J. Eur. Math. Soc. 16, 2097–2164

DOI 10.4171/JEMS/484

Pandelis Dodos · Vassilis Kanellopoulos · Konstantinos Tyros

A density version of the Carlson–Simpson theorem

Received October 4, 2012

Abstract. We prove a density version of the Carlson–Simpson Theorem. Specifically we show the following.

For every integer $k \geq 2$ and every set A of words over k satisfying

$$\limsup_{n\to\infty} |A \cap [k]^n|/k^n > 0$$

there exist a word c over k and a sequence (w_n) of left variable words over k such that the set

$$\{c\} \cup \{c^\frown w_0(a_0)^\frown \dots ^\frown w_n(a_n) : n \in \mathbb{N} \text{ and } a_0, \dots, a_n \in [k]\}$$

is contained in A.

While the result is infinite-dimensional, its proof is based on an appropriate finite and quantitative version, also obtained in the paper.

Keywords. Words, left variable words, density

Contents

P. Dodos: Department of Mathematics, University of Athens, Panepistimiopolis 157 84, Athens, Greece; e-mail: pdodos@math.uoa.gr

V. Kanellopoulos: National Technical University of Athens, Faculty of Applied Sciences, Department of Mathematics, Zografou Campus, 157 80, Athens, Greece; e-mail: bkanel@math.ntua.gr

K. Tyros Department of Mathematics, University of Toronto, Toronto, Canada M5S 2E4; e-mail: ktyros@math.toronto.edu

Mathematics Subject Classification (2010): Primary 05D10

1. Introduction

1.1. Overview. Our topic is Ramsey theory, the general area of combinatorics that studies the basic pigeonhole principles of discrete structures and organizes, in a systematic way, the results obtained by iterating them.

1.1.1. *The coloring versions.* The first pigeonhole principle relevant to our discussion in this paper is the Hales–Jewett Theorem [20]. To state it we need to introduce some pieces of notation and some terminology. For every integer $k \geq 2$ let $[k]^{<\mathbb{N}}$ be the set of all finite sequences having values in $[k] := \{1, \ldots, k\}$. The elements of $[k]^{<\mathbb{N}}$ are referred to as *words over* k, or simply *words* if k is understood. If $n \in \mathbb{N}$, then $[k]^n$ stands for the set of words of length n. We fix a letter v that we regard as a variable. A *variable word over* k is a finite sequence having values in $[k] \cup \{v\}$ where the letter v appears at least once. If w is a variable word and $a \in [k]$, then $w(a)$ is the word obtained by replacing all appearances of the letter v in w by a. A *combinatorial line* of $[k]^n$ is a set of the form $\{w(a) : a \in [k]\}$ where w is a variable word over k of length n.

Hales–Jewett Theorem. *For every $k, r \in \mathbb{N}$ with $k \geq 2$ and $r \geq 1$ there exists an integer N with the following property. If $n \geq N$, then for every r-coloring of $[k]^n$ there exists a combinatorial line of $[k]^n$ which is monochromatic. The least integer N with this property will be denoted by* $\mathrm{HJ}(k, r)$.

The Hales–Jewett Theorem is the bread and butter of Ramsey theory and is often regarded as an abstract version of the van der Waerden Theorem [38]. The exact asymptotics of the numbers $\mathrm{HJ}(k, r)$ are still unknown. The best known upper bounds are primitive recursive and are due to S. Shelah [33].

The second pigeonhole principle relevant to our discussion is the Halpern–Läuchli Theorem [21], a rather deep result that concerns partitions of finite products of infinite trees.

Halpern–Läuchli Theorem. *For every finite tuple $(T_1, \ldots, T_d)$ of uniquely rooted and finitely branching trees without maximal nodes and every finite coloring of the level product*

$$\bigcup_{n \in \mathbb{N}} T_1(n) \times \cdots \times T_d(n)$$

of $(T_1, \ldots, T_d)$ there exist strong subtrees $(S_1, \ldots, S_d)$ of $(T_1, \ldots, T_d)$ having a common level set such that the level product of $(S_1, \ldots, S_d)$ is monochromatic.

We recall that a subtree S of a tree $(T, <)$ is said to be *strong* if: (a) S is uniquely rooted, (b) there exists an infinite subset $L_T(S) = \{l_0 < l_1 < \cdots\}$ of $\mathbb{N}$, called the *level set* of S, such that for every $n \in \mathbb{N}$ the n-level $S(n)$ of S is a subset of $T(l_n)$, and (c) for every $s \in S$ and every immediate successor t of s in T there exists a unique immediate successor s' of s in S with $t \leq s'$. The notion of a strong subtree was highlighted by the work of K. Milliken [25, 26] who used the Halpern–Läuchli Theorem to show that the family of strong subtrees of a uniquely rooted and finitely-branching tree is partition regular.

The Hales–Jewett Theorem and the Halpern–Läuchli Theorem are pigeonhole principles of quite different nature. Nevertheless, they do admit a common extension which is due to T. J. Carlson and S. G. Simpson [6]. To state it we recall that a *left variable word over k* is a variable word over k whose leftmost letter is the variable v. The concatenation of two words x and y over k is denoted by $x^\frown y$.

Carlson–Simpson Theorem. *For every integer $k \geq 2$ and every finite coloring of the set of all words over k there exist a word c over k and a sequence (w_n) of left variable words over k such that the set*

$$\{c\} \cup \{c^\frown w_0(a_0)^\frown \dots ^\frown w_n(a_n) : n \in \mathbb{N} \text{ and } a_0, \dots, a_n \in [k]\}$$

is monochromatic.

The Carlson–Simpson Theorem belongs to the circle of results that provide information on the structure of the wildcard[1] set of the variable word obtained by the Hales–Jewett Theorem; see, e.g., [3, 22, 24, 34, 40]. This extra information (namely, that the sequence (w_n) consists of left variable words) can be used to derive the Halpern–Läuchli Theorem when the trees $T_1, \dots, T_d$ are homogeneous,[2] a special case which is sufficient for all known combinatorial applications of the Halpern–Läuchli Theorem (see [29]).

1.1.2. *The density versions.* It is a remarkably fruitful phenomenon that many pigeonhole principles have a density version. These density versions are strengthenings of their coloristic counterparts and assert that every large subset of a "structure" must contain a "substructure". In fact, the first pigeonhole principle we discussed so far, namely the Hales–Jewett Theorem, admits a density version which is due to H. Furstenberg and Y. Katznelson [15].

Density Hales–Jewett Theorem. *For every integer $k \geq 2$ and every $0 < \delta \leq 1$ there exists an integer N with the following property. If $n \geq N$, then every subset A of $[k]^n$ with $|A| \geq \delta k^n$ contains a combinatorial line of $[k]^n$. The least integer N with this property will be denoted by* $\mathrm{DHJ}(k, \delta)$.

The density Hales–Jewett Theorem is a fundamental result of Ramsey theory. It has several strong results as consequences, most notably the famous Szemerédi Theorem on arithmetic progressions [36] and its multidimensional version [13]. The best known upper bounds for the numbers $\mathrm{DHJ}(k, \delta)$ are obtained in [28] and have an Ackermann-type dependence with respect to k.

It turns out that the Halpern–Läuchli Theorem also has a density version that was obtained relatively recently in [8].

[1] We recall that if $w = (w_i)_{i=0}^{n-1}$ is a variable word over k of length n, then its *wildcard set* is defined to be the set $\{i \in \{0, \dots, n-1\} : w_i = v\}$.

[2] A tree T is *homogeneous* if it is uniquely rooted and there exists an integer $b \geq 2$ such that every $t \in T$ has exactly b immediate successors; e.g., every dyadic or triadic tree is homogeneous.

Density Halpern–Läuchli Theorem. *For every finite tuple $(T_1, \dots, T_d)$ of homogeneous trees and every subset A of the level product of $(T_1, \dots, T_d)$ satisfying*

$$\limsup_{n\to\infty} \frac{|A \cap \big(T_1(n) \times \cdots \times T_d(n)\big)|}{|T_1(n) \times \cdots \times T_d(n)|} > 0$$

there exist strong subtrees $(S_1, \dots, S_d)$ of $(T_1, \dots, T_d)$ having a common level set such that the level product of $(S_1, \dots, S_d)$ is a subset of A.

We should point out that the assumption in the above result that the trees $T_1, \dots, T_d$ are homogeneous is not redundant. On the contrary, various examples given in [4] show that it is essentially optimal.

1.2. The main results. In view of the above it is natural to ask whether the Carlson–Simpson Theorem has a density analogue which would extend, among others, both the density Hales–Jewett Theorem and the density Halpern–Läuchli Theorem. Our goal in this paper is to answer this question affirmatively. Specifically we show the following theorem.

Theorem A. *For every integer $k \geq 2$ and every set A of words over k satisfying*

$$\limsup_{n\to\infty} |A \cap [k]^n|/k^n > 0$$

there exist a word c over k and a sequence (w_n) of left variable words over k such that

$$\{c\} \cup \{c^\frown w_0(a_0)^\frown \dots ^\frown w_n(a_n) : n \in \mathbb{N} \text{ and } a_0, \dots, a_n \in [k]\} \subseteq A.$$

The proof of Theorem A follows a strategy that was already applied in a closely related context and was described in some detail in [10, §1.3]. It consists in reducing Theorem A to an appropriate finite version. This finite version, which represents the combinatorial core of Theorem A, is the content of the following theorem which is the second main result of the paper.

Theorem B. *For every integer $k \geq 2$, every integer $m \geq 1$ and every $0 < \delta \leq 1$ there exists an integer N with the following property. If L is a finite subset of $\mathbb{N}$ of cardinality at least N and A is a set of words over k satisfying $|A \cap [k]^n| \geq \delta k^n$ for every $n \in L$, then there exist a word c over k and a finite sequence $(w_n)_{n=0}^{m-1}$ of left variable words over k such that*

$$\{c\} \cup \big\{c^\frown w_0(a_0)^\frown \dots ^\frown w_n(a_n) : n \in \{0, \dots, m-1\} \text{ and } a_0, \dots, a_n \in [k]\big\} \subseteq A.$$

The least integer N with this property will be denoted by $\mathrm{DCS}(k, m, \delta)$.

The main point in Theorem B is that the result is independent of the position of the finite set L. Its proof is based on a density increment strategy—a powerful method pioneered by K. F. Roth [31]—and yields explicit upper bounds for the numbers $\mathrm{DCS}(k, m, \delta)$. These upper bounds are admittedly rather weak. They are in line, however, with several other bounds obtained recently in the area; see, e.g., [9, 17, 28, 30].

Although Theorem B refers to left variable words, it can be used to obtain variable words with quite divergent structure. Specifically, given two sequences (p_n) and (w_n) of variable words over k, we say that the sequence (w_n) is of *pattern* (p_n) if p_n is an initial segment of w_n for every $n \in \mathbb{N}$. So, for instance, if $q_n = (v)$ for every $n \in \mathbb{N}$, then a sequence (w_n) of variable words over k is of pattern (q_n) if and only if it consists of left variable words. We prove the following theorem.

Theorem C. *Let $k \in \mathbb{N}$ with $k \geq 2$ and (p_n) be an arbitrary sequence of variable words over k. Then for every set A of words over k satisfying*

$$\limsup_{n\to\infty} |A \cap [k]^n|/k^n > 0$$

there exist a word c over k and a sequence (w_n) of variable words over k of pattern (p_n) such that

$$\{c\} \cup \{c^\frown w_0(a_0)^\frown \dots ^\frown w_n(a_n) : n \in \mathbb{N} \text{ and } a_0, \dots, a_n \in [k]\} \subseteq A.$$

Of course, there is also a finite version of Theorem C in the spirit of Theorem B. This is the content of Theorem 11.1 in the main text.

1.3. Structure of the paper. The paper is organized as follows. In §2 we set up our notation and terminology, and we recall some tools needed for the proof of the main results. Of particular importance is the notion of a *Carlson–Simpson tree* introduced in §2.5. It is the analogue, within the context of left variable words, of the notion of a combinatorial subspace.

The next four sections contain several preparatory results needed for the proof of Theorem B. This material is not only independent of the rest of the paper but also of independent interest. In §3 we state and prove a "regularity lemma" for subsets of $[k]^{<\mathbb{N}}$. The lemma asserts that every dense subset of $[k]^{<\mathbb{N}}$ is inherently pseudorandom and is proved via an energy increment strategy, an influential method introduced by E. Szemerédi [37]. In the next section, §4, we present a partition result for Carlson–Simspon trees which is, essentially, a variant of the classical Graham–Rothschild Theorem [19]. Finally, in §5 and §6 we develop a method of "gluing" a pair x and y of words over k. The method can be thought of as a natural extension of the familiar practice of concatenating x and y. It is encoded by what we call a *convolution operation* which is introduced and studied in §5. Iterations of convolution operations are studied in §6. We emphasize that the results in §6 are invoked only in §9. However, the material in §3, §4 and §5 is heavily used and the reader is advised to gain some familiarity with the contents of these sections before reading the rest of the paper.

The next four sections are devoted to the proof of Theorem B. The results in §7 are independent of the rest of the argument. In particular, this section can be read separately. The main part of the proof is contained in §8 and §9. The reader will find a detailed outline and an exposition of the key ideas in §8.1 and §9.1. The proof of Theorem B is completed in §10.

Finally, the last section of the paper contains a discussion of some consequences of Theorem B, including the proofs of Theorem A and Theorem C.

2. Background material

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the natural numbers. For every integer $n \geq 1$ we set $[n] = \{1, \dots, n\}$. If X is a nonempty finite set, then $\mathbb{E}_{x\in X}$ will denote the average $|X|^{-1}\sum_{x\in X}$ where, as usual, $|X|$ stands for the cardinality of X. For every function $f : \mathbb{N} \to \mathbb{N}$ and every $\ell \in \mathbb{N}$ we shall denote by $f^{(\ell)} : \mathbb{N} \to \mathbb{N}$ the ℓ-th iteration of f defined recursively by $f^{(0)}(n) = n$ and $f^{(\ell+1)}(n) = f(f^{(\ell)}(n))$ for every $n \in \mathbb{N}$.

Let X be a nonempty (possibly infinite) set and A be a subset of X. For every nonempty finite subset Y of X the *density* of A in Y is defined by

$$\mathrm{dens}_Y(A) = |A \cap Y|/|Y|.$$

If it is clear from the context which set Y we are referring to (for instance, if Y coincides with X), then we shall drop the subscript Y and denote the above quantity simply by $\mathrm{dens}(A)$.

2.1. Words. For every $k \in \mathbb{N}$ with $k \geq 2$ and every $n \in \mathbb{N}$ let $[k]^n$ be the set of all sequences of length n having values in $[k]$. Precisely, $[k]^0$ contains just the empty sequence, while if $n \geq 1$, then

$$[k]^n = \big\{(s_0, \dots, s_{n-1}) : s_i \in [k] \text{ for every } i \in \{0, \dots, n-1\}\big\}.$$

Also let

$$[k]^{<n} = \bigcup_{\{i\in\mathbb{N}:\, i<n\}} [k]^i.$$

Notice, in particular, that $[k]^{<0}$ is empty. We set

$$[k]^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} [k]^n.$$

The elements of $[k]^{<\mathbb{N}}$ are called *words over* k, or simply *words* if k is understood. The *length* of a word x over k, denoted by $|x|$, is defined to be the unique natural number n such that $x \in [k]^n$. For every $i \in \mathbb{N}$ with $i \leq |x|$ we denote by $x|_i$ the word of length i which is an initial segment of x. The concatenation of two words x, y will be denoted by $x^\frown y$.

2.2. Located words. For every $k \in \mathbb{N}$ with $k \geq 2$ and every (possibly empty) finite subset J of $\mathbb{N}$ we denote by $[k]^J$ the set of all functions from J into $[k]$. An element of the set

$$\bigcup_{J\subseteq\mathbb{N} \text{ finite}} [k]^J$$

will be called a *located word over* k. If $x \in [k]^J$ is a located word over k, and S is a subset of J, then $x|_S$ stands for the restriction of x to S; notice that $x|_S \in [k]^S$. Moreover, for every $x \in [k]^I$ and every $y \in [k]^J$, where I and J are two finite subsets of $\mathbb{N}$ with $I \cap J = \emptyset$, we denote by (x, y) the unique element z of $[k]^{I\cup J}$ satisfying $z|_I = x$ and $z|_J = y$.

Of course, every word over k is a located word over k. Indeed, notice that

$$[k]^{\{m\in\mathbb{N}:\, m<n\}} = [k]^n$$

for every $n \in \mathbb{N}$. Conversely, we may identify located words over k with words over k as follows. Let J be a nonempty finite subset of $\mathbb{N}$. We set $j = |J|$ and we write the set J in increasing order as $\{n_0 < \dots < n_{j-1}\}$. The *canonical isomorphism* associated to J is the bijection $\mathrm{I}_J : [k]^j \to [k]^J$ defined by

$$\mathrm{I}_J(x)(n_i) = x(i)$$

for every $i \in \{0, \dots, j-1\}$. Observing that $[k]^{\emptyset} = [k]^0 = \{\emptyset\}$, we define the canonical isomorphism $\mathrm{I}_{\emptyset}$ associated to the empty set to be the identity.

2.3. Variable words. Let $k, m \in \mathbb{N}$ with $k \geq 2$ and $m \geq 1$, and fix a tuple $v_0, \dots, v_{m-1}$ of distinct letters. An *m-variable word over k* is a finite sequence having values in $[k] \cup \{v_0, \dots, v_{m-1}\}$ such that: (a) for every $i \in \{0, \dots, m-1\}$ the letter v_i appears at least once, and (b) if $m \geq 2$, then for every $i, j \in \{0, \dots, m-1\}$ with $i < j$ all occurrences of v_i precede all occurrences of v_j. For every m-variable word w over k and every $a_0, \dots, a_{m-1} \in [k]$ we denote by $w(a_0, \dots, a_{m-1})$ the unique word over k obtained by replacing in w all appearances of the letter v_i with a_i for every $i \in \{0, \dots, m-1\}$. A *left variable word over k* is a 1-variable word over k whose leftmost letter is the variable v.

2.4. Combinatorial subspaces. Let $k, m \in \mathbb{N}$ with $k \geq 2$ and $m \geq 1$. An *m-dimensional combinatorial subspace* of $[k]^{<\mathbb{N}}$ is a set of the form

$$V = \big\{ w(a_0, \dots, a_{m-1}) : a_0, \dots, a_{m-1} \in [k] \big\}$$

where w is an m-variable word over k. 1-dimensional combinatorial subspaces are called *combinatorial lines*.

For every m-dimensional combinatorial subspace V of $[k]^{<\mathbb{N}}$ and every $\ell \in [m]$ let $\mathrm{Subs}_{\ell}(V)$ be the set of all ℓ-dimensional combinatorial subspaces of $[k]^{<\mathbb{N}}$, contained in V. We will need the following special case of the Graham–Rothschild Theorem [19].

Theorem 2.1. *For every integer $k \geq 2$, every pair of integers $d \geq m \geq 1$ and every integer $r \geq 1$ there exists an integer N with the following property. If $n \geq N$ and V is an n-dimensional combinatorial subspace of $[k]^{<\mathbb{N}}$, then for every r-coloring of the set* $\mathrm{Subs}_m(V)$ *there exists $W \in \mathrm{Subs}_d(V)$ such that the set* $\mathrm{Subs}_m(W)$ *is monochromatic. The least integer N with this property will be denoted by* $\mathrm{GR}(k, d, m, r)$.

Detailed expositions as well as infinite extensions of Theorem 2.1 can be found in various places in the literature; see, e.g., [2, 5, 14, 23, 29]. Also there exist primitive recursive upper bounds for the numbers $\mathrm{GR}(k, d, m, r)$ which are due to S. Shelah [33].

2.5. Carlson–Simpson trees. We now introduce a family of combinatorial objects which will be of particular importance throughout the paper.

Definition 2.2. Let $k \in \mathbb{N}$ with $k \geq 2$. A *Carlson–Simpson tree* of $[k]^{<\mathbb{N}}$ is a set of the form

$$W = \{c\} \cup \big\{c^\frown w_0(a_0)^\frown \dots ^\frown w_n(a_n) : n \in \{0, \dots, m-1\} \text{ and } a_0, \dots, a_n \in [k]\big\} \tag{2.1}$$

where c is a word over k and $(w_n)_{n=0}^{m-1}$ is a nonempty finite sequence of left variable words over k.

It is easy to see that the sequence $(c, w_0, \dots, w_{m-1})$ that generates a Carlson–Simpson tree W via (2.1) is unique. It will be called the *generating* sequence of W. The corresponding natural number m will be called the *dimension* of W and denoted by $\dim(W)$. 1-dimensional Carlson–Simpson trees will be called *Carlson–Simpson lines*.

Let W be an m-dimensional Carlson–Simpson tree of $[k]^{<\mathbb{N}}$ and $(c, w_0, \dots, w_{m-1})$ be its generating sequence. The 0-*level* $W(0)$ of W is defined by

$$W(0) = \{c\}.$$

Observe that $W(0)$ is contained in $[k]^{\ell_0}$ where ℓ_0 is the length of c. Moreover, for every $n \in [m]$ the n-*level* $W(n)$ of W is defined by

$$W(n) = \{c^\frown w_0(a_0)^\frown \dots ^\frown w_{n-1}(a_{n-1}) : a_0, \dots, a_{n-1} \in [k]\}.$$

Notice that $W(n)$ is an n-dimensional combinatorial subspace of $[k]^{<\mathbb{N}}$ and is contained in $[k]^{\ell_n}$ where ℓ_n is the sum of the lengths of $c, w_0, \dots, w_{n-1}$. The set $\{\ell_0 < \dots < \ell_m\}$ will be called the *level set* of W and denoted by $L(W)$.

For every m-dimensional Carlson–Simpson tree W of $[k]^{<\mathbb{N}}$ and every $\ell \in [m]$ we denote by $\mathrm{Subtr}_\ell(W)$ the set of all ℓ-dimensional Carlson–Simpson trees of $[k]^{<\mathbb{N}}$ which are contained in W. An element of $\mathrm{Subtr}_\ell(W)$ will be called an *ℓ-dimensional Carlson–Simpson subtree* of W, or simply a *Carlson–Simpson subtree* of W if the dimension ℓ is understood.

The archetypical example of a Carlson–Simpson tree of $[k]^{<\mathbb{N}}$ of dimension m is $[k]^{<m+1}$. In fact, every Carlson–Simpson tree of dimension m can be thought of as a "copy" of $[k]^{<m+1}$ inside $[k]^{<\mathbb{N}}$. Specifically, let W be an m-dimensional Carlson–Simpson tree of $[k]^{<\mathbb{N}}$ and $(c, w_0, \dots, w_{m-1})$ be its generating sequence. The *canonical isomorphism* associated to W is the bijection $\mathrm{I}_W : [k]^{<m+1} \to W$ defined by $\mathrm{I}_W(\emptyset) = c$ and

$$\mathrm{I}_W((a_0, \dots, a_{n-1})) = c^\frown w_0(a_0)^\frown \dots ^\frown w_{n-1}(a_{n-1})$$

for every $n \in [m]$ and every $(a_0, \dots, a_{n-1}) \in [k]^n$. The canonical isomorphism I_W preserves all structural properties one is interested in while working in the category of Carlson–Simpson trees. For instance, if $\ell \in [m]$ and V is a Carlson–Simpson subtree of $[k]^{<m+1}$ of dimension ℓ, then its image $\mathrm{I}_W(V)$ under the canonical isomorphism is an ℓ-dimensional Carlson–Simpson subtree of W. Thus, for most practical purposes, we may identify W with $[k]^{<m+1}$ via the canonical isomorphism I_W.

More generally, let W and U be two Carlson–Simpson trees of $[k]^{<\mathbb{N}}$ of the same dimension. The *canonical isomorphism* associated to the pair W, U is the bijection $\mathrm{I}_{W,U} : W \to U$ defined by

$$\mathrm{I}_{W,U}(t) = (\mathrm{I}_U \circ \mathrm{I}_W^{-1})(t)$$

where I_W and I_U are the canonical isomorphisms associated to W and U. Of course, the map $\mathrm{I}_{W,U}$ will be used to transfer information from W to U and vice versa.

Finally, for every m-dimensional Carlson–Simpson tree W of $[k]^{<\mathbb{N}}$ and every k' in $\{2,\dots,k\}$ we define the *k'-restriction* $W\upharpoonright k'$ of W to be the set

$$\{c\} \cup \big\{c^\frown w_0(a'_0)^\frown \dots ^\frown w_n(a'_n) : n \in \{0,\dots,m-1\} \text{ and } a'_0,\dots,a'_n \in [k']\big\} \tag{2.2}$$

where $(c, w_0, \dots, w_{m-1})$ stands for the generating sequence of W. Notice that the canonical isomorphism of W maps $[k']^{<m+1}$ onto $W\upharpoonright k'$. Therefore, $W\upharpoonright k'$ can be naturally identified as a Carlson–Simpson tree of $[k']^{<\mathbb{N}}$.

2.6. Insensitive sets. Let $k \in \mathbb{N}$ with $k \geq 2$ and x, y be two words over k. Also let $i, j \in [k]$ with $i \neq j$. We say that x and y are *(i, j)-equivalent* if: (a) x and y have common length, and (b) if n is the common length of x and y, then for every $s \in [k]\setminus\{i, j\}$ and every $r \in \mathbb{N}$ with $r < n$ we have $x(r) = s$ if and only if $y(r) = s$.

If $n \in \mathbb{N}$ and A is a subset of $[k]^n$, then A is said to be *(i, j)-insensitive* if for every $x \in A$ and every $y \in [k]^n$, if x and y are (i, j)-equivalent, then $y \in A$. The notion of an (i, j)-insensitive set was introduced by S. Shelah [33] and highlighted in the polymath proof [28] of the density Hales–Jewett Theorem. It can be naturally extended to subsets of $[k]^{<\mathbb{N}}$ as follows.

Definition 2.3. Let $k \in \mathbb{N}$ with $k \geq 2$ and $i, j \in [k]$ with $i \neq j$. Also let A be a subset of $[k]^{<\mathbb{N}}$. We say that A is *(i, j)-insensitive* if for every $n \in \mathbb{N}$ the set $A \cap [k]^n$ is (i, j)-insensitive. If W is Carlson–Simpson tree of $[k]^{<\mathbb{N}}$, then we say that A is *(i, j)-insensitive in W* if $\mathrm{I}_W^{-1}(A \cap W)$ is an (i, j)-insensitive subset of $[k]^{<\mathbb{N}}$ where I_W is the canonical isomorphism associated to W.

It is easy to see that the family of all (i, j)-insensitive subsets of $[k]^{<\mathbb{N}}$ is closed under intersections, unions and complements. The same remark, of course, applies to the family of all (i, j)-insensitive sets in a Carlson–Simpson tree W of $[k]^{<\mathbb{N}}$.

2.7. Furstenberg–Weiss measures. Let $k, m \in \mathbb{N}$ with $k \geq 2$ and $m \geq 1$. The *Furstenberg–Weiss measure* $\mathrm{d}^m_{\mathrm{FW}}$ associated to $[k]^{<m+1}$ is the probability measure on $[k]^{<\mathbb{N}}$ defined by

$$\mathrm{d}^m_{\mathrm{FW}}(A) = \mathbb{E}_{n\in\{0,\dots,m\}} \,\mathrm{dens}_{[k]^n}(A).$$

This class of measures was introduced by H. Furstenberg and B. Weiss [16] and has proven to be useful in various problems of Ramsey theory (see, e.g., [9, 27]). We will need the following two variants.

Definition 2.4. Let $k \in \mathbb{N}$ with $k \geq 2$.

(i) For every Carlson–Simpson tree W of $[k]^{<\mathbb{N}}$ the *Furstenberg–Weiss measure* $\mathrm{d}^W_{\mathrm{FW}}$ associated to W is the probability measure on $[k]^{<\mathbb{N}}$ defined by

$$\mathrm{d}^W_{\mathrm{FW}}(A) = \mathbb{E}_{n\in\{0,\dots,\dim(W)\}} \,\mathrm{dens}_{W(n)}(A).$$

(ii) For every nonempty finite subset L of $\mathbb{N}$ the *generalized Furstenberg–Weiss measure* d_L associated to L is the probability measure on $[k]^{<\mathbb{N}}$ defined by

$$\mathrm{d}_L(A) = \mathbb{E}_{n\in L}\,\mathrm{dens}_{[k]^n}(A).$$

It is, of course, clear that if L is an initial interval of $\mathbb{N}$ of cardinality $\ell \geq 2$, then the generalized Furstenberg–Weiss measure d_L associated to L coincides with the Furstenberg–Weiss measure $\mathrm{d}^{\ell-1}_{\mathrm{FW}}$.

2.8. Probabilistic preliminaries. We record, for future use, three probabilistic facts. The first one is an immediate consequence of Markov's inequality.

Lemma 2.5. *Let (Ω, Σ, μ) be a probability space and $0 < \delta \leq 1$. Also let $(A_i)_{i=1}^n$ be a finite family of measurable events in (Ω, Σ, μ) such that $\mu(A_i) \geq \delta$ for every $i \in [n]$. Then, setting $L_\omega = \{i \in [n] : \omega \in A_i\}$ for every $\omega \in \Omega$, we have*

$$\mu(\{\omega : |L_\omega| \geq (\delta/2)n\}) \geq \delta/2.$$

Proof. For every $i \in [n]$ let $\mathbf{1}_{A_i}$ be the indicator function of the event A_i and set $Z = n^{-1}\sum_{i=1}^n \mathbf{1}_{A_i}$. Then $\mathbb{E}[Z] \geq \delta$ and the result follows. □

To state the second result we recall that if (Ω, Σ, μ) is a probability space and $Y \in \Sigma$ with $\mu(Y) > 0$, then μ_Y stands for the conditional probability measure of μ relative to Y defined by

$$\mu_Y(A) = \frac{\mu(A \cap Y)}{\mu(Y)} \quad \text{for every } A \in \Sigma.$$

Lemma 2.6. *Let (Ω, Σ, μ) be a probability space and $0 < \lambda, \beta, \varepsilon \leq 1$. Let A and B be two measurable events in (Ω, Σ, μ) such that $A \subseteq B$, $\mu(A) \geq \lambda\mu(B)$ and $\mu(B) \geq \beta$. Suppose that $\mathcal{Q} = (Q_i)_{i=1}^n$ is a finite family of pairwise disjoint measurable events in (Ω, Σ, μ) such that $\mu(B \setminus \bigcup \mathcal{Q}) \leq \varepsilon\beta/2$ and $\mu(Q_i) > 0$ for every $i \in [n]$. Set*

$$I = \{i \in [n] : \mu_{Q_i}(A) \geq (\lambda - \varepsilon)\mu_{Q_i}(B) \text{ and } \mu_{Q_i}(B) \geq \beta\varepsilon/4\}.$$

Then

$$\sum_{i\in I} \mu(Q_i) \geq \beta\varepsilon/4.$$

In particular, if $\mu(Q_i) = \mu(Q_j)$ for every $i, j \in [n]$, then $|I| \geq (\beta\varepsilon/4)n$.

Proof. Notice first that $\mu(A \setminus \bigcup \mathcal{Q}) \leq \varepsilon\beta/2$. This is easily seen to imply that

$$\sum_{i=1}^n \frac{\mu(A \cap Q_i)}{\mu(B)} \geq \lambda - \varepsilon/2. \tag{2.3}$$

For every $i \in [n]$ let $a_i = \mu_{Q_i}(A)/\mu_{Q_i}(B)$, $b_i = \mu_{Q_i}(B)$ and $c_i = \mu(Q_i)/\mu(B)$ with the convention that $a_i = 0$ if $\mu(B \cap Q_i) = 0$. Then inequality (2.3) can be reformulated as

$$\sum_{i=1}^n a_i b_i c_i \geq \lambda - \varepsilon/2. \tag{2.4}$$

Notice that

$$\sum_{i=1}^{n} b_i c_i \leq 1 \quad \text{and} \quad \sum_{i=1}^{n} c_i \leq \frac{1}{\beta}. \tag{2.5}$$

Also observe that $I = \{i \in [n] : a_i \geq \lambda - \varepsilon$ and $b_i \geq \beta\varepsilon/4\}$. Since $0 \leq a_i, b_i \leq 1$ for every $i \in [n]$, combining (2.4), (2.5) and the previous remarks, we see that $\sum_{i \in I} c_i \geq \varepsilon/4$, and the proof is complete. □

The final result of this subsection is the following.

Lemma 2.7. *Let $0 < \theta < \varepsilon \leq 1$ and $n \in \mathbb{N}$ with $n \geq (\varepsilon^2 - \theta^2)^{-1}$. If $(A_i)_{i=1}^n$ is a family of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_i) \geq \varepsilon$ for every $i \in [n]$, then there exist $i, j \in [n]$ with $i \neq j$ such that $\mu(A_i \cap A_j) \geq \theta^2$.*

Proof. We set $X = \sum_{i=1}^n \mathbf{1}_{A_i}$ where $\mathbf{1}_{A_i}$ is the indicator function of the event A_i for every $i \in [n]$. Then $\mathbb{E}[X] \geq \varepsilon n$ so, by convexity,

$$\sum_{i \in [n]} \sum_{j \in [n] \setminus \{i\}} \mu(A_i \cap A_j) = \mathbb{E}[X(X-1)] \geq \varepsilon n(\varepsilon n - 1).$$

Therefore, there exist $i, j \in [n]$ with $i \neq j$ such that $\mu(A_i \cap A_j) \geq \theta^2$. □

3. A regularity lemma for subsets of $[k]^{<\mathbb{N}}$

3.1. Statement of the main result. Our goal in this section is to prove a "regularity lemma" for subsets of $[k]^{<\mathbb{N}}$. Roughly speaking, the lemma asserts that if n is large enough and A is a subset of $[k]^n$, then we may find a set of coordinates $I \subseteq \{m \in \mathbb{N} : m < n\}$ of preassigned cardinality such that the set A, viewed as a subset of the product $[k]^I \times [k]^{\{m \in \mathbb{N} : m < n\} \setminus I}$, behaves like a randomly chosen set.

To put things in a proper perspective we need, first, to determine the kind of randomness we are referring to. This is the content of the following definition.

Definition 3.1. Let $k \in \mathbb{N}$ with $k \geq 2$ and let $\mathcal{F}$ be a family of subsets of $[k]^{<\mathbb{N}}$. Also let $0 < \varepsilon \leq 1$ and L be a nonempty finite subset of $\mathbb{N}$. The family $\mathcal{F}$ will be called (ε, L)-*regular* provided that for every $A \in \mathcal{F}$, every $n \in L$, every (possibly empty) subset I of $\{l \in L : l < n\}$ and every $y \in [k]^I$ we have

$$\Big|\mathrm{dens}\big(\{w \in [k]^{\{m \in \mathbb{N} : m < n\} \setminus I} : (y, w) \in A \cap [k]^n\}\big) - \mathrm{dens}(A \cap [k]^n)\Big| \leq \varepsilon.$$

Notice that for every $y \in [k]^I$ the set $\{w \in [k]^{\{m \in \mathbb{N} : m < n\} \setminus I} : (y, w) \in A \cap [k]^n\}$ is just the section $A \cap [k]^n$ at y. So what Definition 3.1 guarantees is that for every $n \in L$ and every $I \subseteq \{l \in L : l < n\}$ the densities of the sections of $A \cap [k]^n$ along elements of $[k]^I$ are essentially equal to the density of $A \cap [k]^n$.

