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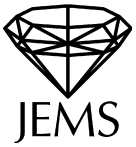

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The Cauchy problem for the two-dimensional Euler–Poisson system

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Abstract. The Euler–Poisson system is a fundamental two-fluid model to describe the dynamics of the plasma consisting of compressible electrons and a uniform ion background. In the 3D case Guo [7] first constructed a global smooth irrotational solution by using the dispersive Klein–Gordon effect. It has been conjectured that the same result should hold in the two-dimensional case. In our recent work [13], we proved the existence of a family of smooth solutions by constructing the wave operators for the 2D system. In this work we completely settle the 2D Cauchy problem.

Keywords. Euler–Poisson, Klein–Gordon, normal form, global well-posedness

1. Introduction

The Euler–Poisson system is one of the simplest two-fluid models used to describe the dynamics of a plasma consisting of moving electrons and ions. In this model the heavy ions are assumed to be immobile and uniformly distributed in space, providing only a background of positive charge. The light electrons are modeled as a charged compressible fluid moving against the ionic forces. Neglecting magnetic effects, the governing dynamics of the electron fluid is given by the following Euler–Poisson system in $(t, x) \in [0, \infty) \times \mathbb{R}^d$:

$$\begin{cases} \partial_t n + \nabla \cdot (n\mathbf{u}) = 0, \\ m_e n(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}) + \nabla p(n) = en\nabla\phi, \\ \Delta\phi = 4\pi e(n - n_0). \end{cases} \tag{1.1}$$

Here $n = n(t, x)$ and $\mathbf{u} = \mathbf{u}(t, x)$ denote the density and average velocities of the electrons respectively. The symbol e and m_e denote the unit charge and mass of electrons. The pressure term $p(n)$ is assumed to obey the *polytropic γ-law*, i.e.

$$p(n) = An^\gamma, \tag{1.2}$$

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where A is the entropy constant and $\gamma \geq 1$ is called the *adiabatic index*. The term $en\nabla\phi = (-ne)\cdot(-\nabla\phi)$ quantifies the electric force acting on the electron fluid by the positive ion background. Note that the electrons carry negative charge $-ne$. We assume that at equilibrium the density of ions and electrons are both a constant denoted by n_0. To ensure charge neutrality it is natural to impose the condition

$$\int_{\mathbb{R}^d} (n - n_0)\, dx = 0.$$

The boundary condition for the electric potential ϕ is decay at infinity, i.e.

$$\lim_{|x|\to\infty} \phi(t, x) = 0. \tag{1.3}$$

The first and second equations in (1.1) represent the mass conservation and momentum balance of the electron fluid respectively. The third equation in (1.1) is the usual Gauss law in electrostatics. It computes the electric potential self-consistently through the charge distribution $n_0 e - ne$. The Euler–Poisson system is one of the simplest two-fluid models in the sense that the ions are treated as uniformly distributed sources in space and they appear only as a constant n_0 in the Poisson equation. This is a very physical approximation since $m_{\text{ion}} \gg m_e$ and the heavy ions move much more slowly than the light electrons.

Throughout the rest of this paper, we shall consider an irrotational flow,

$$\nabla \times \mathbf{u} = 0, \tag{1.4}$$

which is preserved in time. For flows with nonzero curl the magnetic field is no longer negligible and it is more physical to consider the full Euler-Maxwell system.

We are interested in constructing a smooth global solution around the equilibrium $(n, \mathbf{u}) \equiv (n_0, 0)$. To do this we first transform the system (1.1) in terms of certain perturbed variables. For simplicity set all physical constants e, m_e, 4π and A to be 1. To simplify the presentation, we also set $\gamma = 3$ although other cases of γ can be easily treated as well. Define the rescaled functions

$$u(t, x) = \frac{n(t/c_0, x) - n_0}{n_0}, \qquad \mathbf{v}(t, x) = \frac{1}{c_0}\mathbf{u}(t/c_0, x), \qquad \psi(t, x) = 3\phi(t/c_0, x),$$

where the sound speed is $c_0 = \sqrt{3}\, n_0$. For convenience we set $n_0 = 1/3$ so that the characteristic wave speed is unity. The Euler–Poisson system (1.1) in the new variables takes the form

$$\begin{cases} \partial_t u + \nabla\cdot\mathbf{v} + \nabla\cdot(u\mathbf{v}) = 0, \\ \partial_t \mathbf{v} + \nabla u + \nabla\big(\frac{1}{2}u^2 + \frac{1}{2}|\mathbf{v}|^2\big) = \nabla\psi, \\ \Delta\psi = u. \end{cases} \tag{1.5}$$

Taking one more time derivative and using (1.4) then transforms (1.5) into the following quasi-linear Klein–Gordon system:

$$\begin{cases} (\Box + 1)u = \Delta\big(\frac{1}{2}u^2 + \frac{1}{2}|\mathbf{v}|^2\big) - \partial_t\nabla\cdot(u\mathbf{v}), \\ (\Box + 1)\mathbf{v} = -\partial_t\nabla\big(\frac{1}{2}u^2 + \frac{1}{2}|\mathbf{v}|^2\big) + (1 - \Delta^{-1})\nabla\nabla\cdot(u\mathbf{v}). \end{cases} \tag{1.6}$$

For the above system, in the 3D case Guo [7] first constructed a global smooth irrotational solution by using dispersive Klein–Gordon effect and adapting Shatah's normal form method. It has been conjectured that the same results should hold in the two-dimensional case. In our recent work [13], we proved the existence of a family of smooth solutions by constructing the wave operators for the 2D system. The 2D problem with radial data was studied in [12]. Note that for radial data,[1] one has

$$\Delta^{-1}\nabla\nabla\cdot(u\mathbf{v}) = u\mathbf{v},$$

and the result follows easily from [18].

In this work we completely settle the 2D Cauchy problem for general non-radial data. The approach we take is inspired by a new set-up of normal form transformation developed by Gustafson, Nakanishi, Tsai [10] and also Germain, Masmoudi and Shatah [4, 5, 3]. Roughly speaking (and over-simplifying quite a bit), the philosophy of the normal form method is that *one should integrate by parts whenever one can in either (frequency) space or time*. The part where one cannot integrate by parts is called the set of space-time resonances which can often be controlled by some finer analysis provided the set is not too large or satisfies some frequency separation properties. The implementation of such ideas is often challenging and depends heavily on the problem under study. In fact the heart of the whole analysis is to choose appropriate function spaces utilizing the fine structure of the equations. The main obstructions in the 2D Euler–Poisson system are slow (nonintegrable) $\langle t\rangle^{-1}$ dispersion, quasilinearity and nonlocality caused by the Riesz transform. Nevertheless we overcome all such difficulties in this paper. Shortly after our work was completed, a similar result requiring at least 30+ derivatives was obtained in [11]. To put things in perspective, we review below some related literature as well as some technical developments on this problem.

The main difficulty in constructing time-global smooth solutions for the Euler–Poisson system comes from the fact that the Euler–Poisson system is a hyperbolic conservation law with zero dissipation for which no general theory is available. The "Euler" part of the Euler–Poisson system is the well-known compressible Euler equations. Indeed in (1.1), if the electric field term $\nabla\phi$ is dropped, one recovers the usual Euler equations for compressible fluids. In [21], Sideris considered the 3D compressible Euler equation for a classical polytropic ideal gas with adiabatic index $\gamma > 1$. For a class of initial data which coincide with a constant state outside a ball, he proved that the lifespan of the corresponding C^1 solution must be finite. In [19] Rammaha extended this result to the 2D case. For the Euler–Poisson system, Guo and Tahvildar-Zadeh [9] established a "Siderian" blow-up result for spherically symmetric initial data. Recently Chae and Tadmor [1] proved finite-time blow-up for C^1 solutions of a class of pressureless attractive Euler–Poisson equations in $\mathbb{R}^n$, $n \geq 1$. These negative results show the abundance of shock waves for large solutions.

The "Poisson" part of the Euler–Poisson system has a stabilizing effect which makes the whole analysis of (1.1) quite different from the pure compressible Euler equations. This is best understood in analyzing small irrotational perturbations of the equilibrium

[1] The vector function $\mathbf{v}$ is *radial* if it is the gradient of a scalar radial function.

state $n \equiv n_0$, $\mathbf{u} \equiv 0$. For the 3D compressible Euler equation with irrotational initial data $(n_\epsilon(0), \mathbf{u}_\epsilon(0)) = (\epsilon\rho_0 + n_0, \epsilon\mathbf{v}_0)$, where $\rho_0 \in \mathcal{S}(\mathbb{R}^3)$, $\mathbf{v}_0 \in \mathcal{S}(\mathbb{R}^3)^3$ are fixed functions (with ϵ sufficiently small), Sideris [22] proved that the lifespan T_ϵ of the classical solution satisfies $T_\epsilon > \exp(C/\epsilon)$. As for the upper bound, it follows from Sideris's previous paper [21] that $T_\epsilon < \exp(C/\epsilon^2)$. Sharper results were obtained by Godin [6] who showed that for radial initial data like a smooth compact ϵ-perturbation of the constant state, the precise asymptotic of the lifespan T_ϵ is exponential in the sense that

$$\lim_{\epsilon \to 0+} \epsilon \log T_\epsilon = T^*,$$

where T^* is a constant. All these results rely crucially on the observation that after some simple reductions, the compressible Euler equation in rescaled variables is given by a vectorial nonlinear wave equation with pure quadratic nonlinearities. The linear part of the wave equation decays at most as $t^{-(d-1)/2}$, which in 3D is not integrable. Unless the nonlinearity has some additional nice structure such as the null condition [2, 15], one cannot in general expect global existence of small solutions. On the other hand, the situation for the Euler–Poisson system (1.1) is quite different due to the additional Poisson coupling term. As was already explained before, the Euler–Poisson system (1.1) expressed in rescaled variables is given by the quasi-linear Klein–Gordon system (1.6) for which the linear solutions have an enhanced decay of $(1 + t)^{-d/2}$. This is in sharp contrast with the pure Euler case for which the decay is only $t^{-(d-1)/2}$. Note that for $d = 3$, $(1 + t)^{-d/2} = (1 + t)^{-3/2}$, which is integrable in t. In a seminal paper [7], by exploiting the crucial decay property of the Klein–Gordon flow in 3D, Guo modified Shatah's normal form method [20] and constructed a smooth irrotational global solution to (1.1) around the equilibrium state $(n_0, 0)$ for which the perturbations decay at a rate of $C_p \cdot (1 + t)^{-p}$ for any $1 < p < 3/2$ (here C_p denotes a constant depending on the parameter p). Note in particular that the sharp decay $t^{-3/2}$ is marginally missed here due to a technical complication caused by the nonlocal Riesz operator in the nonlinearity.

Construction of smooth global solutions to (1.1) in the two-dimensional case has been an open problem since Guo's work. The first obstacle comes from slow dispersion since the linear solution to the Klein–Gordon system in $d = 2$ decays only as $(1 + t)^{-1}$, which is not integrable, in particular making the strategy of [7] difficult to apply. The other main technical difficulty comes from the nonlocal nonlinearity in (1.6) which involves a Riesz-type singular operator. For general scalar quasi-linear Klein–Gordon equations in 3D with quadratic type nonlinearities, global small smooth solutions were first constructed independently by Klainerman [14] using the invariant vector field method and by Shatah [20] using a normal form method. Even in 3D there are essential technical difficulties in employing Klainerman's invariant vector field method due to the Riesz type nonlocal term in (1.6). The Klainerman invariant vector fields consist of infinitesimal generators which commute well with the linear operator $\partial_{tt} - \Delta + 1$. The most problematic part comes from the Lorentz boost $\Omega_{0j} = t\partial_{x_j} + x_j\partial_t$. While the first part $t\partial_{x_j}$ commutes naturally with the Riesz operator $R_{ij} = (-\Delta)^{-1}\partial_{x_i}\partial_{x_j}$, the second part $x_j\partial_t$ interacts rather badly with R_{ij}, producing a commutator which scales as

$$[x_j\partial_t, R_{ij}] \sim \partial_t|\nabla|^{-1}.$$

After repeated commutation of these operators one in general obtains terms of the form $|\nabla|^{-N}$, which makes the low frequency part of the solution out of control. It is for this reason that in the 3D case Guo [7] adopted Shatah's method of normal form in the L^p $(p > 1)$ setting for which the Riesz term R_{ij} causes no trouble.

We turn now to the 2D Klein–Gordon equations with pure quadratic nonlinearities. In this case, direct applications of either Klainerman's invariant vector field method or Shatah's normal form method are not possible since the linear solutions only decay as $(1 + t)^{-1}$, which is not integrable and makes the quadratic nonlinearity quite resonant. In [23], Simon and Taflin constructed wave operators for the 2D semilinear Klein–Gordon system with quadratic nonlinearities. In [18], Ozawa, Tsutaya and Tsutsumi considered the Cauchy problem and constructed smooth global solutions by first transforming the quadratic nonlinearity into a cubic one using Shatah's normal form method and then applying Klainerman's invariant vector field method to obtain decay of intermediate norms. Due to the nonlocal complication with the Lorentz boost which we explained earlier, this approach seems difficult to apply to the 2D Euler–Poisson system.

As was already mentioned, the purpose of this work is to settle the Cauchy problem for (1.1) in the two-dimensional case. Before we state our main results, we need to make some further simplifications. Since $\mathbf{v}$ is irrotational, we can write $\mathbf{v} = \nabla\phi_1$ and obtain from (1.5) (here $\langle\nabla\rangle = \sqrt{1-\Delta}$, see (2.1))

$$\begin{cases} \partial_t u + \Delta\phi_1 + \nabla\cdot(u\nabla\phi_1) = 0, \\ \partial_t\phi_1 + |\nabla|^{-2}\langle\nabla\rangle^2 u + \frac{1}{2}(u^2 + |\nabla\phi_1|^2) = 0. \end{cases} \tag{1.7}$$

We can diagonalize the system (1.7) by introducing the complex scalar function

$$h(t) = \frac{\langle\nabla\rangle}{|\nabla|}u - i|\nabla|\phi_1 = \frac{\langle\nabla\rangle}{|\nabla|}u + i\frac{\nabla}{|\nabla|}\cdot\mathbf{v}. \tag{1.8}$$

Note that since $\mathbf{v}$ is irrotational, we have

$$\mathbf{v} = -\frac{\nabla}{|\nabla|}\operatorname{Im}(h). \tag{1.9}$$

By (1.5), we have

$$h(t) = e^{it\langle\nabla\rangle}h_0 + \int_0^t e^{i(t-s)\langle\nabla\rangle}\left(-\frac{\langle\nabla\rangle\nabla}{|\nabla|}\cdot(u\mathbf{v}) + \frac{i}{2}|\nabla|(u^2 + |\mathbf{v}|^2)\right)ds, \tag{1.10}$$

where h_0 is the initial data given by

$$h_0 = \frac{\langle\nabla\rangle}{|\nabla|}u_0 + i\frac{\nabla}{|\nabla|}\cdot\mathbf{v}_0.$$

Here u_0 is the initial density (perturbation) and $\mathbf{v}_0$ is the initial velocity.

For $T \geq 0$, $\delta > 0$, $N \geq 8$, $N' = N - 3/2$, we introduce the norms

$$\begin{aligned} \|h\|_{\tilde{X}_T} := {} & \|\langle t\rangle|\nabla|^{\delta}\langle\nabla\rangle h(t)\|_{L^\infty_{t,x}([0,T])} + \|\langle t\rangle^{1-2\delta}\langle\nabla\rangle h(t)\|_{L^\infty_t L^{1/\delta}_x([0,T])} \\ & + \|x(1-\Delta)e^{-it\langle\nabla\rangle}h(t)\|_{L^\infty_t L^{2+\delta}_x([0,T])} \end{aligned}$$

and

$$\|h\|_{X_T} := \|h\|_{\tilde{X}_T} + \|h(t)\|_{C_t^0 H^{N'}([0,T])} + \|\langle t\rangle^{-\delta} h(t)\|_{C_t^0 H^N([0,T])}.$$

Here for simplicity we have suppressed the notational dependence of the X_T norm on δ. We will use the notation X_∞ (resp. $\tilde{X}_\infty$) when the norms are evaluated on the time interval $[0, \infty)$.

Our result is stated in the following

Theorem 1.1 (Smooth global solutions for the Cauchy problem). *There exists an absolute constant $\delta_* > 0$ sufficiently small such that the following hold:*

For any $0 < \delta < \delta_*$, there exists $\epsilon > 0$ sufficiently small such that if the initial data h_0 satisfies $\|e^{it\langle\nabla\rangle} h_0\|_{X_\infty} \le \epsilon$, then there exists a unique smooth global solution to the 2D Euler–Poisson system (1.8)–(1.10) satisfying $\|h\|_{X_\infty} \le \text{const} \cdot \epsilon$. Moreover the solution scatters in the energy space $H^{N'}$.

Remark 1.2. A simple inspection of our proof shows that it suffices to take $\delta_* = 1/500$. We do not make much effort to lower the regularity assumption ($N \ge 8$) on the initial data although the result here is already better than many existing methods. The main point here is to construct a smooth and global in time classical solution.

To prove Theorem 1.1, we shall establish an a priori estimate of the form

$$\|h\|_{X_t} \lesssim \|e^{i\tau\langle\nabla\rangle} h_0\|_{X_\infty} + \|h\|_{X_t}^2 + \|h\|_{X_t}^3 + \|h\|_{X_t}^4, \tag{1.11}$$

where the implied constant depends only on δ and N. The function can be shown to be continuous in t (see Step 2 below). By a standard continuity argument, if $\|e^{i\tau\langle\nabla\rangle} h_0\|_{X_\infty}$ is sufficiently small, then $\|h\|_{X_t}$ remains bounded for all $t \ge 0$, which yields global well-posedness easily. Therefore our main work is to show (1.11). We sketch its proof in the following steps.

Step 1: Preliminary transformations and normal form. In this step, we introduce $f(t) = e^{-it\langle\nabla\rangle} h(t)$ and rewrite (1.10) as

$$\hat{f}(t,\xi) = \hat{h}_0(\xi) + \int_0^t \int e^{-is\phi_0(\xi,\eta)} \langle\xi\rangle \frac{\xi}{|\xi|} \widehat{\mathcal{R}f}(s,\xi-\eta) \widehat{\mathcal{R}f}(s,\eta)\, d\eta\, ds, \tag{1.12}$$

where $\mathcal{R}$ is some Riesz-type operator and

$$\phi_0(\xi,\eta) = \langle\xi\rangle \pm \langle\xi-\eta\rangle \pm \langle\eta\rangle.$$

By using the fact that the Klein–Gordon phase $\phi_0(\xi,\eta)$ never vanishes, we perform a normal form transformation and integrate by parts in the time variable s. After some simplifications, we arrive at an equation of the form

$$\hat{f}(t,\xi) = \text{“initial data”} + \text{“quadratic boundary terms”} + \hat{f}_{\text{cubic}}(t,\xi),$$

where f_{cubic} is cubic in h and has the form $f_{\text{cubic}} = \mathcal{R} f_3$ with

$$\hat{f}_3(t,\xi) = \int_0^t \int e^{-is\phi(\xi,\eta,\sigma)} \frac{\langle\xi\rangle\cdot\langle\eta\rangle}{\phi_0(\xi,\eta)} \cdot \frac{\eta}{|\eta|} \widehat{\mathcal{R}f}(s,\xi-\eta)\cdot\widehat{\mathcal{R}f}(s,\eta-\sigma)\cdot\widehat{\mathcal{R}f}(s,\sigma)\, d\sigma\, d\eta\, ds. \tag{1.13}$$

Here

$$\phi(\xi,\eta,\sigma) = \langle\xi\rangle \pm \langle\xi-\eta\rangle \pm \langle\eta-\sigma\rangle \pm \langle\sigma\rangle.$$

The estimates of the initial data part and the boundary terms are given in Section 5.

Step 2: Local theory, continuity of the X norm along the flow and $H^{N'}$ estimation. First we carry out the (standard) H^N energy estimation and obtain an estimate of the form

$$\frac{d}{dt}(\|h(t)\|_{H^N}^2) \lesssim \big(\|u(t)\|_\infty + \|\nabla u(t)\|_\infty + \|\nabla \mathbf{v}(t)\|_\infty\big)\cdot \|h(t)\|_{H^N}^2.$$

The subtle point here is that $\|\mathbf{v}(t)\|_\infty$ does not appear in the energy estimate.

Due to the slow $(1/t)$ decay in 2D, we need to have a slight $\langle t\rangle^\delta$ growth of the norm $\|h(t)\|_{H^N}$ in order to close the estimates. Note that $u = \frac{|\nabla|}{\langle\nabla\rangle}\,\mathrm{Re}(h)$ and $\mathbf{v} = -\frac{\nabla}{|\nabla|}\,\mathrm{Im}(h)$, hence

$$\|u(t)\|_\infty + \|\nabla u(t)\|_\infty + \|\nabla \mathbf{v}(t)\|_\infty \lesssim \big\||\nabla|^\delta\langle\nabla\rangle h(t)\big\|_\infty.$$

It remains to prove the sharp $1/t$ decay of the L^∞-norm $\big\||\nabla|^\delta\langle\nabla\rangle h(t)\big\|_\infty$. For this and later estimates, we need to show the time-continuity of the norm $\|x(1-\Delta)e^{-it\langle\nabla\rangle}h(t)\|_{2+\delta}$. This is done in Section 4. The main idea there is a bootstrap estimate exploiting the finite speed propagation property of the Klein–Gordon flow. In the last part of Section 4, we complete the $H^{N'}$ estimation of h. To lower the regularity assumption, we first introduce frequency cut-offs $\chi_{\geq\langle s\rangle^{\delta_0}}$ and $\chi_{<\langle s\rangle^{\delta_0}}$ in (1.12). For the high frequency part, we estimate it using energy smoothing (recall $N' = N - 3/2$) and dispersive decay. For the low frequency piece, we use the normal form and obtain a cubic nonlinearity localized to low frequencies. The $H^{N'}$ estimate is used in controlling some boundary terms in Section 5.

Step 3: Reduction to low frequencies and the $(2+\delta)$-trick. This is an important step in controlling the X norm of h. We use a multiscale argument and introduce the parameter $\delta_0 = 20\delta$. We then decompose the cubic nonlinear term $f_{\text{cubic}} = \mathcal{R} f_3$ (see (1.13)) into two pieces:

$$\begin{aligned}\hat{f}_3(t,\xi) = \int_0^t \int e^{-is\phi(\xi,\eta,\sigma)} &\cdot \frac{\langle\xi\rangle\cdot\langle\eta\rangle}{\phi_0(\xi,\eta)} \cdot \frac{\eta}{|\eta|} \\ &\cdot (m_{\text{low}}(\xi,\eta,\sigma,s) + m_{\text{high}}(\xi,\eta,\sigma,s))\cdot \widehat{\mathcal{R}f}(s,\xi-\eta) \\ &\cdot \widehat{\mathcal{R}f}(s,\eta-\sigma)\cdot\widehat{\mathcal{R}f}(s,\sigma)\, d\sigma\, d\eta\, ds \\ =: \widehat{f_3^{(1)}} + \widehat{f_3^{(2)}},&\end{aligned}$$

where

$$\begin{aligned} m_{\text{low}}(\xi,\eta,\sigma,s) &= \chi_{|\xi-\eta|\leq\langle s\rangle^{\delta_0}}\cdot\chi_{|\eta-\sigma|\leq\langle s\rangle^{\delta_0}}\cdot\chi_{|\sigma|\leq\langle s\rangle^{\delta_0}}, \\ m_{\text{high}}(\xi,\eta,\sigma,s) &= 1 - m_{\text{low}}(\xi,\eta,\sigma,s).\end{aligned}$$

We first show that the high frequency piece has good decay properties, namely

$$\|e^{i\tau\langle\nabla\rangle}\mathcal{R}f_3^{(2)}(\tau)\|_{\tilde{X}_t} \lesssim \|h\|_{X_t}^3. \tag{1.14}$$

Thanks to the frequency cut-off m_{high}, we must have either $|\xi-\eta| \gtrsim \langle s\rangle^{\delta_0}$, $|\eta-\sigma| \gtrsim \langle s\rangle^{\delta_0}$, or $|\sigma| \gtrsim \langle s\rangle^{\delta_0}$. This frequency localization coupled with the energy norm and dispersive effects then produces strong decay estimates for the $\tilde{X}_t$ norm of $e^{i\tau\langle\nabla\rangle}\mathcal{R}f_3^{(2)}(\tau)$. By a delicate analysis we are able to prove (1.14) under the weak assumption that $N \geq 8$. We emphasize that this is the main place where the high derivative assumption is needed.

To control the X norm of the low frequency piece, we must estimate several quantities including $\big\||\nabla|^{\delta}\langle\nabla\rangle e^{i\tau\langle\nabla\rangle}\mathcal{R}f_3^{(1)}(\tau)\big\|_{\infty}$, $\|\langle\nabla\rangle e^{i\tau\langle\nabla\rangle}\mathcal{R}f_3^{(1)}(\tau)\|_{1/\delta}$, and $\|x(1-\Delta)\mathcal{R}f_3^{(1)}(\tau)\|_{2+\delta}$. To do this we show that all the above norms can be bounded by the L^{2-} norm of some weighted integral produced from f_3. More precisely, we show that

$$\|e^{i\tau\langle\nabla\rangle}\mathcal{R}f_3^{(1)}(\tau)\|_{\tilde{X}_t} \lesssim \|f_{\text{low}}(\tau)\|_{L_\tau^\infty L_x^{2-\delta/100}([0,t])} + \|h\|_{X_t}^3, \tag{1.15}$$

where

$$\begin{aligned}\hat{f}_{\text{low}}(t,\xi) = \int_0^t \int e^{-is\phi}\cdot\frac{s\partial_\xi\phi}{\phi_0(\xi,\eta)}\cdot\langle\xi\rangle^{4+2\delta}\cdot\langle\eta\rangle\cdot\frac{\eta}{|\eta|}\cdot m_{\text{low}}(\xi,\eta,\sigma,s)\\ \cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\cdot\widehat{\mathcal{R}f}(s,\eta-\sigma)\cdot\widehat{\mathcal{R}f}(s,\sigma)\,d\sigma\,d\eta\,ds.\end{aligned} \tag{1.16}$$

We stress that the choice of the norm $\|x(1-\Delta)e^{-it\langle\nabla\rangle}h(t)\|_{2+\delta}$ (the $2+\delta$ *trick*) comes from this part of analysis. In particular, when bounding the quantity $\|x\mathcal{R}f_3^{(1)}(\tau)\|_{2+\delta}$, we have to control the commutator

$$\|[x,\mathcal{R}]f_3^{(1)}(\tau)\|_{2+\delta} \sim \big\||\nabla|^{-1}f_3^{(1)}(\tau)\big\|_{2+\delta}.$$

This latter quantity can be bounded by $\|f_{\text{low}}(\tau)\|_{2-\delta/100}$ thanks to the assumption $\delta > 0$.

Step 4: Control of the low frequency piece. The goal is to prove the bound

$$\|f_{\text{low}}(\tau)\|_{L_\tau^\infty L_x^{2-\delta/100}([0,t])} \lesssim \|h\|_{X_t}^3 + \|h\|_{X_t}^4. \tag{1.17}$$

The main difficulty in establishing this bound is the slow $(1/\langle s\rangle)$ decay in (1.16). To see this point, we can perform a rough estimation as follows: the integral in (1.16) can be written as (see (2.2))

$$f_{\text{low}}(t) = \int_0^t s e^{-is\langle\nabla\rangle} T_{\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+\delta}}\big(P_{\lesssim\langle s\rangle^{\delta_0}}\mathcal{R}h, \mathcal{R}\big(P_{\lesssim\langle s\rangle^{\delta_0}}\mathcal{R}h\cdot P_{\lesssim\langle s\rangle^{\delta_0}}\mathcal{R}h\big)\big)\,ds.$$

Ignoring the linear flow ($e^{-is\langle\nabla\rangle}$) and multiplier issues for the moment, one has

$$\begin{aligned}\|f_{\text{low}}(t)\|_{2-\delta/100} &\lesssim \int_0^t \langle s\rangle\cdot\|h(s)\|_{2+}\|h(s)\|_{\infty-}^2\,ds \lesssim \int_0^t \langle s\rangle^{1-2(1-O(\delta))}\,ds\,\|h\|_{X_t}^3\\ &\lesssim \int_0^t \langle s\rangle^{-1+O(\delta)}\,ds\,\|h\|_{X_t}^3.\end{aligned} \tag{1.18}$$

Clearly this shows that the decay in s is not enough to make the above time integral converge. To resolve this difficulty we have to appeal to the specific form of the phase function $\phi = \phi(\xi, \eta, \sigma)$ in (1.16) and exploit some subtle cancelations in various cases. The main goal is to obtain a strong decay $\langle s\rangle^{-1-\epsilon+O(\delta)}$ with $\epsilon \gg O(\delta)$ in (1.18). For this we shall use some new ideas and devices, which are discussed below.

