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Variation for the Riesz transform and uniform rectifiability

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Abstract. For $1 \le n < d$ integers and $\rho > 2$, we prove that an n-dimensional Ahlfors–David regular measure μ in $\mathbb{R}^d$ is uniformly n-rectifiable if and only if the ρ-variation for the Riesz transform with respect to μ is a bounded operator in $L^2(\mu)$. This result can be considered as a partial solution to a well known open problem posed by G. David and S. Semmes which relates the $L^2(\mu)$ boundedness of the Riesz transform to the uniform rectifiability of μ.

Keywords. ρ-variation and oscillation, Calderón–Zygmund singular integrals, Riesz transform, uniform rectifiability

1. Introduction

In this paper we characterize the notion of uniform rectifiability in the sense of David and Semmes [DS2] in terms of the L^2 boundedness of the ρ-variation for the Riesz transform, with $\rho > 2$.

Given integers $1 \le n < d$ and a Radon measure μ in $\mathbb{R}^d$, one defines the *n-dimensional Riesz transform* of a function $f \in L^1(\mu)$ by $R^\mu f(x) = \lim_{\epsilon\searrow 0} R^\mu_\epsilon f(x)$ (whenever the limit exists), where

$$R^\mu_\epsilon f(x) = \int_{|x-y|>\epsilon} \frac{x-y}{|x-y|^{n+1}} f(y)\, d\mu(y), \qquad x \in \mathbb{R}^d.$$

We will use the notation $\mathcal{R}^\mu f(x) := \{R^\mu_\epsilon f(x)\}_{\epsilon>0}$. When $d = 2$ (i.e., μ is a Radon measure in $\mathbb{C}$), one defines the *Cauchy transform* of $f \in L^1(\mu)$ by $C^\mu f(x) = \lim_{\epsilon\searrow 0} C^\mu_\epsilon f(x)$ (whenever the limit exists), where

$$C^\mu_\epsilon f(x) = \int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\, d\mu(y), \qquad x \in \mathbb{C}.$$

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To avoid the problem of existence of the preceding limits, it is useful to consider the maximal operators $R_*^\mu f(x) = \sup_{\epsilon>0} |R_\epsilon^\mu f(x)|$ and $C_*^\mu f(x) = \sup_{\epsilon>0} |C_\epsilon^\mu f(x)|$. Notice that the Cauchy transform coincides with the 1-dimensional Riesz transform in $\mathbb{R}^2$ modulo conjugation, since $1/x = \overline{x}/|x|^2$ for all $x \in \mathbb{C} \setminus \{0\}$.

The Cauchy and Riesz transforms are two very important examples of singular integral operators with a Calderón–Zygmund kernel. Given $d \geq 2$, the kernels $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ that we consider in this paper satisfy

$$|K(x)| \leq \frac{C}{|x|^n}, \qquad |\partial_{x^i} K(x)| \leq \frac{C}{|x|^{n+1}}, \qquad |\partial_{x^i}\partial_{x^j} K(x)| \leq \frac{C}{|x|^{n+2}}, \tag{1}$$

for all $1 \leq i, j \leq d$ and $x = (x^1, \ldots, x^d) \in \mathbb{R}^d \setminus \{0\}$, where $1 \leq n < d$ is some integer and $C > 0$ is some constant; and moreover $K(-x) = -K(x)$ for all $x \neq 0$ (i.e. K is odd). Notice that the n-dimensional Riesz transform corresponds to the vector kernel $(x^1, \ldots, x^d)/|x|^{n+1}$, and the Cauchy transform to $(x^1, -x^2)/|x|^2$ (so, we may consider K to be any scalar component of these vector kernels). For $f \in L^1(\mu)$ and $x \in \mathbb{R}^d$, we set

$$T_\epsilon^\mu f(x) \equiv T_\epsilon(f\mu)(x) := \int_{|x-y|>\epsilon} K(x-y) f(y)\, d\mu(y),$$

and we denote $\mathcal{T}^\mu f(x) = \{T_\epsilon^\mu f(x)\}_{\epsilon>0}$.

Definition 1.1 (ρ-variation and oscillation). Let $\mathcal{F} := \{F_\epsilon\}_{\epsilon>0}$ be a family of functions defined on $\mathbb{R}^d$. Given $\rho > 0$, the *ρ-variation* of $\mathcal{F}$ at $x \in \mathbb{R}^d$ is defined by

$$\mathcal{V}_\rho(\mathcal{F})(x) := \sup_{\{\epsilon_m\}} \Big(\sum_{m\in\mathbb{Z}} |F_{\epsilon_{m+1}}(x) - F_{\epsilon_m}(x)|^\rho \Big)^{1/\rho},$$

where the pointwise supremum is taken over all decreasing sequences $\{\epsilon_m\}_{m\in\mathbb{Z}} \subset (0,\infty)$. Fix a decreasing sequence $\{r_m\}_{m\in\mathbb{Z}} \subset (0,\infty)$. The *oscillation* of $\mathcal{F}$ at $x \in \mathbb{R}^d$ is defined by

$$\mathcal{O}(\mathcal{F})(x) := \sup_{\{\epsilon_m\},\{\delta_m\}} \Big(\sum_{m\in\mathbb{Z}} |F_{\epsilon_m}(x) - F_{\delta_m}(x)|^2 \Big)^{1/2},$$

where the pointwise supremum is taken over all sequences $\{\epsilon_m\}_{m\in\mathbb{Z}}$ and $\{\delta_m\}_{m\in\mathbb{Z}}$ such that $r_{m+1} \leq \epsilon_m \leq \delta_m \leq r_m$ for all $m \in \mathbb{Z}$.

The ρ-variation and oscillation for martingales and some families of operators have been studied in many recent papers on probability, ergodic theory, and harmonic analysis (see [Lp], [Bo], [JKRW], [CJRW1], [JSW], [LT], and [OSTTW], for example). In this paper we are interested in the ρ-variation and oscillation of the family $\mathcal{T}^\mu f$. That is, given a Radon measure μ in $\mathbb{R}^d$ and $f \in L^1(\mu)$ we will deal with

$$(\mathcal{V}_\rho \circ \mathcal{T}^\mu) f(x) := \mathcal{V}_\rho(\mathcal{T}^\mu f)(x), \quad (\mathcal{O} \circ \mathcal{T}^\mu) f(x) := \mathcal{O}(\mathcal{T}^\mu f)(x).$$

We are specially interested in the case $\mathcal{T}^\mu = \mathcal{R}^\mu$. Notice, by the way, that $T_*^\mu f(x) \leq (\mathcal{V}_\rho \circ \mathcal{T}^\mu) f(x)$ for any compactly supported function $f \in L^1(\mu)$ and all $x \in \mathbb{R}^d$.

When μ coincides with the Lebesgue measure on the real line and $K(x) = 1/x$ is the kernel of the Hilbert transform, Campbell, Jones, Reinhold and Wierdl [CJRW1] showed that $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ and $\mathcal{O} \circ \mathcal{T}^\mu$ are bounded in $L^p(\mu)$ for $1 < p < \infty$, and of weak type $(1,1)$. This result was extended to other singular integral operators in higher dimensions in [CJRW2]. The case of the Cauchy transform and other odd Calderón–Zygmund operators on Lipschitz graphs was studied recently in [MT].

Let us now turn our attention to uniform rectifiability. Recall that a Radon measure μ in $\mathbb{R}^d$ is called *n-rectifiable* if there exists a countable family $\{M_i\}_{i\in\mathbb{N}}$ of n-dimensional C^1 submanifolds in $\mathbb{R}^d$ such that $\mu(E \setminus \bigcup_{i\in\mathbb{N}} M_i) = 0$. Moreover, μ is said to be *n-dimensional Ahlfors–David regular*, or simply AD regular, if there exists some constant $C > 0$ such that $C^{-1}r^n \le \mu(B(x,r)) \le Cr^n$ for all $x \in \operatorname{supp}\mu$ and $0 < r \le \operatorname{diam}(\operatorname{supp}\mu)$. One also says that μ is *uniformly n-rectifiable* if there exist $\theta, M > 0$ so that, for each $x \in \operatorname{supp}\mu$ and $r > 0$, there is a Lipschitz mapping g from the n-dimensional ball $B^n(0,r) \subset \mathbb{R}^n$ into $\mathbb{R}^d$ such that $\operatorname{Lip}(g) \le M$ and $\mu(B(x,r) \cap g(B^n(0,r))) \ge \theta r^n$, where $\operatorname{Lip}(g)$ stands for the Lipschitz constant of g. In particular, uniform rectifiability implies rectifiability. Given a set $E \subset \mathbb{R}^d$, we denote by $\mathcal{H}^n_E$ the n-dimensional Hausdorff measure restricted to E. Then E is called, respectively, n-rectifiable, AD regular, or uniformly n-rectifiable if $\mathcal{H}^n_E$ is so. By the Lebesgue differentiation theorem, any n-dimensional AD regular measure μ is of the form $\mu = f\mathcal{H}^n_{\operatorname{supp}\mu}$ with $C^{-1} \le f(x) \le C$ for some constant $C > 0$ and all $x \in \operatorname{supp}\mu$.

More than twenty years ago G. David and S. Semmes asked the following question, which is still open (see, for example, [Pa, Chapter 7]):

Question 1.2. *Is it true that an n-dimensional AD regular measure μ is uniformly n-rectifiable if and only if R^μ_* is bounded in $L^2(\mu)$?*

Some comments are in order. By the results in [DS1], the "only if" implication of the question above is already known to hold. Also in [DS1], G. David and S. Semmes gave a positive answer to Question 1.2 if one replaces the L^2 boundedness of R^μ_* by the L^2 boundedness of T^μ_* for a wide class of odd kernels K. In the case $n = 1$ (in particular, for the Cauchy transform), the "if" implication was proved by P. Mattila, M. Melnikov and J. Verdera [MMV] using the notion of curvature of measures. Later on, G. David and J. C. Léger [Lé] proved that the L^2 boundedness C^μ_* implies that μ is rectifiable, even without the AD regularity assumption (with $n = 1$).

When μ is the n-dimensional Hausdorff measure on a set $E \subset \mathbb{R}^d$ such that $\mu(E) < \infty$, the rectifiability of μ is also related to the existence μ-a.e. of the principal value of the Riesz transform of μ, that is, the existence of $R^\mu 1(x) = \lim_{\epsilon \searrow 0} R^\mu_\epsilon 1(x)$ for μ-a.e. $x \in E$. In [MPr], P. Mattila and D. Preiss proved that, under the additional assumption that

$$\liminf_{r\to 0} r^{-n}\mu(B(x,r)) > 0 \quad \text{for } \mu\text{-a.e. } x \in E, \tag{2}$$

the rectifiability of E is equivalent to the existence of $R^\mu 1(x)$ μ-a.e. $x \in E$. Later on, in [To3] X. Tolsa removed the assumption (2) and proved the result in full generality: a set $E \subset \mathbb{R}^d$ with $\mu(E) < \infty$ is rectifiable if and only if $R^\mu 1(x)$ exists for μ-a.e. $x \in E$. Let us mention that, for the case $n = 1$ and $d = 2$ (that is, for the Cauchy transform), the analogous results had been obtained previously by [Ma2] under the assumption (2), and in [To1], in full generality, by using the notion of curvature of measures.