We are now ready to state the main result of this section.

Lemma 3.2. *For every $0 < \varepsilon \leq 1$ and every $k, \ell, q \in \mathbb{N}$ with $k \geq 2$ and $\ell, q \geq 1$ there exists an integer n with the following property. If N is a finite subset of $\mathbb{N}$ with $|N| \geq n$ and $\mathcal{F}$ is a family of subsets of $[k]^{<\mathbb{N}}$ with $|\mathcal{F}| = q$, then there exists a subset L of N with $|L| = \ell$ such that $\mathcal{F}$ is (ε, L)-regular. The least integer n with this property will be denoted by* $\mathrm{Reg}(k, \ell, q, \varepsilon)$.

The proof of Lemma 3.2 will be given in §3.2. It is based on an energy increment strategy, a powerful method introduced by E. Szemerédi in his proof of the celebrated regularity lemma [37]. The argument is, of course, effective and yields explicit upper bounds for the numbers $\mathrm{Reg}(k, \ell, q, \varepsilon)$.

3.2. Proof of Lemma 3.2. We begin with the following definition which is the most important ingredient of the proof.

Definition 3.3. Let $k, n \in \mathbb{N}$ with $k \geq 2$ and let I be a (possibly empty) subset of $\{m \in \mathbb{N} : m < n\}$. For every subset A of $[k]^n$ we define the *energy* of A with respect to I to be the quantity

$$\mathrm{e}_I(A) = \mathbb{E}_{y\in[k]^I}\, \mathrm{dens}(A_y)^2$$

where $A_y = \{w \in [k]^{\{m\in\mathbb{N}:\, m<n\}\setminus I} : (y, w) \in A\}$ is the section of A at y.

We will isolate some basic properties of the energy which are needed for the proof. To this end, we need to introduce some pieces of notation. Let $k, n \in \mathbb{N}$ with $k \geq 2$ and I, J be two subsets of $\{m \in \mathbb{N} : m < n\}$ with $I \cap J = \emptyset$. We set $M = \{m \in \mathbb{N} : m < n\} \setminus (I \cup J)$. If we are given a subset A of $[k]^n$, we may view it as a subset of the product $[k]^I \times [k]^J \times [k]^M$ and so we may define the section $A_{(y,z)} = \{v \in [k]^M : (y, z, v) \in A\}$ for every $(y, z) \in [k]^I \times [k]^J$. Notice that

$$\mathrm{dens}(A_y) = \mathbb{E}_{z\in[k]^J}\, \mathrm{dens}(A_{(y,z)}) \quad \text{and} \quad \mathrm{dens}(A_z) = \mathbb{E}_{y\in[k]^I}\, \mathrm{dens}(A_{(y,z)}) \tag{3.1}$$

for every $y \in [k]^I$ and every $z \in [k]^J$. We have the following.

Fact 3.4. *Let $k, n \in \mathbb{N}$ with $k \geq 2$ and let I be a subset of $\{m \in \mathbb{N} : m < n\}$. Then for every subset A of $[k]^n$ we have $\mathrm{e}_I(A) \leq 1$. Moreover, if J is a subset of $\{m \in \mathbb{N} : m < n\}$ with $I \cap J = \emptyset$, then*

$$\mathrm{e}_{I\cup J}(A) - \mathrm{e}_J(A) = \mathbb{E}_{z\in[k]^J}\, \mathbb{E}_{y\in[k]^I} \big(\mathrm{dens}(A_{(y,z)}) - \mathbb{E}_{y\in[k]^I}\, \mathrm{dens}(A_{(y,z)})\big)^2. \tag{3.2}$$

In particular, $\mathrm{e}_J(A) \leq \mathrm{e}_{I\cup J}(A)$.

Proof. The fact that $\mathrm{e}_I(A) \leq 1$ follows immediately from Definition 3.3. Observe that

$$\mathrm{e}_{I\cup J}(A) \;=\; \mathbb{E}_{z\in[k]^J} \big(\mathbb{E}_{y\in[k]^I}\, \mathrm{dens}(A_{(y,z)})^2\big), \tag{3.3}$$

$$\mathrm{e}_J(A) \overset{(3.1)}{=} \mathbb{E}_{z\in[k]^J} \big(\mathbb{E}_{y\in[k]^I}\, \mathrm{dens}(A_{(y,z)})\big)^2. \tag{3.4}$$

Combining (3.3) and (3.4) yields the result. □

The first step towards the proof of Lemma 3.2 is the following.

Sublemma 3.5. *Let $k, n \in \mathbb{N}$ with $k \geq 2$ and let I and J be two subsets of $\{m \in \mathbb{N} : m < n\}$ with $I \cap J = \emptyset$. Finally let $A \subseteq [k]^n$ and $0 < \varepsilon < k^{-|I|}$. If $\mathrm{e}_{I\cup J}(A) - \mathrm{e}_J(A) \leq \varepsilon^4$, then*

$$\mathrm{dens}\big(\big\{z \in [k]^J : |\mathrm{dens}(A_{(y,z)}) - \mathrm{dens}(A_z)| \leq \varepsilon \text{ for every } y \in [k]^I\big\}\big) \geq 1 - \varepsilon.$$

Proof. We set $Y = [k]^I$ and $Z = [k]^J$. For every $z \in Z$ let $f_z : Y \to [0, 1]$ be the random variable defined by $f_z(y) = \mathrm{dens}(A_{(y,z)})$. Let $E(f_z) = \mathbb{E}_{y\in Y} f_z(y)$ be the expected value of f_z and $\mathrm{Var}(f_z) = E(f_z^2) - E(f_z)^2$ be its variance. Notice that $E(f_z) = \mathrm{dens}(A_z)$. By (3.2), we see that $\mathbb{E}_{z\in Z} \mathrm{Var}(f_z) = \mathrm{e}_{I\cup J}(A) - \mathrm{e}_J(A)$. Hence, by our assumptions,

$$\mathbb{E}_{z\in Z} \mathrm{Var}(f_z) \leq \varepsilon^4$$

and so, by Markov's inequality,

$$\mathrm{dens}(\{z \in Z : \mathrm{Var}(f_z) \leq \varepsilon^3\}) \geq 1 - \varepsilon.$$

Fix $z_0 \in Z$ with $\mathrm{Var}(f_{z_0}) \leq \varepsilon^3$. By Chebyshev's inequality, we have

$$\mathrm{dens}(\{y \in Y : |f_{z_0}(y) - E(f_{z_0})| \leq \varepsilon\}) \geq 1 - \varepsilon$$

and since $\varepsilon < |Y|^{-1}$ we get $|f_{z_0}(y) - E(f_{z_0})| \leq \varepsilon$ for every $y \in Y$. This is equivalent to $|\mathrm{dens}(A_{(y,z_0)}) - \mathrm{dens}(A_{z_0})| \leq \varepsilon$ for every $y \in [k]^I$, and the proof is complete. □

Sublemma 3.5 will be used in the following form.

Corollary 3.6. *Let $k, n \in \mathbb{N}$ with $k \geq 2$ and let I and J be two subsets of $\{m \in \mathbb{N} : m < n\}$ with $I \cap J = \emptyset$. Finally let $A \subseteq [k]^n$ and $0 < \varepsilon < k^{-|I|}$. If $\mathrm{e}_{I\cup J}(A) - \mathrm{e}_J(A) \leq \varepsilon^4/16$, then $|\mathrm{dens}(A_y) - \mathrm{dens}(A)| \leq \varepsilon$ for every $y \in [k]^I$.*

Proof. We set $\varepsilon_0 = \varepsilon/2$ and

$$Z_0 = \{z \in [k]^J : |\mathrm{dens}(A_{(y,z)}) - \mathrm{dens}(A_z)| \leq \varepsilon_0 \text{ for every } y \in [k]^I\}. \tag{3.5}$$

By Sublemma 3.5, we have $\mathrm{dens}([k]^J \setminus Z_0) \leq \varepsilon_0$. Hence, for every $y \in [k]^I$,

$$\begin{aligned}|\mathrm{dens}(A_y) - \mathrm{dens}(A)| &\overset{(3.1)}{\leq} \mathbb{E}_{z\in[k]^J} |\mathrm{dens}(A_{(y,z)}) - \mathrm{dens}(A_z)| \\ &\leq \mathbb{E}_{z\in Z_0} |\mathrm{dens}(A_{(y,z)}) - \mathrm{dens}(A_z)| + \varepsilon_0 \overset{(3.5)}{\leq} \varepsilon_0 + \varepsilon_0 = \varepsilon.\end{aligned}$$ □

We proceed to the second step of the proof of Lemma 3.2.

Sublemma 3.7. *Let $k, m, q \in \mathbb{N}$ with $k \geq 2$ and $q \geq 1$ and let $0 < \varepsilon < k^{-m}$. Also let N be a finite subset of $\mathbb{N}$ with $|N| \geq (q\lfloor 16\varepsilon^{-4}\rfloor + 1)m + 1$ and $\mathcal{F}$ be a family of subsets of $[k]^{\max(N)}$ with $|\mathcal{F}| = q$. Set $N' = N \setminus \{\max(N)\}$. Then there exists a subinterval M of N' (i.e., $M = J \cap N'$ for some interval J of $\mathbb{N}$) with $|M| = m$ and such that for every $A \in \mathcal{F}$, every subset I of M and every $y \in k^I$ we have $|\mathrm{dens}(A_y) - \mathrm{dens}(A)| \leq \varepsilon$.*

Proof. Clearly we may assume that $m \geq 1$. We set $r_0 = q\lfloor 16\varepsilon^{-4}\rfloor + 1$. Write the first $r_0 \cdot m$ elements of N' in increasing order as $\{n_0 < n_1 < \cdots < n_{r_0\cdot m-1}\}$. For every $p \in \{0, \ldots, r_0 - 1\}$ let

$$I_p = \{n_{p\cdot m+j} : j \in \{0, \ldots, m - 1\}\} \quad \text{and} \quad J_p = \{j \in \mathbb{N} : j < n_{p\cdot m}\}.$$

Notice that $\max(J_p) < \min(I_p) < \max(N)$. Moreover, $I_p \cup J_p \subseteq J_{p+1}$ if $p \leq r_0 - 2$. Hence, by Fact 3.4, we have

$$\mathrm{e}_{J_p}(A) \leq \mathrm{e}_{I_p\cup J_p}(A) \leq \mathrm{e}_{J_{p+1}}(A) \leq \mathrm{e}_{I_{p+1}\cup J_{p+1}}(A) \leq 1 \tag{3.6}$$

for every $p \in \{0, \dots, r_0 - 2\}$ and every $A \in \mathcal{F}$. For every $A \in \mathcal{F}$ let

$$P_A = \big\{p \in \{0, \dots, r_0 - 1\} : \mathrm{e}_{I_p \cup J_p}(A) - \mathrm{e}_{J_p}(A) > \varepsilon^4/16\big\}.$$

The previous discussion implies that the set P_A has cardinality at most $\lfloor 16\varepsilon^{-4} \rfloor$. Therefore, we may select $p_0 \in \{0, \dots, r_0 - 1\}$ such that $p_0 \notin P_A$ for every $A \in \mathcal{F}$; in particular, $\mathrm{e}_{I_{p_0} \cup J_{p_0}}(A) - \mathrm{e}_{J_{p_0}}(A) \le \varepsilon^4/16$. Since $I_{p_0} \cap J_{p_0} = \emptyset$, by Corollary 3.6, we conclude that

$$|\mathrm{dens}(A_y) - \mathrm{dens}(A)| \le \varepsilon \tag{3.7}$$

for every $y \in [k]^{I_{p_0}}$ and every $A \in \mathcal{F}$.

We set $M = I_{p_0}$. We will show that with this choice all requirements of the sublemma are satisfied. Indeed, notice that M is a subinterval of N' with $|M| = m$. We fix $A \in \mathcal{F}$. Also let $I \subseteq M$ and $y \in [k]^I$. Observe that $(y, z) \in [k]^{I_{p_0}}$ for every $z \in [k]^{M \setminus I}$. Hence,

$$\begin{aligned} |\mathrm{dens}(A_y) - \mathrm{dens}(A)| &= |\,\mathbb{E}_{z \in [k]^{M \setminus I}} \mathrm{dens}(A_{(y,z)}) - \mathrm{dens}(A)| \\ &\le \mathbb{E}_{z \in [k]^{M \setminus I}} |\mathrm{dens}(A_{(y,z)}) - \mathrm{dens}(A)| \overset{(3.7)}{\le} \varepsilon. \end{aligned}$$ ◻

We are in a position to complete the proof of Lemma 3.2. To this end, we need to introduce some numerical invariants. For every $0 < \varepsilon \le 1$ and every $k, \ell, q \in \mathbb{N}$ with $k \ge 2$ and $\ell, q \ge 1$ let

$$\rho = \rho(k, \ell, q, \varepsilon) = \min\{\varepsilon, k^{-\ell}/2\}$$

and define $F_{k,\ell,q,\varepsilon} : \mathbb{N} \to \mathbb{N}$ by

$$F_{k,\ell,q,\varepsilon}(m) = (q \lfloor 16\rho^{-4} \rfloor + 1)m + 1.$$

Proof of Lemma 3.2. We will show that

$$\mathrm{Reg}(k, \ell, q, \varepsilon) \le F^{(\ell)}_{k,\ell,q,\varepsilon}(0)$$

for every $0 < \varepsilon \le 1$ and every $k, \ell, q \in \mathbb{N}$ with $k, \ell \ge 2$ and $q \ge 1$. Indeed, let N be a finite subset of $\mathbb{N}$ with $|N| \ge F^{(\ell)}_{k,\ell,q,\varepsilon}(0)$ and fix a family $\mathcal{F}$ of subsets of $[k]^{<\mathbb{N}}$ with $|\mathcal{F}| = q$. We select a subset M_0 of N with $|M_0| = F^{(\ell)}_{k,\ell,q,\varepsilon}(0)$. By repeated application of Sublemma 3.7, we may construct a family $\{M_1, \dots, M_{\ell-1}\}$ of finite subsets of M_0 such that for every $i \in [\ell - 1]$:

(a) $|M_i| = F^{(\ell-i)}_{k,\ell,q,\varepsilon}(0)$,
(b) M_i is a subinterval of $M_{i-1} \setminus \{\max(M_{i-1})\}$, and
(c) for every $A \in \mathcal{F}$, every subset I of M_i and every $y \in k^I$ we have

$$\big|\mathrm{dens}\big(\{w \in [k]^C : (y, w) \in A \cap [k]^{\max(M_{i-i})}\}\big) - \mathrm{dens}(A \cap [k]^{\max(M_{i-1})})\big| \le \varepsilon$$

where $C = \{m \in \mathbb{N} : m < \max(M_{i-1})\} \setminus I$.

We set $L = \{\max(M_{\ell-1}) < \cdots < \max(M_0)\big\}$. Using properties (b) and (c) it is easy to check that the family $\mathcal{F}$ is (ε, L)-regular, as desired. ◻

4. A variant of the Graham–Rothschild Theorem for left variable words

Recall that for every Carlson–Simpson tree V of $[k]^{<\mathbb{N}}$ and every $\ell \in [\dim(V)]$ we denote by $\mathrm{Subtr}_\ell(V)$ the set of all ℓ-dimensional Carlson–Simpson subtrees of V. This section is devoted to the proof of the following partition result.

Theorem 4.1. *For every integer $k \geq 2$, every pair of integers $d \geq m \geq 1$ and every integer $r \geq 1$ there exists an integer N with the following property. If $n \geq N$ and W is an n-dimensional Carlson–Simpson tree of $[k]^{<\mathbb{N}}$, then for every r-coloring of the set $\mathrm{Subtr}_m(W)$ there exists $U \in \mathrm{Subtr}_d(W)$ such that the set $\mathrm{Subtr}_m(U)$ is monochromatic. The least integer N with this property will be denoted by $\mathrm{CS}(k, d, m, r)$.*

Theorem 4.1 is, of course, a variant of Theorem 2.1. It can hardly be called new since it follows using fairly standard arguments. Nevertheless, we have decided to include a proof for two reasons. The first one is self-containedness. Secondly, we want to emphasize the bounds we get from the argument for the numbers $\mathrm{CS}(k, d, m, r)$.

We start by introducing some pieces of notation. Let $k, d, m \in \mathbb{N}$ with $k \geq 2$ and $d \geq m \geq 1$. Also let W be a d-dimensional Carlson–Simpson tree of $[k]^{<\mathbb{N}}$ and $V \in \mathrm{Subtr}_m(W)$. The *depth* of V in W, denoted by $\mathrm{depth}_W(V)$, is defined to be the unique integer $i \in \{m, \dots, d\}$ such that the m-level $V(m)$ of V is contained in the i-level $W(i)$ of W, or equivalently, $V(m) \in \mathrm{Subs}_m(W(i))$. We set

$$\mathrm{Subtr}_m^{\max}(W) = \{V \in \mathrm{Subtr}_m(W) : \mathrm{depth}_W(V) = \dim(W)\}.$$

That is, $\mathrm{Subtr}_m^{\max}(W)$ is the set of all m-dimensional Carlson–Simpson subtrees of W of maximal depth. Our interest in this subclass is partly justified by the following simple, though important fact, which is a rather straightforward consequence of the relevant definitions.

Fact 4.2. *For every integer $d \geq 1$, every d-dimensional Carlson–Simpson tree W of $[k]^{<\mathbb{N}}$ and every $m \in [d]$ the map*

$$\mathrm{Subtr}_m^{\max}(W) \ni V \mapsto V(m) \in \mathrm{Subs}_m(W(d))$$

is a bijection.

Combining Theorem 2.1 and Fact 4.2 we get the following corollary.

Corollary 4.3. *Let $k \in \mathbb{N}$ with $k \geq 2$ and let $d, m, r \in \mathbb{N}$ with $d \geq m \geq 1$ and $r \geq 1$. If $n \geq \mathrm{GR}(k, d, m, r)$, then for every n-dimensional Carlson–Simpson tree W of $[k]^{<\mathbb{N}}$ and every r-coloring of the set $\mathrm{Subtr}_m^{\max}(W)$ there exists $U \in \mathrm{Subtr}_d^{\max}(W)$ such that the set $\mathrm{Subtr}_m^{\max}(U)$ is monochromatic.*

The proof of Theorem 4.1 is based on a strengthening of Corollary 4.3. To state it, it is convenient to introduce the following definition. For every $k, m, r \in \mathbb{N}$ with $k \geq 2$ and $m, r \geq 1$ we define the function $g_{k,m,r} : \mathbb{N} \to \mathbb{N}$ by $g_{k,m,r}(n) = 0$ if $n < m - 1$ and

$$g_{k,m,r}(n) = \mathrm{GR}(k, n + 1, m, r) \tag{4.1}$$

if $n \geq m - 1$. We have the following lemma.

Lemma 4.4. *Let $k, m, r \in \mathbb{N}$ with $k \geq 2$ and $m, r \geq 1$. Also let $q, n \in \mathbb{N}$ with $q \geq 1$ and $n \geq g^{(q)}_{k,m,r}(m)$. Then for every n-dimensional Carlson–Simpson tree W of $[k]^{<\mathbb{N}}$ and every r-coloring of the set $\mathrm{Subtr}_m(W)$ there exists $U \in \mathrm{Subtr}^{\max}_{m+q}(W)$ with the following property. For every pair $S, T \in \mathrm{Subtr}_m(U)$ with $\mathrm{depth}_U(S) = \mathrm{depth}_U(T)$ the Carlson–Simpson trees S and T have the same color.*

Proof. We fix a coloring $c : \mathrm{Subtr}_m(W) \to [r]$. For every $i \in \{0, \dots, q\}$ we set $n_i = g^{(q-i)}_{k,m,r}(m)$. Notice that, by (4.1), for every $i \in \{0, \dots, q-1\}$ we have

$$n_i = \mathrm{GR}(k, n_{i+1} + 1, m, r) \geq n_{i+1} + 1 \geq n_q = m. \tag{4.2}$$

We select $U_0 \in \mathrm{Subtr}^{\max}_{n_0}(W)$. By (4.2) and Corollary 4.3, we may construct a family $\{U_1, \dots, U_q\}$ of Carlson–Simpson subtrees of U_0 with the following properties:

(a) For every $i \in [q]$ we have $\dim(U_i) = n_i + 1$.
(b) $U_1 \in \mathrm{Subtr}^{\max}_{n_1+1}(U_0)$. Moreover, if $q \geq 2$, then for every $i \in [q-1]$ we have $U_{i+1} \in \mathrm{Subtr}^{\max}_{n_{i+1}}(U'_i)$ where $U'_i = U_i \setminus U_i(n_i + 1)$.
(c) For every $i \in [q]$ the set $\mathrm{Subtr}^{\max}_m(U_i)$ is monochromatic with respect to c.

For every $i \in [q]$ let $(c_i, w^{(i)}_0, \dots, w^{(i)}_{n_i})$ be the generating sequence of U_i. We define U to be the Carlson–Simpson tree of $[k]^{<\mathbb{N}}$ generated by the sequence

$$(c_q, w^{(q)}_0, \dots, w^{(q)}_{n_q})^\frown(w^{(q-1)}_{n_{q-1}}, \dots, w^{(2)}_{n_2}, w^{(1)}_{n_1}).$$

We will show that U is as desired. Indeed, notice first that

$$\dim(U) = (n_q + 1) + (q - 1) = m + q.$$

Also observe that $U \in \mathrm{Subtr}^{\max}_{m+q}(U_0)$ and so $U \in \mathrm{Subtr}^{\max}_{m+q}(W)$. Finally let $\ell \in [q]$ be arbitrary and set $i_\ell = q - \ell + 1 \in [q]$. By the definition of U and (b) above, we see that $U(m + \ell)$ is contained in $U_{i_\ell}(n_{i_\ell} + 1)$. Hence, for every pair $S, T \in \mathrm{Subtr}_m(U)$ with $\mathrm{depth}_U(S) = \mathrm{depth}_U(T) = m + \ell$ we have $S, T \in \mathrm{Subtr}^{\max}_m(U_{i_\ell})$. Invoking (c), we conclude that $c(S) = c(T)$, and the proof is complete. □

We are ready to proceed to the proof of Theorem 4.1.

Proof of Theorem 4.1. Let $k \in \mathbb{N}$ with $k \geq 2$. Also let $d, m, r \in \mathbb{N}$ with $d \geq m \geq 1$ and $r \geq 1$. We will show that

$$\mathrm{CS}(k, d, m, r) \leq g^{(d \cdot r - m)}_{k,m,r}(m).$$

Indeed, let $n \geq g^{(d \cdot r - m)}_{k,m,r}(m)$ and W be an arbitrary n-dimensional Carlson–Simpson tree of $[k]^{<\mathbb{N}}$. We fix a coloring $c : \mathrm{Subtr}_m(W) \to [r]$. By Lemma 4.4, there exists a Carlson–Simpson subtree R of W with $\dim(R) = d \cdot r$ such that for every $S \in \mathrm{Subtr}_m(R)$ the color $c(S)$ of S depends only on the depth of S in R. Therefore, by the classical pigeonhole principle, there exist a subset I of $\{0, \dots, d \cdot r\}$ with $|I| = d + 1$ and $r_0 \in [r]$ such that for every $i \in I$ and every $S \in \mathrm{Subtr}_m(R)$ with $\mathrm{depth}_R(S) = i$ we have $c(S) = r_0$. Let U be any d-dimensional Carlson–Simpson subtree of R which is contained in the set $\bigcup_{i \in I} R(i)$. By the previous discussion, we see that the coloring c restricted to $\mathrm{Subtr}_m(U)$ is constantly equal to r_0. The proof of Theorem 4.1 is thus complete. □

5. The convolution operation

The concatenation of two finite sequences provides us with a canonical way to "glue" a pair of elements of $[k]^{<\mathbb{N}}$. Our goal in this section is to describe a different "gluing" method which will be of fundamental importance throughout the paper.

The method is particularly easy to grasp for pairs of sequences of given length. Let $n, m \geq 1$ and fix a subset L of $\{0, \dots, n+m-1\}$ of cardinality n. Given an element x of $[k]^n$ and an element y of $[k]^m$, the outcome of the "gluing" method for the pair x, y is the unique element z of $[k]^{n+m}$ which is "equal" to x on L and to y on the rest of the coordinates. This simple process can, of course, be extended to arbitrary pairs of $[k]^{<\mathbb{N}}$. This is the content of the following definition.

Definition 5.1. Let $k \in \mathbb{N}$ with $k \geq 2$ and let $L = \{l_0 < \cdots < l_{|L|-1}\}$ be a nonempty finite subset of $\mathbb{N}$. For every $i \in \{0, \dots, |L|-1\}$ we set

$$L_i = \{l \in L : l < l_i\} \quad \text{and} \quad \overline{L}_i = \{n \in \mathbb{N} : n < l_i \text{ and } n \notin L_i\}. \tag{5.1}$$

Also let $n_L = \max(L) - |L| + 1$ and set

$$X_L = [k]^{n_L}.$$

We define the *convolution operation* $\mathrm{c}_L : [k]^{<|L|} \times X_L \to [k]^{<\mathbb{N}}$ associated to L as follows. For every $i \in \{0, \dots, |L|-1\}$, every $t \in [k]^i$ and every $x \in X_L$ we set

$$\mathrm{c}_L(t, x) = (\mathrm{I}_{L_i}(t), \mathrm{I}_{\overline{L}_i}(x|_{|\overline{L}_i|})) \in [k]^{l_i} \tag{5.2}$$

where I_{L_i} and $\mathrm{I}_{\overline{L}_i}$ are the canonical isomorphisms defined in §2.2.

More generally, let V be a Carlson–Simpson tree of $[k]^{<\mathbb{N}}$ and assume that L is contained in $\{0, \dots, \dim(V)\}$. The *convolution operation* $\mathrm{c}_{L,V} : [k]^{<|L|} \times X_L \to V$ associated to (L, V) is defined by

$$\mathrm{c}_{L,V}(t, x) = \mathrm{I}_V(\mathrm{c}_L(t, x)) \tag{5.3}$$

where I_V is the canonical isomorphism defined in §2.5.

Before we proceed let us give a specific example. Let $k = 5$ and $L = \{1, 3, 7, 9\}$, and notice that $X_L = [5]^6$. In particular, the convolution operation c_L associated to the set L is defined for pairs in $[5]^{<4} \times [5]^6$. Then for $t = (1, 2)$ and $x = (3, 5, 4, 2, 4, 1)$ we have

$$\mathrm{c}_L(t, x) = (3, \mathbf{1}, 5, \mathbf{2}, 4, 2, 4)$$

where we indicated with boldface letters the contribution of t.

The rest of this section is devoted to the study of convolution operations. We notice that all properties described below follow by carefully manipulating the relevant definitions. In fact, once the basic definitions have been properly understood, most of the material of this section should be regarded as fairly straightforward.

We begin with the following fact.

Fact 5.2. *Let $k \in \mathbb{N}$ with $k \geq 2$. Let V be a Carlson–Simpson tree of $[k]^{<\mathbb{N}}$ and $L = \{l_0 < \cdots < l_{|L|-1}\}$ be a nonempty finite subset of $\{0, \ldots, \dim(V)\}$. For every $t \in [k]^{<|L|}$ set*

$$\Omega_t = \{\mathrm{c}_{L,V}(t, x) : x \in X_L\}. \tag{5.4}$$

Then for every $t, t' \in [k]^{<|L|}$ with $t \neq t'$ we have $\Omega_t \cap \Omega_{t'} = \emptyset$. Moreover, for every $i \in \{0, \ldots, |L| - 1\}$ the family $\{\Omega_t : t \in [k]^i\}$ forms an equipartition of $V(l_i)$.

Proof. By the definition of the convolution operation, we see that $\Omega_t \cap \Omega_{t'} = \emptyset$ if $t \neq t'$. It is also easy to check that the family $\{\Omega_t : t \in [k]^i\}$ forms a partition of $V(l_i)$. Therefore, to complete the proof it is enough to observe that for every $i \in \{0, \ldots, |L|-1\}$ and every $t \in [k]^i$ we have

$$\mathrm{I}_V^{-1}(\Omega_t) = \{x \in [k]^{l_i} : x|_{L_i} = \mathrm{I}_{L_i}(t)\}.$$

Clearly this implies that $|\Omega_t| = |\Omega_{t'}|$ for every $t, t' \in [k]^i$. □

Using similar elementary observations we get the following.

Fact 5.3. *Let k, V and L be as in Fact 5.2. For every $t \in [k]^{<|L|}$ and every $s \in [k]^{<\mathbb{N}}$ we set*

$$Y_s^t = \{x \in X_L : \mathrm{c}_{L,V}(t, x) = s\}. \tag{5.5}$$

Then for every $t \in [k]^{<|L|}$ and every $s, s' \in \Omega_t$ with $s \neq s'$, where Ω_t is as in (5.4), the sets Y_s^t and $Y_{s'}^t$ are nonempty disjoint subsets of X_L. Moreover, the family $\{Y_s^t : s \in \Omega_t\}$ forms an equipartition of X_L.

We will also need the following fact.

Fact 5.4. *Let k, V and L be as in Fact 5.2. For every $t \in [k]^{<|L|}$ and every $s \in [k]^{<\mathbb{N}}$ let Ω_t and Y_s^t be as in (5.4) and (5.5) respectively. Then for every $i \in \{0, \ldots, |L| - 1\}$ and every $t, t' \in [k]^i$ there exists a map $g_{t,t'} : \Omega_t \to \Omega_{t'}$ with the following properties:*

(i) *For every $s \in \Omega_t$ we have $Y_s^t = Y_{g_{t,t'}(s)}^{t'}$.*
(ii) *$g_{t,t'}$ is a bijection.*
(iii) *$g_{t,t'}$ preserves the lexicographical order.*
(iv) *If t and t' are (r, r')-equivalent for some $r, r' \in [k]$ with $r \neq r'$ (see* §2.6*), then s and $g_{t,t'}(s)$ are (r, r')-equivalent for every $s \in \Omega_t$.*

Proof. For every $s \in \Omega_t$ we select $x_s \in Y_s^t$ and we set $g_{t,t'}(s) = \mathrm{c}_{L,V}(t', x_s)$. It is easy to check that $g_{t,t'}$ is well-defined and satisfies the above properties. □

We proceed with the following lemma.

Lemma 5.5. *Let $k \in \mathbb{N}$ with $k \geq 2$. Let V be a Carlson–Simpson tree of $[k]^{<\mathbb{N}}$ and $L = \{l_0 < \cdots < l_{|L|-1}\}$ be a nonempty finite subset of $\{0, \ldots, \dim(V)\}$. Also let $t \in [k]^{<|L|}$ and $A \subseteq [k]^{<\mathbb{N}}$ and set $B = \mathrm{c}_{L,V}^{-1}(A)$. Then:*

(i) $\mathrm{dens}_{\Omega_t}(A) = \mathrm{dens}_{\{t\} \times X_L}(B)$ *where Ω_t is as in (5.4).*
(ii) *For every $i \in \{0, \ldots, |L| - 1\}$ we have* $\mathrm{dens}_{V(l_i)}(A) = \mathrm{dens}_{[k]^i \times X_L}(B)$.

Proof. For every $s \in \Omega_t$ let Y_s^t be as in (5.5). By the definition of B and Ω_t,

$$B \cap (\{t\} \times X_L) = \{(t,x) : \mathrm{c}_{L,V}(t,x) \in A \cap \Omega_t\} = \bigcup_{s \in A \cap \Omega_t} \{(t,x) : \mathrm{c}_{L,V}(t,x) = s\}$$
$$= \bigcup_{s \in A \cap \Omega_t} \{t\} \times Y_s^t. \tag{5.6}$$

By Fact 5.2, for every $s \in \Omega_t$ we have

$$|Y_s^t|/|X_L| = 1/|\Omega_t|. \tag{5.7}$$

Therefore,

$$\mathrm{dens}_{\{t\}\times X_L}(B) = \frac{|B \cap (\{t\} \times X_L)|}{|\{t\} \times X_L|} \overset{(5.6)}{=} \sum_{s \in A \cap \Omega_t} \frac{|\{t\} \times Y_s^t|}{|\{t\} \times X_L|}$$
$$= \sum_{s \in A \cap \Omega_t} \frac{|Y_s^t|}{|X_L|} \overset{(5.7)}{=} \frac{|A \cap \Omega_t|}{|\Omega_t|} = \mathrm{dens}_{\Omega_t}(A).$$

This completes the proof (i).

For (ii) is satisfied, let $i \in \{0, \dots, |L|-1\}$. By Fact 5.2, we see that $\mathrm{dens}_{V(l_i)}(A) = \mathbb{E}_{t \in [k]^i} \mathrm{dens}_{\Omega_t}(A)$. Hence by (i), the result follows. □

The final three lemmas of this section contain some coherence properties of convolution operations. The first one shows that convolution operations preserve Carlson–Simpson trees.

Lemma 5.6. *Let k, V and L be as in Lemma 5.5. Also let W be a Carlson–Simpson subtree of V and $x \in X_L$. Set*

$$W_x = \{\mathrm{c}_{L,V}(w,x) : w \in W\}. \tag{5.8}$$

Then W_x is a Carlson–Simpson subtree of $[k]^{<|L|}$ of dimension $\dim(W)$. *Moreover, for every $i \in \{0, \dots, \dim(W)\}$ we have*

$$W_x(i) = \{\mathrm{c}_{L,V}(w,x) : w \in W(i)\}.$$

Proof. Let $\ell = |L|$ and let $\{l_0 < \dots < l_{\ell-1}\}$ be the increasing enumeration of L. Set

$$J_0 = \{n \in \mathbb{N} : n < l_0\} \quad \text{and} \quad J_i = \{n \in \mathbb{N} : l_{i-1} - i + 1 \le n \le l_i - i - 1\}$$

for every $i \in [\ell-1]$. Observe that the family $\{J_0, J_1, \dots, J_{\ell-1}\}$ forms a partition of $\{0, \dots, l_{\ell-1} - \ell\}$ into successive intervals some of which may be empty. Let

$$c = \mathrm{I}_{J_0}^{-1}(x|_{J_0}).$$

Also, for every $i \in \{0, \dots, \ell-2\}$ we define

$$w_i = v^\frown \mathrm{I}_{J_{i+1}}^{-1}(x|_{J_{i+1}}).$$

Clearly w_i is a left variable word over k for every $i \in \{0, \dots, \ell - 2\}$. Let S_x be the Carlson–Simpson tree generated by the sequence $(c, w_0, \dots, w_{\ell-2})$ and observe that $\mathrm{c}_L(t, x) = \mathrm{I}_{S_x}(t)$ for every $t \in [k]^{<|L|}$. Hence, for every $t \in [k]^{<|L|}$ we have

$$\mathrm{c}_{L,V}(t, x) = \mathrm{I}_V(\mathrm{I}_{S_x}(t)). \tag{5.9}$$

Using (5.9) and invoking the definition of W_x in (5.8) yields the result. □

The next result enables us to transfer quantitative information from the space $[k]^{<\mathbb{N}}$ to the space on which the convolution operations are acting.