• **Hidden derivatives.** The first observation is that for phases of the form $\phi(\xi, \eta, \sigma) = \langle\xi\rangle - \langle\xi - \eta\rangle \pm \langle\eta - \sigma\rangle \pm \langle\sigma\rangle$, we have

$$\partial_\xi \phi = \frac{\xi}{\langle\xi\rangle} - \frac{\xi - \eta}{\langle\xi - \eta\rangle} = Q(\xi, \eta)\eta, \tag{1.19}$$

where Q is smooth in (ξ, η). For $|\eta| \lesssim \langle s\rangle^{-C\delta_0}$, the factor η in (1.19) corresponds to a derivative and produces an extra decay $\langle s\rangle^{-C\delta_0}$ which will be enough to make the time integral in (1.18) converge. Similarly for the phases $\phi(\xi, \eta, \sigma) = \langle\xi\rangle + \langle\xi - \eta\rangle \pm \langle\eta - \sigma\rangle \pm \langle\sigma\rangle$, the factor $\partial_\xi\phi$ will also produce an extra decay $\langle s\rangle^{-C\delta_0}$ in the low frequency regime $|\xi| \lesssim \langle s\rangle^{-C\delta_0}$, $|\eta| \lesssim \langle s\rangle^{-C\delta_0}$.

• **Normal form and the $\eta/|\eta|$ problem.** Consider phases of the form $\phi(\xi, \eta, \sigma) = \langle\xi\rangle + \langle\xi - \eta\rangle + \langle\eta - \sigma\rangle \pm \langle\sigma\rangle$. They have the property

$$\phi(\xi, \sigma, \sigma) \gtrsim \frac{1}{\langle\xi\rangle + \langle\xi - \eta\rangle + \langle\eta - \sigma\rangle + \langle\sigma\rangle}.$$

By using this fact we can integrate by parts in the variable s in (1.16). Dropping boundary terms, we arrive at an expression of the form

$$\begin{aligned}\hat{f}_{\text{low}}(t, \xi) \sim \int_0^t \int e^{-is\phi} \cdot \frac{s\partial_\xi\phi}{\phi_0(\xi, \eta)} \cdot \frac{\langle\xi\rangle^{4+2\delta}}{\phi(\xi, \eta, \sigma)} \cdot \langle\eta\rangle \cdot \frac{\eta}{|\eta|} \\ \cdot m_{\text{low}}(\xi, \eta, \sigma, s) \cdot \partial_s(\widehat{\mathcal{R}f}(s, \xi - \eta)) \cdot \widehat{\mathcal{R}f}(s, \eta - \sigma) \cdot \widehat{\mathcal{R}f}(s, \sigma)\, d\sigma\, d\eta\, ds \\ + \text{similar terms.}\end{aligned}$$

Note that by (1.12), $\partial_s(\widehat{\mathcal{R}f}) \sim O((\mathcal{R}f)^2)$, which is quadratic in f. By this fact one may hope to get $\langle s\rangle^{-2+O(\delta)}$ decay in (1.18). However this argument is only correct in the regime $|\eta| \gtrsim \langle s\rangle^{-\delta_0}$. In the low frequency regime $|\eta| \lesssim \langle s\rangle^{-\delta_0}$, the symbol $\frac{1}{\phi(\xi,\eta,\sigma)} \cdot \frac{\eta}{|\eta|}$ is no longer smooth and one has to deal with it separately.

• **Partial normal form transform.** To solve the $\eta/|\eta|$ problem, we will integrate by parts using only *part of the phase*, to which we refer as *partial normal form transform*. Consider for example the phase $\phi(\xi, \eta, \sigma) = \langle\xi\rangle + \langle\xi - \eta\rangle + \langle\eta - \sigma\rangle - \langle\sigma\rangle$. We use the identity

$$e^{-is(\langle\xi\rangle+\langle\xi-\eta\rangle)} = \frac{i}{\langle\xi\rangle + \langle\xi - \eta\rangle} \frac{\partial}{\partial s}(e^{-is(\langle\xi\rangle+\langle\xi-\eta\rangle)})$$

to do integration by parts in s. When the derivative ∂_s hits the term $e^{-is(\langle\eta-\sigma\rangle-\langle\sigma\rangle)}$, we obtain a factor $\langle\eta - \sigma\rangle - \langle\sigma\rangle \approx Q(\eta, \sigma)\eta$, which gains extra decay $\langle s\rangle^{-C\delta_0}$. When the

derivative hits the other terms, we obtain a quintic nonlinearity. Note that in this case all symbols are separable in the sense that they can be written as

$$\tilde{m}(\xi, \eta, \sigma) = a(\xi, \eta)b(\eta, \sigma)$$

for some functions a and b. The Riesz factor $\eta/|\eta|$ then causes no problem since we can deal with the multipliers corresponding to (ξ, η) and (η, σ) separately.

• **Transformation of phase derivatives and frequency separation.** Consider for example the phase $\phi(\xi, \eta, \sigma) = \langle\xi\rangle + \langle\xi - \eta\rangle - \langle\eta - \sigma\rangle - \langle\sigma\rangle$. By Lemma 2.8 we can write, for some smooth Q_1, Q_2,

$$\partial_\xi \phi = Q_1(\xi, \eta, \sigma)\partial_\eta \phi + Q_2(\xi, \eta, \sigma)\partial_\sigma \phi,$$
$$ise^{is\phi}\partial_\xi \phi = Q_1(\xi, \eta, \sigma)\partial_\eta(e^{is\phi}) + Q_2(\xi, \eta, \sigma)\partial_\sigma(e^{is\phi}).$$

Consequently, one can integrate by parts in η and σ respectively, which boosts the decay in s to $\langle s\rangle^{-2+O(\delta)}$. Note that there is still a subtle issue when we perform the above argument and integrate by parts in η. Namely the ∂_η derivative may hit the Riesz term $\eta/|\eta|$ and produce an operator $|\nabla|^{-1}$ which is hard to control for $|\eta| \lesssim \langle s\rangle^{-\delta_0}$. To solve this problem we have to do a multi-scale partition of the (ξ, η, σ)-phase space and discuss several subcases (cf. Subcases 3a to 3d in Case 3). In particular for the low frequency regime $|\eta| \lesssim \langle s\rangle^{-\delta_0}$, we have to discuss several situations and use the hidden derivatives and partial normal form together with several other tricks to treat these cases (see in particular Subcases 3a to 3c in Case 3). This part of the analysis is quite involved and uses the nonlinear structure in an essential way.

The above ideas together with some further delicate analysis complete the proof of Theorem 1.1. The rest of this paper is organized as follows. In Section 2 we gather some preliminary linear estimates. In Section 3 we perform some preliminary transformations and decompose the solution into three parts: the initial data, the boundary term g and the cubic interaction term f_{cubic}. In Section 4 we establish local theory, prove continuity of the X norm along the flow and give the $H^{N'}$ estimate of h. Section 5 is devoted to the estimation of the boundary terms g arising from the normal form transformation. In Section 6 we control the high frequency part of cubic interactions. In Section 7 we control the low frequency part of cubic interactions, which is the most delicate part of our analysis. In Section 8 we complete the proof of our main theorem.

2. Preliminaries

2.1. Some notation

We write $X \lesssim Y$ or $Y \gtrsim X$ to indicate $X \leq CY$ for some constant $C > 0$. We use $O(Y)$ to denote any quantity X such that $|X| \lesssim Y$. We use the notation $X \sim Y$ whenever $X \lesssim Y \lesssim X$. If C depends upon some additional parameters, we will indicate this with subscripts; for example, $X \lesssim_u Y$ denotes the assertion that $X \leq C_u Y$ for some C_u depending on u. Sometimes when the context is clear, we will suppress the dependence

on u and write $X \lesssim_u Y$ as $X \lesssim Y$. We will write $C = C(Y_1, \ldots, Y_n)$ to stress that the constant C depends on the quantities $Y_1, \ldots, Y_n$. We denote by $X\pm$ any quantity of the form $X \pm \epsilon$ for any $\epsilon > 0$.

We use the "Japanese bracket" convention $\langle x \rangle := (1 + |x|^2)^{1/2}$. It is convenient to use the notation $\langle \nabla \rangle = \sqrt{1 - \Delta}$ where

$$\widehat{\langle \nabla \rangle f}(\xi) = (1 + |\xi|^2)^{1/2} \hat{f}(\xi). \tag{2.1}$$

In a similar manner one can define $\langle \nabla \rangle^s$ and $|\nabla|^s$ for any $s \in \mathbb{R}$.

For any function f on $\mathbb{R}^d$, we shall use the notation $\|f\|_{L^p}$ or $\|f\|_p$ to denote the usual Lebesgue norm for $1 \le p \le \infty$.

We write $L^q_t L^r_x$ to denote the Banach space with norm

$$\|u\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} |u(t,x)|^r \, dx \right)^{q/r} dt \right)^{1/q},$$

with the usual modifications when q or r are equal to infinity, or when the domain $\mathbb{R} \times \mathbb{R}^d$ is replaced by a smaller region of spacetime such as $I \times \mathbb{R}^d$. When $q = r$ we abbreviate $L^q_t L^q_x$ as $L^q_{t,x}$.

We will use $\phi \in C^\infty(\mathbb{R}^d)$, a radial bump function supported in the ball $\{x \in \mathbb{R}^d : |x| \le 25/24\}$ and equal to one on the ball $\{x \in \mathbb{R}^d : |x| \le 1\}$. For any constant $C > 0$, we denote $\phi_{\le C}(x) := \phi(x/C)$ and $\phi_{>C} := 1 - \phi_{\le C}$. We also write $\chi_{|x|>C} = \chi_{>C} = \phi_{>C}$ (resp. $\chi_{|x|\le C}$) sometimes.

We will often need the Fourier multiplier operators defined by

$$\begin{aligned} \mathcal{F}\big(T_{m(\xi,\eta)}(f,g)\big)(\xi) &= \int m(\xi,\eta) \hat{f}(\xi-\eta) \hat{g}(\eta) \, d\eta, \\ \mathcal{F}\big(T_{m(\xi,\eta,\sigma)}(f,g,h)\big)(\xi) &= \int m(\xi,\eta,\sigma) \hat{f}(\xi-\eta) \hat{g}(\eta-\sigma) \hat{h}(\sigma) \, d\eta \, d\sigma. \end{aligned} \tag{2.2}$$

Similarly one can define $T_m(f_1, \ldots, f_n)$ for functions $f_1, \ldots, f_n$ and a general symbol $m = m(\xi, \eta_1, \ldots, \eta_{n-1})$.

2.2. *Basic harmonic analysis*

For each number $N > 0$, we define the Fourier multipliers

$$\begin{aligned} \widehat{P_{\le N} f}(\xi) &:= \phi_{\le N}(\xi) \hat{f}(\xi), \qquad \widehat{P_{>N} f}(\xi) := \phi_{>N}(\xi) \hat{f}(\xi), \\ \widehat{P_N f}(\xi) &:= (\phi_{\le N} - \phi_{\le N/2})(\xi) \hat{f}(\xi), \end{aligned}$$

and similarly for $P_{<N}$ and $P_{\ge N}$. We also define

$$P_{M<\cdot\le N} := P_{\le N} - P_{\le M} = \sum_{M<N'\le N} P_{N'}$$

whenever $M < N$. We will usually use these multipliers when M and N are *dyadic numbers* (that is, of the form 2^n for some integer n); in particular, all summations over N

or M are understood to be over dyadic numbers. Nevertheless, it will occasionally be convenient to allow M and N not to be a power of 2. As P_N is not truly a projection, $P_N^2 \neq P_N$, we will occasionally need to use fattened Littlewood–Paley operators

$$\tilde{P}_N := P_{N/2} + P_N + P_{2N}. \tag{2.3}$$

These obey $P_N \tilde{P}_N = \tilde{P}_N P_N = P_N$.

Like all Fourier multipliers, the Littlewood–Paley operators commute with the propagator $e^{it\Delta}$, as well as with differential operators such as $i\partial_t + \Delta$. We will use basic properties of these operators many times, including

Lemma 2.1 (Bernstein estimates). *For $1 \le p \le q \le \infty$,*

$$\begin{aligned} \big\| |\nabla|^{\pm s} P_M f \big\|_{L_x^p(\mathbb{R}^d)} &\sim M^{\pm s} \|P_M f\|_{L_x^p(\mathbb{R}^d)}, \\ \|P_{\le M} f\|_{L_x^q(\mathbb{R}^d)} &\lesssim M^{d/p-d/q} \|P_{\le M} f\|_{L_x^p(\mathbb{R}^d)}, \\ \|P_M f\|_{L_x^q(\mathbb{R}^d)} &\lesssim M^{d/p-d/q} \|P_M f\|_{L_x^p(\mathbb{R}^d)}. \end{aligned}$$

We shall repeatedly use the following lemma which allows us to commute the L^p estimates with the linear flow $e^{it\langle\nabla\rangle}$. Roughly speaking, it says that for $t \gtrsim 1$,

$$\|P_{<t^C} e^{it\langle\nabla\rangle} f\|_p \lesssim t^{0+} \|f\|_p, \quad p = 2+ \text{ or } p = 2-.$$

Lemma 2.2. *For any $1 \le p \le \infty$, $g \in L^p(\mathbb{R}^2)$ and dyadic $M > 0$, we have*

$$\|e^{it\langle\nabla\rangle} P_{<M} g\|_p \lesssim \langle Mt\rangle^{|1-2/p|} \|g\|_p. \tag{2.4}$$

Also for any $1 \le p \le \infty$, $s > |1-2/p|$, we have

$$\|e^{it\langle\nabla\rangle} g\|_p \lesssim \langle t\rangle^{|1-2/p|} \|\langle\nabla\rangle^s g\|_p. \tag{2.5}$$

In particular for any $0 \le \epsilon < 1$, we have

$$\|e^{it\langle\nabla\rangle} g\|_{2+\epsilon} \lesssim_\epsilon \langle t\rangle^{\epsilon/(2+\epsilon)} \|\langle\nabla\rangle^{\epsilon/2} g\|_{2+\epsilon}, \qquad \|e^{it\langle\nabla\rangle} g\|_{2-\epsilon} \lesssim_\epsilon \langle t\rangle^{\epsilon/(2-\epsilon)} \|\langle\nabla\rangle^{\epsilon} g\|_{2-\epsilon}. \tag{2.6}$$

Proof. We first prove (2.4). The idea is to use interpolation between $p = 1$, $p = 2$ and $p = \infty$. We consider only the case $p = \infty$. The other case $p = 1$ is similar. To establish the inequality it suffices to bound the L_x^1 norm of the kernel $e^{it\langle\nabla\rangle} P_{<M}$.

Note that $e^{it\langle\nabla\rangle} P_{<M} f = K * f$, where $\hat{K}(\xi) = e^{it\langle\xi\rangle}\phi(\xi/M)$. Observe $\|K\|_{L_x^2} \lesssim M$ and for $t > 0$,

$$\big\| |x|^2 K(x) \big\|_{L_x^2} = \|\partial_\xi^2(\hat{K}(\xi))\|_{L_\xi^2} \lesssim t^2 M + t + 1/M.$$

Then

$$\|K\|_{L_x^1} \lesssim \|K\|_{L_x^2}^{1/2} \big\| |x|^2 K \big\|_{L_x^2}^{1/2} \lesssim \langle Mt\rangle.$$

The desired inequality then follows from Young's inequality.

Next we show (2.5). By (2.4) and the inequality $\langle Mt\rangle \le \langle M\rangle\langle t\rangle$, we have

$$\begin{aligned}\|e^{it\langle\nabla\rangle}g\|_p &\lesssim \|e^{it\langle\nabla\rangle}P_{<1}g\|_p + \sum_{M>1}\|e^{it\langle\nabla\rangle}P_M g\|_p\\ &\lesssim \langle t\rangle^{|1-2/p|}\|g\|_p + \sum_{M>1} M^{|1-2/p|}\langle t\rangle^{|1-2/p|}\|P_M g\|_p \lesssim \langle t\rangle^{|1-2/p|}\|\langle\nabla\rangle^s g\|_p.\end{aligned}$$

□

Lemma 2.3. *Suppose $m = m(\xi,\eta) \in C^3(\mathbb{R}^2\times\mathbb{R}^2)$ satisfies*

$$|m| + |\partial_\xi^3 m| + |\partial_\eta^3 m| \in L^2_{\xi,\eta}(\mathbb{R}^2\times\mathbb{R}^2). \tag{2.7}$$

Then

$$\|T_m(f,g)\|_r \lesssim \|f\|_{p_1}\|g\|_{p_2} \tag{2.8}$$

for any $1/r = 1/p_1 + 1/p_2$, $1 \le r, p_1, p_2 \le \infty$.

Proof of Lemma 2.3. Let

$$K(x,y) = \frac{1}{(2\pi)^4}\int m(\xi,\eta)e^{i(x\cdot\xi+y\cdot\eta)}\,d\xi\,d\eta.$$

By (2.7), it is easy to check that

$$\begin{aligned}\|K\|_{L^1_{x,y}(\mathbb{R}^2\times\mathbb{R}^2)} &\lesssim \|(1+|x|^3+|y|^3)K(x,y)\|_{L^2_{x,y}(\mathbb{R}^2\times\mathbb{R}^2)}\\ &\lesssim \|m\|_{L^2_{\xi,\eta}(\mathbb{R}^2\times\mathbb{R}^2)} + \|\partial_\xi^3 m\|_{L^2_{\xi,\eta}(\mathbb{R}^2\times\mathbb{R}^2)} + \|\partial_\eta^3 m\|_{L^2_{\xi,\eta}(\mathbb{R}^2\times\mathbb{R}^2)} < \infty.\end{aligned}$$

Define

$$F(x,y) = \frac{1}{(2\pi)^4}\int m(\xi,\eta)\hat f(\xi-\eta)\hat g(\eta)e^{i(x\cdot\xi+y\cdot\eta)}\,d\xi\,d\eta.$$

By Fourier transform,

$$F(x,y) = \int K(x-x',y-y')h(x',y')\,dx'\,dy',$$

where

$$h(x',y') = \frac{1}{(2\pi)^4}\int \hat f(\xi-\eta)\hat g(\eta)e^{i(x'\cdot\xi+y'\cdot\eta)}\,d\xi\,d\eta = f(x')g(x'+y').$$

By Young's and Hölder's inequalities, we then have

$$\begin{aligned}\|(T_m(f,g))(x)\|_{L^r_x} &= \|F(x,0)\|_{L^r_x}\\ &\le \int\left\|\int K(x-x',y-y')f(x')g(x'+y')\,dx'\right\|_{L^r_x}dy'\\ &\le \int \|K(\cdot,y-y')\|_{L^1_x}\|f\|_{L^{p_1}_x}\|g\|_{L^{p_2}_x}\,dy' = \|K\|_{L^1_{x,y}}\|f\|_{p_1}\|g\|_{p_2}.\end{aligned}$$

□

By a similar proof we have

Corollary 2.4. *Suppose* $m = m(\xi, \eta, \sigma) \in C^4(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2)$ *satisfies*

$$\|m\|_{L^2_{\xi,\eta,\sigma}} + \|\partial_\xi^4 m\|_{L^2_{\xi,\eta,\sigma}} + \|\partial_\eta^4 m\|_{L^2_{\xi,\eta,\sigma}} + \|\partial_\sigma^4 m\|_{L^2_{\xi,\eta,\sigma}} \le A < \infty. \tag{2.9}$$

Then

$$\|T_m(f, g, h)\|_r \le C \cdot A \cdot \|f\|_{p_1} \cdot \|g\|_{p_2} \cdot \|h\|_{p_3}$$

for any $1/r = 1/p_1 + 1/p_2 + 1/p_3$, $1 \le r, p_1, p_2, p_3 \le \infty$. *Here* $C > 0$ *is an absolute constant.*

We shall need the following simple Sobolev embedding lemma.

Lemma 2.5. *Let the numbers* (r, p) *satisfy* $2 < r < \infty$, $r > p$, $p \ge (1/2 + 1/r)^{-1}$. *Then for any smooth* f *on* $\mathbb{R}^2$, *we have*

$$\big\| |\nabla|^{-1} f \big\|_r \lesssim \|\langle x\rangle f\|_p. \tag{2.10}$$

In particular, (2.10) *holds for any* $2 \le p < r < \infty$.

Proof of Lemma 2.5. We only need to prove (2.10). By Sobolev embedding and Hölder,

$$\big\| |\nabla|^{-1} f \big\|_r \lesssim \|f\|_{(1/2+1/r)^{-1}} \lesssim \|\langle x\rangle f\|_p \cdot \|\langle x\rangle^{-1}\|_{(1/2+1/r-1/p)^{-1}} \lesssim \|\langle x\rangle f\|_p. \quad \square$$

Lemma 2.6 (Bounds on the phase function). *Let* $\psi(x, y) = \frac{1}{\langle x\rangle + \langle y\rangle - \langle x+y\rangle}$ *for* $x, y \in \mathbb{R}^2$. *Then*

$$|\partial_x^\alpha \partial_y^\beta \psi(x, y)| \lesssim_{\alpha,\beta} \min\{\langle x\rangle, \langle y\rangle, \langle x + y\rangle\}, \qquad \forall x, y \in \mathbb{R}^2. \tag{2.11}$$

Proof. Write

$$\begin{aligned} \psi(x, y) &= \frac{\langle x\rangle + \langle y\rangle + \langle x + y\rangle}{(\langle x\rangle + \langle y\rangle)^2 - (\langle x + y\rangle)^2} \\ &= \frac{\langle x\rangle + \langle y\rangle + \langle x + y\rangle}{1 + 2(\langle x\rangle\langle y\rangle - x \cdot y)} =: \frac{\langle x\rangle + \langle y\rangle + \langle x + y\rangle}{B}. \end{aligned} \tag{2.12}$$

We first show that

$$|\partial_x^\alpha \partial_y^\beta (1/B)| \lesssim_{\alpha,\beta} 1/B. \tag{2.13}$$

We begin with the estimate

$$|\partial_x B|/B \lesssim 1. \tag{2.14}$$

This is equivalent to

$$\left| \frac{x}{\langle x\rangle} \langle y\rangle - y \right| \lesssim 1 + (\langle x\rangle\langle y\rangle - x \cdot y). \tag{2.15}$$

Denote $\theta = \frac{x \cdot y}{|x||y|}$. It is obvious that (2.13) holds for $-1 \le \theta \le 0$. Therefore we only need to consider the case $0 < \theta \le 1$. Taking the square of both sides of (2.15), we see

that it suffices to prove for some $0 < \epsilon < 1$ the inequality

$$\frac{|x|^2}{\langle x\rangle^2}\langle y\rangle^2 + |y|^2 - 2\frac{\langle y\rangle}{\langle x\rangle}|x|\,|y|\theta \le \frac{1}{\epsilon}\big(1 + (\langle x\rangle\langle y\rangle - |x|\,|y|\theta)^2\big). \tag{2.16}$$

Now consider the function

$$F(\theta) = |x|^2|y|^2\theta^2 - 2|x|\,|y|\langle y\rangle(\langle x\rangle - \epsilon/\langle x\rangle)\theta.$$

By using the obvious inequality $\langle x\rangle - |x| \ge \frac{1}{2\langle x\rangle}$, it is not difficult to check that for $0 < \epsilon \le 1/2$,

$$\frac{\langle y\rangle(\langle x\rangle - \epsilon/\langle x\rangle)}{|x|\,|y|} > 1.$$

Since $0 \le \theta \le 1$, clearly $F(\theta)$ achieves its minimum at $\theta = 1$. Therefore it suffices to prove (2.16) or equivalently (2.15) for $\theta = 1$.

Consider (2.15) for $\theta = 1$. We have

$$\left|\frac{x}{\langle x\rangle}\langle y\rangle - y\right| = \left|\frac{|x|}{\langle x\rangle}\langle y\rangle - |y|\right| \lesssim 1 + |y|\cdot\left|\frac{|x|}{\langle x\rangle} - 1\right| = 1 + \frac{|y|}{\langle x\rangle}(\langle x\rangle - |x|).$$

On the other hand

$$\langle x\rangle\langle y\rangle - |x|\,|y| \ge (\langle x\rangle - |x|)|y|. \tag{2.17}$$

Therefore (2.15) holds and consequently (2.14) is proved. By using an estimate similar to (2.17), we have

$$\langle x\rangle\langle y\rangle - |x|\,|y| \gtrsim \max\{|y|/\langle x\rangle, |x|/\langle y\rangle\}. \tag{2.18}$$

This together with (2.14) obviously implies that

$$\frac{|\partial_x B| + |\partial_y B| + \langle x\rangle/\langle y\rangle + \langle y\rangle/\langle x\rangle}{B} \lesssim 1. \tag{2.19}$$

It is easy to check that

$$|\partial_x^\alpha\partial_y^\beta B| \lesssim_{\alpha,\beta} \frac{\langle y\rangle}{\langle x\rangle} + \frac{\langle x\rangle}{\langle y\rangle}, \quad \forall |\alpha| + |\beta| \ge 2. \tag{2.20}$$

The estimate (2.13) now follows from (2.19), (2.20) and an induction argument.

By (2.12) and (2.13), we have

$$|\partial_x^\alpha\partial_y^\beta\psi(x, y)| \lesssim \psi(x, y).$$

It remains to prove (2.11) for $\alpha = \beta = 0$. If $\langle x + y\rangle \ll \langle x\rangle$ or $\langle x + y\rangle \ll \langle y\rangle$, then the estimate is obvious. Without loss of generality assume that $\langle y\rangle \ge \langle x\rangle$ and $\min\{\langle x\rangle, \langle y\rangle, \langle x + y\rangle\} \sim \langle x\rangle$. Then by (2.18) and (2.12), we have

$$\psi(x, y) \le \frac{\langle x\rangle + \langle y\rangle}{1 + |y|/\langle x\rangle} \lesssim \langle x\rangle.$$

Therefore (2.11) is proved. □

We need a simple lemma from vector algebra.

Lemma 2.7. *For any $x, y \in \mathbb{R}^2$, we have*

$$\frac{x}{\langle x\rangle} - \frac{y}{\langle y\rangle} = Q(x, y)(x - y), \tag{2.21}$$

where $Q(x, y) = Q$ is the matrix given by

$$Q_{ij} = \frac{1}{\langle y\rangle}\left(I - \frac{x(x+y)^T}{\langle x\rangle(\langle x\rangle + \langle y\rangle)}\right)_{ij} = \frac{1}{\langle y\rangle}\left(\delta_{ij} - \frac{x_i(x_j + y_j)}{\langle x\rangle(\langle x\rangle + \langle y\rangle)}\right), \quad 1 \le i, j \le 2. \tag{2.22}$$

Denote $\tilde{x} = (-x_2, x_1)^T$, $\tilde{y} = (-y_2, y_1)^T$. Then

$$Q^{-1} = \langle x\rangle\langle y\rangle(\langle x\rangle + \langle y\rangle)(1 + \langle x\rangle\langle y\rangle - x \cdot y)^{-1}\left(I - \frac{(\tilde{x} + \tilde{y})(\tilde{x})^T}{\langle x\rangle(\langle x\rangle + \langle y\rangle)}\right). \tag{2.23}$$

We have the pointwise bounds

$$\begin{aligned} |\partial_x^\alpha \partial_y^\beta Q(x, y)| &\lesssim_{\alpha,\beta} \langle y\rangle^{-1}, && \forall \alpha, \beta, \\ |\partial_x^\alpha \partial_y^\beta (Q^{-1}(x, y))| &\lesssim_{\alpha,\beta} \langle x\rangle^3 + \langle y\rangle^3, && \forall \alpha, \beta. \end{aligned} \tag{2.24}$$

Proof. We first show (2.21):

$$\begin{aligned} \frac{x}{\langle x\rangle} - \frac{y}{\langle y\rangle} &= x\left(\frac{1}{\langle x\rangle} - \frac{1}{\langle y\rangle}\right) + \frac{1}{\langle y\rangle}(x - y) = x\frac{(y + x)^T(y - x)}{\langle x\rangle\langle y\rangle(\langle x\rangle + \langle y\rangle)} + \frac{1}{\langle y\rangle}(x - y) \\ &= \frac{1}{\langle y\rangle}\left(I - \frac{x(x+y)^T}{\langle x\rangle(\langle x\rangle + \langle y\rangle)}\right)(x - y). \end{aligned}$$

Since Q is a two by two matrix, the expression for Q^{-1} is a straightforward computation. The bounds (2.24) follow easily from (2.22), (2.23) and a similar estimate to (2.13). □

We shall need to exploit some subtle cancelations of the phases. The following lemma will be useful in our nonlinear estimates.

