aligned} g_{ij} &= \delta_{ij} - 2x_N H_{ij} + \tfrac{1}{3} R_{istj} x_s x_t + x_N^2 (H^2)_{ij} + O(|x|^3), \qquad 1 \le i, j \le N-1; \\ g_{aj} &= -x_N (H_{aj} + \tilde g_{ac} H_{cj}) + O(|x|^2), \qquad 1 \le a \le k, 1 \le j \le N-1; \\ g_{ab} &= \tilde g_{ab} - \{\tilde g_{ac} \Gamma_{bi}^c + \tilde g_{bc} \Gamma_{ai}^c\} x_i - x_N \{H_{ac} \tilde g_{bc} + H_{bc} \tilde g_{ac}\} + [R_{sabl} + \tilde g_{cd} \Gamma_{as}^c \Gamma_{dl}^b] x_s x_l \\ &\quad + x_N^2 (H^2)_{ab} + x_N x_k \big[H_{ac}\{\tilde g_{bf} \Gamma_{ck}^f + \tilde g_{cf} \Gamma_{bk}^f\} + H_{bc}\{\tilde g_{af} \Gamma_{ck}^f + \tilde g_{cf} \Gamma_{ak}^f\}\big] + O(|x|^3), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \le a, b \le k; \end{aligned}$$

$$g_{aN} \equiv 0, \quad a = 1, \ldots, k; \qquad g_{iN} \equiv 0, \quad i = 1, \ldots, N-1; \qquad g_{NN} \equiv 1.$$

In the above expressions $H_{\alpha\beta}$ denotes the components of the matrix tensor H defined in (2.9), R_{istj} are the components of the curvature tensor as defined in (2.6), and $\Gamma_a^b(E_i)$ are defined in (2.4). Here we have set

$$(A^2)_{\alpha\beta} = A_{\alpha i} A_{i\beta} + \tilde g_{cd} A_{\alpha c} A_{\beta d}.$$

Furthermore, we have the following expansion:

$$\begin{aligned} \log(\det g) &= \log(\det \tilde g) - 2x_N \operatorname{tr}(H) - 2\Gamma_{bk}^b x_k + \tfrac{1}{3} R_{miil} x_m x_l \\ &\quad + (\tilde g^{ab} R_{mabl} - \Gamma_{am}^c \Gamma_{cl}^a) x_m x_l - x_N^2 \operatorname{tr}(H^2) + O(|x|^3). \end{aligned}$$

Recall first that K minimal implies that $\Gamma_{bk}^b = 0$. The expansions of the metric in the above lemma follow from Lemma 2.1 and formulas (2.13)–(2.14) while the expansion of the log of the determinant of g follows from the fact that $g = G + M$ with

$$G = \begin{pmatrix} \tilde g & 0 \\ 0 & \mathrm{Id}_{\mathbb{R}^n} \end{pmatrix} \quad \text{and} \quad M = O(|x|).$$

Then we have the following expansion:

$$\log(\det g) = \log(\det G) + \operatorname{tr}(G^{-1} M) - \tfrac{1}{2} \operatorname{tr}((G^{-1} M)^2) + O(\|M\|^3).$$

We are now in a position to give the expansion of the Laplace–Beltrami operator. Recall that

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \partial_\alpha \big(\sqrt{\det g}\, g^{\alpha\beta} \partial_\beta u\big),$$

where summation over repeated indices is understood and where $g^{\alpha\beta}$ denotes the entries of the inverse of the metric $(g_{\alpha\beta})$. The above formula can be rewritten as

$$\Delta_g u = g^{\alpha\beta} \partial_{\alpha\beta}^2 u + \partial_\alpha (g^{\alpha\beta}) \partial_\beta u + \tfrac{1}{2} \partial_\alpha (\log(\det g)) g^{\alpha\beta} \partial_\beta u.$$

Using the expansions in Lemma 2.2 we have the following expansion for the Laplace–Beltrami operator:

Lemma 2.3. *In the above coordinates the Laplace–Beltrami operator can be expanded as*

$$
\begin{aligned}
\Delta_g u(x,y) = {} & \Delta_K u + \partial^2_{ii} u + \partial^2_{NN} u - \mathrm{tr}(H)\partial_N u + 2x_N H_{ij}\partial^2_{ij} u \\
& + x_N^2 Q(H)_{ij}\partial^2_{ij} u - x_N \,\mathrm{tr}(H^2)\partial_N u + 2x_N H_{ab}\Gamma^b_{ai}\partial_i u \\
& + \left(\tfrac{2}{3} R_{mlli} + \tilde g^{ab} R_{iabm} - \Gamma^c_{am}\Gamma^a_{ci}\right) x_m \partial_i u - \tfrac{1}{3} R_{islj} x_s x_l \partial^2_{ij} u \\
& + 2x_N (H_{aj} + \tilde g^{ac} H_{cj})\partial^2_{aj} u + \left(O(|x|^2) + O(|x|)\partial_a \mathfrak{F}^{a\beta}(y,x)\right)\partial_\beta u \\
& + \{\tilde g^{ac}\Gamma^c_{bi} + \tilde g^{bc}\Gamma^c_{ai}\} x_i \partial^2_{ab} u + x_N \{H_{ac}\tilde g^{bc} + H_{bc}\tilde g^{ac}\}\partial^2_{ab} u \\
& + O(|x|^3)\partial^2_{ij} u + O(|x|^2)\partial^2_{aj} u + O(|x|^2)\partial^2_{ab} u.
\end{aligned}
$$

Here $Q(H)$ is a quadratic term in H given by

$$
Q(H)_{ij} = 3x_N^2 H_{ik} H_{kj} + x_N^2 (2H_{ia} H_{aj} + \tilde g^{ab} H_{ia} H_{bj}), \tag{2.15}
$$

while the term $\mathfrak{F}^{a\beta}$ ($\beta = b$ or $\beta = j$) is given by

$$
\begin{aligned}
O(|x|)\mathfrak{F}^{ab}(y,x) &= (g^{ab} - \tilde g^{ab}) + \tfrac{1}{2}\left(\log(\det g) g^{ab} - \log(\det \tilde g)\tilde g^{ab}\right), \\
O(|x|)\mathfrak{F}^{aj}(y,x) &= g^{aj} + \tfrac{1}{2}\log(\det \tilde g) g^{aj}.
\end{aligned}
$$

Proof. Using the expansion of Lemma 2.2 and the fact that if $g = G + M$ with

$$
G = \begin{pmatrix} \tilde g & 0 \\ 0 & \mathrm{Id}_{\mathbb{R}^N} \end{pmatrix} \qquad \text{and} \qquad M = O(|x|)
$$

then

$$
g^{-1} = G^{-1} - G^{-1} M G^{-1} + G^{-1} M G^{-1} M G^{-1} + O(\|M\|^3),
$$

it is easy to check that the following expansions hold true:

$$
\begin{aligned}
g^{ij} &= \delta_{ij} + 2x_N H_{ij} - \tfrac{1}{3} R_{istj} x_s x_t + x_N^2 Q(H)_{ij} + O(|x|^3), \qquad 1 \le i, j \le N-1; \\
g^{aj} &= x_N (H_{aj} + \tilde g^{ac} H_{cj}) + O(|x|^2), \qquad 1 \le a \le k,\ 1 \le j \le N-1; \\
g^{ab} &= \tilde g^{ab} + \{\tilde g^{ac}\Gamma^c_{bi} + \tilde g^{bc}\Gamma^c_{ai}\} x_i + x_N \{H_{ac}\tilde g^{bc} + H_{bc}\tilde g^{ac}\} + O(|x|^2), \qquad 1 \le a, b \le k; \\
g^{aN} &\equiv 0, \quad a = 1, \dots, k; \qquad g^{iN} \equiv 0, \quad i = 1, \dots, N-1; \qquad g^{NN} \equiv 1.
\end{aligned}
$$

The lemma follows at once. □

2.3. Nondegenerate minimal submanifolds

Denote by $C^\infty(NK)$ the space of smooth normal vector fields on K. Then, for $\Phi \in C^\infty(NK)$, we define the one-parameter family of submanifolds $t \mapsto K_{t,\Phi}$ by

$$
K_{t,\Phi} := \{\exp^{\partial\Omega}_y (t\Phi(y)) : y \in K\}. \tag{2.16}
$$

The first variation formula for the volume is the equation

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Vol}(K_{t,\Phi}) = \int_K \langle \Phi, \mathbf{h}\rangle_N \, dV_K, \tag{2.17}$$

where $\mathbf{h}$ stands for the *mean curvature* (vector) of K in $\partial\Omega$, $\langle\cdot,\cdot\rangle_N$ denotes the restriction of the metric $\overline{g}$ to NK, and dV_K is the volume element of K.

A submanifold K is said to be *minimal* if it is a critical point for the volume functional, that is,

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Vol}(K_{t,\Phi}) = 0 \quad \text{for any } \Phi \in C^\infty(NK) \tag{2.18}$$

or, equivalently by (2.17), if the mean curvature $\mathbf{h}$ is identically zero on K. One can prove that condition (2.18) is equivalent to (2.5).

The *Jacobi operator* $\mathfrak{J}$ appears in the expression of the second variation of the volume functional for a minimal submanifold K:

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Vol}(K_{t,\Phi}) = -\int_K \langle \mathfrak{J}\Phi, \Phi\rangle_N \, dV_K, \qquad \Phi \in C^\infty(NK), \tag{2.19}$$

and it is given by

$$\mathfrak{J}\Phi := -\Delta_K^N \Phi + \mathfrak{R}^N\Phi - \mathfrak{B}^N\Phi, \tag{2.20}$$

where $\mathfrak{R}^N, \mathfrak{B}^N : NK \to NK$ are defined as

$$\mathfrak{R}^N\Phi = (R(E_a, \Phi)E_a)^N, \qquad \overline{g}(\mathfrak{B}^N\Phi, n_K) := \Gamma_b^a(\Phi)\Gamma_a^b(n_K),$$

for any unit normal vector n_K to K. The Jacobi operator defined in (2.20) expressed in Fermi coordinates takes the form

$$(\mathfrak{J}\Phi)^l = -\Delta_K\Phi^l + \big(\tilde{g}^{ab}R_{mabl} - \Gamma_a^c(E_m)\Gamma_c^a(E_l)\big)\Phi^m, \qquad l = 1, \ldots, N-1, \tag{2.21}$$

where R_{maal} and $\Gamma_a^c(E_m)$ are smooth functions on K and they are defined respectively in (2.6) and (2.4). A submanifold K is said to be *nondegenerate* if the Jacobi operator $\mathfrak{J}$ is invertible, or equivalently if the equation $\mathfrak{J}\Phi = 0$ has only the trivial solution in sections of NK.

3. Expressing the equation in coordinates

We recall from (1.14) that we want to find a solution to the problem

$$\begin{cases} \Delta v - \varepsilon v + v_+^{\frac{N+2}{N-2}} = 0 & \text{in } \Omega_\varepsilon, \\ \partial v/\partial \nu = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{3.1}$$

The first element to construct an approximate solution to our problem is the standard bubble

$$w_0(\xi) = \frac{\alpha_N}{(1+|\xi|^2)^{(N-2)/2}}, \qquad \alpha_N = (N(N-2))^{(N-2)/4} \quad \text{for all } \xi \in \mathbb{R}^N, \tag{3.2}$$

which solves

$$\Delta w + w^{\frac{N+2}{N-2}} = 0 \quad \text{in } \mathbb{R}^N_+, \qquad \frac{\partial w}{\partial \xi_N} = 0 \quad \text{in } \partial\mathbb{R}^N_+. \tag{3.3}$$

It is well known that all positive and bounded solutions to (3.3) are given by the family of functions

$$\mu^{-(N-2)/2} w_0\left(\frac{x-P}{\mu}\right),$$

for any $\mu > 0$ and any point $P = (P_1, \ldots, P_{N-1}, 0) \in \partial\mathbb{R}^N_+$. The solution we are building will have at main order the shape of a copy of w_0, centered and translated along the k-dimensional manifold K inside $\partial\Omega$. In the original variables in Ω, this approximation will be scaled by a small factor, so that it will turn out to be very much concentrated around the manifold K.

To describe this approximation, it will be useful to introduce the following change of variables. Let $(y, x) \in \mathbb{R}^{k+N}$ be the local coordinates along K introduced in (2.8). Let $z = y/\varepsilon \in K_\varepsilon$ and $X = x/\varepsilon \in \mathbb{R}^N$. A parameterization of a neighborhood (in Ω_ε) of $q/\varepsilon \in K_\varepsilon \subset \partial\Omega_\varepsilon$ close to K_ε is given by the map Υ_ε defined by

$$\Upsilon_\varepsilon(z, \bar{X}, X_N) = \frac{1}{\varepsilon}\Upsilon(\varepsilon z, \varepsilon X), \qquad X = (\bar{X}, X_N) \in \mathbb{R}^{N-1} \times \mathbb{R}^+, \tag{3.4}$$

where Υ is the parameterization given in (2.8).

Given a positive smooth function $\mu_\varepsilon = \mu_\varepsilon(y)$ defined on K and a smooth function $\Phi_\varepsilon : K \to \mathbb{R}^{N-1}$ defined by $\Phi_\varepsilon(y) = (\Phi^1_\varepsilon(y), \ldots, \Phi^{N-1}_\varepsilon(y))$, $y \in K$, we consider the change of variables

$$v(z, \overline{X}, X_N) = \mu_\varepsilon^{-(N-2)/2}(\varepsilon z) W\big(z, \mu_\varepsilon^{-1}(\varepsilon z)(\overline{X} - \Phi_\varepsilon(\varepsilon z)), \mu_\varepsilon^{-1}(\varepsilon z) X_N\big), \tag{3.5}$$

with the new W being a function

$$W = W(z, \xi), \qquad z = \frac{y}{\varepsilon}, \qquad \overline{\xi} = \frac{\overline{X} - \Phi_\varepsilon}{\mu_\varepsilon}, \qquad \xi_N = \frac{X_N}{\mu_\varepsilon}. \tag{3.6}$$

To emphasize the dependence of the above change of variables on μ_ε and Φ_ε, we will use the notation

$$v = \mathcal{T}_{\mu_\varepsilon, \Phi_\varepsilon}(W) \;\Leftrightarrow\; v \text{ and } W \text{ satisfy (3.5)}. \tag{3.7}$$

We assume now that the functions μ_ε and Φ_ε are uniformly bounded, as $\varepsilon \to 0$, on K. Since the original variables $(y, x) \in \mathbb{R}^{k+N}$ are local coordinates along K, we let the variables (z, ξ) vary in the set

$$\mathcal{D} = \{(z, \bar{\xi}, \xi_N) : \varepsilon z \in K,\ |\bar{\xi}| < \delta/\varepsilon,\ 0 < \xi_N < \delta/\varepsilon\} \tag{3.8}$$

for some fixed positive number δ. We will also use the notation $\mathcal{D} = K_\varepsilon \times \hat{\mathcal{D}}$, where $K_\varepsilon = \varepsilon^{-1} K$ and

$$\hat{\mathcal{D}} = \{(\bar{\xi}, \xi_N) : |\bar{\xi}| < \delta/\varepsilon,\ 0 < \xi_N < \delta/\varepsilon\}.$$

Having the expansion of the metric coefficients obtained in Section 2, we easily get the expansion of the metric in the expanded variables: letting $g^{\varepsilon}_{\alpha,\beta}$ be the coefficients of the metric g^{ε}, we have

$$g^{\varepsilon}_{\alpha,\beta}(z,x) = g_{\alpha,\beta}(\varepsilon z, \varepsilon x),$$

where $g_{\alpha,\beta}$ are given in Lemma 2.2. By an easy computation we deduce

Lemma 3.1. *For the (Euclidean) metric g^{ε} in the above coordinates (z, X) we have the expansions*

$$\begin{aligned}
g^{\varepsilon}_{ij} &= \delta_{ij} - 2\varepsilon X_N H_{ij} + \tfrac{1}{3}\varepsilon^2 R_{istj} X_s X_t + \varepsilon^2 X_N^2 (H^2)_{ij} + O(\varepsilon^3(|X|^3), \qquad 1 \le i, j \le N-1;\\
g^{\varepsilon}_{aj} &= -\varepsilon X_N (H_{aj} + \tilde g^{\varepsilon}_{ac} H_{cj}) + O(\varepsilon^2 |X|^2) \qquad 1 \le a \le k,\ 1 \le j \le N-1;\\
g^{\varepsilon}_{ab} &= \tilde g^{\varepsilon}_{ab} - \varepsilon\{\tilde g^{\varepsilon}_{ac}\Gamma^c_{bi} + \tilde g^{\varepsilon}_{bc}\Gamma^c_{ai}\}X_i - \varepsilon X_N\{H_{ac}\tilde g^{\varepsilon}_{bc} + H_{bc}\tilde g^{\varepsilon}_{ac}\}\\
&\quad + \varepsilon^2[R_{sabl} + \tilde g^{\varepsilon}_{cd}\Gamma^c_{as}\Gamma^b_{dl}]X_s X_l + \varepsilon^2 X_N^2 (H^2)_{ab}\\
&\quad + \varepsilon^2 X_N X_k\big[H_{ac}\{\tilde g^{\varepsilon}_{bf}\Gamma^f_{ck} + \tilde g^{\varepsilon}_{cf}\Gamma^f_{bk}\} + H_{bc}\{\tilde g^{\varepsilon}_{af}\Gamma^f_{ck} + \tilde g^{\varepsilon}_{cf}\Gamma^f_{ak}\}\big] + O(\varepsilon^3|X|^3),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \le a, b \le k;\\
g^{\varepsilon}_{aN} &\equiv 0, \quad a = 1,\dots,k; \qquad g^{\varepsilon}_{iN} \equiv 0, \quad i = 1,\dots,N-1; \qquad g^{\varepsilon}_{NN} \equiv 1.
\end{aligned}$$

In the above expressions, $H_{\alpha\beta}$ denotes the components of the matrix tensor H defined in (2.9), R_{istj} are the components of the curvature tensor as defined in (2.6), Γ^b_{ai} are defined in (2.4) and $\tilde g^{\varepsilon}_{ab}(z) = \tilde g_{ab}(\varepsilon z)$.

Lemma 3.2. *We have the following expansions:*

$$\begin{aligned}
\sqrt{\det g^{\varepsilon}} &= \sqrt{\det \tilde g^{\varepsilon}}\big\{1 - \varepsilon X_N \operatorname{tr}(H) + \tfrac{1}{6}\varepsilon^2 R_{miil} X_m X_l + \tfrac{1}{2}\varepsilon^2((\tilde g^{\varepsilon})^{ab} R_{mabl} - \Gamma^c_{am}\Gamma^a_{cl}) X_m X_l\\
&\quad + \tfrac{1}{2}\varepsilon^2 X_N^2 \operatorname{tr}(H)^2 - \varepsilon^2 X_N^2 \operatorname{tr}(H^2)\big\} + \varepsilon^3 O(|X|^3),
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
\log(\det g^{\varepsilon}) &= \log(\det \tilde g^{\varepsilon}) - 2\varepsilon X_N \operatorname{tr}(H) + \tfrac{1}{3}\varepsilon^2 R_{miil} X_m X_l\\
&\quad + \varepsilon^2((\tilde g^{\varepsilon})^{ab} R_{mabl} - \Gamma^c_{am}\Gamma^a_{cl}) X_m X_l - \varepsilon^2 X_N^2 \operatorname{tr}(H^2) + O(\varepsilon^3|X|^3).
\end{aligned}$$

We are now in a position to expand the Laplace–Beltrami operator in the new variables (z, ξ) in terms of the parameter ε and the functions $\mu_\varepsilon(y)$ and $\Phi_\varepsilon(y)$. This is the content of the next lemma, whose proof is postponed to Section 8.

Lemma 3.3. *Given the change of variables (3.5), the following expansion holds true:*

$$\mu_\varepsilon^{(N+2)/2}\Delta v = \mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon}(W) := \mu_\varepsilon^2 \Delta_{K_\varepsilon} W + \Delta_\xi W + \sum_{\ell=0}^{5} \mathcal{A}_\ell W + B(W). \tag{3.10}$$

Here, the $\mathcal{A}_k$ are the following differential operators:

$$\begin{aligned}\mathcal{A}_0 W &= \varepsilon^2 \mu_\varepsilon D_{\overline{\xi}} W[\Delta_K \Phi_\varepsilon] - \varepsilon^2 \mu_\varepsilon \Delta_K \mu_\varepsilon (\gamma W + D_\xi W[\xi]) \\ &\quad + \varepsilon^2 |\nabla_K \mu_\varepsilon|^2 \big[D_{\xi\xi} W[\xi]^2 + 2(1+\gamma) D_\xi W[\xi] + \gamma(1+\gamma) W \big] \\ &\quad + \varepsilon^2 \nabla_K \mu_\varepsilon \cdot \{ 2 D_{\overline{\xi}\overline{\xi}} W[\overline{\xi}] + N D_{\overline{\xi}} W \} [\nabla_K \Phi_\varepsilon] + \varepsilon^2 D_{\overline{\xi}\overline{\xi}} W [\nabla_K \Phi_\varepsilon]^2 \\ &\quad - 2\varepsilon \mu_\varepsilon \tilde{g}^{ab} \big[D_\xi (\partial_{\bar{a}} W)[\partial_b \mu_\varepsilon \xi] + D_{\overline{\xi}} (\partial_{\bar{a}} W)[\partial_b \Phi_\varepsilon] + \gamma \partial_a \mu_\varepsilon \partial_{\bar{b}} W \big], \end{aligned} \tag{3.11}$$

where we have set $\gamma = (N-2)/2$, and

$$\begin{aligned}\mathcal{A}_1 W = \sum_{i,j} \Big[& 2\mu_\varepsilon \varepsilon H_{ij} \xi_N - \frac{1}{3} \varepsilon^2 \sum_{m,l} R_{mijl} (\mu_\varepsilon \xi_m + \Phi_\varepsilon^m)(\mu_\varepsilon \xi_l + \Phi_\varepsilon^l) \\ & + \mu_\varepsilon^2 \varepsilon^2 \xi_N^2 Q(H)_{ij} + \mu_\varepsilon \varepsilon^2 \xi_N \sum_l \mathfrak{D}_{Nl}^{ij} (\mu_\varepsilon \xi_l + \Phi_\varepsilon^l) \Big] \partial_{ij}^2 W, \end{aligned} \tag{3.12}$$

where the $\mathfrak{D}_{Nk}^{ij}$ are smooth functions of $z = y/\varepsilon$ and are uniformly bounded. Furthermore,

$$\mathcal{A}_2 W = \varepsilon^2 \mu_\varepsilon \sum_j \Big[\sum_s \tfrac{2}{3} R_{mssj} + \sum_{m,a,b} (\tilde{g}_\varepsilon^{ab} R_{mabj} - \Gamma_{am}^b \Gamma_{bj}^a) \Big] (\mu_\varepsilon \xi_m + \Phi_\varepsilon^m) \partial_j W, \tag{3.13}$$

$$\mathcal{A}_3 W = \Big[-\varepsilon \operatorname{tr}(H) - 2\mu_\varepsilon \varepsilon^2 \operatorname{tr}(H^2) \xi_N - 2\varepsilon^2 \sum_{i,a,b} (\mu_\varepsilon \xi_i + \Phi_\varepsilon^i) H_{ab} \Gamma_{bi}^a \Big] \mu_\varepsilon \partial_N W. \tag{3.14}$$

Moreover,

$$\begin{aligned}\mathcal{A}_4 W &= 4\varepsilon \mu_\varepsilon \xi_N \\ &\quad \times \sum_{a,j} H_{aj} \big(-\varepsilon D_y (\partial_j W)[\partial_a \Phi_\varepsilon] + \mu_\varepsilon \partial_{\bar{a}j}^2 W - \varepsilon \partial_a \mu_\varepsilon (\gamma \partial_j W + D_\xi (\partial_j W)[\xi]) \big), \end{aligned} \tag{3.15}$$

$$\begin{aligned}\mathcal{A}_5 W &= \Big(\sum_{a,j} \mathfrak{D}_j^a \varepsilon^2 [\mu_\varepsilon \xi_j + \Phi_\varepsilon^j] + \varepsilon^2 \mu_\varepsilon \mathfrak{D}_N^a \xi_N \Big) \\ &\quad \times \big\{ \mu_\varepsilon \big[-\varepsilon D_{\overline{\xi}} W[\partial_a \Phi_\varepsilon] + \mu_\varepsilon \partial_{\bar{a}} W - \varepsilon \partial_a \mu_\varepsilon (\gamma W + D_\xi W[\xi]) \big] \big\}, \end{aligned} \tag{3.16}$$

where $\mathfrak{D}_j^a$ and $\mathfrak{D}_N^a$ are smooth functions of $z = y/\varepsilon$. Finally, the operator $B(v)$ is defined below in (8.1).

We recall that ∂_a, $\partial_{\overline{a}}$ and ∂_i denote ∂_{y_a}, ∂_{z_a} and ∂_{ξ_i} respectively.