In this paper we prove the following:

Theorem 1.3. *Let $1 \leq n < d$ and $\rho > 2$. An n-dimensional AD regular Radon measure μ in $\mathbb{R}^d$ is uniformly n-rectifiable if and only if $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is a bounded operator in $L^2(\mu)$. Moreover, if μ is uniformly n-rectifiable, then for any kernel K satisfying (1), the operator $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ is bounded in $L^2(\mu)$.*

Let us compare this result with the David–Semmes Question 1.2. Notice that the preceding theorem asserts that if we replace the $L^2(\mu)$ boundedness of R^μ_* by the stronger assumption that $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is bounded in $L^2(\mu)$, then μ must be uniformly rectifiable. On the other hand, the theorem claims that the variation for odd singular integral operators with any kernel satisfying (1), in particular for the n-dimensional Riesz transforms, is bounded in $L^2(\mu)$.

A natural question then arises. Given an arbitrary measure μ on $\mathbb{R}^d$, without atoms say, does the $L^2(\mu)$ boundedness of R^μ_* implies the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho \circ \mathcal{R}^\mu$, for $\rho > 2$? By the results of [MMV] and Theorem 1.3, this is true in the case $n = 1$ if μ is AD regular 1-dimensional. Clearly, a positive answer in the general case $n \geq 1$ would solve the David–Semmes problem in the affirmative. Nevertheless, such an approach to try to solve this problem looks quite difficult. In fact, we recall that it is not even known if the $L^2(\mu)$ boundedness of R^μ_* ensures the μ-a.e. existence of the principal values of $R^\mu 1$, which is a necessary condition for the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho \circ \mathcal{R}^\mu$.

Concerning the proof of Theorem 1.3, in our previous paper [MT] we showed that, if μ stands for the n-dimensional Hausdorff-measure on an n-dimensional Lipschitz graph, then the ρ-variation for Riesz transforms and odd Calderón–Zygmund operators with smooth truncations are bounded in $L^2(\mu)$. This is a fundamental step to prove that $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ and, more generally, $\mathcal{V}_\rho \circ \mathcal{T}^\mu$, are bounded in $L^2(\mu)$ if μ is uniformly n-rectifiable. Another basic tool in our arguments is the geometric corona decomposition of uniformly rectifiable measures introduced by David and Semmes in [DS1], which, roughly speaking, describes how $\operatorname{supp}\mu$ can be approximated at different scales by n-dimensional Lipschitz graphs.

The proof of the fact that the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ implies the uniform rectifiability of μ is not so laborious as the one of the converse implication. As remarked above, if $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is bounded in $L^2(\mu)$, then the principal values of $R^\mu 1$ exist μ-a.e., which implies the n-rectifiability of μ, by the results of [MPr] or [To3]. However, this is not enough to ensure the *uniform* n-rectifiability of μ. We will prove the uniform n-rectifiability by arguments partially inspired by some of the techniques in [To4].

Finally, let us remark that Theorem 1.3 follows from a more general result, namely Theorem 2.3 below, which also deals with the variation for Riesz transforms and odd Calderón–Zygmund operators with smooth truncations.

As usual, the letter 'C' stands for some constant which may change its value at different occurrences, and which quite often only depends on n and d. Given two families of constants $A(t)$ and $B(t)$, where t stands for all the explicit or implicit parameters determining $A(t)$ and $B(t)$, the notation $A(t) \lesssim B(t)$ (or $A(t) \gtrsim B(t)$) means that there is some fixed constant C such that $A(t) \leq C B(t)$ (resp. $A(t) \geq C B(t)$) for all t, with C as above. Also, $A(t) \approx B(t)$ is equivalent to $A(t) \lesssim B(t) \lesssim A(t)$.

2. Preliminaries

2.1. The main theorem

Definition 2.1 (families of truncations). Let $\chi_{\mathbb{R}} := \chi_{[1,\infty)}$ and let $\varphi_{\mathbb{R}} : [0,\infty) \to [0,\infty)$ be a nondecreasing $\mathcal{C}^2$ function with $\chi_{[4,\infty)} \le \varphi_{\mathbb{R}} \le \chi_{[1/4,\infty)}$. Suppose moreover that $|\varphi'_{\mathbb{R}}|$ is bounded below away from zero in $[1/3,3]$, i.e., $\chi_{[1/3,3]} \le C|\varphi'_{\mathbb{R}}|$ for some $C > 0$.

Given $x \in \mathbb{R}^d$, and $0 < \epsilon \le \delta$, we set

$$\chi_\epsilon(x) := \chi_{\mathbb{R}}(|x|/\epsilon), \qquad \chi_\epsilon^\delta(x) := \chi_\epsilon(x) - \chi_\delta(x),$$
$$\varphi_\epsilon(x) := \varphi_{\mathbb{R}}(|x|^2/\epsilon^2), \qquad \varphi_\epsilon^\delta(x) := \varphi_\epsilon(x) - \varphi_\delta(x).$$

Notice that, for any finite Radon measure μ, $T_\epsilon\mu(x) = (K\chi_\epsilon * \mu)(x)$. Given $x = (x^1,\ldots,x^d) \in \mathbb{R}^d$, we denote $\widetilde{x} = (x^1,\ldots,x^n,0,\ldots,0) \in \mathbb{R}^d$, and we set $\widetilde{\varphi}_\epsilon(x) := \varphi_\epsilon(\widetilde{x})$ and $\widetilde{\varphi}_\epsilon^\delta(x) := \varphi_\epsilon^\delta(\widetilde{x})$. Finally, for $f \in L^1(\mu)$ we set $\mathcal{T}^\mu f \equiv \mathcal{T}(f\mu) := \{T_\epsilon^\mu f\}_{\epsilon>0}$,

$$T^\mu_{\varphi_\epsilon} f(x) \equiv T_{\varphi_\epsilon}(f\mu)(x) := (K\varphi_\epsilon * \mu)(x), \qquad \mathcal{T}^\mu_\varphi f \equiv \mathcal{T}_\varphi(f\mu) := \{T^\mu_{\varphi_\epsilon} f\}_{\epsilon>0},$$
$$T^\mu_{\widetilde{\varphi}_\epsilon} f(x) \equiv T_{\widetilde{\varphi}_\epsilon}(f\mu)(x) := (K\widetilde{\varphi}_\epsilon * \mu)(x), \qquad \mathcal{T}^\mu_{\widetilde{\varphi}} f \equiv \mathcal{T}_{\widetilde{\varphi}}(f\mu) := \{T^\mu_{\widetilde{\varphi}_\epsilon} f\}_{\epsilon>0}.$$

Remark 2.2. In the definition, the choice of $[4,\infty)$, $[1/4,\infty)$, and $[1/3,3]$ is not specially relevant, it is just for definiteness. One can replace those intervals by other suitable intervals, and all the proofs remain almost the same.

We will prove the following.

Theorem 2.3 (Main Theorem). *Let $1 \le n < d$ be integers. Let μ be an n-dimensional AD regular Radon measure on $\mathbb{R}^d$. The following are equivalent:*

(a) *μ is uniformly n-rectifiable.*
(b) *For any K satisfying (1) and any $\rho > 2$, the operator $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi$ is bounded in $L^p(\mu)$ for all $1 < p < \infty$, and from $L^1(\mu)$ into $L^{1,\infty}(\mu)$.*
(c) *For any K satisfying (1) and any $\rho > 2$, the operator $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ is bounded in $L^2(\mu)$.*
(d) *For some $\rho > 0$, the operator $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is bounded in $L^2(\mu)$.*
(e) *For $K(x) = x/|x|^{n+1}$ and some $\rho > 0$, the operator $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi$ is bounded in $L^2(\mu)$.*

Clearly, Theorem 1.3 is a direct consequence of the preceding result.

Remark 2.4. Let $\{r_m\}_{m\in\mathbb{Z}} \subset (0,\infty)$ be a fixed decreasing sequence defining $\mathcal{O}$. Then the implications (a)⇒(b), ..., (e) in the theorem above still hold if one replaces $\mathcal{V}_\rho$ by $\mathcal{O}$. If there exists $C > 0$ such that $C^{-1}r_m \le r_m - r_{m+1} \le Cr_m$ for all $m \in \mathbb{Z}$, then the implications (b), ..., (e)⇒(a) also hold (so Theorem 2.3 remains true after replacing $\mathcal{V}_\rho$ by $\mathcal{O}$), but we do not know if they are still true without this additional assumption (see Remark 6.9).

Notice that, by Theorem 2.3, besides $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ and $\mathcal{O} \circ \mathcal{R}^\mu$, the operators $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$ and $\mathcal{O} \circ \mathcal{T}_\varphi^\mu$ for $K(x) = x/|x|^{n+1}$ characterize completely the n-AD regular measures μ which are uniformly n-rectifiable.

One of the main ingredients for the proof of Theorem 2.3 is the following result, which strengthens one of the endpoint estimates obtained in [MT]. Let $M(\mathbb{R}^d)$ be the space of finite real Radon measures on $\mathbb{R}^d$, with the norm induced by the variation of measures.

Theorem 2.5. *Let $\rho > 2$ and let μ be the n-dimensional Hausdorff measure restricted to an n-dimensional Lipschitz graph. Then $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is a bounded operator from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$, i.e., there exists $C > 0$ such that, for all $\lambda > 0$ and all $\nu \in M(\mathbb{R}^d)$,*

$$\mu(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) > \lambda\}) \leq \frac{C}{\lambda}\|\nu\|.$$

In particular, $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$ is of weak type $(1,1)$. The constant C only depends on n, d, K, ρ, $\varphi_\mathbb{R}$, and the maximal slope of Γ.

By an *n-dimensional Lipschitz graph* $\Gamma \subset \mathbb{R}^d$ we mean any translation and rotation of a set of the type $\{x \in \mathbb{R}^d : x = (y, A(y)),\ y \in \mathbb{R}^n\}$, where $A : \mathbb{R}^n \to \mathbb{R}^{d-n}$ is some Lipschitz function with Lipschitz constant $\mathrm{Lip}(A)$, which coincides with the maximal slope of Γ.

Remark 2.6. The theorem above remains valid if one replaces $\mathcal{V}_\rho$ by $\mathcal{O}$. Moreover, the norm of $\mathcal{O} \circ \mathcal{T}_\varphi^\mu$ is bounded independently of the sequence that defines $\mathcal{O}$.

The plan to prove Theorem 2.3 is the following: in Section 3 we deal with Theorem 2.5, which is used in Section 4 to obtain the implication (a)⇒(b) of Theorem 2.3. In Section 5 we prove (a)⇒(c) in Theorem 5.1, and in Section 6 we prove Theorem 6.8, which gives (d)⇒(a) and (e)⇒(a), and finishes the proof of Theorem 2.3, taking into account that the implications (b)⇒(e) and (c)⇒(d) are trivial.

Theorems 2.3 and 2.5 are stated in terms of $\mathcal{V}_\rho$, but they also hold for $\mathcal{O}$, as remarked above. However, we will only give the proof for $\mathcal{V}_\rho$, because the case of $\mathcal{O}$ follows by very similar arguments and computations.

2.2. *Calderón–Zygmund decomposition for measures*

Given a cube $Q \subset \mathbb{R}^d$ and $a > 0$, we denote by $\ell(Q)$ the side length of Q and by aQ the cube concentric with Q with side length $a\ell(Q)$. The cubes that we consider in this paper have sides parallel to the coordinate axes in $\mathbb{R}^d$.

A proof of the following result can be found in [To5, Chapter 2] or [M, Lemma 5.1.2].