Lemma 2.8 (Transformation of phase derivatives). *Consider the following phases:*

$$\begin{aligned} \phi_1(\xi, \eta, \sigma) &= \langle\xi\rangle + \langle\xi - \eta\rangle - \langle\eta - \sigma\rangle - \langle\sigma\rangle, \\ \phi_2(\xi, \eta, \sigma) &= \langle\xi\rangle - \langle\xi - \eta\rangle + \langle\eta - \sigma\rangle - \langle\sigma\rangle, \\ \phi_3(\xi, \eta, \sigma) &= \langle\xi\rangle - \langle\xi - \eta\rangle - \langle\eta - \sigma\rangle + \langle\sigma\rangle. \end{aligned}$$

There exist smooth matrix functions $Q_{11} = Q_{11}(\xi, \eta, \sigma)$, $Q_{12} = Q_{12}(\xi, \eta, \sigma)$, $Q_{21} = Q_{21}(\xi, \eta)$, $Q_{22} = Q_{22}(\eta, \sigma)$, $Q_{31} = Q_{31}(\xi, \eta)$, $Q_{32} = Q_{32}(\eta, \sigma)$ such that

$$\begin{aligned}\partial_\xi \phi_1 &= Q_{11}(\xi,\eta,\sigma)\partial_\eta\phi_1 + Q_{12}(\xi,\eta,\sigma)\partial_\sigma\phi_1,\\ \partial_\xi \phi_2 &= Q_{21}(\xi,\eta)Q_{22}(\eta,\sigma)\partial_\sigma\phi_2,\\ \partial_\xi \phi_3 &= Q_{31}(\xi,\eta)Q_{32}(\eta,\sigma)\partial_\sigma\phi_3.\end{aligned}$$

Moreover we have the pointwise bounds

$$\begin{aligned}|\partial_\xi^\alpha\partial_\eta^\beta\partial_\sigma^\gamma Q_{11}(\xi,\eta,\sigma)| + |\partial_\xi^\alpha\partial_\eta^\beta\partial_\sigma^\gamma Q_{12}(\xi,\eta,\sigma)| &\lesssim_{\alpha,\beta,\gamma} \langle|\xi|+|\eta|+|\sigma|\rangle^3, \quad \forall\alpha,\beta,\gamma,\\ |\partial_\xi^\alpha\partial_\eta^\beta Q_{21}(\xi,\eta)| + |\partial_\xi^\alpha\partial_\eta^\beta Q_{31}(\xi,\eta)| &\lesssim_{\alpha,\beta} 1, \quad \forall\alpha,\beta,\\ |\partial_\eta^\alpha\partial_\sigma^\beta Q_{22}(\eta,\sigma)| + |\partial_\eta^\alpha\partial_\sigma^\beta Q_{32}(\eta,\sigma)| &\lesssim_{\alpha,\beta} \langle|\eta|+|\sigma|\rangle^3, \quad \forall\alpha,\beta.\end{aligned} \tag{2.25}$$

Proof. We prove the statements for ϕ_1. The other two cases are simpler. By Lemma 2.7, we write

$$\begin{aligned}\partial_\xi\phi_1 &= \frac{\xi}{\langle\xi\rangle} + \frac{\xi-\eta}{\langle\xi-\eta\rangle} = \tilde{Q}_1(\xi,\eta)\cdot(2\xi-\eta),\\ \partial_\eta\phi_1 &= \frac{\eta-\xi}{\langle\eta-\xi\rangle} - \frac{\eta-\sigma}{\langle\eta-\sigma\rangle} = \tilde{Q}_2(\xi,\eta,\sigma)\cdot(\xi-\sigma),\\ \partial_\sigma\phi_1 &= \frac{\eta-\sigma}{\langle\eta-\sigma\rangle} - \frac{\sigma}{\langle\sigma\rangle} = \tilde{Q}_3(\eta,\sigma)\cdot(\eta-2\sigma).\end{aligned}$$

Hence

$$\partial_\xi\phi_1 = \tilde{Q}_1(2\tilde{Q}_2^{-1}\partial_\eta\phi_1 - \tilde{Q}_3^{-1}\partial_\sigma\phi_1) =: Q_{11}\partial_\eta\phi_1 + Q_{12}\partial_\sigma\phi_1.$$

The bound (2.25) is obvious. □

3. Preliminary transformations

Since the function $h = h(t,x)$ is complex-valued, we write it as $h(t,x) = h_1(t,x) + ih_2(t,x)$. By (1.8) and (1.9), we have

$$u = \frac{|\nabla|}{\langle\nabla\rangle}h_1, \qquad \mathbf{v} = -\frac{\nabla}{|\nabla|}h_2.$$

In Fourier space, (1.10) then takes the form

$$\begin{aligned}\hat{h}(t,\xi) = e^{it\langle\xi\rangle}\hat{h}_0(\xi) &- \int_0^t\int e^{i(t-s)\langle\xi\rangle}\langle\xi\rangle\langle\eta\rangle^{-1}|\eta|\frac{\xi\cdot(\xi-\eta)}{|\xi|\,|\xi-\eta|}\hat{h}_1(s,\eta)\hat{h}_2(s,\xi-\eta)\,d\eta\,ds\\ &+ \frac{i}{2}\int_0^t\int e^{i(t-s)\langle\xi\rangle}|\xi|\frac{|\eta|\,|\xi-\eta|}{\langle\eta\rangle\langle\xi-\eta\rangle}\hat{h}_1(s,\eta)\hat{h}_1(s,\xi-\eta)\,d\eta\,ds\\ &- \frac{i}{2}\int_0^t\int e^{i(t-s)\langle\xi\rangle}|\xi|\frac{\eta\cdot(\xi-\eta)}{|\eta||\xi-\eta|}\hat{h}_2(s,\eta)\hat{h}_2(s,\xi-\eta)\,d\eta\,ds.\end{aligned}$$

Denote $f(t) = e^{-it\langle\nabla\rangle}h(t)$. Then after a tedious calculation,

$$\begin{aligned}\hat f(t,\xi)=\hat h_0(\xi)&+\int_0^t\int e^{-is(\langle\xi\rangle-\langle\eta\rangle-\langle\xi-\eta\rangle)}\Big(\frac{i}{4}\langle\xi\rangle\langle\eta\rangle^{-1}|\eta|\frac{\xi\cdot(\xi-\eta)}{|\xi|\,|\xi-\eta|}\\&+\frac{i}{8}|\xi|\frac{|\eta|\,|\xi-\eta|}{\langle\eta\rangle\langle\xi-\eta\rangle}+\frac{i}{8}|\xi|\frac{\eta\cdot(\xi-\eta)}{|\eta|\,|\xi-\eta|}\Big)\hat f(s,\eta)\,\hat f(s,\xi-\eta)\,d\eta\,ds\\&+\int_0^t\int e^{-is(\langle\xi\rangle-\langle\eta\rangle+\langle\xi-\eta\rangle)}\Big(-\frac{i}{4}\langle\xi\rangle\langle\eta\rangle^{-1}|\eta|\frac{\xi\cdot(\xi-\eta)}{|\xi|\,|\xi-\eta|}\\&+\frac{i}{8}|\xi|\frac{|\eta|\,|\xi-\eta|}{\langle\eta\rangle\langle\xi-\eta\rangle}-\frac{i}{8}|\xi|\frac{\eta\cdot(\xi-\eta)}{|\eta|\,|\xi-\eta|}\Big)\hat f(s,\eta)\,\overline{\hat f(s,\eta-\xi)}\,d\eta\,ds\\&+\int_0^t\int e^{-is(\langle\xi\rangle+\langle\eta\rangle-\langle\xi-\eta\rangle)}\Big(\frac{i}{4}\langle\xi\rangle\langle\eta\rangle^{-1}|\eta|\frac{\xi\cdot(\xi-\eta)}{|\xi|\,|\xi-\eta|}\\&+\frac{i}{8}|\xi|\frac{|\eta|\,|\xi-\eta|}{\langle\eta\rangle\langle\xi-\eta\rangle}-\frac{i}{8}|\xi|\frac{\eta\cdot(\xi-\eta)}{|\eta|\,|\xi-\eta|}\Big)\overline{\hat f(s,-\eta)}\,\hat f(s,\xi-\eta)\,d\eta\,ds\\&+\int_0^t\int e^{-is(\langle\xi\rangle+\langle\eta\rangle+\langle\xi-\eta\rangle)}\Big(-\frac{i}{4}\langle\xi\rangle\langle\eta\rangle^{-1}|\eta|\frac{\xi\cdot(\xi-\eta)}{|\xi|\,|\xi-\eta|}\\&+\frac{i}{8}|\xi|\frac{|\eta|\,|\xi-\eta|}{\langle\eta\rangle\langle\xi-\eta\rangle}+\frac{i}{8}|\xi|\frac{\eta\cdot(\xi-\eta)}{|\eta|\,|\xi-\eta|}\Big)\overline{\hat f(s,-\eta)}\,\overline{\hat f(s,\eta-\xi)}\,d\eta\,ds.\end{aligned}\tag{3.1}$$

Here $\overline{\hat f}$ denotes the complex conjugate of $\hat f$. Note that

$$\overline{\hat f(t,-\xi)}=e^{it\langle\xi\rangle}\hat{\bar h}(t,\xi),\qquad \hat f(t,\xi)=e^{-it\langle\xi\rangle}\hat h(t,\xi).$$

To simplify matters, we shall write (3.1) collectively as

$$\hat f(t,\xi)=\hat h_0(\xi)+\int_0^t\int e^{-is\phi_0(\xi,\eta)}m_0(\xi,\eta)\,\hat f(s,\xi-\eta)\,\hat f(s,\eta)\,d\eta\,ds,\tag{3.2}$$

where

$$\phi_0(\xi,\eta)=\langle\xi\rangle\pm\langle\xi-\eta\rangle\pm\langle\eta\rangle,\tag{3.3}$$

and $m_0(\xi,\eta)$ is given by (after some symmetrization between η and $\xi-\eta$)

$$\begin{aligned}m_0(\xi,\eta)&=\text{const}\cdot\langle\xi\rangle\frac{\xi\cdot\eta}{|\xi|\,|\eta|}\,\frac{|\xi-\eta|}{\langle\xi-\eta\rangle}+\text{const}\cdot\langle\xi\rangle\frac{\xi\cdot(\xi-\eta)}{|\xi||\xi-\eta|}\,\frac{|\eta|}{\langle\eta\rangle}\\&\quad+\text{const}\cdot|\xi|\frac{|\eta|}{\langle\eta\rangle}\,\frac{|\xi-\eta|}{\langle\xi-\eta\rangle}+\text{const}\cdot|\xi|\frac{(\xi-\eta)\cdot\eta}{|\xi-\eta|\,|\eta|}\\&:=\sum_{i=1}^4 m_i(\xi,\eta).\end{aligned}$$

Hereafter we shall slightly abuse notation and write $\hat f(t,\xi)$ for both itself and its complex conjugate (i.e. $\overline{\hat f(t,-\xi)}$, see (3.1)). Note that in the expression of $m_0(\xi,\eta)$ there are four types of symbols. For $w=(w_1,w_2)\in\mathbb{R}^2$, define

$$r_1(w)=w_1/|w|,\qquad r_2(w)=w_2/|w|,\qquad r_3(w)=|w|/\langle w\rangle.$$

We write $m_0(\xi,\eta)$ collectively as

$$m_0(\xi,\eta)=\sum_{1\le j,k,l\le 3} a_{jkl}\cdot\langle\xi\rangle\cdot r_j(\xi)r_k(\xi-\eta)r_l(\eta), \tag{3.4}$$

where a_{jkl} are some constant coefficients. For example

$$m_3(\xi,\eta)=\text{const}\cdot\langle\xi\rangle\cdot\frac{|\xi|}{\langle\xi\rangle}\cdot\frac{|\xi-\eta|}{\langle\xi-\eta\rangle}\cdot\frac{|\eta|}{\langle\eta\rangle}=\text{const}\cdot\langle\xi\rangle\cdot r_3(\xi)r_3(\xi-\eta)r_3(\eta).$$

Although the frequency variables (ξ,η) are vectors, this fact will play no role in our analysis. The actual value of the constants a_{jkl} will not be important either. Therefore we shall suppress the subscript notation and summation in (3.4), pretend everything is scalar, and regard $m_0(\xi,\eta)$ as any one of the summands in (3.4). Observe that $m_0(\xi,\eta)$ is symmetric in the sense that

$$m_0(\xi,\eta)=m_0(\xi,\xi-\eta). \tag{3.5}$$

The nice feature of Klein–Gordon is (cf. Lemma 2.6)

$$|\phi_0(\xi,\eta)|\gtrsim 1/\langle|\xi|+|\eta|\rangle \qquad \text{for any } (\xi,\eta).$$

By the simple identity

$$e^{-is\phi_0(\xi,\eta)}=\frac{1}{-i\phi_0(\xi,\eta)}\frac{\partial}{\partial s}(e^{-is\phi_0(\xi,\eta)}),$$

we can then integrate by parts in the time variable s in (3.2). By (3.5),

$$\begin{aligned}\int_0^t\int e^{-is\phi_0(\xi,\eta)}\frac{m_0(\xi,\eta)}{\phi_0(\xi,\eta)}\partial_s\hat f(s,\xi-\eta)\hat f(s,\eta)\,d\eta\\ =\int_0^t\int e^{-is\phi_0(\xi,\eta)}\frac{m_0(\xi,\eta)}{\phi_0(\xi,\eta)}\partial_s\hat f(s,\eta)\hat f(s,\xi-\eta)\,d\eta\end{aligned} \tag{3.6}$$

using the change of variable $\eta\to\xi-\eta$. In the above equality we have again abused notation and denoted $\phi_0(\xi,\eta)=\phi_0(\xi,\xi-\eta)$ since it will remain of the same form as (3.3). By (3.2), we have

$$\partial_s\hat f(s,\eta)=\int e^{-is\phi_0(\eta,\sigma)}m_0(\eta,\sigma)\hat f(s,\eta-\sigma)\hat f(s,\sigma)\,d\sigma. \tag{3.7}$$

Integrating by parts in the time variable s in (3.2), using (3.7) and (3.6), we obtain

$$\begin{aligned}\hat f(t,\xi)&=\hat{\hat h}_0(\xi)+\hat g(t,\xi)\\ &\quad+\int_0^t\int e^{-is\phi(\xi,\eta,\sigma)}m_1(\xi,\eta,\sigma)\hat f(s,\xi-\eta)\hat f(s,\eta-\sigma)\hat f(s,\sigma)\,d\sigma\,d\eta\,ds\\ &=:\hat{\hat h}_0(\xi)+\hat g(t,\xi)+\hat f_{\text{cubic}}(t,\xi),\end{aligned} \tag{3.8}$$

where $\tilde{h}_0$ collects the contribution from the boundary term $s = 0$ and data h_0:

$$\hat{\tilde{h}}_0(\xi) = \hat{h}_0(\xi) + \int \frac{m_0(\xi,\eta)}{i\phi_0(\xi,\eta)} \hat{h}_0(\xi-\eta)\hat{h}_0(\eta)\,d\eta = \hat{h}_0(\xi) - \hat{g}(0,\xi); \tag{3.9}$$

the term g denotes the boundary term arising from $s = t$:

$$\hat{g}(t,\xi) = \int e^{-it\phi_0(\xi,\eta)} \cdot \frac{m_0(\xi,\eta)}{-i\phi_0(\xi,\eta)} \hat{f}(t,\xi-\eta)\hat{f}(t,\eta)\,d\eta; \tag{3.10}$$

and

$$\begin{aligned} m_1(\xi,\eta,\sigma) &= \frac{m_0(\xi,\eta)m_0(\eta,\sigma)}{i\phi_0(\xi,\eta)}, \\ \phi(\xi,\eta,\sigma) &= \langle\xi\rangle \pm \langle\xi-\eta\rangle \pm \langle\eta-\sigma\rangle \pm \langle\sigma\rangle. \end{aligned}$$

Note that

$$\begin{aligned} m_0(\xi,\eta)m_0(\eta,\sigma) &= \sum_{1\le j,k,l,j',k',l'\le 3} \langle\xi\rangle\langle\eta\rangle r_j(\xi)r_k(\xi-\eta)r_l(\eta)r_{j'}(\eta)r_{k'}(\eta-\sigma)r_{l'}(\sigma) \\ &= \sum_{1\le j,k,l,j',k',l'\le 3} \langle\xi\rangle\langle\eta\rangle r_l(\eta)r_{j'}(\eta)r_j(\xi)r_k(\xi-\eta)r_{k'}(\eta-\sigma)r_{l'}(\sigma). \end{aligned}$$

We shall slightly abuse notation and write

$$\begin{aligned} &\widehat{\mathcal{R}f}(\xi) = r(\xi)\hat{f}(\xi), \qquad r(\xi) = r_1(\xi),\ r_2(\xi),\ r_3(\xi),\ \text{or } r_j(\xi)r_{j'}(\xi), \\ &\eta/|\eta| = r_l(\eta)r_{j'}(\eta). \end{aligned}$$

The notations $\mathcal{R}$ and $\eta/|\eta|$ suggest that the functions r_j and $r_l r_{j'}$ are essentially the symbols of some Riesz-type operators or better. Their estimates are the same and the actual form plays no role in the proof. By adopting the above notation we can greatly simplify the presentation and also the analysis. In this notation, we shall write

$$\hat{f}_{\text{cubic}}(t,\xi) = \text{const} \cdot \widehat{\mathcal{R}f_3}(t,\xi),$$

and

$$\begin{aligned} \hat{f}_3(t,\xi) = \int_0^t \int e^{-is\phi(\xi,\eta,\sigma)} \cdot \frac{\langle\xi\rangle\cdot\langle\eta\rangle}{\phi_0(\xi,\eta)} \widehat{\mathcal{R}f}(s,\xi-\eta) \\ \cdot \frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds. \end{aligned} \tag{3.11}$$

In a similar way, we write the boundary terms as

$$\begin{aligned} \hat{g}(t,\xi) &= \text{const} \cdot \widehat{\mathcal{R}g_1}(t,\xi), \\ \hat{g}_1(t,\xi) &= \int e^{-it\phi_0(\xi,\eta)} \cdot \frac{\langle\xi\rangle}{\phi_0(\xi,\eta)} \widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{\mathcal{R}f}(t,\eta)\,d\eta. \end{aligned} \tag{3.12}$$

4. Local theory, continuity of X norm and $H^{N'}$ estimate

We recall that

$$\partial_t h = i\langle\nabla\rangle h - \frac{\langle\nabla\rangle\nabla}{|\nabla|}\cdot(u\mathbf{v}) + \frac{i}{2}|\nabla|(u^2+|\mathbf{v}|^2), \tag{4.1}$$

where $h = h_1 + ih_2$, and

$$u = \frac{|\nabla|}{\langle\nabla\rangle}h_1, \qquad \mathbf{v} = -\frac{\nabla}{|\nabla|}h_2.$$

Theorem 4.1. *For any $k \geq 4$ and $h_0 \in H^k(\mathbb{R}^2)$, there exists $T_0 = T_0(\|h_0\|_{H^k}) > 0$ and a unique smooth local solution $h \in C_t^0 H^k([0,T_0]\times\mathbb{R}^2)$ to (1.10). Moreover, if $h_0 \in H^7(\mathbb{R}^2)$ and $\|x(1-\Delta)h_0\|_{2+\delta} < \infty$, then*

$$\tilde{a}(t) := \|x(1-\Delta)e^{-it\langle\nabla\rangle}h(t)\|_{2+\delta} < \infty$$

for any $0 \leq t \leq T_0$, and $\tilde{a}(t)$ is a continuous function of t. Furthermore,

$$\|h(\tau)\|_{C_\tau^0 H^{N'}([0,t])} \lesssim \|h_0\|_{H^{N'}} + \|h\|_{X_t}^2 + \|h\|_{X_t}^3.$$

The rest of this section is devoted to the proof of this theorem. We begin with the H^k local well-posedness theory, which is quite standard. We sketch the details here for the sake of completeness.

4.1. Energy estimates

Let m be an integer. By (4.1), we compute

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \partial^m h\partial^m\bar{h}\\
&\quad= -\int \partial^m\left(\frac{\langle\nabla\rangle}{|\nabla|}\nabla\cdot(u\mathbf{v})\right)\partial^m\frac{\langle\nabla\rangle}{|\nabla|}u + \frac{1}{2}\int \partial^m|\nabla|(u^2+|\mathbf{v}|^2)\partial^m\left(\frac{\nabla}{|\nabla|}\cdot\mathbf{v}\right)\\
&\quad= -\int \partial^m\nabla\cdot(u\mathbf{v})\partial^m\frac{1-\Delta}{-\Delta}u + \frac{1}{2}\int \partial^m\left(u^2+|\mathbf{v}|^2\right)\partial^m\left(\nabla\cdot\mathbf{v}\right).
\end{aligned} \tag{4.2}$$

L^2 estimate. Taking $m = 0$ in (4.2), we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|h(t)\|_{L^2}^2) &= \int u\mathbf{v}\cdot\nabla\frac{1-\Delta}{-\Delta}u + \frac{1}{2}\int(u^2+|\mathbf{v}|^2)(\nabla\cdot\mathbf{v})\\
&\lesssim \|u\|_\infty\|\mathbf{v}\|_{L^2}\left\|\frac{\langle\nabla\rangle^2}{|\nabla|}u\right\|_2 + \|\nabla\cdot\mathbf{v}\|_\infty(\|u\|_2^2+\|\mathbf{v}\|_2^2)\\
&\lesssim (\|u\|_\infty + \|\nabla\cdot\mathbf{v}\|_\infty)\|h\|_{H^k}^2.
\end{aligned}$$

H^k estimate. Taking $m = k$ in (4.2), we have

$$\frac{1}{2}\frac{d}{dt}(\|\partial^k h(t)\|_{L^2}^2) = -\int \partial^k \nabla\cdot(u\mathbf{v})(\partial^k(-\Delta)^{-1}u) \tag{4.3}$$

$$-\int \partial^k \nabla\cdot(u\mathbf{v})\partial^k u \tag{4.4}$$

$$+\frac{1}{2}\int \partial^k(u^2)\partial^k(\nabla\cdot\mathbf{v}) \tag{4.5}$$

$$+\frac{1}{2}\int \partial^k(|\mathbf{v}|^2)\partial^k(\nabla\cdot\mathbf{v}). \tag{4.6}$$

For (4.3), we estimate

$$\begin{aligned}
(4.3) =& -\int \partial^k\nabla u\cdot\mathbf{v}(\partial^k(-\Delta)^{-1}u) - \int u(\partial^k\nabla\cdot\mathbf{v})\partial^k(-\Delta)^{-1}u\\
&+\sum_{1\le l\le k} O\Big(\int \partial^l u\partial^{k+1-l}\mathbf{v}\partial^k(-\Delta)^{-1}u\Big)\\
=&\frac{1}{2}\int |\partial^k(-\Delta)^{-1}\nabla u|^2(\nabla\cdot\mathbf{v}) + \int u(\partial^k\mathbf{v}\cdot\nabla\partial^k(-\Delta)^{-1}u)\\
&+O\Big(\int (-\Delta)^{-1}\partial^{k+2}u\cdot\partial\mathbf{v}\partial^k(-\Delta)^{-1}u\Big)\\
&+\sum_{1\le l\le k} O\Big(\int \partial^l u\partial^{k+1-l}\mathbf{v}\partial^k(-\Delta)^{-1}u\Big)\\
\lesssim& \|\nabla\cdot\mathbf{v}\|_\infty\|u\|_{H^k}^2 + \|u\|_\infty\|\mathbf{v}\|_{H^k}\|u\|_{H^k} + \|\partial\mathbf{v}\|_\infty\|u\|_{H^k}^2\\
&+\sum_{l=1}^k \|\partial^l u\|_{\frac{2(k-1)}{l-1}}\|\partial^{k+1-l}\mathbf{v}\|_{\frac{2(k-1)}{k-l}}\|u\|_{H^k}\\
\lesssim& (\|u\|_\infty + \|\partial\mathbf{v}\|_\infty)(\|u\|_{H^k}^2 + \|\mathbf{v}\|_{H^k}^2)\\
&+\sum_{l=1}^k \|\partial^k u\|_2^{\frac{l-1}{k-1}}\|\partial u\|_\infty^{\frac{k-l}{k-1}}\|\partial^k\mathbf{v}\|_2^{\frac{k-l}{k-1}}\|\partial\mathbf{v}\|_\infty^{\frac{l-1}{k-1}}\|u\|_{H^k}\\
\lesssim& (\|u\|_\infty + \|\partial u\|_\infty + \|\partial\mathbf{v}\|_\infty)(\|u\|_{H^k}^2 + \|\mathbf{v}\|_{H^k}^2).
\end{aligned}$$

For (4.4), we write

$$\begin{aligned}
(4.4) =& -\int (\partial^k\nabla\cdot\mathbf{v})u(\partial^k u) - \int (\partial^k\nabla u\cdot\mathbf{v})(\partial^k u)\\
&+\sum_{1\le l\le k} O\Big(\int \partial^l u\partial^{k+1-l}\mathbf{v}\partial^k u\,dx\Big)\\
=& -\int (\partial^k\nabla\cdot\mathbf{v})u\partial^k u + \frac{1}{2}\int \nabla\cdot\mathbf{v}\big(\partial^k u\big)^2 + \cdots,
\end{aligned}$$

where "$\cdots$" denotes terms which can be estimated in a similar way to (4.3).

Similarly,

$$(4.5) = \int (\partial^k \nabla \cdot \mathbf{v}) u \partial^k u + \cdots .$$

Also, using the fact that $\operatorname{curl} \mathbf{v} = 0$,

$$(4.6) = \frac{1}{4} \int |\partial^k \mathbf{v}|^2 (\nabla \cdot \mathbf{v}) + \cdots .$$

Collecting all the estimates, we obtain

$$\frac{1}{2} \frac{d}{dt} (\|h(t)\|_{H^k}^2) \lesssim (\|u\|_\infty + \|\partial u\|_\infty + \|\partial \mathbf{v}\|_\infty) \|h(t)\|_{H^k}^2.$$

This concludes the energy estimates.

4.2. *Continuity of X norm along the flow*

Now we show that

$$\tilde{a}(t) = \|x(1-\Delta) e^{-it\langle\nabla\rangle} h(t)\|_{2+\delta}$$

is a continuous function of t (so that we can use the continuity argument later). Without loss of generality we shall assume $0 \le t \le 1$.

Step 1. For any dyadic R, define

$$A_R = \left\| \chi_{R/2 \le |x| \le 2R} \begin{pmatrix} u \\ \mathbf{v} \end{pmatrix} \right\|_p + \left\| \chi_{R/2 \le |x| \le 2R} \begin{pmatrix} \nabla u \\ \nabla \mathbf{v} \end{pmatrix} \right\|_p,$$

where we fix some p such that $2 + \delta < p < 2(2+\delta)$. Here

$$\chi_{R/2 \le |x| \le 2R} = \chi_{|x| \le 2R} - \chi_{|x| \le R/2}.$$

We first show that

$$A_R \lesssim 1/R \qquad \text{for } R \ge R_0, \tag{4.7}$$

and R_0 as sufficiently large.