After performing the change of variables in (3.5), the original equation in v reduces locally close to $K_\varepsilon = \varepsilon^{-1} K$ to the following equation in W:

$$-\mathcal{A}_{\mu_\varepsilon, \Phi_\varepsilon} W + \varepsilon \mu_\varepsilon^2 W - W^p = 0, \tag{3.17}$$

where $\mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon}$ is defined in (3.10) and $p=\frac{N+2}{N-2}$. We denote by $\mathcal{S}_\varepsilon$ the operator given by (3.17), that is,

$$\mathcal{S}_\varepsilon(v) := -\mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon} v + \varepsilon\mu_\varepsilon^2 v - v^p. \tag{3.18}$$

In order to study equation (3.17) in the set of (z,ξ) with $z\in K_\varepsilon$, $|\bar\xi|\le\delta/\varepsilon$ and $0<\xi_N\le\delta/\varepsilon$, we will first construct an approximate solution to (3.17) in the whole space $K_\varepsilon\times\mathbb{R}^{N-1}\times[0,\infty)$ (see Section 4). Then, by using proper cut-off functions, we will build a first approximation to (3.17) in the original region $z\in K_\varepsilon$, $|\bar\xi|\le\delta/\varepsilon$ and $0<\xi_N\le\delta/\varepsilon$.

The basic tool for this construction is a linear theory we describe below.

Let us recall the well known fact that, due to the invariance under translations and dilations of equation (3.3), and since w_0 is a nondegenerate solution for (3.3), the functions

$$Z_j(\xi)=\frac{\partial w_0}{\partial\xi_j},\quad j=1,\dots,N-1,\quad \text{and}\quad Z_0(\xi)=\xi\cdot\nabla w_0(\xi)+\frac{N-2}{2}w_0(\xi) \tag{3.19}$$

are the only bounded solutions to the linearized equation around w_0 of problem (3.3),

$$-\Delta\phi - pw_0^{p-1}\phi=0\quad \text{in } \mathbb{R}^{N-1}\times\mathbb{R}^+,\qquad \frac{\partial\phi}{\partial\xi_N}=0\quad \text{on } \{\xi_N=0\}.$$

Let us now consider a smooth function $a:K\to\mathbb{R}$ with $a(y)\ge\lambda>0$ for all $y\in K$ and a function $g:K\times\mathbb{R}^{N-1}\times\mathbb{R}^+\to\mathbb{R}$ that depends smoothly on $y\in K$. Recall that a variable $z\in K_\varepsilon$ has the form $\varepsilon z=y\in K$.

We want to find a linear theory for the following linear problem:

$$\begin{cases} -\Delta_{\mathbb{R}^N}\phi - pw_0^{p-1}\phi+\varepsilon a(\varepsilon z)\phi=g & \text{in } \mathbb{R}^N_+,\\ \partial\phi/\partial\xi_N=0 & \text{on } \{\xi_N=0\},\\ \int_{\mathbb{R}^{N-1}\times[0,\infty)}\phi(\varepsilon z,\xi)Z_j(\xi)\,d\xi=0 & \text{for all } z\in K_\varepsilon,\ j=0,\dots,N-1. \end{cases} \tag{3.20}$$

To do so we first define the following norms: Let $\delta>0$ be a small fixed number and r be a positive number. For a function w defined in $K_\varepsilon\times\mathbb{R}^{N-1}\times[0,\infty)$, we define

$$\|w\|_{\varepsilon,r} := \sup_{(z,\xi)\in K_\varepsilon\times\{|\xi|\le\delta/\sqrt{\varepsilon}\}}(1+|\xi|^2)^{r/2}|w(z,\xi)| + \sup_{(z,\xi)\in K_\varepsilon\times\{|\xi|\ge\delta/\sqrt{\varepsilon}\}}\varepsilon^{-r/2}|w(z,\xi)|. \tag{3.21}$$

Let $\sigma\in(0,1)$. We further define

$$\begin{aligned}\|w\|_{\varepsilon,r,\sigma} := \|w\|_{\varepsilon,r} &+ \sup_{(z,\xi)\in K_\varepsilon\times\{|\xi|\le\delta/\sqrt{\varepsilon}\}}(1+|\xi|^2)^{(r+\sigma)/2}[w]_{\sigma,B(\xi,1)}\\ &+ \sup_{(z,\xi)\in K_\varepsilon\times\{|\xi|\ge\delta/\sqrt{\varepsilon}\}}\varepsilon^{-(r+\sigma)/2}[w]_{\sigma,B(\xi,1)},\end{aligned} \tag{3.22}$$

where we have set

$$[w]_{\sigma,B(\xi,1)} := \sup_{\xi_1,\xi_2\in B(\xi,1)}\frac{|w(z,\xi_1)-w(z,\xi_2)|}{|\xi_1-\xi_2|^\sigma}. \tag{3.23}$$

Lemma 3.4. *Let $2 < r < N$ and $\sigma \in (0, 1)$. Let $a : K \to \mathbb{R}$ be a smooth function such that $a(y) \ge \lambda > 0$ for all $y \in K$. Let $g : K \times \mathbb{R}^{N-1} \times [0, \infty) \to \mathbb{R}$ depend smoothly on $y \in K$ with $\|g\|_{\varepsilon,r} < \infty$ and*

$$\int_{\mathbb{R}^{N-1}\times[0,\infty)} g(\varepsilon z, \xi) Z_j(\xi)\, d\xi = 0 \quad \text{for all } z \in K_\varepsilon,\ j = 0, \dots, N-1.$$

Then there exists a positive constant C such that for all sufficiently small ε there is a solution ϕ to problem (3.20) such that

$$\|D^2_\xi \phi\|_{\varepsilon,r,\sigma} + \|D_\xi \phi\|_{\varepsilon,r-1,\sigma} + \|\phi\|_{\varepsilon,r-2,\sigma} \le C\|g\|_{\varepsilon,r,\sigma}. \tag{3.24}$$

Furthermore, ϕ depends smoothly on εz, and for any integer l there exists a positive constant C_l such that

$$\|D^l_z \phi\|_{\varepsilon,r-2,\sigma} \le C_l \sum_{k\le l} \|D^k_z g\|_{\varepsilon,r,\sigma}. \tag{3.25}$$

Proof. The proof is divided into several steps.

Step 1. We start by proving an a priori external estimate for a solution ϕ to problem (3.20). Given $R > 0$, we claim that

$$|\phi(z, \xi)| \le C\big(\|\phi\|_{L^\infty(|\xi|=\delta R\varepsilon^{-1/2})} + \varepsilon^{(r-2)/2}\|g\|_{\varepsilon,r}\big) \tag{3.26}$$

for all $z \in K_\varepsilon$ and $|\xi| > R\delta\varepsilon^{-1/2}$.

Fix $R > 0$, independent of ε. In the region $|\xi| > R\delta\varepsilon^{-1/2}$ the function ϕ solves

$$-\Delta\phi + \varepsilon b(\varepsilon z, \xi)\phi = g,$$

where

$$b(\varepsilon z, \xi) = a(\varepsilon z) - p w_0^{p-1}/\varepsilon = a(\varepsilon z) + \varepsilon \Theta_\varepsilon(\xi),$$

with Θ_ε uniformly bounded in the region as $\varepsilon \to 0$. Thus b is uniformly positive and bounded as $\varepsilon \to 0$. Using the maximum principle, we get

$$\begin{aligned}\|\phi\|_{L^\infty(|\xi|>\delta R\varepsilon^{-1/2})} &\le C\big(\varepsilon^{-1}\|g\|_{L^\infty(|\xi|>\delta R\varepsilon^{-1/2})} + \|\phi\|_{L^\infty(|\xi|=\delta R\varepsilon^{-1/2})}\big)\\ &\le C(\varepsilon^{r-2/2}\|g\|_{\varepsilon,r} + \|\phi\|_{L^\infty(|\xi|=\delta R\varepsilon^{-1/2})}),\end{aligned}$$

which gives (3.26).

Step 2. We will now prove that there exists $C > 0$ such that

$$\|\phi\|_{\varepsilon,r-2} \le C\|g\|_{\varepsilon,r}. \tag{3.27}$$

Towarrds a contradiction, assume that there exist sequences $\varepsilon_n \to 0$, g_n with $\|g_n\|_{\varepsilon_n,r} \to 0$ and solutions ϕ_n to (3.20) with $\|\phi_n\|_{\varepsilon_n,r-2} = 1$. Let $z_n \in K_{\varepsilon_n}$ and ξ_n be such that

$$|\phi_n(\varepsilon_n z_n, \xi_n)| = \sup |\phi_n(y, \xi)|.$$

We may assume that, up to subsequences, $\varepsilon_n z_n \to \bar{y}$ in K. On the other hand, from Step 1,

$$\sup_{z \in K_{\varepsilon_n},\, |\xi| > \delta R \varepsilon_n^{-1/2}} \varepsilon_n^{(r-2)/2} |\phi_n(\varepsilon_n z, \xi)| \le C R^{-(r-2)/2};$$

thus choosing R sufficiently large, but independent of ε_n, we see that

$$\sup_{z \in K_{\varepsilon_n},\, |\xi| > \delta R \varepsilon_n^{-1/2}} \varepsilon_n^{(r-2)/2} |\phi_n(z, \xi)|$$

is arbitrarily small. In particular, $|\xi_n| \le C\varepsilon_n^{-1/2}$ for some positive constant C independent of ε_n.

Now assume that there exists a positive constant M such that $|\xi_n| \le M$. In this case, up to subsequences, one gets $\xi_n \to \xi_0$. We then consider the functions $\tilde{\phi}_n(z, \xi) = \phi_n(z, \xi + \xi_n)$. This is a sequence of uniformly bounded functions, and $(\tilde{\phi}_n)$ converges uniformly over compact subsets of $K \times \mathbb{R}^{N-1} \times \mathbb{R}^+$ to a function $\tilde{\phi}$ solving

$$\begin{cases} -\Delta\tilde{\phi} - p w_0^{p-1} \tilde{\phi} = 0 & \text{in } \mathbb{R}^N_+, \\ \partial\tilde{\phi}/\partial\xi_N = 0 & \text{on } \{\xi_N = 0\}. \end{cases}$$

Since the orthogonality conditions pass to the limit, we get furthermore

$$\int_{\mathbb{R}^{N-1} \times [0,\infty)} \tilde{\phi}(y, \xi) Z_j(\xi)\, d\xi = 0 \quad \text{for all } y \in K \text{ and } j = 0, \dots, N-1.$$

These facts imply that $\tilde{\phi} \equiv 0$, a contradiction.

Assume now that $\lim_{n\to\infty} |\xi_n| = \infty$ and define $\tilde{\phi}_n(z, \xi) = \phi_n(z, |\xi_n|\xi + \xi_n)$. Clearly $\tilde{\phi}_n$ satisfies the equation

$$\Delta\tilde{\phi}_n + pC_N \frac{|\xi_n|^2}{\left(1 + \big||\xi_n|\xi + \xi_n\big|^2\right)^2} \tilde{\phi}_n - |\xi_n|^2 \varepsilon_n a \tilde{\phi}_n = |\xi_n|^2 g(z, |\xi_n|\xi + \xi_n).$$

Consider first the case in which $\lim_{n\to\infty} \varepsilon_n |\xi_n|^2 = 0$. Under our assumptions, $\tilde{\phi}_n$ is uniformly bounded and it converges uniformly over compact sets to $\tilde{\phi}$ solving

$$\Delta\tilde{\phi} = 0 \quad \text{in } \mathbb{R}^N, \qquad |\tilde{\phi}| \le C|\xi|^{2-r}.$$

Since $2 < r < N$, we conclude that $\tilde{\phi} \equiv 0$, a contradiction.

Consider now the other possible case, namely that $\lim_{n\to\infty} \varepsilon_n |\xi_n|^2 = \beta > 0$. Then, up to subsequences, $\tilde{\phi}_n$ converges uniformly over compact sets to $\tilde{\phi}$ solving

$$\Delta\tilde{\phi} - \beta a \tilde{\phi} = 0 \quad \text{in } \mathbb{R}^N, \qquad |\tilde{\phi}| \le C|\xi|^{2-r}.$$

This implies that $\tilde{\phi} \equiv 0$, a contradiction. Taking into account the result of Step 1, the proof of (3.27) is complete.

Step 3. We now show that there exists $C > 0$ such that if ϕ is a solution to (3.20), then

$$\|D_\xi \phi\|_{\varepsilon, r-1} + \|\phi\|_{\varepsilon, r-2} \le C\|g\|_{\varepsilon, r}. \tag{3.28}$$

First assume we are in the region $|\xi| < \delta\varepsilon^{-1/2}$, and $z \in K_\varepsilon$. Then, using estimate (3.27), we find that ϕ solves $-\Delta\phi = \hat{g}$ in $|\xi| < \delta\varepsilon^{-1/2}$ with $|\hat{g}| \le \|g\|_{\varepsilon,r}/(1+|\xi|^r)$.

Now fix $e \in \mathbb{R}^N$ with $|e| = 1$ and $R > 0$. Perform the change of variables $\tilde{\phi}(z,t) = R^{r-2}\phi(z, Rt + 3Re)$; then

$$\Delta\tilde{\phi} = \tilde{g} \quad \text{in } |t| \le 1,$$

where $\tilde{g}(t,z) = R^r \hat{g}(z, Rt + 3Re)$; then

$$\|\tilde{\phi}\|_{L^\infty(B(0,2))} + \|\tilde{g}\|_{L^\infty(B(0,2))} \le \|g\|_{\varepsilon,r}.$$

Elliptic estimates give $\|D\tilde{\phi}\|_{L^\infty(B(0,1))} \le C\|\tilde{g}\|_{L^\infty(B(0,2))}$. This inequality implies that

$$\|(1+|\xi|)^{r-1}D\phi\|_{L^\infty(|\xi|\le\delta\varepsilon^{-1/2})} \le C\|(1+|\xi|)^r g\|_{L^\infty(|\xi|\le 2\delta\varepsilon^{-1/2})}.$$

Assume now that $|\xi| > \delta\varepsilon^{-1/2}$. In this region the function ϕ solves

$$-\Delta\phi = \hat{g},$$

where $|\hat{g}| \le C\|g\|_{\varepsilon,r}\varepsilon^{r/2}$, and $|\phi| \le C\|g\|_{\varepsilon,r}\varepsilon^{(r-2)/2}$. After scaling out $\varepsilon^{1/2}$, elliptic estimates yield $|D\phi| \le C\varepsilon^{(r-1)/2}$. This concludes the proof of (3.28).

Step 4. We now show that

$$\|D_\xi^2\phi\|_{\varepsilon,r,\sigma} + \|D_\xi\phi\|_{\varepsilon,r-1,\sigma} + \|\phi\|_{\varepsilon,r-2,\sigma} \le C\|g\|_{\varepsilon,r,\sigma}. \tag{3.29}$$

First assume we are in the region $|\xi| < \delta\varepsilon^{-1/2}$. Then ϕ solves $-\Delta\phi = \hat{g}$ in $|\xi| < \delta\varepsilon^{-1/2}$, where, thanks to the C^1-estimate of Step 3, $\|\tilde{g}\|_{\varepsilon,r,\sigma} \le C\|g\|_{\varepsilon,r,\sigma}$.

Arguing as in the previous step, we fix $e \in \mathbb{R}^N$ with $|e| = 1$ and $R > 0$, and let $\tilde{\phi}$ and $\tilde{g}$ be as defined in Step 3. Elliptic estimates then give $\|D^2\tilde{\phi}\|_{C^{0,\sigma}(B(0,1))} \le C\|\tilde{g}\|_{C^{0,\sigma}(B(0,2))}$. This inequality gets translated in terms of ϕ as the desired Schauder estimate within $|\xi| \le \delta\varepsilon^{-1/2}$. The Hölder estimate for $D\phi$ follows by interpolation. In the region $|\xi| > \delta\varepsilon^{-1/2}$, we argue exactly as in the proof of Step 3. This concludes the proof of (3.29).

Step 5. Now we prove the existence of the solution ϕ to problem (3.20). We consider first the following auxiliary problem: find $\bar{\phi}$ and $\alpha : K_\varepsilon \to \mathbb{R}$ solving

$$\begin{cases} -\Delta\bar{\phi} - pw_0^{p-1}\bar{\phi} + \varepsilon a(y)\bar{\phi} = g + \alpha(z)Z(\xi) & \text{in } \mathbb{R}^N_+, \\ \partial\bar{\phi}/\partial\xi_N = 0 & \text{on } \{\xi_N = 0\}, \\ \int_{\mathbb{R}^{N-1}\times\mathbb{R}^+} \bar{\phi}(z,\xi)Z_j(\xi)\,d\xi = \int_{\mathbb{R}^{N-1}\times\mathbb{R}^+} \bar{\phi}(z,\xi)Z(\xi)\,d\xi = 0 & \text{for } z \in K_\varepsilon \text{ and} \\ & j = 0,\dots,N-1, \end{cases} \tag{3.30}$$

where Z is the first eigenfunction, with corresponding first eigenvalue $\lambda_0 > 0$, in $L^2(\mathbb{R}^N)$ of the problem

$$\Delta_\xi\bar{\phi} + pw_0(\xi)^{p-1}\bar{\phi} = \lambda\bar{\phi} \quad \text{in } \mathbb{R}^N. \tag{3.31}$$

The above problem is variational: for any fixed $z \in K_\varepsilon$, solutions to (3.30) correspond to critical points of the energy functional

$$E(\bar\phi) = \frac{1}{2}\int_{\mathbb{R}^{N-1}\times\mathbb{R}^+} |\nabla\bar\phi|^2 - pw_0^{p-1}\bar\phi^2 + \varepsilon a\bar\phi^2 - \int_{\mathbb{R}^{N-1}\times\mathbb{R}^+} g\bar\phi$$

for functions $\bar\phi \in H^1(\mathbb{R}^{N-1}\times\mathbb{R}^+)$ that satisfy

$$\int_{\mathbb{R}^{N-1}\times\mathbb{R}^+} \bar\phi Z_j = \int_{\mathbb{R}^{N-1}\times\mathbb{R}^+} \bar\phi Z = 0$$

for all $j = 0, \ldots, N-1$. This functional is smooth, uniformly bounded from below, and satisfies the Palais–Smale condition. We thus conclude that E has a minimum, which gives a solution to (3.30).

Observe now that multiplying the equation in (3.30) by Z, integrating over $\mathbb{R}^{N-1}\times\mathbb{R}^+$, and using the orthogonality conditions in (3.30), one easily gets

$$\alpha(z) = \int_{\mathbb{R}^{N-1}\times\mathbb{R}^+} g(z,\xi)Z(\xi)\,d\xi \quad \text{for all } z \in K_\varepsilon. \tag{3.32}$$

Given a solution $(\bar\phi, \alpha)$ to (3.30), we define

$$\phi = \bar\phi + \beta Z \quad \text{with} \quad \beta(z) = \frac{\int g(z,\xi)Z(\xi)\,d\xi}{\lambda_0 + \varepsilon a(\varepsilon z)}.$$

A straightforward computation shows that ϕ is a solution to (3.30).

Finally, estimate (3.25) follows by direct differentiation of (3.20) and using (3.24). This concludes the proof of Lemma 3.4. □

4. Construction of approximate solutions

This section will be devoted to building an approximate solution to problem (3.17) locally close to K_ε, using an iterative method that we describe below. Let I be an integer. The expanded variables (z, ξ) will be defined as in (3.6) with

$$\mu_\varepsilon(\varepsilon z) = \mu_0 + \mu_{1,\varepsilon} + \cdots + \mu_{I,\varepsilon}, \tag{4.1}$$

where $\mu_0, \mu_{1,\varepsilon}, \ldots, \mu_{I,\varepsilon}$ are smooth functions on K, with μ_0 positive, and

$$\Phi_\varepsilon(\varepsilon z) = \Phi_{1,\varepsilon} + \cdots + \Phi_{I,\varepsilon}, \tag{4.2}$$

where $\Phi_{1,\varepsilon}, \ldots, \Phi_{I,\varepsilon}$ are smooth functions defined along K with values in $\mathbb{R}^{N-1}$. In the (z, ξ) variables, the shape of the approximate solution will be given by

$$W_{I+1,\varepsilon}(z, \bar\xi, \xi_N) = w_0(\xi) + w_{1,\varepsilon}(z,\xi) + \cdots + w_{I+1,\varepsilon}(z,\xi), \quad \xi = (\bar\xi, \xi_N), \tag{4.3}$$

where w_0 is given by (3.2) and the functions $w_{j,\varepsilon}$ for $j \geq 1$ are to be determined so that the above function $W_{I+1,\varepsilon}$ satisfies formally

$$-\mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon} W_{I+1,\varepsilon} + \varepsilon\mu_\varepsilon^2 W_{I+1,\varepsilon} - W_{I+1,\varepsilon}^{\frac{N+2}{N-2}} = O(\varepsilon^{1+I/2}) \quad \text{in } K_\varepsilon \times \mathbb{R}^{N-1} \times \mathbb{R}^+.$$

This can be done by expanding equation (3.17) formally in powers of ε (and in terms of μ_ε and Φ_ε) for $W = W_{I+1,\varepsilon}$ (using basically Lemma 3.3) and analyzing each term separately. For example, looking at the coefficient of ε in the expansion we will determine μ_0 and $w_{1\varepsilon}$, while looking at the coefficient of $\varepsilon^{1+j/2}$ we will determine $w_{j,\varepsilon}$, $\mu_{j-1,\varepsilon}$ and $\Phi_{j-1,\varepsilon}$, for $j = 2, \ldots, I+1$. In this procedure we use crucially the nondegeneracy assumption on K (which implies the invertibility of the Jacobi operator) when considering the projection on some elements of the kernel of the linearization of (1.14) at w_0, while when projecting on the remaining part of the kernel we have to choose the functions $\mu_{j,\varepsilon}$. This section is devoted to presenting this construction.

Lemma 4.1. *For any integer $I \in \mathbb{N}$ there exist smooth functions $\mu_\varepsilon : K \to \mathbb{R}$ and $\Phi_\varepsilon : K \to \mathbb{R}^{N-1}$ such that*

$$\|\mu_\varepsilon\|_{L^\infty(K)} + \|\partial_a \mu_\varepsilon\|_{L^\infty(K)} + \|\partial_a^2 \mu_\varepsilon\|_{L^\infty(K)} \leq C, \tag{4.4}$$

$$\|\Phi_\varepsilon\|_{L^\infty(K)} + \|\partial_a \Phi_\varepsilon\|_{L^\infty(K)} + \|\partial_a^2 \Phi_\varepsilon\|_{L^\infty(K)} \leq C, \tag{4.5}$$

for some positive constant C, independent of ε. In particular,

$$\|\mu_\varepsilon - \mu_0\|_{L^\infty(K)} \to 0, \qquad \|\Phi_\varepsilon - \Phi_0\|_{L^\infty(K)} \to 0, \tag{4.6}$$

where μ_0 is the function defined explicitly as

$$\mu_0(y) = \frac{\int_{\mathbb{R}^N_+} \xi_N |\partial_1 w_0|^2}{\int_{\mathbb{R}^N_+} w_0^2} [2H_{aa}(y) + H_{ii}(y)].$$

By assumption (1.13), the function μ_0 is strictly positive along K. Moreover Φ_0 is a smooth function along K with values in $\mathbb{R}^{N-1}$. Furthermore there exists a positive function $W_{I+1,\varepsilon} : K_\varepsilon \times \mathbb{R}^N_+ \to \mathbb{R}$ such that

$$\mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon}(W_{I+1,\varepsilon}) - \varepsilon\mu_\varepsilon^2 W_{I+1,\varepsilon} + W_{I+1,\varepsilon}^p = \mathcal{E}_{I+1,\varepsilon} \quad \text{in } K_\varepsilon \times \mathbb{R}^N_+,$$
$$\frac{\partial W_{I+1,\varepsilon}}{\partial \nu} = 0 \quad \text{on } \partial\mathbb{R}^N_+$$

with

$$\|W_{I+1,\varepsilon} - W_{I,\varepsilon}\|_{\varepsilon,N-4,\sigma} \leq C\varepsilon^{1+I/2}, \tag{4.7}$$

$$\|\mathcal{E}_{I+1,\varepsilon}\|_{\varepsilon,N-2,\sigma} \leq C\varepsilon^{1+(I+1)/2}. \tag{4.8}$$

We should emphasize that $\int w_0^2$ is indeed finite thanks to the fact that $N \geq 5$, and we are actually assuming that $N \geq 7$.