Lemma 5.7. *Let k, V and L be as in Lemma 5.5. Also let W be a Carlson–Simpson subtree of $[k]^{<|L|}$ and $x \in X_L$ and define W_x as in (5.8). If $A \subseteq [k]^{<\mathbb{N}}$ and $B = \mathrm{c}_{L,V}^{-1}(A)$, then for every $i \in \{0, \dots, \dim(W)\}$ we have*

$$\mathrm{dens}_{W_x(i)}(A) = \mathrm{dens}_{W(i)\times\{x\}}(B). \tag{5.10}$$

Proof. We define the Carlson–Simpson tree S_x exactly as in the proof of Lemma 5.6. By (5.9), for every $t \in [k]^{<|L|}$ we see that $(t, x) \in B$ if and only if $\mathrm{I}_V(\mathrm{I}_{S_x}(t)) \in A$, and the result follows. □

We close this section with the following lemma.

Lemma 5.8. *Let k, V and L be as in Lemma 5.5. Also let W be a Carlson–Simpson subtree of $[k]^{<|L|}$ and let $A \subseteq [k]^{<\mathbb{N}}$. Then for every $i \in \{0, \dots, \dim(W)\}$ we have*

$$\mathrm{dens}_{\mathrm{c}_{L,V}(W(i)\times X_L)}(A) = \mathbb{E}_{x\in X_L}\,\mathrm{dens}_{W_x(i)}(A) \tag{5.11}$$

where W_x is as in (5.8). In particular, for every $i \in \{0, \dots, |L| - 1\}$ we have

$$\mathrm{dens}_{V(l_i)}(A) = \mathbb{E}_{x\in X_L}\,\mathrm{dens}_{R_x(i)}(A) \tag{5.12}$$

where $R_x = \{\mathrm{c}_{L,V}(t, x) : t \in [k]^{<|L|}\}$ for every $x \in X_L$.

Proof. Let $i \in \{0, \dots, \dim(W)\}$. There exists a unique $l \in \{0, \dots, |L| - 1\}$ such that $W(i) \subseteq [k]^l$. By Fact 5.2, the family $\{\Omega_t : t \in W(i)\}$ forms an equipartition of $\mathrm{c}_{L,V}(W(i) \times X_L)$. Therefore, setting $B = \mathrm{c}_{L,V}^{-1}(A)$, by Lemma 5.5,

$$\begin{aligned}
\mathrm{dens}_{\mathrm{c}_{L,V}(W(i)\times X_L)}(A) &= \mathbb{E}_{t\in W(i)}\,\mathrm{dens}_{\Omega_t}(A) = \mathbb{E}_{t\in W(i)}\,\mathrm{dens}_{\{t\}\times X_L}(B) \\
&= \mathrm{dens}_{W(i)\times X_L}(B) = \mathbb{E}_{x\in X_L}\,\mathrm{dens}_{W(i)\times\{x\}}(B) \\
&\overset{(5.10)}{=} \mathbb{E}_{x\in X_L}\,\mathrm{dens}_{W_x(i)}(A).
\end{aligned}$$

Finally notice that $V(l_i) = \mathrm{c}_{L,V}([k]^i \times X_L)$ for every $i \in \{0, \dots, |L| - 1\}$. Therefore, equality (5.12) follows from (5.11). □

6. Iterated convolutions

Our goal in this section is to study iterations of convolution operations. We remark that this material will be used only in §9. The exact statements that we need are isolated in §6.2.

6.1. Definitions and basic properties. We start with the following definition.

Definition 6.1. Let $\mathbf{L} = (L_n)_{n=0}^d$ be a finite sequence of nonempty finite subsets of $\mathbb{N}$. Also let $k \in \mathbb{N}$ with $k \geq 2$ and $\mathbf{V} = (V_n)_{n=0}^d$ be a finite sequence of Carlson–Simpson trees of $[k]^{<\mathbb{N}}$ with the same length as $\mathbf{L}$. We say that the pair $(\mathbf{L}, \mathbf{V})$ is *k-compatible*, or simply *compatible* if k is understood, provided that for every $n \in \{0, \dots, d\}$ we have $L_n \subseteq \{0, \dots, \dim(V_n)\}$ and, if $n < d$, then $V_{n+1} \subseteq [k]^{<|L_n|}$.

Notice that if $(\mathbf{L}, \mathbf{V})$ is a compatible pair and $\mathbf{L}'$,$\mathbf{V}'$ are initial subsequences of $\mathbf{L}$, $\mathbf{V}$ with common length, then the pair $(\mathbf{L}', \mathbf{V}')$ is also compatible. Also observe that for every compatible pair $(\mathbf{L}, \mathbf{V}) = \big((L_n)_{n=0}^d, (V_n)_{n=0}^d\big)$ and every $n \in \{0, \dots, d\}$ we can define the convolution operation $\mathrm{c}_{L_n, V_n} : [k]^{<|L_n|} \times X_{L_n} \to V_n$ associated to (L_n, V_n) as described in §5. What Definition 6.1 guarantees is that for compatible pairs we can iterate these operations. This is the content of the following definition.

Definition 6.2. Let $k \in \mathbb{N}$ with $k \geq 2$ and let $(\mathbf{L}, \mathbf{V}) = ((L_n)_{n=0}^d, (V_n)_{n=0}^d)$ be a k-compatible pair. We set

$$X_{\mathbf{L}} = \prod_{n=0}^{d} X_{L_n}.$$

By recursion on d we define the *iterated convolution operation*

$$\mathrm{c}_{\mathbf{L},\mathbf{V}} : [k]^{<|L_d|} \times X_{\mathbf{L}} \to V_0$$

associated to $(\mathbf{L}, \mathbf{V})$ as follows. For $d = 0$ this is the c_{L_0, V_0} convolution operation defined in (5.3).

If $d \geq 1$, then let $\mathbf{L}' = (L_n)_{n=0}^{d-1}$ and $\mathbf{V}' = (V_n)_{n=0}^{d-1}$ and assume that the operation $\mathrm{c}_{\mathbf{L}',\mathbf{V}'}$ has been defined. We set

$$\mathrm{c}_{\mathbf{L},\mathbf{V}}(s, x_0, \dots, x_d) = \mathrm{c}_{\mathbf{L}',\mathbf{V}'}\big(\mathrm{c}_{L_d,V_d}(s, x_d), x_0, \dots, x_{d-1}\big)$$

for every $s \in [k]^{<|L_d|}$ and every $(x_0, \dots, x_d) \in X_{\mathbf{L}}$. In this case, the *quotient map*

$$\mathrm{q}_{\mathbf{L},\mathbf{V}} : [k]^{<|L_d|} \times X_{\mathbf{L}} \to [k]^{<|L_{d-1}|} \times X_{\mathbf{L}'}$$

associated to $(\mathbf{L}, \mathbf{V})$ is defined by

$$\mathrm{q}_{\mathbf{L},\mathbf{V}}(t, \mathbf{x}, x) = (\mathrm{c}_{L_d,V_d}(t, x), \mathbf{x}) \tag{6.1}$$

for every $t \in [k]^{<|L_d|}$ and every $(\mathbf{x}, x) \in X_{\mathbf{L}'} \times X_{L_d}$.

The rest of this subsection is devoted to several lemmas establishing properties of iterated convolutions. Just as in §5, all this material follows by carefully manipulating the relevant definitions. We begin with the following elementary fact.

Fact 6.3. *Let* $k \geq 2$ *and* $(\mathbf{L}, \mathbf{V}) = ((L_n)_{n=0}^{d}, (V_n)_{n=0}^{d})$ *be a* k*-compatible pair. If* $d \geq 1$ *and* $(\mathbf{L}', \mathbf{V}') = ((L_n)_{n=0}^{d-1}, (V_n)_{n=0}^{d-1})$*, then* $\mathrm{c}_{\mathbf{L},\mathbf{V}} = \mathrm{c}_{\mathbf{L}',\mathbf{V}'} \circ \mathrm{q}_{\mathbf{L},\mathbf{V}}$.

The next two lemmas are multidimensional analogues of Lemmas 5.6 and 5.7.

Lemma 6.4. *Let* $k \geq 2$ *and* $(\mathbf{L}, \mathbf{V}) = ((L_n)_{n=0}^{d}, (V_n)_{n=0}^{d})$ *be a* k*-compatible pair. Also let* W *be a Carlson–Simpson subtree of* $[k]^{<|L_d|}$ *and let* $\mathbf{x} \in X_{\mathbf{L}}$*. Then the set*

$$W_{\mathbf{x}} = \{\mathrm{c}_{\mathbf{L},\mathbf{V}}(w, \mathbf{x}) : w \in W\}$$

is a Carlson–Simpson subtree of V_0 *with the same dimension as* W*. Moreover, for every* $i \in \{0, \dots, \dim(W)\}$ *we have*

$$W_{\mathbf{x}}(i) = \{\mathrm{c}_{\mathbf{L},\mathbf{V}}(w, \mathbf{x}) : w \in W(i)\}.$$

Proof. Both assertions are proved by induction on d and using similar arguments. We will give the details only for the first one. The case $d = 0$ is the content of Lemma 5.6. So, let $d \geq 1$ and assume that the result has been proved up to $d-1$. Fix a compatible pair $(\mathbf{L}, \mathbf{V}) = ((L_n)_{n=0}^{d}, (V_n)_{n=0}^{d})$ and let W and $\mathbf{x}$ be as in the statement of the lemma. Write $\mathbf{x} = (x_0, \dots, x_d)$ and set $\mathbf{x}' = (x_0, \dots, x_{d-1})$ and $S = \{\mathrm{c}_{L_d,V_d}(w, x_d) : w \in W\}$. Also let $\mathbf{L}' = (L_n)_{n=0}^{d-1}$ and $\mathbf{V}' = (V_n)_{n=0}^{d-1}$ and observe that the pair $(\mathbf{L}', \mathbf{V}')$ is compatible. By Lemma 5.6, S is a Carlson–Simpson subtree of V_d with $\dim(S) = \dim(W)$. By Definition 6.1, S is contained in $[k]^{<|L_{d-1}|}$. Therefore, applying the inductive assumptions for the pair $(\mathbf{L}', \mathbf{V}')$, we see that $S_{\mathbf{x}'}$ is a Carlson–Simpson subtree of V_0 of dimension $\dim(W)$. Noticing that $S_{\mathbf{x}'}$ coincides with $W_{\mathbf{x}}$ yields the result. □

Lemma 6.5. *Let* $k, \mathbf{L}, \mathbf{V}, W$ *and* $\mathbf{x}$ *be as Lemma 6.4. Also let* $A \subseteq [k]^{<\mathbb{N}}$ *and set* $B = \mathrm{c}_{\mathbf{L},\mathbf{V}}^{-1}(A)$*. Then for every* $i \in \{0, \dots, \dim(W)\}$ *we have*

$$\mathrm{dens}_{W(i)\times\{\mathbf{x}\}}(B) = \mathrm{dens}_{W_{\mathbf{x}}(i)}(A).$$

Proof. By induction on d. The case $d = 0$ follows from Lemma 5.7. Let $d \geq 1$ and assume that the result has been proved up to $d - 1$. Fix a k-compatible pair $(\mathbf{L}, \mathbf{V}) = ((L_n)_{n=0}^{d}, (V_n)_{n=0}^{d})$ and let $W, \mathbf{x}, A$ and B be as in the statement of the lemma. Write $\mathbf{x} = (x_0, \dots, x_d)$ and define $\mathbf{x}', S, \mathbf{L}'$ and $\mathbf{V}'$ precisely as in the proof of Lemma 6.4. We set $C = \mathrm{c}_{\mathbf{L}',\mathbf{V}'}^{-1}(A)$. For every $\mathbf{y} \in X_{\mathbf{L}'}$ let $C_{\mathbf{y}}$ and $B_{\mathbf{y}}$ be the sections of C and B at $\mathbf{y}$. By Fact 6.3, we see that $B_{\mathbf{y}} = \mathrm{c}_{L_d,V_d}^{-1}(C_{\mathbf{y}})$. Hence,

$$\begin{aligned}\mathrm{dens}_{W(i)\times\{\mathbf{x}\}}(B) &= \mathrm{dens}_{W(i)\times\{\mathbf{x}'\}\times\{x_d\}}(B) = \mathrm{dens}_{W(i)\times\{\mathbf{x}'\}\times\{x_d\}}(B_{\mathbf{x}'} \times \{\mathbf{x}'\}) \\ &= \mathrm{dens}_{W(i)\times\{x_d\}}(B_{\mathbf{x}'}) = \mathrm{dens}_{W(i)\times\{x_d\}}(\mathrm{c}_{L_d,V_d}^{-1}(C_{\mathbf{x}'}))\end{aligned} \tag{6.2}$$

Invoking Lemma 5.7 we have

$$\mathrm{dens}_{W(i)\times\{x_d\}}(\mathrm{c}_{L_d,V_d}^{-1}(C_{\mathbf{x}'})) = \mathrm{dens}_{S(i)}(C_{\mathbf{x}'}). \tag{6.3}$$

Next observe that

$$\mathrm{dens}_{S(i)}(C_{\mathbf{x}'}) = \mathrm{dens}_{S(i)\times\{\mathbf{x}'\}}(C_{\mathbf{x}'} \times \{\mathbf{x}'\}) = \mathrm{dens}_{S(i)\times\{\mathbf{x}'\}}(C). \tag{6.4}$$

As we have already pointed out in the proof of Lemma 6.4, the set $S_{\mathbf{x}'}$ coincides with $W_{\mathbf{x}}$. Since $C = \mathrm{c}^{-1}_{\mathbf{L}',\mathbf{V}'}(A)$, we may apply our inductive hypothesis to the Carlson–Simpson tree S, the element $\mathbf{x}'$ and the set A to infer that

$$\mathrm{dens}_{S(i)\times\{\mathbf{x}'\}}(C) = \mathrm{dens}_{S_{\mathbf{x}'}(i)}(A) = \mathrm{dens}_{W_{\mathbf{x}}(i)}(A). \tag{6.5}$$

Combining equalities (6.2) to (6.5) gives the result. □

We proceed with the following lemma.

Lemma 6.6. *Let $k \geq 2$ and $(\mathbf{L},\mathbf{V}) = ((L_n)_{n=0}^{d}, (V_n)_{n=0}^{d})$ be a k-compatible pair. Assume that $d \geq 1$ and let $\mathbf{L}' = (L_n)_{n=0}^{d-1}$ and $\mathbf{V}' = (V_n)_{n=0}^{d-1}$. Let $C \subseteq [k]^{<|L_{d-1}|} \times X_{\mathbf{L}'}$ and set $B = \mathrm{q}^{-1}_{\mathbf{L},\mathbf{V}}(C)$. Finally let $t \in [k]^{<|L_d|}$ and set*

$$\Omega_t = \{\mathrm{c}_{L_d,V_d}(t,x) : x \in X_{L_d}\}. \tag{6.6}$$

Then $\mathrm{q}^{-1}_{\mathbf{L},\mathbf{V}}(\Omega_t \times X_{\mathbf{L}'}) = \{t\} \times X_{\mathbf{L}}$ and

$$\mathrm{dens}_{\Omega_t\times X_{\mathbf{L}'}}(C) = \mathrm{dens}_{\{t\}\times X_{\mathbf{L}}}(B). \tag{6.7}$$

Proof. By the definition of the quotient map $\mathrm{q}_{\mathbf{L},\mathbf{V}}$ in (6.1), we have

$$\mathrm{q}^{-1}_{\mathbf{L},\mathbf{V}}(\Omega_t \times X_{\mathbf{L}'}) = \mathrm{c}^{-1}_{L_d,V_d}(\Omega_t) \times X_{\mathbf{L}'}. \tag{6.8}$$

Since $\mathrm{c}^{-1}_{L_d,V_d}(\Omega_t) = \{t\} \times X_{L_d}$, by (6.8) we see that $\mathrm{q}^{-1}_{\mathbf{L},\mathbf{V}}(\Omega_t \times X_{\mathbf{L}'}) = \{t\} \times X_{\mathbf{L}}$.

Now for every $\mathbf{x}' \in X_{\mathbf{L}'}$ let $C_{\mathbf{x}'}$ and $B_{\mathbf{x}'}$ be the sections of C and B at $\mathbf{x}'$ respectively. Observe that $C_{\mathbf{x}'} \subseteq [k]^{<|L_{d-1}|}$ and $B_{\mathbf{x}'} \subseteq [k]^{<|L_d|} \times X_{L_d}$. Also notice that $B_{\mathbf{x}'} = \mathrm{c}^{-1}_{L_d,V_d}(C_{\mathbf{x}'})$ for every $\mathbf{x}' \in X_{\mathbf{L}'}$. Hence, by Lemma 5.5,

$$\mathrm{dens}_{\{t\}\times X_{L_d}}(B_{\mathbf{x}'}) = \mathrm{dens}_{\Omega_t}(C_{\mathbf{x}'}) \tag{6.9}$$

for every $\mathbf{x}' \in X_{\mathbf{L}'}$. Therefore,

$$\mathrm{dens}_{\{t\}\times X_{\mathbf{L}}}(B) = \mathbb{E}_{\mathbf{x}'\in X_{\mathbf{L}'}} \mathrm{dens}_{\{t\}\times X_{L_d}}(B_{\mathbf{x}'}) \overset{(6.9)}{=} \mathbb{E}_{\mathbf{x}'\in X_{\mathbf{L}'}} \mathrm{dens}_{\Omega_t}(C_{\mathbf{x}'}) = \mathrm{dens}_{\Omega_t\times X_{\mathbf{L}'}}(C).$$

□

We close this subsection with the following consequence of Lemma 6.6.

Corollary 6.7. *Let $k \geq 2$ and $(\mathbf{L},\mathbf{V}) = ((L_n)_{n=0}^{d}, (V_n)_{n=0}^{d})$ be a k-compatible pair. Assume that $d \geq 1$ and let $\mathbf{L}' = (L_n)_{n=0}^{d-1}$ and $\mathbf{V}' = (V_n)_{n=0}^{d-1}$. Let $A \subseteq [k]^{<\mathbb{N}}$ and set $A^d = \mathrm{c}^{-1}_{\mathbf{L},\mathbf{V}}(A)$ and $A^{d-1} = \mathrm{c}^{-1}_{\mathbf{L}',\mathbf{V}'}(A)$. Then for every $t \in [k]^{<|L_d|}$,*

$$\mathrm{dens}_{X_{\mathbf{L}}}(A^d_t) = \mathbb{E}_{s\in\Omega_t} \mathrm{dens}_{X_{\mathbf{L}'}}(A^{d-1}_s)$$

where A^d_t is the section of A^d at t, $\Omega_t \subseteq V_d \subseteq [k]^{<|L_{d-1}|}$ is as in (6.6) and A^{d-1}_s is the section of A^{d-1} at s.

Proof. We fix $t \in [k]^{<|L_d|}$. Notice that

$$A^d = \mathrm{c}^{-1}_{\mathbf{L},\mathbf{V}}(A) = \mathrm{q}^{-1}_{\mathbf{L},\mathbf{V}}(\mathrm{c}^{-1}_{\mathbf{L}',\mathbf{V}'}(A)) = \mathrm{q}^{-1}_{\mathbf{L},\mathbf{V}}(A^{d-1}). \tag{6.10}$$

By (6.10) and Lemma 6.6 applied to the sets "$B = A^d$" and "$C = A^{d-1}$" we obtain

$$\begin{aligned}\mathrm{dens}_{X_{\mathbf{L}}}(A^d_t) &= \mathrm{dens}_{\{t\}\times X_{\mathbf{L}}}(A^d) \\ &\overset{(6.7)}{=} \mathrm{dens}_{\Omega_t \times X_{\mathbf{L}'}}(A^{d-1}) = \mathbb{E}_{s\in\Omega_t}\, \mathrm{dens}_{X_{\mathbf{L}'}}(A^{d-1}_s).\end{aligned}$$ □

6.2. Consequences. As already mentioned, in this subsection we will collect some results which will be of particular importance in §9. The first two of them follow by repeated applications of Corollary 6.7. The details are left to the reader.

Corollary 6.8. *Let $k \geq 2$ and $(\mathbf{L}, \mathbf{V}) = ((L_n)^d_{n=0}, (V_n)^d_{n=0})$ be a k-compatible pair. Let $0 < \gamma \leq 1$ and $A \subseteq [k]^{<\mathbb{N}}$. Set $A^d = \mathrm{c}^{-1}_{\mathbf{L},\mathbf{V}}(A)$ and $A^0 = \mathrm{c}^{-1}_{L_0,V_0}(A)$. Suppose that $\mathrm{dens}_{X_{L_0}}(A^0_s) \geq \gamma$ for every $s \in [k]^{<|L_0|}$ where A^0_s is the section of A^0 at s. If $t \in [k]^{<|L_d|}$ and A^d_t is the section of A^d at t, then $\mathrm{dens}_{X_{\mathbf{L}}}(A^d_t) \geq \gamma$.*

Corollary 6.9. *Let $k \geq 2$ and $(\mathbf{L}, \mathbf{V}) = ((L_n)^d_{n=0}, (V_n)^d_{n=0})$ be a k-compatible pair. Let $0 < \lambda \leq 1$ and $A, B \subseteq [k]^{<\mathbb{N}}$ with $A \subseteq B$. Set $A^d = \mathrm{c}^{-1}_{\mathbf{L},\mathbf{V}}(A)$, $A^0 = \mathrm{c}^{-1}_{L_0,V_0}(A)$, $B^d = \mathrm{c}^{-1}_{\mathbf{L},\mathbf{V}}(B)$ and $B^0 = \mathrm{c}^{-1}_{L_0,V_0}(B)$. Suppose that $\mathrm{dens}_{X_{L_0}}(A^0_s) \geq \lambda \cdot \mathrm{dens}_{X_{L_0}}(B^0_s)$ for every $s \in [k]^{<|L_0|}$ where A^0_s and B^0_s are the sections of A^0 and B^0 at s. If $t \in [k]^{<|L_d|}$, then $\mathrm{dens}_{X_{\mathbf{L}}}(A^d_t) \geq \lambda \cdot \mathrm{dens}_{X_{\mathbf{L}}}(B^d_t)$ where A^d_t and B^d_t are the sections of A^d and B^d at t.*

The final result is a consequence of Lemma 6.6.

Corollary 6.10. *Let $k \geq 2$ and $(\mathbf{L}, \mathbf{V}) = ((L_n)^d_{n=0}, (V_n)^d_{n=0})$ be a k-compatible pair. Assume that $d \geq 1$ and let $\mathbf{L}' = (L_n)^{d-1}_{n=0}$ and $\mathbf{V}' = (V_n)^{d-1}_{n=0}$. Let $t \in [k]^{<|L_d|}$ and C_0 be a nonempty subset of $\Omega_t \times X_{\mathbf{L}'}$ where Ω_t is as in (6.6). Also let $C_1 \subseteq [k]^{<|L_{d-1}|} \times X_{\mathbf{L}'}$. Set $B_i = \mathrm{q}^{-1}_{\mathbf{L},\mathbf{V}}(C_i)$ for every $i \in \{0, 1\}$. Then $\mathrm{dens}_{C_0}(C_1) = \mathrm{dens}_{B_0}(B_1)$.*

Proof. Notice that $\mathrm{q}^{-1}_{\mathbf{L},\mathbf{V}}(C_0 \cap C_1) = B_0 \cap B_1$. Since $\mathrm{q}^{-1}_{\mathbf{L},\mathbf{V}}(\Omega_t \times X_{\mathbf{L}'}) = \{t\} \times X_{\mathbf{L}}$, we see that $B_0 \subseteq \{t\} \times X_{\mathbf{L}}$. Therefore,

$$\begin{aligned}\mathrm{dens}_{C_0}(C_1) &= \frac{|C_0 \cap C_1|}{|C_0|} = \frac{\mathrm{dens}_{\Omega_t\times X_{\mathbf{L}'}}(C_0 \cap C_1)}{\mathrm{dens}_{\Omega_t\times X_{\mathbf{L}'}}(C_0)} \\ &\overset{(6.7)}{=} \frac{\mathrm{dens}_{\{t\}\times X_{\mathbf{L}}}(B_0 \cap B_1)}{\mathrm{dens}_{\{t\}\times X_{\mathbf{L}}}(B_0)} = \frac{|B_0 \cap B_1|}{|B_0|} = \mathrm{dens}_{B_0}(B_1).\end{aligned}$$ □

7. Preliminary tools for the proof of Theorem B

In this section we will gather some results that are part of the proof of Theorem B but are not directly related to the main argument. Specifically, in §7.1 we prove the first instance of Theorem B which can be seen as a variant of the classical Sperner Theorem [35]. In §7.2 we show how one can estimate the number $\mathrm{DCS}(k, m + 1, \delta)$ assuming that the numbers $\mathrm{DCS}(k, m, \beta)$ have been defined for every $0 < \beta \le 1$. This result is part of an inductive scheme that we will discuss in detail in §8.1. Finally, in §7.3 we present some consequences.

7.1. Estimating the numbers $\mathrm{DCS}(2, 1, \delta)$**.** We have the following proposition.

Proposition 7.1. *Let $0 < \delta \le 1$. Also let A be a subset of $[2]^{<\mathbb{N}}$ and N be a finite subset of $\mathbb{N}$ such that*

$$|N| \ge \mathrm{Reg}\big(2, \mathrm{CS}(2, \lceil 17\delta^{-2}\rceil, 1, 2) + 1, 1, \delta/4\big).$$

If $|A \cap [2]^n| \ge \delta 2^n$ for every $n \in N$, then there exists a Carlson–Simpson line R of $[2]^{<\mathbb{N}}$ which is contained in A. In particular,

$$\mathrm{DCS}(2, 1, \delta) \le \mathrm{Reg}\big(2, \mathrm{CS}(2, \lceil 17\delta^{-2}\rceil, 1, 2) + 1, 1, \delta/4\big).$$

We should point out that the estimate for the numbers $\mathrm{DCS}(2, 1, \delta)$ obtained in Proposition 7.1 is rather weak and far from being optimal. However, the proof of Proposition 7.1 is conceptually close to the proof of the general case of Theorem B, and as such, should serve as a motivating introduction to the main argument.

We start with the following lemma.

Lemma 7.2. *Let A and N be as in Proposition 7.1. Then there exists $L \subseteq N$ with*

$$|L| = \mathrm{CS}(2, \lceil 17\delta^{-2}\rceil, 1, 2) + 1 \tag{7.1}$$

and satisfying the following property. Let $\mathrm{c}_L : [2]^{<|L|} \times X_L \to [2]^{<\mathbb{N}}$ be the convolution operation associated to L and set $B = \mathrm{c}_L^{-1}(A)$. Then for every $t \in [2]^{<|L|}$ we have $\mathrm{dens}(B_t) \ge 3\delta/4$ where $B_t = \{x \in X_L : (t, x) \in B\}$ is the section of B at t.

Proof. By Lemma 3.2 and our assumptions on the size of the set N, there exists a subset L of N with $|L| = \mathrm{CS}(2, \lceil 17\delta^{-2}\rceil, 1, 2) + 1$ and such that the family $\mathcal{F} := \{A\}$ is $(\delta/4, L)$-regular. Write the set L in increasing order as $\{l_0 < \dots < l_{|L|-1}\}$ and for every $i \in \{0, \dots, |L| - 1\}$ let L_i and $\overline{L}_i$ be as in (5.1). Since $|A \cap [2]^n| \ge \delta 2^n$ for every $n \in L$ and the singleton $\{A\}$ is $(\delta/4, L)$-regular, we see that

$$\mathrm{dens}\big(\{w \in [2]^{\overline{L}_i} : (y, w) \in A \cap [2]^{l_i}\}\big) \ge 3\delta/4 \tag{7.2}$$

for every $i \in \{0, \dots, |L| - 1\}$ and every $y \in [2]^{L_i}$. By the definition of the convolution operation in (5.2), for every $i \in \{0, \dots, |L| - 1\}$ and every $t \in [2]^i$ we have

$$\Omega_t \overset{(5.4)}{=} \{\mathrm{c}_L(t, x) : x \in X_L\} = \{z \in [2]^{l_i} : z|_{L_i} = \mathrm{I}_{L_i}(t)\}.$$

Thus, by (7.2) applied to "$y = \mathrm{I}_{L_i}(t)$", we obtain

$$\mathrm{dens}_{\Omega_t}(A) = \mathrm{dens}\big(\{w \in [2]^{\overline{L}_i} : (\mathrm{I}_{L_i}(t), w) \in A \cap [2]^{l_i}\}\big) \ge 3\delta/4. \tag{7.3}$$

Finally, by Lemma 5.5, we have

$$\mathrm{dens}_{\Omega_t}(A) = \mathrm{dens}_{\{t\}\times X_L}(B) = \mathrm{dens}_{X_L}(B_t). \tag{7.4}$$

Combining (7.3) and (7.4) yields the result. □

For the next step of the proof of Proposition 7.1 we need to introduce some terminology. Let W be an m-dimensional Carlson–Simpson tree of $[2]^{<\mathbb{N}}$ and $(c, w_0, \dots, w_{m-1})$ be its generating sequence. Also let $t, t' \in W$. We say that t' is a *successor* of t in W if there exist $i, j \in \{0, \dots, m-1\}$ with $i \le j$ as well as $a_i, \dots, a_j \in [2]$ such that $t' = t^\frown w_i(a_i)^\frown \dots ^\frown w_j(a_j)$. If, in addition, we have $a_i = 1$, then we say that t' is a *left successor* of t in W.

Lemma 7.3. *Let L and $\{B_t : t \in [2]^{<|L|}\}$ be as in Lemma 7.2. Then there exists a Carlson–Simpson subtree W of $[2]^{<|L|}$ with $\dim(W) = \lceil 17\delta^{-2} \rceil$ and such that*

$$\mathrm{dens}(B_t \cap B_{t'}) \ge \delta^2/16$$

for every $t, t' \in W$ with t' a left successor of t in W.

Proof. For every Carlson–Simpson line S of $[2]^{<\mathbb{N}}$ let us denote by (c_S, w_S) its generating sequence. We set

$$\mathcal{L} = \{S \in \mathrm{Subtr}_1([2]^{<|L|}) : \mathrm{dens}(B_{c_S} \cap B_{c_S^\frown w_S(1)}) \ge \delta^2/16\}.$$

By Theorem 4.1 and (7.1), there exists a Carlson–Simpson subtree W of $[2]^{<|L|}$ with $\dim(W) = \lceil 17\delta^{-2} \rceil$ such that either $\mathrm{Subtr}_1(W) \subseteq \mathcal{L}$ or $\mathrm{Subtr}_1(W) \cap \mathcal{L} = \emptyset$. Observe that for every $t, t' \in W$, t' is a left successor of t in W if and only if there exists a Carlson–Simpson line S of W such that $t = c_S$ and $t' = c_S^\frown w_S(1)$. Therefore, the proof will be completed once we show that $\mathrm{Subtr}_1(W) \cap \mathcal{L} \ne \emptyset$. To this end we argue as follows. Let $d = \dim(W) = \lceil 17\delta^{-2} \rceil$ and $(c, w_0, \dots, w_{d-1})$ be the generating sequence of W. We set $t_0 = c$ and $t_i = c^\frown w_0(1)^\frown \dots ^\frown w_{i-1}(1)$ for every $i \in [d]$. By our assumptions, we have $\mathrm{dens}(B_{t_i}) \ge 3\delta/4$ for every $i \in \{0, \dots, d\}$. Hence, by Lemma 2.7 applied for $\varepsilon = 3\delta/4$ and $\theta = \delta/4$, there exist $i, j \in \{0, \dots, d\}$ with $i < j$ and such that $\mathrm{dens}(B_{t_i} \cap B_{t_j}) \ge \delta^2/16$. If R is the unique Carlson–Simpson line of W with $c_R = t_i$ and $w_R = w_i^\frown \dots ^\frown w_{j-1}$, then the previous discussion implies that $R \in \mathcal{L}$, as desired. □

The following lemma is the last step towards the proof of Proposition 7.1.

Lemma 7.4. *Let W be the Carlson–Simpson tree obtained in Lemma 7.3. Then W contains a Carlson–Simpson line S such that*

$$\bigcap_{t \in S} B_t \ne \emptyset.$$

Proof. As in Lemma 7.3, let $d = \dim(W) = \lceil 17\delta^{-2} \rceil$ and $(c, w_0, \dots, w_{d-1})$ be the generating sequence of W. For every $i \in \{0, \dots, d-2\}$ we set

$$t_i = c^\frown w_0(2)^\frown \dots ^\frown w_i(2) \quad \text{and} \quad s_i = t_i^\frown w_{i+1}(1)^\frown \dots ^\frown w_{d-1}(1).$$

Observe that s_i is a left successor of t_i in W. Therefore, by Lemma 7.3, setting $C_i = B_{t_i} \cap B_{s_i}$ we have $\mathrm{dens}(C_i) \ge \delta^2/16$ for every $i \in \{0, \dots, d-2\}$. Also let $s_{d-1} = c^\frown w_0(2)^\frown \dots ^\frown w_{d-1}(2)$ and $C_{d-1} = B_{s_{d-1}}$ and notice that, by Lemma 7.2, we have $\mathrm{dens}(C_{d-1}) \ge 3\delta/4 \ge \delta^2/16$. Since $d > 16/\delta^2$ there exist $0 \le i < j \le d-1$ such that $C_i \cap C_j \ne \emptyset$. We define

$$c' = t_i \quad \text{and} \quad w' = w_{i+1}^\frown \dots ^\frown w_j^\frown y$$

where $y = w_{j+1}(1)^\frown \dots ^\frown w_{d-1}(1)$ if $j < d-1$ and $y = \emptyset$ otherwise. Let S be the Carlson–Simpson line of W generated by the sequence (c', w') and observe that $S = \{t_i\} \cup \{s_i, s_j\}$. Hence, $\bigcap_{t\in S} B_t \supseteq C_i \cap C_j \neq \emptyset$. □

We are ready to proceed to the proof of Proposition 7.1.