Linear flow estimate. For $0 \le t \le 1$, by Lemmas 2.2 and 2.5, we have

$$\begin{aligned}
&\left\| \chi_{R/2 \le |x| \le 2R} \frac{|\nabla|}{\langle\nabla\rangle} e^{it\langle\nabla\rangle} h_0 \right\|_p + \left\| \chi_{R/2 \le |x| \le 2R} \frac{\nabla}{|\nabla|} e^{it\langle\nabla\rangle} h_0 \right\|_p \\
&\quad \lesssim \frac{1}{R} \left(\left\| x \frac{|\nabla|}{\langle\nabla\rangle} e^{it\langle\nabla\rangle} h_0 \right\|_p + \left\| x \frac{\nabla}{|\nabla|} e^{it\langle\nabla\rangle} h_0 \right\|_p \right) \\
&\quad \lesssim \frac{1}{R} \left(\left\| |\nabla|^{-1} e^{it\langle\nabla\rangle} h_0 \right\|_p + \langle t \rangle \|e^{it\langle\nabla\rangle} h_0\|_p + \|e^{it\langle\nabla\rangle} (x h_0)\|_p \right) \\
&\quad \lesssim \frac{1}{R} \langle t \rangle^{|1-2/p|} \left(\left\| |\nabla|^{-1} \langle\nabla\rangle^{|1-2/p|+} h_0 \right\|_p + \langle t \rangle \|\langle\nabla\rangle h_0\|_p + \|\langle\nabla\rangle (x h_0)\|_p \right) \\
&\quad \lesssim \frac{1}{R} \left(\|\langle x \rangle h_0\|_{2+\delta} + \|h_0\|_{H^3} + \|x \Delta h_0\|_{2+\delta} \right).
\end{aligned}$$

Similarly,

$$\begin{aligned}\Big\|\chi_{R/2\le|x|\le2R}\frac{|\nabla|}{\langle\nabla\rangle}\nabla e^{it\langle\nabla\rangle}h_0\Big\|_p+\Big\|\chi_{R/2\le|x|\le2R}\frac{\nabla}{|\nabla|}\nabla e^{it\langle\nabla\rangle}h_0\Big\|_p&\\ \lesssim\frac{1}{R}\Big(\Big\|x\frac{|\nabla|}{\langle\nabla\rangle}\nabla e^{it\langle\nabla\rangle}h_0\Big\|_p+\Big\|x\frac{\nabla}{|\nabla|}\nabla e^{it\langle\nabla\rangle}h_0\Big\|_p\Big)&\\ \lesssim\frac{1}{R}\big(\|e^{it\langle\nabla\rangle}h_0\|_p+\|\nabla e^{it\langle\nabla\rangle}h_0\|_p+\|\nabla e^{it\langle\nabla\rangle}(xh_0)\|_p\big)&\\ \lesssim\frac{1}{R}\big(\|h_0\|_{H^3}+\|\nabla\langle\nabla\rangle^{(1-2/p)+}(xh_0)\|_p\big).&\end{aligned}$$

Now note that by Sobolev embedding,

$$\|\nabla\langle\nabla\rangle^{(1-2/p)+}(xh_0)\|_p\lesssim\|\nabla\langle\nabla\rangle^{(1-2/p)+}\langle\nabla\rangle^{\frac{2}{2+\delta}-\frac{2}{p}}(xh_0)\|_{2+\delta}.$$

Since $2+\frac{2}{2+\delta}-\frac{4}{p}<2$, we get

$$\|\nabla\langle\nabla\rangle^{(1-2/p)+}(xh_0)\|_p\lesssim\|h_0\|_{H^3}+\|x\Delta h_0\|_{2+\delta}.$$

So the contribution from the linear flow is $\lesssim 1/R$.

Nonlinear flow estimate. Denote

$$\begin{aligned}\mathcal{N}_u(t)&=\int_0^t e^{i(t-s)\langle\nabla\rangle}\Big[-\nabla\cdot(u\mathbf{v})+\frac{i}{2}\frac{-\Delta}{\langle\nabla\rangle}(u^2+|\mathbf{v}|^2)\Big]ds,\\ \mathcal{N}_{\mathbf{v}}(t)&=\int_0^t e^{i(t-s)\langle\nabla\rangle}\left[\frac{\nabla}{|\nabla|}\left(\frac{\langle\nabla\rangle}{|\nabla|}\nabla\cdot(u\mathbf{v})\right)-\frac{i}{2}\nabla(u^2+|\mathbf{v}|^2)\right]ds.\end{aligned}$$

We discuss two cases.

Low frequency piece. First note that by using the finite speed propagation of the Klein–Gordon propagators $\cos\tau\langle\nabla\rangle$, $\frac{\sin\tau\langle\nabla\rangle}{\langle\nabla\rangle}$ we have, for all $0\le\tau\le1$ and $R\ge100$,

$$\begin{aligned}\chi_{R/2\le|\cdot|\le2R}\cos\tau\langle\nabla\rangle&=\chi_{R/2\le|\cdot|\le2R}\cos\tau\langle\nabla\rangle[\chi_{(2/5)R\le|\cdot|\le(5/2)R}],\\ \chi_{R/2\le|\cdot|\le2R}\frac{\sin\tau\langle\nabla\rangle}{\langle\nabla\rangle}&=\chi_{R/2\le|\cdot|\le2R}\frac{\sin\tau\langle\nabla\rangle}{\langle\nabla\rangle}[\chi_{(2/5)R\le|\cdot|\le(5/2)R}].\end{aligned}\tag{4.8}$$

Consider the operators

$$\begin{aligned}K^{(1)}_{<1}f&=\chi_{(2/5)R\le|x|\le(5/2)R}\nabla P_{<1}(\widetilde{\chi}f),\\ K^{(2)}_{<1}f&=\chi_{(2/5)R\le|x|\le(5/2)R}\frac{\Delta}{\langle\nabla\rangle}P_{<1}(\widetilde{\chi}f),\\ K^{(3)}_{<1}f&=\chi_{(2/5)R\le|x|\le(5/2)R}\frac{\nabla}{|\nabla|}\frac{\nabla}{|\nabla|}\langle\nabla\rangle P_{<1}(\widetilde{\chi}f),\end{aligned}$$

where $\widetilde{\chi}=\chi_{\le R/4}$ or $\chi_{\ge4R}$. We claim that

$$\|K^j_{<1}f\|_p\lesssim\frac{1}{R}\|f\|_{(1/2+1/p)^{-1}}\quad\text{for any } j=1,2,3.\tag{4.9}$$

Indeed, we shall prove it for $j = 3$ and $\widetilde{\chi} = \chi_{\le R/4}$. The other cases are similar. For any dyadic $N < 1$, it is not difficult to check that for some $\Psi(\xi) = \phi_{\le 1}(\xi) - \phi_{\le 1/2}(\xi)$,

$$\begin{aligned}\left[\mathcal{F}^{-1}\left(\frac{\nabla}{|\nabla|}\frac{\nabla}{|\nabla|}\langle\nabla\rangle P_N\right)\right](z) &= \int e^{i\xi\cdot z}\Psi\left(\frac{\xi}{N}\right)\frac{\xi}{|\xi|}\frac{\xi}{|\xi|}\langle\xi\rangle\, d\xi \\ &= N^2\int e^{iN\xi\cdot z}\Psi(\xi)\frac{\xi}{|\xi|}\frac{\xi}{|\xi|}\langle N\xi\rangle\, d\xi = N^2\tilde{\phi}(N,z),\end{aligned}$$

where $\tilde{\phi} \in C^\infty$ satisfies

$$|\tilde{\phi}(N,z)| \lesssim_k \langle Nz\rangle^{-k} \quad \text{for any } z \in \mathbb{R}^2,\ N < 1.$$

We then have

$$\begin{aligned}\|K^{(3)}_{<1} f\|_p &\lesssim \sum_{N<1}\|\chi_{2/5R\le|x|\le 5/2R}\frac{\nabla}{|\nabla|}\frac{\nabla}{|\nabla|}\langle\nabla\rangle P_N(\widetilde{\chi} f)\|_p \\ &\lesssim \sum_{N<1}\langle NR\rangle^{-10}N\|f\|_{(1/2+1/p)^{-1}} \lesssim \frac{1}{R}\|f\|_{(1/2+1/p)^{-1}}.\end{aligned}$$

This settles the estimate (4.9). By using (4.8) and (4.9), we have

$$\begin{aligned}&\|\chi_{R/2\le|x|\le 2R}P_{<1}\mathcal{N}_u\|_p + \|\chi_{R/2\le|x|\le 2R}P_{<1}\mathcal{N}_{\mathbf{v}}\|_p \\ &\qquad + \|\chi_{R/2\le|x|\le 2R}\nabla P_{<1}\mathcal{N}_u\|_p + \|\chi_{R/2\le|x|\le 2R}\nabla P_{<1}\mathcal{N}_{\mathbf{v}}\|_p \\ &\le \eta(T_0)\|\chi_{R/4\le|x|\le 4R}u^2\|_p + \eta(T_0)\|\chi_{R/4\le|x|\le 4R}|\mathbf{v}|^2\|_p + C/R \\ &\le \eta(T_0)\big(\|\chi_{R/4\le|x|\le 4R}u\|_p + \|\chi_{R/4\le|x|\le 4R}\mathbf{v}\|_p\big) + C/R.\end{aligned}$$

Here $\eta(T_0) \to 0$ as $T_0 \to 0$.

High frequency piece. By (4.8) and a similar computation to the one in the low frequency case, we have

$$\begin{aligned}&\|\chi_{R/2\le|x|\le 2R}P_{>1}\mathcal{N}_u\|_p + \|\chi_{R/2\le|x|\le 2R}P_{>1}\mathcal{N}_{\mathbf{v}}\|_p \\ &\qquad + \|\chi_{R/2\le|x|\le 2R}\nabla P_{>1}\mathcal{N}_u\|_p + \|\chi_{R/2\le|x|\le 2R}\nabla P_{>1}\mathcal{N}_{\mathbf{v}}\|_p \\ &\le \eta(T_0)\big[\|\langle\nabla\rangle^3(\chi_{R/4\le|x|\le 4R}u\,\chi_{R/4\le|x|\le 4R}\mathbf{v})\|_2 \\ &\qquad + \|\langle\nabla\rangle^3[(\chi_{R/4\le|x|\le 4R}u)^2]\|_2 + \|\langle\nabla\rangle^3[(\chi_{R/4\le|x|\le 4R}\mathbf{v})^2]\|_2\big] \\ &\qquad + \sum_{N>1}(NR)^{-10}N^4(\|u\|^2_{H^4} + \|\mathbf{v}\|^2_{H^4}) \\ &\le \eta(T_0)\big(\|\chi_{R/4\le|x|\le 4R}u\|_p + \|\chi_{R/4\le|x|\le 4R}\mathbf{v}\|_p\big) + C/R.\end{aligned}$$

Collecting the estimates, we obtain

$$A_R \lesssim \eta(T_0)\big(\|\chi_{R/4\le|x|\le 4R}u\|_p + \|\chi_{R/4\le|x|\le 4R}\mathbf{v}\|_p\big) + C/R.$$

Now denote

$$a_m = \left\|\chi_{2^{m-1}\le|x|\le 2^{m+1}}\begin{pmatrix}u\\ \mathbf{v}\end{pmatrix}\right\|_p + \left\|\chi_{2^{m-1}\le|x|\le 2^{m+1}}\begin{pmatrix}\nabla u\\ \nabla\mathbf{v}\end{pmatrix}\right\|_p.$$

Clearly by choosing T_0 sufficiently small, we have

$$a_m \le \tfrac{1}{8}(a_{m-1} + a_m + a_{m+1}) + C \cdot 2^{-m}. \tag{4.10}$$

Note that $a_m \lesssim 1$ for any m. Iterating (4.10) gives $a_m \lesssim 2^{-m}$. Therefore (4.7) is proved.

Step 2. We show that $\|x(1-\Delta)e^{-it\langle\nabla\rangle}h(t)\|_{2+\delta}$ is continuous in t. We first prove that

$$\left\| x \begin{pmatrix} u \\ \mathbf{v} \end{pmatrix} \right\|_\infty \lesssim 1. \tag{4.11}$$

This is equivalent to

$$\left\| \chi_{|x|\sim R} \begin{pmatrix} u \\ \mathbf{v} \end{pmatrix} \right\|_\infty \lesssim \frac{1}{R} \qquad \text{for any } R \ge 100.$$

From Step 1 and Sobolev embedding, we have

$$\left\| \chi_{|x|\sim R} \begin{pmatrix} u \\ \mathbf{v} \end{pmatrix} \right\|_\infty \lesssim \left\| \chi_{|x|\sim R} \begin{pmatrix} u \\ \mathbf{v} \end{pmatrix} \right\|_p + \left\| \nabla \left[\chi_{|x|\sim R} \begin{pmatrix} u \\ \mathbf{v} \end{pmatrix} \right] \right\|_p \lesssim 1/R.$$

Hence (4.11) holds.

To continue we need a simple lemma.

Lemma 4.2. *For any $s \ge 0$,*

$$\|x\langle\nabla\rangle^s(fg)\|_{2+\delta} \lesssim \|xf\|_\infty \|g\|_{H^{s+3}} + \|xg\|_\infty \|f\|_{H^{s+3}} + \|f\|_{H^{s+3}}\|g\|_{H^{s+3}}. \tag{4.12}$$

Proof. We write

$$\begin{aligned}
&\widehat{\langle\nabla\rangle^s(fg)}(\xi) \\
&\quad = \langle\xi\rangle^s \int \chi_{\langle\eta\rangle/\langle\xi-\eta\rangle\le 1} \hat f(\xi-\eta)\hat g(\eta)\,d\eta + \langle\xi\rangle^s \int \chi_{\langle\eta\rangle/\langle\xi-\eta\rangle>1} \hat f(\xi-\eta)\hat g(\eta)\,d\eta \\
&\quad = \langle\xi\rangle^s \int \chi_{\langle\xi-\eta\rangle/\langle\eta\rangle\le 1} \hat f(\eta)\hat g(\xi-\eta)\,d\eta + \langle\xi\rangle^s \int \chi_{\langle\eta\rangle/\langle\xi-\eta\rangle>1} \hat f(\xi-\eta)\hat g(\eta)\,d\eta.
\end{aligned}$$

Differentiating in ξ gives

$$\begin{aligned}
\mathcal{F}\big(x\langle\nabla\rangle^s(fg)\big)(\xi) = {} & O(\langle\xi\rangle^{s-1}) \int \chi_{\langle\xi-\eta\rangle/\langle\eta\rangle\le 1} \hat f(\eta)\hat g(\xi-\eta)\,d\eta && (4.13)\\
& + \langle\xi\rangle^s \int \partial_\xi \chi_{\langle\xi-\eta\rangle/\langle\eta\rangle\le 1} \hat f(\eta)\hat g(\xi-\eta)\,d\eta && (4.14)\\
& + \langle\xi\rangle^s \int \chi_{\langle\xi-\eta\rangle/\langle\eta\rangle\le 1} \hat f(\eta)\widehat{xg}(\xi-\eta)\,d\eta && (4.15)\\
& + \langle\xi\rangle^s \int \chi_{\langle\eta\rangle/\langle\xi-\eta\rangle>1} \widehat{xf}(\xi-\eta)\hat g(\eta)\,d\eta && (4.16)\\
& + \cdots,
\end{aligned}$$

where "$\cdots$" denotes similar terms.

It is not difficult to show that

$$\|\mathcal{F}^{-1}((4.13))\|_{2+\delta} + \|\mathcal{F}^{-1}((4.14))\|_{2+\delta} \lesssim \|f\|_{H^{s+3}}\|g\|_{H^{s+3}}.$$

We shall only estimate (4.15). The estimation of (4.16) is similar. By Lemma 2.3, we have

$$\begin{aligned}\|\mathcal{F}^{-1}((4.15))\|_{2+\delta} &\lesssim \|T_{\chi_{\langle\xi-\eta\rangle/\langle\eta\rangle\le 1}\langle\xi\rangle^s\langle\eta\rangle^{-(s+2)-}}(\langle\nabla\rangle^{(s+2)+}f, xg)\|_{2+\delta}\\ &\lesssim \|\langle\nabla\rangle^{(s+2)+}f\|_{2+\delta}\|xg\|_\infty \lesssim \|xg\|_\infty\|f\|_{H^{s+3}}.\end{aligned}$$

The lemma is proved. □

By (1.10), observe that

$$\begin{aligned}(1-\Delta)e^{-it\langle\nabla\rangle}h = (1-\Delta)h_0\\ + \int_0^t e^{-is\langle\nabla\rangle}(1-\Delta)\left(-\frac{\langle\nabla\rangle\nabla}{|\nabla|}\cdot(u\mathbf{v}) + \frac{i}{2}|\nabla|(u^2+|\mathbf{v}|^2)\right)ds.\end{aligned}$$

By Lemma 4.2 and (4.11), we have

$$\begin{aligned}&\left\|x\big((1-\Delta)e^{-it\langle\nabla\rangle}h(t)\big) - x(1-\Delta)h_0\right\|_{2+\delta}\\ &\qquad\lesssim |t|\,\|u\|_{L_t^\infty H^6}\|\mathbf{v}\|_{L_t^\infty H^6} + \int_0^t \|x\langle\nabla\rangle^4(u\mathbf{v})\|_{2+\delta}\,ds\\ &\qquad\quad + \int_0^t \big(\|x\langle\nabla\rangle^4(u^2)\|_{2+\delta} + \|x\langle\nabla\rangle^4(|\mathbf{v}|^2)\|_{2+\delta}\big)\,ds\\ &\qquad\lesssim |t| + |t|\big(\|xu\|_\infty(\|u\|_{H^7}+\|\mathbf{v}\|_{H^7})\\ &\qquad\quad + \|x\mathbf{v}\|_\infty(\|u\|_{H^7}+\|\mathbf{v}\|_{H^7}) + (\|u\|_{H^7}+\|\mathbf{v}\|_{H^7})^2\big)\\ &\qquad\lesssim |t|.\end{aligned}$$

Clearly this gives continuity in t.

4.3. $H^{N'}$ estimate of h

By (1.12), we decompose f as

$$\begin{aligned}&\hat f(t,\xi)\\ &= \hat h_0(\xi) + \int_0^t\int e^{-is\phi_0}\frac{\xi}{|\xi|}\langle\xi\rangle\chi_{|\xi-\eta|\le\langle s\rangle^{10\delta}}\chi_{|\eta|\le\langle s\rangle^{10\delta}}\widehat{\mathcal{R}f}(s,\xi-\eta)\widehat{\mathcal{R}f}(s,\eta)\,d\eta\,ds \qquad (4.17)\\ &\quad + \int_0^t\int e^{-is\phi_0}\frac{\xi}{|\xi|}\langle\xi\rangle\chi_{|\xi-\eta|>\langle s\rangle^{10\delta}}\chi_{|\eta|\le\langle s\rangle^{10\delta}}\widehat{\mathcal{R}f}(s,\xi-\eta)\widehat{\mathcal{R}f}(s,\eta)\,d\eta\,ds \qquad (4.18)\\ &\quad + \int_0^t\int e^{-is\phi_0}\frac{\xi}{|\xi|}\langle\xi\rangle\chi_{|\eta|>\langle s\rangle^{10\delta}}\widehat{\mathcal{R}f}(s,\xi-\eta)\widehat{\mathcal{R}f}(s,\eta)\,d\eta\,ds. \qquad (4.19)\end{aligned}$$

For (4.19), we compute

$$\begin{aligned}
&\|\mathcal{F}^{-1}((4.19))\|_{H^{N'}}\\
&\quad\lesssim \int_0^t \|P_{>\langle s\rangle^{10\delta}}\mathcal{R}h(s)\cdot\mathcal{R}h(s)\|_{H^{N'+1}}\,ds\\
&\quad\lesssim \int_0^t \big(\|P_{>\langle s\rangle^{10\delta}}h(s)\|_{H^{N'+1}}\|\mathcal{R}h(s)\|_\infty + \|h(s)\|_{H^{N'+1}}\|P_{>\langle s\rangle^{10\delta}}\mathcal{R}h(s)\|_\infty\big)\,ds\\
&\quad\lesssim \int_0^t \langle s\rangle^{-5\delta}\|h(s)\|_{H^{N'+3/2}}\cdot\|\langle\nabla\rangle h(s)\|_{1/\delta}\,ds\\
&\quad\lesssim \int_0^t \langle s\rangle^{-5\delta+\delta-(1-2\delta)}\,ds\,\|h\|_{X_t}^2 \lesssim \|h\|_{X_t}^2.
\end{aligned}$$

Here we have used the fact that $N' = N - 3/2$.

Similarly

$$\|\mathcal{F}^{-1}((4.18))\|_{H^{N'}} \lesssim \|h\|_{X_t}^2.$$

For (4.17), we use the identity $e^{-is\phi_0} = \frac{i}{\phi_0}\frac{\partial}{\partial s}(e^{-is\phi_0})$ to integrate by parts in s, which gives

$$(4.17) = i\int \frac{e^{-is\phi_0}}{\phi_0}\frac{\xi}{|\xi|}\langle\xi\rangle\chi_{|\xi-\eta|\le\langle s\rangle^{10\delta}}\chi_{|\eta|\le\langle s\rangle^{10\delta}}\widehat{\mathcal{R}f}(s,\xi-\eta)\widehat{\mathcal{R}f}(s,\eta)\,d\eta\Big|_{s=0}^{s=t} \tag{4.20}$$

$$-\int_0^t\int \frac{ie^{-is\phi_0}}{\phi_0}\frac{\xi}{|\xi|}\langle\xi\rangle\partial_s(\chi_{|\xi-\eta|\le\langle s\rangle^{10\delta}}\chi_{|\eta|\le\langle s\rangle^{10\delta}})\widehat{\mathcal{R}f}(s,\xi-\eta)\widehat{\mathcal{R}f}(s,\eta)\,d\eta\,ds \tag{4.21}$$

$$-\int_0^t\int \frac{ie^{-is\phi_0}}{\phi_0}\frac{\xi}{|\xi|}\langle\xi\rangle\chi_{|\xi-\eta|\le\langle s\rangle^{10\delta}}\chi_{|\eta|\le\langle s\rangle^{10\delta}}\partial_s\big(\widehat{\mathcal{R}f}(s,\xi-\eta)\widehat{\mathcal{R}f}(s,\eta)\big)\,d\eta\,ds. \tag{4.22}$$

For (4.20), we have

$$\begin{aligned}
\|\mathcal{F}^{-1}((4.20))\|_{H^{N'}} &\lesssim \|T_{1/\phi_0}(P_{\le1}\mathcal{R}h_0, P_{\le1}\mathcal{R}h_0)\|_{H^{N'+1}}\\
&\quad+\|T_{1/\phi_0}(P_{\le\langle t\rangle^{10\delta}}\mathcal{R}h(t), P_{\le\langle t\rangle^{10\delta}}\mathcal{R}h(t))\|_{H^{N'+1}}\\
&\lesssim \|h_0\|_2^2 + \|P_{\le\langle t\rangle^{10\delta}}h(t)\|_{H^{N'+4+\delta}}\cdot\|\mathcal{R}h(t)\|_\infty\\
&\lesssim \|h_0\|_2^2 + \langle t\rangle^{40\delta-(1-2\delta)}\|h\|_{X_t}^2 \lesssim \|h\|_{X_t}^2.
\end{aligned}$$

For (4.21), we note that

$$\begin{aligned}
\partial_s(\chi_{|\xi-\eta|\le\langle s\rangle^{10\delta}}\chi_{|\eta|\le\langle s\rangle^{10\delta}}) &= \chi^{(1)}_{|\xi-\eta|\lesssim\langle s\rangle^{10\delta}}\chi_{|\eta|\le\langle s\rangle^{10\delta}}\langle s\rangle^{-1}\\
&\quad+\chi_{|\xi-\eta|\le\langle s\rangle^{10\delta}}\chi^{(2)}_{|\eta|\lesssim\langle s\rangle^{10\delta}}\langle s\rangle^{-1},
\end{aligned}$$

where $\chi^{(1)}$, $\chi^{(2)}$ are some modified cut-offs. Therefore

$$\begin{aligned}\|\mathcal{F}^{-1}((4.21))\|_{H^{N'}} &\lesssim \int_0^t \langle s\rangle^{-1}\|\langle\nabla\rangle^{N'+4+\delta}P_{\le\langle s\rangle^{10\delta}}h(s)\|_2\|\mathcal{R}P_{\le\langle s\rangle^{10\delta}}h(s)\|_\infty\,ds\\ &\lesssim \int_0^t \langle s\rangle^{-1-(1-2\delta)+40\delta}\,ds\,\|h\|_{X_t}^2 \lesssim \|h\|_{X_t}^2.\end{aligned}$$

For (4.22), we observe that (see (3.7))

$$e^{is\langle\nabla\rangle}\mathcal{F}^{-1}(\partial_s(\widehat{\mathcal{R}f}(s))) = \mathcal{R}\langle\nabla\rangle(\mathcal{R}h(s)\cdot\mathcal{R}h(s)).$$

Therefore

$$\|\mathcal{F}^{-1}((4.22))\|_{H^{N'}} \lesssim \int_0^t \langle s\rangle^{-2(1-2\delta)+50\delta}\,ds\,\|h\|_{X_t}^3 \lesssim \|h\|_{X_t}^3.$$

5. Estimates of the boundary term g

In this section we control the boundary term g coming from integration by parts in the time variable s (see (3.10)). We have the following

Proposition 5.1.

$$\|\langle\tau\rangle(1-\Delta)e^{i\tau\langle\nabla\rangle}g(\tau)\|_{L_\tau^\infty L_x^{1/\delta}([0,t])} + \|x(1-\Delta)g(\tau)\|_{L_\tau^\infty L_x^{2+\delta}([0,t])} \lesssim \|h\|_{X_t}^2.$$

By Proposition 5.1 and Sobolev embedding, it is easy to show that

$$\|\langle\tau\rangle e^{i\tau\langle\nabla\rangle}g(\tau)\|_{L_{\tau,x}^\infty([0,t])} \lesssim \|h\|_{X_t}^2.$$

The rest of this section is devoted to the proof of Proposition 5.1. We begin with a simple lemma.

Lemma 5.2. *For any $1 \le s' \le 7$ and $t \ge 0$, we have*

$$\|\langle\nabla\rangle^{s'}h(t)\|_{16/s'} \lesssim \langle t\rangle^{-(1-s'/8-\delta)}\|h\|_{X_t}. \tag{5.1}$$

Similarly for any $1 \le s' \le 6$ and $t \ge 0$, we have

$$\|\langle\nabla\rangle^{s'}h(t)\|_{13/s'} \lesssim \langle t\rangle^{-(1-s'/6.5)}\|h\|_{X_t}. \tag{5.2}$$

Proof of Lemma 5.2. Observe that by interpolation we have

$$\|\langle\nabla\rangle^{s'}P_{<1}h(t)\|_{16/s'} \lesssim \|h(t)\|_{16/s'} \lesssim \langle t\rangle^{-(1-s'/8)}\|h\|_{X_t}.$$

On the other hand, for any dyadic $M \ge 1$,

$$\begin{aligned}\|\langle\nabla\rangle^{s'}P_Mh(t)\|_{16/s'} &\lesssim M^{-(1-s'/8)}\big(M^8\|P_Mh(t)\|_2\big)^{s'/8}\big(M\|P_Mh(t)\|_\infty\big)^{1-s'/8}\\ &\lesssim M^{-(1-s'/8)}\langle t\rangle^{-(1-s'/8-\delta)}\|h\|_{X_t}.\end{aligned}$$

Summing over M gives (5.1).