Construction of $w_{1,\varepsilon}$**.** Using Lemma 3.3, we *formally* have

$$
\begin{aligned}
-\mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon} W_{1,\varepsilon} + \varepsilon\mu_\varepsilon^2 W_{1,\varepsilon} - W_{1,\varepsilon}^p = {} & -\Delta_{\mathbb{R}^N} w_0 - w_0^p \\
& + (-\Delta_{\mathbb{R}^N} w_{1,\varepsilon} - p w_0^{p-1} w_{1,\varepsilon} + \varepsilon\mu_0^2 w_{1,\varepsilon}) \\
& + \varepsilon\mu_0[H_{\alpha\alpha}\partial_{\xi_N} w_0 - 2\xi_N H_{ij}\partial_{ij}^2 w_0 + \mu_0(y) w_0] \\
& + \mathcal{E}_{1,\varepsilon} + Q_\varepsilon(w_{1,\varepsilon}),
\end{aligned}
$$

where $\mathcal{E}_{1,\varepsilon}$ is a sum of functions of the form

$$
\varepsilon\mu_0(\varepsilon\mu_0 + \varepsilon\partial_a\mu_0 + \varepsilon\partial_a^2\mu_0) a(z) b(\xi)
$$

and $a(\varepsilon z)$ is a smooth function uniformly bounded, together with its derivatives, as $\varepsilon \to 0$, while the function b is such that

$$
\sup_\xi (1 + |\xi|^{N-2}) |b(\xi)| < \infty.
$$

The term $Q_\varepsilon(w_{1,\varepsilon})$ is quadratic in $w_{1,\varepsilon}$, in fact it is explicitly given by

$$
-(w_0 + w_{1,\varepsilon})^p + w_0^p + p w_0^{p-1} w_{1,\varepsilon}.
$$

Observe now that the term of order 0 (in the power expansion in ε) vanishes because of the equation satisfied by w_0. In order to make the coefficient of ε vanish, $w_{1,\varepsilon}$ must satisfy

$$
\begin{cases}
-\Delta w_{1,\varepsilon} - p w_0^{p-1} w_{1,\varepsilon} + \varepsilon\mu_0^2 w_{1,\varepsilon} = \varepsilon g_{1,\varepsilon}(\varepsilon z, \xi) & \text{in } \mathbb{R}_+^N, \\
\partial w_{1,\varepsilon}/\partial\xi_N = 0 & \text{on } \{\xi_N = 0\},
\end{cases} \tag{4.9}
$$

where

$$
g_{1,\varepsilon}(\varepsilon z, \xi) = -\mu_0(y)[H_{\alpha\alpha}\partial_{\xi_N} w_0 - 2\xi_N H_{ij}\partial_{ij}^2 w_0 + \mu_0(y) w_0]. \tag{4.10}
$$

Using Lemma 3.4, we see that (4.9) is solvable if the right hand side is L^2-orthogonal to the functions Z_j for $j = 0, \dots, N-1$. These conditions, for $j = 1, \dots, N-1$, are clearly satisfied since both $\partial_N w_0$ and $\partial_{ij}^2 w_0$ are even in $\bar{\xi}$, while the Z_i's are odd in $\bar{\xi}$ for every i. It remains to compute the L^2 product of the right hand side with Z_0. We claim that

$$
\int_{\mathbb{R}_+^N} (H_{\alpha\alpha}\partial_N w_0 - 2H_{ij}\xi_N\partial_{ij}^2 w_0) Z_0 = \mathfrak{A}_0 H_{\alpha\alpha} - \mathfrak{A}_1 H_{ii}, \tag{4.11}
$$

where $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are the constants defined by

$$
\mathfrak{A}_0 = \frac{1}{2}\int_{\mathbb{R}_+^N} \xi_N |\nabla w_0|^2 - \frac{N-2}{2N}\int_{\mathbb{R}_+^N} \xi_N w_0^{\frac{2N}{N-2}}, \tag{4.12}
$$

$$
\mathfrak{A}_1 = \int_{\mathbb{R}_+^N} \xi_N |\partial_1 w_0|^2 > 0. \tag{4.13}
$$

Furthermore,

$$
\int_{\mathbb{R}_+^N} w_0 Z_0 = -\int_{\mathbb{R}_+^N} w_0^2. \tag{4.14}
$$

Indeed, since $(\partial_\mu w_\mu)_{|\mu=1} = -Z_0$, we have

$$\int_{\mathbb{R}^N_+} w_0 Z_0 = -\int_{\mathbb{R}^N_+} w_0 (\partial_\mu w_\mu)_{|\mu=1} = -\frac{1}{2}\partial_\mu \left(\int_{\mathbb{R}^N_+} w_\mu^2 \right)_{|\mu=1}$$
$$= -\frac{1}{2}\partial_\mu \left(\mu^2 \int_{\mathbb{R}^N_+} w_0^2 \right)_{|\mu=1}.$$

We postpone the proof of (4.11). We turn now to the solvability in $w_{1,\varepsilon}$. Assuming the quantity on the right hand side of (4.11) is negative, we define

$$\mu_0(y) := \frac{\mathfrak{A}_0 H_{\alpha\alpha} - \mathfrak{A}_1 H_{ii}}{\int_{\mathbb{R}^N_+} w_0^2}. \tag{4.15}$$

With this choice for μ_0, the integral of the right hand side in (4.9) against Z_0 vanishes on K and this implies the existence of $w_{1,\varepsilon}$, thanks to Lemma 3.4. Moreover, it is straightforward to check that

$$\|g_{1,\varepsilon}\|_{\varepsilon,N-2,\sigma} \le C$$

for some $\sigma \in (0,1)$. Lemma 3.4 thus implies that

$$\|D_\xi^2 w_{1,\varepsilon}\|_{\varepsilon,N-2,\sigma} + \|D_\xi w_{1,\varepsilon}\|_{\varepsilon,N-3,\sigma} + \|w_{1,\varepsilon}\|_{\varepsilon,N-4,\sigma} \le C\varepsilon \tag{4.16}$$

and that there exists a positive constant β (depending only on Ω, K and N) such that for any integer ℓ,

$$\|\nabla_z^{(\ell)} w_{1,\varepsilon}(z,\cdot)\|_{\varepsilon,N-2,\sigma} \le \beta C_l \varepsilon, \qquad z \in K_\varepsilon, \tag{4.17}$$

where C_l depends only on l, p, K and Ω.

Proof of (4.11)–(4.13). We first compute $\int_{\mathbb{R}^N_+} w_0 \partial_N w_0$. To do so, for any $\mu > 0$ we denote by w_μ the scaled function

$$w_\mu(x) = \mu^{-(N-2)/2} w_0(\mu^{-1}x).$$

Since $(\partial_\mu w_\mu)_{|\mu=1} = -Z_0$, a direct differentiation and integration by parts gives

$$\int_{\mathbb{R}^N_+} Z_0 \partial_N w_0 = \partial_\mu \left[\frac{1}{2} \int_{\mathbb{R}^N_+} \xi_N |\nabla w_\mu|^2 - \frac{N-2}{2N} \xi_N w_\mu^{\frac{2N}{N-2}} \right]_{|\mu=1}.$$

Now changing variables $\xi \mapsto \mu\xi$, a direct computation gives

$$\frac{1}{2}\int_{\mathbb{R}^N_+} \xi_N |\nabla w_\mu|^2 - \frac{N-2}{2N} \int_{\mathbb{R}^N_+} \xi_N w_\mu^{\frac{2N}{N-2}} = \mu \left[\frac{1}{2}\int_{\mathbb{R}^N_+} \xi_N |\nabla w_0|^2 - \frac{N-2}{2N} \int_{\mathbb{R}^N_+} \xi_N w_0^{\frac{2N}{N-2}} \right],$$

from which (4.12) follows. Next, we compute $-2\int_{\mathbb{R}^N_+} \xi_N \partial^2_{ij} w_0 Z_0$. By symmetry we have $2\int_{\mathbb{R}^N_+} \xi_N \partial^2_{ij} w_0 Z_0 = 0$ if $i \ne j$. Assume then that $i = j$ is fixed. Integration by parts and direct differentiation yields

$$-2\int_{\mathbb{R}^N_+} \xi_N \partial^2_{ii} w_0 Z_0 = 2\int_{\mathbb{R}^N_+} \xi_N \partial_i w_0 \partial_i Z_0 = -\partial_\mu \left[\int_{\mathbb{R}^N_+} \xi_N |\partial_1 w_\mu|^2 \right]_{|\mu=1}$$

and also

$$\int_{\mathbb{R}^N_+} \xi_N |\partial_1 w_\mu|^2 = \mu \int_{\mathbb{R}^N_+} \xi_N |\partial_1 w_0|^2.$$

The above facts yield (4.11).

Now we claim that $\mathfrak{A}_0 = 2\mathfrak{A}_1$. Indeed,

$$\begin{aligned}\int_{\mathbb{R}^N_+} \xi_N |\nabla w_0|^2 &= \int_{\mathbb{R}^N_+} \xi_N |\partial_N w_0|^2 + \int_{\mathbb{R}^N_+} \xi_N |\partial_i w_0|^2\\ &= \int_{\mathbb{R}^N_+} \xi_N |\partial_N w_0|^2 + (N-1) \int_{\mathbb{R}^N_+} \xi_N |\partial_1 w_0|^2\\ &= (N+1) \int_{\mathbb{R}^N_+} \xi_N |\partial_1 w_0|^2 = (N+1)\mathfrak{A}_1.\end{aligned}$$

Here we have used the fact that $\int_{\mathbb{R}^N_+} \xi_N |\partial_N w_0|^2 = 2 \int_{\mathbb{R}^N_+} \xi_N |\partial_1 w_0|^2$ since

$$\begin{aligned}\int_{\mathbb{R}^N_+} \xi_N |\partial_N w_0|^2 &= \alpha_N^2 \int_{\mathbb{R}^N_+} \xi_N^2 \frac{\xi_N}{(1+|\xi|^2)^N} = -\alpha_N^2 \frac{1}{2(N-1)} \int_{\mathbb{R}^N_+} \xi_N^2 \partial_N \left(\frac{1}{(1+|\xi|^2)^{N-1}} \right)\\ &= \alpha_N^2 \frac{1}{N-1} \int_{\mathbb{R}^N_+} \frac{\xi_N}{(1+|\xi|^2)^{N-1}}\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N_+} \xi_N |\partial_1 w_0|^2 &= \alpha_N^2 \int_{\mathbb{R}^N_+} \xi_N \xi_1 \frac{\xi_1}{(1+|\xi|^2)^N} = -\alpha_N^2 \frac{1}{2(N-1)} \int_{\mathbb{R}^N_+} \xi_N \xi_1 \partial_1 \left(\frac{1}{(1+|\xi|^2)^{N-1}} \right)\\ &= \alpha_N^2 \frac{1}{2(N-1)} \int_{\mathbb{R}^N_+} \frac{\xi_N}{(1+|\xi|^2)^{N-1}}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^N_+} \xi_N w_0^{\frac{2N}{N-2}} &= \frac{1}{2} \int_{\mathbb{R}^N_+} \partial_N(\xi_N^2) w_0^{\frac{2N}{N-2}} = -\frac{1}{2} \int_{\mathbb{R}^N_+} \frac{2N}{N-2} \xi_N^2 w_0^{\frac{N+2}{N-2}} \partial_N w_0\\ &= -\frac{N}{N-2} \int_{\mathbb{R}^N_+} \xi_N^2 \partial_N w_0 \underbrace{w_0^{\frac{N+2}{N-2}}}_{-\Delta w_0} = \frac{N}{N-2} \int_{\mathbb{R}^N_+} \xi_N^2 \partial_N w_0 (\partial_{NN}^2 w_0 + \partial_{ii} w_0).\end{aligned}$$

Now we use the fact that

$$\int_{\mathbb{R}^N_+} \xi_N^2 \partial_N w_0 \partial_{NN}^2 w_0 = -2 \int_{\mathbb{R}^N_+} \xi_N \partial_N w_0 \partial_N w_0 - \int_{\mathbb{R}^N_+} \xi_N^2 \partial_{NN}^2 w_0 \partial_N w_0,$$

which implies that

$$\int_{\mathbb{R}^N_+} \xi_N^2 \partial_N w_0 \partial_{NN}^2 w_0 = -\int_{\mathbb{R}^N_+} \xi_N \partial_N w_0 \partial_N w_0 = -2\mathfrak{A}_1.$$

Now integrating first in ξ_1 and then in ξ_N yields

$$I := \int_{\mathbb{R}^N_+} \xi_N^2 \partial_N w_0 \partial^2_{11} w_0 = -\int_{\mathbb{R}^N_+} \xi_N^2 \partial^2_{N1} w_0 \partial_1 w_0 = \int_{\mathbb{R}^N_+} \partial_1 w_0 \partial_N (\xi_N^2 \partial_1 w_0)$$
$$= \int_{\mathbb{R}^N_+} \partial_1 w_0 (2\xi_N \partial_1 w_0 + \xi_N^2 \partial^2_{1N} w_0) = 2\mathfrak{A}_1 - I.$$

This implies that

$$I = \mathfrak{A}_1 \quad \text{and} \quad \int_{\mathbb{R}^N_+} \xi_N^2 \partial_N w_0 \partial^2_{ii} w_0 = (N-1) \int_{\mathbb{R}^N_+} \xi_N^2 \partial_N w_0 \partial^2_{11} w_0 = (N-1)\mathfrak{A}_1.$$

We deduce that

$$\int_{\mathbb{R}^N_+} \xi_N w_0^{\frac{2N}{N-2}} = \frac{N}{N-2}(-2\mathfrak{A}_1 + (N-1)\mathfrak{A}_1) = \frac{N(N-3)}{N-2}\mathfrak{A}_1.$$

Hence

$$\mathfrak{A}_0 = \frac{1}{2}\int_{\mathbb{R}^N_+} \xi_N |\nabla w_0|^2 - \frac{N-2}{2N}\int_{\mathbb{R}^N_+} \xi_N w_0^{\frac{2N}{N-2}}$$
$$= \frac{N+1}{2}\mathfrak{A}_1 - \frac{N-2}{2N}\,\frac{N(N-3)}{N-2}\mathfrak{A}_1 = 2\mathfrak{A}_1.$$

This proves the claim. In particular, equation (4.15) can be written as

$$\mu_0(y) := \mathfrak{A}_1 \frac{2H_{\alpha\alpha} - H_{ii}}{\int_{\mathbb{R}^N_+} w_0^2} = \mathfrak{A}_1 \frac{2H_{aa} + H_{ii}}{\int_{\mathbb{R}^N_+} w_0^2}. \tag{4.18}$$

Construction of $w_{2,\varepsilon}$**.** We take $I = 2$, $\mu_\varepsilon = \mu_0 + \mu_{1,\varepsilon}$, $\Phi_\varepsilon = \Phi_{1,\varepsilon}$ and $W_{2,\varepsilon}(z,\xi) = w_0(\xi) + w_{1,\varepsilon}(z,\xi) + w_{2,\varepsilon}(z,\xi)$, where μ_0 and $w_{1,\varepsilon}$ have already been constructed in the previous step. Computing $\mathcal{S}(W_{2,\varepsilon})$ (see (3.18)) we get

$$-\Delta w_{2,\varepsilon} + \varepsilon\mu_0^2 w_{2,\varepsilon} - p w_0^{p-1} w_{2,\varepsilon} = \varepsilon g_{2,\varepsilon} + \mathcal{E}_{2,\varepsilon} + Q_\varepsilon(w_{2,\varepsilon}). \tag{4.19}$$

In (4.19) the function $g_{2,\varepsilon}$ is given by

$$\begin{aligned} g_{2,\varepsilon} &= \mu_{1,\varepsilon}(y)[-H_{\alpha\alpha}\partial_{\xi_N} w_0 + 2\xi_N H_{ij}\partial^2_{ij} w_0 - 2\mu_0 w_0] \\ &\quad + \mu_0(y)[-H_{\alpha\alpha}\partial_{\xi_N} w_{1,\varepsilon} + 2\xi_N H_{ij}\partial^2_{ij} w_{1,\varepsilon}] + \varepsilon\mathfrak{G}_{2,\varepsilon}(\xi, z, w_0, \mu_0) \\ &\quad - \varepsilon\mu_0\Delta_K \Phi^j_{1,\varepsilon}\partial_j w_0 - \tfrac{1}{3}\varepsilon\mu_0 R_{mijl}(\xi_m\Phi^l_{1,\varepsilon} + \xi_l\Phi^m_{1,\varepsilon})\partial^2_{ij} w_0 \\ &\quad + \tfrac{2}{3}\varepsilon\mu_0 R_{mssj}\Phi^m_{1,\varepsilon}\partial_j w_0 + \varepsilon\mu_0\big((\tilde{g}^\epsilon)^{ab}R_{maaj} - \Gamma^c_a(E_m)\Gamma^a_c(E_j)\big)\Phi^m_{1,\varepsilon}\partial_j w_0. \end{aligned} \tag{4.20}$$

In (4.20), $\mathfrak{G}_{2,\varepsilon}(\xi, z, w_0, w_{1,\varepsilon}, \mu_0)$ is a sum of functions of the form

$$\mathcal{Q}(\mu_0, \partial_a\mu_0, \partial^2_a\mu_0)a(\varepsilon z)b(\xi),$$

where $\mathcal{Q}$ denotes a quadratic function of its arguments, $a(\varepsilon z)$ is a smooth function uniformly bounded, together with its derivatives, in ε as $\varepsilon \to 0$, while b is such that

$$\sup_{\xi} (1 + |\xi|^{N-2})|b(\xi)| < \infty.$$

In (4.19) the term $\mathcal{E}_{2,\varepsilon}$ can be described as a sum of functions of the form

$$(\varepsilon \mathcal{L}(\mu_1, \Phi_1) + \mathcal{Q}(\mu_1, \Phi_1))a(\varepsilon z)b(\xi),$$

where $(\mu_1, \Phi_1) = (\mu_{1,\varepsilon}, \partial_a \mu_{1,\varepsilon}, \partial_a^2 \mu_{1,\varepsilon}, \Phi_{1,\varepsilon}, \partial_a \Phi_{1,\varepsilon}, \partial_a^2 \Phi_{1,\varepsilon})$, $\mathcal{L}$ denotes a linear function of its arguments, $\mathcal{Q}$ denotes a quadratic function of its arguments, and $a(\varepsilon z)$ and b have the same properties as above. Finally, the term $Q_\varepsilon(w_{2,\varepsilon})$ is a sum of quadratic terms in $w_{2,\varepsilon}$ like

$$-(w_0 + w_{1,\varepsilon} + w_{2,\varepsilon})^p + (w_0 + w_{1,\varepsilon})^p + p(w_0 + w_{1,\varepsilon})^{p-1} w_{2,\varepsilon}$$

and linear terms in $w_{2,\varepsilon}$ multiplied by a term of order ε, like

$$p((w_0 + w_{1,\varepsilon})^{p-1} - w_0^{p-1}) w_{2,\varepsilon}.$$

We will choose $w_{2,\varepsilon}$ to satisfy

$$\begin{cases} -\Delta w_{2,\varepsilon} - p w_0^{p-1} w_{2,\varepsilon} + \varepsilon \mu_0^2 w_{2,\varepsilon} = \varepsilon g_{2,\varepsilon} & \text{on } \mathbb{R}^N_+, \\ \partial w_{2,\varepsilon}/\partial \xi_N = 0 & \text{on } \{\xi_N = 0\}. \end{cases} \tag{4.21}$$

Again by Lemma 3.4, the above equation is solvable if $g_{2,\varepsilon}$ is L^2-orthogonal to Z_j, $j = 0, 1, \dots, N-1$. These orthogonality conditions will define the parameters $\mu_{1,\varepsilon}$ and the normal section $\Phi_{1,\varepsilon}$.

Projection onto Z_0 and choice of $\mu_{1,\varepsilon}$. Recalling that by definition of μ_0 one has

$$\int_{\mathbb{R}^{N-1}\times\mathbb{R}^+} [-H_{\alpha\alpha} \partial_{\xi_N} w_0 + 2\xi_N H_{ij} \partial_{ij}^2 w_0 - \mu_0(y) w_0] Z_0 \, d\xi = 0, \tag{4.22}$$

and using the fact that w_0 is an even function in $\bar{\xi}$, we have

$$\begin{aligned} \int_{\mathbb{R}^N_+} g_{2,\varepsilon} Z_0 = \mu_0 & \int_{\mathbb{R}^N_+} [-H_{\alpha\alpha} \partial_{\xi_N} w_{1,\varepsilon} + 2\xi_N H_{ij} \partial_{ij}^2 w_{1,\varepsilon}] Z_0 \, d\xi \\ & - \mu_0 \mu_{1,\varepsilon} \int_{\mathbb{R}^N_+} w_0 Z_0 \, d\xi + \varepsilon \int_{\mathbb{R}^N_+} \mathfrak{G}_{2,\varepsilon}(\xi, z, w_0, \mu_0) Z_0 \, d\xi. \end{aligned}$$

We observe that the term that factors like $\mu_{1,\varepsilon}$ in the definition of $g_{2,\varepsilon}$ in (4.20) disappears after integration thanks to relation (4.22). Here and later, $\mathfrak{G}_{i,\varepsilon}$ designates a quantity that may change from line to line and which is uniformly bounded in ε and depends smoothly on its arguments.

Then, we define $\mu_{1,\varepsilon}$ to make the above quantity zero. The above relation defines $\mu_{1,\varepsilon}$ as a smooth function of y in K. From estimates (4.16) for $w_{1,\varepsilon}$ we get

$$\|\mu_{1,\varepsilon}\|_{L^\infty(K)} + \|\partial_a \mu_{1,\varepsilon}\|_{L^\infty(K)} + \|\partial_a^2 \mu_{1,\varepsilon}\|_{L^\infty(K)} \le C\varepsilon. \tag{4.23}$$

Projection onto Z_l and choice of $\Phi_{1,\varepsilon}$. Multiplying $g_{2,\varepsilon}$ with $\partial_l w_0$, integrating over $\mathbb{R}^N_+$ and using the fact w_0 is even in the variable $\overline{\xi}$, one obtains

$$\begin{aligned}
&(\varepsilon\mu_0)^{-1} \int_{\mathbb{R}^N_+} g_{2,\varepsilon} \partial_l w_0 \\
&\quad = -\Delta_K \Phi^j_{1,\varepsilon} \int_{\mathbb{R}^N_+} \partial_j w_0 \partial_l w_0 + \mu_o^{-1} \int_{\mathbb{R}^N_+} \mathfrak{G}_{2,\varepsilon}(\xi, z, w_0, \mu_0, w_{1,\varepsilon}) \partial_l w_0 \\
&\qquad - \tfrac{1}{3} R_{mijs} \int_{\mathbb{R}^N_+} (\xi_m \Phi^s_{1,\varepsilon} + \xi_s \Phi^m_{1,\varepsilon}) \partial^2_{ij} w_0 \partial_l w_0 \\
&\qquad + \left[\tfrac{2}{3} R_{mssj} \Phi^m_{1,\varepsilon} + ((\tilde{g}^\varepsilon)^{ab} R_{mabj} - \Gamma^c_a(E_m)\Gamma^a_c(E_j))\Phi^m_{1,\varepsilon}\right] \int_{\mathbb{R}^N_+} \partial_j w_0 \partial_l w_0.
\end{aligned} \tag{4.24}$$

First of all, observe that by oddness in $\overline{\xi}$ we have

$$\int_{\mathbb{R}^N_+} \partial_j w_0 \partial_l w_0 = \delta_{lj} C_0 \quad \text{with} \quad C_0 := \int_{\mathbb{R}^N_+} |\partial_1 w_0|^2.$$

On the other hand, the integral $\int_{\mathbb{R}^N_+} \xi_m \partial^2_{ij} w_0 \partial_l w_0$ is nonzero only if either $i = j$ and $m = l$, or $i = l$ and $j = m$, or $i = m$ and $j = l$. In the latter case we have $R_{mijs} = 0$ (by the antisymmetry of the curvature tensor in the first two indices). Therefore, the first term of the second line of the above formula becomes simply

$$\begin{aligned}
\tfrac{1}{3} R_{mijs} \int_{\mathbb{R}^N_+} \xi_m \Phi^s_{1,\varepsilon} \partial^2_{ij} w_0 \partial_l w_0 = {}& \tfrac{1}{3} R_{liis} \Phi^s_{1,\varepsilon} \int_{\mathbb{R}^N_+} \xi_l \partial_l w_0 \partial^2_{ii} w_0 \, d\xi \\
&+ \tfrac{1}{3} R_{jijs} \Phi^s_{1,\varepsilon} \int_{\mathbb{R}^N_+} \xi_j \partial_i w_0 \partial^2_{ij} w_0 \, d\xi.
\end{aligned}$$

Observe that, integrating by parts, when $l \neq i$ (otherwise $R_{liis} = 0$) we have

$$\int_{\mathbb{R}^N_+} \xi_l \partial_l w_0 \partial^2_{ii} w_0 \, d\xi = - \int_{\mathbb{R}^N_+} \xi_l \partial_i w_0 \partial^2_{li} w_0 \, d\xi.$$

Hence, still by the antisymmetry of the curvature tensor, we are left with

$$-\tfrac{2}{3} R_{ijjs} \Phi^s_{1,\varepsilon} \int_{\mathbb{R}^N_+} \xi_j \partial_i w_0 \partial^2_{ij} w_0 \, d\xi.$$

The last integral can be computed with a further integration by parts and is equal to $-\frac{1}{2} C_0$, so we get

$$\tfrac{1}{3} C_0 R_{ijjs} \Phi^s_{1,\varepsilon}.$$

In a similar way (permuting the indices s and m in the above argument), one obtains

$$\tfrac{1}{3}R_{sijm}\int_{\mathbb{R}^N_+}\xi_s\Phi^m_{1,\varepsilon}\partial^2_{ij}w_0\partial_l w_0=\tfrac{1}{3}C_0\sum_i R_{ijjm}\Phi^m_{1,\varepsilon}.$$

Collecting the above computations yields

$$-\tfrac{1}{3}R_{mijs}\int_{\mathbb{R}^N_+}(\xi_m\Phi^s_{1,\varepsilon}+\xi_s\Phi^m_{1,\varepsilon})\partial^2_{ij}w_0\partial_l w_0+\tfrac{2}{3}R_{mssj}\Phi^m_{1,\varepsilon}\int_{\mathbb{R}^N_+}\partial_j w_0\partial_l w_0=0.$$

Hence formula (4.24) becomes simply

$$\int_{\mathbb{R}^N_+}g_{2,\varepsilon}\partial_l w_0=-\varepsilon\mu_0C_0\Delta_K\Phi^l_{1,\varepsilon}+\varepsilon\mu_0C_0\big((\tilde g^{\epsilon})^{ab}R_{maal}-\Gamma^c_a(E_m)\Gamma^a_c(E_l)\big)\Phi^m_{1,\varepsilon}$$
$$+\varepsilon\int_{\mathbb{R}^N_+}\mathfrak{G}_{2,\varepsilon}\partial_l w_0.$$

We then conclude that $g_{2,\varepsilon}(z,\xi,w_0,\dots,w_{1,\varepsilon})$, on the right hand side of (4.21), is L^2-orthogonal to Z_l $(l=1,\dots,N-1)$ if and only if $\Phi_{1,\varepsilon}$ satisfies an equation of the form

$$\Delta_K\Phi^l_{1,\varepsilon}-\big((\tilde g^{\varepsilon})^{ab}R_{mabj}-\Gamma^c_a(E_m)\Gamma^a_c(E_l)\big)\Phi^m_{1,\varepsilon}=G_{2,\varepsilon}(\varepsilon z), \tag{4.25}$$

for some expression $G_{2,\varepsilon}$ smooth in its argument. Observe that the operator acting on $\Phi_{1,\varepsilon}$ on the left hand side is nothing but the Jacobi operator (see (2.21)), which is invertible by the nondegeneracy of K. This implies the solvability of the above equation in $\Phi_{1,\varepsilon}$.