Proof of Proposition 7.1. Let S be the Carlson–Simpson line obtained in Lemma 7.4. We select $x_0 \in X_L$ such that $x_0 \in B_t$ for every $t \in S$ and we set $R = \{\mathrm{c}_L(t, x_0) : t \in S\}$. By Lemma 5.6, R is a Carlson–Simpson line of $[2]^{<\mathbb{N}}$. Next, recall that $B = \mathrm{c}_L^{-1}(A)$. Since $(t, x_0) \in B$ for every $t \in S$, we conclude that R is contained in A, as desired. □

7.2. Estimating the numbers $\mathrm{DCS}(k, m+1, \delta)$**.** Let $k, m \in \mathbb{N}$ with $k \geq 2$ and $m \geq 1$ and assume that for every $0 < \beta \leq 1$ the number $\mathrm{DCS}(k, m, \beta)$ has been defined. This assumption, of course, implies that for every $\ell \in [m]$ and every $0 < \beta \leq 1$ the number $\mathrm{DCS}(k, \ell, \beta)$ has been defined. Therefore, for every $\ell \in [m]$ and every $0 < \delta \leq 1$ we may set

$$\Lambda(k, \ell, \delta) = \lceil \delta^{-1} \mathrm{DCS}(k, \ell, \delta) \rceil, \tag{7.5}$$

$$\Theta(k, \ell, \delta) = \frac{2\delta}{|\mathrm{Subtr}_\ell([k]^{<\Lambda(k,\ell,\delta)})|}. \tag{7.6}$$

Moreover, let

$$\Lambda_0 = \Lambda_0(k, \delta) = \Lambda(k, 1, \delta^2/16) \quad \text{and} \quad \Theta_0 = \Theta_0(k, \delta) = \Theta(k, 1, \delta^2/16) \tag{7.7}$$

and define $h_\delta : \mathbb{N} \to \mathbb{N}$ by

$$h_\delta(n) = \Lambda_0 + \lceil 2\Theta_0^{-1} n \rceil. \tag{7.8}$$

It is, of course, clear that for the definition of Λ_0, Θ_0 and h_δ we only need to have the number $\mathrm{DCS}(k, 1, \delta^2/16)$ at our disposal.

The main result of this section is the following dichotomy.

Proposition 7.5. *Let $k \in \mathbb{N}$ with $k \geq 2$ and assume that for every $0 < \beta \leq 1$ the number $\mathrm{DCS}(k, 1, \beta)$ has been defined. Let $0 < \delta \leq 1$ and define Λ_0 and Θ_0 as in (7.7). Also let L be a nonempty finite subset of $\mathbb{N}$ and $A \subseteq [k]^{<\mathbb{N}}$ such that $\mathrm{dens}_{[k]^\ell}(A) \geq \delta$ for every $\ell \in L$. Finally, let $n \in \mathbb{N}$ with $n \geq 1$ and assume that $|L| \geq h_\delta(n)$ where h_δ is as in (7.8). Define L_0 to be the set of the first Λ_0 elements of L. Then either*

(i) *there exist a subset L' of $L \setminus L_0$ with $|L'| \geq n$ and a word $t_0 \in [k]^{\ell_0}$ for some $\ell_0 \in L_0$ such that*
$$\mathrm{dens}_{[k]^{\ell-\ell_0}}(\{s \in [k]^{<\mathbb{N}} : t_0^\frown s \in A\}) \geq \delta + \delta^2/8$$
for every $\ell \in L'$, or

(ii) *there exist a subset L'' of $L \setminus L_0$ with $|L''| \geq n$ and a Carlson–Simpson line V of $[k]^{<\mathbb{N}}$ contained in A with $L(V) \subseteq L_0$ and such that, setting ℓ_1 to be the unique integer with $V(1) \subseteq [k]^{\ell_1}$, for every $\ell \in L''$ we have*
$$\mathrm{dens}_{[k]^{\ell-\ell_1}}\big(\{s \in [k]^{<\mathbb{N}} : t^\frown s \in A \text{ for every } t \in V(1)\}\big) \geq \Theta_0/2.$$

Proposition 7.5 can be used to estimate the numbers $\mathrm{DCS}(k, m+1, \delta)$ via a standard iteration. In particular, we have the following corollary.

Corollary 7.6. *Let $k, m \in \mathbb{N}$ with $k \geq 2$ and $m \geq 1$ and assume that for every $0 < \beta \leq 1$ the number $\mathrm{DCS}(k, m, \beta)$ has been defined. Then, for every $0 < \delta \leq 1$,*

$$\mathrm{DCS}(k, m+1, \delta) \leq h_\delta^{(\lceil 8\delta^{-2} \rceil)}(\mathrm{DCS}(k, m, \Theta_0/2)).$$

Proof. We fix $0 < \delta \leq 1$. For notational convenience we set $N_0 = \mathrm{DCS}(k, m, \Theta_0/2)$. Let L be an arbitrary finite subset of $\mathbb{N}$ with $|L| \geq h_\delta^{(\lceil 8\delta^{-2} \rceil)}(N_0)$ and A be a subset of $[k]^{<\mathbb{N}}$ such that $\mathrm{dens}_{[k]^\ell}(A) \geq \delta$ for every $\ell \in L$. By our assumptions on the size of L and repeated application of Proposition 7.5, it is possible to find a subset L'' of L with $|L''| \geq N_0$ and a Carlson–Simpson line V of $[k]^{<\mathbb{N}}$ contained in A such that, setting ℓ_1 to be the unique integer with $V(1) \subseteq [k]^{\ell_1}$, we have $\ell_1 < \min(L'')$ and

$$\mathrm{dens}_{[k]^{\ell - \ell_1}}\big(\{s \in [k]^{<\mathbb{N}} : t^\frown s \in A \text{ for every } t \in V(1)\}\big) \geq \Theta_0/2 \tag{7.9}$$

for every $\ell \in L''$. By the choice of N_0 and (7.9), there exists an m-dimensional Carlson–Simpson tree U of $[k]^{<\mathbb{N}}$ such that

$$U \subseteq \{s \in [k]^{<\mathbb{N}} : t^\frown s \in A \text{ for every } t \in V(1)\}.$$

Therefore, setting

$$S = V(0) \cup \bigcup_{t \in V(1)} \{t^\frown u : u \in U\},$$

we see that S is a Carlson–Simpson tree of $[k]^{<\mathbb{N}}$ of dimension $m+1$ which is contained in A. The proof is thus complete. □

For the proof of Proposition 7.5 we need some preparations. Let L be a finite subset of $\mathbb{N}$ with $|L| \geq 2$ and consider the convolution operation $\mathrm{c}_L : [k]^{<|L|} \times X_L \to [k]^{<\mathbb{N}}$ associated to L. For every $x \in X_L$ let $R_x = \{\mathrm{c}_L(t, x) : t \in [k]^{<|L|}\}$ and recall that, by Lemma 5.6, the set R_x is a Carlson–Simpson tree of $[k]^{<\mathbb{N}}$ of dimension $|L| - 1$. Therefore, we may consider the Furstenberg–Weiss measure $\mathrm{d}_{\mathrm{FW}}^{R_x}$ associated to R_x defined in §2.7. On the other hand, we may also consider the generalized Furstenberg–Weiss measure d_L associated to L. The following lemma relates these classes of measures.

Lemma 7.7. *Let $k \in \mathbb{N}$ with $k \geq 2$ and let L be a finite subset of $\mathbb{N}$ with $|L| \geq 2$. Then for every subset A of $[k]^{<\mathbb{N}}$ we have*

$$\mathbb{E}_{x \in X_L}\, \mathrm{d}_{\mathrm{FW}}^{R_x}(A) = \mathrm{d}_L(A). \tag{7.10}$$

In particular, there exists a Carlson–Simpson tree W of $[k]^{<\mathbb{N}}$ of dimension $|L| - 1$ with $L(W) = L$ and such that $\mathrm{d}_{\mathrm{FW}}^W(A) \geq \mathrm{d}_L(A)$.

Lemma 7.7 follows from Lemma 5.8. More precisely, (7.10) follows from (5.12) by averaging over all $i \in \{0, \dots, |L| - 1\}$.

The next fact is straightforward.

Fact 7.8. *Let $k, m \in \mathbb{N}$ with $k \geq 2$ and $m \geq 1$. Also let $0 < \eta \leq 1/2$ and assume that the number $\mathrm{DCS}(k, m, \eta)$ has been defined. Finally, let W be a Carlson–Simpson tree of $[k]^{<\mathbb{N}}$ with $\dim(W) \geq \Lambda(k, m, \eta) - 1$ where $\Lambda(k, m, \eta)$ is as in (7.5). Then every subset B of $[k]^{<\mathbb{N}}$ with $\mathrm{d}_{\mathrm{FW}}^W(B) \geq 2\eta$ contains a Carlson–Simpson tree of dimension m.*

The final ingredient of the proof of Proposition 7.5 is the following lemma.

Lemma 7.9. *Let $k, m \in \mathbb{N}$ with $k \geq 2$ and $m \geq 1$ and assume that for every $0 < \beta \leq 1$ the number* $\mathrm{DCS}(k, m, \beta)$ *has been defined. Let $0 < \rho, \gamma \leq 1$ and L be a finite subset of $\mathbb{N}$ with $|L| \geq \Lambda(k, m, \rho\gamma/4)$ where $\Lambda(k, m, \rho\gamma/4)$ is as in (7.5). Also let $B \subseteq [k]^{<\mathbb{N}}$ with $\mathrm{d}_L(B) \geq \rho$. If $\{A_t : t \in B\}$ is a family of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_t) \geq \gamma$ for every $t \in B$, then there exists an m-dimensional Carlson–Simpson tree V of $[k]^{<\mathbb{N}}$ which is contained in B and such that*

$$\mu\Big(\bigcap_{t\in V} A_t\Big) \geq \Theta(k, m, \rho\gamma/4)$$

where $\Theta(k, m, \rho\gamma/4)$ is as in (7.6).

Proof. We set $\eta = \rho\gamma/4$ and $\Lambda = \Lambda(k, m, \eta)$. Notice that, by passing to an appropriate subset of L if necessary, we may assume that $|L| = \Lambda$. By Lemma 7.7, there exists a Carlson–Simpson tree W of $[k]^{<\mathbb{N}}$ of dimension $\Lambda - 1$ with $L(W) = L$ and such that $\mathrm{d}^W_{\mathrm{FW}}(B) \geq \mathrm{d}_L(B) \geq \rho$. For every $\omega \in \Omega$ let $B_\omega = \{t \in B \cap W : \omega \in A_t\}$ and set

$$Y = \{\omega \in \Omega : \mathrm{d}^W_{\mathrm{FW}}(B_\omega) \geq \rho\gamma/2\}.$$

Since $\mathrm{d}^W_{\mathrm{FW}}(B) \geq \rho$ and $\mu(A_t) \geq \gamma$ for every $t \in B$, we see that $\mu(Y) \geq \rho\gamma/2$. By Fact 7.8, for every $\omega \in Y$ there exists an m-dimensional Carlson–Simpson tree V_ω of $[k]^{<\mathbb{N}}$ such that $V_\omega \subseteq B_\omega$. Hence, there exist $V \in \mathrm{Subtr}_m(W)$ and $G \in \Sigma$ with $V_\omega = V$ for every $\omega \in G$ and such that

$$\mu(G) \geq \frac{\mu(Y)}{|\mathrm{Subtr}_m(W)|} \geq \frac{2\eta}{|\mathrm{Subtr}_m([k]^{<\Lambda})|} \overset{(7.6)}{=} \Theta(k, m, \eta).$$

It is easy to see that V satisfies the conclusion of the lemma. □

We are now ready to proceed to the proof of Proposition 7.5.

Proof of Proposition 7.5. Let $M = L \setminus L_0$ and set $M_\ell = \{m - \ell : m \in M\}$ for every $\ell \in L_0$. Moreover, for every $\ell \in L_0$ and every $t \in [k]^\ell$ let

$$A_t = \{s \in [k]^{<\mathbb{N}} : t^\frown s \in A\}. \tag{7.11}$$

By (7.6) and (7.7), we see that $\Theta_0 \leq \delta^2/8$. Hence, for every $\ell \in L_0$ we have

$$|M_\ell| = |M| \geq 2\Theta_0^{-1} n \geq 8\delta^{-2} n. \tag{7.12}$$

Also observe that for every $\ell \in L_0$ we have

$$\mathbb{E}_{t\in[k]^\ell}\, \mathrm{d}_{M_\ell}(A_t) = \mathrm{d}_M(A). \tag{7.13}$$

Next recall that $\mathrm{dens}_{[k]^\ell}(A) \geq \delta$ for every $\ell \in L$. Therefore,

$$\mathrm{d}_{L_0}(A) \geq \delta \quad \text{and} \quad \mathrm{d}_M(A) \geq \delta. \tag{7.14}$$

We consider the following cases.

Case 1: There exist $\ell_0 \in L_0$ and $t_0 \in [k]^{\ell_0}$ such that $\mathrm{d}_{M_{\ell_0}}(A_{t_0}) \geq \delta + \delta^2/4$. In this case

$$|\{m \in M_{\ell_0} : \mathrm{dens}_{[k]^m}(A_{t_0}) \geq \delta + \delta^2/8\}| \geq (\delta^2/8)|M_{\ell_0}| \overset{(7.12)}{\geq} n. \tag{7.15}$$

We set $L' = \{m \in M : \mathrm{dens}_{[k]^{m-\ell_0}}(A_{t_0}) \geq \delta + \delta^2/8\}$. By (7.15), we see that with this choice the first alternative of Proposition 7.5 holds true.

Case 2: For every $\ell \in L_0$ and every $t \in [k]^\ell$ we have $\mathrm{d}_{M_\ell}(A_t) < \delta + \delta^2/4$. Combining (7.13), (7.14) and taking into account our assumptions, in this case we see that for every $\ell \in L_0$ we have

$$|\{t \in [k]^\ell : \mathrm{d}_{M_\ell}(A_t) \geq \delta/2\}| \geq (1 - \delta/2)k^\ell. \tag{7.16}$$

We set

$$B = \bigcup_{\ell \in L_0} \{t \in A \cap [k]^\ell : \mathrm{d}_{M_\ell}(A_t) \geq \delta/2\}. \tag{7.17}$$

By (7.14) and (7.16), we get $\mathrm{d}_{L_0}(B) \geq \delta/2$. Let

$$(\Omega, \mu) = \prod_{\ell \in L_0} ([k]^{<\mathbb{N}}, \mathrm{d}_{M_\ell})$$

be the product of the discrete probability spaces $([k]^{<\mathbb{N}}, \mathrm{d}_{M_\ell})$. For every $t \in B$ we define a measurable event $\widetilde{A}_t$ of Ω as follows:

$$\widetilde{A}_t = \prod_{\ell \in L_0} X_t^\ell \tag{7.18}$$

where $X_t^\ell = A_t$ if $\ell = |t|$ and $X_t^\ell = [k]^{<\mathbb{N}}$ otherwise. Notice that for every $\ell \in L_0$ and every $t \in B \cap [k]^\ell$ we have

$$\mu(\widetilde{A}_t) = \mathrm{d}_{M_\ell}(A_t) \overset{(7.17)}{\geq} \delta/2.$$

Recall that $|L_0| = \Lambda_0 \overset{(7.7)}{=} \Lambda(k, 1, \delta^2/16)$. Since $\mathrm{d}_{L_0}(B) \geq \delta/2$, by Lemma 7.9 applied for $\rho = \gamma = \delta/2$, there exists a Carlson–Simpson line V of $[k]^{<\mathbb{N}}$ which is contained in B and such that

$$\mu\Big(\bigcap_{t \in V} \widetilde{A}_t\Big) \geq \Theta(k, 1, \delta^2/16) \overset{(7.7)}{=} \Theta_0. \tag{7.19}$$

Notice, in particular, that the level set $L(V)$ of V is contained in L_0. Let ℓ_1 be the unique integer with $V(1) \subseteq [k]^{\ell_1}$ and observe that $\ell_1 \in L_0$. By the definition of the events $\{\widetilde{A}_t : t \in B\}$ in (7.18) and by (7.19), we get

$$\mathrm{d}_{M_{\ell_1}}\Big(\bigcap_{t \in V(1)} A_t\Big) = \mu\Big(\bigcap_{t \in V(1)} \widetilde{A}_t\Big) \geq \Theta_0. \tag{7.20}$$

Let

$$M'_{\ell_1} = \Big\{m \in M_{\ell_1} : \mathrm{dens}_{[k]^m}\Big(\bigcap_{t \in V(1)} A_t\Big) \geq \Theta_0/2\Big\}. \tag{7.21}$$

By (7.20), we have

$$|M'_{\ell_1}| \geq (\Theta_0/2)|M_{\ell_1}| \overset{(7.12)}{\geq} n.$$

We set

$$L'' = \{\ell_1 + m : m \in M'_{\ell_1}\}.$$

It is clear that $|L''| \geq n$. Moreover, L'' is contained in $L \setminus L_0$ and so $\ell_1 < \min(L'')$. Finally, notice that for every $\ell \in L''$ we have $\ell - \ell_1 \in M'_{\ell_1}$. Therefore, by (7.11) and (7.21), we conclude that

$$\mathrm{dens}_{[k]^{\ell-\ell_1}}\big(\{s \in [k]^{<\mathbb{N}} : t^\frown s \in A \text{ for every } t \in V(1)\}\big) \geq \Theta_0/2.$$

That is, the second alternative of Proposition 7.5 is satisfied. The above cases are exhaustive and so the proof is complete. □

7.3. Consequences. Let $k, m \in \mathbb{N}$ with $k \geq 2$ and $m \geq 1$ and assume that for every $0 < \beta \leq 1$ the number $\mathrm{DCS}(k, m, \beta)$ has been defined. For every $0 < \gamma \leq 1$ we set

$$\theta(k, m, \gamma) = \Theta(k, m, \gamma/4) \tag{7.22}$$

where $\Theta(k, m, \gamma/4)$ is as in (7.6). Recall that for every Carlson–Simpson tree W of $[k]^{<\mathbb{N}}$ and every $k' \in \{2, \dots, k\}$ we denote by $W \upharpoonright k'$ the k'-restriction of W defined in (2.2). We have the following corollary.

Corollary 7.10. *Let $k, m \in \mathbb{N}$ with $k \geq 2$ and $m \geq 1$ and assume that for every $0 < \beta \leq 1$ the number $\mathrm{DCS}(k, m, \beta)$ has been defined. Let $0 < \gamma \leq 1$ and $d \in \mathbb{N}$ with $d \geq \Lambda(k, m, \gamma/4) - 1$ where $\Lambda(k, m, \gamma/4)$ is as in (7.5). Also let V be a Carlson–Simpson tree of $[k+1]^{<\mathbb{N}}$ with*

$$\dim(V) \geq \mathrm{CS}(k+1, d, m, 2). \tag{7.23}$$

If $\{A_t : t \in V\}$ is a family of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_t) \geq \gamma$ for every $t \in V$, then there exists $W \in \mathrm{Subtr}_d(V)$ such that for every U in $\mathrm{Subtr}_m(W)$ we have

$$\mu\Big(\bigcap_{v \in U \upharpoonright k} A_v\Big) \geq \theta(k, m, \gamma)$$

where $\theta(k, m, \gamma)$ is as in (7.22).

Proof. We set $\theta = \theta(k, m, \gamma)$ and we define

$$\mathcal{U} = \Big\{U \in \mathrm{Subtr}_m(V) : \mu\Big(\bigcap_{v \in U \upharpoonright k} A_v\Big) \geq \theta\Big\}.$$

By (7.23) and Theorem 4.1, we can select $W \in \mathrm{Subtr}_d(V)$ such that either $\mathrm{Subtr}_m(W) \subseteq \mathcal{U}$ or $\mathrm{Subtr}_m(W) \cap \mathcal{U} = \emptyset$. Therefore, it is enough to show that $\mathrm{Subtr}_m(W) \cap \mathcal{U} \neq \emptyset$.

To this end we argue as follows. Let $\mathrm{I}_W : [k+1]^{<d+1} \to W$ be the canonical isomorphism associated to W and for every $t \in [k]^{<d+1}$ set $A'_t = A_{\mathrm{I}_W(t)}$. By Lemma 7.9, there exists an m-dimensional Carlson–Simpson subtree R of $[k]^{<d+1}$ such that

$$\mu\Big(\bigcap_{t \in V} A'_t\Big) \geq \theta. \tag{7.24}$$

Let S be the unique element of $\mathrm{Subtr}_m(W)$ such that $S \upharpoonright k = \mathrm{I}_W(R)$. Then, by (7.24), we conclude that $S \in \mathcal{U}$, and the proof is complete. □

The final result of this section is the following corollary.

Corollary 7.11. *Let $k, m \in \mathbb{N}$ with $k \geq 2$ and $m \geq 1$ and assume that for every $0 < \beta \leq 1$ the number* $\mathrm{DCS}(k, m, \beta)$ *has been defined. Let $0 < \gamma \leq 1$ and $d \in \mathbb{N}$ with $d \geq \Lambda(k, m, \gamma/4) - 1$ where $\Lambda(k, m, \gamma/4)$ is as in* (7.5). *Also let V be a Carlson–Simpson tree of $[k+1]^{<\mathbb{N}}$ with*

$$\dim(V) \geq \mathrm{CS}(k+1, d, m, 2).$$

Finally let $\{A_t : t \in V\}$ be a family of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_t) \geq \gamma$ for every $t \in V$. Assume, in addition, that there exists $r \in [k]$ such that $A_t = A_{t'}$ for every $t, t' \in V$ which are $(r, k+1)$-equivalent (see §2.6*). Then there exists $W \in \mathrm{Subtr}_d(V)$ such that for every $U \in \mathrm{Subtr}_m(W)$ we have*

$$\mu\Big(\bigcap_{v \in U} A_v\Big) \geq \theta(k, m, \gamma)$$

where $\theta(k, m, \gamma)$ is as in (7.22).

Proof. Since $A_t = A_{t'}$ for every $t, t' \in V$ which are $(r, k+1)$-equivalent, for every $\ell \in [\dim(V)]$ and every $R \in \mathrm{Subtr}_\ell(V)$ we have $\bigcap_{t \in R} A_t = \bigcap_{t \in R \upharpoonright k} A_t$. Now the result follows from Corollary 7.10. □

8. A probabilistic version of Theorem B

8.1. Overview. In this subsection we give an outline of the proof of Theorem B. Very briefly, and oversimplifying dramatically, the proof proceeds by induction on k and is based on a density increment strategy.

The first step is given in Corollary 7.6. Indeed, by Corollary 7.6, the proof of Theorem B reduces to the task of estimating the numbers $\mathrm{DCS}(k, 1, \delta)$. To achieve this goal we follow an inductive scheme that can be described as follows:

$$\mathrm{DCS}(k, m, \beta) \text{ for every } m \text{ and } \beta \;\Rightarrow\; \mathrm{DCS}(k+1, 1, \delta). \tag{8.1}$$

Precisely, in order to estimate the numbers $\mathrm{DCS}(k+1, 1, \delta)$ we need to have at our disposal the numbers $\mathrm{DCS}(k, m, \beta)$ for every integer $m \geq 1$ and every $0 < \beta \leq 1$. The base case—that is, the estimation of the numbers $\mathrm{DCS}(2, 1, \delta)$—is, of course, the content of Proposition 7.1.

At this point it is useful to recall the philosophy of the density increment method. One starts with a subset A of a "structured" set $\mathcal{S}$ of density δ and assumes that A does not contain a subset of a certain kind. The goal is then to find a sufficiently large "substructure" $\mathcal{S}'$ of $\mathcal{S}$ such that the density of A inside $\mathcal{S}'$ is at least $\delta + \gamma$, where γ is a positive constant that depends only on δ. Usually this task is rather difficult to achieve at once, and so, one first tries to increase the density of A inside a relatively "simple" subset of $\mathcal{S}$. We refer to the essay [18] of W. T. Gowers for a thorough exposition of this method.

The proof of the inductive scheme described in (8.1) follows the strategy just mentioned above. Specifically, fix the parameters k and δ and let A be a subset of $[k+1]^{<\mathbb{N}}$ not containing a Carlson–Simpson line such that $\mathrm{dens}_{[k+1]^n}(A) \geq \delta$ for sufficiently many

$n \in \mathbb{N}$. What we find is a Carlson–Simpson tree W of $[k+1]^{<\mathbb{N}}$ such that the density of A has been significantly increased in sufficiently many levels of W. This is done in two steps. Firstly we show that there exists a Carlson–Simpson tree V of $[k+1]^{<\mathbb{N}}$ and a subset D of V which is the intersection of relatively few insensitive sets and correlates with the set A more than expected in many levels of V (it is useful to view D as a "simple" subset of V). This is the content of Corollary 8.6 below. In the second step we use this information to achieve the density increment. We will not comment at this point on the second step, since we will do so in §9.1; here we simply mention that the statement of main interest is Corollary 9.15.

We will, however, discuss in detail the proof of the first step which is analogous to the first part of the polymath proof of the density Hales–Jewett Theorem. In fact, this is more than an analogy since we are using a beautiful argument from the polymath proof (see [28, §7.2]) to reduce the proof of Corollary 8.6—the main result of this step—to a "probabilistic" version of Theorem B. The analogy, however, with the polymath proof breaks down at this point and the main bulk of the argument is quite different.

The aforementioned "probabilistic" version of Theorem B refers to the question whether a dense subset of $[k+1]^{<\mathbb{N}}$ not only will contain a Carlson–Simpson line but, actually, a nontrivial portion of them. Results of this type figure prominently in Ramsey theory and have found significant applications (see, e.g., [7, 32] and the references therein). A classical result in this direction is the "probabilistic" version of Szemerédi's Theorem, essentially due to P. Varnavides [39], asserting that for every integer $k \geq 3$ and every $0 < \delta \leq 1$ there exists a constant $c(k, \delta) > 0$ such that every subset A of $[n]$ with $|A| \geq \delta n$ contains at least $c(k, \delta)n^2$ arithmetic progressions of length k, as long as n is sufficiently large.

However, some density results do not admit a "probabilistic" version of the form stated above. The most well-known example is the density Hales–Jewett Theorem. Indeed, if n is large enough, then one can find a highly dense subset of, say, $[2]^n$ containing just a tiny portion of combinatorial lines (see [28, §3.1]). This example can modified, in a straightforward way, to show that Theorem B also fails to admit a naive "probabilistic" version.

This phenomenon appears to be quite discouraging, but it can be bypassed. So far there has been only one method in the literature dealing with this problem. It was introduced by the participants of the polymath project and was based on the technique of changing the measure. Part of the novelty of the present paper is the development of a new method which not only is conceptually easy to grasp but also appears to be quite robust (the clearest sign for this is that it can be combined with the arguments in [28] to give a very simple proof [11] of the density Hales–Jewett Theorem). The idea is to avoid the pathological behavior by passing to an appropriate "substructure". This is done by applying the following three basic steps. We will describe them in an abstract setting since we feel that no clarity will be gained by restricting our discussion to the specifics of Theorem B.

Step 1. By an application of Szemerédi's regularity method [37], we show that a given dense set A of our "structured" set $\mathcal{S}$ is sufficiently pseudorandom. This enables us to model the set A as a family of measurable events $\{B_t : t \in \mathcal{R}\}$ in a probability space (Ω, Σ, μ) indexed by a Ramsey space $\mathcal{R}$ closely related, of course, to $\mathcal{S}$. The measure of the events is controlled by the density of A.

Step 2. We apply coloring arguments and our basic density result to show that there exists a "substructure" $\mathcal{R}'$ of $\mathcal{R}$ such that the events in the subfamily $\{B_t : t \in \mathcal{R}'\}$ are highly correlated. The reasoning can be traced to an old paper of P. Erdős and A. Hajnal [12]. Also we notice that it is precisely in this step that we need to pass to a "substructure". As can be seen from the examples mentioned above, this is a necessity rather than a coincidence.

Step 3. We use a double counting argument to locate a "substructure" $\mathcal{S}'$ of $\mathcal{S}$ such that the set A contains a nontrivial portion of subsets of $\mathcal{S}'$ of the desired kind (combinatorial lines, Carlson–Simpson lines, etc.).

Some final comments on the computational effectiveness of the method. In all cases of interest known to the authors, the tools used in the three steps described above have primitive recursive (and fairly reasonable) bounds. However, the argument yields very poor lower bounds for the correlation of the events $\{B_t : t \in \mathcal{R}'\}$ in the second step. These lower bounds are partly responsible for the Ackermannian behavior of the numbers $\mathrm{DCS}(k, m, \delta)$.

8.2. The main dichotomy. Let $k, m \in \mathbb{N}$ with $k \geq 2$ and $m \geq 1$ and assume that for every $0 < \beta \leq 1$ the number $\mathrm{DCS}(k, m, \beta)$ has been defined. Hence, for every $0 < \delta \leq 1$ we may set

$$\vartheta = \vartheta(k, m, \delta) = \Theta(k, m, \delta/8) \quad \text{and} \quad \eta = \eta(k, m, \delta) = \frac{\delta\vartheta}{30k} \tag{8.2}$$

where $\Theta(k, m, \delta/8)$ is as in (7.6). Moreover let

$$\Lambda' = \Lambda(k, m, \delta/8) \overset{(7.5)}{=} \lceil 8\delta^{-1}\mathrm{DCS}(k, m, \delta/8)\rceil \tag{8.3}$$

and for every $n \in \mathbb{N}$ set

$$\ell(n, m) = \mathrm{CS}(k + 1, n + \Lambda', m, 2) + 1.$$

We define the map $G : \mathbb{N} \times \mathbb{N} \times (0, 1] \to \mathbb{N}$ by

$$G(n, m, \varepsilon) = \mathrm{Reg}(k + 1, \ell(n, m), 1, \varepsilon). \tag{8.4}$$

Also for every Carlson–Simpson tree V of $[k]^{<\mathbb{N}}$ and every $1 \leq m \leq i \leq \dim(V)$ let

$$\mathrm{Subtr}^0_m(V, i) = \{R \in \mathrm{Subtr}_m(V) : R(0) = V(0) \text{ and } R(m) \subseteq V(i)\}.$$

As pointed out in §2.5, if W is a Carlson–Simpson tree of $[k+1]^{<\mathbb{N}}$, then its k-restriction $W\restriction k$ can be identified as a Carlson–Simpson tree of $[k]^{<\mathbb{N}}$. Hence, we may also consider the set $\mathrm{Subtr}^0_m(W\restriction k, i)$ whenever W is a Carlson–Simpson tree of $[k+1]^{<\mathbb{N}}$.

We are ready to state the main result of this section.

Proposition 8.1. *Let $k, m \in \mathbb{N}$ with $k \geq 2$ and $m \geq 1$ and assume that for every $0 < \beta \leq 1$ the number $\mathrm{DCS}(k, m, \beta)$ has been defined. Let $0 < \delta \leq 1$ and define ϑ, η and Λ' as in (8.2) and (8.3) respectively. Also let $n \in \mathbb{N}$ with $n \geq 1$ and let N be a finite subset of $\mathbb{N}$ such that*

$$|N| \geq G(\lceil \eta^{-4} n \rceil, m, \eta^2/2) \tag{8.5}$$

where G is as in (8.4). If $A \subseteq [k+1]^{<\mathbb{N}}$ satisfies $|A \cap [k+1]^l| \geq \delta(k+1)^l$ for every $l \in N$, then there exist a Carlson–Simpson tree W of $[k+1]^{<\mathbb{N}}$ with $\dim(W) = \lceil \eta^{-4} n \rceil + \Lambda'$ and $I \subseteq \{m, \dots, \dim(W)\}$ with $|I| \geq n$ such that either

(i) *for every $i \in I$ we have $\mathrm{dens}_{W(i)}(A) \geq \delta + \eta^2/2$, or*
(ii) *for every $i \in I$ we have $\mathrm{dens}_{W(i)}(A) \geq \delta - 2\eta$ and moreover*

$$\mathrm{dens}\big(\{V \in \mathrm{Subtr}^0_m(W \upharpoonright k, i) : V \subseteq A\}\big) \geq \vartheta/2.$$

For the proof of Proposition 8.1 we need to do some preparatory work. We start with the following lemma.

Lemma 8.2. *Let $k, m, \delta, \vartheta, \eta, \Lambda', n$ and N be as in Proposition 8.1. If A is a subset of $[k+1]^{<\mathbb{N}}$ satisfying $|A \cap [k+1]^l| \geq \delta(k+1)^l$ for every $l \in N$, then there exist a subset L of N with $|L| = \mathrm{CS}(k+1, \lceil \eta^{-4} n \rceil + \Lambda', m, 2) + 1$ and a Carlson–Simpson subtree W of $[k+1]^{<|L|}$ of dimension $\lceil \eta^{-4} n \rceil + \Lambda'$ such that, setting $\mathrm{c}_L : [k+1]^{<|L|} \times X_L \to [k+1]^{<\mathbb{N}}$ to be the convolution operation associated to L and $B = \mathrm{c}_L^{-1}(A)$, the following properties hold:*

(i) *For every $t \in [k+1]^{<|L|}$ we have $\mathrm{dens}(B_t) \geq \delta - \eta^2/2$ where B_t is the section of B at t.*
(ii) *For every $U \in \mathrm{Subtr}_m(W \upharpoonright k)$ we have $\mathrm{dens}(\bigcap_{t \in U} B_t) \geq \vartheta$.*

Proof. By (8.5) and the definition of the function G in (8.4), we may apply Lemma 3.2 to get $L \subseteq N$ with $|L| = \mathrm{CS}(k+1, \lceil \eta^{-4} n \rceil + \Lambda', m, 2) + 1$ and such that the family $\mathcal{F} := \{A\}$ is $(\eta^2/2, L)$-regular. Using this information and arguing as in the proof of Lemma 7.2 we see that the first part of the lemma is satisfied.

For (ii), set $V = [k+1]^{<|L|}$. Also let $\gamma = \delta/2$ and $d = \lceil \eta^{-4} n \rceil + \Lambda'$. Notice that $d \geq \Lambda' - 1 = \Lambda(k, m, \gamma/4) - 1$. Moreover, by (i), we have

$$\dim(V) = |L| - 1 = \mathrm{CS}(k+1, d, m, 2)$$

and $\mathrm{dens}(B_t) \geq \delta - \eta^2/2 \geq \gamma$ for every $t \in [k+1]^{<|L|}$. Hence, by Corollary 7.10, we get a d-dimensional Carlson–Simpson subtree W of $[k+1]^{<|L|}$ such that

$$\mathrm{dens}\Big(\bigcap_{t \in U \upharpoonright k} B_t\Big) \geq \theta(k, m, \gamma)$$

for every $U \in \mathrm{Subtr}_m(W)$. Since $\mathrm{Subtr}_m(W \upharpoonright k) \subseteq \{U \upharpoonright k : U \in \mathrm{Subtr}_m(W)\}$ and $\theta(k, m, \gamma) = \Theta(k, m, \gamma/4) = \vartheta$ the result follows. □

To state the next result towards the proof Proposition 8.1 we need to recall some notation introduced in §5. Let L be a nonempty finite subset of $\mathbb{N}$ and consider the convolution operation $\mathrm{c}_L : [k+1]^{<|L|} \times X_L \to [k+1]^{<\mathbb{N}}$ associated to L. As in (5.8), for every Carlson–Simpson subtree W of $[k+1]^{<|L|}$ and every $x \in X_L$ we set

$$W_x = \{\mathrm{c}_L(w,x) : w \in W\}. \tag{8.6}$$

Recall that, by Lemma 5.6, W_x is a Carlson–Simpson tree of $[k+1]^{<\mathbb{N}}$ with $\dim(W_x) = \dim(W)$. We have the following lemma.