The estimation of (5.2) is similar except that we use $\|h(t)\|_{H^{6.5}} \lesssim 1$ for all $t \ge 0$. □

We begin by estimating $\|(1-\Delta)e^{it\langle\nabla\rangle}g(t)\|_{1/\delta}$. By (3.12) and Lemmas 2.3 and 2.6 and 5.2, we have

$$\begin{aligned}\|(1-\Delta)e^{it\langle\nabla\rangle}g(t)\|_{1/\delta} &\lesssim \|T_{\langle\xi\rangle^3/\phi_0}(\mathcal{R}h(t),\mathcal{R}h(t))\|_{1/\delta}\\ &\lesssim \|\langle\nabla\rangle^{5+\delta}\mathcal{R}h(t)\|_\infty\|\langle\nabla\rangle\mathcal{R}h(t)\|_{1/\delta}\\ &\lesssim \|\langle\nabla\rangle^6 h(t)\|_{13/6}\|\langle\nabla\rangle h(t)\|_{1/\delta}\lesssim \frac{1}{\langle t\rangle}\|h\|_{X_t}^2.\end{aligned}$$

It remains to control $\|x(1-\Delta)g(t)\|_{2+\delta}$. By (3.12), we have

$$\|x(1-\Delta)g\|_{2+\delta}\lesssim \|x(1-\Delta)\mathcal{R}g_1\|_{2+\delta}.$$

Note that

$$\partial_\xi\left(\frac{\xi}{|\xi|}\langle\xi\rangle^2\hat{g}_1(\xi)\right)\sim\frac{\langle\xi\rangle^2}{|\xi|}\hat{g}_1(\xi)+\frac{\xi}{|\xi|}\langle\xi\rangle\hat{g}_1(\xi)+\frac{\xi}{|\xi|}\langle\xi\rangle^2\widehat{xg_1}(\xi).$$

Therefore by Lemma 2.5,

$$\begin{aligned}\|x(1-\Delta)\mathcal{R}g_1\|_{2+\delta} &\lesssim \big\||\nabla|^{-1}\langle\nabla\rangle^2 g_1\big\|_{2+\delta}+\|\langle\nabla\rangle g_1\|_{2+\delta}+\|\langle\nabla\rangle^2(xg_1)\|_{2+\delta}\\ &\lesssim \|g_1\|_{H^2}+\|x\langle\nabla\rangle^2 g_1\|_2+\|\langle\nabla\rangle^2(xg_1)\|_{2+\delta}\\ &\lesssim \|g_1\|_{H^2}+\|\langle\nabla\rangle^{2+\delta}(xg_1)\|_2.\end{aligned}$$

It is easy to check that $\|g_1\|_{H^2}\lesssim\|h\|_{X_t}^2$. We only need to estimate $\langle\nabla\rangle^{2+\delta}(xg_1)$. We decompose g_1 as

$$\hat{g}_1(t,\xi)=\int e^{-it\phi_0}\frac{\langle\xi\rangle}{\phi_0}\chi_{|\xi-\eta|/\langle\eta\rangle\le1}\widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{\mathcal{R}f}(t,\eta)\,d\eta \tag{5.3}$$

$$+\int e^{-it\phi_0}\frac{\langle\xi\rangle}{\phi_0}\chi_{|\xi-\eta|/\langle\eta\rangle>1}\widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{\mathcal{R}f}(t,\eta)\,d\eta. \tag{5.4}$$

We shall only estimate the contribution of (5.3). The term (5.4) can be dealt with in the same way as (5.3) using the change of variable $\eta\to\xi-\eta$.

Now we have

$$\begin{aligned}&\langle\xi\rangle^{2+\delta}\widehat{xg_1}(t,\xi)\\ &=(-it)\cdot\int\partial_\xi\phi_0 e^{-it\phi_0}\cdot\frac{\langle\xi\rangle^{3+\delta}}{\phi_0}\cdot\chi_{|\xi-\eta|/\langle\eta\rangle\le1}\widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{\mathcal{R}f}(t,\eta)\,d\eta &(5.5)\\ &\quad+\int e^{-it\phi_0}\langle\xi\rangle^{2+\delta}\partial_\xi\left(\frac{\langle\xi\rangle}{\phi_0}\chi_{|\xi-\eta|/\langle\eta\rangle\le1}\right)\widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{\mathcal{R}f}(t,\eta)\,d\eta &(5.6)\\ &\quad+\int e^{-it\phi_0}\frac{\langle\xi\rangle^{3+\delta}}{\phi_0}\chi_{|\xi-\eta|/\langle\eta\rangle\le1}\partial_\xi(\widehat{\mathcal{R}f}(t,\xi-\eta))\widehat{\mathcal{R}f}(t,\eta)\,d\eta &(5.7)\\ &\quad+\cdots,\end{aligned}$$

where $\cdots$ denotes similar terms.

By Lemmas 2.3 and 5.2, we estimate (5.5) as

$$\begin{aligned}\|\mathcal{F}^{-1}((5.5))\|_2 &\lesssim |t|\Big\| T_{\frac{\langle\xi\rangle^{3+\delta}}{\phi_0}\chi_{|\xi-\eta|/\langle\eta\rangle\le 1}\partial_\xi\phi_0}(\mathcal{R}h(t),\mathcal{R}h(t))\Big\|_2\\ &\lesssim \|\langle\nabla\rangle^{5+2\delta}h(t)\|_{13/6}\|\langle\nabla\rangle h(t)\|_{13/0.5}\\ &\lesssim |t|\langle t\rangle^{-(1-6/6.5)}\langle t\rangle^{-6/6.5}\|h\|_{X_t}^2 \lesssim \|h\|_{X_t}^2.\end{aligned}$$

Similarly

$$\|\mathcal{F}^{-1}((5.6))\|_2 \lesssim \|h\|_{X_t}^2.$$

For (5.7), we note that by Lemmas 2.2 and 2.5,

$$\begin{aligned}&\|\langle\nabla\rangle^{2-20\delta}e^{it\langle\nabla\rangle}\mathcal{F}^{-1}(\partial_\xi(\widehat{\mathcal{R}f}))\|_{2+2\delta}\\ &\qquad\lesssim \langle t\rangle^{\delta}\big(\|\langle\nabla\rangle^{2-19\delta}|\nabla|^{-1}f\|_{2+2\delta}+\|\langle\nabla\rangle^{2-19\delta}\mathcal{R}(xf)\|_{2+2\delta}\big)\\ &\qquad\lesssim \langle t\rangle^{\delta}\big(\|x(1-\Delta)f\|_{2+\delta}+\|f\|_{H^2}\big).\end{aligned}$$

Therefore

$$\|\mathcal{F}^{-1}((5.7))\|_2 \lesssim \|h\|_{X_t}\langle t\rangle^{\delta}\|\langle\nabla\rangle^{4+22\delta}h(t)\|_{(\frac12-\frac{1}{2+2\delta})^{-1}} \lesssim \|h\|_{X_t}^2.$$

The proposition is proved.

6. Reduction to low frequency

In this section we control the high frequency part of the solution. The main result of this section is

Proposition 6.1.

$$\|e^{i\tau\langle\nabla\rangle}f_{\text{cubic}}(\tau)\|_{\tilde{X}_t} \lesssim \|h\|_{X_t}^3+\|f_{\text{low}}(\tau)\|_{L_\tau^\infty L_x^{2-\delta/100}([0,t])},$$

where

$$\begin{aligned}\hat{f}_{\text{low}}(t,\xi) = \int_0^t\int e^{-is\phi}\frac{s\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle m_{\text{low}}(\xi,\eta,\sigma)\\ \cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds \qquad (6.1)\end{aligned}$$

and

$$m_{\text{low}}(\xi,\eta,\sigma) = \chi_{|\xi-\eta|\le\langle s\rangle^{\delta_0}}\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\sigma|\le\langle s\rangle^{\delta_0}}.$$

Here $\delta_0 = 20\delta$.

The rest of this section is devoted to the proof of this proposition.

Estimate of $\big\||\nabla|^\delta\langle\nabla\rangle(e^{it\langle\nabla\rangle}f_{\text{cubic}})\big\|_\infty$ *and* $\|\langle\nabla\rangle(e^{it\langle\nabla\rangle}f_{\text{cubic}})\|_{1/\delta}$. By using the dispersive inequality and noting that $f_{\text{cubic}} = \text{const}\cdot\mathcal{R}f_3$ (see (3.11)), we have

$$\begin{aligned}\big\||\nabla|^\delta\langle\nabla\rangle(e^{it\langle\nabla\rangle}f_{\text{cubic}}(t))\big\|_\infty &\lesssim \sum_{M<1} M^\delta\|P_M e^{it\langle\nabla\rangle}f_3(t)\|_\infty + \sum_{M\geq 1} M^{1+\delta}\|P_M e^{it\langle\nabla\rangle}f_3(t)\|_\infty\\ &\lesssim \frac{1}{\langle t\rangle}\|f_3\|_1 + \frac{1}{\langle t\rangle}\sum_{M\geq 1} M^{3+\delta}\|P_M f_3\|_1 \lesssim \frac{1}{\langle t\rangle}\|\langle\nabla\rangle^{3+2\delta}f_3\|_1.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\langle\nabla\rangle(e^{it\langle\nabla\rangle}f_{\text{cubic}}(t))\|_{1/\delta} &\lesssim \|\langle\nabla\rangle(e^{it\langle\nabla\rangle}f_3(t))\|_{1/\delta} \lesssim \langle t\rangle^{-(1-2\delta)}\|\langle\nabla\rangle^3 f_3\|_{(1-\delta)^{-1}}\\ &\lesssim \langle t\rangle^{-(1-2\delta)}\|\langle\nabla\rangle^{3+2\delta}f_3\|_1.\end{aligned}$$

Since

$$\|\langle\nabla\rangle^{3+2\delta}f_3\|_1 \lesssim \|\langle x\rangle(\langle\nabla\rangle^{3+2\delta}f_3)\|_{2-\delta/100},$$

we obtain

$$\begin{aligned}\langle t\rangle\big\||\nabla|^\delta\langle\nabla\rangle(e^{it\langle\nabla\rangle}f_{\text{cubic}}(t))\big\|_\infty + \langle t\rangle^{1-2\delta}\|\langle\nabla\rangle(e^{it\langle\nabla\rangle}f_{\text{cubic}}(t))\|_{1/\delta}&\\ \lesssim \|\langle x\rangle(\langle\nabla\rangle^{3+2\delta}f_3(t))\|_{2-\delta/100}.&\end{aligned}$$

Estimate of $\|x(1-\Delta)f_{\text{cubic}}\|_{2+\delta}$. By Lemma 2.5, we have

$$\begin{aligned}\|x(1-\Delta)f_{\text{cubic}}\|_{2+\delta} &\lesssim \|x\mathcal{R}\langle\nabla\rangle^2 f_3\|_{2+\delta}\\ &\lesssim \big\||\nabla|^{-1}\langle\nabla\rangle^2 f_3\big\|_{2+\delta} + \|\langle\nabla\rangle f_3\|_{2+\delta} + \|\langle\nabla\rangle^2(xf_3)\|_{2+\delta}\\ &\lesssim \big\||\nabla|^{-1}\langle\nabla\rangle^{3+2\delta}f_3\big\|_{2+\delta} + \|\langle x\rangle\langle\nabla\rangle^{3+2\delta}f_3\|_{2-\delta/100}\\ &\lesssim \|\langle x\rangle\langle\nabla\rangle^{3+2\delta}f_3\|_{2-\delta/100}.\end{aligned}$$

Estimate of $\|\langle x\rangle\langle\nabla\rangle^{3+2\delta}f_3\|_{2-\delta/100}$. We shall only estimate $\|x\langle\nabla\rangle^{3+2\delta}f_3\|_{2-\delta/100}$. The estimation of $\|\langle\nabla\rangle^{3+2\delta}f_3\|_{2-\delta/100}$ is simpler and is omitted.

Observe that by (3.11),

$$\begin{aligned}\mathcal{F}(\langle\nabla\rangle^{3+2\delta}f_3)(\xi) = \int_0^t\int e^{-is\phi}\frac{1}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle&\\ \cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds.&\end{aligned}$$

Differentiating in ξ gives us

$$\begin{aligned}\mathcal{F}((-i)x\langle\nabla\rangle^{3+2\delta}f_3) &= \partial_\xi\big(\mathcal{F}(\langle\nabla\rangle^{3+2\delta}f_3)(\xi)\big)\\ &= \int_0^t\int e^{-is\phi}(-is\partial_\xi\phi)\frac{1}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\\ &\qquad\cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds \qquad (6.2)\end{aligned}$$

$$+\int_0^t\int e^{-is\phi}\partial_\xi\left(\frac{\langle\xi\rangle^{4+2\delta}}{\phi_0(\xi,\eta)}\right)\langle\eta\rangle \cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds \tag{6.3}$$

$$+\int_0^t\int e^{-is\phi}\frac{\langle\xi\rangle^{4+2\delta}}{\phi_0(\xi,\eta)}\langle\eta\rangle \cdot\partial_\xi\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds. \tag{6.4}$$

We first deal with (6.2). We have

$$(6.2)=\int_0^t\int(-is)e^{-is\phi}\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\chi_{|\xi-\eta|>\langle s\rangle^{\delta_0}} \cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds \tag{6.5}$$

$$+\int_0^t\int(-is)e^{-is\phi}\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\chi_{|\xi-\eta|\le\langle s\rangle^{\delta_0}} \cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds. \tag{6.6}$$

For (6.5), we further decompose it as

$$(6.5)=\int_0^t\int(-is)e^{-is\phi}\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\chi_{\langle\eta\rangle/\langle\xi-\eta\rangle\le1}\chi_{|\xi-\eta|>\langle s\rangle^{\delta_0}} \cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds \tag{6.7}$$

$$+\int_0^t\int(-is)e^{-is\phi}\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\chi_{\langle\eta\rangle/\langle\xi-\eta\rangle>1}\chi_{|\xi-\eta|>\langle s\rangle^{\delta_0}} \cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds. \tag{6.8}$$

We estimate (6.7) as

$$\|\mathcal{F}^{-1}((6.7))\|_{2-\delta/100} \lesssim\int_0^t s\left\|e^{is\langle\nabla\rangle}\left(T_{\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\chi_{\langle\eta\rangle/\langle\xi-\eta\rangle\le1}}\big(\mathcal{R}P_{>\langle s\rangle^{\delta_0}}h,\mathcal{R}(\mathcal{R}h\,\mathcal{R}h)\big)\right)\right\|_{2-\delta/100}ds. \tag{6.9}$$

By Lemma 2.2,

$$\|\langle\nabla\rangle^{-\delta/100}e^{is\langle\nabla\rangle}\|_{L_x^{2-\delta/100}\to L_x^{2-\delta/100}}\lesssim\langle s\rangle^{\delta/100}.$$

Therefore by Lemmas 2.3 and 2.6, we have

$$
\begin{aligned}
&\Big\| e^{is\langle\nabla\rangle}\Big(T_{\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\chi_{\langle\eta\rangle/\langle\xi-\eta\rangle\le 1}}\big(\mathcal{R}P_{>\langle s\rangle^{\delta_0}}h,\mathcal{R}(\mathcal{R}h\,\mathcal{R}h)\big)\Big)\Big\|_{2-\delta/100}\\
&=\Big\|\langle\nabla\rangle^{-\delta/100}e^{is\langle\nabla\rangle}\Big(T_{\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta+\delta/100}\langle\eta\rangle\chi_{\langle\eta\rangle/\langle\xi-\eta\rangle\le 1}}\big(\mathcal{R}P_{>\langle s\rangle^{\delta_0}}h,\mathcal{R}(\mathcal{R}h\,\mathcal{R}h)\big)\Big)\Big\|_{2-\delta/100}\\
&\lesssim\langle s\rangle^{\delta/100}\Big\|T_{\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta+\delta/100}\langle\eta\rangle\chi_{\langle\eta\rangle/\langle\xi-\eta\rangle\le 1}}\big(\mathcal{R}P_{>\langle s\rangle^{\delta_0}}h,\mathcal{R}(\mathcal{R}h\,\mathcal{R}h)\big)\Big\|_{2-\delta/100}\\
&\lesssim\langle s\rangle^{\delta/100}\Big\|T_{\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta+\delta/100}\langle\eta\rangle\chi_{\langle\eta\rangle/\langle\xi-\eta\rangle\le 1}\langle\xi-\eta\rangle^{-(7+3\delta)}\langle\eta\rangle^{-1}}\big(\langle\nabla\rangle^{7+3\delta}\mathcal{R}P_{>\langle s\rangle^{\delta_0}}h,\\
&\qquad\qquad\langle\nabla\rangle\mathcal{R}(\mathcal{R}h\,\mathcal{R}h)\big)\Big\|_{2-\delta/100}\\
&\lesssim\langle s\rangle^{\delta/100}\|\langle\nabla\rangle^{7+3\delta}\mathcal{R}P_{>\langle s\rangle^{\delta_0}}h\|_{(1/2-\delta/100-2\delta)^{-1}}\|\langle\nabla\rangle\mathcal{R}(\mathcal{R}h\,\mathcal{R}h)\|_{1/2\delta}\\
&\lesssim\langle s\rangle^{\delta/100}\|\langle\nabla\rangle^{7+7\delta}P_{>\langle s\rangle^{\delta_0}}h\|_2\|\langle\nabla\rangle h\|_{1/\delta}^2\lesssim\langle s\rangle^{\delta/100}\langle s\rangle^{-\delta_0(N-7-7\delta)}\langle s\rangle^{\delta}\langle s\rangle^{-2(1-2\delta)}\|h\|_{X_t}^3\\
&\lesssim\langle s\rangle^{-2+7\delta-\delta_0(N-7)}\|h\|_{X_t}^3\lesssim\langle s\rangle^{-2-}\|h\|_{X_t}^3,
\end{aligned}
$$

where we have used the fact that $N>7$ and $7\delta<\delta_0(N-7)$. This clearly implies that

$$
\|\mathcal{F}^{-1}((6.7))\|_{2-\delta/100}\lesssim\int_0^t\langle s\rangle^{-1-}\,ds\cdot\|h\|_{X_t}^3\lesssim\|h\|_{X_t}^3.
$$

Similarly

$$
\begin{aligned}
&\|\mathcal{F}^{-1}((6.8))\|_{2-\delta/100}\\
&\qquad\lesssim\int_0^t\langle s\rangle^{1+\delta/100}\|\langle\nabla\rangle^{7+3\delta}P_{>\langle s\rangle^{\delta_0}}(\mathcal{R}h\,\mathcal{R}h)\|_{(\frac{1}{2-\delta/100}-\delta)^{-1}}\|\langle\nabla\rangle h\|_{1/\delta}\,ds\\
&\qquad\lesssim\int_0^t\langle s\rangle^{-1-}\,ds\cdot\|h\|_{X_t}^3\lesssim\|h\|_{X_t}^3.
\end{aligned}
$$

Therefore

$$
\|\mathcal{F}^{-1}((6.5))\|_{2-\delta/100}\lesssim\|h\|_{X_t}^3.
$$

For (6.6), we decompose it as

$$
\begin{aligned}
(6.6)=&\int_0^t\int(-is)e^{-is\phi}\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\chi_{|\xi-\eta|\le\langle s\rangle^{\delta_0}}\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\sigma|\le\langle s\rangle^{\delta_0}}\\
&\cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds\qquad(6.10)\\
&+\int_0^t\int(-is)e^{-is\phi}\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\chi_{|\xi-\eta|\le\langle s\rangle^{\delta_0}}\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\sigma|>\langle s\rangle^{\delta_0}}\\
&\cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds\qquad(6.11)\\
&+\int_0^t\int(-is)e^{-is\phi}\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\chi_{|\xi-\eta|\le\langle s\rangle^{\delta_0}}\chi_{|\eta-\sigma|>\langle s\rangle^{\delta_0}}\\
&\cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds.\qquad(6.12)
\end{aligned}
$$

The estimation of (6.11) is similar to that in (6.7). We have

$$\begin{aligned}
&\|\mathcal{F}^{-1}((6.11))\|_{2-\delta/100}\\
&\lesssim \int_0^t \langle s\rangle^{1+\delta/100}\|\langle\nabla\rangle h\|_{1/\delta}\|\langle\nabla\rangle^{7+3\delta}(P_{\le\langle s\rangle^{\delta_0}}\mathcal{R}h P_{>\langle s\rangle^{\delta_0}}\mathcal{R}h)\|_{(\frac{1}{2-\delta/100}-\delta)^{-1}}\,ds\\
&\lesssim \int_0^t \langle s\rangle^{1+\delta/100}\|\langle\nabla\rangle h\|_{1/\delta}\big(\|\langle\nabla\rangle^{7+3\delta}P_{\le\langle s\rangle^{\delta_0}}\mathcal{R}h\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}\|P_{>\langle s\rangle^{\delta_0}}h\|_{1/\delta}\\
&\qquad\qquad + \|\langle\nabla\rangle^{7+3\delta}P_{>\langle s\rangle^{\delta_0}}\mathcal{R}h\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}\|P_{\le\langle s\rangle^{\delta_0}}h\|_{1/\delta}\big)\,ds\\
&\lesssim \int_0^t \langle s\rangle^{1+\delta/100}\|\langle\nabla\rangle h\|_{1/\delta}\big(\|\langle\nabla\rangle^{N}h\|_2\langle s\rangle^{-\delta_0}\|\langle\nabla\rangle h\|_{1/\delta}\\
&\qquad\qquad + \langle s\rangle^{-\delta_0(N-7-7\delta)}\|\langle\nabla\rangle^{N}h\|_2\|\langle\nabla\rangle h\|_{1/\delta}\big)\,ds\\
&\lesssim \int_0^t \langle s\rangle^{1+\delta/100-(1-2\delta)}\big(\langle s\rangle^{-\delta_0}+\langle s\rangle^{-\delta_0(N-7-7\delta)}\big)\langle s\rangle^{\delta}\langle s\rangle^{-(1-2\delta)}\,ds\|h\|_{X_t}^3\\
&\lesssim \int_0^t \langle s\rangle^{-1-}\,ds\|h\|_{X_t}^3 \lesssim \|h\|_{X_t}^3.
\end{aligned}$$

The estimation of (6.12) is the same as (6.11), and we have

$$\|\mathcal{F}^{-1}((6.12))\|_{2-\delta/100}\lesssim\|h\|_{X_t}^3.$$

The piece (6.10) is exactly in the form given by (6.1). Hence we have finished the estimation of (6.6) and consequently the estimation of (6.2).

We now estimate (6.3). Note that

$$\partial_\xi\left(\frac{\langle\xi\rangle^{4+2\delta}}{\phi_0}\right)\sim\frac{\langle\xi\rangle^{3+2\delta}}{\phi_0}+\langle\xi\rangle^{4+2\delta}\partial_\xi\left(\frac{1}{\phi_0}\right)\sim\langle\xi\rangle^{4+2\delta}\left[\frac{1}{\langle\xi\rangle}\,\frac{1}{\phi_0}+\partial_\xi\left(\frac{1}{\phi_0}\right)\right].$$

By Lemma 2.6, obviously

$$\left|\partial_\xi^\alpha\partial_\eta^\beta\left[\frac{1}{\langle\xi\rangle}\,\frac{1}{\phi_0}+\partial_\xi\left(\frac{1}{\phi_0}\right)\right]\right|\lesssim_{\alpha,\beta}\min\{\langle\xi-\eta\rangle,\langle\eta\rangle\},\qquad\forall\xi,\eta\in\mathbb{R}^2.$$

We then write

$$\begin{aligned}
\partial_\xi\left(\frac{\langle\xi\rangle^{4+2\delta}}{\phi_0}\right)&=\chi_{\langle\xi-\eta\rangle/\langle\eta\rangle\le1}\partial_\xi\left(\frac{\langle\xi\rangle^{4+2\delta}}{\phi_0}\right)+\chi_{\langle\xi-\eta\rangle/\langle\eta\rangle>1}\partial_\xi\left(\frac{\langle\xi\rangle^{4+2\delta}}{\phi_0}\right)\\
&=:\widetilde{m_1}(\xi,\eta)+\widetilde{m_2}(\xi,\eta).
\end{aligned}$$

It is not difficult to check that the functions

$$\begin{aligned}
&\langle\xi\rangle^{\delta}\tilde{m}_1(\xi,\eta)\langle\eta\rangle\langle\eta\rangle^{-(6+3\delta)}\cdot\langle\eta\rangle^{-(1+\delta)}\cdot\langle\xi-\eta\rangle^{-1},\\
&\langle\xi\rangle^{\delta}\tilde{m}_2(\xi,\eta)\langle\eta\rangle\langle\xi-\eta\rangle^{-(6+3\delta)}\cdot\langle\xi-\eta\rangle^{-(1+\delta)}\cdot\langle\eta\rangle^{-1}
\end{aligned}$$

satisfy (2.7). Therefore, by Lemma 2.3, we have

$$\begin{aligned}
\|\mathcal{F}^{-1}((6.3))\|_{2-\delta/100} &\lesssim \int_0^t \langle s\rangle^{\delta/100} \\
&\quad \cdot \big\| T_{\langle\xi\rangle^{\delta}\tilde{m}_1(\xi,\eta)\langle\eta\rangle\langle\eta\rangle^{-(6+3\delta)}\cdot\langle\eta\rangle^{-(1+\delta)}\cdot\langle\xi-\eta\rangle^{-1}}\big(\langle\nabla\rangle\mathcal{R}h, \mathcal{R}\langle\nabla\rangle^{7+4\delta}(\mathcal{R}h\,\mathcal{R}h)\big)\big\|_{2-\delta/100}\,ds \\
&\quad + \int_0^t \langle s\rangle^{\delta/100} \\
&\quad \cdot \big\| T_{\langle\xi\rangle^{\delta}\tilde{m}_2(\xi,\eta)\langle\eta\rangle\langle\xi-\eta\rangle^{-(6+3\delta)}\cdot\langle\xi-\eta\rangle^{-(1+\delta)}\cdot\langle\eta\rangle^{-1}}\big(\langle\nabla\rangle^{7+4\delta}\mathcal{R}h, \mathcal{R}\langle\nabla\rangle(\mathcal{R}h\,\mathcal{R}h)\big)\big\|_{2-\delta/100}\,ds \\
&\lesssim \int_0^t \langle s\rangle^{\delta/100}\|\langle\nabla\rangle h\|_{1/\delta}\|\langle\nabla\rangle^{7+4\delta}(\mathcal{R}h\,\mathcal{R}h)\|_{(\frac{1}{2-\delta/100}-\delta)^{-1}}\,ds \\
&\quad + \int_0^t \langle s\rangle^{\delta/100}\|\langle\nabla\rangle^{7+4\delta}h\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}\|\langle\nabla\rangle(\mathcal{R}h\,\mathcal{R}h)\|_{1/(2\delta)}\,ds \\
&\lesssim \int_0^t \langle s\rangle^{-1-}\,ds\,\|h\|_{X_t}^3 \lesssim \|h\|_{X_t}^3.
\end{aligned}$$

Finally we estimate (6.4). We decompose it as

$$\begin{aligned}
(6.4) = \int_0^t \int e^{-is\phi}\frac{\langle\xi\rangle^{4+2\delta}}{\phi_0(\xi,\eta)}\langle\eta\rangle\chi_{|\xi-\eta|\le\langle s\rangle^{\delta_0}} \\
\cdot \partial_\xi\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds \qquad (6.13) \\
+ \int_0^t \int e^{-is\phi}\frac{\langle\xi\rangle^{4+2\delta}}{\phi_0(\xi,\eta)}\langle\eta\rangle\chi_{|\xi-\eta|>\langle s\rangle^{\delta_0}} \\
\cdot \partial_\xi\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds. \qquad (6.14)
\end{aligned}$$

For (6.13), we note that by Lemma 2.6 the function

$$\tilde{m}(\xi,\eta) = \frac{\langle\xi\rangle^{4+3\delta}}{\phi_0(\xi,\eta)}\langle\eta\rangle\chi_{|\xi-\eta|\le\langle s\rangle^{\delta_0}}\langle\xi-\eta\rangle^{-(6+14\delta)}\langle\eta\rangle^{-(6+14\delta)}$$

satisfies (2.7). Therefore, by Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
\|\mathcal{F}^{-1}((6.13))\|_{2-\delta/100} &\lesssim \int_0^t \langle s\rangle^{\delta/100}\|T_{\tilde{m}(\xi,\eta)}\big(\langle\nabla\rangle^{6+4\delta}P_{\le\langle s\rangle^{\delta_0}}e^{is\langle\nabla\rangle}\mathcal{F}^{-1}(\partial_\xi(\widehat{\mathcal{R}f})), \\
&\qquad \langle\nabla\rangle^{6+4\delta}\mathcal{R}(\mathcal{R}h\,\mathcal{R}h)\big)\|_{2-\delta/100}\,ds \\
&\lesssim \int_0^t \langle s\rangle^{\delta/100}\|\langle\nabla\rangle^{6+4\delta}P_{\le\langle s\rangle^{\delta_0}}e^{is\langle\nabla\rangle}(\mathcal{F}^{-1}(\partial_\xi(\widehat{\mathcal{R}f})))\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}} \\
&\qquad \cdot \|\langle\nabla\rangle^{6+4\delta}(\mathcal{R}h\,\mathcal{R}h)\|_{1/(2\delta)}\,ds.
\end{aligned}$$

To continue we need a lemma.