Furthermore, equation (4.25) defines $\Phi_{1,\varepsilon}$ as a smooth function on K, of order ε; more precisely we have

$$\|\Phi_{1,\varepsilon}\|_{L^\infty(K)}+\|\partial_a\Phi_{1,\varepsilon}\|_{L^\infty(K)}+\|\partial^2_a\Phi_{1,\varepsilon}\|_{L^\infty(K)}\le C. \tag{4.26}$$

By our choice of $\mu_{1,\varepsilon}$ and $\Phi_{1,\varepsilon}$ we have solvability of (4.21) in $w_{2,\varepsilon}$. Moreover, it is straightforward to check that

$$|\varepsilon g_{2,\varepsilon}(\varepsilon z,\xi)|\le C\varepsilon|\partial_{\xi_N}w_{1,\varepsilon}|\le C\frac{\varepsilon^{3/2}}{(1+|\xi|)^{N-2}}.$$

Furthermore, for a given $\sigma\in(0,1)$ we have

$$\|\varepsilon g_{2,\varepsilon}\|_{\varepsilon,N-2,\sigma}\le C\varepsilon^{3/2}.$$

Lemma 3.4 then shows that

$$\|D^2_\xi w_{2,\varepsilon}\|_{\varepsilon,N-2,\sigma}+\|D_\xi w_{2,\varepsilon}\|_{\varepsilon,N-3,\sigma}+\|w_{2,\varepsilon}\|_{\varepsilon,N-4,\sigma}\le C\varepsilon^{3/2} \tag{4.27}$$

and that there exists a positive constant β (depending only on Ω, K and n) such that for any integer ℓ,

$$\|\nabla^{(\ell)}_z w_{2,\varepsilon}(z,\cdot)\|_{\varepsilon,N-2,\sigma}\le\beta C_l\varepsilon^{3/2},\qquad z\in K_\varepsilon, \tag{4.28}$$

where C_l depends only on l, p, K and Ω.

Expansion at an arbitrary order. We now take an arbitrary integer I. Let

$$\mu_\varepsilon = \mu_0 + \mu_{1,\varepsilon} + \cdots + \mu_{I-1,\varepsilon} + \mu_{I,\varepsilon}, \tag{4.29}$$
$$\Phi = \Phi_{1,\varepsilon} + \cdots + \Phi_{I-1,\varepsilon} + \Phi_{I,\varepsilon}, \tag{4.30}$$
$$W_{I+1,\varepsilon} = w_0(\xi) + w_{1,\varepsilon}(z,\xi) + \cdots + w_{I,\varepsilon}(z,\xi) + w_{I+1,\varepsilon}(z,\xi), \tag{4.31}$$

where $\mu_0, \mu_{1,\varepsilon}, \ldots, \mu_{I-1,\varepsilon}, \Phi_{1,\varepsilon}, \ldots, \Phi_{I-1,\varepsilon}$ and $w_{1,\varepsilon}, \ldots, w_{I,\varepsilon}$ have already been constructed following an iterative scheme, as described in the previous steps.

In particular, for any $i = 1, \ldots, I-1$,

$$\|\mu_{i,\varepsilon}\|_{L^\infty(K)} + \|\partial_a \mu_{i,\varepsilon}\|_{L^\infty(K)} + \|\partial_a^2 \mu_{i,\varepsilon}\|_{L^\infty(K)} \le C\varepsilon^{1+(i-1)/2}, \tag{4.32}$$
$$\|\Phi_{i,\varepsilon}\|_{L^\infty(K)} + \|\partial_a \Phi_{i,\varepsilon}\|_{L^\infty(K)} + \|\partial_a^2 \Phi_{i,\varepsilon}\|_{L^\infty(K)} \le C\varepsilon^{(i-1)/2}, \tag{4.33}$$

and, for $i = 0, \ldots, I-1$,

$$\|D_\xi^2 w_{i+1,\varepsilon}\|_{\varepsilon,N-2,\sigma} + \|D_\xi w_{i+1,\varepsilon}\|_{\varepsilon,N-3,\sigma} + \|w_{i+1,\varepsilon}\|_{\varepsilon,N-4,\sigma} \le C\varepsilon^{1+i/2}, \tag{4.34}$$

and, for any integer ℓ,

$$\|\nabla_z^{(\ell)} w_{i+1,\varepsilon}(z,\cdot)\|_{\varepsilon,N-2,\sigma} \le \beta C_l \varepsilon^{1+i/2}, \qquad z \in K_\varepsilon. \tag{4.35}$$

The new triplet $(\mu_{I,\varepsilon}, \Phi_{I,\varepsilon}, w_{I+1,\varepsilon})$ will be found by reasoning as in the construction of $(\mu_{1,\varepsilon}, \Phi_{1,\varepsilon}, w_{2,\varepsilon})$. Computing $\mathcal{S}(W_{I+1,\varepsilon})$ (see (3.18)) we get

$$-\Delta w_{I+1,\varepsilon} + \varepsilon\mu_0^2 w_{I+1,\varepsilon} - p w_0^{p-1} w_{I+1,\varepsilon} = \varepsilon g_{I+1,\varepsilon} + \mathcal{E}_{I+1,\varepsilon} + Q_\varepsilon(w_{I+1,\varepsilon}). \tag{4.36}$$

In (4.36) the function $g_{I+1,\varepsilon}$ is given by

$$\begin{aligned} g_{I+1,\varepsilon} = {} & \mu_{I,\varepsilon}(y)[-H_{\alpha\alpha}\partial_{\xi_N} w_0 + 2\xi_N H_{ij}\partial_{ij}^2 w_0 - 2\mu_0 w_0] \\ & + \mu_0(y)[-H_{\alpha\alpha}\partial_{\xi_N} w_{I,\varepsilon} + 2\xi_N H_{ij}\partial_{ij}^2 w_{I,\varepsilon}] \\ & + \varepsilon\mathfrak{G}_{I+1,\varepsilon}(\xi, z, w_0, \ldots, e_{I,\varepsilon}, \mu_0, \ldots, \mu_{I-1,\varepsilon}, \Phi_{1,\varepsilon}, \ldots, \Phi_{I-1,\varepsilon}) \\ & - \varepsilon\mu_0 \Delta_K \Phi_{I,\varepsilon}^j \partial_j w_0 - \tfrac{1}{3}\varepsilon\mu_0 R_{mijl}(\xi_m \Phi_{I,\varepsilon}^l + \xi_l \Phi_{I,\varepsilon}^m)\partial_{ij}^2 w_0 \\ & + \tfrac{2}{3}\varepsilon\mu_0 R_{mssj}\Phi_{I,\varepsilon}^m \partial_j w_0 + \mu_0\varepsilon\big((\tilde{g}^\varepsilon)^{ab} R_{mabj} - \Gamma_a^c(E_m)\Gamma_c^a(E_j)\big)\Phi_{I,\varepsilon}^m \partial_j w_0. \end{aligned} \tag{4.37}$$

In (4.37), $\mathfrak{G}_{I+1,\varepsilon}(\xi, z, \ldots)$ is a smooth function with

$$\|\mathfrak{G}_{I+1,\varepsilon}\|_{\varepsilon,N-2,\sigma} \le C\varepsilon^{1+I/2}. \tag{4.38}$$

In (4.36) the term $\mathcal{E}_{I+1,\varepsilon}$ can be described as a sum of functions of the form

$$(\varepsilon\mathcal{L}(\mu_I, \Phi_I) + \mathcal{Q}(\mu_I, \Phi_I))a(\varepsilon z)b(\xi),$$

where $(\mu_I, \Phi_I) = (\mu_{I,\varepsilon}, \partial_a \mu_{I,\varepsilon}, \partial_a^2 \mu_{I,\varepsilon}, \Phi_{I,\varepsilon}, \partial_a \Phi_{I,\varepsilon}, \partial_a^2 \Phi_{I,\varepsilon})$, $\mathcal{L}$ denotes a linear function of its arguments, $\mathcal{Q}$ denotes a quadratic function of its arguments, $a(\varepsilon z)$ is a smooth function uniformly bounded, together with its derivatives, in ε as $\varepsilon \to 0$, while b is such that

$$\sup_{\xi} (1 + |\xi|^{N-2})|b(\xi)| < \infty.$$

Finally the term $Q_\varepsilon(w_{I+1,\varepsilon})$ is a sum of quadratic terms in $w_{I+1,\varepsilon}$ like

$$(w_0 + w_{1,\varepsilon} + \cdots + w_{I+1,\varepsilon})^p - (w_0 + w_{1,\varepsilon} + \cdots + w_{I+1,\varepsilon})^p \\ - p(w_0 + w_{1,\varepsilon} + \cdots + w_{I,\varepsilon})^{p-1} w_{I+1,\varepsilon}$$

and linear terms in $w_{I+1,\varepsilon}$ multiplied by a term of order ε^2, like

$$p((w_0 + w_{1,\varepsilon})^{p-1} - w_0^{p-1}) w_{I+1,\varepsilon}.$$

We define $w_{I+1,\varepsilon}$ to satisfy

$$\begin{cases} -\Delta w_{I+1,\varepsilon} - p w_0^{p-1} w_{I+1,\varepsilon} + \varepsilon \mu_0^2 w_{I+1,\varepsilon} = \varepsilon g_{I+1,\varepsilon} & \text{on } \mathbb{R}^N_+, \\ \partial w_{I+1,\varepsilon}/\partial \xi_N = 0 & \text{on } \{\xi_N = 0\}. \end{cases} \tag{4.39}$$

Again by Lemma 3.4, the above equation is solvable if $g_{I+1,\varepsilon}$ is L^2-orthogonal to Z_j, $j = 0, 1, \ldots, N-1$. These orthogonality conditions will define the parameters $\mu_{I,\varepsilon}$ and the normal section $\Phi_{I,\varepsilon}$.

Projection onto Z_0 and choice of $\mu_{I,\varepsilon}$. Thanks to the definition of μ_0 one has

$$\int_{\mathbb{R}^N_+} g_{I+1,\varepsilon} Z_0 = \mu_0 \int_{\mathbb{R}^N_+} [-H_{\alpha\alpha} \partial_{\xi_N} w_{I,\varepsilon} + 2\xi_N H_{ij} \partial^2_{ij} w_{I,\varepsilon}] Z_0 \, d\xi \\ - \mu_0 \mu_{I,\varepsilon} \int_{\mathbb{R}^N_+} w_0 Z_0 \, d\xi + \varepsilon \int_{\mathbb{R}^N_+} \mathfrak{G}_{I+1,\varepsilon}(\xi, z) Z_0 \, d\xi.$$

We define $\mu_{I,\varepsilon}$ to make the above quantity zero. The above relation defines $\mu_{I,\varepsilon}$ as a smooth function of εz in K. From estimates (4.34) for $w_{I,\varepsilon}$ we get

$$\|\mu_{I,\varepsilon}\|_{L^\infty(K)} + \|\partial_a \mu_{I,\varepsilon}\|_{L^\infty(K)} + \|\partial^2_a \mu_{I,\varepsilon}\|_{L^\infty(K)} \le C \varepsilon^{1+(I-1)/2}. \tag{4.40}$$

Projection onto Z_l and choice of $\Phi_{I,\varepsilon}$. Multiplying $g_{I+1,\varepsilon}$ with $\partial_l w_0$, integrating over $\mathbb{R}^N_+$ and arguing as in the construction of $\Phi_{1,\varepsilon}$, we get

$$\int_{\mathbb{R}^N_+} g_{I+1,\varepsilon} \partial_l w_0 = -\varepsilon \mu_0 \Delta_K \Phi^l_{I,\varepsilon} + \varepsilon \mu_0 \big((\tilde{g}^\varepsilon)^{ab} R_{mabl} - \Gamma^c_a(E_m) \Gamma^a_c(E_l) \big) \Phi^m_{I,\varepsilon} \\ + \varepsilon \int_{\mathbb{R}^N_+} \mathfrak{G}_{I+1,\varepsilon} \partial_l w_0.$$

We then conclude that $g_{I+1,\varepsilon}(z, \xi, w_0, \ldots, w_{I,\varepsilon})$, on the right hand side of (4.39), is L^2-orthogonal to Z_l ($l = 1, \ldots, N-1$) if and only if $\Phi_{I,\varepsilon}$ satisfies an equation of the form

$$\Delta_K \Phi^l_{I,\varepsilon} - \big((\tilde{g}^\varepsilon)^{ab} R_{mabl} - \Gamma^c_a(E_m) \Gamma^a_c(E_l) \big) \Phi^m_{I,\varepsilon} = G_{I+1,\varepsilon}(\varepsilon z), \tag{4.41}$$

where $G_{I+1,\varepsilon}$ is a smooth function on K. Using again the nondegeneracy of K we have solvability of the above equation in $\Phi_{I,\varepsilon}$. Furthermore, taking into account (4.38), we get

$$\|\Phi_{I,\varepsilon}\|_{L^\infty(K)} + \|\partial_a \Phi_{I,\varepsilon}\|_{L^\infty(K)} + \|\partial_a^2 \Phi_{I,\varepsilon}\|_{L^\infty(K)} \le C\varepsilon^{1+(I-1)/2}. \tag{4.42}$$

By our choice of $\mu_{I+1,\varepsilon}$ and $\Phi_{I+1,\varepsilon}$ we have solvability of (4.39) in $w_{I+1,\varepsilon}$. Moreover, it is straightforward to check that

$$|\varepsilon g_{I+1,\varepsilon}(\varepsilon z, \xi)| \le C \frac{\varepsilon^{1+I/2}}{(1+|\xi|)^{N-2}}.$$

Furthermore, for a given $\sigma \in (0,1)$ we have

$$\|\varepsilon g_{I+1,\varepsilon}\|_{\varepsilon,N-2,\sigma} \le C\varepsilon^{1+I/2}.$$

Lemma 3.4 then implies that

$$\|D_\xi^2 w_{I+1,\varepsilon}\|_{\varepsilon,N-2,\sigma} + \|D_\xi w_{I+1,\varepsilon}\|_{\varepsilon,N-3,\sigma} + \|w_{I+1,\varepsilon}\|_{\varepsilon,N-4,\sigma} \le C\varepsilon^{1+I/2} \tag{4.43}$$

and that there exists a positive constant β (depending only on Ω, K and n) such that for any integer ℓ,

$$\|\nabla_z^{(\ell)} w_{I+1,\varepsilon}(z,\cdot)\|_{\varepsilon,N-2,\sigma} \le \beta C_l \varepsilon^{1+I/2}, \qquad z \in K_\varepsilon. \tag{4.44}$$

This concludes our construction and proves the validity of Lemma 4.1.

5. A global approximation and expansion of a quadratic functional

Let $\mu_\varepsilon(y)$, $\Phi_\varepsilon(y)$ and $W_{I+1,\varepsilon}$ be the functions whose existence and properties have been established in Lemma 4.1. We define locally around $K_\varepsilon := \varepsilon^{-1}K \subset \partial\Omega_\varepsilon$ in Ω_ε the function

$$\begin{aligned} V_\varepsilon(z,X) := {} & \mu_\varepsilon^{-(N-2)/2}(\varepsilon z) W_{I+1,\varepsilon}\big(z, \mu_\varepsilon^{-1}(\varepsilon z)(\bar X - \Phi_\varepsilon(\varepsilon z)), \mu_\varepsilon^{-1}(\varepsilon z) X_N\big) \\ & \times \chi_\varepsilon(|(\bar X - \Phi_\varepsilon(\varepsilon z), X_N)|), \end{aligned} \tag{5.1}$$

where $z \in K_\varepsilon$. In (5.1) the function χ_ε is a smooth cut-off function with

$$\chi_\varepsilon(r) = \begin{cases} 1 & \text{for } r \in [0, 2\varepsilon^{-\gamma}], \\ 0 & \text{for } r \in [3\varepsilon^{-\gamma}, 4\varepsilon^{-\gamma}], \end{cases} \tag{5.2}$$

and

$$|\chi_\varepsilon^{(l)}(r)| \le C_l \varepsilon^{l\gamma} \qquad \text{for all } l \ge 1,$$

for some $\gamma \in (1/2, 1)$ to be fixed later.

The function V_ε is well defined in a small neighborhood of K_ε inside Ω_ε. We will look for a solution to (1.14) of the form

$$v_\varepsilon = V_\varepsilon + \phi.$$

This translates into the fact that ϕ has to satisfy the nonlinear problem

$$\begin{cases} -\Delta\phi + \varepsilon\phi - pV_\varepsilon^{p-1}\phi = S_\varepsilon(V_\varepsilon) + N_\varepsilon(\phi) & \text{in } \Omega_\varepsilon, \\ \partial\phi/\partial\nu = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{5.3}$$

where

$$S_\varepsilon(V_\varepsilon) = \Delta V_\varepsilon - \varepsilon V_\varepsilon + V_\varepsilon^p, \tag{5.4}$$

$$N_\varepsilon(\phi) = (V_\varepsilon + \phi)^p - V_\varepsilon^p - pV_\varepsilon^{p-1}\phi. \tag{5.5}$$

Define

$$L_\varepsilon(\phi) = -\Delta\phi + \varepsilon\phi - pV_\varepsilon^{p-1}\phi.$$

Our strategy consists in solving the nonlinear problem (5.3) using a fixed point argument based on the contraction mapping principle. To do so, we need to establish some invertibility properties of the linear problem

$$L_\varepsilon(\phi) = f \quad \text{in } \Omega_\varepsilon, \qquad \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on } \partial\Omega_\varepsilon,$$

with $f \in L^2(\Omega_\varepsilon)$. We do this in two steps. First we study the above problem in a strip close to the scaled manifold $K_\varepsilon = \varepsilon^{-1}K$ in $\partial\Omega_\varepsilon$. Then we establish a complete theory for the problem in the whole domain Ω_ε; this is done in Section 7.

Let $\gamma \in (1/2, 1)$ be the number fixed before in (5.2) and consider

$$\Omega_{\varepsilon,\gamma} := \{x \in \Omega_\varepsilon : \operatorname{dist}(x, K_\varepsilon) < 2\varepsilon^{-\gamma}\}. \tag{5.6}$$

We are first interested in solving the following problem: given $f \in L^2(\Omega_{\varepsilon,\gamma})$,

$$\begin{cases} -\Delta\phi + \varepsilon\phi - pV_\varepsilon^{p-1}\phi = f & \text{in } \Omega_{\varepsilon,\gamma}, \\ \partial\phi/\partial\nu = 0 & \text{on } \partial\Omega_\varepsilon \cap \bar{\Omega}_{\varepsilon,\gamma}, \\ \phi = 0 & \text{in } \partial\Omega_{\varepsilon,\gamma} \setminus \partial\Omega_\varepsilon. \end{cases} \tag{5.7}$$

Observe that in the region we are considering, the function V_ε is nothing but $\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(W_{I+1,\varepsilon})$, where $W_{I+1,\varepsilon}$ is the function whose existence and properties are proven in Lemma 4.1. For the argument in this part of our proof it is enough to take $I = 3$, and for simplicity of notation we will denote by $\hat{w}$ the function $W_{I+1,\varepsilon}$ with $I = 3$. Referring to (4.3), we have

$$\hat{w}(z,\xi) = w_0(\xi) + \sum_{i=1}^{4} w_{i,\varepsilon}(z,\xi), \tag{5.8}$$

where w_0 is defined by (3.2) and

$$\|D_\xi^2 w_{i+1,\varepsilon}\|_{\varepsilon,N-2,\sigma} + \|D_\xi w_{i+1,\varepsilon}\|_{\varepsilon,N-3,\sigma} + \|w_{i+1,\varepsilon}\|_{\varepsilon,N-4,\sigma} \leq C\varepsilon^{1+i/2}, \tag{5.9}$$

and, for any integer ℓ,

$$\|\nabla_z^{(\ell)} w_{i+1,\varepsilon}(z,\cdot)\|_{\varepsilon,N-2,\sigma} \leq \beta C_l \varepsilon^{1+i/2}, \qquad z \in K_\varepsilon,$$

for any $i = 0, 1, 2, 3$.

We will establish a solvability theory for problem (5.7) in Section 6. For the moment, we devote the rest of this section to expanding the quadratic functional associated to (5.7).

Define

$$H_\varepsilon^1 = \{u \in H^1(\Omega_{\varepsilon,\gamma}) : u(x) = 0 \text{ for } x \in \partial\Omega_{\varepsilon,\gamma} \setminus \partial\Omega_\varepsilon\} \tag{5.10}$$

and the quadratic functional

$$E(\phi) = \frac{1}{2}\int_{\Omega_{\varepsilon,\gamma}} (|\nabla\phi|^2 + \varepsilon\phi^2 - pV_\varepsilon^{p-1}\phi^2) \tag{5.11}$$

for $\phi \in H_\varepsilon^1$.

Let $(z, X) \in \mathbb{R}^{k+N}$ be the local coordinates along K_ε introduced in (3.4); with abuse of notation we will denote

$$\phi(\Upsilon_\varepsilon(z, X)) = \phi(z, X). \tag{5.12}$$

Since the original variables $(z, X) \in \mathbb{R}^{k+N}$ (see (3.4)) are only local coordinates along K_ε, we let (z, X) vary in the set

$$\mathcal{C}_\varepsilon = \{(z, \bar{X}, X_N) : \varepsilon z \in K,\ 0 < X_N < \varepsilon^{-\gamma},\ |\bar{X}| < \varepsilon^{-\gamma}\}. \tag{5.13}$$

We write $\mathcal{C}_\varepsilon = \varepsilon^{-1}K \times \hat{\mathcal{C}}_\varepsilon$ where

$$\hat{\mathcal{C}}_\varepsilon = \{(\bar{X}, X_N) : 0 < X_N < \varepsilon^{-\gamma},\ |\bar{X}| < \varepsilon^{-\gamma}\}. \tag{5.14}$$

Observe that $\hat{\mathcal{C}}_\varepsilon$ approaches, as $\varepsilon \to 0$, the half-space $\mathbb{R}_+^N$.

In these new local coordinates, the energy density associated to the energy E in (5.11) is given by

$$\tfrac{1}{2}(|\nabla_{g^\varepsilon}\phi|^2 + \varepsilon\phi^2 - pV_\varepsilon^{p-1}\phi^2)\sqrt{\det g^\varepsilon}, \tag{5.15}$$

where ∇_{g^ε} denotes the gradient in the new variables and g^ε is the flat metric in $\mathbb{R}^{N+k}$ in the coordinates (z, X).