Lemma 2.7 (Calderón–Zygmund decomposition). *Assume that $\mu := \mathcal{H}^n_{\Gamma\cap B}$, where Γ is an n-dimensional Lipschitz graph and $B \subset \mathbb{R}^d$ is some fixed ball. For any $\nu \in M(\mathbb{R}^d)$ with compact support and any $\lambda > 2^{d+1}\|\nu\|/\|\mu\|$, the following hold:*

(a) *There exists a finite or countable collection* $\{Q_j\}_j \subset \mathbb{R}^d$ *of almost disjoint cubes* (*that is,* $\sum_j \chi_{Q_j} \le C$) *and a function* $f \in L^1(\mu)$ *such that*

$$|\nu|(Q_j) > 2^{-d-1}\lambda\mu(2Q_j), \tag{3}$$

$$|\nu|(\eta Q_j) \le 2^{-d-1}\lambda\mu(2\eta Q_j) \quad \textit{for } \eta > 2, \tag{4}$$

$$\nu = f\mu \quad \textit{in } \mathbb{R}^d \setminus \textstyle\bigcup_j Q_j \quad \textit{with} \quad |f| \le \lambda \ \mu\textit{-a.e.} \tag{5}$$

(b) *For each* j*, let* $R_j := 6Q_j$ *and denote* $w_j := \chi_{Q_j}(\sum_k \chi_{Q_k})^{-1}$. *Then there exists a family* $\{b_j\}_j$ *of functions with* $\operatorname{supp} b_j \subset R_j$ *and with constant sign satisfying*

$$\textstyle\int b_j\,d\mu = \int w_j\,d\nu, \tag{6}$$

$$\|b_j\|_{L^\infty(\mu)}\mu(R_j) \le C|\nu|(Q_j), \tag{7}$$

$$\textstyle\sum_j |b_j| \le C_0\lambda \quad (\textit{where } C_0 \textit{ is some absolute constant}). \tag{8}$$

2.3. Dyadic lattices

For the study of uniformly rectifiable measures we will use the "dyadic cubes" built by G. David [Da, Appendix 1] (see also [DS2, Chapter 3 of Part I]). These are not true cubes, but they play this role with respect to a given n-dimensional AD regular Radon measure μ, in a sense. To distinguish them from the usual cubes, we will call them μ*-cubes*.

Let us explain the precise results and properties related to the lattice of dyadic μ-cubes. Given an n-dimensional AD regular Radon measure μ in $\mathbb{R}^d$ (for simplicity, we may assume $\operatorname{diam}(\operatorname{supp}\mu) = \infty$), for each $j \in \mathbb{Z}$ there exists a family $\mathcal{D}_j$ of Borel subsets of $\operatorname{supp}\mu$ (the dyadic μ-cubes of the jth generation) such that:

(a) each $\mathcal{D}_j$ is a partition of $\operatorname{supp}\mu$, i.e. $\operatorname{supp}\mu = \bigcup_{Q\in\mathcal{D}_j} Q$ (a disjoint union);
(b) if $Q \in \mathcal{D}_j$ and $Q' \in \mathcal{D}_k$ with $k \le j$, then either $Q \subset Q'$ or $Q \cap Q' = \emptyset$;
(c) for all $j \in \mathbb{Z}$ and $Q \in \mathcal{D}_j$, we have $2^{-j} \lesssim \operatorname{diam}(Q) \le 2^{-j}$ and $\mu(Q) \approx 2^{-jn}$;
(d) there exists $C > 0$ such that, for all $j \in \mathbb{Z}$, $Q \in \mathcal{D}_j$, and $0 < \tau < 1$,

$$\mu(\{x \in Q : \operatorname{dist}(x, \operatorname{supp}\mu \setminus Q) \le \tau 2^{-j}\}) + \mu(\{x \in \operatorname{supp}\mu \setminus Q : \operatorname{dist}(x, Q) \le \tau 2^{-j}\}) \le C\tau^{1/C}2^{-jn}. \tag{9}$$

This property is usually called the *small boundaries condition*. From (9), it follows that there is a point $z_Q \in Q$ (the *center* of Q) such that $\operatorname{dist}(z_Q, \operatorname{supp}\mu \setminus Q) \gtrsim 2^{-j}$ (see [DS2, Lemma 3.5 of Part I]).

We denote $\mathcal{D} := \bigcup_{j\in\mathbb{Z}} \mathcal{D}_j$. For $Q \in \mathcal{D}_j$, we define the side length of Q as $\ell(Q) = 2^{-j}$. Notice that $\ell(Q) \lesssim \operatorname{diam}(Q) \le \ell(Q)$. Actually it may happen that a μ-cube Q belongs to $\mathcal{D}_j \cap \mathcal{D}_k$ with $j \ne k$. In this case, $\ell(Q)$ is not well defined. However, this problem can be solved in many ways. For example, the reader may think that a μ-cube is not only a subset of $\operatorname{supp}\mu$, but a couple (Q, j), where Q is a subset of $\operatorname{supp}\mu$ and $j \in \mathbb{Z}$ is such that $Q \in \mathcal{D}_j$.

Given $a > 1$ and $Q \in \mathcal{D}$, we set $aQ := \{x \in \operatorname{supp}\mu : \operatorname{dist}(x, Q) \leq (a-1)\ell(Q)\}$. Observe that $\operatorname{diam}(aQ) \leq \operatorname{diam}(Q) + 2(a-1)\ell(Q) \leq (2a-1)\ell(Q)$.

2.4. *Corona decomposition*

Given an n-dimensional AD regular Radon measure μ on $\mathbb{R}^d$, let $\mathcal{D} := \{Q \in \mathcal{D}_j : j \in \mathbb{Z}\}$ be the dyadic lattice associated to μ introduced in Subsection 2.3. Following [DS2, Definitions 3.13 and 3.19 of Part I], one says that μ admits a *corona decomposition* if, for each $\eta > 0$ and $\theta > 0$, one can find a triple $(\mathcal{B}, \mathcal{G}, \mathrm{Trs})$, where $\mathcal{B}$ and $\mathcal{G}$ are two subsets of $\mathcal{D}$ (the "bad μ-cubes" and the "good μ-cubes") and Trs is a family of subsets $S \subset \mathcal{G}$ (that we will call *trees*), which satisfy the following conditions:

(a) $\mathcal{D} = \mathcal{B} \cup \mathcal{G}$ and $\mathcal{B} \cap \mathcal{G} = \emptyset$.
(b) $\mathcal{B}$ satisfies the *Carleson packing condition*, i.e., $\sum_{Q \in \mathcal{B}:\, Q \subset R} \mu(Q) \lesssim \mu(R)$ for all $R \in \mathcal{D}$.
(c) $\mathcal{G} = \biguplus_{S \in \mathrm{Trs}} S$, i.e., any $Q \in \mathcal{G}$ belongs to only one $S \in \mathrm{Trs}$.
(d) Each $S \in \mathrm{Trs}$ is *coherent*. This means that each $S \in \mathrm{Trs}$ has a unique maximal element Q_S which contains all other elements of S as subsets, that $Q' \in S$ as soon as $Q' \in \mathcal{D}$ satisfies $Q \subset Q' \subset Q_S$ for some $Q \in S$, and that if $Q \in S$ then either all of the children of Q lie in S or none of them does (if $Q \in \mathcal{D}_j$, the *children* of Q are defined as the collection of μ-cubes $Q' \in \mathcal{D}_{j+1}$ such that $Q' \subset Q$).
(e) The maximal μ-cubes Q_S, for $S \in \mathrm{Trs}$, satisfy the Carleson packing condition. That is, $\sum_{S \in \mathrm{Trs}:\, Q_S \subset R} \mu(Q_S) \lesssim \mu(R)$ for all $R \in \mathcal{D}$.
(f) For each $S \in \mathrm{Trs}$, there exists an n-dimensional Lipschitz graph Γ_S with constant smaller than η such that $\operatorname{dist}(x, \Gamma_S) \leq \theta \operatorname{diam}(Q)$ whenever $x \in 2Q$ and $Q \in S$ (one can replace "$x \in 2Q$" by "$x \in C_{\mathrm{cor}}Q$" for any constant $C_{\mathrm{cor}} \geq 2$ given in advance, by [DS2, Lemma 3.31 of Part I]).

It is shown in [DS1] (see also [DS2]) that if μ is uniformly rectifiable then it admits a corona decomposition for all parameters $k > 2$ and $\eta, \theta > 0$. Conversely, the existence of a corona decomposition for a single set of parameters $k > 2$ and $\eta, \theta > 0$ implies that μ is uniformly rectifiable.

2.5. *The α and β coefficients*

Let μ be an n-dimensional AD regular Radon measure in $\mathbb{R}^d$ and $\mathcal{D}$ as in Subsection 2.3. Given $1 \leq p < \infty$ and a μ-cube $Q \in \mathcal{D}$, one sets (see [DS2])

$$\beta_{p,\mu}(Q) = \inf_L \left\{ \frac{1}{\ell(Q)^n} \int_{2Q} \left(\frac{\operatorname{dist}(y, L)}{\ell(Q)} \right)^p d\mu(y) \right\}^{1/p},$$

where the infimum is taken over all n-planes L in $\mathbb{R}^d$. For $p = \infty$ one replaces the L^p norm by the supremum norm. The $\beta_{\infty,\mu}$ coefficients were first introduced by P. Jones in his celebrated work on rectifiability [Jn], while the $\beta_{p,\mu}$'s for $1 \leq p < \infty$ were introduced by G. David and S. Semmes in their pioneering work on uniform rectifiability (see [DS1] for example).

Other coefficients that have proved useful in the study of uniform rectifiability and boundedness of Calderón–Zygmund operators are the α coefficients introduced in [To4]. Let $F \subset \mathbb{R}^d$ be the closure of an open set. Given two finite Radon measures σ, ν on $\mathbb{R}^d$, one sets $\operatorname{dist}_F(\sigma, \nu) := \sup\{|\int f\,d\sigma - \int f\,d\nu| : \operatorname{Lip}(f) \leq 1, \operatorname{supp} f \subset F\}$. Finally, given a μ-cube $Q \in \mathcal{D}$, consider the closed ball $B_Q := B(z_Q, 6\sqrt{d}\,\ell(Q))$, where z_Q denotes the center of Q. Then one defines

$$\alpha_\mu(Q) := \frac{1}{\ell(Q)^{n+1}} \inf_{c \geq 0,\, L} \operatorname{dist}_{B_Q}(\mu, c\mathcal{H}^n_L), \tag{10}$$

where the infimum is taken over all constants $c \geq 0$ and all n-planes L in $\mathbb{R}^d$.

The following result characterizes the uniform rectifiability of μ in terms of the α and β coefficients (see [DS1] for (a)⇔(b) and [To4] for (a)⇔(c)).