Lemma 6.2. *For any dyadic $M \geq 1$, and $2+\delta < p < \infty$, we have*

$$\|P_{<M} e^{it\langle\nabla\rangle} \mathcal{F}^{-1}(\partial_\xi(\widehat{\mathcal{R}f}))\|_p \lesssim M^{1+\frac{2}{2+\delta}-\frac{4}{p}} \langle t\rangle^{1-2/p} \|\langle x\rangle f\|_{2+\delta}.$$

Proof of Lemma 6.2. By Lemmas 2.2 and 2.5, we have

$$\begin{aligned}\|P_{<M} e^{it\langle\nabla\rangle} \mathcal{F}^{-1}(\partial_\xi(\widehat{\mathcal{R}f}))\|_p &\lesssim M^{1-2/p}\langle t\rangle^{1-2/p}\big(\big\||\nabla|^{-1} f\big\|_p + \|P_{<M}(xf)\|_p\big)\\ &\lesssim M^{1-2/p}\langle t\rangle^{1-2/p}(\|\langle x\rangle f\|_{2+\delta} + M^{2(\frac{1}{2+\delta}-\frac{1}{p})}\|xf\|_{2+\delta})\\ &\lesssim M^{1+\frac{2}{2+\delta}-\frac{4}{p}}\langle t\rangle^{1-2/p}\|\langle x\rangle f\|_{2+\delta}. \qquad \square\end{aligned}$$

By Lemma 6.2, we have

$$\begin{aligned}&\|\langle\nabla\rangle^{6+4\delta} P_{\le\langle s\rangle^{\delta_0}} e^{is\langle\nabla\rangle}(\mathcal{F}^{-1}(\partial_\xi(\widehat{\mathcal{R}f})))\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}\\ &\quad\lesssim \langle s\rangle^{\delta_0(6+4\delta)}\langle s\rangle^{\delta_0(1+\frac{2}{2+\delta}-4(\frac{1}{2-\delta/100}-2\delta))}\langle s\rangle^{1-2(\frac{1}{2-\delta/100}-2\delta)}\|\langle x\rangle f\|_{2+\delta}\\ &\quad\lesssim \langle s\rangle^{7\delta_0+4\delta}\|\langle x\rangle f\|_{2+\delta}.\end{aligned}$$

By Sobolev embedding and Lemma 5.2,

$$\begin{aligned}\|\langle\nabla\rangle^{6+4\delta}(\mathcal{R}h\,\mathcal{R}h)\|_{1/(2\delta)} &\lesssim \|\langle\nabla\rangle^{6+4\delta}h\|_{1/\delta}\|h\|_{1/\delta} \lesssim \|\langle\nabla\rangle^7 h\|_{16/7}\|h\|_{1/\delta}\\ &\lesssim \langle s\rangle^{-1/8}\langle s\rangle^{-(1-2\delta)}\langle s\rangle^{\delta}\|h\|_{X_t}^2 = \langle s\rangle^{-9/8+3\delta}\|h\|_{X_t}^2.\end{aligned}$$

Therefore

$$\|\mathcal{F}^{-1}((6.13))\|_{2-\delta/100} \lesssim \int_0^t \langle s\rangle^{\delta/100+4\delta+7\delta_0-9/8+3\delta}\,ds\,\|h\|_{X_t}^3 \lesssim \|h\|_{X_t}^3.$$

For (6.14), we decompose

$$\begin{aligned}(6.14) = \int_0^t\int e^{-is\phi}\frac{\langle\xi\rangle^{4+2\delta}}{\phi_0(\xi,\eta)}\langle\eta\rangle\chi_{|\xi-\eta|>\langle s\rangle^{\delta_0}}\chi_{\langle\xi-\eta\rangle/\langle\eta\rangle\le 1}\\ \cdot\partial_\xi\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds \qquad (6.15)\\ +\int_0^t\int e^{-is\phi}\frac{\langle\xi\rangle^{4+2\delta}}{\phi_0(\xi,\eta)}\langle\eta\rangle\chi_{|\xi-\eta|>\langle s\rangle^{\delta_0}}\chi_{\langle\xi-\eta\rangle/\langle\eta\rangle>1}\\ \cdot\partial_\xi\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds. \qquad (6.16)\end{aligned}$$

For (6.15), we note that by Lemma 2.6 the function

$$\widetilde{m}(\xi,\eta) = \frac{\langle\xi\rangle^{4+3\delta}}{\phi_0(\xi,\eta)}\langle\eta\rangle\chi_{|\xi-\eta|>\langle s\rangle^{\delta_0}}\chi_{\langle\xi-\eta\rangle/\langle\eta\rangle\le 1}\langle\xi-\eta\rangle^{-2+10\delta}\langle\eta\rangle^{-(6+14\delta)}$$

satisfies (2.7). Therefore by Lemmas 2.2 and 2.3, we have

$$\|\mathcal{F}^{-1}((6.15))\|_{2-\delta/100} \lesssim \int_0^t \langle s\rangle^{\delta/100} \big\| T_{\widetilde{m}(\xi,\eta)}\big(\langle\nabla\rangle^{2-10\delta} e^{is\langle\nabla\rangle}\big(\mathcal{F}^{-1}\big(\partial_\xi(\widehat{\mathcal{R}f})\big)\big), \langle\nabla\rangle^{6+14\delta}\mathcal{R}(\mathcal{R}h\,\mathcal{R}h)\big)\big\|_{2-\delta/100}\,ds$$
$$\lesssim \int_0^t \langle s\rangle^{\delta/100} \|\langle\nabla\rangle^{2-10\delta} e^{is\langle\nabla\rangle}(\mathcal{F}^{-1}(\partial_\xi(\widehat{\mathcal{R}f})))\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}} \cdot \|\langle\nabla\rangle^{6+14\delta}(\mathcal{R}h\,\mathcal{R}h)\|_{1/(2\delta)}\,ds.$$

By Lemmas 2.2 and 2.5, we have

$$\|\langle\nabla\rangle^{2-10\delta} e^{is\langle\nabla\rangle}(\mathcal{F}^{-1}(\partial_\xi(\widehat{\mathcal{R}f})))\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}$$
$$\lesssim \langle s\rangle^{4\delta}\big[\|\langle\nabla\rangle^{2-6\delta}|\nabla|^{-1} f\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}} + \|\langle\nabla\rangle^{2-6\delta}(xf)\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}\big]$$
$$\lesssim \langle s\rangle^{4\delta}\|\langle x\rangle\langle\nabla\rangle^2 f\|_{2+\delta}.$$

On the other hand by Lemma 5.2,

$$\|\langle\nabla\rangle^{6+14\delta}(\mathcal{R}h\,\mathcal{R}h)\|_{1/(2\delta)} \lesssim \|\langle\nabla\rangle^{6+14\delta}h\|_{1/\delta}\|h\|_{1/\delta} \lesssim \|\langle\nabla\rangle^7 h\|_{16/7}\|h\|_{1/\delta}$$
$$\lesssim \langle s\rangle^{-1/8}\langle s\rangle^{-(1-2\delta)}\langle s\rangle^{\delta}\|h\|_{X_t}^2 = \langle s\rangle^{-9/8+3\delta}\|h\|_{X_t}^2.$$

Therefore

$$\|\mathcal{F}^{-1}((6.15))\|_{2-\delta/100} \lesssim \int_0^t \langle s\rangle^{\delta/100+4\delta-9/8+3\delta}\,ds\|h\|_{X_t}^3 \lesssim \|h\|_{X_t}^3.$$

For (6.16), we use the identity

$$\partial_\xi(\widehat{\mathcal{R}f}(s,\xi-\eta)) = -\partial_\eta(\widehat{\mathcal{R}f}(s,\xi-\eta))$$

to integrate by parts in η. This gives

$$(6.16) = \int_0^t\int (-is\partial_\eta\phi)e^{-is\phi}\frac{\langle\xi\rangle^{4+2\delta}}{\phi_0}\langle\eta\rangle\chi_{|\xi-\eta|>\langle s\rangle^{\delta_0}}\chi_{\langle\xi-\eta\rangle/\langle\eta\rangle>1} \cdot \widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds \tag{6.17}$$
$$+\int_0^t\int e^{-is\phi}\langle\xi\rangle^{4+2\delta}\partial_\eta\Big(\frac{1}{\phi_0}\langle\eta\rangle\chi_{|\xi-\eta|>\langle s\rangle^{\delta_0}}\chi_{\langle\xi-\eta\rangle/\langle\eta\rangle>1}\Big) \cdot \widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds \tag{6.18}$$
$$+\int_0^t\int e^{-is\phi}\frac{\langle\xi\rangle^{4+2\delta}}{\phi_0}\langle\eta\rangle\chi_{|\xi-\eta|>\langle s\rangle^{\delta_0}}\chi_{\langle\xi-\eta\rangle/\langle\eta\rangle>1} \cdot \widehat{\mathcal{R}f}(s,\xi-\eta)O(1/|\eta|)\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds \tag{6.19}$$
$$+\int_0^t\int e^{-is\phi}\frac{\langle\xi\rangle^{4+2\delta}}{\phi_0}\langle\eta\rangle\chi_{|\xi-\eta|>\langle s\rangle^{\delta_0}}\chi_{\langle\xi-\eta\rangle/\langle\eta\rangle>1} \cdot \widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\partial_\eta\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds. \tag{6.20}$$

The estimation of (6.17) is exactly the same as that of (6.5). The only change is that $\partial_\xi\phi$ is now replaced by $\partial_\eta\phi$. But in the estimates there only the boundedness of $\partial_\xi\phi$ (and its derivatives) is used. Therefore we have

$$\big\|\mathcal{F}^{-1}((6.17))\big\|_{2-\delta/100} \lesssim \|h\|_{X_t}^3.$$

The estimation of (6.18) is similar to the estimation of (6.3), and we have

$$\big\|\mathcal{F}^{-1}((6.18))\big\|_{2-\delta/100} \lesssim \|h\|_{X_t}^3.$$

For (6.19), we can decompose

$$O(1/|\eta|) = O(1/|\eta|)\chi_{|\eta|<1} + O(1/|\eta|)\chi_{|\eta|\geq 1}.$$

The piece corresponding to $O(1/|\eta|)\chi_{|\eta|\geq 1}$ is estimated in the same way as in (6.18). For the low frequency piece, we note that the function

$$\widetilde{m}(\xi,\eta) = \frac{\langle\xi\rangle^{4+3\delta}}{\phi_0}\langle\eta\rangle\chi_{|\xi-\eta|>\langle s\rangle^{\delta_0}}\chi_{\langle\xi-\eta\rangle/\langle\eta\rangle>1}\chi_{|\eta|<1}\cdot\langle\xi-\eta\rangle^{-(5+4\delta)}$$

satisfies (2.7). Therefore by Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
&\|\mathcal{F}^{-1}(6.19)\|_{2-\delta/100}\\
&\quad\lesssim \|h\|_{X_t}^3 + \int_0^t \langle s\rangle^{\delta/100}\big\|T_{\widetilde{m}(\xi,\eta)}\big(\langle\nabla\rangle^{5+4\delta}P_{\gtrsim\langle s\rangle^{\delta_0}}\mathcal{R}h,\,|\nabla|^{-1}(\mathcal{R}h,\mathcal{R}h)\big)\big\|_{2-\delta/100}\\
&\quad\lesssim \|h\|_{X_t}^3 + \int_0^t \langle s\rangle^{\delta/100}\|\langle\nabla\rangle^{5+4\delta}P_{\gtrsim\langle s\rangle^{\delta_0}}e^{is\langle\nabla\rangle}h\|_2\big\||\nabla|^{-1}(\mathcal{R}h\,\mathcal{R}h)\big\|_{(\frac{1}{2-\delta/100}-\frac12)^{-1}}\,ds\\
&\quad\lesssim \|h\|_{X_t}^3 + \int_0^t \langle s\rangle^{\delta/100-\delta_0}\|h\|_{H^{N'}}\|\mathcal{R}h\mathcal{R}h\|_{2-\delta/100}\,ds\\
&\quad\lesssim \|h\|_{X_t}^3 + \int_0^t \langle s\rangle^{\delta/100-\delta_0}\langle s\rangle^{-(1-2\delta)}\,ds\,\|h\|_{X_t}^3 \lesssim \|h\|_{X_t}^3.
\end{aligned}$$

Finally, we deal with (6.20). We decompose it further as

$$\begin{aligned}
(6.20) = &\int_0^t\int e^{-is\phi}\frac{\langle\xi\rangle^{4+2\delta}}{\phi_0}\langle\eta\rangle\chi_{|\xi-\eta|>\langle s\rangle^{\delta_0}}\chi_{\langle\xi-\eta\rangle/\langle\eta\rangle>1}\\
&\cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\partial_\eta\big(\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\leq 1}\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds \qquad (6.21)\\
&+\int_0^t\int e^{-is\phi}\frac{\langle\xi\rangle^{4+2\delta}}{\phi_0}\langle\eta\rangle\chi_{|\xi-\eta|>\langle s\rangle^{\delta_0}}\chi_{\langle\xi-\eta\rangle/\langle\eta\rangle>1}\\
&\cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\partial_\eta\big(\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle>1}\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds. \qquad (6.22)
\end{aligned}$$

We only need to estimate (6.21). The piece (6.22) can be estimated similarly after the change of variable $\sigma\mapsto\eta-\sigma$. Now

$$(6.21) = \int_0^t \int e^{-is\phi} \frac{\langle\xi\rangle^{4+2\delta}}{\phi_0} \langle\eta\rangle \chi_{|\xi-\eta|>\langle s\rangle^{\delta_0}} \chi_{\langle\xi-\eta\rangle/\langle\eta\rangle>1}$$
$$\cdot \widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\partial_\eta(\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le1})\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds \tag{6.23}$$
$$+\int_0^t \int e^{-is\phi} \frac{\langle\xi\rangle^{4+2\delta}}{\phi_0} \langle\eta\rangle \chi_{|\xi-\eta|>\langle s\rangle^{\delta_0}} \chi_{\langle\xi-\eta\rangle/\langle\eta\rangle>1}$$
$$\cdot \widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le1}\partial_\eta[\widehat{\mathcal{R}f}(s,\eta-\sigma)]\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds. \tag{6.24}$$

We first deal with (6.23). Note that the function

$$\widetilde{m}(\xi,\eta) = \frac{\langle\xi\rangle^{4+3\delta}}{\phi_0}\langle\eta\rangle\chi_{|\xi-\eta|>\langle s\rangle^{\delta_0}}\chi_{\langle\xi-\eta\rangle/\langle\eta\rangle>1}\langle\xi-\eta\rangle^{-(7+4\delta)}\langle\eta\rangle^{-1}$$

satisfies (2.7). Therefore by Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
&\|\mathcal{F}^{-1}((6.23))\|_{2-\delta/100}\\
&\lesssim \int_0^t \langle s\rangle^{\delta/100}\big\|T_{\widetilde{m}(\xi,\eta)}\big(\langle\nabla\rangle^{7+4\delta}\mathcal{R}h,\langle\nabla\rangle\mathcal{R}T_{\partial_\eta(\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le1})}(\mathcal{R}h,\mathcal{R}h)\big)\big\|_{2-\delta/100}\,ds\\
&\lesssim \int_0^t \langle s\rangle^{\delta/100}\|\langle\nabla\rangle^{7+4\delta}h\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}\big\|\langle\nabla\rangle T_{\partial_\eta(\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le1})}(\mathcal{R}h,\mathcal{R}h)\big\|_{1/(2\delta)}\,ds.
\end{aligned}$$

Now note that

$$\begin{aligned}
&\|\langle\nabla\rangle T_{\partial_\eta(\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le1})}(\mathcal{R}h,\mathcal{R}h)\|_{1/(2\delta)}\\
&\quad\lesssim \|T_{\partial_\eta(\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le1})}(\mathcal{R}h,\mathcal{R}h)\|_{1/(2\delta)}+\|\nabla T_{\partial_\eta(\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le1})}(\mathcal{R}h,\mathcal{R}h)\|_{1/(2\delta)}\\
&\quad\lesssim \|T_{\partial_\eta(\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le1})}(\mathcal{R}h,\mathcal{R}h)\|_{1/(2\delta)}+\|T_{\partial_\eta(\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le1})}(\nabla\mathcal{R}h,\mathcal{R}h)\|_{1/(2\delta)}\\
&\qquad+\|T_{\partial_\eta(\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le1})}(\mathcal{R}h,\nabla\mathcal{R}h)\|_{1/(2\delta)}.
\end{aligned}$$

It is not difficult to check that

$$|\partial_\eta^\alpha\partial_\sigma^\beta(\partial_\eta(\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le1}))|\lesssim(\langle\eta\rangle+\langle\sigma\rangle)^{-(|\alpha|+|\beta|)}.$$

Therefore, $\partial_\eta(\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le1})$ is a standard Coifman–Meyer multiplier, and we have

$$\|\langle\nabla\rangle T_{\partial_\eta(\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le1})}(\mathcal{R}h,\mathcal{R}h)\|_{1/(2\delta)}\lesssim\|\langle\nabla\rangle h\|_{1/\delta}^2\lesssim\langle s\rangle^{-2(1-2\delta)}\|h\|_{X_t}^2.$$

Hence,

$$\|\mathcal{F}^{-1}((6.23))\|_{2-\delta/100}\lesssim\int_0^t\langle s\rangle^{\delta/100+\delta}\langle s\rangle^{-2(1-2\delta)}\,ds\,\|h\|_{X_t}^3\lesssim\|h\|_{X_t}^3.$$

It remains to estimate (6.24). Note that the function

$$\widetilde{m}(\xi,\eta) = \frac{\langle\xi\rangle^{4+2\delta}}{\phi_0}\langle\eta\rangle\chi_{|\xi-\eta|>\langle s\rangle^{\delta_0}}\chi_{\langle\xi-\eta\rangle/\langle\eta\rangle>1}\langle\xi-\eta\rangle^{-(6+15\delta)}\langle\eta\rangle^{-2+10\delta}$$

satisfies (2.7). Therefore by Lemmas 2.2 and 2.3, we have

$$\begin{aligned}\|\mathcal{F}^{-1}((6.24))\|_{2-\delta/100} &\lesssim \int_0^t \langle s\rangle^{\delta/100}\big\|T_{\widetilde{m}(\xi,\eta)}\big(\langle\nabla\rangle^{6+15\delta}\mathcal{R}h,\\ &\qquad \langle\nabla\rangle^{2-10\delta}\mathcal{R}T_{\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le 1}}(e^{is\langle\nabla\rangle}(\mathcal{F}^{-1}(\partial_\eta(\widehat{\mathcal{R}f}))),\mathcal{R}h)\big)\big\|_{2-\delta/100}\,ds\\ &\lesssim \int_0^t \langle s\rangle^{\delta/100}\|\langle\nabla\rangle^{6+15\delta}h\|_{1/\delta}\\ &\qquad \cdot\|\langle\nabla\rangle^{2-10\delta}T_{\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le 1}}(e^{is\langle\nabla\rangle}\mathcal{F}^{-1}(\partial_\eta(\widehat{\mathcal{R}f})),\mathcal{R}h)\|_{(\frac{1}{2-\delta/100}-\delta)^{-1}}\,ds.\end{aligned}$$

Now we make a Littlewood–Paley decomposition and write

$$\begin{aligned}&\big\|\langle\nabla\rangle^{2-10\delta}T_{\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le 1}}\big(e^{is\langle\nabla\rangle}(\mathcal{F}^{-1}(\partial_\eta(\widehat{\mathcal{R}f}))),\mathcal{R}h\big)\big\|_{(\frac{1}{2-\delta/100}-\delta)^{-1}}\\ &\lesssim \big\|\langle\nabla\rangle^{2-10\delta}T_{\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le 1}}\big(P_{<1}e^{is\langle\nabla\rangle}(\mathcal{F}^{-1}(\partial_\eta(\widehat{\mathcal{R}f}))),\mathcal{R}h\big)\big\|_{(\frac{1}{2-\delta/100}-\delta)^{-1}} \qquad (6.25)\\ &\quad+\sum_{M\ge 1}\big\|\langle\nabla\rangle^{2-10\delta}T_{\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le 1}}\big(P_Me^{is\langle\nabla\rangle}(\mathcal{F}^{-1}(\partial_\eta(\widehat{\mathcal{R}f}))),\mathcal{R}h\big)\big\|_{(\frac{1}{2-\delta/100}-\delta)^{-1}}. \qquad (6.26)\end{aligned}$$

For the low frequency piece (6.25), we note that by the cut-off $\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le 1}$ and $P_{<1}$,

$$\langle\eta\rangle\le\langle\sigma\rangle+\langle\eta-\sigma\rangle\lesssim\langle\eta-\sigma\rangle\lesssim 1.$$

Therefore, using the fact that $\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le 1}$ is a Coifman–Meyer multiplier, we have

$$\begin{aligned}(6.25)&\lesssim \big\|P_{\lesssim 1}T_{\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le 1}}\big(P_{<1}e^{is\langle\nabla\rangle}(\mathcal{F}^{-1}(\partial_\eta(\widehat{\mathcal{R}f}))),\mathcal{R}h\big)\big\|_{(\frac{1}{2-\delta/100}-\delta)^{-1}}\\ &\lesssim \|P_{<1}e^{is\langle\nabla\rangle}\mathcal{F}^{-1}(\partial_\eta(\widehat{\mathcal{R}f}))\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}\|h\|_{1/\delta}\\ &\lesssim \langle s\rangle^{4\delta}\|\langle x\rangle f\|_{2+\delta}\langle s\rangle^{-1+2\delta}\|h\|_{X_t},\end{aligned}$$

where in the last inequality we have used Lemma 6.2. Hence,

$$(6.25)\lesssim\langle s\rangle^{-1+6\delta}\|h\|_{X_t}^2.$$

For (6.26), thanks to the localization P_M and $\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le 1}$, it follows easily that

$$\langle\eta\rangle\le\langle\sigma\rangle+\langle\eta-\sigma\rangle\lesssim\langle\eta-\sigma\rangle\lesssim M.$$

Therefore by Lemma 6.2,

$$\begin{aligned}(6.26)&\lesssim \sum_{M\ge1}M^{2-10\delta}\big\|T_{\chi_{\langle\sigma\rangle/\langle\eta-\sigma\rangle\le 1}}\big(P_Me^{is\langle\nabla\rangle}(\mathcal{F}^{-1}(\partial_\eta(\widehat{\mathcal{R}f}))),\mathcal{R}h\big)\big\|_{(\frac{1}{2-\delta/100}-\delta)^{-1}}\\ &\lesssim \sum_{M\ge1}M^{2-10\delta}\|P_Me^{is\langle\nabla\rangle}(\mathcal{F}^{-1}(\partial_\eta(\widehat{\mathcal{R}f})))\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}\|h\|_{1/\delta}\\ &\lesssim \sum_{M\ge1}M^{2-10\delta}\langle s\rangle^{4\delta}M^{4\delta}\big[\|P_M|\nabla|^{-1}f\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}+\|P_M(xf)\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}\big]\|h\|_{1/\delta}\\ &\lesssim \sum_{M\ge1}M^{-2\delta}\langle s\rangle^{-1+6\delta}[\|\langle\nabla\rangle^2f\|_2+\|\langle\nabla\rangle^2(xf)\|_{2+\delta}]\|h\|_{X_t}\lesssim\langle s\rangle^{-1+6\delta}\|h\|_{X_t}^2.\end{aligned}$$

Collecting the estimates and using Lemma 5.2, we obtain

$$\begin{aligned}\|\mathcal{F}^{-1}((6.24))\|_{2-\delta/100} &\lesssim \int_0^t \langle s\rangle^{\delta/100}\|\langle\nabla\rangle^{6+15\delta}h\|_{1/\delta}\langle s\rangle^{-1+6\delta}\,ds\,\|h\|_{X_t}^2\\ &\lesssim \int_0^t \langle s\rangle^{-1+\delta/100+6\delta}\|\langle\nabla\rangle^{7}h\|_{16/7}\,ds\,\|h\|_{X_t}^2\\ &\lesssim \int_0^t \langle s\rangle^{-1+\delta/100+7\delta}\langle s\rangle^{-1/8}\,ds\,\|h\|_{X_t}^3 \lesssim \|h\|_{X_t}^3.\end{aligned}$$

7. Control of cubic interactions: the low frequency piece

In the previous section, we controlled the high frequency part of the cubic interaction term. In this section, we analyze in detail the low frequency piece. The main result of this section is the following

Proposition 7.1. *We have*

$$\|f_{\text{low}}(\tau)\|_{L_\tau^\infty L_x^{2-\delta/100}([0,t])} \lesssim \|h\|_{X_t}^3 + \|h\|_{X_t}^4,$$

where

$$\begin{aligned}\hat{f}_{\text{low}}(t,\xi) = \int_0^t\int e^{-is\phi}\frac{s\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle m_{\text{low}}(\xi,\eta,\sigma)\\ \cdot\frac{\eta}{|\eta|}\widehat{\mathcal{R}f}(s,\xi-\eta)\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\,d\sigma\,d\eta\,ds \qquad (7.1)\end{aligned}$$

and

$$m_{\text{low}}(\xi,\eta,\sigma) = \chi_{|\xi-\eta|\le\langle s\rangle^{\delta_0}}\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\sigma|\le\langle s\rangle^{\delta_0}}. \qquad (7.2)$$

The rest of this section is devoted to the proof of this proposition. The analysis will depend on the explicit form of the phase function $\phi(\xi,\eta,\sigma)$. We discuss several cases.