Having the expansion of the metric coefficients (see Lemma 3.1), we are in a position to expand the energy (5.11) in the new variables (z, X):

Lemma 5.1. *Let $(y, x) \in \mathbb{R}^{k+N}$ be the local coordinates along the submanifold K introduced in (2.8), and let (z, X) be the expanded variables introduced in (5.12). Assume that (z, X) vary in $\mathcal{C}_\varepsilon$ (see (5.13)). Then the energy functional (5.11) in the new variables (5.12) is given by*

$$\begin{aligned} E(\phi) = &\int_{K_\varepsilon \times \hat{\mathcal{C}}_\varepsilon} \tfrac{1}{2}(|\nabla_X \phi|^2 + \varepsilon \phi^2 - p V_\varepsilon^{p-1} \phi^2) \sqrt{\det g^\varepsilon}\, dz\, dX \\ &+ \int_{K_\varepsilon \times \hat{\mathcal{C}}_\varepsilon} \tfrac{1}{2} \Xi_{ij}(\varepsilon z, X) \partial_i \phi \partial_j \phi \sqrt{\det g^\varepsilon}\, dz\, dX \\ &+ \tfrac{1}{2} \int_{K_\varepsilon \times \hat{\mathcal{C}}_\varepsilon} |\nabla_{K_\varepsilon} \phi|^2 \sqrt{\det g^\varepsilon}\, dz\, dX + \int_{K_\varepsilon \times \hat{\mathcal{C}}_\varepsilon} B(\phi, \phi) \sqrt{\det g^\varepsilon}\, dz\, dX. \end{aligned} \tag{5.16}$$

In the above expression, we have

$$\Xi_{ij}(\varepsilon z, X) = 2\varepsilon H_{ij} X_N - \tfrac{1}{3} \varepsilon^2 R_{islj} X_l X_s, \tag{5.17}$$

we denote by $B(\phi, \phi)$ a quadratic term in ϕ that can be expressed in the form

$$\begin{aligned} B(\phi, \phi) = &\, O\big(\varepsilon^2 X_N^2 + \varepsilon^3 |\bar{X}|^3 + \varepsilon^3 X_N |\bar{X}|^2 + \varepsilon^3 X_N^2 |\bar{X}|\big) \partial_i \phi \partial_j \phi \\ &+ \varepsilon^2 |\nabla_{K_\varepsilon} \phi|^2 O(\varepsilon |X|) + \partial_j \phi \partial_{\bar{a}} \phi O(\varepsilon |\bar{X}| + \varepsilon^2 X_N^2), \end{aligned} \tag{5.18}$$

and we use the Einstein summation convention over repeated indices. Furthermore we use the notation $\partial_a = \partial_{y_a}$ and $\partial_{\bar{a}} = \partial_{z_a}$.

Proof. Our aim is to expand

$$\int \tfrac{1}{2}(|\nabla_{g^\varepsilon} \phi|^2 + \varepsilon \phi^2 - p V_\varepsilon^{p-1} \phi^2) \sqrt{\det g^\varepsilon}.$$

For simplicity we will omit the ε in the notation of g^ε. Recalling our convention about repeated indices, we write $|\nabla_{g^\varepsilon} \phi|^2$ as

$$|\nabla_{g^\varepsilon} \phi|^2 = (g^\varepsilon)^{NN} \partial_N \phi \partial_N \phi + (g^\varepsilon)^{ab} \partial_a \phi \partial_b \phi + (g^\varepsilon)^{ij} \partial_i \phi \partial_j \phi + 2 (g^\varepsilon)^{aj} \partial_a \phi \partial_j \phi,$$

where $(g^\varepsilon)^{\alpha\beta}$ represent the coefficients of the inverse of the metric $g^\varepsilon = (g^\varepsilon_{\alpha\beta})$. Using the expansion of the metric in Lemma 3.1, we see that

$$\begin{aligned} |\nabla_{g^\varepsilon} \phi|^2 = &\, |\partial_N \phi|^2 + |\partial_i \phi|^2 + \big(2\varepsilon H_{ij} X_N - \tfrac{1}{3} \varepsilon^2 R_{islj} X_s X_l\big) \partial_i \phi \partial_j \phi + \partial_{\bar{a}} \phi \partial_{\bar{a}} \phi (1 + \varepsilon |X|) \\ &+ O(\varepsilon^2 X_N^2 + \varepsilon^3 |X|^3) \partial_i \phi \partial_j \phi + O(\varepsilon X_N + \varepsilon^2 O(|X|^2)) \partial_{\bar{a}} \phi \partial_i \phi. \end{aligned}$$

This, together with the expansion of $\sqrt{\det g}$ given in Lemma 3.1, proves Lemma 5.1. □

Given $\phi \in H^1_\varepsilon$ (see (5.10)), we write

$$\phi = \left[\frac{\delta}{\mu_\varepsilon} \mathcal{T}_{\mu_\varepsilon, \Phi_\varepsilon}(Z_0) + \sum_{j=1}^{N-1} \frac{d^j}{\mu_\varepsilon} \mathcal{T}_{\mu_\varepsilon, \Phi_\varepsilon}(Z_j) + \frac{e}{\mu_\varepsilon} \mathcal{T}_{\mu_\varepsilon, \Phi_\varepsilon}(Z) \right] \bar{\chi}_\varepsilon + \phi^\perp, \tag{5.19}$$

where the expression $\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(v)$ is defined in (3.7), the functions Z_0 and Z_j are defined in (3.19) and where Z is the eigenfunction, with $\int_{\mathbb{R}^N} Z^2 = 1$, corresponding to the unique positive eigenvalue λ_0 in $L^2(\mathbb{R}^N)$ of the problem

$$\Delta_{\mathbb{R}^N}\phi + pw_0^{p-1}\phi = \lambda_0\phi \quad \text{in } \mathbb{R}^N. \tag{5.20}$$

It is worth mentioning that $Z(\xi)$ is even and it has exponential decay of order $O(e^{-\sqrt{\lambda_0}|\xi|})$ at infinity. The function $\bar{\chi}_\varepsilon$ is a smooth cut-off function defined by

$$\bar{\chi}_\varepsilon(X) = \hat{\chi}_\varepsilon\left(\left|\left(\frac{\bar{X}-\Phi_\varepsilon}{\mu_\varepsilon}, \frac{X_N}{\mu_\varepsilon}\right)\right|\right), \tag{5.21}$$

with $\hat{\chi}(r) = 1$ for $r \in \left(0, \frac{3}{2}\varepsilon^{-\gamma}\right)$, and $\chi(r) = 0$ for $r > 2\varepsilon^{-\gamma}$. Finally, in (5.19), $\delta = \delta(\varepsilon z)$, $d^j = d^j(\varepsilon z)$ and $e = e(\varepsilon z)$ are functions defined in K such that for all $z \in K_\varepsilon$

$$\int_{\hat{\mathcal{C}}_\varepsilon} \phi^\perp \mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)\bar{\chi}_\varepsilon = \int_{\hat{\mathcal{C}}_\varepsilon} \phi^\perp \mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j)\bar{\chi}_\varepsilon = \int_{\hat{\mathcal{C}}_\varepsilon} \phi^\perp \mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)\bar{\chi}_\varepsilon = 0. \tag{5.22}$$

We will denote by $(H_\varepsilon^1)^\perp$ the subspace of H_ε^1 consisting of the functions that satisfy the orthogonality conditions (5.22).

A direct computation shows that

$$\delta(\varepsilon z) = \frac{\int \phi \mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)}{\mu_\varepsilon \int Z_0^2}(1+O(\varepsilon^2)) + O(\varepsilon^2)\Big(\sum_j d^j(\varepsilon z) + e(\varepsilon z)\Big),$$

$$d^j(\varepsilon z) = \frac{\int \phi \mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j)}{\mu_\varepsilon \int Z_j^2}(1+O(\varepsilon^2)) + O(\varepsilon^2)\Big(\delta(\varepsilon z) + \sum_{i\neq j} d^i(\varepsilon z) + e(\varepsilon z)\Big),$$

$$e(\varepsilon z) = \frac{\int \phi \mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)}{\mu_\varepsilon \int Z^2}(1+O(\varepsilon^2)) + O(\varepsilon^2)\Big(\delta(\varepsilon z) + \sum_j d^j(\varepsilon z)\Big).$$

Observe that since $\phi \in H_\varepsilon^1$, one easily sees that δ, d^j and e belong to the Hilbert space

$$\mathcal{H}^1(K) = \{\zeta \in L^2(K) : \partial_a\zeta \in L^2(K),\ a = 1,\dots,k\}. \tag{5.23}$$

Thanks to the above decomposition (5.19), we have the following expansion for $E(\phi)$.

Theorem 2. *Let $\gamma = 1-\sigma$ for some $\sigma > 0$ small. Write $\phi \in H_\varepsilon^1$ as in (5.19) and let $d = (d^1,\dots,d^{N-1})$. Then there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$,*

$$E(\phi) = E(\phi^\perp) + \varepsilon^{-k}[P_\varepsilon(\delta) + Q_\varepsilon(d) + R_\varepsilon(e)] + \mathcal{M}(\phi^\perp, \delta, d, e). \tag{5.24}$$

Here

$$P_\varepsilon(\delta) = P(\delta) + P_1(\delta) \tag{5.25}$$

with

$$P(\delta) = \frac{A_\varepsilon}{2}\int_K \varepsilon^2|\partial_a(\delta(1+o(\varepsilon^2)\beta_1^\varepsilon(y)))|^2 + \varepsilon\frac{B}{2}\int_K \delta^2 \tag{5.26}$$

with A_ε a real number such that $\lim_{\varepsilon\to 0} A_\varepsilon = A := \int_{\mathbb{R}^N_+} Z_0^2$, $B = -\int_{\mathbb{R}^N_+} w_0 Z_0 > 0$ and β_1^ε is an explicit smooth function defined on K which is uniformly bounded as $\varepsilon \to 0$; furthermore, $P_1(\delta)$ is a small compact perturbation in $\mathcal{H}^1(K)$ which is a sum of quadratic functionals in δ of the form

$$\varepsilon^2 \int_K b(y)|\delta|^2,$$

where $b(y)$ denotes a generic explicit function, smooth and uniformly bounded, as $\varepsilon \to 0$, in K. Moreover, in (5.24),

$$Q_\varepsilon(d) = Q(d) + Q_1(d) \tag{5.27}$$

with

$$Q(d) = \frac{\varepsilon^2}{2} C_\varepsilon \left(\int_K |\partial_a(d(1+o(\varepsilon^2)\beta_2^\varepsilon(y)))|^2 + \int_K \big((\tilde{g}^\varepsilon)^{ab} R_{mabl} - \Gamma_a^c(E_m)\Gamma_c^a(E_l)\big) d^m d^l \right) \tag{5.28}$$

where C_ε is a real number such that $\lim_{\varepsilon\to 0} C_\varepsilon = C := \int_{\mathbb{R}^N_+} Z_1^2$, β_2^ε is an explicit smooth function defined on K which is uniformly bounded as $\varepsilon \to 0$, and the terms R_{mabl} and $\Gamma_a^c(E_m)$ are smooth functions on K defined respectively in (2.6) and (2.4). Furthermore, $Q_1(d)$ is a small compact perturbation in $\mathcal{H}^1(K)$ which is a sum of quadratic functionals in d of the form

$$\varepsilon^3 \int_K b(y) d^i d^j,$$

where again $b(y)$ is a generic explicit function, smooth and uniformly bounded, as $\varepsilon \to 0$, in K. Moreover, in (5.24),

$$R_\varepsilon(e) = R(e) + R_1(e), \tag{5.29}$$

$$R(e) = \varepsilon^{-k}\left[\frac{D_\varepsilon}{2} \left(\varepsilon^2 \int_K |\partial_a(e(1 + e^{-\frac{\lambda_0}{2}\varepsilon^{-\gamma}} \beta_3^\varepsilon(y)))|^2 - \lambda_0 \int_K e^2 \right) \right], \tag{5.30}$$

with D_ε a real number such that $\lim_{\varepsilon\to 0} D_\varepsilon = D := \int_{\mathbb{R}^N_+} Z^2$, β_3^ε an explicit smooth function in K, which is uniformly bounded as $\varepsilon \to 0$, and λ_0 the positive number defined in (5.20). Furthermore, R_1 is a small compact perturbation in $\mathcal{H}^1(K)$ which is a sum of quadratic functionals in e of the form

$$\varepsilon^3 \int_K b(y) e^2$$

where again $b(y)$ is a generic explicit function, smooth and uniformly bounded, as $\varepsilon \to 0$, in K. Finally, in (5.24),

$$\mathcal{M} : (H_\varepsilon^1)^\perp \times (\mathcal{H}^1(K))^{N+1} \to \mathbb{R}$$

is a continuous and differentiable functional with respect to the natural topologies, homogeneous of degree 2:

$$\mathcal{M}(t\phi^\perp, t\delta, td, te) = t^2 \mathcal{M}(\phi^\perp, \delta, d, e) \quad \text{for all } t.$$

The derivative of $\mathcal{M}$ with respect to each of its variables is a small multiple of a linear operator in $(\phi^\perp, \delta, d, e)$ and satisfies

$$\|D_{(\phi^\perp,\delta,d)}\mathcal{M}(\phi_1^\perp, \delta_1, d_1, e_1) - D_{(\phi^\perp,\delta,d)}\mathcal{M}(\phi_2^\perp, \delta_2, d_2, e_2)\| \le C\varepsilon^{\gamma(N-3)}$$
$$\times \left[\|\phi_1^\perp - \phi_2^\perp\| + \varepsilon^{-k}\|\delta_1 - \delta_2\|_{\mathcal{H}^1(K)} + \varepsilon^{-k}\|d_1 - d_2\|_{(\mathcal{H}^1(K))^{N-1}} + \varepsilon^{-k}\|e_1 - e_2\|_{\mathcal{H}^1(K)}\right]. \tag{5.31}$$

Furthermore, there exists a constant $C > 0$ such that

$$|\mathcal{M}(\phi^\perp, \delta, d, e)| \le C\varepsilon^2\left[\|\phi^\perp\|^2 + \varepsilon^{-2k}(\|\delta\|^2_{\mathcal{H}^1(K)} + \|d\|^2_{\mathcal{H}^1(K)} + \|e\|^2_{\mathcal{H}^1(K)})\right]. \tag{5.32}$$

We postpone the proof of Theorem 2 to Appendix 9.

6. Solving a linear problem close to the manifold K

In this section we study the problem of finding $\phi \in H^1_\varepsilon$ (see (5.10)) solving the linear problem (5.7) for a given $f \in L^2(\Omega_{\varepsilon,\gamma})$ (see (5.6)), and we establish a priori bounds for the solution. The result is contained in the following

Theorem 3. *There exist a constant $C > 0$ and a sequence $\varepsilon_l = \varepsilon \to 0$ such that for any $f \in L^2(\Omega_{\varepsilon,\gamma})$ there exists a solution $\phi \in H^1_\varepsilon$ to problem (5.7) such that*

$$\|\phi\|_{H^1_\varepsilon} \le C\varepsilon^{-\max(2,k)}\|f\|_{L^2(\Omega_{\varepsilon,\gamma})}. \tag{6.1}$$

The entire section is devoted to proving Theorem 3.

Given $\phi \in H^1_\varepsilon(\Omega_{\varepsilon,\gamma})$. As in (5.19), we have the decomposition

$$\phi = \left[\frac{\delta}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0) + \sum_{j=1}^{N-1}\frac{d^j}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j) + \frac{e}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)\right]\bar{\chi}_\varepsilon + \phi^\perp.$$

We then define the energy functional associated to problem (5.7),

$$\mathcal{E} : (H^1_\varepsilon)^\perp \times (\mathcal{H}^1(K))^{N+1} \to \mathbb{R},$$

by

$$\mathcal{E}(\phi^\perp, \delta, d, e) = E(\phi) - \mathcal{L}_f(\phi), \tag{6.2}$$

where E is the functional in (5.11) and $\mathcal{L}_f$ is the linear operator given by

$$\mathcal{L}_f(\phi) = \int_{\Omega_{\varepsilon,\gamma}} f\phi.$$

Observe that

$$\mathcal{L}_f(\phi) = \mathcal{L}^1_f(\phi^\perp) + \varepsilon^{-k}[\mathcal{L}^2_f(\delta) + \mathcal{L}^3_f(d) + \mathcal{L}^4_f(e)],$$

where $\mathcal{L}_f^1 : H_\varepsilon^1 \to \mathbb{R}$, $\mathcal{L}_f^2, \mathcal{L}_f^4 : \mathcal{H}^1(K) \to \mathbb{R}$ and $\mathcal{L}_f^3 : (\mathcal{H}^1(K))^{N-1} \to \mathbb{R}$ with

$$\mathcal{L}_f^1(\phi^\perp) = \int_{\Omega_{\varepsilon,\gamma}} f\phi^\perp, \qquad \varepsilon^{-k}\mathcal{L}_f^2(\delta) = \int_{\Omega_{\varepsilon,\gamma}} f\frac{\delta}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)\bar{\chi}_\varepsilon,$$

$$\varepsilon^{-k}\mathcal{L}_f^3(d) = \sum_{j=1}^{N-1}\int_{\Omega_{\varepsilon,\gamma}} f\frac{d^j}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\bar{\Phi}_\varepsilon}(Z_j)\bar{\chi}_\varepsilon, \qquad \varepsilon^{-k}\mathcal{L}_f^4(e) = \int_{\Omega_{\varepsilon,\gamma}} f\frac{e}{\bar{\mu}_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)\bar{\chi}_\varepsilon.$$

Finding a solution $\phi \in H_\varepsilon^1$ to (5.7) reduces to finding a critical point $(\phi^\perp, \delta, d, e)$ for $\mathcal{E}$. This will be done in several steps.

Step 1. We claim that there exist $\sigma > 0$ and ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $\phi^\perp \in (H_\varepsilon^1)^\perp$,

$$E(\phi^\perp) \ge \sigma\|\phi^\perp\|_{L^2}^2. \tag{6.3}$$

Using the local change of variables (3.4) and (5.12), together with Lemma 5.1, we see that, for sufficiently small $\varepsilon > 0$,

$$E(\phi^\perp) \ge \frac{1}{4}E_0(\phi^\perp) \quad \text{with} \quad E_0(\phi^\perp) = \int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon} [|\nabla_X\phi^\perp|^2 - pV_\varepsilon^{p-1}\phi^\perp]\sqrt{\det g^\varepsilon},$$

for any $\phi^\perp = \phi^\perp(\varepsilon z, X)$ with $z \in K_\varepsilon = \varepsilon^{-1}K$. The set $\hat{\mathcal{C}}_\varepsilon$ is defined in (5.14) and the function V_ε is given by (5.1). We recall that $\hat{\mathcal{C}}_\varepsilon \to \mathbb{R}_+^N$ as $\varepsilon \to 0$.

We will establish (6.3) by showing that

$$E_0(\phi^\perp) \ge \sigma\|\phi^\perp\|_{L^2}^2 \quad \text{for all } \phi^\perp. \tag{6.4}$$

To do so, we first observe that if we scale in the z-variable, defining $\varphi^\perp(y, X) = \phi^\perp(y/\varepsilon, X)$, the relation (6.4) becomes

$$E_0(\varphi^\perp) \ge \sigma\|\varphi^\perp\|_{L^2}^2. \tag{6.5}$$

Thus we are led to prove (6.5). For contradiction, assume that for any $n \in \mathbb{N}^*$, there exist $\varepsilon_n \to 0$ and $\varphi_n^\perp \in (H_{\varepsilon_n}^1)^\perp$ such that

$$E_0(\varphi_n^\perp) \le \frac{1}{n}\|\varphi_n^\perp\|_{L^2}^2. \tag{6.6}$$

Without loss of generality we can assume that the sequence $(\|\varphi_n^\perp\|)_n$ is bounded. Hence, up to subsequences, we have

$$\varphi_n^\perp \rightharpoonup \varphi^\perp \quad \text{in } H^1(K\times\mathbb{R}_+^N) \quad \text{and} \quad \varphi_n^\perp \to \varphi^\perp \quad \text{in } L^2(K\times\mathbb{R}_+^N).$$

Furthermore, using the estimate in (4.7) we get

$$\sup_{y\in K,\, X\in\mathbb{R}_+^N}\left|(1+|X|)^{N-4}\left[V_\varepsilon\left(\frac{y}{\varepsilon}, X\right) - \mu_0^{-(N-2)/2}(y)w_0\left(\frac{\bar{X}-\Phi_0(y)}{\mu_0(y)}, \frac{X_N}{\mu_0(y)}\right)\right]\right| \to 0$$

as $\varepsilon \to 0$, where μ_0 and Φ_0 are the smooth explicit functions defined in (4.6) and (4.15).

Letting $n \to \infty$ in (6.6) and applying the dominated convergence theorem, we get

$$\int_{K\times\mathbb{R}^N_+}\left[|\nabla_X\varphi^\perp|^2 - p\left(\mu_0^{-(N-2)/2}(y)w_0\left(\frac{\bar X-\Phi_0(y)}{\mu_0(y)},\frac{X_N}{\mu_0(y)}\right)\right)^{p-1}(\varphi^\perp)^2\right]dy\,dX \le 0. \tag{6.7}$$

Furthermore, passing to the limit in the orthogonality conditions we get, for any $y \in K$,

$$\int_{\mathbb{R}^N_+}\varphi^\perp(y,X)Z_0\left(\frac{\bar X-\Phi_0(y)}{\mu_0(y)},\frac{X_N}{\mu_0(y)}\right)dX = 0, \tag{6.8}$$

$$\int_{\mathbb{R}^N_+}\varphi^\perp(y,X)Z_j\left(\frac{\bar X-\Phi_0(y)}{\mu_0(y)},\frac{X_N}{\mu_0(y)}\right)dX = 0, \qquad j=1,\dots,N-1, \tag{6.9}$$

$$\int_{\mathbb{R}^N_+}\varphi^\perp(y,X)Z\left(\frac{\bar X-\Phi_0(y)}{\mu_0(y)},\frac{X_N}{\mu_0(y)}\right)dX = 0. \tag{6.10}$$

We thus get a contradiction with (6.7), since for any function $\varphi^\perp$ satisfying the orthogonality conditions (6.8)–(6.10) for any $y \in K$ one has

$$\int_{K\times\mathbb{R}^N_+}\left[|\nabla_X\varphi^\perp|^2 - p\left(\mu_0^{-(N-2)/2}(y)w_0\left(\frac{\bar X-\Phi_0(y)}{\mu_0(y)},\frac{X_N}{\mu_0(y)}\right)\right)^{p-1}(\varphi^\perp)^2\right]dy\,dX > 0$$

(see for instance [16, 45]).

Step 2. For all $\varepsilon > 0$ small, the functional $P_\varepsilon(\delta)$ defined in (5.25) is continuous and differentiable in $\mathcal{H}^1(K)$; furthermore, it is strictly convex and bounded from below since

$$P_\varepsilon(\delta) \ge \frac14\left[\frac{A}{2}\varepsilon^2\int_K|\partial_a\delta|^2 + \frac{B}{2}\varepsilon\int_K\delta^2\right] \ge \sigma\varepsilon^2\|\delta\|^2_{\mathcal{H}^1(K)} \tag{6.11}$$

for some small but fixed $\sigma > 0$. A direct consequence of these properties is that

$$\mathcal{H}^1(K) \ni \delta \mapsto P_\varepsilon(\delta) - L_f^2(\delta)$$

has a unique minimum δ, and furthermore

$$\varepsilon^{-k/2}\|\delta\|_{\mathcal{H}^1(K)} \le \mathcal{C}\varepsilon^{-2}\|f\|_{L^2(\Omega_{\varepsilon,\gamma})}$$

for a given positive constant $\mathcal{C}$.

Step 3. For all $\varepsilon > 0$ small, the functional Q_ε defined in (5.27) is a small perturbation in $(\mathcal{H}^1(K))^{N-1}$ of the quadratic form $\varepsilon^2 Q_0(d)$, defined by

$$\varepsilon^2 Q_0(d) = \frac{\varepsilon^2}{2}C\left[\int_K|\partial_a d|^2 + \int_K\left((\tilde g^\varepsilon)^{ab}R_{mabl} - \Gamma_a^c(E_m)\Gamma_c^a(E_l)\right)d^m d^l\right]$$

with $C := \int_{\mathbb{R}^N_+} Z_1^2$ and the terms R_{mabl} and $\Gamma_a^c(E_m)$ are smooth functions on K defined respectively in (2.6) and (2.4). Recall that the nondegeneracy assumption on the minimal submanifold K is equivalent to the invertibility of the operator $Q_0(d)$.

A consequence is that, for each $f \in L^2(\Omega_{\varepsilon,\gamma})$, the map

$$(\mathcal{H}^1(K))^{N-1} \to \mathbb{R}, \quad d \mapsto Q_\varepsilon(d) - \mathcal{L}^3_f(d),$$

has a unique critical point d, which satisfies

$$\varepsilon^{-k/2}\|d\|_{(\mathcal{H}^1(K))^{N-1}} \le \widetilde{\sigma}\varepsilon^{-2}\|f\|_{L^2(\Omega_{\varepsilon,\gamma})}$$

for some $\widetilde{\sigma} > 0$.