Lemma 8.3. *Let $k \in \mathbb{N}$ with $k \geq 2$, L be a nonempty finite subset of $\mathbb{N}$ and consider the convolution operation $\mathrm{c}_L : [k+1]^{<|L|} \times X_L \to [k+1]^{<\mathbb{N}}$ associated to L. Also let W be a Carlson–Simpson subtree of $[k+1]^{<|L|}$ and $A \subseteq [k+1]^{<\mathbb{N}}$. Set $B = \mathrm{c}_L^{-1}(A)$ and for every $t \in [k+1]^{<|L|}$ let B_t be the section of B at t. Moreover, for every $x \in X_L$ let W_x be as in (8.6). Then:*

(i) *For every $i \in \{0, \dots, \dim(W)\}$ we have*

$$\mathbb{E}_{x\in X_L} \mathrm{dens}_{W_x(i)}(A) = \mathbb{E}_{t\in W(i)} \mathrm{dens}(B_t). \tag{8.7}$$

(ii) *For every $1 \leq m \leq i \leq \dim(W)$ we have*

$$\mathbb{E}_{x\in X_L} \mathrm{dens}\big(\{V \in \mathrm{Subtr}^0_m(W_x\upharpoonright k, i) : V \subseteq A\}\big) = \mathbb{E}_{U\in \mathrm{Subtr}^0_m(W\upharpoonright k,i)} \mathrm{dens}\Big(\bigcap_{t\in U} B_t\Big).$$

Proof. (i) Fix $i \in \{0, \dots, \dim(W)\}$. By Lemma 5.8, we have

$$\mathbb{E}_{x\in X_L} \mathrm{dens}_{W_x(i)}(A) = \mathrm{dens}_{\mathrm{c}_L(W(i)\times X_L)}(A). \tag{8.8}$$

Also notice that

$$\mathrm{c}_L(W(i) \times X_L) = \bigcup_{t\in W(i)} \mathrm{c}_L(\{t\} \times X_L) \overset{(5.4)}{=} \bigcup_{t\in W(i)} \Omega_t.$$

Next observe that $W(i) \subseteq [k+1]^l$ for some $l \in \{0, \dots, |L|-1\}$. Therefore, by Fact 5.2, we see that $|\Omega_t| = |\Omega_{t'}|$ for every $t, t' \in W(i)$. Hence,

$$\mathrm{dens}_{\mathrm{c}_L(W(i)\times X_L)}(A) = \mathbb{E}_{t\in W(i)} \mathrm{dens}_{\Omega_t}(A). \tag{8.9}$$

Finally, by Lemma 5.5, for every $t \in W(i)$ we have

$$\mathrm{dens}_{\Omega_t}(A) = \mathrm{dens}_{\{t\}\times X_L}(B) = \mathrm{dens}(B_t). \tag{8.10}$$

Combining (8.8), (8.9) and (8.10) we conclude that (8.7) is satisfied.

(ii) We fix $1 \leq m \leq i \leq \dim(W)$ and set

$$\mathbb{A} = \{(U, x) \in \mathrm{Subtr}^0_m(W\upharpoonright k, i) \times X_L : U_x \subseteq A\}$$

where, as in (8.6), $U_x = \{\mathrm{c}_L(u,x) : u \in U\}$. Also for every $U \in \mathrm{Subtr}^0_m(W\upharpoonright k, i)$ and every $x \in X_L$ let

$$\mathbb{A}_U = \{x \in X_L : (U,x) \in \mathbb{A}\}, \qquad \mathbb{A}_x = \{U \in \mathrm{Subtr}^0_m(W\upharpoonright k, i) : (U,x) \in \mathbb{A}\}$$

be the sections of $\mathbb{A}$ at U and x respectively. Notice that

$$x \in \mathbb{A}_U \Leftrightarrow (U,x) \in \mathbb{A} \Leftrightarrow U_x \subseteq A \Leftrightarrow x \in \bigcap_{t\in U} B_t.$$

This, of course, implies that for every $U \in \mathrm{Subtr}^0_m(W\upharpoonright k, i)$ we have

$$\mathbb{A}_U = \bigcap_{t\in U} B_t. \tag{8.11}$$

Moreover, it is easy to see that for every $x \in X_L$ the map

$$\mathrm{Subtr}^0_m(W\upharpoonright k, i) \ni U \mapsto U_x \in \mathrm{Subtr}^0_m(W_x\upharpoonright k, i)$$

is a bijection. Therefore, for every $x \in X_L$ we have

$$\begin{aligned}\mathrm{dens}\big(\{V \in \mathrm{Subtr}^0_m(W_x\upharpoonright k, i) : V \subseteq A\}\big) &= \frac{|\{V \in \mathrm{Subtr}^0_m(W_x\upharpoonright k, i) : V \subseteq A\}|}{|\mathrm{Subtr}^0_m(W_x\upharpoonright k, i)|} \\ &= \frac{|\{U \in \mathrm{Subtr}^0_m(W\upharpoonright k, i) : U_x \subseteq A\}|}{|\mathrm{Subtr}^0_m(W\upharpoonright k, i)|} = \mathrm{dens}(\mathbb{A}_x).\end{aligned}$$

Taking into account (8.11) and the above equalities we conclude that

$$\begin{aligned}\mathbb{E}_{x\in X_L}\,\mathrm{dens}\big(\{V \in \mathrm{Subtr}^0_m(W_x\upharpoonright k, i) : V \subseteq A\}\big) &= \mathbb{E}_{x\in X_L}\,\mathrm{dens}(\mathbb{A}_x) \\ = \mathbb{E}_{U\in\mathrm{Subtr}^0_m(W\upharpoonright k,i)}\,\mathrm{dens}(\mathbb{A}_U) &= \mathbb{E}_{U\in\mathrm{Subtr}^0_m(W\upharpoonright k,i)}\,\mathrm{dens}\Big(\bigcap_{t\in U} B_t\Big). \qquad \square\end{aligned}$$

We are ready to proceed to the proof of Proposition 8.1.

Proof of Proposition 8.1. We fix $A \subseteq [k+1]^{<\mathbb{N}}$ such that $|A \cap [k+1]^l| \geq \delta(k+1)^l$ for every $l \in N$. For notational convenience, we set $d = \lceil \eta^{-4} n \rceil + \Lambda'$. Let L and W be as in Lemma 8.2 when applied to the fixed set A. Notice that $\dim(W) = d$. Invoking the first parts of Lemmas 8.2 and 8.3, for every $i \in \{0, \dots, d\}$ we have

$$\mathbb{E}_{x\in X_L}\,\mathrm{dens}_{W_x(i)}(A) \geq \delta - \eta^2/2. \tag{8.12}$$

On the other hand, by the second parts of the aforementioned lemmas, we see that

$$\mathbb{E}_{x\in X_L}\,\mathrm{dens}\big(\{V \in \mathrm{Subtr}^0_m(W_x\upharpoonright k, i) : V \subseteq A\}\big) \geq \vartheta \tag{8.13}$$

for every $i \in \{m, \dots, d\}$.

Let $J = \{m, \dots, d\}$ and notice that

$$|J| = d - m + 1 \geq \lceil \eta^{-4} n \rceil. \tag{8.14}$$

Also for every $i \in J$ set

$$\begin{aligned}X_i^0 &= \{x \in X_L : \mathrm{dens}_{W_x(i)}(A) \geq \delta + \eta^2/2\}, \\ X_i^1 &= \{x \in X_L : \mathrm{dens}_{W_x(i)}(A) \geq \delta - 2\eta\}, \\ X_i^2 &= \big\{x \in X_L : \mathrm{dens}\big(\{V \in \mathrm{Subtr}^0_m(W_x\upharpoonright k, i) : V \subseteq A\}\big) \geq \vartheta/2\big\}.\end{aligned}$$

Finally let $J_0 = \{i \in J : \mathrm{dens}(X_i^0) \geq \eta^3\}$. We distinguish the following cases.

Case 1: $|J_0| \geq |J|/2$. By Lemma 2.5, there exists $x_0 \in X_L$ such that

$$|\{i \in J_0 : x_0 \in X_i^0\}| \geq \frac{\eta^3 |J_0|}{2} \geq \frac{\eta^3 |J|}{4} \overset{(8.14)}{\geq} \frac{\eta^3 \lceil \eta^{-4} n \rceil}{4} \geq n.$$

We set $I = \{i \in J : x_0 \in X_i^0\}$ and $W = W_{x_0}$. With these choices it is easy to see that the first part of the proposition is satisfied.

Case 2: $|J_0| < |J|/2$. In this case we set $\bar{J}_0 = J \setminus J_0$. Let $i \in \bar{J}_0$ and notice that

$$\mathrm{dens}(X_i^0) = \mathrm{dens}\big(\{x \in X_L : \mathrm{dens}_{W_x(i)}(A) \geq \delta + \eta^2/2\}\big) < \eta^3. \tag{8.15}$$

Combining (8.12) and (8.15) we see that $\mathrm{dens}(X_i^1) \geq 1 - \eta$. On the other hand, by (8.13), we have $\mathrm{dens}(X_i^2) \geq \vartheta/2$. Therefore, by the choice of η in (8.2), we conclude that $\mathrm{dens}(X_i^1 \cap X_i^2) \geq \vartheta/4$ for every $i \in \bar{J}_0$. By a second application of Lemma 2.5, we find $x_1 \in X_L$ such that

$$|\{i \in \bar{J}_0 : x_1 \in X_i^1 \cap X_i^2\}| \geq \frac{\vartheta|\bar{J}_0|}{8} \geq \frac{\vartheta|J|}{16} \overset{(8.14)}{\geq} \frac{\vartheta\lceil\eta^{-4}n\rceil}{16} \overset{(8.2)}{\geq} n.$$

We set $I = \{i \in \bar{J}_0 : x_1 \in X_i^1 \cap X_i^2\}$ and $W = W_{x_1}$ and we notice that with these choices the second part of the proposition is satisfied. The above cases are exhaustive and the proof is complete. □

8.3. Obtaining insensitive sets. Let $k \in \mathbb{N}$ with $k \geq 2$ and assume that for every $0 < \beta \leq 1$ the number $\mathrm{DCS}(k, 1, \beta)$ has been defined. For every $0 < \delta \leq 1$ let

$$\vartheta_1 = \vartheta_1(k, \delta) = \vartheta(k, 1, \delta) \quad \text{and} \quad \eta_1 = \eta_1(k, \delta) = \eta(k, 1, \delta) \tag{8.16}$$

where $\vartheta(k, 1, \delta)$ and $\eta(k, 1, \delta)$ are as in (8.2). Also define $G_1 : \mathbb{N} \times (0, 1] \to \mathbb{N}$ by

$$G_1(n, \varepsilon) = G(n, 1, \varepsilon) \tag{8.17}$$

where G is as in (8.4). We have the following lemma.

Lemma 8.4. *Let $k \in \mathbb{N}$ with $k \geq 2$ and assume that for every $0 < \beta \leq 1$ the number $\mathrm{DCS}(k, 1, \beta)$ has been defined. Let $0 < \delta \leq 1$ and define ϑ_1 and η_1 as in (8.16). Also let $n \in \mathbb{N}$ with $n \geq 1$ and N be a finite subset of $\mathbb{N}$ such that*

$$|N| \geq G_1(\lceil\eta_1^{-4}(k + 1)n\rceil, \eta_1^2/2)$$

where G_1 is as in (8.17). Finally let $A \subseteq [k+1]^{<\mathbb{N}}$ be such that $|A \cap [k+1]^l| \geq \delta(k+1)^l$ for every $l \in N$ and assume that A contains no Carlson–Simpson line of $[k + 1]^{<\mathbb{N}}$. Assume, moreover, that for every Carlson–Simpson tree W of $[k + 1]^{<\mathbb{N}}$ we have

$$\Big|\{i \in \{0, \ldots, \dim(W)\} : \mathrm{dens}_{W(i)}(A) \geq \delta + \eta_1^2/2\}\Big| < n. \tag{8.18}$$

Then there exist a Carlson–Simpson tree V of $[k + 1]^{<\mathbb{N}}$, a subset C of V and a subset J of $\{0, \ldots, \dim(V)\}$ with the following properties:

(i) $|J| \geq n$.
(ii) $C = \bigcap_{r=1}^{k} C_r$ *where the set C_r is $(r, k + 1)$-insensitive in V for every $r \in [k]$. Moreover,* $\mathrm{dens}_{V(j)}(C) \geq \vartheta_1/2$ *for every $j \in J$.*
(iii) *The sets A and C are disjoint.*
(iv) *For every $j \in J$ we have* $\mathrm{dens}_{V(j)}(A) \geq \delta - 5k\eta_1$.

Lemma 8.4 will be reduced to Proposition 8.1. The reduction will be achieved using essentially the same arguments as in [28, §7.2].

First we need to introduce some pieces of notation. Let W be a Carlson–Simpson tree of $[k+1]^{<\mathbb{N}}$ and set $d = \dim(W)$. Consider the canonical isomorphism $\mathrm{I}_W : [k+1]^{<d+1} \to W$ associated to W (see §2.5) and for every $r \in [k+1]$ let

$$W[r] = \{\mathrm{I}_W(s) : s \in [k+1]^{<d+1} \text{ with } |s| \geq 1 \text{ and } s(0) = r\}.$$

Notice that if $\dim(W) \geq 2$, then $W[r]$ is a Carlson–Simpson subtree of W with $\dim(W[r]) = \dim(W) - 1$. On the other hand, if W is a Carlson–Simpson line, then $W[r]$ is the singleton $\{\mathrm{I}_W(r)\}$; we will in this case identify $W[r]$ with $\mathrm{I}_W(r)$. Next observe that for every Carlson–Simpson subtree U of the k-restriction $W\upharpoonright k$ of W there exists a unique Carlson–Simpson tree R of $[k+1]^{<\mathbb{N}}$ such that $R\upharpoonright k = U$. We will call this unique Carlson–Simpson tree R the *extension* of U and we will denote it by $\bar{U}$. Notice that the extension of U is a Carlson–Simpson subtree of W. Also we will need the following elementary fact.

Fact 8.5. *Let W be a Carlson–Simpson tree of $[k+1]^{<\mathbb{N}}$ with $\dim(W) \geq 2$ and set $V = W[k+1]$. If $j \in \{0, \dots, \dim(V)\}$ and $U \in \mathrm{Subtr}_1^0(W\upharpoonright k, j+1)$, then $\bar{U}[k+1] \in V(j)$. Moreover, the map*

$$\mathrm{Subtr}_1^0(W\upharpoonright k, j+1) \ni U \mapsto \bar{U}[k+1] \in V(j)$$

is a bijection.

We are ready to give the proof of Lemma 8.4.

Proof of Lemma 8.4. By our assumptions we may apply Proposition 8.1 for $m = 1$ to get a Carlson–Simpson tree W of $[k+1]^{<\mathbb{N}}$ and $I \subseteq \{1, \dots, \dim(W)\}$ with $|I| \geq (k+1)n$ and such that $\mathrm{dens}_{W(i)}(A) \geq \delta - 2\eta_1$ and

$$\mathrm{dens}(\{U \in \mathrm{Subtr}_1^0(W\upharpoonright k, i) : U \subseteq A\}) \geq \vartheta_1/2 \tag{8.19}$$

for every $i \in I$. For every $r \in [k+1]$ let $V_r = W[r]$. We set

$$V = V_{k+1}, \tag{8.20}$$

$$C = \bigcup_{i \in I} \{\bar{U}[k+1] : U \in \mathrm{Subtr}_1^0(W\upharpoonright k, i) \text{ with } U \subseteq A\}, \tag{8.21}$$

$$J = \big\{j \in \{0, \dots, \dim(V)\} : \mathrm{dens}_{V(j)}(A) \geq \delta - 5k\eta_1 \text{ and } j+1 \in I\big\}. \tag{8.22}$$

We claim that V, C and J are as desired. First we show that $|J| \geq n$. We define $J_0 = \{j \in \{0, \dots, \dim(V)\} : j+1 \in I\}$. Observe that $J \subseteq J_0$ and

$$|J_0| \geq (k+1)n. \tag{8.23}$$

Let $j \in J_0 \setminus J$ and notice that $\mathrm{dens}_{V(j)}(A) < \delta - 5k\eta_1$. On the other hand, $j+1 \in I$ and so

$$\mathbb{E}_{r \in [k+1]} \mathrm{dens}_{V_r(j)}(A) = \mathrm{dens}_{W(j+1)}(A) \geq \delta - 2\eta_1.$$

Thus, there exists $r_j \in [k]$ such that $\mathrm{dens}_{V_{r_j}(j)}(A) \geq \delta + \eta_1$. Therefore, by the classical pigeonhole principle, there exists $r_0 \in [k]$ such that

$$|\{j \in J_0 \setminus J : \mathrm{dens}_{V_{r_0}(j)}(A) \geq \delta + \eta_1\}| \geq \frac{|J_0 \setminus J|}{k} \overset{(8.23)}{\geq} n + \frac{n - |J|}{k}. \tag{8.24}$$

Moreover, by (8.18), we have

$$|\{j \in J_0 \setminus J : \mathrm{dens}_{V_{r_0}(j)}(A) \geq \delta + \eta_1\}| < n. \tag{8.25}$$

Combining (8.24) and (8.25) we conclude that $|J| \geq n$.

We continue with the proof of (ii). Let $r \in [k]$. For every $l \in \mathbb{N}$ and every $s \in [k+1]^l$ let $s^{k+1 \to r}$ be the unique element of $[k]^l$ obtained by replacing all appearances of $k+1$ in s by r. We set

$$C_r = \Big\{ \mathrm{I}_V(s) : s \in \bigcup_{i \in I} [k+1]^{i-1} \text{ and } s^{k+1 \to r} \in \mathrm{I}_{V_r}^{-1}(A) \Big\}$$

where I_V and I_{V_r} are the canonical isomorphisms associated to V and V_r respectively. It is easy to check that C_r is $(r, k+1)$-insensitive in V. Next we show that C coincides with $C_1 \cap \cdots \cap C_k$. First we notice that $C \subseteq C_1 \cap \cdots \cap C_k$. To see the other inclusion, let $t \in C_1 \cap \cdots \cap C_k$ and set $s = \mathrm{I}_V^{-1}(t)$. Let i be the unique element of I such that $s \in [k+1]^{i-1}$ and set

$$U_t = \{W(0)\} \cup \{\mathrm{I}_{V_r}(s^{k+1 \to r}) : r \in [k]\}.$$

Notice that $U_t \in \mathrm{Subtr}_1^0(W \upharpoonright k, i)$. Next observe that, by (8.19), we have $W(0) \in A$. On the other hand, we have $\mathrm{I}_{V_r}(s^{k+1 \to r}) \in A$ for every $r \in [k]$ since $t \in C_1 \cap \cdots \cap C_k$. This shows that $U_t \subseteq A$. Observing that $t = \bar{U}_t[k+1]$ we conclude that $t \in C$. Finally let $j \in J$. Notice that

$$\mathrm{dens}_{V(j)}(C) = \mathrm{dens}_{V(j)}\big(\{\bar{U}[k+1] : U \in \mathrm{Subtr}_1^0(W \upharpoonright k, j+1) \text{ with } U \subseteq A\}\big)$$

and recall that $j + 1 \in I$. Hence, by Fact 8.5, we have

$$\mathrm{dens}_{V(j)}(C) = \mathrm{dens}\big(\{U \in \mathrm{Subtr}_1^0(W \upharpoonright k, j+1) : U \subseteq A\}\big) \overset{(8.19)}{\geq} \vartheta_1/2$$

as desired.

The fact that A and C are disjoint follows from our assumption that A contains no Carlson–Simpson line of $[k+1]^{<\mathbb{N}}$ and from the definition of C. Finally, (iv) is an immediate consequence of (8.22). □

8.4. Consequences. In this subsection we summarize what we have achieved in Proposition 8.1 and Lemma 8.4. We remark that the resulting statement is the first main step towards the proof of Theorem B.

Corollary 8.6. *Let $k \in \mathbb{N}$ with $k \geq 2$ and assume that for every $0 < \beta \leq 1$ the number* $\mathrm{DCS}(k, 1, \beta)$ *has been defined. Let $0 < \delta \leq 1$ and define η_1 as in (8.16). Also let $n \in \mathbb{N}$ with $n \geq 1$ and N be a finite subset of $\mathbb{N}$ such that*

$$|N| \geq G_1(\lceil \eta_1^{-4}(k+1)kn \rceil, \eta_1^2/2)$$

where G_1 is as in (8.17). Finally let $A \subseteq [k+1]^{<\mathbb{N}}$ be such that $|A \cap [k+1]^l| \geq \delta(k+1)^l$ for every $l \in N$ and assume that A contains no Carlson–Simpson line of $[k+1]^{<\mathbb{N}}$. Then there exist a Carlson–Simpson tree V of $[k+1]^{<\mathbb{N}}$, a subset D of V and a subset I of $\{0, \dots, \dim(V)\}$ with the following properties:

(i) $|I| \geq n$.
(ii) $D = \bigcap_{r=1}^{k} D_r$ *where D_r is $(r, k+1)$-insensitive in V for every $r \in [k]$.*
(iii) *For every $i \in I$ we have* $\mathrm{dens}_{V(i)}(A \cap D) \geq (\delta + \eta_1^2/2)\,\mathrm{dens}_{V(i)}(D)$ *and* $\mathrm{dens}_{V(i)}(D) \geq \eta_1^2/2$.

Proof. First assume that there exists a Carlson–Simpson tree W of $[k+1]^{<\mathbb{N}}$ such that

$$\left|\left\{i \in \{0, \dots, \dim(W)\} : \mathrm{dens}_{W(i)}(A) \geq \delta + \eta_1^2/2\right\}\right| \geq kn.$$

In this case we set $V = W$, $I = \{i \in \{0, \dots, \dim(W)\} : \mathrm{dens}_{W(i)}(A) \geq \delta + \eta_1^2/2\}$ and $D_r = V$ for every $r \in [k]$. It is clear that with these choices the result follows.

Otherwise, by Lemma 8.4, there exist a Carlson–Simpson tree V of $[k+1]^{<\mathbb{N}}$, a subset J of $\{0, \dots, \dim(V)\}$ with $|J| \geq kn$ and a set $C = C_1 \cap \dots \cap C_k$, where C_r is $(r, k+1)$-insensitive in V for every $r \in [k]$, such that

(a) $A \cap C = \emptyset$,
(b) $\mathrm{dens}_{V(j)}(C) \geq \vartheta_1/2$ for every $j \in J$, and
(c) $\mathrm{dens}_{V(j)}(A) \geq \delta - 5k\eta_1$ for every $j \in J$

where ϑ_1 and η_1 are as in (8.16). In particular, for every $j \in J$ we have

$$\frac{\mathrm{dens}_{V(j)}(A)}{\mathrm{dens}_{V(j)}(V \setminus C)} \geq \frac{\delta - 5k\eta_1}{1 - \vartheta_1/2} \geq (\delta - 5k\eta_1)(1 + \vartheta_1/2) \geq \delta + 7k\eta_1.$$

We set $Q_1 = V \setminus C_1$ and $Q_r = (V \setminus C_r) \cap C_1 \cap \dots \cap C_{r-1}$ if $r \in \{2, \dots, k\}$. Clearly the family $(Q_r)_{r=1}^k$ forms a partition of $V \setminus C$. Let $j \in J$. Applying Lemma 2.6 for $\varepsilon = k\eta_1$ we see that there exists $r_j \in [k]$ such that

$$\mathrm{dens}_{V(j)}(A \cap Q_{r_j}) \geq (\delta + 6k\eta_1)\,\mathrm{dens}_{V(j)}(Q_{r_j}), \tag{8.26}$$

$$\mathrm{dens}_{V(j)}(Q_{r_j}) \geq (\delta - 5k\eta_1)\eta_1/4. \tag{8.27}$$

Hence, there exist $r_0 \in [k]$ and a subset I of J with $|I| \geq |J|/k \geq n$ such that $r_i = r_0$ for every $i \in I$. We set $D = Q_{r_0}$. Also let $D_r = C_r$ if $r < r_0$, $D_{r_0} = V \setminus C_{r_0}$ and $D_r = V$ if $r > r_0$. Clearly D_r is $(r, k+1)$-insensitive in V for every $r \in [k]$ and $D_1 \cap \dots \cap D_k = D$. Moreover, by the choice of η_1, for every $i \in I$ we have

$$\mathrm{dens}_{V(i)}(A \cap D) \overset{(8.26)}{\geq} (\delta + 6k\eta_1)\,\mathrm{dens}_{V(i)}(D) \geq (\delta + \eta_1^2/2)\,\mathrm{dens}_{V(i)}(D),$$

$$\mathrm{dens}_{V(i)}(D) \overset{(8.27)}{\geq} (\delta - 5k\eta_1)\eta_1/4 \geq \eta_1^2/2.$$

□

9. An exhaustion procedure: achieving the density increment

9.1. Motivation. As we have seen in Corollary 8.6, if a dense subset A of $[k+1]^{<\mathbb{N}}$ fails to contain a Carlson–Simpson line, then there exist a Carlson–Simpson tree V of $[k+1]^{<\mathbb{N}}$ and a structured subset D of V (recall that D is the intersection of relatively few insensitive sets) that correlates with the set A more than expected in many levels of V. Our goal in this section is to use this information to achieve density increment for the set A. A natural strategy for doing so—initiated by M. Ajtai and E. Szemerédi [1]—is to produce an "almost tiling" of the set D, that is, to construct a collection $\mathcal{V}$ of pairwise disjoint Carlson–Simpson trees of sufficiently large dimension which are all contained in D and are such that the set $D \setminus \bigcup \mathcal{V}$ is essentially negligible. Once this is done, one then expects to be able to find a Carlson–Simpson tree W belonging to the "almost tiling" $\mathcal{V}$ such that the density of A has been significantly increased in sufficiently many levels of W. However, as is shown below, this is not possible in general.

Example 9.1. Let $m \in \mathbb{N}$ with $m \geq 1$ and $0 < \varepsilon \leq 1$. Also let $q, \ell \in \mathbb{N}$ with $q \geq \ell \geq 1$ and such that $2^{\ell}(2^{\ell}-1)2^{-q} \leq \varepsilon$. With these choices it is possible to select a family $\{x_t \in [2]^{q-\ell} : t \in [2]^{<\ell}\}$ such that for every $t, t' \in [2]^{<\ell}$ with $t \neq t'$ we have $x_t \neq x_{t'}$. For every $i \in \{0, \dots, \ell-1\}$ and every $t \in [2]^i$ we set $z_t^1 = t^\frown(1^{\ell-i})^\frown x_t$ and $z_t^2 = t^\frown(2^{\ell-i})^\frown x_t$ and we define

$$\mathcal{F} = \{z_t^j : t \in [2]^{<\ell} \text{ and } j \in [2]\}.$$

Notice that $\mathcal{F} \subseteq [2]^q$ and $\mathrm{dens}(\mathcal{F}) \leq \varepsilon$. Also observe that $\{z_t^1, z_t^2\} \cap \{z_{t'}^1, z_{t'}^2\} = \emptyset$ provided that $t \neq t'$. We set $D = [2]^{<\ell} \cup [2]^q \cup \dots \cup [2]^{q+m-1}$ and

$$A = [2]^{<\ell} \cup \{y^\frown s : y \in [2]^q \setminus \mathcal{F} \text{ and } s \in [2]^{<m}\}.$$

It is clear that D is a highly structured subset of $[2]^{<\mathbb{N}}$—it is the union of certain levels of $[2]^{<\mathbb{N}}$—and A is a subset of D of relative density at least $1-\varepsilon$. Next for every $t \in [2]^{<\ell}$ let

$$V_t = \{t\} \cup \{z_t^{j\frown} s : j \in [2] \text{ and } s \in [2]^{<m}\}.$$

Observe that $\mathcal{V} = \{V_t : t \in [2]^{<\ell}\}$ is a family of pairwise disjoint m-dimensional Carlson–Simpson trees which are all contained in D. Also notice that, no matter how large ℓ is, $\mathcal{V}$ is maximal, that is, the set $D \setminus \bigcup \mathcal{V}$ contains no Carlson–Simpson tree of dimension m. However, $V_t \cap A$ is the singleton $\{t\}$ for every $t \in [2]^{<\ell}$.

The above example shows that, in our context, the problem of achieving the density increment cannot be solved by merely producing an arbitrary "almost tiling" of the structured set D. To overcome this obstacle we devise a refined exhaustion procedure that can be roughly described as follows. At each step of the process we are given a subset D' of D and we produce a collection $\mathcal{E}$ of Carlson–Simpson trees of sufficiently large dimension which are all contained in D'. These Carlson–Simpson trees are *not* pairwise disjoint since we are not aiming at producing a tiling. Instead, what we are really interested in is whether a sufficient portion of them behaves "as expected". If this is the case, then we can easily achieve the density increment. Otherwise, using coloring arguments, we can show that for "almost every" Carlson–Simpson tree V of the collection $\mathcal{E}$, the restriction of our set A in V is quite "thin" and in a very canonical way. We then remove from D' an appropriately chosen subset of $\bigcup\mathcal{E}$ and we repeat the argument for the resulting set. The above process is shown to eventually terminate, thus completing the proof of this step.

At a technical level, in order to execute the steps described above we need to represent any subset of $[k+1]^{<\mathbb{N}}$ as a family of measurable events indexed by an appropriately chosen Carlson–Simpson tree of $[k+1]^{<\mathbb{N}}$. The philosophy is identical to that in §8.2. However, due to the recursive nature of the process, we need to work with iterated convolutions. In particular, the reader is advised to review the material in §6 before studying this section.

9.2. The main result. Let $k \in \mathbb{N}$ with $k \geq 2$ and assume that for every integer $l \geq 1$ and every $0 < \beta \leq 1$ the number $\mathrm{DCS}(k, l, \beta)$ has been defined. This assumption permits us to introduce some numerical invariants. For every integer $m \geq 1$ and every $0 < \gamma \leq 1$ we set

$$\bar{m} = \bar{m}(m, \gamma) = \left\lceil \frac{512}{\gamma^3} m \right\rceil, \tag{9.1}$$

$$M = \Lambda(k, \bar{m}, \gamma^2/32) \overset{(7.5)}{=} \left\lceil \frac{32}{\gamma^2} \mathrm{DCS}(k, \bar{m}, \gamma^2/32) \right\rceil. \tag{9.2}$$

Also let

$$\alpha = \alpha(k, m, \gamma) = \theta(k, \bar{m}, \gamma^2/8) \quad \text{and} \quad p_0 = p_0(k, m, \gamma) = \lfloor \alpha^{-1} \rfloor \tag{9.3}$$

where $\theta(k, \bar{m}, \gamma^2/8)$ is as in (7.22). Finally we define, recursively, three sequences (n^1_p), (n^2_p) and (N_p) in $\mathbb{N}$—also depending on the parameters m and γ—by $n^1_0 = n^2_0 = N_0 = 0$ and

$$\begin{cases} n^1_{p+1} = (N_p + 1)\bar{m} + N_p, \\ n^2_{p+1} = \mathrm{CS}(k+1, n^1_{p+1}, \bar{m}, \bar{m}+1), \\ N_{p+1} = \mathrm{CS}(k+1, \max\{n^2_{p+1}, M\}, \bar{m}, 2). \end{cases} \tag{9.4}$$

We are mainly interested in the sequence (N_p). The sequences (n^1_p) and (n^2_p) are auxiliary ones which will be used in the proof of the following lemma.

Lemma 9.1. *Let $k \in \mathbb{N}$ with $k \geq 2$ and assume that for every integer $l \geq 1$ and every $0 < \beta \leq 1$ the number* $\mathrm{DCS}(k, l, \beta)$ *has been defined. Let $0 < \gamma, \delta \leq 1$ and $r \in [k]$. Also let V be a Carlson–Simpson tree of $[k+1]^{<\mathbb{N}}$ and I be a nonempty subset of* $\{0, \dots, \dim(V)\}$. *Assume that we are given subsets $A, D_r, \dots, D_k$ of $[k+1]^{<\mathbb{N}}$ with the following properties:*

(a) *D_r is $(r, k+1)$-insensitive in V.*
(b) $\mathrm{dens}_{V(i)}(D_r \cap \dots \cap D_k \cap A) \geq (\delta + 2\gamma)\, \mathrm{dens}_{V(i)}(D_r \cap \dots \cap D_k)$ *and* $\mathrm{dens}_{V(i)}(D_r \cap \dots \cap D_k) \geq 2\gamma$ *for every $i \in I$.*

Finally, let $m \in \mathbb{N}$ with $m \geq 1$ and suppose that

$$|I| \geq \mathrm{Reg}(k+1, N_{p_0} + 1, 2, \gamma^2/2)$$

where p_0 and N_{p_0} are defined in (9.3) *and* (9.4) *respectively for the parameters m and γ.*

Then there exist a Carlson–Simpson subtree W of V and a subset I' of $\{0,\dots,\dim(W)\}$ of cardinality m with the following properties. If $r<k$, then

$$\mathrm{dens}_{W(i)}(D_{r+1}\cap\cdots\cap D_k\cap A)\geq(\delta+\gamma/2)\,\mathrm{dens}_{W(i)}(D_{r+1}\cap\cdots\cap D_k),$$
$$\mathrm{dens}_{W(i)}(D_{r+1}\cap\cdots\cap D_k)\geq\frac{\gamma^3}{256},$$

for every $i\in I'$. On the other hand if $r=k$, then

$$\mathrm{dens}_{W(i)}(A)\geq\delta+\gamma/2$$

for every $i\in I'$.

The proof of Lemma 9.1 will be given in §9.3. As the reader might have already guessed, Lemma 9.1 is the main result of this section and incorporates the exhaustion procedure outlined in §9.1. It will be used in §9.4 where we shall achieve the density increment.

9.3. Proof of Lemma 9.1. The first step of the proof relies on an application of the regularity lemma presented in §3. Let $B=D_{r+1}\cap\cdots\cap D_k$ if $r<k$; otherwise, let $B=V$. Identifying the Carlson–Simpson tree V with $[k+1]^{<\dim(V)+1}$ via the canonical isomorphism I_V (see §2.5), we may apply Lemma 3.2 to the family $\mathcal{F}=\{D_r\cap B, D_r\cap B\cap A\}$ to get a subset L of I of cardinality $N_{p_0}+1$ such that $\mathcal{F}$ is $(\gamma^2/2,L)$-regular. We set

$$A^0=\mathrm{c}^{-1}_{L,V}(A),\qquad B^0=\mathrm{c}^{-1}_{L,V}(B),\qquad D^0=\mathrm{c}^{-1}_{L,V}(D_r).$$

The fact that $\mathcal{F}$ is $(\gamma^2/2,L)$-regular and conditions (a) and (b) in the statement of the lemma have some consequences which are isolated in the following fact. Its proof is similar to the proof of Lemma 7.2 and is left to the reader.