Theorem 2.8. *Let $p \in [1, 2]$ and let μ be an n-dimensional AD regular Radon measure in $\mathbb{R}^d$. The following are equivalent:*

(a) *μ is uniformly n-rectifiable.*
(b) $\sum_{Q \in \mathcal{D}:\, Q \subset R} \beta_{p,\mu}(Q)^2 \ell(Q)^n \lesssim \ell(R)^n$ *for all μ-cubes $R \in \mathcal{D}$.*
(c) $\sum_{Q \in \mathcal{D}:\, Q \subset R} \alpha_\mu(Q)^2 \ell(Q)^n \lesssim \ell(R)^n$ *for all μ-cubes $R \in \mathcal{D}$.*

For the case of $\mu = \mathcal{H}^n_\Gamma$ for some Lipschitz graph $\Gamma = \{x \in \mathbb{R}^d : x = (y, A(y)), y \in \mathbb{R}^n\}$, one can take $\mathcal{D} = \{\widetilde{Q} \times \mathbb{R}^{d-n} \cap \Gamma : \widetilde{Q} \in \mathcal{D}(\mathbb{R}^n)\}$, where $\mathcal{D}(\mathbb{R}^n)$ denotes the standard dyadic lattice of $\mathbb{R}^n$. For $Q = (\widetilde{Q} \times \mathbb{R}^{d-n}) \cap \Gamma \in \mathcal{D}$, we set

$$\widetilde{\alpha}_\mu(Q) := \frac{1}{\ell(\widetilde{Q})^{n+1}} \inf_{c \geq 0,\, L} \operatorname{dist}_{6\widetilde{Q} \times \mathbb{R}^{d-n}}(\mu, c\mathcal{H}^n_L), \tag{11}$$

where the infimum is taken over all constants $c \geq 0$ and all n-planes L in $\mathbb{R}^d$. Then it is easy to show that $\widetilde{\alpha}_\mu(Q) \approx \alpha_\mu(Q)$ for all $Q \in \mathcal{D}$.

One can also define $\widetilde{\beta}_{p,\mu}(Q)$ in an analogous manner. By Theorem 2.8,

$$\sum_{Q \in \mathcal{D}:\, Q \subset R} (\widetilde{\beta}_{p,\mu}(Q)^2 + \widetilde{\alpha}_\mu(Q)^2)\ell(Q)^n \leq C\ell(R)^n \tag{12}$$

for all $R \in \mathcal{D}$, with C independent of R. Moreover, one can also show that this last inequality also holds after replacing Q and R by $k_1 Q$ and $k_2 R$ for any $k_1, k_2 \geq 1$ given in advance, where $kQ := (k\widetilde{Q} \times \mathbb{R}^{d-n}) \cap \Gamma$ for $k > 0$.

3. If Γ is an n-dimensional Lipschitz graph, then $\mathcal{V}_\rho \circ \mathcal{T}_\varphi : M(\mathbb{R}^d) \to L^{1,\infty}(\mathcal{H}^n_\Gamma)$ is a bounded operator

The following result is contained in [MT, Theorem 1.1] (see also [M, Main Theorem 3.0.1]).

Theorem 3.1. *Let $\rho > 2$ and let μ be the n-dimensional Hausdorff measure restricted to an n-dimensional Lipschitz graph. Then the operator $\mathcal{V}_\rho \circ \mathcal{T}^\mu_{\widetilde{\varphi}}$ is bounded in $L^2(\mu)$. The bound of the norm only depends on n, d, K, ρ, $\varphi_{\mathbb{R}}$, and the slope of the graph.*

By very similar techniques to the ones used in the proof of the theorem above, one can prove the following.

Theorem 3.2. *Let $\rho > 2$ and let μ be the n-dimensional Hausdorff measure restricted to an n-dimensional Lipschitz graph. Then the operator $\mathcal{V}_\rho \circ \mathcal{T}^\mu_{\varphi}$ is bounded in $L^2(\mu)$. The bound of the norm only depends on n, d, K, ρ, $\varphi_{\mathbb{R}}$, and the slope of the graph.*

Sketch of proof. The first step consists in obtaining the following basic estimate: Fix a cube $\widetilde{P} \subset \mathbb{R}^n$. Set $\Gamma := \{x \in \mathbb{R}^d : x = (y, A(y)),\ y \in \mathbb{R}^n\}$, where $A : \mathbb{R}^n \to \mathbb{R}^{d-n}$ is a Lipschitz function supported in $\widetilde{P}$, and set $P := (\widetilde{P} \times \mathbb{R}^{d-n}) \cap \Gamma$. Set $\mu := f\mathcal{H}^n_\Gamma$, where $f(x) = 1$ for all $x \in \Gamma \setminus P$ and $C_0^{-1} \leq f(x) \leq C_0$ for all $x \in P$, for some constant $C_0 > 0$.

For each $x \in \Gamma$, define

$$W\mu(x)^2 := \sum_{m\in\mathbb{Z}} |(K\varphi_{2^{-m}} * \mu)(x) - (K\widetilde{\varphi}_{2^{-m}} * \mu)(x)|^2, \tag{13}$$

$$S\mu(x)^2 := \sup_{\{\epsilon_m\}} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}:\, \epsilon_m,\epsilon_{m+1}\in I_j} |(K\varphi^{\epsilon_m}_{\epsilon_{m+1}} * \mu)(x)|^2, \tag{14}$$

where $I_j = [2^{-j-1}, 2^{-j})$ and the supremum is taken over all decreasing sequences $\{\epsilon_m\}_{m\in\mathbb{Z}}$ of positive numbers. Then we claim that

$$\|W\mu\|^2_{L^2(\mu)} + \|S\mu\|^2_{L^2(\mu)} \lesssim \sum_{Q\in\mathcal{D}} \big(\widetilde{\alpha}_\mu(C_1 Q)^2 + \widetilde{\beta}_{2,\mu}(Q)^2\big)\ell(Q)^n, \tag{15}$$

where $C_1 > 0$ only depends on C_0, n, d, K, $\varphi_{\mathbb{R}}$, and $\mathrm{Lip}(A)$, and where $\mathcal{D}$ denotes the dyadic lattice associated to $\mathcal{H}^n_\Gamma$ defined after Theorem 2.8.

Let us prove the claim. If we define $\widetilde{S}\mu$ like $S\mu$ but replacing $\varphi^{\epsilon_m}_{\epsilon_{m+1}}$ by $\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}$, in the proof of Theorem 3.1 in [MT] it is shown that $\|\widetilde{S}\mu\|^2_{L^2(\mu)}$ is bounded above by the right hand side of (15). The proof for $\|S\mu\|^2_{L^2(\mu)}$ is almost the same.

Let us deal now with $W\mu$. Fix $D := (\widetilde{D} \times \mathbb{R}^{d-n}) \cap \Gamma \in \mathcal{D}$ with $\ell(D) = 2^{-m}$ and $x \in D$. Let L_D be an n-plane that minimizes $\widetilde{\alpha}_\mu(C_1 D)$ in (11), where $C_1 > 0$ is some constant large enough which will be fixed later, and let $\sigma_D := c_D\mathcal{H}^n_{L_D}$ be a minimizing measure for $\widetilde{\alpha}_\mu(C_1 D)$. Let L^x_D be the n-plane parallel to L_D which contains x, and set $\sigma^x_D := c_D\mathcal{H}^n_{L^x_D}$.

Since $x \in D$ and $\ell(D) = 2^{-m}$, $(\varphi_{2^{-m}}(x - \cdot) - \widetilde{\varphi}_{2^{-m}}(x - \cdot))K(x - \cdot)$ is a function supported in $C_1\widetilde{D} \times \mathbb{R}^{d-n}$ (for some constant C_1 large enough) and with Lipschitz constant smaller than $C2^{m(n+1)}$. Moreover, by the antisymmetry of the function $(\varphi_{2^{-m}}(x - \cdot) - \widetilde{\varphi}_{2^{-m}}(x - \cdot))K(x - \cdot)$, and since σ^x_D is a multiple of the n-dimensional Hausdorff measure on an n-plane which contains x, we have $(K\varphi_{2^{-m}} * \sigma_D^x)(x) - (K\widetilde{\varphi}_{2^{-m}} * \sigma_D^x)(x) = 0$. Therefore,

$$\begin{aligned}(K\varphi_{2^{-m}} * \mu)(x) - (K\widetilde{\varphi}_{2^{-m}} * \mu)(x) &= (K(\varphi_{2^{-m}} - \widetilde{\varphi}_{2^{-m}}) * \mu)(x)\\ &= (K(\varphi_{2^{-m}} - \widetilde{\varphi}_{2^{-m}}) * (\mu - \sigma_D))(x) + (K(\varphi_{2^{-m}} - \widetilde{\varphi}_{2^{-m}}) * (\sigma_D - \sigma_D^x))(x). \quad (16)\end{aligned}$$

Using the definition of $\widetilde{\alpha}_\mu$, we get

$$|(K(\varphi_{2^{-m}} - \widetilde{\varphi}_{2^{-m}}) * (\mu - \sigma_D))(x)| \lesssim 2^{m(n+1)} \operatorname{dist}_{C_1\widetilde{D}\times\mathbb{R}^{d-n}}(\mu, \sigma_D) \lesssim \widetilde{\alpha}_\mu(C_1 D). \quad (17)$$

Since L_D^x is a translation of L_D, by standard estimates it is not hard to show that

$$|(K(\varphi_{2^{-m}} - \widetilde{\varphi}_{2^{-m}}) * (\sigma_D - \sigma_D^x))(x)| \lesssim 2^m \operatorname{dist}(x, L_D) = \operatorname{dist}(x, L_D)/\ell(D). \quad (18)$$

Let $\operatorname{dist}_{\mathcal{H}}(E, F)$ denote the Hausdorff distance of sets $E, F \subset \mathbb{R}^d$, and set $\widetilde{B}_D := 6\widetilde{D} \times \mathbb{R}^{d-n}$. If L_D^1 and L_D^2 denote a minimizing n-plane for $\widetilde{\beta}_{1,\mu}(D)$ and $\widetilde{\beta}_{2,\mu}(D)$, respectively, one can show that $\operatorname{dist}_{\mathcal{H}}(L_D \cap \widetilde{B}_D, L_D^1 \cap \widetilde{B}_D) \lesssim \widetilde{\alpha}_\mu(D)\ell(D)$ and that $\operatorname{dist}_{\mathcal{H}}(L_D^1 \cap \widetilde{B}_D, L_D^2 \cap \widetilde{B}_D) \lesssim \widetilde{\beta}_{2,\mu}(D)\ell(D)$. This easily implies that $\operatorname{dist}(x, L_D) \lesssim \operatorname{dist}(x, L_D^2) + \widetilde{\beta}_{2,\mu}(D)\ell(D) + \widetilde{\alpha}_\mu(D)\ell(D)$ for all $x \in D$. Applying this to (18), and using also (17) and (16), we obtain

$$\begin{aligned}\|W\mu\|_{L^2(\mu)}^2 &= \int \sum_{m\in\mathbb{Z}} |(K(\varphi_{2^{-m}} - \widetilde{\varphi}_{2^{-m}}) * \mu)(x)|^2 \, d\mu(x)\\ &= \sum_{m\in\mathbb{Z}} \sum_{D\in\mathcal{D}:\, \ell(D)=2^{-m}} \int_D |(K(\varphi_{2^{-m}} - \widetilde{\varphi}_{2^{-m}}) * \mu)(x)|^2 \, d\mu(x)\\ &\lesssim \sum_{m\in\mathbb{Z}} \sum_{D\in\mathcal{D}:\, \ell(D)=2^{-m}} \int_D \big(\operatorname{dist}(x, L_D^2)/\ell(D) + \widetilde{\beta}_{2,\mu}(D) + \widetilde{\alpha}_\mu(C_1 D)\big)^2 \, d\mu(x)\\ &\lesssim \sum_{D\in\mathcal{D}} \big(\widetilde{\alpha}_\mu(C_1 D)^2 + \widetilde{\beta}_{2,\mu}(D)^2\big)\ell(D)^n,\end{aligned}$$

which proves (15).