Step 4. Let $f \in L^2(\Omega_{\varepsilon,\gamma})$ and assume that e is a given (fixed) function in $\mathcal{H}^1(K)$. We claim that for all $\varepsilon > 0$ small enough, the functional $\mathcal{G} : (H^1_\varepsilon)^\perp \times (\mathcal{H}^1(K))^N \to \mathbb{R}$ given by

$$(\phi^\perp, \delta, d) \mapsto \mathcal{E}(\phi^\perp, \delta, d, e)$$

has a critical point $(\phi^\perp, \delta, d)$. Furthermore there exists a positive constant C, independent of ε, such that

$$\|\phi^\perp\| + \varepsilon^{-k/2}[\|\delta\|_{\mathcal{H}^1(K)} + \|d\|_{(\mathcal{H}^1(K))^{N-1}}] \le C\varepsilon^{-2}[\|f\|_{L^2(\Omega_{\varepsilon,\gamma})} + \varepsilon^{-k/2}\varepsilon^2\|e\|_{\mathcal{H}^1(K)}]. \tag{6.12}$$

To prove the above assertion, we first consider the functional

$$\mathcal{G}_0(\phi^\perp, \delta, d) = \mathcal{G}(\phi^\perp, \delta, d, e) - \mathcal{M}(\phi^\perp, \delta, d, e),$$

where $\mathcal{M}$ is the functional that collects all mixed terms, as defined in (5.24). A direct consequence of Steps 1–3 is that $\mathcal{G}_0$ has a critical point $(\phi^\perp = \phi^\perp(f), \delta = \delta(f), d = d(f))$, that is, the system

$$D_{\phi^\perp}E(\phi^\perp) = D_{\phi^\perp}\mathcal{L}^1_f(\phi^\perp), \quad \varepsilon^{-k/2}D_\delta P_\varepsilon(\delta) = D_\delta\mathcal{L}^2_f(\delta), \quad \varepsilon^{-k/2}D_d Q_\varepsilon(d) = D_d\mathcal{L}^3_f(d)$$

is uniquely solvable in $(H^1_\varepsilon)^\perp \times (\mathcal{H}^1(K))^N$, and furthermore

$$\|\phi^\perp\|_{H^1_\varepsilon} + \varepsilon^{-k/2}\|\delta\|_{\mathcal{H}^1(K)} + \varepsilon^{-k/2}\|d\|_{(\mathcal{H}^1(K))^{N-1}} \le C\varepsilon^{-2}\|f\|_{L^2(\Omega_{\varepsilon,\gamma})}$$

for some constant $C > 0$, independent of ε.

If we now consider the complete functional $\mathcal{G}$, a critical point of $\mathcal{G}$ will satisfy the system

$$\begin{cases} D_{\phi^\perp}E(\phi^\perp) = D_{\phi^\perp}\mathcal{L}^1_f(\phi^\perp) + D_{\phi^\perp}\mathcal{M}(\phi^\perp, \delta, d, e), \\ D_\delta P_\varepsilon(\delta) = D_\delta\mathcal{L}^2_f(\delta) + D_\delta\mathcal{M}(\phi^\perp, \delta, d, e), \\ D_d Q_\varepsilon(d) = D_d\mathcal{L}^3_f(d) + D_d\mathcal{M}(\phi^\perp, \delta, d, e). \end{cases} \tag{6.13}$$

On the other hand as already observed in Theorem 2,

$$\|D_{(\phi^\perp,\delta,d)}\mathcal{M}(\phi^\perp_1, \delta_1, d_1, e_1) - D_{(\phi^\perp,\delta,d)}\mathcal{M}(\phi^\perp_2, \delta_2, d_2, e_2)\| \le C\varepsilon^2$$
$$\times\big[\|\phi^\perp_1 - \phi^\perp_2\| + \varepsilon^{-k/2}\|\delta_1 - \delta_2\|_{\mathcal{H}^1(K)} + \varepsilon^{-k/2}\|d_1 - d_2\|_{(\mathcal{H}^1(K))^{N-1}} + \varepsilon^{-k/2}\|e_1 - e_2\|_{\mathcal{H}^1(K)}\big].$$

Thus the contraction mapping theorem guarantees the existence of a unique solution $(\bar\phi^\perp, \bar\delta, \bar d)$ to (6.13) in the set

$$\|\phi^\perp\|_{H^1_\varepsilon} + \varepsilon^{-k/2}\|\delta\|_{\mathcal{H}^1(K)} + \varepsilon^{-k/2}\|d\|_{(\mathcal{H}^1(K))^{N-1}} \le C[\varepsilon^{-2}\|f\|_{L^2(\Omega_{\varepsilon,\gamma})} + \varepsilon^2\varepsilon^{-k/2}\|e\|_{\mathcal{H}^1(K)}].$$

Furthermore, the solution $\bar\phi^\perp = \bar\phi^\perp(f,e)$, $\bar\delta = \bar\delta(f,e)$ and $\bar d = \bar d(f,e)$ depends on e in a smooth and nonlocal way.

Step 5. Given $f \in L^2(\Omega_{\varepsilon,\gamma})$, we insert the critical point $(\bar\phi^\perp = \bar\phi^\perp(f,e), \bar\delta = \bar\delta(f,e), \bar d = \bar d(f,e))$ of $\mathcal{G}$ obtained in the previous step into the functional $\mathcal{E}(\phi^\perp, \delta, d, e)$, thus getting a new functional depending only on $e \in \mathcal{H}^1(K)$, which we denote by $\mathcal{F}_\varepsilon(e)$, given by

$$\mathcal{F}_\varepsilon(e) = \varepsilon^{-k}[R_\varepsilon(e) - \mathcal{L}^4_f(e)] + E(\bar\phi^\perp(e)) - \varepsilon^{-k}\mathcal{L}^1_f(\bar\phi^\perp(e)) + \varepsilon^{-k}[P_\varepsilon(\bar\delta(e)) - \mathcal{L}^2_f(\bar\delta(e))] + \varepsilon^{-k}[Q_\varepsilon(\bar d(e)) - \mathcal{L}^3_f(\bar d(e))] + \mathcal{M}(\bar\phi^\perp(e), \bar\delta(e), \bar d(e), e).$$

The rest of the proof is devoted to showing that there exists a sequence $\varepsilon = \varepsilon_l \to 0$ such that

$$D_e\mathcal{F}_\varepsilon(e) = 0 \tag{6.14}$$

is solvable. Using the fact that $(\bar\phi^\perp, \bar\delta, \bar d)$ is a critical point for $\mathcal{G}$ (see Step 4 for the definition), we find that

$$D_e\mathcal{F}_\varepsilon(e) = \varepsilon^{-k}D_e[R_\varepsilon(e) - \mathcal{L}^4_f(e)] + D_e\mathcal{M}(\bar\phi^\perp(e), \bar\delta(e), \bar d(e), e). \tag{6.15}$$

Define

$$\mathcal{L}_\varepsilon := \varepsilon^{-k}D_eR_\varepsilon(e) + D_e\mathcal{M}(\bar\phi^\perp(e), \bar\delta(e), \bar d(e), e), \tag{6.16}$$

regarded as a self-adjoint operator in $L^2(K)$. The work to solve the equation $D_e\mathcal{F}_\varepsilon(e) = 0$ consists in showing the existence of a sequence $\varepsilon_l \to 0$ such that 0 lies suitably far away from the spectrum of $\mathcal{L}_{\varepsilon_l}$.

We recall now that the map

$$(\phi^\perp, \delta, d, e) \mapsto D_e\mathcal{M}(\phi^\perp, \delta, d, e)$$

is a linear operator in the variables $\phi^\perp, \delta, d$, while it is constant in e. This is stated in Theorem 2. If we furthermore take into account that the terms $\bar\phi^\perp$, $\bar\delta$ and $\bar d$ depend smoothly and in a nonlocal way on e, we conclude that, for any $e \in \mathcal{H}^1(K)$,

$$D_e\mathcal{M}(\bar\phi^\perp(e), \bar\delta(e), \bar d(e), e)[e] = \varepsilon^{\gamma(N-3)}\varepsilon^{-k}\int_K (\varepsilon\eta_1(e)\partial_a e + \eta_2(e)e)^2, \tag{6.17}$$

where η_1 and η_2 are nonlocal operators in e, which are bounded, as $\varepsilon \to 0$, on bounded subsets of $L^2(K)$. Thanks to Theorem 2 and the above observation, we conclude that the quadratic from

$$\Upsilon_\varepsilon(e) := \varepsilon^{-k}D_eR_\varepsilon(e)[e] + D_e\mathcal{M}(\bar\phi^\perp(e), \bar\delta(e), \bar d(e), e)[e]$$

can be written as follows:

$$\tilde{\Upsilon}_\varepsilon(e) = \varepsilon^k \Upsilon_\varepsilon(e) = \Upsilon_\varepsilon^0(e) - \bar{\lambda}_0 \int_K e^2 + \varepsilon \Upsilon_\varepsilon^1(e), \tag{6.18}$$

where

$$\Upsilon_\varepsilon^0(e) = \varepsilon^2 \int_K (1 + \varepsilon^{\gamma(N-3)}\eta_1(e))|\partial_a(e(1 + e^{-\varepsilon^{-\lambda'}}\beta_3^\varepsilon(y)))|^2. \tag{6.19}$$

In the above expression, $\bar{\lambda}_0$ is the positive number defined by

$$\bar{\lambda}_0 = \left(\int_{\mathbb{R}_+^N} Z_1^2\right)\lambda_0,$$

$\Upsilon_e^1(e)$ is a compact quadratic form in $\mathcal{H}^1(K)$, and β_3^ε is a smooth and bounded (as $\varepsilon \to 0$) function on K, given by (5.30). Finally, η_1 is a nonlocal operator in e, which is uniformly bounded as $\varepsilon \to 0$ on bounded sets of $L^2(K)$.

Thus, for any $\varepsilon > 0$, the eigenvalues of

$$\mathcal{L}_\varepsilon e = \lambda e, \qquad e \in \mathcal{H}^1(K),$$

form a sequence $\lambda_j(\varepsilon)$, characterized by the Courant–Fisher formulas

$$\lambda_j(\varepsilon) = \sup_{\dim(M)=j-1} \inf_{e\in M^\perp\setminus\{0\}} \frac{\tilde{\Upsilon}_\varepsilon(e)}{\int_K e^2} = \inf_{\dim(M)=j} \sup_{e\in M\setminus\{0\}} \frac{\tilde{\Upsilon}_\varepsilon(e)}{\int_K e^2}. \tag{6.20}$$

The proof of Theorem 3 and of the inequality (6.1) will then follow from Step 4 and formula (6.12), together with

Lemma 6.1. *There exist a sequence $\varepsilon_l \to 0$ and a constant $c > 0$ such that, for all j,*

$$|\lambda_j(\varepsilon_l)| \geq c\varepsilon_l^k. \tag{6.21}$$

The proof of this lemma follows closely the proof of a related result established in [15], but we reproduce it for completeness. We shall thus devote the rest of this section to proving Lemma 6.1.

We write $\Sigma_\varepsilon(e) = \tilde{\Upsilon}_\varepsilon(e)/\int_K e^2$.

For notational convenience, we let $\sigma = \varepsilon^2$. We are thus interested in the eigenvalue problem

$$\mathcal{L}_\sigma \eta = \lambda\eta, \qquad \eta \in \mathcal{H}^1(K). \tag{6.22}$$

With this notation and using (6.18) and (6.19) together with the fact that $\gamma(N-3) > 2$, we have

$$\Sigma_\sigma(e) = \frac{\sigma\int_K (1 + o(\sigma)\eta_1(e))|\partial_a(e(1 + o(\sigma)\beta_3^\sigma(y)))|^2}{\int_K e^2} - \bar{\lambda}_0 + \sqrt{\sigma}\,\frac{\Upsilon_\sigma^1(e)}{\int_K e^2},$$

where $o(\sigma) \to 0$ as $\sigma \to 0$.

We claim that there exists a number $\delta > 0$ such that for any $\sigma_2 > 0$ and for any $j \geq 1$ such that

$$\sigma_2 + |\lambda_j(\sigma_2)| < \delta$$

and any σ_1 with $\sigma_2/2 < \sigma_1 < \sigma_2$, we have

$$\lambda_j(\sigma_1) < \lambda_j(\sigma_2). \tag{6.23}$$

To prove this, we start by observing that, since β_3^σ is an explicit, smooth and bounded (as $\sigma \to 0$) function on K, given by (5.30), and since η_1 is a nonlocal operator in e, which is uniformly bounded as $\sigma \to 0$ on bounded sets of $L^2(K)$, we have

$$\lim_{\sigma\to 0} \frac{\int_K (1 + o(\sigma)\eta_1(e))|\partial_a(e(1 + o(\sigma)\beta_3^\sigma(y)))|^2}{\int_K e^2} = \frac{\int_K |\partial_a e|^2}{\int_K e^2} \tag{6.24}$$

and

$$\lim_{\sigma\to 0} \sqrt{\sigma}\, \frac{\Upsilon_\sigma^1(e)}{\int_K e^2} = 0 \tag{6.25}$$

uniformly for any e.

Consider now two numbers $0 < \sigma_1 < \sigma_2$. Then for any e with $\int_K e^2 = 1$, we have

$$\begin{aligned}\sigma_1^{-1}\Sigma_{\sigma_1}(e) - \sigma_2^{-1}\Sigma_{\sigma_2}(e) = -\bar{\lambda}_0 \frac{\sigma_2 - \sigma_1}{\sigma_1\sigma_2} + \sigma_1^{-1}\Upsilon_{\sigma_1}^0(e) - \sigma_2^{-1}\Upsilon_{\sigma_2}^0(e)\\ + \sigma_1^{-1/2}\Upsilon_{\sigma_1}^1(e) - \sigma_1^{-1/2}\Upsilon_{\sigma_1}^1(e).\end{aligned}$$

A consequence of (6.24) and (6.25) is that there exists $\sigma^* > 0$ such that, for all $\sigma_1 < \sigma_2 < \sigma^*$,

$$|\sigma_1^{-1}\Upsilon_{\sigma_1}^0(e) - \sigma_2^{-1}\Upsilon_{\sigma_2}^0(e)| \leq c(\sigma_2 - \sigma_1)$$

and

$$|\sigma_1^{-1/2}\Upsilon_{\sigma_1}^1(e) - \sigma_2^{-1/2}\Upsilon_{\sigma_2}^1(e)| \leq c\frac{\sigma_2 - \sigma_1}{\sqrt{\sigma_1\sigma_2}(\sigma_1 + \sigma_2)}$$

for some constant c, and uniformly for any e with $\int_K e^2 = 1$. Thus, for some $\gamma_-, \gamma_+ > 0$,

$$\sigma_1^{-1}\Sigma_{\sigma_1}(e) + (\sigma_2 - \sigma_1)\frac{\gamma_-}{2\sigma_2^2} \leq \sigma_2^{-1}\Sigma_{\sigma_2}(e) \leq \sigma_1^{-1}\Sigma_{\sigma_1}(e) + (\sigma_2 - \sigma_1)\frac{2\gamma_+}{\sigma_1^2}$$

for any $\sigma_1 < \sigma_2 < \sigma^*$ and any e with $\int_K e^2 = 1$. In particular, there exists σ^* such that for all $0 < \sigma_1 < \sigma_2 < \sigma^*$ and $j \geq 1$,

$$(\sigma_2 - \sigma_1)\frac{\gamma_-}{2\sigma_2^2} \leq \sigma_2^{-1}\lambda_j(\sigma_2) - \sigma_1^{-1}\lambda_j(\sigma_1) \leq 2(\sigma_2 - \sigma_1)\frac{\gamma_+}{\sigma_1^2}. \tag{6.26}$$

From (6.26) it follows directly that, for all $j \geq 1$, the function $(0, \sigma^*) \ni \sigma \mapsto \lambda_j(\sigma)$ is continuous. If we now assume that $\sigma_1 \geq \sigma_2/2$, formula (6.26) gives

$$\lambda_j(\sigma_1) \leq \lambda_j(\sigma_2) + \frac{\sigma_1 - \sigma_2}{\sigma_2}\left[\lambda_j(\sigma_2) + \gamma\frac{\sigma_1}{\sigma_2}\right] \tag{6.27}$$

for some $\gamma > 0$. This proves (6.23).

We will find a sequence $\sigma_l \in (2^{-(l+1)}, 2^{-l})$ for l large as in the statement of the lemma.

Define

$$Ł = \{\sigma \in (2^{-(l+1)}, 2^{-l}) : \ker \mathcal{L}_\sigma \neq \{0\}\}.$$

If $\sigma \in Ł$ then $\lambda_j(\sigma) = 0$ for some j. Choosing l sufficiently large, the continuity of the function $\sigma \mapsto \lambda_j(\sigma)$ together with (6.23) implies that $\lambda_j(2^{-(l+1)}) < 0$. In other words, for all l sufficiently large,

$$\mathrm{card}(Ł) \leq N(2^{-(l+1)}), \tag{6.28}$$

where $N(\sigma)$ denotes the number of negative eigenvalues of problem (6.22). We next estimate $N(\sigma)$ for small values of σ. To do so, let $a > 0$ be a positive constant such that $a > \bar{\lambda}_0$ and consider the operator

$$\mathcal{L}_\sigma^+ = -\Delta_K - a/\sigma. \tag{6.29}$$

We call its eigenvalues $\lambda_j^+(\sigma)$. The Courant–Fisher characterization of eigenvalues gives $\lambda_j(\sigma) \leq \lambda_j^+(\sigma)$ for all j and all σ small. Thus $N(\sigma) \leq N_+(\sigma)$, where $N_+(\sigma)$ is the number of negative eigenvalues of (6.29).

Denote now by μ_j the eigenvalues of $-\Delta_K$ (ordered so as to be nondecreasing in j, and counted with multiplicity). Weyl's asymptotic formula (see for instance [11]) states that

$$\mu_j = C_K j^{2/k} + o(j^{2/k}) \quad \text{ as } j \to \infty$$

for some positive constant C_K depending only on the dimension k of K. Since $\lambda_j^+ = \mu_j - a/\sigma$, we get

$$N_+(\sigma) = C\sigma^{-k/2} + o(\sigma^{-k/2}) \quad \text{ as } \sigma \to 0.$$

This fact, together with (6.28), gives $\mathrm{card}(Ł) \leq C2^{lk/2}$. Hence there exists an interval $(a_l, b_l) \subset (2^{-(l+1)}, 2^{-l})$ such that $a_l, b_l \in Ł_l$, and all σ with $\ker \mathcal{L}_\sigma \neq \{0\}$ are in (a_l, b_l) so that

$$b_l - a_l \geq \frac{2^{-l} - 2^{-(l+1)}}{\mathrm{card}(Ł_l)} \geq C2^{-l(1+k/2)}. \tag{6.30}$$

Let $\sigma_l = (a_l + b_l)/2$. We will show that this sequence satisfies the statement of Lemma 6.1 and the corresponding estimate (6.21). For contradiction, assume that for some j we have

$$|\lambda_j(\sigma_l)| \leq \delta\sigma_l^{k/2} \tag{6.31}$$

for some $\delta > 0$ arbitrarily small. Assume first that $0 < \lambda_j(\sigma_l) < \delta\sigma_l^{k/2}$. Then from (6.27) we get

$$\lambda_j(a_l) \leq \lambda_j(\sigma_l) - \frac{\sigma_l - a_l}{\sigma_l}\left[\lambda_j(\sigma_l) + \gamma\frac{a_l}{2\sigma_l}\right]$$

and using (6.30)–(6.31) we get

$$\lambda_j(a_l) \leq \delta\sigma_l^{k/2} - C\frac{2^{-l(1+k/2)}}{2\sigma_l}\left[\lambda_j(\sigma_l) + \frac{\gamma a_l}{2\sigma_l}\right] < 0,$$

for δ small. From this it follows that $\lambda_j(\sigma)$ must vanish at some $\sigma \in (a_l, b_l)$, contrary to the choice of the interval (a_l, b_l).

The case $-\delta\sigma_l^{k/2} < \lambda_j(\sigma_l) < 0$ can be treated in a very similar way.

This concludes the proof of Lemma 6.1.

7. Proof of the result

In this section, we show the existence of a solution to problem (1.14) of the form

$$v_\varepsilon = V_\varepsilon + \phi,$$

where V_ε is defined in (5.1). As already observed at the end of Section 4, this reduces to finding a solution ϕ to

$$\begin{cases} -\Delta\phi + \varepsilon\phi - pV_\varepsilon^{p-1}\phi = S_\varepsilon(V_\varepsilon) + N_\varepsilon(\phi) & \text{in } \Omega_\varepsilon, \\ \partial\phi/\partial\nu = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{7.1}$$

where $S_\varepsilon(V_\varepsilon)$ is defined in (5.4), and $N_\varepsilon(\phi)$ in (5.5).

Given the result of Lemma 4.1, a first fact is that

$$\|S_\varepsilon(V_\varepsilon)\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon^{1+(I+1)/2} \tag{7.2}$$

as a direct consequence of estimate (4.8).

Define $L_\varepsilon\phi := -\Delta\phi + \varepsilon\phi - pV_\varepsilon^{p-1}\phi$. We claim that there exist a sequence $\varepsilon_l \to 0$ and a positive constant $C > 0$ such that, for any $f \in L^2(\Omega_{\varepsilon_l})$, there exists a solution $\phi \in H^1(\Omega_{\varepsilon_l})$ to the equation

$$L_{\varepsilon_l}\phi = f \quad \text{in } \Omega_{\varepsilon_l}, \qquad \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on } \partial\Omega_{\varepsilon_l}.$$

Furthermore,

$$\|\phi\|_{H^1(\Omega_{\varepsilon_l})} \le C\varepsilon_l^{-\max\{2,k\}}\|f\|_{L^2(\Omega_{\varepsilon_l})}. \tag{7.3}$$

We postpone the proof of this fact for the moment. For simplicity of notation we omit the dependence of ε on l setting $\varepsilon_l = \varepsilon$. Thus $\phi \in H^1(\Omega_\varepsilon)$ is a solution to (7.1) if and only if

$$\phi = L_\varepsilon^{-1}(S_\varepsilon(V_\varepsilon) + N_\varepsilon(\phi)).$$

Notice that

$$\|N_\varepsilon(\phi)\|_{L^2(\Omega_\varepsilon)} \le C \begin{cases} \|\phi\|^p_{H^1(\Omega_\varepsilon)} & \text{for } p \le 2, \\ \|\phi\|^2_{H^1(\Omega_\varepsilon)} & \text{for } p > 2, \end{cases} \qquad \|\phi\|_{H^1(\Omega_\varepsilon)} \le 1, \tag{7.4}$$

and

$$\begin{aligned} &\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\|_{L^2(\Omega_\varepsilon)} \\ &\quad \le C \begin{cases} (\|\phi_1\|^{p-1}_{H^1(\Omega_\varepsilon)} + \|\phi_2\|^{p-1}_{H^1(\Omega_\varepsilon)})\|\phi_1 - \phi_2\|_{H^1(\Omega_\varepsilon)} & \text{for } p \le 2, \\ (\|\phi_1\|_{H^1(\Omega_\varepsilon)} + \|\phi_2\|_{H^1(\Omega_\varepsilon)})\|\phi_1 - \phi_2\|_{H^1(\Omega_\varepsilon)} & \text{for } p > 2, \end{cases} \end{aligned} \tag{7.5}$$

for any ϕ_1, ϕ_2 in $H^1(\Omega_\varepsilon)$ with $\|\phi_1\|_{H^1(\Omega_\varepsilon)}, \|\phi_2\|_{H^1(\Omega_\varepsilon)} \le 1$.

Defining $T_\varepsilon : H^1(\Omega_\varepsilon) \to H^1(\Omega_\varepsilon)$ as

$$T_\varepsilon(\phi) = L_\varepsilon^{-1}(S_\varepsilon(V_\varepsilon) + N_\varepsilon(\phi))$$

we will show that T_ε is a contraction in some small ball in $H^1(\Omega_\varepsilon)$. A direct consequence of (7.2)–(7.3) is that

$$\|T_\varepsilon(\phi)\|_{H^1(\Omega_\varepsilon)} \le C\varepsilon^{-\max\{2,k\}} \begin{cases} (\varepsilon^{1+(I+1)/2} + \|\phi\|^p_{H^1(\Omega_\varepsilon)}) & \text{for } p \le 2, \\ (\varepsilon^{1+(I+1)/2} + \|\phi\|^2_{H^1(\Omega_\varepsilon)}) & \text{for } p > 2. \end{cases}$$

Now we choose integers d and I so that

$$d > \begin{cases} \max\{2,k\}/p - 1 & \text{for } p \le 2, \\ \max\{2,k\} & \text{for } p > 2, \end{cases} \qquad I > d - 1 + \max\{2,k\}.$$

Then one easily sees that T_ε has a unique fixed point in

$$\mathcal{B} = \{\phi \in H^1(\Omega_\varepsilon) : \|\phi\|_{H^1(\Omega_\varepsilon)} \le \varepsilon^d\},$$

as a direct application of the contraction mapping theorem. This concludes the proof of Theorem 1.

We next prove the claim ending with (7.3). For contradiction, assume that for all $\varepsilon \to 0$ there exists a solution $(\phi_\varepsilon, \lambda_\varepsilon)$, $\phi_\varepsilon \ne 0$, to

$$L_\varepsilon(\phi_\varepsilon) := \Delta\phi_\varepsilon - \varepsilon\phi_\varepsilon + pV_\varepsilon^{p-1}\phi_\varepsilon = \lambda_\varepsilon\phi_\varepsilon \quad \text{in } \Omega_\varepsilon, \qquad \frac{\partial\phi_\varepsilon}{\partial\nu} = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{7.6}$$

with

$$|\lambda_\varepsilon|\varepsilon^{-\max\{2,k\}} \to 0 \quad \text{as } \varepsilon \to 0. \tag{7.7}$$

Let η_ε be a smooth cut-off function (like the one defined in (5.2)) so that $\eta_\varepsilon = 1$ if $\operatorname{dist}(y, K_\varepsilon) < \varepsilon^{-\gamma}/2$ and $\eta_\varepsilon = 0$ if $\operatorname{dist}(y, K_\varepsilon) > \varepsilon^{-\gamma}$. In particular $|\nabla\eta_\varepsilon| \le c\varepsilon^\gamma$ and $|\Delta\eta_\varepsilon| \le c\varepsilon^{2\gamma}$, in the whole domain.