Fact 9.2. *For every $t\in[k+1]^{<|L|}$ let A^0_t, B^0_t and D^0_t be the sections at t of A^0, B^0 and D^0 respectively. Then for every $t,t'\in[k+1]^{<|L|}$ the following hold:*

(i) *If t,t' are $(r,k+1)$-equivalent (see §2.6), then D^0_t and $D^0_{t'}$ coincide.*
(ii) $\mathrm{dens}_{X_L}(D^0_t\cap B^0_t\cap A^0_t)\geq(\delta+\gamma)\,\mathrm{dens}_{X_L}(D^0_t\cap B^0_t)$.
(iii) $\mathrm{dens}_{X_L}(D^0_t\cap B^0_t)\geq\gamma$.

We are ready to proceed to the main part of the proof. In particular, assuming that the lemma is not satisfied, we shall determine an integer $d\in[p_0]$ and construct

(1) a $(k+1)$-compatible pair $((L_n)_{n=0}^d,(V_n)_{n=0}^d)$ with $L_0=L$ and $V_0=V$, and
(2) for every $p\in[d]$, every $\ell\in[p]$ and every $s\in[k+1]^{<|L_p|}$ a family $\mathcal{Q}^{\ell,p}_s$ of subsets of $X_{\mathbf{L}_p}$, where $\mathbf{L}_p=(L_n)_{n=0}^p$ and $\mathbf{V}_p=(V_n)_{n=0}^p$.

The construction is done recursively so that, setting

$$A^p=\mathrm{c}^{-1}_{\mathbf{L}_p,\mathbf{V}_p}(A),\qquad B^p=\mathrm{c}^{-1}_{\mathbf{L}_p,\mathbf{V}_p}(B),\qquad D^p=\mathrm{c}^{-1}_{\mathbf{L}_p,\mathbf{V}_p}(D_r),$$

for every $p\in[d]$ the following conditions are satisfied:

(C1) The set L_p has cardinality $N_{p_0-p}+1$.
(C2) For every $\ell\in[p]$ and every $s\in[k+1]^{<|L_p|}$ the family $\mathcal{Q}^{\ell,p}_s$ consists of pairwise disjoint subsets of the section D^p_s of D^p at s.

(C3) For every $s \in [k+1]^{<|L_p|}$ the sets $\bigcup \mathcal{Q}_s^{1,p}, \dots, \bigcup \mathcal{Q}_s^{p,p}$ are pairwise disjoint.

(C4) For every $\ell \in [p]$, every pair $s, s' \in [k+1]^{<|L_p|}$ with the same length and every $Q \in \mathcal{Q}_s^{\ell,p}$ and $Q' \in \mathcal{Q}_{s'}^{\ell,p}$ we have $\mathrm{dens}_{X_{\mathbf{L}_p}}(Q) = \mathrm{dens}_{X_{\mathbf{L}_p}}(Q')$.

(C5) For every $\ell \in [p]$ and every $s \in [k+1]^{<|L_p|}$ we say that an element Q of $\mathcal{Q}_s^{\ell,p}$ is *good* provided that $\mathrm{dens}_Q(D_s^p \cap B_s^p \cap A_s^p) \geq (\delta + \gamma/2)\,\mathrm{dens}_Q(D_s^p \cap B_s^p)$ and $\mathrm{dens}_Q(D_s^p \cap B_s^p) \geq \gamma^3/256$. Then, setting

$$\mathcal{G}_s^{\ell,p} = \{Q \in \mathcal{Q}_s^{\ell,p} : Q \text{ is good}\},$$

we have

$$|\mathcal{G}_s^{\ell,p}|/|\mathcal{Q}_s^{\ell,p}| < \gamma^3/256. \tag{9.5}$$

(C6) For every $\ell \in [p]$ and every $s \in [k+1]^{<|L_p|}$ we have $\mathrm{dens}_{X_{\mathbf{L}_p}}(\bigcup \mathcal{Q}_s^{\ell,p}) \geq \alpha$ where α is as in (9.3).

(C7) For every $\ell \in [p]$ and every $s, s' \in [k+1]^{<|L_p|}$ we have $\mathcal{Q}_s^{\ell,p} = \mathcal{Q}_{s'}^{\ell,p}$ if s and s' are $(r, k+1)$-equivalent.

(C8) If $p = d$, then there exists $s_0 \in [k+1]^{<|L_d|}$ such that

$$\mathrm{dens}_{X_{\mathbf{L}_d}}\Big(D_{s_0}^d \setminus \bigcup_{\ell=1}^{d} \bigcup \mathcal{Q}_{s_0}^{\ell,d}\Big) < \gamma^2/8. \tag{9.6}$$

Assuming that the above construction has been carried out, let us derive a contradiction. Let s_0 be as in (C8). By Corollary 6.8, Corollary 6.9 and Fact 9.2, we see that

$$\mathrm{dens}_{X_{\mathbf{L}_d}}(D_{s_0}^d \cap B_{s_0}^d \cap A_{s_0}^d) \geq (\delta + \gamma)\,\mathrm{dens}_{X_{\mathbf{L}_d}}(D_{s_0}^d \cap B_{s_0}^d), \tag{9.7}$$

$$\mathrm{dens}_{X_{\mathbf{L}_d}}(D_{s_0}^d \cap B_{s_0}^d) \geq \gamma. \tag{9.8}$$

For every $\ell \in [d]$ we set $C_\ell = \bigcup \mathcal{Q}_{s_0}^{\ell,d}$. By (9.6), the family $\{C_\ell : \ell \in [d]\}$ is an "almost cover" of $D_{s_0}^d \cap B_{s_0}^d$. Hence, invoking (9.7) and (9.8) and applying Lemma 2.6 for $\varepsilon = \gamma/4$, we may find $\ell_0 \in [d]$ such that

$$\mathrm{dens}_{C_{\ell_0}}(D_{s_0}^d \cap B_{s_0}^d \cap A_{s_0}^d) \geq (\delta + 3\gamma/4)\,\mathrm{dens}_{C_{\ell_0}}(D_{s_0}^d \cap B_{s_0}^d), \tag{9.9}$$

$$\mathrm{dens}_{C_{\ell_0}}(D_{s_0}^d \cap B_{s_0}^d) \geq \gamma^2/16. \tag{9.10}$$

Next observe that, by conditions (C2) and (C4), the family $\mathcal{Q}_{s_0}^{\ell_0,d}$ is a partition of C_{ℓ_0} into sets of equal size. Taking into account this observation and the estimates in (9.9) and (9.10), by a second application of Lemma 2.6 for $\varepsilon = \gamma/4$, we conclude that

$$|\mathcal{G}_{s_0}^{\ell_0,d}|/|\mathcal{Q}_{s_0}^{\ell_0,d}| \geq \gamma^3/256.$$

This contradicts (9.5), as desired.

The rest of the proof is devoted to the description of the recursive construction. For $p = 0$ we set $L_0 = L$ and $V_0 = V$. Let $p \in \{0, \dots, p_0\}$ and assume that the construction has been carried out up to p so that conditions (C1)–(C8) are satisfied. We distinguish the following cases.

Case 1: $p = p_0$. Notice first that, by (C1), the set L_p is a singleton. Therefore, $[k+1]^{<|L_p|}$ is the singleton $\{\emptyset\}$. We set $s_0 = \emptyset$ and $d = p_0$. With these choices, the recursive construction will be completed once we show that the estimate in (9.6) is satisfied. This is, however, an immediate consequence of (C3) and (C6) and the choice of α and p_0 in (9.3).

Case 2: We have $p \geq 1$ and there exists $s_0 \in [k+1]^{<|L_p|}$ such that

$$\mathrm{dens}_{X_{\mathbf{L}_p}}\Big(D^p_{s_0} \setminus \bigcup_{\ell=1}^{p} \bigcup \mathcal{Q}^{\ell,p}_{s_0}\Big) < \gamma^2/8.$$

In this case we set $d = p$ and we terminate the construction.

Case 3: We have $p < p_0$ and either $p = 0$, or $p \geq 1$ and

$$\mathrm{dens}_{X_{\mathbf{L}_p}}\Big(D^p_{s} \setminus \bigcup_{\ell=1}^{p} \bigcup \mathcal{Q}^{\ell,p}_{s}\Big) \geq \gamma^2/8 \tag{9.11}$$

for every $s \in [k+1]^{<|L_p|}$. If $p \geq 1$, then for every $s \in [k+1]^{<|L_p|}$ we set

$$\Gamma_s = D^p_s \setminus \bigcup_{\ell=1}^{p} \bigcup \mathcal{Q}^{\ell,p}_s. \tag{9.12}$$

Otherwise, let $\Gamma_s = D^0_s$. The following fact follows from (9.11), condition (a) in the statement of the lemma and condition (C7) if $p \geq 1$, and from Fact 9.2 if $p = 0$.

Fact 9.3. *For every $s \in [k+1]^{<|L_p|}$ we have $\mathrm{dens}_{X_{\mathbf{L}_p}}(\Gamma_s) \geq \gamma^2/8$. Moreover, if $s, s' \in [k+1]^{<|L_p|}$ are $(r, k+1)$-equivalent, then $D^p_s = D^p_{s'}$ and $\Gamma_s = \Gamma_{s'}$.*

By Fact 9.3, condition (C1), the choice of the sequence (N_p) in (9.4) and the choice of α in (9.3), we may apply Corollary 7.11 to get a Carlson–Simpson subtree S of $[k+1]^{<|L_p|}$ with $\dim(S) = n^2_{p_0-p}$ such that for every $\bar{m}$-dimensional Carlson–Simpson subtree U of S, setting

$$\Gamma_U = \bigcap_{s \in U} \Gamma_s, \tag{9.13}$$

we have $\mathrm{dens}_{X_{\mathbf{L}_p}}(\Gamma_U) \geq \alpha$.

For every $i \in \{0, \dots, \bar{m}\}$ and every Carlson–Simpson subtree U of S of dimension $\bar{m}$ let $G_{i,U}$ be the set of all $\mathbf{x} \in \Gamma_U$ satisfying

(P1) $\mathrm{dens}_{U(i)\times\{\mathbf{x}\}}(D^p \cap B^p \cap A^p) \geq (\delta + \gamma/2)\, \mathrm{dens}_{U(i)\times\{\mathbf{x}\}}(D^p \cap B^p)$ and
(P2) $\mathrm{dens}_{U(i)\times\{\mathbf{x}\}}(D^p \cap B^p) \geq \gamma^3/256$.

Our assumption that the lemma is not satisfied reduces to the following property of the sets $G_{i,U}$.

Claim 9.4. *For every $\bar{m}$-dimensional Carlson–Simpson subtree U of S there exists i in $\{0, \dots, \bar{m}\}$ such that $\mathrm{dens}_{\Gamma_U}(G_{i,U}) < \gamma^3/256$.*

Proof. For contradiction, assume that there exists a Carlson–Simpson subtree U of S of dimension $\bar{m}$ such that for every $i \in \{0, \dots, \bar{m}\}$ we have $\mathrm{dens}_{\Gamma_U}(G_{i,U}) \geq \gamma^3/256$. For every $\mathbf{x} \in \Gamma_U$ let

$$I_{\mathbf{x}} = \big\{ i \in \{0, \dots, \bar{m}\} : \mathbf{x} \in G_{i,U} \big\}.$$

By Lemma 2.5, setting $G = \{\mathbf{x} \in \Gamma_U : |I_{\mathbf{x}}| \geq (\bar{m}+1)\gamma^3/512\}$, we have $\mathrm{dens}_{\Gamma_U}(G) \geq \gamma^3/512$. This implies, in particular, that the set G is nonempty. We select $\mathbf{x} \in G$. By the choice of $\bar{m}$ in (9.1) we have $|I_{\mathbf{x}}| \geq m$. We define $W = \{\mathrm{c}_{\mathbf{L}_p,\mathbf{V}_p}(s,\mathbf{x}) : s \in U\}$. By Lemma 6.4, W is a Carlson–Simpson subtree of V of dimension $\bar{m}$. Applying Lemma 6.5 twice for the sets $D_r \cap B \cap A$ and $D_r \cap B$, for every $i \in \{0,\dots,\bar{m}\}$ we have

$$\mathrm{dens}_{W(i)}(D_r \cap B \cap A) = \mathrm{dens}_{U(i)\times\{\mathbf{x}\}}(D^p \cap B^p \cap A^p),$$
$$\mathrm{dens}_{W(i)}(D_r \cap B) = \mathrm{dens}_{U(i)\times\{\mathbf{x}\}}(D^p \cap B^p).$$

The above equalities and the fact that $\mathbf{x} \in G_{i,U}$ for every $i \in I_{\mathbf{x}}$ yield

$$\mathrm{dens}_{W(i)}(D_r \cap B \cap A) \geq (\delta + \gamma/2)\,\mathrm{dens}_{W(i)}(D_r \cap B),$$
$$\mathrm{dens}_{W(i)}(D_r \cap B) \geq \gamma^3/256,$$

for every $i \in I_{\mathbf{x}}$. Finally, observe that W is a subset of D_r since $\mathbf{x} \in \Gamma_U$. It is then clear that W and $I_{\mathbf{x}}$ satisfy the conclusion of the lemma, contrary to our assumption. □

We are now in a position to start the process of selecting the new objects of the recursive construction.

Step 1: Selection of V_{p+1} and L_{p+1}. Firstly, we will use a coloring argument to control the integer i obtained by Claim 9.4. Specifically, by the choice of the sequence (n^2_p) in (9.4) and Claim 9.4, we may apply Theorem 4.1 to obtain $i_0 \in \{0,\dots,\bar{m}\}$ and a Carlson–Simpson subtree T of S of dimension $n^1_{p_0-p}$ such that for every $\bar{m}$-dimensional Carlson–Simpson subtree U of T we have $\mathrm{dens}_{\Gamma_U}(G_{i_0,U}) < \gamma^3/256$. We define

$$V_{p+1} = T \quad \text{and} \quad L_{p+1} = \big\{i_0 + j(i_0+1) : j \in \{0,\dots,N_{p_0-(p+1)}\}\big\}. \tag{9.14}$$

Notice that the pair $((L_n)_{n=0}^{p+1}, (V_n)_{n=0}^{p+1})$ is $(k+1)$-compatible. This follows by our inductive assumptions and the choice of the sequence (n^1_p) in (9.4). Moreover, the cardinality of L_{p+1} is $N_{p_0-(p+1)} + 1$. Hence, with these choices, condition (C1) is satisfied. For notational simplicity, we shall denote by q_{p+1} the quotient map associated to the pair $(\mathbf{L}_{p+1}, \mathbf{V}_{p+1})$ defined in (6.1).

Step 2: Selection of $\mathcal{Q}_t^{p+1,p+1}$. This is the most important part of the recursive selection. The members of the families $\mathcal{Q}_t^{p+1,p+1}$ are, essentially, the sets Γ_U where U ranges over all $\bar{m}$-dimensional Carlson–Simpson subtrees of V_{p+1}. However, in order to carry out the construction, we have to group them in a canonical way. We proceed to the details.

Consider the canonical isomorphism $\mathrm{I}_{V_{p+1}} : [k+1]^{<\dim(V_{p+1})+1} \to V_{p+1}$ defined in §2.5; for convenience it will be denoted by I. For every $j \in \{0,\dots,|L_{p+1}|-1\}$ and every $t \in [k+1]^j$ we set

$$\Omega_t = \{\mathrm{c}_{L_{p+1},V_{p+1}}(t,x) : x \in X_{L_{p+1}}\} \subseteq V_{p+1}(i_0 + j(i_0+1)).$$

If $j \geq 1$, then let t_* be the unique initial segment of t of length $j-1$ and set

$$K_t = \{\mathrm{I}^{-1}(w)^\frown t(j-1) : w \in \Omega_{t_*}\} \subseteq [k+1]^{j(i_0+1)}; \tag{9.15}$$

otherwise, let $K_\emptyset = \{\emptyset\}$. Notice that $K_t = \{\mathrm{c}_{L_{p+1}}(t_*, x)^\frown t(j-1) : x \in X_{L_{p+1}}\}$. Finally for every $s \in K_t$ let $C_s = s^\frown[k+1]^{<\bar{m}+1}$ and set

$$\mathcal{P}_t = \{\mathrm{I}(C_s) : s \in K_t\}. \tag{9.16}$$

Before we analyze the above definitions, let us give a specific example. For concreteness take $k = 3$ and assume, for notational simplicity, that V_{p+1} is of the form $[4]^{<n}$ where n is large enough compared to i_0 (hence, the map I is the identity). Consider the sequence $t = (1, 2, 1)$ and observe that $t_* = (1, 2)$. Notice that Ω_t is the subset of $[4]^{4i_0+3}$ consisting of all finite sequences x such that $x(i_0) = t(0) = 1$, $x(2i_0 + 1) = t(1) = 2$ and $x(3i_0 + 2) = t(2) = 1$. On the other hand, K_t is the subset of $[4]^{3i_0+3}$ consisting of all sequences x such that $x(i_0) = t(0) = 1$, $x(2i_0+1) = t(1) = 2$ and $x(3i_0+2) = t(2) = 1$. It is then easy to see that in this specific case the family $\{U(i_0) : U \in \mathcal{P}_t\}$ forms a partition of Ω_t. This is, actually, a general property, as is shown by the following fact.

Fact 9.5. *Let $t \in [k+1]^{<|L_{p+1}|}$. Then the family $\mathcal{P}_t$ consists of pairwise disjoint $\bar{m}$-dimensional Carlson–Simpson subtrees of V_{p+1}. Moreover,*

$$\Omega_t = \bigcup_{U \in \mathcal{P}_t} U(i_0).$$

Proof. It is clear that $\mathcal{P}_t$ consists of pairwise disjoint Carlson–Simpson trees. Moreover, by the choice of L_{p+1} in (9.14), we have

$$\mathrm{I}^{-1}(\Omega_t) = \{s^\frown y : s \in K_t \text{ and } y \in [k+1]^{i_0}\} = \bigcup_{s \in K_t} C_s(i_0) = \bigcup_{U \in \mathcal{P}_t} \mathrm{I}^{-1}(U)(i_0). \qquad \square$$

We record, for future use, another property of the family $\mathcal{P}_t$.

Fact 9.6. *Let $t, t' \in [k+1]^{<|L_{p+1}|}$ with the same length. Then for every $U \in \mathcal{P}_t$ and every $U' \in \mathcal{P}_{t'}$ we have* $\mathrm{dens}_{\Omega_t}(U(i_0)) = \mathrm{dens}_{\Omega_{t'}}(U'(i_0))$.

Proof. By Fact 9.5, we have $U(i_0) \subseteq \Omega_t$ and $U'(i_0) \subseteq \Omega_{t'}$ while, by Fact 5.2, we have $|\Omega_t| = |\Omega_{t'}|$. Noticing that $|U(i_0)| = |U'(i_0)| = (k+1)^{i_0}$ we obtain the result. $\square$

We are ready to define the families $\mathcal{Q}_t^{p+1,p+1}$. Fix $t \in [k+1]^{<|L_{p+1}|}$. For every $U \in \mathcal{P}_t$ and every $\mathbf{x} \in \Gamma_U$ let

$$Q_t^{\mathbf{x},U} = \{\mathbf{x}\} \times \{x \in X_{L_{p+1}} : \mathrm{c}_{L_{p+1},V_{p+1}}(t, x) \in U(i_0)\} \subseteq X_{\mathbf{L}_{p+1}}. \tag{9.17}$$

By Fact 9.5, the set $U(i_0)$ is contained in Ω_t. Hence,

$$\{t\} \times Q_t^{\mathbf{x},U} = \mathrm{q}_{p+1}^{-1}(U(i_0) \times \{\mathbf{x}\}). \tag{9.18}$$

We define

$$\mathcal{Q}_t^{p+1,p+1} = \{Q_t^{\mathbf{x},U} : U \in \mathcal{P}_t \text{ and } \mathbf{x} \in \Gamma_U\}. \tag{9.19}$$

This completes the second step of the recursive selection.

Step 3: Selection of $\mathcal{Q}_t^{\ell,p+1}$ for every $\ell \in [p]$. In this step we will not introduce anything new but rather "copy" in the space $X_{\mathbf{L}_{p+1}}$ what we have constructed so far. In particular, this step is meaningful only if $p \geq 1$.

Let $p \geq 1$ and fix $t \in [k+1]^{<|L_{p+1}|}$ and $\ell \in [p]$. For every $s \in \Omega_t$ and every $Q \in \mathcal{Q}_s^{\ell,p}$ let

$$C_t^{s,\ell,Q} = Q \times \{x \in X_{L_{p+1}} : \mathrm{c}_{L_{p+1},V_{p+1}}(t,x) = s\} \subseteq X_{\mathbf{L}_{p+1}}. \tag{9.20}$$

Notice that

$$\{t\} \times C_t^{s,\ell,Q} = \mathrm{q}_{p+1}^{-1}(\{s\} \times Q). \tag{9.21}$$

We define

$$\mathcal{Q}_t^{\ell,p+1} = \{C_t^{s,\ell,Q} : s \in \Omega_t \text{ and } Q \in \mathcal{Q}_s^{\ell,p}\}. \tag{9.22}$$

This completes the third, and final, step of the recursive selection.

Step 4: Verification of the inductive assumptions. Recall that condition (C1) has already been verified in Step 1. Also observe that condition (C8) is meaningless in this case. Conditions (C2) up to (C7) will be verified in the following claims.

Claim 9.7. *For every $\ell \in [p+1]$ and every $t \in [k+1]^{<|L_{p+1}|}$ the family $\mathcal{Q}_t^{\ell,p+1}$ consists of pairwise disjoint subsets of D_t^{p+1}. That is, condition* (C2) *is satisfied.*

Proof. Fix $t \in [k+1]^{<|L_{p+1}|}$. First assume that $\ell \in [p]$. Invoking (9.20), (9.22) and our inductive assumptions, we see that the family $\mathcal{Q}_t^{\ell,p+1}$ consists of pairwise disjoint sets. Fix $C_t^{s,\ell,Q} \in \mathcal{Q}_t^{\ell,p+1}$ for some $s \in \Omega_t$ and $Q \in \mathcal{Q}_s^{\ell,p}$. Using our inductive assumptions once again, we see that $Q \subseteq D_s^p$ or equivalently $\{s\} \times Q \subseteq D^p$. Hence, by (9.21) and Fact 6.3,

$$\{t\} \times C_t^{s,\ell,Q} \subseteq \mathrm{q}_{p+1}^{-1}(D^p) = D^{p+1}.$$

Now we treat the case $\ell = p+1$. Let $U, U' \in \mathcal{P}_t$, $\mathbf{x} \in \Gamma_U$ and $\mathbf{x}' \in \Gamma_{U'}$. We will show that the sets $Q_t^{\mathbf{x},U}$ and $Q_t^{\mathbf{x}',U'}$ are disjoint provided that $(U,\mathbf{x}) \neq (U',\mathbf{x}')$. Indeed, if $U = U'$, then necessarily $\mathbf{x} \neq \mathbf{x}'$, which implies that $Q_t^{\mathbf{x},U}$ and $Q_t^{\mathbf{x}',U'}$ are disjoint. Otherwise, by Fact 9.5, the Carlson–Simpson trees U and U' are disjoint. In particular, the sets $U(i_0)$ and $U'(i_0)$ are disjoint, yielding $Q_t^{\mathbf{x},U} \cap Q_t^{\mathbf{x}',U'} = \emptyset$. What remains is to check that $Q_t^{\mathbf{x},U} \subseteq D_t^{p+1}$ for every $U \in \mathcal{P}_t$ and every $\mathbf{x} \in \Gamma_U$. So, fix $U \in \mathcal{P}_t$ and $\mathbf{x} \in \Gamma_U$. By (9.18) and Fact 6.3, we conclude that $\{t\} \times Q_t^{\mathbf{x},U} \subseteq \mathrm{q}_{p+1}^{-1}(D^p) = D^{p+1}$. □

Claim 9.8. *For every $t \in [k+1]^{<|L_{p+1}|}$ the sets $\bigcup \mathcal{Q}_t^{1,p+1}, \dots, \bigcup \mathcal{Q}_t^{p+1,p+1}$ are pairwise disjoint. That is, condition* (C3) *is satisfied.*

Proof. Fix $t \in [k+1]^{<|L_{p+1}|}$ and $\ell \in [p]$. Let $\ell' \in [p+1]$ with $\ell' \neq \ell$. We need to show that $\bigcup \mathcal{Q}_t^{\ell,p+1}$ and $\bigcup \mathcal{Q}_t^{\ell',p+1}$ are disjoint. If $\ell' \leq p$, then this follows immediately from (9.20) and our inductive assumptions. So, assume that $\ell' = p+1$ and let $U \in \mathcal{P}_t$ and $\mathbf{x} \in \Gamma_U$. By (9.12) and (9.13), we have $\mathbf{x} \notin \bigcup \mathcal{Q}_s^{\ell,p}$ for every $s \in U$. Using this observation, we deduce the result from (9.18) and (9.21). □

Claim 9.9. *For every $\ell \in [p+1]$, every $t, t' \in [k+1]^{<|L_{p+1}|}$ with the same length and every $\Theta \in \mathcal{Q}_t^{\ell,p+1}$ and $\Theta' \in \mathcal{Q}_{t'}^{\ell,p+1}$ we have $\mathrm{dens}_{X_{\mathbf{L}_{p+1}}}(\Theta) = \mathrm{dens}_{X_{\mathbf{L}_{p+1}}}(\Theta')$. That is, condition* (C4) *is satisfied.*

Proof. If $\ell \in [p]$, then by (9.21) there exist $s \in \Omega_t$ and $s' \in \Omega_{t'}$ as well as $Q \in \mathcal{Q}_s^{\ell,p}$ and $Q' \in \mathcal{Q}_{s'}^{\ell,p}$ such that $\{t\} \times \Theta = \mathrm{q}_{p+1}^{-1}(\{s\} \times Q)$ and $\{t'\} \times \Theta' = \mathrm{q}_{p+1}^{-1}(\{s'\} \times Q')$. By Fact 5.2, s and s' have the same length and $|\Omega_t| = |\Omega_{t'}|$. Therefore, by our inductive assumptions,

$$\begin{aligned}
\mathrm{dens}_{\Omega_t \times X_{\mathbf{L}_p}}(\{s\} \times Q) &= \frac{1}{|\Omega_t|}\,\mathrm{dens}_{\{s\} \times X_{\mathbf{L}_p}}(\{s\} \times Q) \\
&= \frac{1}{|\Omega_t|}\,\mathrm{dens}_{X_{\mathbf{L}_p}}(Q) = \frac{1}{|\Omega_{t'}|}\,\mathrm{dens}_{X_{\mathbf{L}_p}}(Q') \\
&= \frac{1}{|\Omega_{t'}|}\,\mathrm{dens}_{\{s'\} \times X_{\mathbf{L}_p}}(\{s'\} \times Q') \\
&= \mathrm{dens}_{\Omega_{t'} \times X_{\mathbf{L}_p}}(\{s'\} \times Q'). \qquad (9.23)
\end{aligned}$$

Applying Lemma 6.6 we conclude that

$$\begin{aligned}
\mathrm{dens}_{X_{\mathbf{L}_{p+1}}}(\Theta) &= \mathrm{dens}_{\{t\} \times X_{\mathbf{L}_{p+1}}}(\{t\} \times \Theta) = \mathrm{dens}_{\Omega_t \times X_{\mathbf{L}_p}}(\{s\} \times Q) \\
&\overset{(9.23)}{=} \mathrm{dens}_{\Omega_{t'} \times X_{\mathbf{L}_p}}(\{s'\} \times Q') = \mathrm{dens}_{\{t'\} \times X_{\mathbf{L}_{p+1}}}(\{t'\} \times \Theta') = \mathrm{dens}_{X_{\mathbf{L}_{p+1}}}(\Theta').
\end{aligned}$$

If $\ell = p+1$, then by (9.18) there exist $U \in \mathcal{P}_t$ and $U' \in \mathcal{P}_{t'}$ as well as $\mathbf{x} \in \Gamma_U$ and $\mathbf{x}' \in \Gamma_{U'}$ such that $\{t\} \times \Theta = \mathrm{q}_{p+1}^{-1}(U(i_0) \times \{\mathbf{x}\})$ and $\{t'\} \times \Theta = \mathrm{q}_{p+1}^{-1}(U'(i_0) \times \{\mathbf{x}'\})$. By Fact 9.6, we have

$$\begin{aligned}
\mathrm{dens}_{\Omega_t \times X_{\mathbf{L}_p}}(U(i_0) \times \{\mathbf{x}\}) &= \frac{1}{|X_{\mathbf{L}_p}|}\,\mathrm{dens}_{\Omega_t}(U(i_0)) = \frac{1}{|X_{\mathbf{L}_p}|}\,\mathrm{dens}_{\Omega_{t'}}(U'(i_0)) \\
&= \mathrm{dens}_{\Omega_{t'} \times X_{\mathbf{L}_p}}(U'(i_0) \times \{\mathbf{x}'\}).
\end{aligned}$$

Using this Lemma 6.6 and arguing precisely as in the previous case, we see that $\mathrm{dens}_{X_{\mathbf{L}_{p+1}}}(\Theta) = \mathrm{dens}_{X_{\mathbf{L}_{p+1}}}(\Theta')$. □

Claim 9.10. *For every $\ell \in [p+1]$ and every $t \in [k+1]^{<|L_{p+1}|}$ we have*

$$|\mathcal{G}_t^{\ell,p+1}|/|\mathcal{Q}_t^{\ell,p+1}| < \gamma^3/256.$$

That is, condition (C5) *is satisfied.*

Proof. Fix $t \in [k+1]^{<|L_{p+1}|}$. Assume first that $\ell \in [p]$. For every $s \in \Omega_t$ let

$$\mathcal{Q}_s = \{C_t^{s,\ell,Q} : Q \in \mathcal{Q}_s^{\ell,p}\} \quad \text{and} \quad \mathcal{G}_s = \mathcal{G}_t^{\ell,p+1} \cap \mathcal{Q}_s.$$

By (9.22), the family $\{\mathcal{Q}_s : s \in \Omega_t\}$ is a partition of $\mathcal{Q}_t^{\ell,p+1}$. The family $\{\mathcal{G}_s : s \in \Omega_t\}$ is the induced partition of $\mathcal{G}_t^{\ell,p+1}$. Moreover, for every $s \in \Omega_t$ let

$$\mathcal{B}_s = \{C_t^{s,\ell,Q} : Q \in \mathcal{G}_s^{\ell,p}\}.$$

Subclaim 9.11. *For every $s \in \Omega_t$ we have $\mathcal{G}_s = \mathcal{B}_s$.*

Proof of Subclaim 9.11. Fix $s \in \Omega_t$ and let $\Theta \in \mathcal{Q}_s$. By (9.21), the map $\mathcal{Q}_s^{\ell,p} \ni Q \mapsto C_t^{s,\ell,Q} \in \mathcal{Q}_s$ is a bijection. Hence, there exists a unique $Q \in \mathcal{Q}_s^{\ell,p}$ such that $\Theta = C_t^{s,\ell,Q}$. Using (9.21) once again, we see that $\{t\}\times\Theta = \mathrm{q}_{p+1}^{-1}(\{s\}\times Q)$. Since $s \in \Omega_t$, by Corollary 6.10 we get

$$\begin{aligned}\mathrm{dens}_\Theta(D_t^{p+1}\cap B_t^{p+1}) &= \mathrm{dens}_{\{t\}\times\Theta}(D^{p+1}\cap B^{p+1}) = \mathrm{dens}_{\mathrm{q}_{p+1}^{-1}(\{s\}\times Q)}(\mathrm{q}_{p+1}^{-1}(D^p\cap B^p)) \\ &= \mathrm{dens}_{\{s\}\times Q}(D^p\cap B^p) = \mathrm{dens}_Q(D_s^p\cap B_s^p). \qquad (9.24)\end{aligned}$$

Similarly,

$$\mathrm{dens}_\Theta(D_t^{p+1}\cap B_t^{p+1}\cap A_t^{p+1}) = \mathrm{dens}_Q(D_s^p\cap B_s^p\cap A_s^p). \qquad (9.25)$$

Using (9.24) and (9.25) and invoking the definition of a good set in condition (C5), we conclude that $\Theta \in \mathcal{G}_s$ if and only if $Q \in \mathcal{G}_s^{\ell,p}$. This is equivalent to $\mathcal{G}_s = \mathcal{B}_s$. □

We are ready to complete the proof for the case $\ell \in [p]$. Indeed, we have already pointed out that the map $\mathcal{Q}_s^{\ell,p} \ni Q \mapsto C_t^{s,\ell,Q} \in \mathcal{Q}_s$ is a bijection. Therefore, using our inductive assumptions, we see that

$$\begin{aligned}\frac{|\mathcal{G}_t^{\ell,p+1}|}{|\mathcal{Q}_t^{\ell,p+1}|} &= \sum_{s\in\Omega_t}\frac{|\mathcal{G}_s|}{|\mathcal{Q}_s|}\cdot\frac{|\mathcal{Q}_s|}{|\mathcal{Q}_t^{\ell,p+1}|} = \sum_{s\in\Omega_t}\frac{|\mathcal{B}_s|}{|\mathcal{Q}_s|}\cdot\frac{|\mathcal{Q}_s|}{|\mathcal{Q}_t^{\ell,p+1}|} \\ &= \sum_{s\in\Omega_t}\frac{|\mathcal{G}_s^{\ell,p}|}{|\mathcal{Q}_s^{\ell,p}|}\cdot\frac{|\mathcal{Q}_s|}{|\mathcal{Q}_t^{\ell,p+1}|} < \frac{\gamma^3}{256}\sum_{s\in\Omega_t}\frac{|\mathcal{Q}_s|}{|\mathcal{Q}_t^{\ell,p+1}|} = \frac{\gamma^3}{256}.\end{aligned}$$

Now we treat the case $\ell = p+1$. The argument is similar. For every $U \in \mathcal{P}_t$ let

$$\mathcal{Q}_U = \{Q_t^{\mathbf{x},U} : \mathbf{x}\in\Gamma_U\} \quad \text{and} \quad \mathcal{G}_U = \mathcal{G}_t^{\ell,p+1}\cap\mathcal{Q}_U.$$

By (9.19), the family $\{\mathcal{Q}_U : U \in \mathcal{P}_t\}$ is a partition of $\mathcal{Q}_t^{\ell,p+1}$ and the family $\{\mathcal{G}_U : U \in \mathcal{P}_t\}$ is the induced partition of $\mathcal{G}_t^{\ell,p+1}$. Also, for every $U \in \mathcal{P}_t$ let

$$\mathcal{B}_U = \{Q_t^{\mathbf{x},U} : \mathbf{x}\in G_{i_0,U}\}.$$

Recall that $G_{i_0,U}$ is the set of all $\mathbf{x} \in \Gamma_U$ satisfying properties (P1) and (P2). Moreover, by the choice of i_0 in Step 1, we have $\mathrm{dens}_{\Gamma_U}(G_{i_0,U}) < \gamma^3/256$. We have the following analogue of Subclaim 9.11.