Case 1:

$$\phi(\xi,\eta,\sigma) = \langle\xi\rangle - \langle\xi-\eta\rangle + \langle\eta-\sigma\rangle - \langle\sigma\rangle. \qquad (7.3)$$

By Lemma 2.8, we have

$$\partial_\xi\phi = Q_1(\xi,\eta)Q_2(\eta,\sigma)\partial_\sigma\phi,$$

where

$$|\partial_\xi^\alpha\partial_\eta^\beta Q_1(\xi,\eta)| \lesssim_{\alpha,\beta} 1, \qquad |\partial_\eta^\alpha\partial_\sigma^\beta Q_2(\eta,\sigma)| \lesssim_{\alpha,\beta} \langle|\eta|+|\sigma|\rangle^3. \qquad (7.4)$$

We now write

$$s\partial_\xi\phi e^{-is\phi} = iQ_1(\xi,\eta)Q_2(\eta,\sigma)\partial_\sigma(e^{-is\phi}). \qquad (7.5)$$

Plugging (7.5) into (7.1) and integrating by parts in σ, we then obtain

$$\begin{aligned}
&\hat{f}_{\mathrm{low}}(t,\xi)\\
&= -i\int_0^t\int e^{-is\phi}\frac{Q_1(\xi,\eta)}{\phi_0(\xi,\eta)}\partial_\sigma\big(Q_2(\eta,\sigma)\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\sigma|\le\langle s\rangle^{\delta_0}}\big)\\
&\quad\cdot\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\chi_{|\xi-\eta|\le\langle s\rangle^{\delta_0}}\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds \qquad (7.6)\\
&\quad -i\int_0^t\int e^{-is\phi}\frac{Q_1(\xi,\eta)}{\phi_0(\xi,\eta)}Q_2(\eta,\sigma)\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\xi-\eta|\le\langle s\rangle^{\delta_0}}\\
&\qquad\cdot\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\frac{\eta}{|\eta|}\widehat{\mathcal{R}f}(s,\xi-\eta)(\partial_\sigma\widehat{\mathcal{R}f}(s,\eta-\sigma))\widehat{\mathcal{R}f}(s,\sigma)\,d\sigma\,d\eta\,ds \qquad (7.7)\\
&\quad -i\int_0^t\int e^{-is\phi}\frac{Q_1(\xi,\eta)}{\phi_0(\xi,\eta)}Q_2(\eta,\sigma)\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\xi-\eta|\le\langle s\rangle^{\delta_0}}\\
&\qquad\cdot\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\frac{\eta}{|\eta|}\widehat{\mathcal{R}f}(s,\xi-\eta)\widehat{\mathcal{R}f}(s,\eta-\sigma)\partial_\sigma\widehat{\mathcal{R}f}(s,\sigma)\,d\sigma\,d\eta\,ds. \qquad (7.8)
\end{aligned}$$

We first estimate (7.6). By Lemma 2.2, we have

$$\begin{aligned}
&\|\mathcal{F}^{-1}((7.6))\|_{2-\delta/100}\lesssim\int_0^t\langle s\rangle^{\delta/100+\delta_0\delta/100+\delta_0(4+2\delta)}\\
&\cdot\Big\|T_{\frac{Q_1(\xi,\eta)}{\phi_0(\xi,\eta)}\langle\eta\rangle}\big(P_{\le\langle s\rangle^{\delta_0}}\mathcal{R}h,\mathcal{R}T_{\partial_\sigma(Q_2(\eta,\sigma)\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\sigma|\le\langle s\rangle^{\delta_0}})}(\mathcal{R}h,\mathcal{R}h)\big)\Big\|_{2-\delta/100}\,ds. \qquad (7.9)
\end{aligned}$$

By (7.4) and Lemma 2.6, it is easy to check that the functions

$$\begin{aligned}
\widetilde{m_1}(\xi,\eta)&=\frac{Q_1(\xi,\eta)}{\phi_0(\xi,\eta)}\langle\eta\rangle\langle\xi-\eta\rangle^{-2-\delta/200}\langle\eta\rangle^{-2-\delta/200},\\
\widetilde{m_2}(\eta,\sigma)&=\partial_\sigma\big(Q_2(\eta,\sigma)\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\sigma|\le\langle s\rangle^{\delta_0}}\big)\langle\eta-\sigma\rangle^{-4-\delta/200}\langle\sigma\rangle^{-4-\delta/200}
\end{aligned}$$

satisfy (2.7). By Lemma 2.3, we have

$$\begin{aligned}
&\Big\|T_{\frac{Q_1(\xi,\eta)}{\phi_0(\xi,\eta)}\langle\eta\rangle}(P_{\le\langle s\rangle^{\delta_0}}\mathcal{R}h,\mathcal{R}T_{\partial_\sigma(Q_2(\eta,\sigma)\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\sigma|\le\langle s\rangle^{\delta_0}})}(\mathcal{R}h,\mathcal{R}h))\Big\|_{2-\delta/100}\\
&\quad=\Big\|T_{\widetilde{m_1}(\xi,\eta)}(P_{\le\langle s\rangle^{\delta_0}}\langle\nabla\rangle^{2+\delta/200}\mathcal{R}h,\,P_{\lesssim\langle s\rangle^{\delta_0}}\mathcal{R}\langle\nabla\rangle^{2+\delta/200}\\
&\qquad\qquad\cdot T_{\widetilde{m_2}(\eta,\sigma)}\big(P_{\le\langle s\rangle^{\delta_0}}\langle\nabla\rangle^{4+\delta/200}\mathcal{R}h,\,P_{\lesssim\langle s\rangle^{\delta_0}}\langle\nabla\rangle^{4+\delta/200}\mathcal{R}h\big)\Big)\Big\|_{2-\delta/100}\\
&\quad\lesssim\|P_{\le\langle s\rangle^{\delta_0}}\langle\nabla\rangle^{2+\delta/200}\mathcal{R}h\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}\langle s\rangle^{(2+\delta/200)\delta_0}\|\langle\nabla\rangle^{4+\delta/200}P_{\le\langle s\rangle^{\delta_0}}h\|_{1/\delta}^2\\
&\quad\lesssim\langle s\rangle^{(2+\delta/200)\delta_0+2(3+\delta/200)\delta_0-2(1-2\delta)}\|h\|_{X_t}^3\lesssim\langle s\rangle^{(8+\delta/100)\delta_0-2+5\delta}\|h\|_{X_t}^3.
\end{aligned}$$

Plugging the above estimate into (7.9), we obtain

$$\|\mathcal{F}^{-1}((7.6))\|_{2-\delta/100}\lesssim\int_0^t\langle s\rangle^{(12+\delta/50+2\delta)\delta_0+(5+1/100)\delta-2}\,ds\,\|h\|_{X_t}^3\lesssim\|h\|_{X_t}^3.$$

The estimation of (7.7) is similar. By Lemma 6.2 we have, for some $\tilde{m}_3(\eta,\sigma)$ similar to $\tilde{m}_2(\eta,\sigma)$,

$$\begin{aligned}
&\|\mathcal{F}^{-1}((7.7))\|_{2-\delta/100}\\
&\quad\lesssim \int_0^t \langle s\rangle^{\delta/100+\delta_0\delta/100+\delta_0(4+2\delta)} \big\| T_{\widetilde{m_1}(\xi,\eta)}(P_{\le\langle s\rangle^{\delta_0}}\langle\nabla\rangle^{2+\delta/200}\mathcal{R}h,\\
&\qquad\qquad P_{\lesssim\langle s\rangle^{\delta_0}}\mathcal{R}\langle\nabla\rangle^{2+\delta/200} T_{\widetilde{m_3}(\eta,\sigma)}(P_{\le\langle s\rangle^{\delta_0}}\langle\nabla\rangle^{4+\delta/200} e^{is\langle\nabla\rangle}\mathcal{F}^{-1}(\partial_\sigma(\widehat{\mathcal{R}f})),\\
&\qquad\qquad\qquad P_{\le\langle s\rangle^{\delta_0}}\langle\nabla\rangle^{4+\delta/200}\mathcal{R}h))\big\|_{2-\delta/100}\,ds\\
&\quad\lesssim \int_0^t \langle s\rangle^{\delta/100+\delta_0\delta/100+(4+2\delta)\delta_0}\|\langle\nabla\rangle^{2+\delta/200}P_{\le\langle s\rangle^{\delta_0}}h\|_{1/\delta}\\
&\qquad\cdot\langle s\rangle^{(2+\delta/200)\delta_0}\langle s\rangle^{(4+\delta/200)\delta_0}\|P_{\le\langle s\rangle^{\delta_0}}e^{is\langle\nabla\rangle}\mathcal{F}^{-1}(\partial_\sigma(\widehat{\mathcal{R}f}))\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}\\
&\qquad\cdot\|\langle\nabla\rangle^{4+\delta/200}P_{\le\langle s\rangle^{\delta_0}}h\|_{1/\delta}\,ds\\
&\quad\lesssim \int_0^t \langle s\rangle^{\delta/100+\delta_0(4+2\delta+\delta/100)}\langle s\rangle^{\delta_0(4+\delta/100)-2(1-2\delta)}\langle s\rangle^{\delta_0(6+\delta/100)}\\
&\qquad\cdot\langle s\rangle^{[1+\frac{2}{2+\delta}-4(\frac{1}{2-\delta/100}-2\delta)]\delta_0+1-2(\frac{1}{2-\delta/100}-2\delta)}\,ds\,\|h\|_{X_t}^3\\
&\quad\lesssim \int_0^t \langle s\rangle^{-1-}\,ds\,\|h\|_{X_t}^3\lesssim\|h\|_{X_t}^3.
\end{aligned}$$

Similarly,

$$\|\mathcal{F}^{-1}((7.8))\|_{2-\delta/100}\lesssim\|h\|_{X_t}^3.$$

This concludes Case 1.

Case 2:

$$\phi(\xi,\eta,\sigma)=\langle\xi\rangle-\langle\xi-\eta\rangle-\langle\eta-\sigma\rangle+\langle\sigma\rangle.$$

This is exactly the same as Case 1 after the change of variable $\sigma\to\eta-\sigma$.

Case 3:

$$\phi(\xi,\eta,\sigma)=\langle\xi\rangle+\langle\xi-\eta\rangle-\langle\eta-\sigma\rangle-\langle\sigma\rangle. \tag{7.10}$$

For this case, we will have to exploit some delicate cancelations of the phases. Let $N_1=4$. We now introduce several frequency cut-offs and write (7.1) as

$$\begin{aligned}
(7.1)&=\sum_{i=1}^4\int_0^t\int e^{-is\phi}\frac{s\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle m_i(\xi,\eta,\sigma,s)\\
&\qquad\cdot\frac{\eta}{|\eta|}\widehat{\mathcal{R}f}(s,\xi-\eta)\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds\\
&=:\sum_{i=1}^4 I_i,
\end{aligned}$$

where

$$
\begin{aligned}
m_1(\xi,\eta,\sigma,s) &= \chi_{|\xi-\eta|\le\langle s\rangle^{\delta_0}}\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\eta|\le\langle s\rangle^{-\delta_0}}\chi_{|\xi|\le\langle s\rangle^{-\delta_0/N_1}},\\
m_2(\xi,\eta,\sigma,s) &= \chi_{|\xi-\eta|\le\langle s\rangle^{\delta_0}}\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\eta|\le\langle s\rangle^{-\delta_0}}\chi_{|\xi|>\langle s\rangle^{-\delta_0/N_1}}\chi_{|\sigma|\le 2\langle s\rangle^{-\delta_0}},\\
m_3(\xi,\eta,\sigma,s) &= \chi_{|\xi-\eta|\le\langle s\rangle^{\delta_0}}\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\eta|\le\langle s\rangle^{-\delta_0}}\chi_{|\xi|>\langle s\rangle^{-\delta_0/N_1}}\chi_{|\sigma|>2\langle s\rangle^{-\delta_0}},\\
m_4(\xi,\eta,\sigma,s) &= \chi_{|\xi-\eta|\le\langle s\rangle^{\delta_0}}\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\eta|>\langle s\rangle^{-\delta_0}}.
\end{aligned}
$$

Subcase 3a: estimation of I_1. By (7.10), we have

$$\partial_\xi\phi = \frac{\xi}{\langle\xi\rangle} + \frac{\xi-\eta}{\langle\xi-\eta\rangle}.$$

Since on the support of $m_1(\xi,\eta,\sigma,s)$ both ξ and η are localized to low frequencies, we gain one derivative by using the above identity. Therefore

$$
\begin{aligned}
\|\mathcal{F}^{-1}(I_1)\|_{2-\delta/100} &\lesssim \int_0^t \langle s\rangle^{\delta/100+1-\delta_0/N_1}\|P_{\lesssim\langle s\rangle^{-\delta_0/N_1}}h\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}\|P_{\le\langle s\rangle^{\delta_0}}h\|_{1/\delta}^2\,ds\\
&\lesssim \int_0^t \langle s\rangle^{\delta/100+1-\delta_0/N_1-2(1-2\delta)}\,ds\,\|h\|_{X_t}^3\\
&\lesssim \int_0^t \langle s\rangle^{-1-}\,ds\,\|h\|_{X_t}^3 \lesssim \|h\|_{X_t}^3,
\end{aligned}
$$

where we require that $\delta_0/N_1 > 4.01\delta$.

Subcase 3b: estimation of I_2. Note that in this subcase we have $|\xi|\ge\langle s\rangle^{-\delta_0/N_1}$, $|\eta|\le\frac{25}{24}\langle s\rangle^{-\delta_0}$, and $|\sigma|\le 2\cdot\frac{25}{24}\cdot\langle s\rangle^{-\delta_0}$ on the support of $m_2(\xi,\eta,\sigma,s)$. Hence

$$
\begin{aligned}
&\langle\xi\rangle+\langle\xi-\eta\rangle-2 \ge \langle\xi\rangle-1 = \frac{|\xi|^2}{\langle\xi\rangle+1} \gtrsim \langle s\rangle^{-2\delta_0/N_1} \quad \text{if } |\xi|\le 3,\\
&\langle\xi\rangle+\langle\xi-\eta\rangle-2 \gtrsim \langle\xi-\eta\rangle \quad \text{if } |\xi|>3,\\
&\langle\eta-\sigma\rangle+\langle\sigma\rangle-2 = \langle\eta-\sigma\rangle-1+\langle\sigma\rangle-1 = \frac{(\eta-\sigma)\cdot(\eta-\sigma)}{\langle\eta-\sigma\rangle+1}+\frac{\sigma\cdot\sigma}{\langle\sigma\rangle+1}.
\end{aligned}
\tag{7.11}
$$

We now perform a **partial normal form** transform. Namely, we write

$$e^{-is\phi} = e^{-is(\langle\xi\rangle+\langle\xi-\eta\rangle-2)}e^{is(\langle\eta-\sigma\rangle+\langle\sigma\rangle-2)}.$$

Using the identity

$$e^{-is(\langle\xi\rangle+\langle\xi-\eta\rangle-2)} = \frac{i}{\langle\xi\rangle+\langle\xi-\eta\rangle-2}\partial_s(e^{-is(\langle\xi\rangle+\langle\xi-\eta\rangle-2)})$$

and integrating by parts in the time variable s, we obtain

$$
\begin{aligned}
I_2 = \int_0^t \int \frac{i}{\langle\xi\rangle + \langle\xi-\eta\rangle - 2}\partial_s(e^{-is(\langle\xi\rangle+\langle\xi-\eta\rangle-2)})e^{is(\langle\eta-\sigma\rangle+\langle\sigma\rangle-2)} \\
\cdot \frac{s\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle m_2(\xi,\eta,\sigma,s) \\
\cdot \frac{\eta}{|\eta|}\widehat{\mathcal{R}f}(s,\xi-\eta)\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\,d\sigma\,d\eta\,ds \\
= \int e^{-it\phi}\frac{i}{\langle\xi\rangle + \langle\xi-\eta\rangle - 2}\frac{t\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle m_2(\xi,\eta,\sigma,t) \\
\cdot \frac{\eta}{|\eta|}\widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{\mathcal{R}f}(t,\eta-\sigma)\widehat{\mathcal{R}f}(t,\sigma)\,d\sigma\,d\eta \qquad (7.12) \\
- \int_0^t \int e^{-is\phi}\frac{i}{\langle\xi\rangle + \langle\xi-\eta\rangle - 2}\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\partial_s(sm_2(\xi,\eta,\sigma,s)) \\
\cdot \frac{\eta}{|\eta|}\widehat{\mathcal{R}f}(s,\xi-\eta)\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\,d\sigma\,d\eta\,ds \qquad (7.13) \\
+ \int_0^t \int e^{-is\phi}\frac{\langle\eta-\sigma\rangle+\langle\sigma\rangle-2}{\langle\xi\rangle + \langle\xi-\eta\rangle - 2}\frac{s\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle m_2(\xi,\eta,\sigma,s) \\
\cdot \frac{\eta}{|\eta|}\widehat{\mathcal{R}f}(s,\xi-\eta)\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\,d\sigma\,d\eta\,ds \qquad (7.14) \\
- \int_0^t \int e^{-is\phi}\frac{i}{\langle\xi\rangle + \langle\xi-\eta\rangle - 2}\frac{s\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle m_2(\xi,\eta,\sigma,s) \\
\cdot \frac{\eta}{|\eta|}\partial_s\widehat{\mathcal{R}f}(s,\xi-\eta)\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\,d\sigma\,d\eta\,ds \qquad (7.15) \\
- \int_0^t \int e^{-is\phi}\frac{i}{\langle\xi\rangle + \langle\xi-\eta\rangle - 2}\frac{s\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle m_2(\xi,\eta,\sigma,s) \\
\cdot \frac{\eta}{|\eta|}\widehat{\mathcal{R}f}(s,\xi-\eta)\partial_s\big[\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big]\,d\sigma\,d\eta\,ds. \qquad (7.16)
\end{aligned}
$$

For (7.12), by using (7.11) and Lemma 2.6, it is not difficult to check that the functions

$$
\begin{aligned}
\widetilde{m_1}(\xi,\eta) &= \frac{i}{\langle\xi\rangle + \langle\xi-\eta\rangle - 2}\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)} \\
&\quad \cdot \langle\xi\rangle^{4+2\delta+\delta/100}\langle\eta\rangle\chi_{|\xi|\geq 3}\chi_{|\eta|\lesssim 1}\cdot\langle\xi-\eta\rangle^{-(4+2\delta+\delta/99)}, \\
\widetilde{m_2}(\xi,\eta) &= \frac{i}{\langle\xi\rangle + \langle\xi-\eta\rangle - 2}\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)} \\
&\quad \cdot \langle\xi\rangle^{4+2\delta+\delta/100}\langle\eta\rangle\chi_{|\xi|<3}\chi_{|\eta|\lesssim 1}\langle t\rangle^{-6\delta_0/N_1}\chi_{\langle\xi\rangle+\langle\xi-\eta\rangle-2\gtrsim\langle t\rangle^{-2\delta_0/N_1}}
\end{aligned}
$$

satisfy (2.7). Therefore by Lemma 2.3, we have

$$
\begin{aligned}
\|\mathcal{F}^{-1}((7.12))\|_{2-\delta/100} \lesssim \langle t\rangle^{1+\delta/100}\big\| T_{\widetilde{m_1}(\xi,\eta)}\big(\langle\nabla\rangle^{4+2\delta+\delta/99}P_{\leq\langle t\rangle^{\delta_0}}h, \\
\mathcal{R}P_{\leq\langle t\rangle^{-\delta_0}}(\mathcal{R}P_{\leq\langle t\rangle^{\delta_0}}h\cdot\mathcal{R}P_{\leq\langle t\rangle^{\delta_0}}P_{\leq 2\langle t\rangle^{-\delta_0}}h)\big)\big\|_{2-\delta/100}
\end{aligned}
$$

$$
\begin{aligned}
&+ \langle t\rangle^{1+\delta/100+6\delta_0/N_1} \big\| T_{\widetilde{m_2}(\xi,\eta)}\big(P_{\le\langle t\rangle^{\delta_0}} P_{\lesssim 1} h,\\
&\qquad\qquad \mathcal{R}P_{\le\langle t\rangle^{-\delta_0}}\big(\mathcal{R}P_{\le\langle t\rangle^{\delta_0}} h \cdot \mathcal{R}P_{\le\langle t\rangle^{\delta_0}} P_{\le 2\langle t\rangle^{-\delta_0}} h\big)\big)\big\|_{2-\delta/100}\\
&\lesssim \langle t\rangle^{1+\delta/100} \|\langle\nabla\rangle^{4+2\delta+\delta/99} P_{\lesssim\langle t\rangle^{\delta_0}} h\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}} \|h\|_{1/\delta}^2\\
&\quad + \langle t\rangle^{1+\delta/100+6\delta_0/N_1} \|P_{\lesssim 1} h\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}} \|h\|_{1/\delta}^2\\
&\lesssim \langle t\rangle^{1+\delta/100+6\delta_0/N_1-2(1-2\delta)} \|h\|_{X_t}^3 \lesssim \|h\|_{X_t}^3. \qquad (7.17)
\end{aligned}
$$

To estimate (7.13), we need a simple fact: if $\psi = \psi(x)$ is a smooth cut-off function localized to $\{x : |x| \le 1\}$, then for any real number α,

$$
\frac{\partial}{\partial s}\left(\psi\left(\frac{x}{\langle s\rangle^\alpha}\right)\right) = \left[\frac{x}{\langle s\rangle^\alpha}\cdot\nabla\psi\left(\frac{x}{\langle s\rangle^\alpha}\right)\right]\cdot O\left(\frac{1}{\langle s\rangle}\right) = \chi_{\le\langle s\rangle^\alpha}\cdot O\left(\frac{1}{\langle s\rangle}\right),
$$

i.e. the function $\partial_s(\psi(x/\langle s\rangle^\alpha))$ has the same support as $\psi(x/\langle s\rangle^\alpha)$ and picks up a decay factor $1/\langle s\rangle$. Using this fact, we can write

$$
\partial_s(s m_2(\xi,\eta,\sigma,s)) = \widetilde{m_2}(\xi,\eta,\sigma,s),
$$

where $\widetilde{m_2}$ has essentially the same form as m_2. By essentially repeating the estimation as in (7.12) (see (7.17)), we have

$$
\begin{aligned}
\|\mathcal{F}^{-1}((7.12))\|_{2-\delta/100} &\lesssim \int_0^t \langle s\rangle^{1+\delta/100+6\delta_0/N_1-2(1-2\delta)}\, ds\, \|h\|_{X_t}^3\\
&\lesssim \int_0^t \langle s\rangle^{-1-}\, ds\, \|h\|_{X_t}^3 \lesssim \|h\|_{X_t}^3.
\end{aligned}
$$

For (7.14), we need to use the third identity in (7.11). Note that $|\eta| \le \frac{25}{24}\langle s\rangle^{-\delta_0}$ and $|\sigma| \le 3\langle s\rangle^{-\delta_0}$, and we can insert a fattened cut-off $P_{\lesssim\langle s\rangle^{-\delta_0}}$ when needed. By an estimation similar to that in (7.17), we have

$$
\begin{aligned}
\|\mathcal{F}^{-1}((7.14))\|_{2-\delta/100} &\lesssim \int_0^t \langle s\rangle^{1+\delta/100+6\delta_0/N_1} \|h\|_{X_t} \left\|\frac{\Delta}{\langle\nabla\rangle+1} P_{\lesssim\langle s\rangle^{-\delta_0}} h\right\|_{1/\delta} \|h\|_{1/\delta}\, ds\\
&\lesssim \int_0^t \langle s\rangle^{1+\delta/100+6\delta_0/N_1-2\delta_0} \langle s\rangle^{-2(1-2\delta)}\, ds\, \|h\|_{X_t}^3\\
&\lesssim \int_0^t \langle s\rangle^{-1-}\, ds\, \|h\|_{X_t}^3 \lesssim \|h\|_{X_t}^3,
\end{aligned}
$$

where we need $(2 - 6/N_1)\delta_0 > (4 + 1/100)\delta$.

We turn now to the estimation of (7.15). For this we need a lemma.

Lemma 7.2. *For any $\beta \ge 0$ and $2 \le p < 1/\delta$, we have*

$$
\|\langle\nabla\rangle^\beta e^{it\langle\nabla\rangle}\partial_t(\mathcal{R}f(t))\|_p \lesssim \|\langle\nabla\rangle^{\beta+1} h(t)\|_{(1/p-\delta)^{-1}} \|h(t)\|_{1/\delta}.
$$

Proof of Lemma 7.2. By (3.7), we have

$$e^{it\langle\nabla\rangle}\partial_t(\mathcal{R}f(t)) = \langle\nabla\rangle\mathcal{R}(\mathcal{R}h(t)\,\mathcal{R}h(t)).$$

Then the result follows from the product rule. □

Now we continue the estimation of (7.15). By Lemma 7.2 and a similar computation to (7.17), we have

$$\begin{aligned}\|\mathcal{F}^{-1}((7.15))\|_{2-\delta/100} &\lesssim \int_0^t \langle s\rangle^{1+\delta/100+6\delta_0/N_1}\|\langle\nabla\rangle^{5+2\delta+\delta/99}h\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}\|h\|_{1/\delta}^3\,ds\\ &\lesssim \int_0^t \langle s\rangle^{1+\delta/100+6\delta_0/N_1}\langle s\rangle^{-3(1-2\delta)}\,ds\,\|h\|_{X_t}^4\\ &\lesssim \int_0^t \langle s\rangle^{-1-}\,ds\,\|h\|_{X_t}^4 \lesssim \|h\|_{X_t}^4.\end{aligned}$$

In a similar way, we bound (7.16) as

$$\begin{aligned}\|\mathcal{F}^{-1}((7.16))\|_{2-\delta/100} &\lesssim \int_0^t \langle s\rangle^{1+\delta/100+6\delta_0/N_1}\|h\|_{X_t}\|e^{is\langle\nabla\rangle}\partial_s(\mathcal{R}f)\|_{1/(2\delta)}\|h\|_{1/\delta}\,ds\\ &\lesssim \int_0^t \langle s\rangle^{1+\delta/100+6\delta_0/N_1}\|\langle\nabla\rangle h\|_{1/\delta}\|h\|_{1/\delta}\|\langle\nabla\rangle h\|_{1/\delta}\,ds\,\|h\|_{X_t}\\ &\lesssim \int_0^t \langle s\rangle^{1+\delta/100+6\delta_0/N_1}\langle s\rangle^{-3(1-2\delta)}\,ds\,\|h\|_{X_t}^4\\ &\lesssim \int_0^t \langle s\rangle^{-1-}\,ds\,\|h\|_{X_t}^4 \lesssim \|h\|_{X_t}^4.\end{aligned}$$

Subcase 3c: estimation of I_3. In this subcase, we have $|\eta| \le \frac{25}{24}\langle s\rangle^{-\delta_0}$ and $2\langle s\rangle^{-\delta_0} \le |\sigma| \le \frac{25}{24}\langle s\rangle^{\delta_0}$ on the support of $m_3(\xi,\eta,\sigma,s)$. Then clearly,

$$|2\sigma-\eta| \ge \tfrac{1}{2}|\sigma|.$$

By (7.10) and (2.23), we then have

$$|\partial_\sigma\phi| = \left|\frac{\sigma-\eta}{\langle\sigma-\eta\rangle} - \frac{\sigma}{\langle\sigma\rangle}\right| \gtrsim \frac{|\sigma|}{\langle\sigma\rangle^2} \gtrsim \langle s\rangle^{-2\delta_0}. \tag{7.18}$$

Using the identity $se^{-is\phi} = i\frac{\partial_\sigma\phi}{|\partial_\sigma\phi|^2}\cdot\partial_\sigma(e^{-is\phi})$, we integrate by parts in σ in I_3 to obtain

$$\begin{aligned}I_3 = &-i\int_0^t\int e^{-is\phi}\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\partial_\sigma\cdot\left(\frac{\partial_\sigma\phi}{|\partial_\sigma\phi|^2}m_3(\xi,\eta,\sigma,s)\right)\\ &\cdot\frac{\eta}{|\eta|}\widehat{\mathcal{R}f}(s,\xi-\eta)\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\,d\sigma\,d\eta\,ds \qquad (7.19)\\ &-i\int_0^t\int e^{-is\phi}\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\frac{\partial_\sigma\phi}{|\partial_\sigma\phi|^2}m_3(\xi,\eta,\sigma,s)\\ &\cdot\frac{\eta}{|\eta|}\widehat{\mathcal{R}f}(s,\xi-\eta)\partial_\sigma\big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds. \qquad (7.20)\end{aligned}$$

For (7.19), note that

$$\begin{aligned}
&\partial_\sigma\cdot\left(\frac{\partial_\sigma\phi}{|\partial_\sigma\phi|^2}\chi_{|\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\sigma|>2\langle s\rangle^{-\delta_0}}\right)\\
&\qquad=\partial_\sigma\cdot\left(\frac{\partial_\sigma\phi}{|\partial_\sigma\phi|^2}\right)\chi_{|\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\sigma|>2\langle s\rangle^{-\delta_0}}\\
&\qquad\quad+\frac{\partial_\sigma\phi}{|\partial_\sigma\phi|^2}\langle s\rangle^{-\delta_0}\widetilde{\chi}_{|\sigma|\sim\langle s\rangle^{\delta_0}}\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\sigma|>2\langle s\rangle^{-\delta_0}}\\
&\qquad\quad+\frac{\partial_\sigma\phi}{|\partial_\sigma\phi|^2}\chi_{|\sigma|\le\langle s\rangle^{\delta_0}}\langle s\rangle^{-\delta_0}\widetilde{\chi}_{|\eta-\sigma|\sim\langle s\rangle^{\delta_0}}\chi_{|\sigma|>2\langle s\rangle^{-\delta_0}}\\
&\qquad\quad+\frac{\partial_\sigma\phi}{|\partial_\sigma\phi|^2}\chi_{|\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\langle s\rangle^{\delta_0}\widetilde{\chi}_{|\sigma|\sim 2\langle s\rangle^{-\delta_0}},
\end{aligned}$$

where $\widetilde{\chi}$ are some modified cut-offs.