Define $\tilde\phi_\varepsilon = \phi_\varepsilon\eta_\varepsilon$. Then $\tilde\phi_\varepsilon$ solves

$$\begin{cases} L_\varepsilon(\tilde\phi_\varepsilon) = \lambda_\varepsilon\tilde\phi_\varepsilon - \nabla\eta_\varepsilon\nabla\phi_\varepsilon - \Delta\eta_\varepsilon\phi_\varepsilon & \text{in } \Omega_{\varepsilon,\gamma} \\ \partial\phi_\varepsilon/\partial\nu = 0 & \text{on } \partial\Omega_\varepsilon \setminus \overline{\Omega}_{\varepsilon,\gamma}, \\ \phi_\varepsilon = 0 & \text{in } \partial\Omega_\varepsilon \cap \partial\Omega_{\varepsilon,\gamma}, \end{cases} \tag{7.8}$$

where $\Omega_{\varepsilon,\gamma}$ is defined in (5.6). Now Theorem 3 guarantees the existence of a sequence $\varepsilon_l \to 0$ and a constant c such that

$$\|\tilde\phi_{\varepsilon_l}\|_{H^1_{\varepsilon_l}} \le c\varepsilon_l^{-\max\{2,k\}}\big[\lambda_{\varepsilon_l}\|\tilde\phi_{\varepsilon_l}\|_{L^2} + \|\nabla\eta_{\varepsilon_l}\nabla\phi_{\varepsilon_l}\|_{L^2} + \|\Delta\eta_{\varepsilon_l}\phi_{\varepsilon_l}\|_{L^2}\big]. \tag{7.9}$$

Observe now that, in the region where $\nabla\eta_{\varepsilon_l} \ne 0$ and $\Delta\eta_{\varepsilon_l} \ne 0$, the function V_{ε_l} can be uniformly bounded as $|V_\varepsilon(y)| \le c\varepsilon$, with a positive constant c; this follows directly from (5.1) and (4.7). Furthermore, since we are assuming (7.7), we see that in the region we are considering, namely where $\nabla\eta_{\varepsilon_l} \ne 0$ and $\Delta\eta_{\varepsilon_l} \ne 0$, the function ϕ_{ε_l} satisfies $-\Delta\phi_{\varepsilon_l} + \varepsilon_l a_{\varepsilon_l}(y)\phi_{\varepsilon_l} = 0$ for a certain smooth function a_{ε_l} which is uniformly positive and bounded as $\varepsilon_l \to 0$. Elliptic estimates show that, in this region, $|\phi_{\varepsilon_l}| \leq ce^{-\varepsilon_l^{\gamma'}}$, and $|\nabla\phi_{\varepsilon_l}| \leq ce^{-\varepsilon_l^{\gamma'}}$ for some $\gamma' > 0$ and $c > 0$. Inserting this in (7.9), it is easy to see that

$$\|\tilde{\phi}_{\varepsilon_l}\|_{H^1_{\varepsilon_l}} \leq c\varepsilon_l^{-\max\{2,k\}}\lambda_{\varepsilon_l}\|\tilde{\phi}_{\varepsilon_l}\|_{H^1_{\varepsilon_l}}(1+o(1)),$$

where $o(1) \to 0$ as $\varepsilon_l \to 0$. Taking into account (7.7), the above inequality contradicts the fact that ϕ_ε is never identically zero. This concludes the proof of the claim.

8. Appendix: Proof of Lemma 3.3

The proof is simply based on a Taylor expansion of the metric coefficients in terms of the geometric properties of $\partial\Omega$ and K, as in Lemma 3.1. Recall that the Laplace–Beltrami operator is given by

$$\Delta_{g^\varepsilon} = \frac{1}{\sqrt{\det g^\varepsilon}}\partial_A\left(\sqrt{\det g^\varepsilon}(g^\varepsilon)^{AB}\partial_B\right),$$

where A and B run between 1 and $n = N + k$. We can write

$$\mathcal{A}_{\mu_\varepsilon,\Phi_\varepsilon} = (g^\varepsilon)^{AB}\partial^2_{AB} + \partial_A(g^\varepsilon)^{AB}\partial_B + \partial_A\left(\log\sqrt{\det g^\varepsilon}\right)(g^\varepsilon)^{AB}\partial_B.$$

Now, if v and W are defined as in (3.5), one has

$$\begin{aligned}
\mu_\varepsilon^{(N+2)/2}\partial^2_{X_iX_j}v &= \partial^2_{\xi_i\xi_j}W(z,\xi) = \partial^2_{ij}W,\\
\mu_\varepsilon^{(N+2)/2}\partial^2_{X_N}v &= \partial^2_{\xi_N}W(z,\xi) = \partial^2_{NN}W,\\
\mu_\varepsilon^{(N+2)/2}\partial^2_{z_aX_l}v &= -\frac{N}{2}\partial_{\bar a}\mu_\varepsilon\partial_l W + \mu_\varepsilon\partial^2_{\bar a l}W - \partial_{\bar a}\mu_\varepsilon\xi_J\partial^2_{lJ}W - \partial_{\bar a}\Phi^j\partial^2_{lj}W
\end{aligned}$$

and

$$\begin{aligned}
\mu_\varepsilon^{(N+2)/2}\partial^2_{z_az_b}v = {}& \frac{N(N-2)}{4}\partial_{\bar a}\mu_\varepsilon\partial_{\bar b}\mu_\varepsilon W - \frac{N-2}{2}\mu_\varepsilon(\partial_{\bar a}\mu_\varepsilon\partial_{\bar b}W + \partial_{\bar b}\mu_\varepsilon\partial_{\bar a}W)\\
&+ N\partial_{\bar a}\mu_\varepsilon\partial_{\bar b}\mu_\varepsilon\xi_J\partial_J W + \frac{N}{2}\{\partial_{\bar a}\mu_\varepsilon\partial_{\bar b}\Phi^j + \partial_{\bar b}\mu_\varepsilon\partial_{\bar a}\Phi^j\}\partial_j W\\
&- \frac{N-2}{2}\mu_\varepsilon\partial^2_{\bar a\bar b}\mu_\varepsilon W + \mu_\varepsilon^2\partial^2_{\bar a\bar b}W - \mu_\varepsilon\partial^2_{\bar a\bar b}\mu_\varepsilon\xi_J\partial_J W - \mu_\varepsilon\partial^2_{\bar a\bar b}\Phi^j\partial_j W\\
&- \mu_\varepsilon(\partial_{\bar b}\mu_\varepsilon\xi_J\partial^2_{J\bar a}W + \partial_{\bar a}\mu_\varepsilon\xi_J\partial^2_{J\bar b}W) - \mu_\varepsilon(\partial_{\bar b}\Phi^j\partial^2_{j\bar a}W + \partial_{\bar a}\Phi^j\partial^2_{j\bar b}W)\\
&+ \partial_{\bar a}\mu_\varepsilon\partial_{\bar b}\mu_\varepsilon\xi_J\xi_L\partial^2_{JL}W + \{\partial_{\bar a}\mu_\varepsilon\partial_{\bar b}\Phi^l + \partial_{\bar b}\mu_\varepsilon\partial_{\bar a}\Phi^l\}\xi_J\partial^2_{Jl}W\\
&+ \partial_{\bar a}\Phi^l\partial_{\bar b}\Phi^j\partial^2_{jl}W\\
:= {}& \mathcal{A}_{ab},
\end{aligned}$$

where $\partial_a = \partial_{y_a}$ and $\partial_{\bar a} = \partial_{z_a}$ and J, L run between 1 and N, while as before j, l run between 1 and $N-1$.

Using the above expansions of the metric coefficients, we easily see that

$$\mu_\varepsilon^{(N+2)/2}(g^\varepsilon)^{AB}\partial^2_{AB}v = \partial^2_{ii}W + \partial^2_{NN}W + \mu_\varepsilon^2(\tilde g^\varepsilon)^{ab}\partial^2_{ab}W + 2\mu_\varepsilon\varepsilon\xi_N H_{ij}\partial^2_{ij}W$$
$$+ \left\{3\varepsilon^2\mu_\varepsilon^2\xi_N^2(H^2)_{ij} - \tfrac{1}{3}\varepsilon^2 R_{mijl}(\mu_\varepsilon\xi_l + \Phi^l)(\mu_\varepsilon\xi_m + \Phi^m)\right\}\partial^2_{ij}W$$
$$+ 2\mu_\varepsilon\varepsilon\xi_N(H_{ja}+(\tilde g^\varepsilon)^{ac}H_{cj})\left(-\frac{N-2}{2}\partial_{\bar a}\mu_\varepsilon\partial_j W + \frac{\mu_\varepsilon}{\varepsilon}\partial^2_{\bar a j}W - \partial_{\bar a}\mu_\varepsilon\xi_L\partial^2_{jL}W - \partial_{\bar a}\Phi^l\partial^2_{jl}W\right)$$
$$+ \mathcal{A}_{aa} + \mathcal{B}_1(W),$$

where

$$\mathcal{B}_1(W) = O\left(\varepsilon^2(\mu_\varepsilon\bar y + \Phi)^2 + \varepsilon^2\mu_\varepsilon\xi_N(\mu_\varepsilon\bar\xi + \Phi) + \varepsilon^2\mu_\varepsilon^2\xi_N^2\right)$$
$$\times\left(-\frac{N}{2}\partial_{\bar a}\mu_\varepsilon\partial_l W + \frac{\mu_\varepsilon}{\varepsilon}\partial^2_{\bar a l}W - \partial_{\bar a}\mu_\varepsilon\xi_J\partial^2_{lJ}W - \partial_{\bar a}\Phi^j\partial^2_{lj}W\right)$$
$$+ O\left(\varepsilon^3|\mu_\varepsilon\bar y + \Phi|^3 + \varepsilon^3\mu_\varepsilon\xi_N|\mu_\varepsilon\bar y + \Phi|^2 + \varepsilon^3\mu_\varepsilon^2\xi_N^2|\mu_\varepsilon\bar y + \Phi| + \varepsilon^3\mu_\varepsilon^3\xi_N^3\right)\partial^2_{ij}W$$
$$+ O(|\mu_\varepsilon\bar y + \Phi|\varepsilon + \varepsilon\mu_\varepsilon\xi_N))\mathcal{A}_{ab}.$$

Now recall the expansion of $\log(\det g^\varepsilon)$ given in Lemma 3.2:

$$\log(\det g^\varepsilon) = \log(\det\tilde g^\varepsilon) - 2\varepsilon X_N\,\mathrm{tr}(H) - 2\varepsilon\Gamma^b_{bk}X_k + \tfrac{1}{3}\varepsilon^2 R_{miil}x_m X_l$$
$$+ \varepsilon^2((\tilde g^\varepsilon)^{ab}R_{mabl} - \Gamma^c_{am}\Gamma^a_{cl})X_m X_l - \varepsilon^2 X_N^2\,\mathrm{tr}(H^2) + O(\varepsilon^3|X|^3).$$

Hence, differentiating with respect to X_i, X_N and z_a (and performing the change of variables $z = y/\varepsilon$ and $\bar\xi = (\bar X - \Phi)/\mu_\varepsilon$ and $\xi_N = X_N/\mu_\varepsilon$) one has

$$\partial_{X_N}\log\sqrt{\det g^\varepsilon} = -\varepsilon\,\mathrm{tr}(H) - 2\mu_\varepsilon\varepsilon^2\xi_N\,\mathrm{tr}(H^2) + O(|(\mu_\varepsilon\xi + \phi)|^2\varepsilon^3),$$
$$\partial_{X_j}\log\sqrt{\det g^\varepsilon} = \varepsilon^2\left(\tfrac{1}{3}R_{mssj} + (\tilde g^\varepsilon)^{ab}R_{mabj} - \Gamma^c_a(E_m)\Gamma^a_c(E_j)\right)(\mu_\varepsilon\xi_m + \Phi^m)$$
$$+ O(\varepsilon^3|(\mu_\varepsilon\xi + \Phi)|^2),$$

and

$$\partial_{z_a}\log\sqrt{\det g^\varepsilon} = \varepsilon\mu_\varepsilon\xi_N\partial_{\bar a}\,\mathrm{tr}(H) + O(\varepsilon^2|(\mu_\varepsilon\xi + \Phi)|^2).$$

It follows that

$$\mu_\varepsilon^{(N+2)/2}\partial_A(\log\sqrt{\det g^\varepsilon})(g^\varepsilon)^{AB}\partial_B v = \varepsilon\mu_\varepsilon\,\mathrm{tr}(H)\partial_N W + 2\mu_\varepsilon\varepsilon^2(-\mu_\varepsilon\xi_N\,\mathrm{tr}(H^2))\partial_N W$$
$$+ \mu_\varepsilon\varepsilon^2\left(\tfrac{1}{3}R_{mssj}(\mu_\varepsilon\xi_m + \Phi^m) + \{(\tilde g^\varepsilon)^{ab}R_{mabj} - \Gamma^c_a(E_m)\Gamma^a_c(E_j)\}(\mu_\varepsilon\xi_m + \Phi^m)\right)\partial_j W$$
$$+ \mathcal{A}_{51} + \mathcal{B}_2(W),$$

where

$$\mathcal{A}_{51} = -\varepsilon^2\mu_\varepsilon^2\xi_N\partial_a\,\mathrm{tr}(H)\left(-\frac{N-2}{2}\partial_{\bar a}\mu_\varepsilon W + \mu_\varepsilon\partial_{\bar a}W - (\partial_{\bar a}\mu_\varepsilon\xi_J\partial_J v + \partial_{\bar a}\Phi^j\partial_j W)\right),$$
$$\mathcal{B}_2(W) = O\left(\varepsilon^2(\mu_\varepsilon\bar\xi + \Phi)^2 + \varepsilon^2\mu_\varepsilon\xi_N(\mu_\varepsilon\bar\xi + \Phi) + \varepsilon^2\mu_\varepsilon^2\xi_N^2\right)(\varepsilon\mu_\varepsilon\partial_j W + \varepsilon\mu_\varepsilon\partial_N W)$$
$$+ O\left(\varepsilon^2(\mu_\varepsilon\bar\xi + \Phi)^2 + \varepsilon^2\mu_\varepsilon\xi_N(\mu_\varepsilon\bar\xi + \Phi) + \varepsilon^2\mu_\varepsilon^2\xi_N^2\right)$$
$$\times\left(-\frac{N}{2}\partial_{\bar a}\mu_\varepsilon\partial_l W + \mu_\varepsilon\partial^2_{\bar a l}W - (\partial_{\bar a}\mu_\varepsilon\xi_J\partial^2_{lJ}W + \partial_{\bar a}\Phi^j\partial^2_{lj}W)\right).$$

Finally, using the properties of the curvature tensor ($R_{iilj} = 0$ and $R_{smsj} = -R_{mssj}$) we get

$$\begin{aligned}\mu_\varepsilon^{(N+2)/2}\partial_A((g^\varepsilon)^{AB})\partial_B v &= \partial_i((g^\varepsilon)^{ij})\partial_j v + \partial_a((g^\varepsilon)^{ab})\partial_b v + \partial_a((g^\varepsilon)^{aj})\partial_j v \\ &\quad + \partial_j((g^\varepsilon)^{aj})\partial_a v \\ &= \tfrac{1}{3}\varepsilon^2\mu_\varepsilon R_{liij}(\mu_\varepsilon\xi_l + \Phi^l)\partial_j W + \mathcal{A}_{52}W + \mathcal{B}_3(W),\end{aligned}$$

where we have set

$$\begin{aligned}\mathcal{A}_{52}W &= \big(\mathfrak{D}_j^a[\mu_\varepsilon\xi_j + \Phi^j]\varepsilon^2 + \varepsilon\mu_\varepsilon\xi_N\mathfrak{D}_N^a\big) \\ &\quad\times\left\{\mu_\varepsilon\left[-\varepsilon D_{\bar\xi}W[\partial_{\bar a}\Phi] + \mu_\varepsilon\partial_{\bar a}W - \varepsilon\partial_{\bar a}\mu_\varepsilon\left(\frac{N-2}{2}W + D_\xi W[\xi]\right)\right]\right\},\end{aligned}$$

where $\mathfrak{D}_N^j$ and $\mathfrak{D}_N^a$ are smooth functions in z, and

$$\begin{aligned}\mathcal{B}_3(W) &= O\big(\varepsilon^2(\mu_\varepsilon\bar\xi + \Phi)^2 + \varepsilon^2\mu_\varepsilon\xi_N(\mu_\varepsilon\bar\xi + \Phi) + \varepsilon^2\mu_\varepsilon^2\xi_N^2\big)\mu_\varepsilon\varepsilon(\partial_j W + \partial_{\bar a}W) \\ &\quad + \big((\mu_\varepsilon\bar\xi + \Phi)^2 + \mu_\varepsilon\xi_N(\mu_\varepsilon\bar\xi + \Phi) + \mu_\varepsilon^2\xi_N^2\big) \\ &\quad\times\left\{\mu_\varepsilon\left[-\varepsilon D_{\bar\xi}W[\partial_{\bar a}\Phi] + \mu_\varepsilon\partial_{\bar a}W - \varepsilon\partial_{\bar a}\mu_\varepsilon\left(\frac{N}{2}W + D_\xi W[\xi]\right)\right]\right\}.\end{aligned}$$

Collecting these formulas together and setting

$$\mathcal{A}_0 = \sum_{a=1}^{k}\mathcal{A}_{aa}, \qquad \mathcal{A}_5 = \mathcal{A}_{51} + \mathcal{A}_{52},$$

and

$$B(v) = \mathcal{B}_1(v) + \mathcal{B}_2(v) + \mathcal{B}_3(v), \tag{8.1}$$

the result follows at once.

9. Appendix: Proof of Theorem 2

The main ingredient to prove Theorem 2 is the following

Lemma 9.1. *Under the assumptions and notation of Theorem 2, there exists* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$,

$$E\left(\frac{\delta}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)\bar\chi_\varepsilon\right) = \varepsilon^{-k}P_\varepsilon(\delta), \tag{9.1}$$

$$E\left(\frac{d^j}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j)\bar\chi_\varepsilon\right) = \varepsilon^{-k}Q_\varepsilon(d^j), \tag{9.2}$$

$$E\left(\frac{e}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)\bar\chi_\varepsilon\right) = \varepsilon^{-k}R_\varepsilon(e). \tag{9.3}$$

Proof. Define

$$F(u) := \int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon}\left(\frac{1}{2}|\nabla_X u|^2+\frac{1}{2}\varepsilon u^2-\frac{1}{p+1}u^{p+1}\right)\sqrt{\det g^\varepsilon}\,dz\,dX$$
$$+\int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon}\frac{1}{2}\Xi_{ij}(\varepsilon z,X)\partial_i u\partial_j u\sqrt{\det g^\varepsilon}\,dz\,dX$$
$$+\frac{1}{2}\int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon}\partial_{\bar a}u\partial_{\bar a}u\sqrt{\det g^\varepsilon}\,dz\,dX+\int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon}B(u,u)\sqrt{\det g^\varepsilon}\,dz\,dX. \tag{9.4}$$

We refer to Lemma 5.1 for the definitions of the objects appearing in (9.4).

Step 1: *Proof of (9.1).* Given a small $t\neq 0$, Taylor expansion gives

$$[DF(\mathcal{T}_{\mu_\varepsilon+t\delta,\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon)-DF(\mathcal{T}_{\mu_\varepsilon,\bar\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon)]\left(\frac{\delta}{\bar\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)\bar\chi_\varepsilon\right)$$
$$=-tD^2F(\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon)\left[\frac{\delta}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)\bar\chi_\varepsilon\right](1+O(t))$$
$$=-2tE\left(\frac{\delta}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)\bar\chi_\varepsilon\right)(1+O(t)). \tag{9.5}$$

On the other hand, we write, for any ψ,

$$[DF(\mathcal{T}_{\mu_\varepsilon+t\delta,\bar\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon)-DF(\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon)](\psi)=\mathfrak{a}(t)-\mathfrak{a}(0)+\mathfrak{b}(t)+\mathfrak{c}(t), \tag{9.6}$$

where

$$\mathfrak{a}(t)=\int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon}(\nabla_X\mathcal{T}_{\mu_\varepsilon+t\delta,\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon)\nabla_X\psi+\varepsilon\mathcal{T}_{\mu_\varepsilon+t\delta,\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon\psi-(\mathcal{T}_{\mu_\varepsilon+t\delta,\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon)^p\psi$$
$$+\int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon}\Xi_{ij}(\varepsilon z,X)\partial_i(\mathcal{T}_{\mu_\varepsilon+t\delta,\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon)\partial_j\psi,$$
$$\mathfrak{b}(t)=\int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon}\partial_{\bar a}(\mathcal{T}_{\mu_\varepsilon+t\delta,\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon)\partial_{\bar a}\psi-\int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon}\partial_{\bar a}(\mathcal{T}_{\bar\mu_\varepsilon,\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon)\partial_{\bar a}\psi,$$
$$\mathfrak{c}(t)=\int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon}B(\mathcal{T}_{\bar\mu_\varepsilon+t\delta,\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon,\psi)-\int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon}B(\mathcal{T}_{\bar\mu_\varepsilon,\bar\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon,\psi).$$

We now compute $\mathfrak{a}(t)$ with $\psi=\frac{\delta}{\bar\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon+t\delta,\Phi_\varepsilon}(Z_0)\bar\chi_\varepsilon$. Performing the change of variables $\bar X=(\bar\mu_\varepsilon+t\delta)\bar\xi+\Phi_\varepsilon$, $X_N=(\mu_\varepsilon+t\delta)\xi_N$ in the integral $\mathfrak{a}(t)$ and using (3.9) together with the definition of $\bar\chi_\varepsilon$ in (5.21), we get

$$\mathfrak{a}(t)=\Bigg[-\int\frac{\delta}{\mu_\varepsilon}[\nabla\hat w\nabla Z_0+\varepsilon(\mu_\varepsilon+t\delta)^2\hat wZ_0-\hat w^pZ_0]\big(1+\varepsilon(\mu_\varepsilon+t\delta)\xi_NH_{\alpha\alpha}+\varepsilon^2O(|\xi|^2)\big)$$
$$-\int\frac{\delta}{\mu_\varepsilon}[-2\varepsilon(\mu_\varepsilon+t\delta)\xi_NH_{ij}+\varepsilon^2O(|\xi|^2)]\partial_i\hat w\partial_jZ_0\Bigg]$$
$$\times(1+O(t))(1+O(\varepsilon)+O(\varepsilon^{\gamma(N-4)})). \tag{9.7}$$

Thus we see immediately from (9.7) that

$$t^{-1}[\mathfrak{a}(t)-\mathfrak{a}(0)]=\Big[2\varepsilon\int\delta^2\hat{w}Z_0-\int\varepsilon\frac{\delta^2}{\mu_\varepsilon}[\nabla\hat{w}\nabla Z_0+\varepsilon\mu_\varepsilon^2\hat{w}Z_0-\hat{w}^pZ_0]\xi_NH_{\alpha\alpha}$$
$$+2\varepsilon\int\frac{\delta^2}{\mu_\varepsilon}\xi_NH_{ij}\partial_i\hat{w}\partial_jZ_0\Big](1+O(t))(1+O(\varepsilon)+O(\varepsilon^{\gamma(N-4)})).$$

Integrating by parts in the ξ variables and using the fact that $\hat{\mathcal{C}}_\varepsilon\to\mathbb{R}^N_+$ as $\varepsilon\to0$ one can write

$$t^{-1}[\mathfrak{a}(t)-\mathfrak{a}(0)]=\Big[2\varepsilon\int\delta^2\hat{w}Z_0-\int\varepsilon\frac{\delta^2}{\mu_\varepsilon}[-\Delta\hat{w}+\varepsilon\mu_\varepsilon^2\hat{w}-\hat{w}^p]Z_0\xi_NH_{\alpha\alpha}$$
$$+\int\varepsilon\frac{\delta^2}{\mu_\varepsilon}\hat{w}Z_0H_{\alpha\alpha}-2\varepsilon\int\frac{\delta^2}{\mu_\varepsilon}\xi_NH_{ij}\partial_{ij}\hat{w}Z_0\Big](1+O(t))(1+O(\varepsilon)+O(\varepsilon^{\gamma(N-4)})).$$

Now using the fact that $\|-\Delta\hat{w}+\varepsilon\mu_\varepsilon^2\hat{w}-\hat{w}^p\|_{\varepsilon,N-2}\le C\varepsilon^3$ we get

$$t^{-1}[\mathfrak{a}(t)-\mathfrak{a}(0)]=\Big[2\varepsilon\int\delta^2\hat{w}Z_0+\varepsilon\Big[\int\frac{\delta^2}{\mu_0}\Big(\int_{\mathbb{R}^N_+}H_{\alpha\alpha}\hat{w}Z_0\xi_N-2\int_{\mathbb{R}^N_+}H_{ij}\partial_{ij}\hat{w}Z_0\Big)\Big]$$
$$+\varepsilon^2Q(\delta)\Big](1+O(t))(1+O(\varepsilon)+O(\varepsilon^{\gamma(N-4)})).$$

Since $N>6$, we can choose $\gamma=1-\sigma$, $\sigma>0$ so that $\gamma(N-4)>2$. Thanks to the definition of μ_0 given in (4.15), we conclude that

$$\mathfrak{a}(t)-\mathfrak{a}(0)=t\varepsilon^{-k}\Big[-B\varepsilon\int_K\delta^2+O(\varepsilon^2)Q(\delta)\Big](1+O(t))(1+O(\varepsilon)),\tag{9.8}$$

where

$$-B=\int_{\mathbb{R}^N_+}wZ_0<0\quad\text{and}\quad Q(\delta)=\int_K\kappa(y)\delta^2$$

for some smooth and uniformly bounded (as $\varepsilon\to0$) function κ defined on K.