Subclaim 9.12. *For every $U \in \mathcal{P}_t$ we have $\mathcal{G}_U = \mathcal{B}_U$.*

Proof of Subclaim 9.12. Fix $U \in \mathcal{P}_t$ and let $\Theta \in \mathcal{Q}_U$. By (9.18), the map $\Gamma_U \ni \mathbf{x} \mapsto Q_t^{\mathbf{x},U} \in \mathcal{Q}_U$ is a bijection. Hence, there exists a unique $\mathbf{x} \in \Gamma_U$ such that $Q_t^{\mathbf{x},U} = \Theta$. Invoking (9.18) once again, $\{t\}\times Q_t^{\mathbf{x},U} = \mathrm{q}_{p+1}^{-1}(U(i_0)\times\{\mathbf{x}\})$. Moreover, by Fact 9.5, we have $U(i_0) \subseteq \Omega_t$. By the previous remarks and Corollary 6.10, and arguing precisely as in the proof of Subclaim 9.11, we see that $\Theta \in \mathcal{G}_U$ if and only if $\mathbf{x} \in G_{i_0,U}$. □

With Subclaim 9.12 at our disposal, we are ready to complete the proof in the case $\ell = p+1$. We have already pointed out that $\Gamma_U \ni \mathbf{x} \mapsto \mathcal{Q}_t^{\mathbf{x},U} \in \mathcal{Q}_U$ is a bijection. Therefore, using our inductive assumptions, we conclude that

$$\frac{|\mathcal{G}_t^{p+1,p+1}|}{|\mathcal{Q}_t^{p+1,p+1}|} = \sum_{U\in\mathcal{P}_t} \frac{|\mathcal{G}_U|}{|\mathcal{Q}_U|} \cdot \frac{|\mathcal{Q}_U|}{|\mathcal{Q}_t^{p+1,p+1}|} = \sum_{U\in\mathcal{P}_t} \frac{|\mathcal{B}_U|}{|\mathcal{Q}_U|} \cdot \frac{|\mathcal{Q}_U|}{|\mathcal{Q}_t^{p+1,p+1}|}$$
$$= \sum_{U\in\mathcal{P}_t} \frac{|G_{i_0,U}|}{|\Gamma_U|} \cdot \frac{|\mathcal{Q}_U|}{|\mathcal{Q}_t^{p+1,p+1}|} < \frac{\gamma^3}{256} \sum_{U\in\mathcal{P}_t} \frac{|\mathcal{Q}_U|}{|\mathcal{Q}_t^{p+1,p+1}|} = \frac{\gamma^3}{256}.$$

The proof of Claim 9.10 is complete. □

Claim 9.13. *For every $\ell \in [p+1]$ and every $t \in [k+1]^{<|L_{p+1}|}$ we have $\mathrm{dens}_{X_{\mathbf{L}_{p+1}}}(\bigcup \mathcal{Q}_t^{\ell,p+1}) \geq \alpha$. That is, condition* (C6) *is satisfied.*

Proof. Let $t \in [k+1]^{<|L_{p+1}|}$. Assume first that $\ell \in [p]$. By the inductive assumptions,

$$\mathrm{dens}_{\Omega_t \times X_{\mathbf{L}_p}}\Big(\bigcup_{s\in\Omega_t} \{s\} \times \bigcup \mathcal{Q}_s^{\ell,p}\Big) \geq \alpha. \tag{9.26}$$

On the other hand,

$$\mathrm{q}_{p+1}^{-1}\Big(\bigcup_{s\in\Omega_t} \{s\} \times \bigcup \mathcal{Q}_s^{\ell,p}\Big) = \mathrm{q}_{p+1}^{-1}\Big(\bigcup_{s\in\Omega_t} \bigcup_{Q\in\mathcal{Q}_s^{\ell,p}} \{s\} \times Q\Big) = \bigcup_{s\in\Omega_t} \bigcup_{Q\in\mathcal{Q}_s^{\ell,p}} \mathrm{q}_{p+1}^{-1}(\{s\} \times Q)$$
$$\overset{(9.21)}{=} \bigcup_{s\in\Omega_t} \bigcup_{Q\in\mathcal{Q}_s^{\ell,p}} \{t\} \times C_t^{s,\ell,Q} \overset{(9.22)}{=} \{t\} \times \bigcup \mathcal{Q}_t^{\ell,p+1}. \tag{9.27}$$

By (9.26), (9.27) and Lemma 6.6, we conclude that

$$\mathrm{dens}_{X_{\mathbf{L}_{p+1}}}(\bigcup \mathcal{Q}_t^{\ell,p+1}) = \mathrm{dens}_{\{t\}\times X_{\mathbf{L}_{p+1}}}(\{t\} \times \bigcup \mathcal{Q}_t^{\ell,p+1}) \geq \alpha.$$

Next assume that $\ell = p+1$. By Fact 9.5, the family $\{U(i_0) : U \in \mathcal{P}_t\}$ forms a partition Ω_t. Moreover, $\mathrm{dens}_{X_{\mathbf{L}_p}}(\Gamma_U) \geq \alpha$ as already pointed out immediately after (9.13). Hence,

$$\mathrm{dens}_{\Omega_t \times X_{\mathbf{L}_p}}\Big(\bigcup_{U\in\mathcal{P}_t} U(i_0) \times \Gamma_U\Big) \geq \alpha. \tag{9.28}$$

Notice that

$$\begin{aligned}
\mathrm{q}_{p+1}^{-1}\Big(\bigcup_{U\in\mathcal{P}_t} U(i_0) \times \Gamma_U\Big) &= \mathrm{q}_{p+1}^{-1}\Big(\bigcup_{U\in\mathcal{P}_t} \bigcup_{\mathbf{x}\in\Gamma_U} U(i_0) \times \{\mathbf{x}\}\Big) \\
&= \bigcup_{U\in\mathcal{P}_t} \bigcup_{\mathbf{x}\in\Gamma_U} \mathrm{q}_{p+1}^{-1}(U(i_0) \times \{\mathbf{x}\}) \\
&\overset{(9.18)}{=} \bigcup_{U\in\mathcal{P}_t} \bigcup_{\mathbf{x}\in\Gamma_U} \{t\} \times \mathcal{Q}_t^{\mathbf{x},U} \overset{(9.19)}{=} \{t\} \times \bigcup \mathcal{Q}_t^{p+1,p+1}
\end{aligned} \tag{9.29}$$

Combining (9.28), (9.29) and applying Lemma 6.6 we obtain

$$\mathrm{dens}_{X_{\mathbf{L}_{p+1}}}(\bigcup \mathcal{Q}_t^{p+1,p+1}) = \mathrm{dens}_{\{t\}\times X_{\mathbf{L}_{p+1}}}(\{t\} \times \bigcup \mathcal{Q}_t^{p+1,p+1}) \geq \alpha.$$ □

Claim 9.14. *For every $\ell \in [p+1]$ and every $t, t' \in [k+1]^{<|L_{p+1}|}$, if t and t' are $(r, k+1)$-equivalent, then $\mathcal{Q}_t^{\ell,p+1} = \mathcal{Q}_{t'}^{\ell,p+1}$. That is, condition* (C7) *is satisfied.*

Proof. We fix $t, t' \in [k+1]^{<|L_{p+1}|}$ which are $(r, k+1)$-equivalent. For every $s \in \Omega_t$ let $Y_s^t = \{x \in X_{L_{p+1}} : \mathrm{c}_{L_{p+1},V_{p+1}}(t, x) = s\}$, and for every $s' \in \Omega_{t'}$ let $Y_{s'}^{t'} = \{x \in X_{L_{p+1}} : \mathrm{c}_{L_{p+1},V_{p+1}}(t', x) = s'\}$. Let $g_{t,t'} : \Omega_t \to \Omega_{t'}$ be the bijection obtained in Fact 5.4 and recall that for every $s \in \Omega_t$ we have

$$Y_s^t = Y_{g_{t,t'}(s)}^{t'}. \tag{9.30}$$

Since t and t' are $(r, k+1)$-equivalent, by Fact 5.4 again, we see that s and $g_{t,t'}(s)$ are also $(r, k+1)$-equivalent for every $s \in \Omega_t$.

After this preliminary discussion, we are ready for the main argument. First assume that $\ell \in [p]$. For every $s \in \Omega_t$ and every $s' \in \Omega_{t'}$ let

$$\mathcal{Q}_s^t = \{C_t^{s,\ell,Q} : Q \in \mathcal{Q}_s^{\ell,p}\} \quad \text{and} \quad \mathcal{Q}_{s'}^{t'} = \{C_{t'}^{s',\ell,Q'} : Q' \in \mathcal{Q}_{s'}^{\ell,p}\}.$$

By (9.22), the families $\{\mathcal{Q}_s^t : s \in \Omega_t\}$ and $\{\mathcal{Q}_{s'}^{t'} : s' \in \Omega_{t'}\}$ form partitions of $\mathcal{Q}_t^{\ell,p+1}$ and $\mathcal{Q}_{t'}^{\ell,p+1}$ respectively. By (9.20), for every $s \in \Omega_t$ and every $Q \in \mathcal{Q}_s^{\ell,p}$ we have

$$C_t^{s,\ell,Q} = Q \times Y_s^t. \tag{9.31}$$

Of course, we have the same equality for t', that is, for every $s' \in \Omega_{t'}$ and every $Q' \in \mathcal{Q}_{s'}^{\ell,p}$,

$$C_{t'}^{s',\ell,Q'} = Q' \times Y_{s'}^{t'}. \tag{9.32}$$

Moreover, invoking our inductive assumptions, for every $s \in \Omega_t$ we have

$$\mathcal{Q}_s^{\ell,p} = \mathcal{Q}_{g_{t,t'}(s)}^{\ell,p}. \tag{9.33}$$

Hence,

$$\begin{aligned}\mathcal{Q}_s^t &= \{C_t^{s,\ell,Q} : Q \in \mathcal{Q}_s^{\ell,p}\} \overset{(9.31)}{=} \{Q \times Y_s^t : Q \in \mathcal{Q}_s^{\ell,p}\} \overset{(9.30)}{=} \{Q \times Y_{g_{t,t'}(s)}^{t'} : Q \in \mathcal{Q}_s^{\ell,p}\} \\ &\overset{(9.33)}{=} \{Q \times Y_{g_{t,t'}(s)}^{t'} : Q \in \mathcal{Q}_{g_{t,t'}(s)}^{\ell,p}\} \overset{(9.32)}{=} \mathcal{Q}_{g_{t,t'}(s)}^{t'}.\end{aligned}$$

Since $g_{t,t'}$ is a bijection we conclude that $\mathcal{Q}_t^{\ell,p+1} = \mathcal{Q}_{t'}^{\ell,p+1}$.

Before we proceed, we need to introduce some terminology. Let U and U' be two Carlson–Simpson trees of $[k+1]^{<\mathbb{N}}$ of the same dimension and consider the canonical isomorphism $\mathrm{I}_{U,U'}$ associated to the pair U, U' described in §2.5. We say that U and U' are $(r, k+1)$-*equivalent* if for every $s \in U$ the elements s and $\mathrm{I}_{U,U'}(s)$ are $(r, k+1)$-equivalent.

Now assume that $\ell = p+1$ and let K_t and $K_{t'}$ be as in (9.15). Our first goal is to define a map $h : K_t \to K_{t'}$ with the following properties:

(a) h is a bijection.
(b) h preserves the lexicographical order.
(c) For every $s \in K_t$ the elements s and $h(s)$ are $(r, k+1)$-equivalent.

If $t = \emptyset$, then h is the identity. Assume that $|t| = |t'| = j$ for some $j \geq 1$ and recall that t_* and t'_* are the initial segments of t and t' respectively of length $j-1$. Let $g_{t_*,t'_*} : \Omega_{t_*} \to \Omega_{t'_*}$ be obtained Fact 5.4. We define

$$h(\mathrm{I}^{-1}(w)^\frown t(j-1)) = \mathrm{I}^{-1}(g_{t_*,t'_*}(w))^\frown t'(j-1)$$

for every $w \in \Omega_{t_*}$. With this choice the aforementioned properties of h follow readily from the properties of g_{t_*,t'_*}.

The map h induces a function $f : \mathcal{P}_t \to \mathcal{P}_{t'}$ defined by

$$f(\mathrm{I}(C_s)) = \mathrm{I}(C_{h(s)}).$$

We isolate, for future use, the following properties of f, whose verification is straightforward.

(d) f is a bijection.
(e) For every $U, U' \in \mathcal{P}_t$ if $U(0) <_{\mathrm{lex}} U'(0)$, then $f(U)(0) <_{\mathrm{lex}} f(U')(0)$.
(f) For every $U \in \mathcal{P}_t$ the elements U and $f(U)$ are $(r, k+1)$-equivalent.

Also observe that for every $U \in \mathcal{P}_t$ we have

$$\Gamma_U = \Gamma_{f(U)}. \tag{9.34}$$

This follows from Fact 9.3 and property (f) above. More important, however, is the relation of f to $g_{t,t'}$: for every $U \in \mathcal{P}_t$ we have

$$\{g_{t,t'}(s) : s \in U(i_0)\} = f(U)(i_0). \tag{9.35}$$

To see this, notice first, that for every $s \in K_t$ the set $C_s(i_0)$ is an interval, in the lexicographical order, of $[k+1]^l$ for some $l \in \{0, \dots, \dim(V_{p+1})\}$ depending only on the length of t (precisely, $l = i_0 + (i_0+1)|t|$). Hence, by (9.16), for every $U \in \mathcal{P}_t$ the set $U(i_0)$ is an interval of $V_{p+1}(l)$ for the same l. Therefore, (9.34) follows from Fact 9.5 and property (e) isolated above.

For every $U \in \mathcal{P}_t$ and every $U' \in \mathcal{P}_{t'}$ we set

$$\mathcal{Q}^t_U = \{Q^{\mathbf{x},U}_t : \mathbf{x} \in \Gamma_U\} \quad \text{and} \quad \mathcal{Q}^{t'}_{U'} = \{Q^{\mathbf{x}',U'}_{t'} : \mathbf{x}' \in \Gamma_{U'}\}. \tag{9.36}$$

By (9.19), the families $\{\mathcal{Q}^t_U : U \in \mathcal{P}_t\}$ and $\{\mathcal{Q}^{t'}_{U'} : U' \in \mathcal{P}_{t'}\}$ form partitions of $\mathcal{Q}^{p+1,p+1}_t$ and $\mathcal{Q}^{p+1,p+1}_{t'}$ respectively. By (9.17), for every $U \in \mathcal{P}_t$ and $U' \in \mathcal{P}_{t'}$ and every $\mathbf{x} \in \Gamma_U$ and $\mathbf{x}' \in \Gamma_{U'}$ we have

$$Q^{\mathbf{x},U}_t = \{\mathbf{x}\} \times \bigcup_{s \in U(i_0)} Y^t_s \quad \text{and} \quad Q^{\mathbf{x}',U'}_{t'} = \{\mathbf{x}'\} \times \bigcup_{s' \in U'(i_0)} Y^{t'}_{s'}. \tag{9.37}$$

Thus, for every $U \in \mathcal{P}_t$,

$$\begin{aligned}\mathcal{Q}^t_U &\overset{(9.36)}{=} \{Q^{\mathbf{x},U}_t : \mathbf{x} \in \Gamma_U\} \overset{(9.37)}{=} \Big\{\{\mathbf{x}\} \times \bigcup_{s \in U(i_0)} Y^t_s : \mathbf{x} \in \Gamma_U\Big\} \\ &\overset{(9.34)}{=} \Big\{\{\mathbf{x}'\} \times \bigcup_{s \in U(i_0)} Y^t_s : \mathbf{x}' \in \Gamma_{f(U)}\Big\} \overset{(9.30)}{=} \Big\{\{\mathbf{x}'\} \times \bigcup_{s \in U(i_0)} Y^{t'}_{g_{t,t'}(s)} : \mathbf{x}' \in \Gamma_{f(U)}\Big\} \\ &\overset{(9.35)}{=} \Big\{\{\mathbf{x}'\} \times \bigcup_{s' \in f(U)(i_0)} Y^{t'}_{s'} : \mathbf{x}' \in \Gamma_{f(U)}\Big\} = \mathcal{Q}^{t'}_{f(U)}.\end{aligned}$$

Since f is a bijection we conclude that $\mathcal{Q}^{p+1,p+1}_t = \mathcal{Q}^{p+1,p+1}_{t'}$. The proof of Claim 9.14 is thus complete. □

By Claims 9.7 to 9.14, the pair (V_{p+1}, L_{p+1}) and the families $\mathcal{Q}^{\ell,p+1}_t$ constructed in Steps 1, 2 and 3 satisfy all required conditions. This completes the recursive selection, and as we have already indicated, the proof of Lemma 9.1 is also complete.

9.4. Consequences. In this subsection we will isolate what we get by iterating Lemma 9.1. The resulting statement together with Corollary 8.6 form the basis of the proof of Theorem B. We proceed to the details.

Let $k \in \mathbb{N}$ with $k \geq 2$ and assume that for every integer $l \geq 1$ and every $0 < \beta \leq 1$ the number $\mathrm{DCS}(k, l, \beta)$ has been defined. We define $H : \mathbb{N} \times (0, 1] \to \mathbb{N}$ by $H(0, \gamma) = 0$ and

$$H(m, \gamma) = \mathrm{Reg}(k + 1, N_{p_0} + 1, 2, \gamma^2/2)$$

if $m \geq 1$, where p_0 and N_{p_0} are defined in (9.3) and (9.4) respectively for the parameters m and γ. Next for every $n \in \{0, \dots, k\}$ we define $H^{(n)} : \mathbb{N} \times (0, 1] \to \mathbb{N}$ recursively by $H^{(0)}(m, \gamma) = m$ and

$$H^{(n+1)}(m, \gamma) = H(H^{(n)}(m, \gamma), \gamma). \tag{9.38}$$

Finally, for every $0 < \gamma \leq 1$ let

$$\xi = \xi(\gamma) = \frac{\gamma^{3^k}}{(2^{1/2} \cdot 32)^{3^k - 1}}. \tag{9.39}$$

The following corollary is an immediate consequence of Lemma 9.1.

Corollary 9.15. *Let $k \in \mathbb{N}$ with $k \geq 2$ and assume that for every integer $l \geq 1$ and every $0 < \beta \leq 1$ the number $\mathrm{DCS}(k, l, \beta)$ has been defined. Let $0 < \gamma, \delta \leq 1$. Also let V be a Carlson–Simpson tree of $[k + 1]^{<\mathbb{N}}$ and I be a nonempty subset of $\{0, \dots, \dim(V)\}$. Assume that we are given subsets $A, D_1, \dots, D_k$ of $[k + 1]^{<\mathbb{N}}$ with the following properties:*

(a) *For every $r \in [k]$ the set D_r is $(r, k + 1)$-insensitive in V.*
(b) *$\mathrm{dens}_{V(i)}(D_1 \cap \cdots \cap D_k \cap A) \geq (\delta + \gamma)\, \mathrm{dens}_{V(i)}(D_1 \cap \cdots \cap D_k)$ and $\mathrm{dens}_{V(i)}(D_1 \cap \cdots \cap D_k) \geq \gamma$ for every $i \in I$.*

Finally, let $m \in \mathbb{N}$ with $m \geq 1$ and suppose that

$$|I| \geq H^{(k)}(m, \xi)$$

where $H^{(k)}$ and ξ are defined in (9.38) and (9.39) respectively for the parameters m and γ. Then there exist a Carlson–Simpson subtree W of V and a subset I' of $\{0, \ldots, \dim(W)\}$ of cardinality m such that

$$\mathrm{dens}_{W(i)}(A) \geq \delta + \xi$$

for every $i \in I'$.

10. Proof of Theorem B

In this section we will complete the proof of Theorem B following the inductive scheme outlined in §8.1. Notice first that the numbers $\mathrm{DCS}(2, 1, \delta)$ are estimated in Proposition 7.1. It is then easy to see that, by induction on m and Corollary 7.6, we may also estimate the numbers $\mathrm{DCS}(2, m, \delta)$.

Now we argue for the general inductive step. So let $k \in \mathbb{N}$ with $k \geq 2$ and assume that for every integer $l \geq 1$ and every $0 < \beta \leq 1$ the number $\mathrm{DCS}(k, l, \beta)$ has been defined. We fix $0 < \delta \leq 1$. Let η_1 be as in (8.16). Recall that

$$\eta_1 = \frac{\delta^2}{120k \cdot |\mathrm{Subtr}_1([k]^{<\Lambda})|}$$

where $\Lambda = \lceil 8\delta^{-1}\mathrm{DCS}(k, 1, \delta/8)\rceil$. We set

$$\varrho = \xi(\eta_1^2/2) \overset{(9.39)}{=} \frac{(\eta_1^2/2)^{3^k}}{(2^{1/2} \cdot 32)^{3^k - 1}} \tag{10.1}$$

and we define $F_\delta : \mathbb{N} \to \mathbb{N}$ by

$$F_\delta(m) = G_1\big(\lceil \eta_1^{-4}(k+1)k \cdot H^{(k)}(m, \varrho)\rceil, \eta_1^2/2\big) \tag{10.2}$$

where G_1 and $H^{(k)}(m, \varrho)$ are as in (8.17) and (9.38) respectively. The following proposition is the heart of the density increment strategy. It follows immediately from Corollaries 8.6 and 9.15.

Proposition 10.1. *Let $k \in \mathbb{N}$ with $k \geq 2$ and assume that for every integer $l \geq 1$ and every $0 < \beta \leq 1$ the number $\mathrm{DCS}(k, l, \beta)$ has been defined. Let $0 < \delta \leq 1$ and L be a nonempty finite subset of $\mathbb{N}$. Also let $A \subseteq [k+1]^{<\mathbb{N}}$ be such that $|A \cap [k+1]^l| \geq \delta(k+1)^l$ for every $l \in L$ and assume that A contains no Carlson–Simpson line of $[k+1]^{<\mathbb{N}}$. Finally, let $m \in \mathbb{N}$ with $m \geq 1$ and suppose that $|L| \geq F_\delta(m)$ where F_δ is as in (10.2). Then there exist a Carlson–Simpson tree W of $[k+1]^{<\mathbb{N}}$ and a subset I of $\{0, \ldots, \dim(W)\}$ of cardinality m such that $\mathrm{dens}_{W(i)}(A) \geq \delta + \varrho$ for every $i \in I$ where ϱ is as in (10.1).*

Using Proposition 10.1 the numbers $\mathrm{DCS}(k+1, 1, \delta)$ can, of course, be estimated easily. In particular, we have the following corollary.

Corollary 10.2. *Let $k \in \mathbb{N}$ with $k \geq 2$ and assume that for every integer $l \geq 1$ and every $0 < \beta \leq 1$ the number $\mathrm{DCS}(k, l, \beta)$ has been defined. Then for every $0 < \delta \leq 1$ we have*

$$\mathrm{DCS}(k+1, 1, \delta) \leq F_{\delta}^{(\lceil \varrho^{-1} \rceil)}(1).$$

Finally, just as in the case $k = 2$, the numbers $\mathrm{DCS}(k+1, m, \delta)$ can be estimated using Corollaries 10.2 and 7.6. This completes the proof of the general inductive step, and so the entire proof of Theorem B is complete.

11. Consequences

Our goal in this section is to prove several consequences of Theorem B. These include Theorems A and C stated in the introduction, as well as an appropriate finite version of Theorem C. To state this finite version we need to introduce some terminology.

Recall that, given two sequences (p_n) and (w_n) of variable words over k, we say that (w_n) is of *pattern* (p_n) if p_n is an initial segment of w_n for every $n \in \mathbb{N}$. This notion can, of course, be extended to finite sequences of the same length. Specifically, given two finite sequences $(p_n)_{n=0}^{m-1}$ and $(w_n)_{n=0}^{m-1}$ of variable words over k, we say that $(w_n)_{n=0}^{m-1}$ is of *pattern* $(p_n)_{n=0}^{m-1}$ if p_n is an initial segment of w_n for every $n \in \{0, \dots, m-1\}$. In particular, if p and w are variable words over k, then w is of pattern p if p is an initial segment of w. We have the following theorem.

Theorem 11.1. *For every integer $k \geq 2$, every nonempty finite sequence $(\tau_n)_{n=0}^{m-1}$ of positive integers and every $0 < \delta \leq 1$ there exists an integer N with the following property. If $(p_n)_{n=0}^{m-1}$ is a finite sequence of variable words over k such that the length of p_n is τ_n for every $n \in \{0, \dots, m-1\}$, L is a finite subset of $\mathbb{N}$ of cardinality at least N and A is a subset of $[k]^{<\mathbb{N}}$ satisfying $\mathrm{dens}_{[k]^\ell}(A) \geq \delta$ for every $\ell \in L$, then there exist a word c over k and a finite sequence $(w_n)_{n=0}^{m-1}$ of variable words over k of pattern $(p_n)_{n=0}^{m-1}$ such that*

$$\{c\} \cup \big\{ c^\frown w_0(a_0)^\frown \dots ^\frown w_n(a_n) : n \in \{0, \dots, m-1\} \text{ and } a_0, \dots, a_n \in [k] \big\} \subseteq A.$$

The least N with the above property will be denoted by $\mathrm{DP}(k, (\tau_n)_{n=0}^{m-1}, \delta)$.

The proof of Theorem 11.1 will be given in §11.4. All necessary tools (besides, of course, Theorem B) are developed in the previous subsections. The corresponding infinite versions—that is, Theorems A and C—will be proved in §11.5. Finally, in §11.6 we discuss how one can derive the density Hales–Jewett Theorem and the density Halpern–Läuchli Theorem from Theorems B and A respectively.

11.1. Sparse sets and regularity. We begin with the following definition.

Definition 11.2. Let $\tau \in \mathbb{N}$ with $\tau \geq 1$ and L be a nonempty subset of $\mathbb{N}$. We say that L is τ-*sparse* if for every $l, l' \in L$ with $l \neq l'$ we have $|l' - l| \geq \tau$. If L is a τ-sparse subset of $\mathbb{N}$, then we define the τ-*extension* of L to be the set

$$(L)_\tau = L + \{0, \dots, \tau - 1\} = \{l + n : l \in L \text{ and } 0 \leq n \leq \tau - 1\}.$$

Every nonempty subset of $\mathbb{N}$ is 1-sparse and coincides with its 1-extension. Also every subset of $\mathbb{N}$ of cardinality at least $\tau(m-1)+1$ contains a τ-sparse subset of cardinality m. Finally, the notion of τ-sparseness is hereditary, that is, every nonempty subset of a τ-sparse set is τ-sparse.

Much of our interest in sparse sets is related to the following generalized version of Definition 3.1.

Definition 11.3. Let $k \in \mathbb{N}$ with $k \geq 2$ and let $\mathcal{F}$ be a family of subsets of $[k]^{<\mathbb{N}}$. Also let $0 < \varepsilon \leq 1$, $\tau \in \mathbb{N}$ with $\tau \geq 1$ and let L be a τ-sparse finite subset of $\mathbb{N}$. The family $\mathcal{F}$ will be called (ε, τ, L)*-regular* if for every $n \in L$, every $I \subseteq \{l \in L : l < n\}$ and every $y \in [k]^{(I)_\tau}$ we have

$$\Big|\mathrm{dens}\big(\{z \in [k]^{\{m\in\mathbb{N}:\, m<n\}\setminus(I)_\tau} : (y,z) \in A\}\big) - \mathrm{dens}(A \cap [k]^n)\Big| \leq \varepsilon.$$

We have the following analogue of Lemma 3.2. It is the main result of this subsection.

Lemma 11.4. *Let* $0 < \varepsilon \leq 1$ *and* $k, \ell, q, \tau \in \mathbb{N}$ *with* $k \geq 2$ *and* $\ell, q, \tau \geq 1$. *Then there exists an integer n with the following property. If N is a finite* τ*-sparse subset of* $\mathbb{N}$ *with* $|N| \geq n$ *and* $\mathcal{F}$ *is a family of subsets of* $[k]^{<\mathbb{N}}$ *with* $|\mathcal{F}| = q$*, then there exists a subset L of N with* $|L| = \ell$ *such that* $\mathcal{F}$ *is* (ε, τ, L)*-regular. The least integer n with this property will be denoted by* $\mathrm{Reg}_\tau(k, \ell, q, \varepsilon)$.

The proof of Lemma 11.4 is similar to the proof of Lemma 3.2 and is based on the following consequence of Sublemma 3.7.

Corollary 11.5. *Let* $k, m, \tau, q \in \mathbb{N}$ *with* $k \geq 2$ *and* $\tau, q \geq 1$. *Let* $0 < \varepsilon < k^{-\tau(m+1)}$ *and N be a finite* τ*-sparse subset of* $\mathbb{N}$ *with*

$$|N| \geq (q\lfloor 16\varepsilon^{-4}\rfloor + 1)(m+1) + 1.$$

Finally, let $\mathcal{F}$ *be a family of subsets of* $[k]^{\max(N)}$ *with* $|\mathcal{F}| = q$. *Then there exists a subinterval M of* $N \setminus \{\max(N)\}$ *with* $|M| = m$ *such that for every* $A \in \mathcal{F}$*, every subset I of* $(M)_\tau$ *and every* $y \in [k]^I$ *we have* $|\mathrm{dens}(A_y) - \mathrm{dens}(A)| \leq \varepsilon$.

Proof. We set $N' = N \setminus \{\max(N)\}$. Observe that

$$|(N')_\tau \cup \{\max(N)\}| = \tau(N-1) + 1 \geq \tau(m+1)(q\lfloor 16\varepsilon^{-4}\rfloor + 1) + 1.$$

By Sublemma 3.7, there exists a subinterval M' of $(N')_\tau$ with $|M'| = \tau(m+1)$ such that for every $A \in \mathcal{F}$, every subset I of M' and every $y \in [k]^I$ we have

$$|\mathrm{dens}(A_y) - \mathrm{dens}(A)| \leq \varepsilon.$$

Since M' is a subinterval of $(N')_\tau$ of cardinality $\tau(m+1)$, it is possible to select a subinterval M of N' of cardinality m such that $(M)_\tau \subseteq M'$. Clearly M is as desired. □

We are ready to proceed to the proof of Lemma 11.4.

Proof of Lemma 11.4. We set $\varrho = \min\{\varepsilon, k^{-\tau(\ell+1)}/2\}$ and we define $\tilde{F} : \mathbb{N} \to \mathbb{N}$ by

$$\tilde{F}(m) = (q\lfloor 16\varrho^{-4}\rfloor + 1)(m+1) + 1.$$

Arguing precisely as in the proof of Lemma 3.2 and using Corollary 11.5 instead of Sublemma 3.7, we see that $\mathrm{Reg}_\tau(k,\ell,q,\varepsilon) \le \tilde{F}^{(\ell)}(0)$, and the proof is complete. □

11.2. The (p,L)-restriction of $[k]^{<\mathbb{N}}$. We are about to introduce a family of subsets of $[k]^{<\mathbb{N}}$ which are the analogues of Carlson–Simpson trees in the context of variable words of a fixed pattern p.

Definition 11.6. Let $k,\tau\in\mathbb{N}$ with $k\ge 2$ and $\tau\ge 1$. Let p be a variable word over k of length τ and $L=\{l_0<\dots<l_{|L|-1}\}$ a τ-sparse finite subset of $\mathbb{N}$. Recursively, for every $i\in\{0,\dots,|L|-1\}$ we define a subset $R_{p,L}(i)$ of $[k]^{l_i}$ as follows. We set $R_{p,L}(0)=[k]^{l_0}$. Assume that $R_{p,L}(i)$ has been defined for some $i\in\{0,\dots,|L|-2\}$. Then we set

$$R_{p,L}(i+1)=\{x^\frown p(a)^\frown y : x\in R_{p,L}(i),\ a\in[k] \text{ and } y\in[k]^{l_{i+1}-l_i-\tau}\}.$$

We define the (p,L)*-restriction* of $[k]^{<\mathbb{N}}$ to be the set

$$R_{p,L}=\bigcup_{i=0}^{|L|-1} R_{p,L}(i).$$

Notice that $R_{p,L}$ is a rather "thin" subset of $[k]^{<\mathbb{N}}$. So, if we are given a subset A of $[k]^{<\mathbb{N}}$ it is likely that the density of A inside $R_{p,L}$ will be negligible. This phenomenon, however, does not occur for A sufficiently regular. In particular, we have the following lemma, which is a straightforward consequence of the relevant definitions.

Lemma 11.7. *Let $k,\tau\in\mathbb{N}$ with $k\ge 2$ and $\tau\ge 1$, p be a variable word over k of length τ and $L=\{l_0<\dots<l_{|L|-1}\}$ a τ-sparse finite subset of $\mathbb{N}$. Let $0<\varepsilon\le 1$ and $\mathcal{F}$ be a family of subsets of $[k]^{<\mathbb{N}}$ which is (ε,τ,L)-regular. Then*

$$|\mathrm{dens}_{R_{p,L}(i)}(A)-\mathrm{dens}_{[k]^{l_i}}(A)|\le\varepsilon$$

for every $i\in\{0,\dots,|L|-1\}$ and every $A\in\mathcal{F}$.

We will need to parameterize the (p,L)-restriction $R_{p,L}$ of $[k]^{<\mathbb{N}}$ in a "canonical" way. This is, essentially, the content of the following definition.

Definition 11.8. Let $k,\tau\in\mathbb{N}$ with $k\ge 2$ and $\tau\ge 1$. Let p be a variable word over k of length τ and $L=\{l_0<\dots<l_{|L|-1}\}$ a τ-sparse finite subset of $\mathbb{N}$. For every $i\in\{0,\dots,|L|-1\}$ we set $l'_i=l_i-i\tau+i$ and we define

$$L^{(\tau)}=\{l'_i : i=0,\dots,|L|-1\}. \tag{11.1}$$

Recursively we define a bijection

$$\Phi_{p,L}:\bigcup_{l\in L^{(\tau)}}[k]^l\to R_{p,L}$$

as follows. For every $x\in[k]^{l'_0}=[k]^{l_0}$ we set $\Phi_{p,L}(x)=x$. Let $i\in\{0,\dots,|L|-2\}$ and assume that $\Phi_{p,L}(y)$ has been defined for every $y\in[k]^{l'_i}$. Then for every $x\in[k]^{l'_{i+1}}$ we define

$$\Phi_{p,L}(x)=\Phi_{p,L}(x_1)^\frown p(a_x)^\frown x_2$$

where $x_1=(x(0),\dots,x(l'_i-1))$, $a_x=x(l'_i)$ and $x_2=(x(l'_i+1),\dots,x(l'_{i+1}-1))$.

We isolate, for future use, some elementary properties of the map $\Phi_{p,L}$.