Let now μ be as in Theorem 3.2. Using (15) and Theorem 3.1, one can show that there exists $C > 0$ such that, for any cube $\widetilde{D} \subset \mathbb{R}^n$ and any $g \in L^\infty(\mu)$ supported in D (where $D := \widetilde{D} \times \mathbb{R}^{d-n}$),

$$\int_D ((\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu)g)^2 \, d\mu \le C\|g\|_{L^\infty(\mu)}^2 \mu(D).$$

This yields the endpoint estimates $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu : H^1(\mu) \to L^1(\mu)$ and $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu : L^\infty(\mu) \to \mathrm{BMO}(\mu)$, where $H^1(\mu)$ denotes the atomic Hardy space related to μ. Then by interpolation, one deduces that $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$ is bounded in $L^2(\mu)$. Since this part of the proof is analogous to the one in the proof of Theorem 3.1 (see [MT, Theorem 1.1]), we omit it. □

3.1. *Proof of Theorem 2.5*

The proof of Theorem 2.5 uses the Calderón–Zygmund decomposition of Lemma 2.7 and rather standard arguments. Set $\mu := \mathcal{H}^n_{\Gamma\cap B}$, where $B \subset \mathbb{R}^d$ is some fixed ball. Let $\nu \in M(\mathbb{R}^d)$ be a finite Radon measure with compact support and $\lambda > 2^{d+1}\|\nu\|/\|\mu\|$. We will show that

$$\mu(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) > \lambda\}) \leq \frac{C}{\lambda}\|\nu\|, \tag{19}$$

where $C > 0$ depends on n, d, K, ρ and Γ, but not on B. Let us check that this implies that $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mathcal{H}^n_\Gamma)$. First, we show that (19) also holds for ν without compact support. Set $\nu_N = \chi_{B(0,N)}\nu$ and let N_0 be such that $\operatorname{supp}\mu \subset B(0, N_0)$. Then it is not hard to show that, for $x \in \operatorname{supp}\mu$,

$$|(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) - (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_N(x)| \leq C\,\frac{|\nu|(\mathbb{R}^d \setminus B(0,N))}{N - N_0},$$

thus $(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_N(x) \to (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x)$ for all $x \in \operatorname{supp}\mu$, and since the estimate (19) holds by assumption for ν_N, letting $N \to \infty$, we deduce that it also holds for ν. Now, by increasing the size of the ball B and by monotone convergence, we deduce that $\mathcal{H}^n_\Gamma(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) > \lambda\}) \leq C\lambda^{-1}\|\nu\|$, as desired.

To prove (19) for $\nu \in M(\mathbb{R}^d)$ with compact support, let $\{Q_j\}_j$ be the almost disjoint family of cubes of Lemma 2.7, and set $\Omega := \bigcup_j Q_j$ and $R_j := 6Q_j$. Then we can write $\nu = g\mu + \nu_b$, with

$$g\mu = \chi_{\mathbb{R}^d\setminus\Omega}\nu + \sum_j b_j\mu \quad \text{and} \quad \nu_b = \sum_j \nu_b^j := \sum_j (w_j\nu - b_j\mu),$$

where the functions b_j satisfy (6)–(8) and $w_j = \chi_{Q_j}(\sum_k \chi_{Q_k})^{-1}$.

By the subadditivity of $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$, we have

$$\begin{aligned}&\mu(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) > \lambda\})\\ &\leq \mu(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi)g(x) > \lambda/2\}) + \mu(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_b(x) > \lambda/2\}).\end{aligned} \tag{20}$$

Since $\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}^n_\Gamma}_\varphi$ is bounded in $L^2(\mathcal{H}^n_\Gamma)$ by Theorem 3.2, it is easy to show that $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi$ is bounded in $L^2(\mu)$, with a bound independent of B. Notice that $|g| \leq C\lambda$ by (5) and (8). Then using (7),

$$\begin{aligned}\mu(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi)g(x) > \lambda/2\}) &\lesssim \frac{1}{\lambda^2}\int |(\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi)g|^2\,d\mu \lesssim \frac{1}{\lambda^2}\int |g|^2\,d\mu\\ &\lesssim \frac{1}{\lambda}\int |g|\,d\mu \lesssim \frac{1}{\lambda}\left(|\nu|(\mathbb{R}^d \setminus \Omega) + \sum_j \int_{R_j} |b_j|\,d\mu\right)\\ &\lesssim \frac{1}{\lambda}\left(|\nu|(\mathbb{R}^d \setminus \Omega) + \sum_j |\nu|(Q_j)\right) \lesssim \frac{\|\nu\|}{\lambda}.\end{aligned} \tag{21}$$

Let $\widehat{\Omega} := \bigcup_j 2Q_j$. By (3), we have $\mu(\widehat{\Omega}) \le \sum_j \mu(2Q_j) \lesssim \lambda^{-1} \sum_j |\nu|(Q_j) \lesssim \lambda^{-1}\|\nu\|$. We are going to show now that

$$\mu(\{x \in \mathbb{R}^d \setminus \widehat{\Omega} : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_b(x) > \lambda/2\}) \le \frac{C}{\lambda}\|\nu\|; \tag{22}$$

then (19) is a direct consequence of (20)–(22) and the estimate $\mu(\widehat{\Omega}) \lesssim \lambda^{-1}\|\nu\|$. Since $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is sublinear,

$$\begin{aligned}\mu(\{x \in \mathbb{R}^d \setminus \widehat{\Omega} : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_b(x) > \lambda/2\}) &\lesssim \frac{1}{\lambda}\sum_j \int_{\mathbb{R}^d\setminus\widehat{\Omega}} (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_b^j \, d\mu \\ &\le \frac{1}{\lambda}\sum_j \int_{\mathbb{R}^d\setminus 2R_j} (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_b^j \, d\mu + \frac{1}{\lambda}\sum_j \int_{2R_j\setminus 2Q_j} (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_b^j \, d\mu. \end{aligned} \tag{23}$$

We are going to estimate the two terms on the right of (23) separately. Let us start with the first one. Given j and $x \in \operatorname{supp}\mu \setminus 2R_j$, let $\{\epsilon_m\}_{m\in\mathbb{Z}}$ be a decreasing sequence of positive numbers (which depend on j and x, i.e. $\epsilon_m \equiv \epsilon_m(j, x)$) such that

$$(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_b^j(x) \le 2\Big(\sum_{m\in\mathbb{Z}} |(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * \nu_b^j)(x)|^\rho\Big)^{1/\rho}. \tag{24}$$

If we set $I_k := [2^{-k-1}, 2^{-k})$, we can decompose $\mathbb{Z} = \mathcal{S} \cup \mathcal{L}$, where

$$\begin{aligned}&\mathcal{L} := \{m \in \mathbb{Z} : \epsilon_m \in I_k,\ \epsilon_{m+1} \in I_i \text{ for some } i > k\},\\ &\mathcal{S} := \bigcup_{k\in\mathbb{Z}} \mathcal{S}_k, \qquad \mathcal{S}_k := \{m \in \mathbb{Z} : \epsilon_m, \epsilon_{m+1} \in I_k\}.\end{aligned}$$

Let z_j denote the center of Q_j (and of R_j). Then, since $\nu_b^j(R_j) = 0$ and $\operatorname{supp}\nu_b^j \subset R_j$,

$$\begin{aligned}|(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * \nu_b^j)(x)| &= \Big|\int \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)K(x-y)\, d\nu_b^j(y)\Big| \\ &\le \int |\varphi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)K(x-y) - \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x-z_j)K(x-z_j)|\, d|\nu_b^j|(y). \end{aligned} \tag{25}$$

If $m \in \mathcal{L}$, it is easy to see that $|\nabla(\varphi_{\epsilon_{m+1}}^{\epsilon_m}K)(t)| \le |\nabla(\varphi_{\epsilon_{m+1}}K)(t)| + |\nabla(\varphi_{\epsilon_m}K)(t)| \lesssim |t|^{-n-1}$ for all $t \in \mathbb{R}^d \setminus \{0\}$. Moreover, since $x \in \mathbb{R}^d \setminus 2R_j$ and $\operatorname{supp}\nu_b^j \subset R_j$, there are finitely many $m \in \mathcal{L}$ such that $(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * \nu_b^j)(x) \ne 0$, and their number only depends on n and d. On the other hand, if $m \in \mathcal{S}_k$, it is not hard to show that $|\nabla(\varphi_{\epsilon_{m+1}}^{\epsilon_m}K)(t)| \lesssim 2^k|\epsilon_m - \epsilon_{m+1}|\,|t|^{-n-1}$. Actually, this follows from the fact that $(\varphi_{\epsilon_{m+1}}^{\epsilon_m}K)(t) \ne 0$ only if $|t| \approx 2^{-k}$ and the estimates

$$\begin{aligned}|\varphi_{\epsilon_{m+1}}^{\epsilon_m}(t)| &= \Big|\varphi_{\mathbb{R}}\Big(\frac{|t|}{\epsilon_{m+1}}\Big) - \varphi_{\mathbb{R}}\Big(\frac{|t|}{\epsilon_m}\Big)\Big| \le \|\varphi_{\mathbb{R}}'\|_{L^\infty(\mathbb{R})}\Big|\frac{|t|}{\epsilon_{m+1}} - \frac{|t|}{\epsilon_m}\Big| \\ &= \|\varphi_{\mathbb{R}}'\|_\infty |t| \frac{\epsilon_m - \epsilon_{m+1}}{\epsilon_m\epsilon_{m+1}} \lesssim 2^k|\epsilon_m - \epsilon_{m+1}| \end{aligned} \tag{26}$$

and

$$\begin{aligned}|\partial_{t^i}(\varphi_{\epsilon_{m+1}}^{\epsilon_m}(t))| &\leq \left|\varphi_{\mathbb{R}}'\left(\frac{|t|}{\epsilon_m}\right)\frac{1}{\epsilon_m} - \varphi_{\mathbb{R}}'\left(\frac{|t|}{\epsilon_{m+1}}\right)\frac{1}{\epsilon_{m+1}}\right| \\ &\leq \left|\varphi_{\mathbb{R}}'\left(\frac{|t|}{\epsilon_m}\right)\right|\left|\frac{1}{\epsilon_m} - \frac{1}{\epsilon_{m+1}}\right| + \left|\varphi_{\mathbb{R}}'\left(\frac{|t|}{\epsilon_m}\right) - \varphi_{\mathbb{R}}'\left(\frac{|t|}{\epsilon_{m+1}}\right)\right|\frac{1}{\epsilon_{m+1}} \\ &\leq \left(\|\varphi_{\mathbb{R}}'\|_\infty + \|\varphi_{\mathbb{R}}''\|_\infty \frac{|t|}{\epsilon_{m+1}}\right)\frac{\epsilon_m - \epsilon_{m+1}}{\epsilon_m \epsilon_{m+1}} \lesssim 2^k(\epsilon_m - \epsilon_{m+1})|t|^{-1}, \qquad (27)\end{aligned}$$

where $1 \leq i \leq d$ and t^i denotes the ith coordinate of $t \in \mathbb{R}^d$ (recall that $\epsilon_m \approx \epsilon_{m+1} \approx 2^{-k}$ for $m \in \mathcal{S}_k$ and we assumed $|t| \approx 2^{-k}$). Similarly to the case $m \in \mathcal{L}$, there are finitely many $k \in \mathbb{Z}$ such that $\operatorname{supp}\varphi_{2^{-k-1}}^{2^{-k}}(x - \cdot) \cap R_j \neq \emptyset$, and their number only depends on n and d (notice that $\operatorname{supp}\varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - \cdot) \subset \operatorname{supp}\varphi_{2^{-k-1}}^{2^{-k}}(x - \cdot)$ for all $m \in \mathcal{S}_k$).