By (7.18), it is easy to check that the functions

$$\widetilde{m_1}(\eta,\sigma)=\chi_{|\partial_\sigma\phi|\gtrsim\langle s\rangle^{-2\delta_0}}\partial_\sigma\cdot\left(\frac{\partial_\sigma\phi}{|\partial_\sigma\phi|^2}\right)\langle s\rangle^{-10\delta_0}\langle\eta-\sigma\rangle^{-(1+\delta/400)}\langle\sigma\rangle^{-(1+\delta/400)}\quad\text{and}$$

$$\widetilde{m_2}(\eta,\sigma)=\chi_{|\partial_\sigma\phi|\gtrsim\langle s\rangle^{-2\delta_0}}\frac{\partial_\sigma\phi}{|\partial_\sigma\phi|^2}\langle s\rangle^{\delta_0}\langle s\rangle^{-10\delta_0}\langle\eta-\sigma\rangle^{-(1+\delta/400)}\langle\sigma\rangle^{-(1+\delta/400)}$$

satisfy (2.7). Therefore by Lemma 2.3, we have

$$\begin{aligned}
\|\mathcal{F}^{-1}((7.19))\|_{2-\delta/100}&\lesssim\int_0^t\langle s\rangle^{\delta/100}\|\langle\nabla\rangle^{5+2\delta+\delta/100}\mathcal{R}P_{\le\langle s\rangle^{\delta_0}}h\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}\\
&\cdot\langle s\rangle^{10\delta_0}\sum_{i=1}^{2}\big\|T_{\widetilde{m_i}(\eta,\sigma)}\big(\langle\nabla\rangle^{1+\delta/400}\mathcal{R}P_{\lesssim\langle s\rangle^{\delta_0}}h,\langle\nabla\rangle^{1+\delta/400}\mathcal{R}P_{\lesssim\langle s\rangle^{\delta_0}}P_{\gtrsim\langle s\rangle^{-\delta_0}}h\big)\big\|_{1/(2\delta)}\,ds\\
&\lesssim\int_0^t\langle s\rangle^{\delta/100}\|h\|_{X_t}\langle s\rangle^{10\delta_0+\delta_0\delta/200}\|\langle\nabla\rangle h\|_{1/\delta}^2\,ds\\
&\lesssim\int_0^t\langle s\rangle^{\delta/100+(10+\delta/200)\delta_0-2(1-2\delta)}\,ds\,\|h\|_{X_t}^3\lesssim\int_0^t\langle s\rangle^{-1-}\,ds\,\|h\|_{X_t}^3\lesssim\|h\|_{X_t}^3.
\end{aligned}$$

Similarly for (7.20), we use Lemma 6.2 to obtain

$$\begin{aligned}
\|\mathcal{F}^{-1}((7.20))\|_{2-\delta/100}&\lesssim\int_0^t\langle s\rangle^{\delta/100}\|\langle\nabla\rangle^{5+2\delta+\delta/100}\mathcal{R}P_{\lesssim\langle s\rangle^{\delta_0}}h\|_{1/\delta}\\
&\qquad\cdot\langle s\rangle^{8\delta_0}\|\langle\nabla\rangle^{1+\delta/400}P_{\lesssim\langle s\rangle^{\delta_0}}e^{is\langle\nabla\rangle}\mathcal{F}^{-1}(\partial_\sigma(\widehat{\mathcal{R}f}))\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}\\
&\qquad\cdot\|\langle\nabla\rangle^{1+\delta/400}\mathcal{R}P_{\lesssim\langle s\rangle^{\delta_0}}h\|_{1/\delta}\,ds\\
&\lesssim\int_0^t\langle s\rangle^{\delta/100+(4+2\delta+\delta/100)\delta_0}\|\langle\nabla\rangle h\|_{1/\delta}\langle s\rangle^{8\delta_0}\langle s\rangle^{\delta_0(1+\delta/400+\frac{2}{2+\delta}-4(\frac{1}{2-\delta/100}-2\delta))}\\
&\qquad\cdot\langle s\rangle^{1-2(\frac{1}{2-\delta/100}-2\delta)}\|\langle x\rangle f\|_{2+\delta}\langle s\rangle^{\delta\delta_0/400}\|\langle\nabla\rangle h\|_{1/\delta}\,ds\\
&\lesssim\int_0^t\langle s\rangle^{-1-}\,ds\,\|h\|_{X_t}^3\lesssim\|h\|_{X_t}^3.
\end{aligned}$$

This ends the estimation of I_3.

Subcase 3d: estimation of I_4. Note that in this subcase, $|\eta| \gtrsim \langle s\rangle^{-\delta_0}$. By Lemma 2.8, we have

$$\partial_\xi \phi = Q_1(\xi,\eta,\sigma)\partial_\eta\phi + Q_2(\xi,\eta,\sigma)\partial_\sigma\phi,$$

where

$$\left|\partial_\xi^\alpha\partial_\eta^\beta\partial_\sigma^\gamma Q_i(\xi,\eta,\sigma)\right| \lesssim_{\alpha,\beta,\gamma} \langle|\xi|+|\eta|+\sigma\rangle^3, \quad i=1,2.$$

Obviously,

$$s\partial_\xi\phi e^{-is\phi} = i\big(Q_1\partial_\eta(e^{-is\phi}) + Q_2\partial_\sigma(e^{-is\phi})\big).$$

Using the above identity, we shall integrate by parts in η and σ. It is not difficult to check that the functions

$$\begin{aligned}
\widetilde{m_i}(\xi,\eta,\sigma) &= \frac{\partial_\xi\phi\langle\xi\rangle^{4+2\delta}}{\phi_0(\xi,\eta)}\langle\eta\rangle\frac{\eta}{|\eta|}m_4(\xi,\eta,\sigma,s)Q_i(\xi,\eta,\sigma)\langle s\rangle^{-(13+2\delta)\delta_0}, \quad i=1,2,\\
\widetilde{m_3}(\xi,\eta,\sigma) &= \partial_\eta\widetilde{m_i}(\xi,\eta,\sigma,s)\langle s\rangle^{-(14+2\delta)\delta_0}, \quad i=1,2,\\
\widetilde{m_4}(\xi,\eta,\sigma) &= \partial_\sigma\widetilde{m_i}(\xi,\eta,\sigma,s)\langle s\rangle^{-(13+2\delta)\delta_0}, \quad i=1,2,
\end{aligned}$$

satisfy (2.9). Therefore by Corollary 2.4, we have

$$\begin{aligned}
\|\mathcal{F}^{-1}(I_4)\|_{2-\delta/100} &\lesssim \int_0^t \langle s\rangle^{\delta/100+(14+2\delta)\delta_0}\|T_{\widetilde{m_3}+\widetilde{m_4}}(\mathcal{R}h,\mathcal{R}h,\mathcal{R}h)\|_{2-\delta/100}\,ds\\
&\quad+\int_0^t \langle s\rangle^{\delta/100+(13+2\delta)\delta_0}\big\|T_{\widetilde{m_1}}\big(P_{\lesssim\langle s\rangle^{\delta_0}}e^{is\langle\nabla\rangle}\mathcal{F}^{-1}(\partial_\eta(\widehat{\mathcal{R}f})),\mathcal{R}h,\mathcal{R}h\big)\big\|_{2-\delta/100}\,ds\\
&\quad+\int_0^t \langle s\rangle^{\delta/100+(13+2\delta)\delta_0}\big\|T_{\widetilde{m_2}}\big(\mathcal{R}h,\mathcal{R}h,P_{\lesssim\langle s\rangle^{\delta_0}}e^{is\langle\nabla\rangle}\mathcal{F}^{-1}(\partial_\sigma(\widehat{\mathcal{R}f}))\big)\big\|_{2-\delta/100}\,ds\\
&\lesssim \int_0^t \langle s\rangle^{\delta/100+(14+2\delta)\delta_0-2(1-2\delta)}\,ds\,\|h\|_{X_t}^3 \lesssim \int_0^t\langle s\rangle^{-1-}\,ds\,\|h\|_{X_t}^3 \lesssim \|h\|_{X_t}^3.
\end{aligned}$$

Hence Case 3 is finished.

Case 4:

$$\phi(\xi,\eta,\sigma) = \langle\xi\rangle + \langle\xi-\eta\rangle + \langle\eta-\sigma\rangle - \langle\sigma\rangle. \tag{7.21}$$

In this case we decompose (see (7.2))

$$\begin{aligned}
m_{\text{low}}(\xi,\eta,\sigma) &= m_{\text{low}}(\xi,\eta,\sigma)\chi_{|\eta|\le\langle s\rangle^{-\delta_0}} + m_{\text{low}}(\xi,\eta,\sigma)\chi_{|\eta|>\langle s\rangle^{-\delta_0}}\\
&= m_{\text{low}}^{(1)}(\xi,\eta,\sigma) + m_{\text{low}}^{(2)}(\xi,\eta,\sigma),
\end{aligned}$$

and denote the corresponding integrals in (7.1) by I_1 and I_2 respectively.

Subcase 4a: estimation of I_1. We again use the partial normal form trick. Note that

$$\langle\sigma-\eta\rangle - \langle\sigma\rangle = \frac{(2\sigma-\eta)\cdot(-\eta)}{\langle\sigma-\eta\rangle+\langle\sigma\rangle}.$$

Using the identity

$$e^{-is(\langle\xi\rangle+\langle\xi-\eta\rangle)} = \frac{i}{\langle\xi\rangle+\langle\xi-\eta\rangle}\partial_s(e^{-is(\langle\xi\rangle+\langle\xi-\eta\rangle)})$$

and integrating by parts in the time variable s, we get

$$\begin{aligned} I_1 = \int e^{-it\phi}\frac{it\partial_\xi\phi}{\phi_0(\xi,\eta)}\frac{\langle\xi\rangle^{4+2\delta}}{\langle\xi\rangle+\langle\xi-\eta\rangle}\langle\eta\rangle m^{(1)}_{\text{low}} \\ \cdot\frac{\eta}{|\eta|}\widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{\mathcal{R}f}(t,\eta-\sigma)\widehat{\mathcal{R}f}(t,\sigma)\,d\sigma\,d\eta \qquad (7.22) \\ -\int_0^t\int e^{-is\phi}\frac{(2\sigma-\eta)\cdot(-\eta)}{\langle\sigma-\eta\rangle+\langle\sigma\rangle}\frac{\langle\xi\rangle^{4+2\delta}}{\langle\xi\rangle+\langle\xi-\eta\rangle}\langle\eta\rangle\frac{s\partial_\xi\phi}{\phi_0(\xi,\eta)}\frac{\eta}{|\eta|}m^{(1)}_{\text{low}} \\ \cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\,d\sigma\,d\eta\,ds \qquad (7.23) \\ -\int_0^t\int e^{-is\phi}\frac{\langle\xi\rangle^{4+2\delta}}{\langle\xi\rangle+\langle\xi-\eta\rangle}\langle\eta\rangle\frac{i\partial_\xi\phi}{\phi_0(\xi,\eta)}\frac{\eta}{|\eta|}\partial_s(sm^{(1)}_{\text{low}}) \\ \cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\,d\sigma\,d\eta\,ds \qquad (7.24) \\ -\int_0^t\int e^{-is\phi}\frac{\langle\xi\rangle^{4+2\delta}}{\langle\xi\rangle+\langle\xi-\eta\rangle}\langle\eta\rangle\frac{is\partial_\xi\phi}{\phi_0(\xi,\eta)}\frac{\eta}{|\eta|}m^{(1)}_{\text{low}} \\ \cdot\partial_s\big(\widehat{\mathcal{R}f}(s,\xi-\eta)\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds. \qquad (7.25) \end{aligned}$$

The estimation of (7.22) is similar to (7.12), and we have

$$\|\mathcal{F}^{-1}((7.22))\|_{2-\delta/100} \lesssim \|h\|^3_{X_t}.$$

For (7.23), note that $\frac{2\sigma-\eta}{\langle\sigma-\eta\rangle+\langle\sigma\rangle}$ is a Coifman–Meyer multiplier. We compute

$$\begin{aligned} \|\mathcal{F}^{-1}((7.23))\|_{2-\delta/100} &\lesssim \int_0^t \langle s\rangle^{1+\delta/100}\|\langle\nabla\rangle^{4+3\delta}P_{\le\langle s\rangle^{\delta_0}}\mathcal{R}h\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}} \\ &\quad\cdot\|\nabla P_{\lesssim\langle s\rangle^{-\delta_0}}T_{\frac{2\sigma-\eta}{\langle\sigma-\eta\rangle+\langle\sigma\rangle}}(P_{\le\langle s\rangle^{\delta_0}}\mathcal{R}h,\,P_{\le\langle s\rangle^{\delta_0}}\mathcal{R}h)\|^2_{1/(2\delta)}\,ds \\ &\lesssim \int_0^t \langle s\rangle^{1+\delta/100}\|h\|_{H^{N'}}\langle s\rangle^{-\delta_0}\|h\|^2_{1/\delta}\,ds \\ &\lesssim \int_0^t \langle s\rangle^{1+\delta/100-\delta_0-2(1-2\delta)}\,ds\,\|h\|^3_{X_t} \\ &\lesssim \int_0^t \langle s\rangle^{-1-}\,ds\,\|h\|^3_{X_t} \lesssim \|h\|^3_{X_t}. \end{aligned}$$

The estimation of (7.24) is similar to (7.13), and we obtain

$$\|\mathcal{F}^{-1}((7.24))\|_{2-\delta/100} \lesssim \|h\|^3_{X_t}.$$

The estimation of (7.25) is also similar to that of (7.15) and (7.16). We have

$$\|\mathcal{F}^{-1}((7.25))\|_{2-\delta/100} \lesssim \|h\|^3_{X_t}.$$

Subcase 4b: estimation of I_2. It is not difficult to check that

$$\langle\xi\rangle+\langle\xi-\eta\rangle+\langle\eta-\sigma\rangle-\langle\sigma\rangle\gtrsim 1/\langle\xi\rangle,\qquad \forall\xi,\eta,\sigma\in\mathbb{R}^2. \tag{7.26}$$

Using the identity $e^{-is\phi}=\frac{i}{\phi}\partial_s(e^{-is\phi})$, we integrate by parts in the variable s. This gives

$$\begin{aligned} I_2=&\int e^{-it\phi}\frac{i}{\phi}\frac{t\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\frac{\eta}{|\eta|}m_{\text{low}}^{(2)}\\ &\qquad\cdot\widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{\mathcal{R}f}(t,\eta-\sigma)\widehat{\mathcal{R}f}(t,\sigma)\,d\sigma\,d\eta \qquad (7.27)\\ -&\int_0^t\int e^{-is\phi}\frac{i}{\phi}\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\frac{\eta}{|\eta|}\partial_s(sm_{\text{low}}^{(2)})\\ &\qquad\cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\,d\sigma\,d\eta\,ds \qquad (7.28)\\ -&\int_0^t\int e^{-is\phi}\frac{i}{\phi}\frac{s\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\frac{\eta}{|\eta|}m_{\text{low}}^{(2)}\\ &\qquad\cdot\partial_s\big(\widehat{\mathcal{R}f}(s,\xi-\eta)\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\big)\,d\sigma\,d\eta\,ds. \qquad (7.29)\end{aligned}$$

For (7.27), by using (7.26) and Lemma 2.6, it is not difficult to check that the function

$$\begin{aligned}\widetilde{m}(\xi,\eta,\sigma,s)=&\frac{i}{\phi}\frac{\partial_\xi\phi}{\phi_0(\xi,\eta)}\langle\xi\rangle^{4+2\delta}\langle\eta\rangle\frac{\eta}{|\eta|}\\ &\cdot\chi_{|\xi-\eta|\le\langle s\rangle^{\delta_0}}\chi_{|\eta-\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\sigma|\le\langle s\rangle^{\delta_0}}\chi_{|\eta|>\langle s\rangle^{-\delta_0}}\langle s\rangle^{-(14+3\delta)\delta_0}\end{aligned}$$

satisfies (2.9). Therefore by Corollary 2.4, we have

$$\begin{aligned}\|\mathcal{F}^{-1}((7.27))\|_{2-\delta/100}&\lesssim\langle t\rangle^{1+\delta/100+(14+3\delta)\delta_0}\|h(t)\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}\|h(t)\|_{1/\delta}^2\\ &\lesssim\langle t\rangle^{1+\delta/100+(14+3\delta)\delta_0-2(1-2\delta)}\|h\|_{X_t}^3\lesssim\|h\|_{X_t}^3.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\mathcal{F}^{-1}((7.28))\|_{2-\delta/100}&\lesssim\int_0^t\langle s\rangle^{1+\delta/100+(14+3\delta)\delta_0-2(1-2\delta)}\,ds\,\|h\|_{X_t}^3\\ &\lesssim\int_0^t\langle s\rangle^{-1-}\,ds\,\|h\|_{X_t}^3\lesssim\|h\|_{X_t}^3.\end{aligned}$$

In a similar way, using Lemma 7.2, we have

$$\begin{aligned}\|\mathcal{F}^{-1}((7.29))\|_{2-\delta/100}&\lesssim\int_0^t\langle s\rangle^{1+\delta/100+(14+3\delta)\delta_0}\|e^{is\langle\nabla\rangle}\partial_s(\mathcal{R}f)\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}\|h\|_{1/\delta}^2\,ds\\ &\lesssim\int_0^t\langle s\rangle^{1+\delta/100+(14+3\delta)\delta_0}\|h\|_{1/\delta}^3\|h\|_{H^3}\,ds\\ &\lesssim\int_0^t\langle s\rangle^{1+\delta/100+(14+3\delta)\delta_0-3(1-2\delta)}\,ds\,\|h\|_{X_t}^4\\ &\lesssim\int_0^t\langle s\rangle^{-1-}\,ds\,\|h\|_{X_t}^4\lesssim\|h\|_{X_t}^4.\end{aligned}$$

Case 5:

$$\phi(\xi,\eta,\sigma)=\langle\xi\rangle+\langle\xi-\eta\rangle-\langle\eta-\sigma\rangle+\langle\sigma\rangle.$$

This is exactly the same as Case 4 after the change of variable $\sigma\to\eta-\sigma$.

Case 6:

$$\phi(\xi,\eta,\sigma)=\langle\xi\rangle-\langle\xi-\eta\rangle+\langle\eta-\sigma\rangle+\langle\sigma\rangle. \tag{7.30}$$

In this case we decompose (see (7.2))

$$\begin{aligned} m_{\mathrm{low}}(\xi,\eta,\sigma)&=m_{\mathrm{low}}(\xi,\eta,\sigma)\chi_{|\eta|\le\langle s\rangle^{-\delta_0}}+m_{\mathrm{low}}(\xi,\eta,\sigma)\chi_{|\eta|>\langle s\rangle^{-\delta_0}}\\ &=m^{(1)}_{\mathrm{low}}(\xi,\eta,\sigma)+m^{(2)}_{\mathrm{low}}(\xi,\eta,\sigma),\end{aligned}$$

and denote the corresponding integrals in (7.1) by I_1 and I_2 respectively. The estimation of I_2 is exactly the same as in Subcase 4b. Hence we only need to estimate I_1. In this situation, note that

$$\partial_\xi\phi=\frac{\xi}{\langle\xi\rangle}-\frac{\xi-\eta}{\langle\xi-\eta\rangle}=Q(\xi,\eta)\eta,\quad\text{where}\quad|\partial_\xi^\alpha\partial_\eta^\beta Q(\xi,\eta)|\lesssim_{\alpha,\beta}1.$$

Therefore,

$$\begin{aligned}\|\mathcal{F}^{-1}(I_1)\|_{2-\delta/100}&\lesssim\int_0^t\langle s\rangle^{1+\delta/100}\|\langle\nabla\rangle^{5+3\delta}h\|_{(\frac{1}{2-\delta/100}-2\delta)^{-1}}\\&\quad\cdot\big\|P_{\le\langle s\rangle^{-\delta_0}}\nabla\big(P_{\le\langle s\rangle^{\delta_0}}\mathcal{R}(P_{\le\langle s\rangle^{\delta_0}}\mathcal{R}h\cdot P_{\le\langle s\rangle^{\delta_0}}\mathcal{R}h)\big)\big\|_{1/(2\delta)}\,ds\\&\lesssim\int_0^t\langle s\rangle^{1+\delta/100}\|h\|_{H^{N'}}\langle s\rangle^{-\delta_0}\|h\|_{1/\delta}^2\,ds\\&\lesssim\int_0^t\langle s\rangle^{1+\delta/100+\delta-\delta_0-2(1-2\delta)}\,ds\,\|h\|_{X_t}^3\\&\lesssim\int_0^t\langle s\rangle^{-1-}\,ds\,\|h\|_{X_t}^3\lesssim\|h\|_{X_t}^3.\end{aligned}$$

This settles Case 6.

Case 7:

$$\phi(\xi,\eta,\sigma)=\langle\xi\rangle-\langle\xi-\eta\rangle-\langle\eta-\sigma\rangle-\langle\sigma\rangle. \tag{7.31}$$

In this case we again decompose

$$\begin{aligned} m_{\mathrm{low}}(\xi,\eta,\sigma)&=m_{\mathrm{low}}(\xi,\eta,\sigma)\chi_{|\eta|\le\langle s\rangle^{-\delta_0}}+m_{\mathrm{low}}(\xi,\eta,\sigma)\chi_{|\eta|>\langle s\rangle^{-\delta_0}}\\ &=m^{(1)}_{\mathrm{low}}(\xi,\eta,\sigma)+m^{(2)}_{\mathrm{low}}(\xi,\eta,\sigma),\end{aligned}$$

and denote the corresponding integrals in (7.1) by I_1 and I_2 respectively. Note that

$$|\phi(\xi,\eta,\sigma)|\gtrsim 1/\langle\xi\rangle\quad\text{and}\quad\partial_\xi\phi=\frac{\xi}{\langle\xi\rangle}-\frac{\xi-\eta}{\langle\xi-\eta\rangle}.$$

The estimations of I_1 and I_2 are exactly the same as in Case 6. Hence Case 7 is settled.

Case 8:

$$\phi(\xi,\eta,\sigma)=\langle\xi\rangle+\langle\xi-\eta\rangle+\langle\eta-\sigma\rangle+\langle\sigma\rangle. \tag{7.32}$$

In this case we again decompose (see (7.2))

$$\begin{aligned} m_{\text{low}}(\xi,\eta,\sigma)&=m_{\text{low}}(\xi,\eta,\sigma)\chi_{|\eta|\le\langle s\rangle^{-\delta_0}}\chi_{|\sigma|>2\langle s\rangle^{-\delta_0}}\\ &+m_{\text{low}}(\xi,\eta,\sigma)\chi_{|\eta|\le\langle s\rangle^{-\delta_0}}\chi_{|\sigma|\le 2\langle s\rangle^{-\delta_0}}\\ &+m_{\text{low}}(\xi,\eta,\sigma)\chi_{|\eta|>\langle s\rangle^{-\delta_0}}, \end{aligned}$$

and denote the corresponding integrals in (7.1) by I_1, I_2 and I_3 respectively. We discuss three subcases.

Subcase 8a: estimation of I_1. This subcase is exactly the same as Subcase 3c before. Therefore,

$$\|\mathcal{F}^{-1}(I_1)\|_{2-\delta/100}\lesssim\|h\|_{X_t}^3+\|h\|_{X_t}^4.$$

Subcase 8b: estimation of I_2. In this subcase, we shall again use the partial normal form trick. Write

$$e^{-is\phi}=\frac{i}{\langle\xi\rangle+\langle\xi-\eta\rangle+2}\partial_s(e^{-is(\langle\xi\rangle+\langle\xi-\eta\rangle+2)})e^{-is(\langle\eta-\sigma\rangle+\langle\sigma\rangle-2)}.$$

Note that by (7.11),

$$\langle\eta-\sigma\rangle+\langle\sigma\rangle-2=\frac{(\eta-\sigma)\cdot(\eta-\sigma)}{\langle\eta-\sigma\rangle+1}+\frac{\sigma\cdot\sigma}{\langle\sigma\rangle+1}.$$

Integrating by parts in s, we arrive at essentially the same situation as in Subcase 3b before. Hence we have

$$\|\mathcal{F}^{-1}(I_2)\|_{2-\delta/100}\lesssim\|h\|_{X_t}^3+\|h\|_{X_t}^4.$$

Subcase 8c: estimation of I_3. In this subcase we note that $|\eta|\gtrsim\langle s\rangle^{-\delta_0}$ and $\phi(\xi,\eta,\sigma)\gtrsim 1$. We can integrate by parts in the time variable s and use the same estimates as in Subcase 4b. Hence

$$\|\mathcal{F}^{-1}(I_3)\|_{2-\delta/100}\lesssim\|h\|_{X_t}^3+\|h\|_{X_t}^4.$$

We have completed the estimation of all phases. The proposition is now proved.

8. Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1. Define

$$\begin{aligned} a(t)&=\|\langle\tau\rangle^{-\delta}h(\tau)\|_{C_\tau^0H^N([0,t])}+\|h(\tau)\|_{C_\tau^0H^{N'}([0,t])}\\ &+\|\langle\tau\rangle|\nabla|^{\delta}\langle\nabla\rangle h(\tau)\|_{L_\tau^\infty L_x^\infty([0,t])}+\|\langle\tau\rangle^{1-2\delta}\langle\nabla\rangle h(\tau)\|_{L_\tau^\infty L_x^{1/\delta}([0,t])}\\ &+\|x(1-\Delta)e^{-i\tau\langle\nabla\rangle}h(\tau)\|_{L_\tau^\infty L_x^{2+\delta}([0,t])}. \end{aligned}$$

By the local theory in Section 4, $a(t)$ is a continuous function of t. Also from the energy estimates therein, we have

$$\begin{aligned}\frac{d}{d\tau}(\|h(\tau)\|_{H^N}) &\lesssim \big(\|u(\tau)\|_\infty + \|\nabla u(\tau)\|_\infty + \|\nabla \mathbf{v}(\tau)\|_\infty\big)\|h(\tau)\|_{H^N}\\ &\lesssim \big\||\nabla|^\delta\langle\nabla\rangle h(\tau)\big\|_\infty \|h(\tau)\|_{H^N} \lesssim a(\tau)^2\langle\tau\rangle^{-1+\delta}.\end{aligned}$$

Integrating in time and using the monotonicity of $a(\tau)$ gives us

$$\|h(s)\|_{H^N} \lesssim \|h_0\|_{H^N} + a(s)^2\langle s\rangle^\delta,$$

or

$$\|\langle\tau\rangle^{-\delta}h(\tau)\|_{C^0_\tau H^N([0,t])} \lesssim \|e^{i\tau\langle\nabla\rangle}h_0\|_{X_\infty} + a(t)^2.$$

By the analysis in Sections 4–7, we also have

$$\begin{aligned}&\|\langle\tau\rangle|\nabla|^{1/2}\langle\nabla\rangle h(\tau)\|_{L^\infty_\tau L^\infty_x([0,t])} + \|h(\tau)\|_{C^0_\tau H^{N'}_x([0,t])}\\ &\quad + \|\langle\tau\rangle^{1-2\delta}h(\tau)\|_{L^\infty_\tau L^{1/\delta}_x([0,t])} + \|\langle x\rangle(1-\Delta)e^{-i\tau\langle\nabla\rangle}h(\tau)\|_{L^\infty_\tau L^{2+\delta}_x([0,t])}\\ &\qquad\lesssim \|e^{i\tau\langle\nabla\rangle}h_0\|_{X_\infty} + a(t)^2 + a(t)^3 + a(t)^4.\end{aligned}$$

Thus we have proved that for some constant $C > 0$,

$$a(t) \le C\cdot\big(\|e^{i\tau\langle\nabla\rangle}h_0\|_{X_\infty} + a(t)^2 + a(t)^3 + a(t)^4\big).$$

Since $a(t)$ is a continuous function of t and $a(0) \le \|e^{i\tau\langle\nabla\rangle}h_0\|_{X_\infty}$, by a standard argument we conclude that if $\|e^{i\tau\langle\nabla\rangle}h_0\|_{X_\infty}$ is sufficiently small, then $a(t)$ is bounded for all $t \ge 0$. Note that the scattering of the $H^{N'}$ norm is a simple consequence of the analysis in Section 4. This concludes the proof of Theorem 1.1.

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