Lemma 11.9. *Let $k, \tau \in \mathbb{N}$ with $k \geq 2$ and $\tau \geq 1$. Let p be a variable word over k of length τ and $L = \{l_0 < \cdots < l_{|L|-1}\}$ a τ-sparse finite subset of $\mathbb{N}$. Let $L^{(r)} = \{l'_i : i = 0, \dots, |L|-1\}$ be as in (11.1). Then:*

(i) *For every $i \in \{0, \dots, |L|-1\}$ we have $\Phi_{p,L}([k]^{l'_i}) = R_{p,L}(i)$.*
(ii) *For every Carlson–Simpson line W of $[k]^{<\mathbb{N}}$ with $L(W) \subseteq L^{(\tau)}$ its image $\Phi_{p,L}(W)$ is of the form $\{c\} \cup \{c^\frown w(a) : a \in [k]\}$ where c is a word over k and w is a variable word over k of pattern p.*
(iii) *If $\mathcal{F}$ is a family of subsets of $[k]^{<\mathbb{N}}$ which is (ε, τ, L)-regular for some $0 < \varepsilon \leq 1$, then for every $i \in \{0, \dots, |L|-1\}$ and every $A \in \mathcal{F}$ we have*

$$|\mathrm{dens}_{[k]^{l'_i}}(\Phi^{-1}_{p,L}(A)) - \mathrm{dens}_{[k]^{l_i}}(A)| \leq \varepsilon.$$

Parts (i) and (ii) are immediate consequences of Definition 11.8. Part (iii) follows easily from Lemma 11.7. We leave the details to the reader.

11.3. Preliminary lemmas. As in (7.5) and (7.6), for every $k \in \mathbb{N}$ with $k \geq 2$ and every $0 < \delta \leq 1$ we set

$$\Lambda(k, 1, \delta) = \lceil \delta^{-1} \mathrm{DCS}(k, 1, \delta) \rceil, \tag{11.2}$$

$$\Theta(k, 1, \delta) = \frac{2\delta}{|\mathrm{Subtr}_1([k]^{<\Lambda(k,1,\delta)})|}. \tag{11.3}$$

We have the following analogue of Lemma 7.9.

Lemma 11.10. *Let $k \in \mathbb{N}$ with $k \geq 2$ and $0 < \rho, \gamma \leq 1$. Also let $\tau \in \mathbb{N}$ with $\tau \geq 1$ and M be a finite subset of $\mathbb{N}$ with*

$$|M| \geq \tau \cdot \mathrm{Reg}_\tau\big(k, \Lambda(k, 1, \rho\gamma/8), 1, \rho/2\big) \tag{11.4}$$

where $\Lambda(k, 1, \rho\gamma/8)$ is as in (11.2). Finally let $B \subseteq [k]^{<\mathbb{N}}$ with $\mathrm{dens}_{[k]^n}(B) \geq \rho$ for every $n \in M$. If $\{A_t : t \in B\}$ is a family of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_t) \geq \gamma$ for every $t \in B$, then for every variable word p over k of length τ there exist a word c over k and a variable word w over k of pattern p such that, setting $V = \{c\} \cup \{c^\frown w(a) : a \in [k]\}$, we have $V \subseteq B$ and

$$\mu\Big(\bigcap_{t \in V} A_t\Big) \geq \Theta(k, 1, \rho\gamma/8)$$

where $\Theta(k, 1, \rho\gamma/8)$ is as in (11.3).

Proof. By (11.4), we may select a τ-sparse subset N of M with

$$|N| \geq \mathrm{Reg}_\tau\big(k, \Lambda(k, 1, \rho\gamma/8), 1, \rho/2\big).$$

By Lemma 11.4, there exists a subset L of N with $|L| = \Lambda(k, 1, \rho\gamma/8)$ such that the family $\mathcal{F} := \{B\}$ is $(\rho/2, \tau, L)$-regular.

Fix a variable word p over k of length τ. Let $L^{(\tau)}$ and $\Phi_{p,L}$ be as in Definition 11.8. We set $B' = \Phi_{p,L}^{-1}(B)$. Since the singleton $\{B\}$ is $(\rho/2, \tau, L)$-regular and $\mathrm{dens}_{[k]^l}(B) \geq \rho$ for every $l \in L$, by Lemma 11.9(iii) we get $\mathrm{dens}_{[k]^{l'}}(B') \geq \rho/2$ for every $l' \in L^{(\tau)}$. Next for every $t' \in B'$ let $C_{t'} = A_{\Phi_{p,L}(t')}$. By our assumptions, $\mu(A_t) \geq \gamma$ for every $t \in B$, and so $\mu(C_{t'}) \geq \gamma$ for every $t' \in B'$. Finally notice that $|L^{(\tau)}| = |L| = \Lambda(k, 1, \rho\gamma/8)$. By the previous discussion, we may apply Lemma 7.9 and we obtain a Carlson–Simpson line $V' \subseteq B'$ with $L(V') \subseteq L^{(\tau)}$ and such that

$$\mu\Big(\bigcap_{t' \in V'} C_{t'}\Big) \geq \Theta(k, 1, \rho\gamma/8). \tag{11.5}$$

We set $V = \Phi_{p,L}(V')$. By Lemma 11.9(iii) and (11.5), we see that V is as desired. □

To state the next result we need to introduce some numerical invariants. For every $k, \tau \in \mathbb{N}$ with $k \geq 2$ and $\tau \geq 1$ and every $0 < \delta \leq 1$ we set

$$\Lambda_P = \Lambda_P(k, \delta) = \Lambda(k, 1, \delta^2/32) \quad \text{and} \quad \Theta_P = \Theta_P(k, \delta) = \Theta(k, 1, \delta^2/32) \tag{11.6}$$

and we define $h_{\tau,\delta} : \mathbb{N} \to \mathbb{N}$ by

$$h_{\tau,\delta}(n) = \tau \cdot \mathrm{Reg}_\tau(k, \Lambda_P, 1, \delta/4) + \lceil 2\Theta_P^{-1} \cdot n \rceil. \tag{11.7}$$

The following proposition corresponds to Proposition 7.5.

Proposition 11.11. *Let $k \in \mathbb{N}$ with $k \geq 2$ and $0 < \delta \leq 1$ and define Λ_P and Θ_P as in (11.6). Also let $\tau \in \mathbb{N}$ with $\tau \geq 1$, L be a nonempty finite subset of $\mathbb{N}$ and A be a subset of $[k]^{<\mathbb{N}}$ such that $\mathrm{dens}_{[k]^l}(A) \geq \delta$ for every $l \in L$. Finally let $n \in \mathbb{N}$ with $n \geq 1$ and assume that $|L| \geq h_{\tau,\delta}(n)$ where $h_{\tau,\delta}$ is as in (11.7). Let L_0 be the set of the first Λ_P elements of L. Then either*

(i) *there exist a subset L' of $L \setminus L_0$ with $|L'| \geq n$ and a word $t_0 \in [k]^{\ell_0}$ for some $\ell_0 \in L_0$ such that*
$$\mathrm{dens}_{[k]^{\ell-\ell_0}}(\{s \in [k]^{<\mathbb{N}} : t_0^\frown s \in A\}) \geq \delta + \delta^2/8$$
for every $\ell \in L'$, or

(ii) *for every variable word p over k of length τ there exist a word c over k, a variable word w over k of pattern p and a subset L'' of $L \setminus L_0$ with $|L''| \geq n$ such that:*
   (a) *$V = \{c\} \cup \{c^\frown w(a) : a \in [k]\} \subseteq \bigcup_{\ell \in L_0} A \cap [k]^\ell$.*
   (b) *Setting $V(1) = \{c^\frown w(a) : a \in [k]\}$ and letting ℓ_1 be the unique integer with $V(1) \subseteq [k]^{\ell_1}$, for every $\ell \in L''$ we have*
$$\mathrm{dens}_{[k]^{\ell-\ell_1}}\big(\{s \in [k]^{<\mathbb{N}} : t^\frown s \in A \text{ for every } t \in V(1)\}\big) \geq \Theta_P/2.$$

The proof of Proposition 11.11 is identical to the proof of Proposition 7.5, using Lemma 11.10 instead of Lemma 7.9. The details are left to the reader.

We close this subsection with the following consequence of Proposition 11.11. It is the main tool for the proof of Theorem C.

Corollary 11.12. *Let $k \in \mathbb{N}$ with $k \geq 2$ and $0 < \delta \leq 1$. Also let L be an infinite subset of $\mathbb{N}$ and $A \subset [k]^{<\mathbb{N}}$ be such that $\mathrm{dens}_{[k]^l}(A) \geq \delta$ for every $l \in L$. Then for every variable word p over k there exist a word c over k, a variable word w over k of pattern p and an infinite subset L' of L with the following properties:*

(i) *$V = \{c\} \cup \{c^\frown w(a) : a \in [k]\} \subseteq \bigcup_{\ell \in L_0} A \cap [k]^\ell$ where $L_0 = \{\ell \in L : \ell < \min(L')\}$.*
(ii) *Setting $V(1) = \{c^\frown w(a) : a \in [k]\}$ and letting ℓ_1 be the unique integer such that $V(1) \subseteq [k]^{\ell_1}$, for every $\ell \in L'$ we have*

$$\mathrm{dens}_{[k]^{\ell-\ell_1}}\big(\{s \in [k]^{<\mathbb{N}} : t^\frown s \in A \text{ for every } t \in V(1)\}\big) \geq 2^{-1}\Theta_P(k, \delta/2)$$

where $\Theta_P(k, \delta/2)$ is as in (11.6).

Proof. For every $t \in [k]^{<\mathbb{N}}$ let $A_t = \{s \in [k]^{<\mathbb{N}} : t^\frown s \in A\}$ and define

$$\delta_t = \limsup_{\ell \in L} \mathrm{dens}_{[k]^{\ell-|t|}}(A_t).$$

We set $\delta^* = \sup_{t \in [k]^{<\mathbb{N}}} \delta_t$ and we notice that $\delta \leq \delta^* \leq 1$. Hence, we may select $0 < \delta_0 \leq 1$, $t_0 \in [k]^{<\mathbb{N}}$ and an infinite subset M of L with $\min(M) > |t_0|$ such that

$$\delta/2 < \delta_0 < \delta^* < \delta_0 + \delta_0^2/8 \tag{11.8}$$

and $\delta_0 < \mathrm{dens}_{[k]^{\ell-|t_0|}}(A_{t_0})$ for every $\ell \in M$.

Fix a variable word p over k and denote by τ its length. Let M_0 be the initial segment of M of cardinality

$$|M_0| = \tau \cdot \mathrm{Reg}_\tau(k, \Lambda_P(k, \delta_0), 1, \delta/4). \tag{11.9}$$

By the definition of δ^* and (11.8), there exists $q \in M \setminus M_0$ such that for every $s \in \bigcup_{\ell \in M_0} [k]^{\ell-|t_0|}$ and every $\ell \in M$ with $\ell \geq q$ we have

$$\mathrm{dens}_{[k]^{\ell-|t_0^\frown s|}}(A_{t_0^\frown s}) < \delta_0 + \delta_0^2/8. \tag{11.10}$$

We set $d = \lceil 2\Theta_P(k, \delta_0)^{-1} \rceil$ and we select a sequence (E_n) of pairwise disjoint subsets of $\{m \in M : m \geq q\}$ such that $|E_n| = d$ for every $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. We set $F_n = M_0 \cup E_n$ and we observe that

$$|F_n| = |M_0| + |E_n| \overset{(11.9)}{=} \tau \cdot \mathrm{Reg}_\tau(k, \Lambda_P(k, \delta_0), 1, \delta/4) + d \overset{(11.7)}{=} h_{\tau,\delta_0}(1). \tag{11.11}$$

Hence we may apply Proposition 11.11. Notice that the first alternative of Proposition 11.11 contradicts (11.10). Hence, there exist a word c_n over k, a variable word w_n over k of pattern p and $m_n \in E_n$ such that

$$V_n = \{c_n\} \cup \{c_n^\frown w_n(a) : a \in [k]\} \subseteq \bigcup_{\ell \in M_0} A_{t_0} \cap [k]^{\ell-|t_0|} \tag{11.12}$$

and, setting $Q_n = \{s \in [k]^{<\mathbb{N}} : t^\frown s \in A_{t_0} \text{ for every } t \in V_n(1)\}$,

$$\mathrm{dens}_{[k]^{m_n-\ell_n-|t_0|}}(Q_n) \geq 2^{-1}\Theta_P(k, \delta_0) \overset{(11.8)}{\geq} 2^{-1}\Theta_P(k, \delta/2) \tag{11.13}$$

where $V_n(1) = \{c_n^\frown w_n(a) : a \in [k]\}$ and ℓ_n is the unique integer with $V_n(1) \subseteq [k]^{\ell_n}$.

By the classical pigeonhole principle, there exists an infinite subset N of $\mathbb{N}$, a word c' over k and a variable word w over k of pattern p such that $c_n = c'$ and $w_n = w$ for every $n \in N$. We set $L' = \{m_n : n \in N\}$ and $c = t_0^\frown c'$. Using (11.12) and (11.13) it is easy to see that L', c and w are as desired. □

11.4. Proof of Theorem 11.1. The proof proceeds by induction on m. For $m = 1$ we notice that

$$\mathrm{DP}(k,\tau,\delta) \leq \tau \cdot \mathrm{Reg}_\tau(\tau, \Lambda(k,1,\delta/8), 1, \delta/2) \tag{11.14}$$

for every $0 < \delta \leq 1$ and every $k, \tau \in \mathbb{N}$ with $k \geq 2$ and $\tau \geq 1$. Indeed, let M be a finite subset of $\mathbb{N}$ with $|M| \geq \tau \cdot \mathrm{Reg}_\tau(k, \Lambda(k,1,\delta/8), 1, \delta/2)$ and A be a subset of $[k]^{<\mathbb{N}}$ such that $\mathrm{dens}_{[k]^n}(A) \geq \delta$ for every $n \in M$. Let (Ω, Σ, μ) be an arbitrary probability space and set $A_t = \Omega$ for every $t \in A$. By Lemma 11.10 applied for $\rho = \delta$, $\gamma = 1$ and $B = A$, we see that (11.14) is satisfied.

Let $m \in \mathbb{N}$ with $m \geq 1$ and assume that for every integer $k \geq 2$, every $0 < \beta \leq 1$ and every finite sequence $(\sigma_n)_{n=0}^{m-1}$ of positive integers the number $\mathrm{DP}(k, (\sigma_n)_{n=0}^{m-1}, \beta)$ has been defined. Let $k \geq 2$, $0 < \delta \leq 1$ and $\tau_0, \dots, \tau_m \in \mathbb{N}$ with $\tau_0, \dots, \tau_m \geq 1$. We set $\tau'_n = \tau_{n+1}$ for every $n \in \{0, \dots, m-1\}$ and

$$N_0 = \mathrm{DP}(k, (\tau'_n)_{n=0}^{m-1}, \Theta_P/2). \tag{11.15}$$

We claim that

$$\mathrm{DP}(k, (\tau_n)_{n=0}^{m}, \delta) \leq h_{\tau_0,\delta}^{(\lceil 8\delta^{-2}\rceil)}(N_0). \tag{11.16}$$

Clearly this will finish the proof. To see that (11.16) is satisfied let L be a finite subset of $\mathbb{N}$ with $|L| \geq h_{\tau_0,\delta}^{(\lceil 8\delta^{-2}\rceil)}(N_0)$ and A be a subset of $[k]^{<\mathbb{N}}$ with $\mathrm{dens}_{[k]^l}(A) \geq \delta$ for every $l \in L$. Also fix a finite sequence $(p_n)_{n=0}^{m}$ of variable words over k with $|p_n| = \tau_n$ for every $n \in \{0, \dots, m\}$. By our assumptions on the cardinality of the set L and repeated application of Proposition 11.11 for $\tau = \tau_0$ and $p = p_0$, we see that there exist a word c over k, a variable word w over k of pattern p_0 and a subset L'' of L with

$$|L''| \geq N_0 \overset{(11.15)}{=} \mathrm{DP}(k, (\tau'_n)_{n=0}^{m-1}, \Theta_P/2) \tag{11.17}$$

such that the following properties are satisfied:

(a) $V = \{c\} \cup \{c^\frown w(a) : a \in [k]\} \subseteq A$. Moreover, setting $V(1) = \{c^\frown w(a) : a \in [k]\}$ and letting ℓ_1 be the unique integer with $V(1) \subseteq [k]^{\ell_1}$, we have $\ell_1 < \min(L'')$.

(b) For every $\ell \in L''$ we have

$$\mathrm{dens}_{[k]^{\ell-\ell_1}}(\Delta) \geq \Theta_P/2 \tag{11.18}$$

where $\Delta = \{s \in [k]^{<\mathbb{N}} : t^\frown s \in A \text{ for every } t \in V(1)\}$.

For every $n \in \{0, \dots, m-1\}$ we set $p'_n = p_{n+1}$ and we notice that the length of p'_n is τ'_n. Therefore, by (11.17) and (11.18) and our inductive assumptions, there exist a word c' over k and a finite sequence $(w'_n)_{n=0}^{m-1}$ of variable words over k of pattern $(p'_n)_{n=0}^{m-1}$ such that

$$\{c'\} \cup \big\{c^\frown w'_0(a_0)^\frown \dots ^\frown w'_n(a_n) : n \in \{0, \dots, m-1\} \text{ and } a_0, \dots, a_n \in [k]\big\} \subseteq \Delta.$$

We set $w_0 = w^\frown c'$ and $w_n = w'_{n-1}$ for every $n \in [m]$. It is easily verified that

$$\{c\} \cup \big\{c^\frown w_0(a_0)^\frown \dots ^\frown w_n(a_n) : n \in \{0, \dots, m\} \text{ and } a_0, \dots, a_n \in [k]\big\} \subseteq A.$$

This shows that (11.16) is satisfied and so the proof of Theorem 11.1 is complete.

11.5. Proofs of Theorems A and C. As indicated in the introduction, Theorem A is a special case of Theorem C. Indeed, let $k \in \mathbb{N}$ with $k \geq 2$ and set $q_n = (v)$ for every $n \in \mathbb{N}$. Notice that a sequence (w_n) of variable words over k consists of left variable words if and only if it is of pattern (q_n). Thus, Theorem A follows from Theorem C applied to (q_n).

So, we only need to prove Theorem C. To this end, fix an integer $k \geq 2$ and a sequence (p_n) of variable words over k. Let $0 < \delta \leq 1$ and $A \subseteq [k]^{<\mathbb{N}}$ such that

$$\limsup_{n\to\infty} |A \cap [k]^n|/k^n > \delta.$$

We fix an infinite subset L of $\mathbb{N}$ such that $\mathrm{dens}_{[k]^\ell}(A) \geq \delta$ for every $\ell \in L$. Recursively, we define a sequence (δ_n) in $(0, 1]$ by

$$\delta_0 = \delta \quad \text{and} \quad \delta_{n+1} = 2^{-1}\Theta_P(k, \delta_n/2).$$

Using Corollary 11.12 we may select

(i) a sequence (c_n) of words over k,
(ii) a sequence (v_n) of variable words over k of pattern (p_n),
(iii) a sequence (A_n) of subsets of $[k]^{<\mathbb{N}}$ with $A_0 = A$, and
(iv) two sequences (L_n) and (L'_n) of infinite subsets of $\mathbb{N}$

such that for every $n \in \mathbb{N}$ the following conditions are satisfied:

(C1) $L'_n \subseteq L_n$; moreover, $L_0 = L$.
(C2) $V_n = \{c_n\} \cup \{c_n^\frown v_n(a) : a \in [k]\} \subseteq \bigcup_{\ell \in L_n^0} A_n \cap [k]^\ell$ where $L_n^0 = \{\ell \in L_n : \ell < \min(L'_n)\}$.
(C3) Let $V_n(1) = \{c_n^\frown v_n(a) : a \in [k]\}$ and ℓ_n be the unique integer with $V_n(1) \subseteq [k]^{\ell_n}$. Then

$$A_{n+1} = \{s \in [k]^{<\mathbb{N}} : t^\frown s \in A_n \text{ for every } t \in V_n(1)\},$$
$$L_{n+1} = L'_n - \ell_n = \{\ell - \ell_n : \ell \in L'_n\}.$$

(C4) For every $\ell \in L_n$ we have $\mathrm{dens}_{[k]^\ell}(A_n) \geq \delta_n$.

The recursive selection is fairly standard and the details are left to the reader.

We set $c = c_0$ and $w_n = v_n^\frown c_{n+1}$ for every $n \in \mathbb{N}$. By (ii) above, the sequence (w_n) is of pattern (p_n). Moreover, using conditions (C2) and (C3), it is easily verified that

$$\{c\} \cup \{c^\frown w_0(a_0)^\frown \dots ^\frown w_n(a_n) : n \in \mathbb{N} \text{ and } a_0, \dots, a_n \in [k]\} \subseteq A.$$

The proof of Theorem C is complete.

11.6. Further implications. In this subsection we will discuss the relation of Theorems B and A to the density Hales–Jewett Theorem and the density Halpern–Läuchli Theorem respectively. Notice first that the density Hales–Jewett Theorem follows from Theorem A via a standard compactness argument. In fact, we have the following finer quantitative information.

Proposition 11.13. *For every integer $k \geq 2$ and every $0 < \delta \leq 1$ we have*

$$\mathrm{DHJ}(k, \delta) \leq \mathrm{DCS}(k, 1, \delta). \tag{11.19}$$

Proof. Let $n \geq \mathrm{DCS}(k, 1, \delta)$ and fix a subset A of $[k]^n$ with $|A| \geq \delta k^n$. For every $\ell \in [n]$ and every $y \in [k]^{n-\ell}$ let $A_y = \{x \in [k]^\ell : y^\frown x \in A\}$ and observe that

$$\mathbb{E}_{y \in [k]^{n-\ell}} \mathrm{dens}(A_y) = \mathrm{dens}(A) \geq \delta.$$

Hence, for every $\ell \in [n]$ we may select $y_\ell \in [k]^{n-\ell}$ such that $\mathrm{dens}(A_{y_\ell}) \geq \delta$. We set

$$B = \bigcup_{\ell \in [n]} A_{y_\ell}$$

and we notice that $\mathrm{dens}_{[k]^\ell}(B) \geq \delta$ for every $\ell \in [n]$. Since $n \geq \mathrm{DCS}(k, 1, \delta)$, there exists a Carlson–Simpson line R of $[k]^{<\mathbb{N}}$ which is contained in B. Let (c, w) be the generating sequence of R. Also let $\ell_R \in [n]$ be the unique integer such that the 1-level $R(1)$ of R is contained in $[k]^{\ell_R}$. Then, setting $V = \{y_{\ell_R}^\frown c^\frown w(a) : a \in [k]\}$, we see that V is a combinatorial line of $[k]^n$ and $V \subseteq A$. This shows that (11.19) is satisfied, as desired. □

We proceed to discuss how one can deduce the density Halpern–Läuchli Theorem from Theorem A. The argument is well-known (see, e.g., [6, 29]) but we will comment on it for the benefit of the reader.

Recall that a *tree* is a partially ordered set $(T, <)$ such that the set $\{s \in T : s < t\}$ is finite and linearly ordered under $<$ for every $t \in T$. A tree T is said to be *homogeneous* if it is uniquely rooted and there exists an integer $b \geq 2$, called the *branching number* of T, such that every $t \in T$ has exactly b immediate successors. A typical example of a homogeneous tree with branching number $b \geq 2$ is the set consisting of all finite sequence having values in a set $\mathbb{A}$ of cardinality b and equipped with the partial order of end-extension; it is denoted by $\mathbb{A}^{<\mathbb{N}}$ and can, of course, be identified with $[b]^{<\mathbb{N}}$. Part of the interest in homogeneous trees of this form is based on the fact that they can be used to "code" the level product

$$\otimes(T_1, \ldots, T_d) := \bigcup_{n \in \mathbb{N}} T_1(n) \times \cdots \times T_d(n)$$

of a finite sequence $(T_1, \ldots, T_d)$ of homogeneous trees. Specifically, we have the following lemma.

Lemma 11.14. *Let $d \in \mathbb{N}$ with $d \geq 1$. Also let $\mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{N}^d$ with $b_i \geq 2$ for every $i \in [d]$ and set $\mathbb{A}_\mathbf{b} = [b_1] \times \cdots \times [b_d]$. Finally let $(T_1, \ldots, T_d)$ be a finite sequence of homogeneous trees such that the branching number of T_i is b_i for every $i \in [d]$. Then there exists a bijection*

$$\Phi_\mathbf{b} : \mathbb{A}_\mathbf{b}^{<\mathbb{N}} \to \otimes(T_1, \ldots, T_d)$$

with the following properties:

(i) *For every* $n \in \mathbb{N}$ *we have* $\Phi_{\mathbf{b}}(\mathbb{A}_{\mathbf{b}}^n) = T_1(n) \times \cdots \times T_d(n)$.

(ii) *For every* $c \in \mathbb{A}_{\mathbf{b}}^{<\mathbb{N}}$ *and every sequence* (w_n) *of left variable words over* $\mathbb{A}_{\mathbf{b}}$ *there exist strong subtrees* $(S_1, \dots, S_d)$ *of* $(T_1, \dots, T_d)$ *having a common level set such that, setting*

$$\mathcal{S} = \{c\} \cup \{c^\frown w_0(a_0)^\frown \dots ^\frown w_n(a_n) : n \in \mathbb{N} \text{ and } a_0, \dots, a_n \in \mathbb{A}_{\mathbf{b}}\},$$

we have $\Phi_{\mathbf{b}}(\mathcal{S}) = \otimes(S_1, \dots, S_d)$.

Proof. Let $i \in [d]$ and $\pi_i : \mathbb{A}_{\mathbf{b}} \to [b_i]$ be the natural projection. Clearly we may assume that the tree T_i coincides with $[b_i]^{<\mathbb{N}}$ and so we may consider the "extension" $\bar{\pi}_i : \mathbb{A}_{\mathbf{b}}^{<\mathbb{N}} \to T_i$ of π_i defined by $\bar{\pi}_i(\emptyset) = \emptyset$ and

$$\bar{\pi}_i((a_0, \dots, a_{n-1})) = (\pi_i(a_0), \dots, \pi_i(a_{n-1}))$$

for every integer $n \geq 1$ and every $(a_0, \dots, a_{n-1}) \in \mathbb{A}_{\mathbf{b}}^n$. The map $\Phi_{\mathbf{b}}$ is then defined by $\Phi_{\mathbf{b}}(s) = (\bar{\pi}_1(s), \dots, \bar{\pi}_d(s))$. It is easily verified that $\Phi_{\mathbf{b}}$ is a bijection and has all the desired properties. □

With Lemma 11.14 at our disposal, let us see how Theorem A yields the density Halpern–Läuchli Theorem. To this end, fix a finite sequence $(T_1, \dots, T_d)$ of homogeneous trees and a subset A of the level product of $(T_1, \dots, T_d)$ such that

$$\limsup_{n\to\infty} \frac{|A \cap (T_1(n) \times \cdots \times T_d(n))|}{|T_1(n) \times \cdots \times T_d(n)|} > 0. \tag{11.20}$$

Let $\mathbf{b} = (b_1, \dots, b_d)$ where b_i is the brunching number of the tree T_i for every $i \in [d]$ and consider the bijection $\Phi_{\mathbf{b}}$ obtained by Lemma 11.14. We set $B = \Phi_{\mathbf{b}}^{-1}(A)$. By Lemma 11.14(i) and (11.20), we see that

$$\limsup_{n\to\infty} \frac{|B \cap \mathbb{A}_{\mathbf{b}}^n|}{|\mathbb{A}_{\mathbf{b}}|^n} > 0.$$

Hence, by Theorem A, there exist $c \in \mathbb{A}_{\mathbf{b}}^{<\mathbb{N}}$ and a sequence (w_n) of left variable words over $\mathbb{A}_{\mathbf{b}}$ such that

$$\{c\} \cup \{c^\frown w_0(a_0)^\frown \dots ^\frown w_n(a_n) : n \in \mathbb{N} \text{ and } a_0, \dots, a_n \in \mathbb{A}_{\mathbf{b}}\} \subseteq B.$$

Invoking the definition of B and Lemma 11.14(ii), we conclude that there exist strong subtrees $(S_1, \dots, S_d)$ of $(T_1, \dots, T_d)$ having a common level set such that the level product of $(S_1, \dots, S_d)$ is contained in A.

References

[1] Ajtai, M., Szemerédi, E.: Sets of lattice points that form no squares. Studia Sci. Math. Hungar. **9**, 9–11 (1974) Zbl 0303.10046 MR 0369299

[2] Bergelson, V., Blass, A., Hindman, N.: Partition theorems for spaces of variable words. Proc. London Math. Soc. **68**, 449–476 (1994) Zbl 0809.04005 MR 1262304

[3] Bergelson, V., Leibman, A.: Set-polynomials and polynomial extension of the Hales–Jewett theorem. Ann. of Math. **150**, 33–75 (1999) Zbl 0969.05061 MR 1715320

[4] Bicker, R., Voigt, B.: Density theorems for finitistic trees. Combinatorica **3**, 305–313 (1983) Zbl 0528.05002 MR 0729783

[5] Carlson, T. J.: Some unifying principles in Ramsey theory. Discrete Math. **68**, 117–169 (1988) Zbl 0817.04002 MR 0926120

[6] Carlson, T. J., Simpson, S. G.: A dual form of Ramsey's theorem. Adv. Math. **53**, 265–290 (1984) Zbl 0564.05005 MR 0753869

[7] Conlon, D., Gowers, W. T.: Combinatorial theorems in sparse random sets. Preprint (2010)

[8] Dodos, P., Kanellopoulos, V., Karagiannis, N.: A density version of the Halpern–Läuchli theorem. Adv. Math. **244**, 955–978 (2013). Zbl 1282.05209 MR 3077894

[9] Dodos, P., Kanellopoulos, V., Tyros, K.: Dense subsets of products of finite trees. Int. Math. Res. Notices **2013**, 924–970 MR 3024269

[10] Dodos, P., Kanellopoulos, V., Tyros, K.: Measurable events indexed by products of trees. Combinatorica **34**, 427–470 (2014) MR 3259812

[11] Dodos, P., Kanellopoulos, V., Tyros, K.: A simple proof of the density Hales–Jewett theorem. Int. Math. Res. Notices **2014**, 3340–3352 Zbl 06340375 MR 3217664

[12] Erdős, P., Hajnal, A.: Some remarks on set theory, IX. Combinatorial problems in measure theory and set theory. Michigan Math. J. **11**, 107–127 (1964) Zbl 0199.02302 MR 0171713

[13] Furstenberg, H., Katznelson, Y.: An ergodic Szemerédi theorem for commuting transformations. J. Anal. Math. **31**, 275–291 (1978) Zbl 0426.28014 MR 0531279

[14] Furstenberg, H., Katznelson, Y.: Idempotents in compact semigroups and Ramsey theory. Israel J. Math. **68**, 257–270 (1989) Zbl 0714.05059 MR 1039473

[15] Furstenberg, H., Katznelson, Y.: A density version of the Hales–Jewett theorem. J. Anal. Math. **57**, 64–119 (1991) Zbl 0770.05097 MR 1191743

[16] Furstenberg, H., Weiss, B.: Markov processes and Ramsey theory for trees. Combin. Probab. Comput. **12**, 547–563 (2003) Zbl 1061.05094 MR 2037069

[17] Gowers, W. T.: Hypergraph regularity and the multidimensional Szemerédi theorem. Ann. of Math. **166**, 897–946 (2007) Zbl 1159.05052 MR 2373376

[18] Gowers, W. T.: Polymath and the density Hales–Jewett theorem. In: An Irregular Mind: Szemerédi is 70, Springer, New York, 659–687 (2010) Zbl 1223.05305 MR 2815619

[19] Graham, R. L., Rothschild, B. L.: Ramsey's theorem for n-parameter sets. Trans. Amer. Math. Soc. **159**, 257–292 (1971) Zbl 0233.05003 MR 0284352

[20] Hales, A. H., Jewett, R. I.: Regularity and positional games. Trans. Amer. Math. Soc. **106**, 222–229 (1963) Zbl 0113.14802 MR 0143712

[21] Halpern, J. D., Läuchli, H.: A partition theorem. Trans. Amer. Math. Soc. **124**, 360–367 (1966) Zbl 0158.26902 MR 0200172

[22] Hindman, N., McCutcheon, R.: Partition theorems for left and right variable words. Combinatorica **24**, 271–286 (2004) Zbl 1070.05079 MR 2071335

[23] McCutcheon, R.: Elemental Methods in Ergodic Ramsey Theory. Lecture Notes in Math. 1722, Springer (1999) Zbl 0945.05063 MR 1738544

[24] McCutcheon, R.: Two new extensions of the Hales–Jewett theorem. Electron. J. Combin. **7**, R49 (2000) Zbl 0957.05104 MR 1785145

[25] Milliken, K.: A Ramsey theorem for trees. J. Combin. Theory Ser. A **26**, 215–237 (1979) Zbl 0429.05035 MR 0535155

[26] Milliken, K.: A partition theorem for the infinite subtrees of a tree. Trans. Amer. Math. Soc. **263**, 137–148 (1981) Zbl 0471.05026 MR 0590416

[27] Pach, J., Solymosi, J., Tardos, G.: Remarks on a Ramsey theory for trees. Combinatorica **32**, 473–482 (2012) Zbl 1289.05452 MR 2965287

[28] Polymath, D. H. J.: A new proof of the density Hales–Jewett theorem. Ann. of Math. **175**, 1283–1327 (2012) Zbl 1267.11010 MR 2912706

[29] Prömel, H. J., Voigt, B.: Graham–Rothschild parameter sets. In: Mathematics of Ramsey Theory, Springer, Berlin, 113–149 (1990) Zbl 0741.05075 MR 0957064

[30] Rödl, V., Nagle, B., Skokan, J., Schacht, M., Kohayakawa, Y.: The hypergraph regularity method and its applications. Proc. Nat. Acad. Sci. USA **102**, 8109–8113 (2005) Zbl 1135.05307 MR 2167756

[31] Roth, K. F.: On certain sets of integers. J. London Math. Soc. **28**, 104–109 (1953) Zbl 0050.04002 MR 0051853

[32] Schacht, M.: Extremal results for random discrete structures. Preprint (2010)

[33] Shelah, S.: Primitive recursive bounds for van der Waerden numbers. J. Amer. Math. Soc. **1**, 683–697 (1988) Zbl 0649.05010 MR 0929498

[34] Shelah, S.: A partition theorem. Sci. Math. Japon. **56**, 413–438 (2002) Zbl 1020.05067 MR 1922806

[35] Sperner, E.: Ein Satz über Untermengen einer endlichen Menge. Math. Z. **27**, 544–548 (1928) JFM 54.0090.06 MR 1544925

[36] Szemerédi, E.: On sets of integers containing no k elements in arithmetic progression. Acta Arith. **27**, 199–245 (1975) Zbl 0303.10056 MR 0369312

[37] Szemerédi, E.: Regular partitions of graphs. In: Proc. Colloque Internat. CNRS (J.-C. Bermond et al., eds.), CNRS, Paris, 399–401 (1978) Zbl 0413.05055 MR 0540024

[38] van der Waerden, B. L.: Beweis einer Baudetschen Vermutung. Nieuw. Arch. Wisk. **15**, 212–216 (1927) JFM 53.0073.12

[39] Varnavides, P.: On certain sets of positive density. J. London Math. Soc. **34**, 358–360 (1959) Zbl 0088.25702 MR 0106865

[40] Walters, M.: Combinatorial proofs of the polynomial van der Waerden and the polynomial Hales–Jewett theorem. J. London Math. Soc. **61**, 1–12 (2000) Zbl 0948.05056 MR 1745405