From these estimates and remarks, and (24), (25), we obtain

$$\begin{aligned}(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_b^j(x) &\lesssim \sum_{k\in\mathbb{Z}}\sum_{m\in\mathcal{S}_k}|(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * \nu_b^j)(x)| + \sum_{m\in\mathcal{L}}|(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * \nu_b^j)(x)| \\ &\lesssim \sum_{k\in\mathbb{Z}:\, \operatorname{supp}\varphi_{2^{-k-1}}^{2^{-k}}(x-\cdot)\cap R_j\neq\emptyset}\;\sum_{m\in\mathcal{S}_k} 2^k|\epsilon_m - \epsilon_{m+1}|\,|x - z_j|^{-n-1}\ell(R_j)\|\nu_b^j\| \\ &\quad + \sum_{m\in\mathcal{L}:\, \operatorname{supp}\varphi_{\epsilon_{m+1}}^{\epsilon_m}(x-\cdot)\cap R_j\neq\emptyset} |x - z_j|^{-n-1}\ell(R_j)\|\nu_b^j\| \lesssim |x - z_j|^{-n-1}\ell(R_j)\|\nu_b^j\|\end{aligned}$$

for all j and $x \in \operatorname{supp}\mu \setminus 2R_j$. Therefore, since μ has n-dimensional growth and $\|\nu_b^j\| \lesssim |\nu|(Q_j)$, and since the Q_j's are almost disjoint,

$$\begin{aligned}\sum_j \int_{\mathbb{R}^d\setminus 2R_j}(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_b^j\,d\mu &\lesssim \sum_j \ell(R_j)\|\nu_b^j\|\int_{\mathbb{R}^d\setminus 2R_j}|x - z_j|^{-n-1}\,d\mu \\ &\lesssim \sum_j \|\nu_b^j\| \lesssim \|\nu\|. \qquad (28)\end{aligned}$$

Let us now estimate the second term on the right hand side of (23). As above, given j and $x \in 2R_j \setminus 2Q_j$, let $\{\epsilon_m\}_{m\in\mathbb{Z}}$ be a decreasing sequence of positive numbers such that

$$(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(w_j\nu)(x) \leq 2\Big(\sum_{m\in\mathbb{Z}}|(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (w_j\nu))(x)|^\rho\Big)^{1/\rho},$$

where $w_j = \chi_{Q_j}(\sum_k \chi_{Q_k})^{-1}$. Since $\rho > 2$, $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is sublinear, and since $\nu_b^j = w_j\nu - b_j\mu$, for $x \in 2R_j \setminus 2Q_j$ we have

$$\begin{aligned}(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_b^j(x) &\leq (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(w_j\nu)(x) + (\mathcal{V}_\rho \circ \mathcal{T})(b_j\mu)(x) \\ &\leq 2\sum_{m\in\mathbb{Z}}|(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (w_j\nu))(x)| + (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu)b_j(x) \\ &\lesssim |\nu|(Q_j)|x - z_j|^{-n} + (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu)b_j(x).\end{aligned}$$

Since $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$ is bounded in $L^2(\mu)$, using the estimate above and Cauchy–Schwarz we get

$$\begin{aligned}\sum_j \int_{2R_j\setminus 2Q_j} (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_b^j \, d\mu \\ &\lesssim \sum_j \int_{2R_j\setminus 2Q_j} \frac{|\nu|(Q_j)}{|x-z_j|^n}\, d\mu(x) + \sum_j \int_{2R_j\setminus 2Q_j} (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu) b_j \, d\mu \\ &\lesssim \sum_j |\nu|(Q_j) \frac{\mu(2R_j)}{\ell(Q_j)^n} + \sum_j \|(\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu) b_j\|_{L^2(\mu)} \mu(2R_j)^{1/2} \\ &\lesssim \sum_j |\nu|(Q_j) + \sum_j \|b_j\|_{L^\infty(\mu)}\mu(R_j) \lesssim \sum_j |\nu|(Q_j) \lesssim \|\nu\|.\end{aligned}$$

Together with (28) and (23), this proves (22), and Theorem 2.5 follows.

4. If μ is a uniformly n-rectifiable measure, then $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu : L^p(\mu) \to L^p(\mu)$ is a bounded operator for $1 < p < \infty$

The purpose of this section is to prove the following theorem and the subsequent corollary.

Theorem 4.1. *Let μ be an n-dimensional AD regular Radon measure in $\mathbb{R}^d$ and let $\rho > 2$. Assume that there exist constants C_0 and C_1 such that, for each ball B centered in* supp μ*, there is a set $F = F_B$ such that:*

(a) $\mu(F \cap B) \geq C_0 \mu(B)$,
(b) *$\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mathcal{H}^n_F)$ with constant bounded by C_1.*

Then $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$, and $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$ is a bounded operator in $L^p(\mu)$ for all $1 < p < \infty$.

Corollary 4.2. *If μ is an n-dimensional AD regular uniformly n-rectifiable measure, then $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$ is a bounded operator in $L^p(\mu)$ for all $1 < p < \infty$ and $\rho > 2$. Moreover, $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$, so $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$ is also of weak type* $(1, 1)$.

Proof. Recall from [DS2, Definition 1.26] that a Radon measure ν in $\mathbb{R}^d$ has *BPLG* (*big pieces of Lipschitz graphs*) if ν is n-dimensional AD regular and there exist constants $C_1 > 0$ and $\theta > 0$ such that, for any $x \in \operatorname{supp}\nu$ and $0 < r < \operatorname{diam}(\operatorname{supp}\nu)$, there is (a rotation and translation of) an n-dimensional Lipschitz graph Γ with constant less than C_1 such that $\nu(\Gamma \cap B(x,r)) \geq \theta r^n$. Thus, if ν has *BPLG*, the assumption (a) of Theorem 4.1 is satisfied for ν by taking $F = \Gamma$, while Theorem 2.5 implies that the assumption (b) holds with a uniform constant. Therefore, from Theorem 4.1 we deduce that, if ν has *BPLG* and $\rho > 2$, then $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\nu)$.

Similarly, a measure ν has $(BP)^2LG$ (*big pieces of big pieces of Lipschitz graphs*) if there exist constants C_g, θ, and $0 < \alpha \leq 1$ such that, if B is any ball centered in supp ν, then there is an n-dimensional AD regular set $F \subset \mathbb{R}^d$ (with constant bounded by C_g) such that $\nu(F \cap B) \geq \alpha\nu(B)$ and $\mathcal{H}^n_F$ has *BPLG* with uniform constants. So $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is a bounded operator from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mathcal{H}^n_F)$, by the comments above. Hence, we can apply once again Theorem 4.1 to ν (now (b) is satisfied for the big pieces F of ν), and we deduce that, for any measure ν which has $(BP)^2LG$, $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\nu)$. Similar arguments show that $\mathcal{V}_\rho \circ \mathcal{T}^\nu_\varphi$ is a bounded operator in $L^p(\nu)$ for all $1 < p < \infty$.

Finally, from [DS2, p. 22] and the remark in [DS2, p. 16], we know that if μ is n-dimensional AD regular, then being uniformly n-rectifiable is equivalent to having $(BP)^2LG$. Therefore, the corollary is proved by applying the comments above to $\nu = \mu$. □

Since the arguments for proving Theorem 4.1 are more or less standard in Calderón–Zygmund theory, for brevity we will only sketch its proof (see [To5, Chapter 2] or [DS2, Proposition 1.28 of Part I] for a similar argument).

Sketch of proof of Theorem 4.1. The proof follows by the so-called *good* λ *inequality* method. Fix $\rho > 2$ and let M^μ denote the Hardy–Littlewood maximal operator

$$M^\mu\nu(x) := \sup_{r>0} \frac{|\nu|(B(x,r))}{\mu(B(x,r))} \qquad \text{for } \nu \in M(\mathbb{R}^d) \text{ and } x \in \operatorname{supp}\mu.$$

The good λ *inequality*: there exists some absolute constant $\eta > 0$ such that for all $\epsilon > 0$ there exists $\delta := \delta(\epsilon) > 0$ such that

$$\begin{aligned}\mu(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) > (1+\epsilon)\lambda,\ M^\mu\nu(x) \le \delta\lambda\})\\ \le (1-\eta)\mu(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) > \lambda\})\end{aligned} \tag{29}$$

for all $\lambda > 0$ and $\nu \in M(\mathbb{R}^d)$. It is easy to check that this implies that $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$, and that $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi$ is bounded in $L^p(\mu)$ for all $1 < p < \infty$, by standard arguments (recall that M^μ is bounded in these spaces).

The proof of (29) is quite standard. The interested reader may look at [M, Theorem 5.2.1] for the detailed proof, or at [To5, Chapter 2] for similar arguments. The only point we should mention is that, in order to pursue the good λ inequality method, one needs the following estimate: Let $\nu \in M(\mathbb{R}^d)$, consider a ball $B \subset \mathbb{R}^d$ and take $x, z \in B$. Then

$$|(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d\setminus 2B}\nu)(x) - (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d\setminus 2B}\nu)(z)| \lesssim M^\mu\nu(x). \tag{30}$$

We finish the sketch of proof of Theorem 4.1 by showing (30). Since $x, z \in B$ and $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is sublinear and positive, by the mean value theorem we have

$$\begin{aligned}&|(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d\setminus 2B}\nu)(x) - (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d\setminus 2B}\nu)(z)|\\ &\quad\le \sup_{\epsilon_m}\Big(\sum_{m\in\mathbb{Z}} |(K\varphi^{\epsilon_m}_{\epsilon_{m+1}} * (\chi_{\mathbb{R}^d\setminus 2B}\nu))(x) - (K\varphi^{\epsilon_m}_{\epsilon_{m+1}} * (\chi_{\mathbb{R}^d\setminus 2B}\nu))(z)|^\rho\Big)^{1/\rho}\\ &\quad\le \sup_{\epsilon_m}\bigg(\sum_{m\in\mathbb{Z}}\bigg(\int_{B_m(x,z)} |\nabla(\varphi^{\epsilon_m}_{\epsilon_{m+1}}K)(u_{x,z}(y)-y)|\,|x-z|\,d|\nu|(y)\bigg)^\rho\bigg)^{1/\rho},\end{aligned} \tag{31}$$

where $B_m(x,z) := (\mathbb{R}^d \setminus 2B) \cap (\operatorname{supp} \varphi^{\epsilon_m}_{\epsilon_{m+1}}(x-\cdot) \cup \operatorname{supp} \varphi^{\epsilon_m}_{\epsilon_{m+1}}(z-\cdot))$ and $u_{x,z}(y)$ is some point on the segment joining x and z. For each x and z, let $\epsilon_m \equiv \epsilon_m(x,z)$ be a sequence that realizes the supremum on the right hand side of (31). Given $\epsilon_m > 0$, let $j(\epsilon_m)$ denote the integer such that $\epsilon_m \in [2^{-j(\epsilon_m)-1}, 2^{-j(\epsilon_m)})$. For $j \in \mathbb{Z}$ set $I_j := [2^{-j-1}, 2^{-j})$. As usual, we decompose $\mathbb{Z} = \mathcal{S} \cup \mathcal{L}$, where

$$\mathcal{S} := \bigcup_{j \in \mathbb{Z}} \mathcal{S}_j, \qquad \mathcal{S}_j := \{m \in \mathbb{Z} : \epsilon_m, \epsilon_{m+1} \in I_j\},$$
$$\mathcal{L} := \{m \in \mathbb{Z} : \epsilon_m \in I_i,\ \epsilon_{m+1} \in I_j \text{ for some } i < j\}.$$