Observe that $\partial_\mu\mathcal{T}_{\mu,\Phi}(\hat{w})=-\frac{1}{\mu}\mathcal{T}_{\mu,\Phi}(Z_0)(1+\varepsilon R_0(z,\xi))$, where R_0 is a smooth function of the variables (z,ξ), uniformly bounded in z and satisfying

$$|R_0(y,\xi)|\le\frac{C\vartheta(y)}{1+|\xi|^{N-2}}$$

for some positive constant C independent of ε, and some generic function $\vartheta(y)$ defined on K, smooth and uniformly bounded as $\varepsilon\to0$. Hence, recalling the definition of the function $\mathfrak{b}$ above, Taylor expansion gives

$$\mathfrak{b}(t)=-t\int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon}\Big|\partial_{\bar{a}}\Big(\frac{\delta}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\bar{\Phi}_\varepsilon}(Z_0)\bar{\chi}_\varepsilon\Big)\Big|^2(1+O(t)).$$

Observe now that

$$
\begin{aligned}
\partial_{\bar a}\left(\frac{\delta}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)\bar\chi_\varepsilon\right) &= (\partial_{\bar a}\delta)\frac{1}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)\bar\chi_\varepsilon + \delta\partial_{\bar a}\left(\frac{1}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)\bar\chi_\varepsilon\right)\\
&= \varepsilon(\partial_a\delta)\frac{1}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)\bar\chi_\varepsilon + \varepsilon\delta(\partial_a\mu_\varepsilon)\partial_{\mu_\varepsilon}\left(\frac{1}{\bar\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)\bar\chi_\varepsilon\right)\\
&\quad + \varepsilon\delta(\partial_a\Phi_\varepsilon)\partial_{\bar\Phi_\varepsilon}\left(\frac{1}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\bar\Phi_\varepsilon}(Z_0)\bar\chi_\varepsilon\right).
\end{aligned}
$$

Since $\int\left(\frac{1}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)\bar\chi_\varepsilon\right)^2 dX = A(1+o(\varepsilon))$, we conclude that

$$
\mathfrak{b}(t) = -t\varepsilon^{-k}\left[A_\varepsilon\varepsilon^2\int_K |\partial_a(\delta(1+o(\varepsilon^2)\beta_1^\varepsilon(y)))|^2\right], \tag{9.9}
$$

where $A_\varepsilon \in \mathbb{R}$, $\lim_{\varepsilon\to 0} A_\varepsilon = A = \int_{\mathbb{R}^N_+} Z_0^2$ and β_1^ε is an explicit smooth function in K, which is uniformly bounded as $\varepsilon \to 0$. Finally, we observe that the last term $\mathfrak{c}(t)$ defined above is of lower order, and can be absorbed in the terms described in (9.8) and (9.9).

The expansion (9.1) clearly holds from (9.5), (9.6), (9.8) and (9.9).

Step 2: *Proof of (9.2).* To get the expansion in (9.2) we argue in the same spirit as before. Let d be the vector field along K defined by $d(\varepsilon z) = (d^1(\varepsilon z), \dots, d^{N-1}(\varepsilon z))$. For any $t \neq 0$ small, we have (see (9.4))

$$
\begin{aligned}
&[DF(\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon+td}(\hat w)\bar\chi_\varepsilon) - DF(\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon)][\varphi]\\
&\quad = tD^2F(\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon)\left[\sum_l \frac{d^l}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_l)\bar\chi_\varepsilon\right][\varphi](1+O(t))(1+O(\varepsilon))
\end{aligned}
$$

for any $\varphi \in H^1_\varepsilon$. In particular, choosing $\varphi = \frac{d^j}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j)\bar\chi_\varepsilon$ we get (using the fact that $\int_{\mathbb{R}^N_+} Z_j Z_l = C_0\delta_{jl}$)

$$
\begin{aligned}
&[DF(\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon+td}(\hat w)\bar\chi_\varepsilon) - DF(\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon)]\left[\frac{d^j}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j)\bar\chi_\varepsilon\right]\\
&\quad = 2tE\left(\frac{d^j}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j)\bar\chi_\varepsilon\right)(1+O(t))(1+O(\varepsilon)).
\end{aligned} \tag{9.10}
$$

On the other hand, as in the previous step, we write

$$
\begin{aligned}
&[DF(\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon+td}(\hat w)\bar\chi_\varepsilon) - DF(\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon)]\left[\frac{d^j}{\bar\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j)\bar\chi_\varepsilon\right]\\
&\quad = \mathfrak{a}_2(t) - \mathfrak{a}_2(0) + \mathfrak{b}_2(t) + \mathfrak{c}_2(t),
\end{aligned} \tag{9.11}
$$

where we have set, for $\psi = \frac{d^j}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j)\bar\chi_\varepsilon$,

$$\begin{aligned}
\mathfrak{a}_2(t) &= \int (\nabla_X \mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon+td}(\hat w)\bar\chi_\varepsilon)\nabla_X\psi + \varepsilon\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon+td}(\hat w)\bar\chi_\varepsilon\psi - (\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon+td}(\hat w\bar\chi_\varepsilon))^p\psi\\
&\quad + \int \Xi_{ij}(\varepsilon z, X)\partial_i(\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon+td}(\hat w)\bar\chi_\varepsilon)\partial_j\psi,\\
\mathfrak{b}_2(t) &= \int \partial_{\bar a}(\mathcal{T}_{\bar\mu_\varepsilon,\Phi_\varepsilon+td}(\hat w)\bar\chi_\varepsilon)\partial_{\bar a}(\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon+td}(\hat w)\bar\chi_\varepsilon)\\
&\quad - \int \partial_{\bar a}(\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon)\partial_{\bar a}(\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon),\\
\mathfrak{c}_2(t) &= \int B(\mathcal{T}_{\bar\mu_\varepsilon,\bar\Phi_\varepsilon+td}(\hat w)\bar\chi_\varepsilon,\psi) - \int B(\mathcal{T}_{\bar\mu_\varepsilon,\Phi_\varepsilon}(\hat w)\bar\chi_\varepsilon,\psi).
\end{aligned}$$

We now compute $\mathfrak{a}_2(t)$ with $\psi = \frac{\delta}{\bar\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j)$. Defining the tensor $\mathcal{R}_{ml}$ by

$$\mathcal{R}_{ml} = (\tilde g^\varepsilon)^{ab}R_{mabl} - \Gamma_a^c(E_m)\Gamma_c^a(E_l),$$

performing the change of variables $\bar X = (\mu_\varepsilon + t\delta)\bar\xi + \Phi_\varepsilon$, $X_N = (\bar\mu_\varepsilon + t\delta)\xi_N$ in the integral $\mathfrak{a}_2(t)$, using (3.9), (5.16) and recalling the definition of the cut-off function $\bar\chi_\varepsilon$, we obtain

$$\begin{aligned}
\mathfrak{a}_2(t) = \Bigg\{&\int \frac{d^j}{\mu_\varepsilon}[\nabla\hat w\nabla Z_j + \varepsilon\mu_\varepsilon^2\hat w Z_j - p\hat w^p Z_j]\Bigg[1 - \varepsilon\mu_\varepsilon\xi_N H_{\alpha\alpha}\\
&+ \varepsilon^2\left(\frac{R_{miil}}{6} + \frac{\mathcal{R}_{ml}}{2}\right)(\varepsilon\mu_\varepsilon\xi_m + \Phi_{\varepsilon m} + td^m)(\varepsilon\bar\mu_\varepsilon\xi_l + \Phi_{\varepsilon l} + td^l) + O(\varepsilon^3|\xi|^3)\Bigg]\\
&+ \int \frac{d^j}{\mu_\varepsilon}\big[2\varepsilon\xi_N H_{ir} - \tfrac{1}{3}\varepsilon^2 R_{imlr}(\mu_\varepsilon\xi_m + \Phi_{\varepsilon m} + td^m)(\mu_\varepsilon\xi_l + \Phi_{\varepsilon l} + td^l)\\
&+ O(\varepsilon^3|\xi|^3)\big]\partial_i\hat w\partial_r Z_j\Bigg\}(1 + O(t))(1 + O(\varepsilon) + O(\varepsilon^{\gamma(N-3)})).
\end{aligned}$$

Thus we immediately get (using the fact that $\gamma(N-3) > 1$)

$$\begin{aligned}
t^{-1}[\mathfrak{a}_2(t) - \mathfrak{a}_2(0)] = \varepsilon^2\Bigg\{&\int \frac{d^j}{\mu_\varepsilon}[\nabla\hat w\nabla Z_j + \varepsilon\bar\mu_\varepsilon^2\hat w Z_j - p\hat w^p Z_j]\\
&\times\left(\frac{R_{mijl}}{6} + \frac{\mathcal{R}_{lm}}{2}\right)[(\bar\mu_\varepsilon\xi_m + \Phi_{\varepsilon m})d^l + (\mu_\varepsilon\xi_l + \bar\Phi_{\varepsilon l})d^m]\\
&- \int \frac{d^j}{\mu_\varepsilon}\frac{R_{ilmr}}{3}[(\bar\mu_\varepsilon\xi_m + \Phi_{\varepsilon m})d^l + (\mu_\varepsilon\xi_l + \Phi_{\varepsilon l})d^m]\partial_i\hat w\partial_r Z_j\Bigg\}(1 + O(\varepsilon))(1 + O(t)).
\end{aligned}$$

Integration by parts in the ξ variables and using the fact that $\hat{\mathcal{C}}_\varepsilon \to \mathbb{R}^N_+$ as $\varepsilon \to 0$, we get

$$
\begin{aligned}
t^{-1}[\mathfrak{a}_2(t)-\mathfrak{a}_2(0)] = \varepsilon^2 \Bigg\{ \int \frac{d^j}{\mu_\varepsilon}[-\Delta\hat{w} + \varepsilon\mu_\varepsilon\hat{w} - p\hat{w}]Z_j \\
\times \left(\frac{R_{mijl}}{6} + \frac{\mathcal{R}_{lm}}{2}\right)[(\mu_\varepsilon\xi_m + \Phi_{\varepsilon m})d^l + (\mu_\varepsilon\xi_l + \Phi_{\varepsilon l})d^m] \\
- \int \left(\frac{R_{mijl}}{6} + \frac{\mathcal{R}_{lm}}{2}\right) d^j [\partial_l \hat{w} d^m + \partial_m \hat{w} d^l] Z_j + \int d^j \left[\frac{R_{ilrr}}{3} d^l + \frac{R_{irmr}}{3} d^m\right] Z_i Z_j \Bigg\} \\
\times (1+O(\varepsilon))(1+O(t)).
\end{aligned}
$$

Now using the fact that $\| -\Delta\hat{w} + \varepsilon\mu_\varepsilon^2\hat{w} - \hat{w}^p\|_{\varepsilon,N-2} \le C\varepsilon^3$ and $R_{ilrr} = 0$, we deduce that

$$
\begin{aligned}
t^{-1}[\mathfrak{a}_2(t)-\mathfrak{a}_2(0)] &= \varepsilon^2 \left\{ -C \int_{K_e} \left(\frac{R_{miij}}{3} + \frac{\mathcal{R}_{mj}}{2}\right) d^j d^m + C \int_{K_\varepsilon} \frac{R_{jrrm}}{3} d^m d^j \right\} \\
&\quad \times (1+o(\varepsilon))(1+O(t)) \\
&= \varepsilon^{-k}\varepsilon^2 \left[-C \int \frac{\mathcal{R}_{mj}}{2} d^j d^m + O(\varepsilon) Q(d) \right](1+O(t)), \qquad (9.12)
\end{aligned}
$$

where we have set

$$
C = \int_{\mathbb{R}^N_+} Z_1^2 \quad \text{and} \quad Q(d) := \int_K \pi(y) d^i d^j
$$

for some smooth and uniformly bounded (as $\varepsilon \to 0$) function $\pi(y)$. To estimate the term $\mathfrak{b}_2$ above we argue as in (9.9) to get

$$
t^{-1}\mathfrak{b}_2(t) = -\varepsilon^{-k}\left[\varepsilon^2 C_\varepsilon \int_K |\partial_a(d^j(1+\beta_2^\varepsilon(y)o(\varepsilon^2)))|^2\right](1+O(t)). \qquad (9.13)
$$

Finally we observe that the last term $\mathfrak{c}_2(t)$ is of lower order, and can be absorbed in the terms described in (9.12) and (9.13). We get the expansion (9.2) from (9.10)–(9.13).

Step 3: *Proof of (9.3).* To get the expansion in (9.3), we compute

$$
E\left(\frac{e}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)\right) = I + II + III, \qquad (9.14)
$$

where

$$
\begin{aligned}
I &= \int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon} \frac{\delta^2}{\mu_\varepsilon^2}\left(\tfrac{1}{2}(|\nabla_X \mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)|^2 + \varepsilon\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)^2 - pV_\varepsilon^{p-1}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)^2)\right)\sqrt{\det g^\varepsilon}\, dz\, dX \\
&\quad + \int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon} \frac{\delta^2}{\mu_\varepsilon^2}\frac{1}{2}\Xi_{ij}(\varepsilon z, X)\partial_i\mathcal{T}_{\bar{\mu}_\varepsilon,\Phi_\varepsilon}(Z)\partial_j\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)\sqrt{\det g^\varepsilon}\, dz\, dX, \\
II &= \frac{1}{2}\int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon} \partial_{\bar{a}}\left(\frac{e}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)\right)\partial_{\bar{a}}\left(\frac{e}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)\right)\sqrt{\det g^\varepsilon} dz\, dX, \\
III &= \int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon} B\left(\frac{e}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z), \frac{e}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)\right)\sqrt{\det g^\varepsilon} dz\, dX.
\end{aligned}
$$

Using the change of variables $\bar{X} = \mu_\varepsilon \bar{\xi} + \bar{\Phi}_\varepsilon$, $X_N = \mu_\varepsilon \xi_N$ in I, we can write

$$I = \int \frac{1}{2} \frac{\delta^2}{\mu_\varepsilon^2} [|\nabla Z|^2 - p\hat{w}^{p-1} Z^2 + \varepsilon \mu_\varepsilon^2 Z^2](1 + \varepsilon O(e^{-|\xi|})).$$

Then, recalling the definition of λ_0 in (5.20), we get

$$I = \varepsilon^{-k} \left[-\frac{\lambda_0}{2} D \int_K e^2 + \varepsilon Q(e) \right], \tag{9.15}$$

where we have set

$$D = \int_{\mathbb{R}^N_+} Z^2(\xi)\, d\xi \quad \text{and} \quad Q(e) := \int_K \tau(y) e^2 dy,$$

for some smooth and uniformly bounded, as $\varepsilon \to 0$, function τ. Moreover, using a direct computation and arguing as in (9.9), we get

$$II = \frac{D_\varepsilon}{2} \int_{K_\varepsilon} |\partial_{\bar{a}} e + e^{-\lambda_0 \varepsilon^{-\gamma}} \beta_3^\varepsilon(\varepsilon z) e|^2 = \varepsilon^{-k} \left[\frac{D_\varepsilon}{2} \varepsilon^2 \int_K |\partial_a (e(1 + e^{-\lambda' \varepsilon^{-\gamma}} \beta_3^\varepsilon(y)))|^2 \right], \tag{9.16}$$

where β_3^ε is an explicit smooth function on K, which is uniformly bounded as $\varepsilon \to 0$, while λ' is a positive real number. Finally, we observe that the last term *III* is of lower order, and can be absorbed in the terms described in (9.15) and (9.16). This concludes the proof of (9.3). □

Proof of Theorem 2. Given the result in Lemma 9.1, we can write

$$\mathcal{M}(\phi^\perp, \delta, d, e) = E(\phi) - E(\phi^\perp) - E\left(\frac{\delta}{\mu_\varepsilon} \mathcal{T}_{\mu_\varepsilon, \Phi_\varepsilon}(Z_0) \bar{\chi}_\varepsilon \right) - \sum_{j=1}^{N-1} E\left(\frac{d_j}{\mu_\varepsilon} \mathcal{T}_{\mu_\varepsilon, \Phi_\varepsilon}(Z_j) \bar{\chi}_\varepsilon \right)$$
$$- E\left(\frac{e}{\mu_\varepsilon} \mathcal{T}_{\mu_\varepsilon, \Phi_\varepsilon}(Z) \bar{\chi}_\varepsilon \right).$$

Thus it is clear that the term $\mathcal{M}$ collects all the mixed terms in the expansion of $E(\phi)$. Indeed, if we define

$$\begin{aligned} m(f, g) = &\int_{K_\varepsilon \times \hat{\mathcal{C}}_\varepsilon} (\nabla_X f \nabla_X g + \varepsilon f g - p V_\varepsilon^{p-1} f g) \sqrt{\det g^\varepsilon}\, dz\, dX \\ &+ \int_{K_\varepsilon \times \hat{\mathcal{C}}_\varepsilon} \Xi_{ij}(\varepsilon z, X) \partial_i f\, \partial_j g \sqrt{\det g^\varepsilon}\, dz\, dX \\ &+ \int_{K_\varepsilon \times \hat{\mathcal{C}}_\varepsilon} \partial_{\bar{a}} f\, \partial_{\bar{a}} g \sqrt{\det g^\varepsilon}\, dz\, dX + \int_{K_\varepsilon \times \hat{\mathcal{C}}_\varepsilon} B(f, g) \sqrt{\det g^\varepsilon}\, dz\, dX \end{aligned}$$

for f and g in H^1_ε, then

$$\begin{aligned}
\mathcal{M}(\phi^\perp, \delta, d, e) &= m\left(\phi^\perp, \frac{\delta}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)\bar\chi_\varepsilon\right) + \sum_j m\left(\phi^\perp, \frac{d_j}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j)\bar\chi_\varepsilon\right) \\
&\quad + m\left(\phi^\perp, \frac{e}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)\bar\chi_\varepsilon\right) \\
&\quad + \sum_j m\left(\frac{\delta}{\bar\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)\bar\chi_\varepsilon, \frac{d^j}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\bar\Phi_\varepsilon}(Z_j)\bar\chi_\varepsilon\right) \\
&\quad + \sum_{i\neq j} m\left(\frac{d^j}{\mu_\varepsilon}\mathcal{T}_{\bar\mu_\varepsilon,\Phi_\varepsilon}(Z_j)\bar\chi_\varepsilon, \frac{d_i}{\bar\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_i)\bar\chi_\varepsilon\right) \\
&\quad + m\left(\frac{\delta}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\bar\Phi_\varepsilon}(Z_0)\bar\chi_\varepsilon, \frac{e}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)\bar\chi_\varepsilon\right) \\
&\quad + \sum_j m\left(\frac{d^j}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j)\bar\chi_\varepsilon, \frac{e}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z)\bar\chi_\varepsilon\right). \qquad (9.17)
\end{aligned}$$

One can see clearly that $\mathcal{M}$ is homogeneous of degree 2 and that its first derivative with respect to its variables is a linear operator in $(\phi^\perp, \delta, d, e)$.

We now prove estimate (5.32); the validity of (5.31) is shown in a very similar way. To prove (5.32), we should treat each one of the above terms. Since the computations are very similar, we will limit ourselves to the term

$$m := m\left(\frac{\delta}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)\bar\chi_\varepsilon, \frac{d^j}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j)\bar\chi_\varepsilon\right).$$

This term can be written as

$$m = \sum_{i=1}^{5} m_i, \qquad (9.18)$$

where

$$\begin{aligned}
m_1 &= \int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon} (\nabla_X f \nabla_X g - pV_\varepsilon^{p-1} fg)\sqrt{\det g^\varepsilon}\, dz\, dX, \\
m_2 &= \int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon} \varepsilon fg\sqrt{\det g^\varepsilon}\, dz\, dX, \qquad m_3 = \int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon} \Xi_{ij}(\varepsilon z, X)\partial_i f\,\partial_j g\sqrt{\det g^\varepsilon}\, dz\, dX, \\
m_4 &= \int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon} \partial_{\bar a} f\,\partial_{\bar a} g\sqrt{\det g^\varepsilon}\, dz\, dX, \qquad m_5 = \int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon} B(f,g)\sqrt{\det g^\varepsilon}\, dz\, dX,
\end{aligned}$$

with $f = \frac{\delta}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\bar\Phi_\varepsilon}(Z_0)\bar\chi_\varepsilon$ and $g = \frac{d^j}{\mu_\varepsilon}\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j)\bar\chi_\varepsilon$. Using the fact that

$$\Delta Z_0 + p w_0^{p-1} Z_0 = 0 \quad \text{in } \mathbb{R}^N,$$

with $\int_{\mathbb{R}^N_+} \partial_{\xi_N} Z_0 Z_j = 0$, and integrating by parts in the X variable (recalling the expansion of $\sqrt{\det g^\varepsilon}$), one gets

$$m_1 = \left\{ \int \frac{\delta d^j}{\mu_\varepsilon^2} [-\Delta \mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0) - p V_\varepsilon^{p-1} \mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0)] \bar{\chi}_\varepsilon^2 \mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j) \sqrt{\det g^\varepsilon} \right.$$
$$\left. + \int \frac{\delta d^j}{\mu_\varepsilon^2} \partial_{\xi_N} (\mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_0) \bar{\chi}_\varepsilon) \mathcal{T}_{\mu_\varepsilon,\Phi_\varepsilon}(Z_j) \frac{1}{\mu_\varepsilon} (\varepsilon \operatorname{tr}(H) + O(\varepsilon^2)) \bar{\chi}_\varepsilon \right\} (1 + o(1)),$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Thus, the Hölder inequality yields

$$|m_1| \le C \varepsilon^{-k} \varepsilon^{\gamma(N-2)} \|\delta\|_{L^2(K)} \|d^j\|_{L^2(K)}.$$

Moreover, using the orthogonality condition $\int_{\mathbb{R}^N_+} Z_0 Z_j = 0$, we get

$$\begin{aligned} |m_2| &\le C \varepsilon \varepsilon^{-k} \left(\int_{|\xi| > \varepsilon^{-\gamma}} Z_0 Z_j \right) \|\delta\|_{L^2(K)} \|d^j\|_{L^2(K)} \\ &\le C \varepsilon^{-k} \varepsilon^{1+\gamma(N-3)} \|\delta\|_{L^2(K)} \|d^j\|_{L^2(K)}. \end{aligned}$$

Now, since $\int_{\mathbb{R}^N_+} \xi_N \partial_i Z_0 \partial_l Z_j = 0$ for any $i, j, l = 1, \ldots, N-1$, one gets

$$\begin{aligned} |m_3| &\le C \varepsilon \varepsilon^{-k} \left(\int_{|\xi| > \varepsilon^{-\gamma}} \xi_N \partial_i Z_0 \partial_l Z_j \right) \|\delta\|_{L^2(K)} \|d^j\|_{L^2(K)} \\ &\le C \varepsilon^{-k} \varepsilon^{1+\gamma(N-2)} \|\delta\|_{L^2(K)} \|d^j\|_{L^2(K)}. \end{aligned}$$

A direct computation m_4 gives

$$\begin{aligned} |m_4| \le C \varepsilon^{-k} \bigg\{ & \varepsilon^2 \left(\int_{|\xi| > \varepsilon^{-\gamma}} Z_0 Z_j \right) \|\partial_a \delta\|_{L^2(K)} \|\partial_a d^j\|_{L^2(K)} \\ & + \varepsilon \left(\int_{|\xi| > \varepsilon^{-\gamma}} Z_0 Z_j \right) (\|\delta\|_{L^2(K)} \|\partial_a d^j\|_{L^2(K)} + \|\partial_a \delta\|_{L^2(K)} \|d^j\|_{L^2(K)}) \\ & + \left(\int_{|\xi| > \varepsilon^{-\gamma}} Z_0 Z_j \right) \|\delta\|_{L^2(K)} \|d^j\|_{L^2(K)} \bigg\} \\ \le C \varepsilon^{-k} \varepsilon^{\gamma(N-3)} & [\|\delta\|^2_{\mathcal{H}^1(K)} + \|d^j\|^2_{\mathcal{H}^1(K)}]. \end{aligned}$$

Since $|m_5| \le C \sum_{j=1}^4 |m_j|$ we conclude that

$$|m| \le C \varepsilon^{-k} \varepsilon^{\gamma(N-3)} [\|\delta\|^2_{\mathcal{H}^1(K)} + \|d^j\|^2_{\mathcal{H}^1(K)}].$$

Each one of the terms appearing in (9.17) can be estimated to finally get the validity of (5.32). This concludes the proof of Theorem 2. ◻

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