Notice that if $2^{-j+2} < r(B)$, where $r(B)$ denotes the radius of B, then $B_m(x,z) = \emptyset$ for all $m \in \mathcal{S}_j$. Therefore, we can assume that $j \leq \log_2(4/r(B))$. If $m \in \mathcal{S}_j$, then $B_m(x,z) \subset B(x, 2^{-j+3})$, and for $t \in \operatorname{supp}(\varphi^{\epsilon_m}_{\epsilon_{m+1}} K)$ we have $|\nabla(\varphi^{\epsilon_m}_{\epsilon_{m+1}} K)(t)| \lesssim 2^{j(n+2)}|\epsilon_m - \epsilon_{m+1}|$ (see (26) and (27)). If $m \in \mathcal{L}$, we easily see that $|\nabla(\varphi^{\epsilon_m}_{\epsilon_{m+1}} K)(t)| \lesssim |t|^{-n-1}$. Therefore, using (31) and the facts that $\rho > 2$, the sets $B_m(x,z)$ have bounded overlap for $m \in \mathcal{L}$, and $|x - z| \lesssim r(B)$, we get

$$\begin{aligned}
&|(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d \setminus 2B}\nu)(x) - (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d \setminus 2B}\nu)(z)| \\
&\quad \lesssim \sum_{j \leq \log_2(4/r(B))} \sum_{m \in \mathcal{S}_j} |x - z| 2^{j(n+2)} |\epsilon_m - \epsilon_{m+1}| \int_{B(x,2^{-j+3})} d|\nu|(y) \\
&\qquad + |x - z| \sum_{m \in \mathcal{L}} \int_{B_m(x,z)} |x - y|^{-n-1}\, d|\nu|(y) \\
&\quad \lesssim \sum_{j \leq \log_2(4/r(B))} r(B) 2^{j(n+1)} \int_{B(x,2^{-j+3})} d|\nu|(y) + r(B) \int_{\mathbb{R}^d \setminus 2B} \frac{d|\nu|(y)}{|x - y|^{n+1}} \\
&\quad \lesssim \sum_{j \leq \log_2(4/r(B))} \frac{r(B) 2^j}{\mu(B(x, 2^{-j+3}))} \int_{B(x,2^{-j+3})} d|\nu|(y) \\
&\qquad + r(B) \sum_{k \geq 1} \int_{2^{k+2} r(B) \geq |x-y| \geq 2^{k-1} r(B)} \frac{d|\nu|(y)}{|x - y|^{n+1}} \\
&\quad \lesssim M^\mu \nu(x) + \sum_{k \geq 1} \frac{2^{-k}}{\mu(B(x, 2^{k+2} r(B_i)))} \int_{B(x, 2^{k+2} r(B))} d|\nu|(y) \lesssim M^\mu \nu(x). \qquad \square
\end{aligned}$$

Remark 4.3. Notice that, to prove (30), it is a key fact that we are considering smooth truncations (given by $\varphi_{\mathbb{R}}$) in the definition of $\mathcal{T}_\varphi$. These computations are no longer valid if one replaces $\mathcal{T}_\varphi$ by $\mathcal{T}$.

5. If μ is a uniformly n-rectifiable measure, then $\mathcal{V}_\rho \circ \mathcal{T}^\mu : L^2(\mu) \to L^2(\mu)$ is a bounded operator

This section is devoted to the proof of the following result.

Theorem 5.1. *Let $\rho > 2$ and let μ be an n-dimensional AD regular Radon measure on $\mathbb{R}^d$. If μ is uniformly n-rectifiable, then $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ is a bounded operator in $L^2(\mu)$.*

5.1. *Short and long variation*

Given $j \in \mathbb{Z}$, set $I_j := [2^{-j-1}, 2^{-j})$. Then, using the triangle inequality, we can split the variation operator into the so-called *short variation* and *long variation operators*, $(\mathcal{V}_\rho \circ \mathcal{T}^\mu)f(x) \le (\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu)f(x) + (\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu)f(x)$, where

$$\begin{aligned}(\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu)f(x) &:= \sup_{\{\epsilon_m\}}\Big(\sum_{j\in\mathbb{Z}}\sum_{\epsilon_m,\epsilon_{m+1}\in I_j}|(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^\rho\Big)^{1/\rho},\\ (\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu)f(x) &:= \sup_{\{\epsilon_m\}}\Big(\sum_{\substack{m\in\mathbb{Z}:\,\epsilon_m\in I_j,\,\epsilon_{m+1}\in I_k\\ \text{for some } j<k}}|(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^\rho\Big)^{1/\rho}, \qquad (32)\end{aligned}$$

and, in both cases, the pointwise supremum is taken over all sequences $\{\epsilon_m\}_{m\in\mathbb{Z}}$ of positive numbers decreasing to zero. To prove Theorem 5.1 we will show that both the short and long variation operators are bounded in $L^2(\mu)$.

5.2. $L^2(\mu)$ *boundedness of* $\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu$

The $L^2(\mu)$-norm of the long variation operator $\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu$ can be handled by comparing it with its smoothed version $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$, using Corollary 4.2, and estimating the error terms by the short variation operator.

Lemma 5.2. *We have* $\|(\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu)f\|_{L^2(\mu)} \lesssim \|(\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu)f\|_{L^2(\mu)} + \|f\|_{L^2(\mu)}$.

Proof. We decompose

$$\begin{aligned}((\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu)f(x))^\rho &= \sup_{\{\epsilon_m\}}\sum_{\substack{m\in\mathbb{Z}:\,\epsilon_m\in I_j,\,\epsilon_{m+1}\in I_k\\ \text{for some } j<k}}|(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^\rho\\ &\lesssim \sup_{\{\epsilon_m\}}\sum_{\substack{m\in\mathbb{Z}:\\ \epsilon_m\in I_j,\,\epsilon_{m+1}\in I_k\\ \text{for some } j<k}}\big(|(K(\chi_{\epsilon_{m+1}}^{\epsilon_m} - \varphi_{\epsilon_{m+1}}^{\epsilon_m}) * (f\mu))(x)|^\rho + |(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^\rho\big)\\ &\lesssim \sup_{\{\epsilon_m\}}\sum_{\substack{m\in\mathbb{Z}:\,\epsilon_m\in I_j,\,\epsilon_{m+1}\in I_k\\ \text{for some } j<k}}|(K(\chi_{\epsilon_{m+1}}^{\epsilon_m} - \varphi_{\epsilon_{m+1}}^{\epsilon_m}) * (f\mu))(x)|^\rho + ((\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu)f(x))^\rho. \qquad (33)\end{aligned}$$

For simplicity, we denote by $((\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}_{\chi-\varphi}^\mu)f(x))^\rho$ the first term on the right hand side of (33). Notice that, given $\epsilon, \delta > 0$, we have $\chi_\epsilon^\delta - \varphi_\epsilon^\delta = (\chi_\epsilon - \varphi_\epsilon) - (\chi_\delta - \varphi_\delta)$. Recall that, in the definition of $\varphi_{\mathbb{R}}$ in Definition 2.1, we have taken $\chi_{[4,\infty)} \le \varphi_{\mathbb{R}} \le \chi_{[1/4,\infty)}$. Hence, given $t \ge 0$,

$$\begin{aligned}\chi_{\mathbb{R}}(t) - \varphi_{\mathbb{R}}(t) &= \chi_{[1,\infty)}(t) - \int_{1/4}^4 \varphi'_{\mathbb{R}}(s)\chi_{[s,\infty)}(t)\,ds\\ &= \int_{1/4}^4 \varphi'_{\mathbb{R}}(s)(\chi_{[1,\infty)}(t) - \chi_{[s,\infty)}(t))\,ds\end{aligned}$$

(that is, $\chi_{\mathbb{R}} - \varphi_{\mathbb{R}}$ is a convex combination of $\chi_{[1,\infty)} - \chi_{[s,\infty)}$ for $1/4 \le s \le 4$), and thus, by Fubini's theorem,

$$
\begin{aligned}
&(K(\chi_\epsilon - \varphi_\epsilon) * (f\mu))(x) \\
&\quad = \int \big(\chi_{\mathbb{R}}(|x-y|^2/\epsilon^2) - \varphi_{\mathbb{R}}(|x-y|^2/\epsilon^2)\big) K(x-y) f(y)\, d\mu(y) \\
&\quad = \int_{1/4}^{4} \varphi'_{\mathbb{R}}(s) \int \big(\chi_{[1,\infty)}(|x-y|^2/\epsilon^2) - \chi_{[s,\infty)}(|x-y|^2/\epsilon^2)\big) K(x-y) f(y)\, d\mu(y)\, ds \\
&\quad = \int_{1/4}^{4} \varphi'_{\mathbb{R}}(s) \int \chi_\epsilon^{\epsilon\sqrt{s}}(x-y) K(x-y) f(y)\, d\mu(y)\, ds \\
&\quad = \int_{1/4}^{4} \varphi'_{\mathbb{R}}(s) ((K\chi_\epsilon^{\epsilon\sqrt{s}} * (f\mu))(x))\, ds.
\end{aligned}
$$

Therefore, by the triangle inequality and Minkowski's integral inequality, we get

$$
\begin{aligned}
\|(\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu_{\chi-\varphi}) f\|_{L^2(\mu)} &\le 2 \Big\| \sup_{\{\epsilon_m \in I_m : m\in\mathbb{Z}\}} \Big(\sum_{m\in\mathbb{Z}} |(K(\chi_{\epsilon_m} - \varphi_{\epsilon_m}) * (f\mu))(x)|^\rho \Big)^{1/\rho} \Big\|_{L^2(\mu)} \\
&\le 2 \int_{1/4}^{4} \varphi'_{\mathbb{R}}(s) \Big\| \sup_{\{\epsilon_m \in I_m : m\in\mathbb{Z}\}} \Big(\sum_{m\in\mathbb{Z}} |(K\chi_{\epsilon_m}^{\epsilon_m\sqrt{s}} * (f\mu))(x)|^\rho \Big)^{1/\rho} \Big\|_{L^2(\mu)} ds.
\end{aligned}
$$

One can easily verify that $\sup_{\{\epsilon_m\in I_m : m\in\mathbb{Z}\}} (\sum_{m\in\mathbb{Z}} |(K\chi_{\epsilon_m}^{\epsilon_m\sqrt{s}} * (f\mu))(x)|^\rho)^{1/\rho} \lesssim (\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu) f(x)$ for all $s \in [1/4, 4]$ with uniform bounds. Hence

$$
\begin{aligned}
\|(\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu_{\chi-\varphi}) f\|_{L^2(\mu)} &\lesssim \int_{1/4}^{4} \varphi'_{\mathbb{R}}(s) \|(\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu) f\|_{L^2(\mu)}\, ds \\
&\lesssim \|(\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu) f\|_{L^2(\mu)}. \qquad (34)
\end{aligned}
$$

Finally, using (33), (34), and Corollary 4.2, we obtain

$$
\begin{aligned}
\|(\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu) f\|_{L^2(\mu)} &\lesssim \|(\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu_{\chi-\varphi}) f\|_{L^2(\mu)} + \|(\mathcal{V}_\rho \circ \mathcal{T}^\mu_{\varphi}) f\|_{L^2(\mu)} \\
&\lesssim \|(\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu) f\|_{L^2(\mu)} + \|f\|_{L^2(\mu)}. \qquad \square
\end{aligned}
$$

Thus, to prove Theorem 5.1, it only remains to show the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu$.

5.3. *$L^2(\mu)$ boundedness of $\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu$*

We will see that $\mathcal{V}_2^{\mathcal{S}} \circ \mathcal{T}^\mu$ is bounded in $L^2(\mu)$, basically due to the big amount of cancellation given by the kernel defining $\mathcal{T}^\mu$ and the good geometric properties of μ. Since $\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu \le \mathcal{V}_2^{\mathcal{S}} \circ \mathcal{T}^\mu$ for $\rho \ge 2$, we will be done. One could try the same technique for $\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu$; however, $\mathcal{V}_2^{\mathcal{L}} \circ \mathcal{T}^\mu$ is not bounded in $L^2(\mu)$ in general, even for the case of the Hilbert transform or in the setting of martingales (see [JKRW] for a precise example), and this is why we should mantain $\rho > 2$ when we deal with $\mathcal{V}_\rho \circ \mathcal{T}^\mu$. Let us mention that to pass from $\rho > 2$ to $\rho = 2$ in the study of the short variation operator is a rather standard argument (see [CJRW1] for example).

Given $f \in L^2(\mu)$ and $x \in \operatorname{supp}\mu$, let $\{\epsilon_m\}_{m\in\mathbb{Z}}$ be a decreasing sequence of positive numbers (depending on x) such that

$$((\mathcal{V}_2^{\mathcal{S}} \circ \mathcal{T}^\mu)f(x))^2 \leq 2\sum_{j\in\mathbb{Z}} \sum_{\epsilon_m,\epsilon_{m+1}\in I_j} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^2.$$

Given $D \in \mathcal{D}_j$ (see Section 2.3 for the definition of $\mathcal{D}_j$) and $x \in D$, we set $\mathcal{S}_D(x) := \{m \in \mathbb{Z} : \epsilon_m, \epsilon_{m+1} \in I_j\}$. Since $\rho \geq 2$, we have

$$\begin{aligned}\|(\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu)f\|_{L^2(\mu)}^2 &\leq \|(\mathcal{V}_2^{\mathcal{S}} \circ \mathcal{T}^\mu)f\|_{L^2(\mu)}^2\\ &\lesssim \int \sum_{j\in\mathbb{Z}} \sum_{\epsilon_m,\epsilon_{m+1}\in I_j} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^2\,d\mu(x)\\ &= \sum_{D\in\mathcal{D}} \int_D \sum_{m\in\mathcal{S}_D(x)} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^2\,d\mu(x).\end{aligned}$$

Let η and θ be two positive numbers that will be fixed below (see the proofs of Claims 5.5 and 5.6). Consider a corona decomposition of μ with parameters η and θ as in Subsection 2.4. Then we can decompose $\mathcal{D} = \mathcal{B} \cup \bigcup_{S\in\mathrm{Trs}} S$, so that

$$\begin{aligned}\|(\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu)f\|_{L^2(\mu)}^2 \lesssim &\sum_{D\in\mathcal{B}} \int_D \sum_{m\in\mathcal{S}_D(x)} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^2\,d\mu(x)\\ &+ \sum_{S\in\mathrm{Trs}} \sum_{D\in S} \int_D \sum_{m\in\mathcal{S}_D(x)} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^2\,d\mu(x). \qquad (35)\end{aligned}$$

Since the μ-cubes in $\mathcal{B}$ satisfy the Carleson packing condition, we can use Carleson's embedding theorem to estimate the sum on the right hand side of (35) over the μ-cubes in $\mathcal{B}$. Carleson's embedding theorem is a well known result in the area of harmonic analysis (see [To5, Chapter 5] for example), but the most usual "continuous" version of this result can be found in [Du, Theorem 9.5] for example. Thus, if we set $m_D^\mu f := \mu(D)^{-1}\int_D f\,d\mu$ for $D \in \mathcal{D}$, we have

$$\begin{aligned}&\sum_{D\in\mathcal{B}} \int_D \sum_{m\in\mathcal{S}_D(x)} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^2\,d\mu(x)\\ &\quad\leq \sum_{D\in\mathcal{B}} \int_D \sum_{m\in\mathcal{S}_D(x)} \left(\int_{\epsilon_{m+1}\leq|x-y|\leq\epsilon_m} |K(x-y)|\,|f(y)|\,d\mu(y)\right)^2 d\mu(x)\\ &\quad\lesssim \sum_{D\in\mathcal{B}} \int_D \left(\frac{1}{\ell(D)^n}\int_{5D}|f|\,d\mu\right)^2 d\mu \approx \sum_{D\in\mathcal{B}} (m_{5D}^\mu|f|)^2\mu(D) \lesssim \|f\|_{L^2(\mu)}^2. \qquad (36)\end{aligned}$$

Now we are going to estimate the second term on the right hand side of (35), that is, the sum over the μ-cubes in S, for all $S \in \mathrm{Trs}$. To this end, we need to introduce some notation.

Definition 5.3. Given $R \in \mathcal{D}_j$ for some $j \in \mathbb{Z}$, let $P(R)$ denote the μ-cube in $\mathcal{D}_{j-1}$ which contains R (the *parent* of R), and set

$$\mathrm{Ch}(R) := \{Q \in \mathcal{D}_{j+1} : Q \subset R\},$$
$$V(R) := \{Q \in \mathcal{D}_j : Q \cap B(y, \ell(R)) \neq \emptyset \text{ for some } y \in R\}$$

($\mathrm{Ch}(R)$ are the *children* of R, and $V(R)$ stands for the *vicinity* of R). If $R \in S$ for some $S \in \mathrm{Trs}$, we denote by $\mathrm{Tr}(R)$ the set of μ-cubes $Q \in S$ such that $Q \subset R$ (the *tree* of R). Otherwise, i.e., if $R \in \mathcal{B}$, we set $\mathrm{Tr}(R) := \emptyset$. Finally, if $\mathrm{Tr}(R) \neq \emptyset$, let $\mathrm{Stp}(R)$ denote the set of μ-cubes $Q \in \mathcal{B} \cup (\mathcal{G} \setminus \mathrm{Tr}(R))$ such that $Q \subset R$ and $P(Q) \in \mathrm{Tr}(R)$ (the *stopping* μ-cubes relative to R), so actually $Q \subsetneq R$. On the other hand, if $R \in \mathcal{B}$, we set $\mathrm{Stp}(R) := \{R\}$.

Notice that $P(R)$ is a μ-cube but $\mathrm{Ch}(R)$ and $V(R)$ are collections of μ-cubes. It is not hard to show that the number of μ-cubes in $\mathrm{Ch}(R)$ and $V(R)$ is bounded by some constant depending only on n and the AD regularity constant of μ.

Fix $S \in \mathrm{Trs}$, $D \in S$, and $x \in D$. To deal with the second term on the right hand side of (35), we have to estimate the sum $\sum_{m\in\mathcal{S}_D(x)} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^2$. By the definition of $\mathcal{S}_D(x)$, we have

$$\sum_{m\in\mathcal{S}_D(x)} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^2 = \sum_{m\in\mathcal{S}_D(x)} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (\chi_{\widetilde{D}} f\mu))(x)|^2, \tag{37}$$

where $\widetilde{D} := \bigcup_{R\in V(D)} R$. Since this union of μ-cubes is disjoint, we can decompose the function $\chi_{\widetilde{D}} f$ using a Haar basis adapted to $\mathcal{D}$ in the following manner:

$$\chi_{\widetilde{D}} f = \sum_{R\in V(D)} \Big((m_R^\mu f)\chi_R + \sum_{Q\in\mathrm{Tr}(R)} \Delta_Q f + \sum_{Q\in\mathrm{Stp}(R)} \widetilde{\Delta}_Q f \Big), \tag{38}$$

where we have set

$$\Delta_Q f := \sum_{U\in\mathrm{Ch}(Q)} \chi_U (m_U^\mu f - m_Q^\mu f),$$
$$\widetilde{\Delta}_Q f := \sum_{U\in\mathrm{Ch}(Q)} \chi_U (f - m_Q^\mu f) = \chi_Q (f - m_Q^\mu f).$$

Using (38), we split the left hand side of (37) as follows:

$$\begin{aligned}\sum_{m\in\mathcal{S}_D(x)} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^2 &\lesssim \sum_{m\in\mathcal{S}_D(x)} \Big| \sum_{R\in V(D)} (K\chi_{\epsilon_{m+1}}^{\epsilon_m} * ((m_R^\mu f)\chi_R\mu))(x)\Big|^2 \\ &+ \sum_{m\in\mathcal{S}_D(x)} \Big| \sum_{R\in V(D)} \sum_{Q\in\mathrm{Tr}(R)} (K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (\Delta_Q f\mu))(x)\Big|^2 \\ &+ \sum_{m\in\mathcal{S}_D(x)} \Big| \sum_{R\in V(D)} \sum_{Q\in\mathrm{Stp}(R)} (K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (\widetilde{\Delta}_Q f\mu))(x)\Big|^2. \end{aligned} \tag{39}$$

In the following subsections, we will estimate each part separately. We could think that the leading term on the right hand side of (39) is the second one, which corresponds to the μ-cubes $Q \in \mathrm{Tr}(R)$ with $R \in V(D)$. To control it, we will use the fact that in these μ-cubes the measure μ is very close to a sufficiently flat Lipschitz graph, so good estimates can be achieved using approximation arguments. To control the third term on the right hand side of (39), we will basically use the fact that the number of cubes Q_S

with $S \in \mathrm{Trs}$ or which belong to $\mathcal{B}$ is not too large (see the packing conditions (b) and (e) in Subsection 2.4), so we will be able to apply Carleson's embedding theorem. The first term in (39) requires a much more detailed study, and we will need to use intensively the multiscale analysis given by the α_μ coefficients apart from Carleson's embedding theorem and the above-mentioned ideas.

5.3.1. Estimate of $\sum_{m\in\mathcal{S}_D(x)}|\sum_{R\in V(D)}\sum_{Q\in\mathrm{Tr}(R)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*(\Delta_Q f\mu))(x)|^2$ *from (39)*

Lemma 5.4. *Under the notation above, we have*

$$\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D\sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}\sum_{Q\in\mathrm{Tr}(R)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*(\Delta_Q f\mu))(x)\Big|^2 d\mu(x)\lesssim\|f\|_{L^2(\mu)}^2.$$

Proof. Let $C_0>0$ be a small constant to be fixed below. Given $m\in\mathcal{S}_D(x)$ let $A_m(x):=A(x,\epsilon_{m+1},\epsilon_m)=\{y\in\mathbb{R}^d:\epsilon_{m+1}\le|y-x|\le\epsilon_m\}$, and given $R\in V(D)$ let

$$J_m^{1,R}:=\{Q\in\mathrm{Tr}(R):Q\cap A_m(x)\neq\emptyset,\ \ell(Q)>C_0(\epsilon_m-\epsilon_{m+1})\},$$
$$J_m^{2,R}:=\{Q\in\mathrm{Tr}(R):Q\cap A_m(x)\neq\emptyset,\ \ell(Q)\le C_0(\epsilon_m-\epsilon_{m+1})\}.$$

Roughly speaking, $J_m^{1,R}$ contains the μ-cubes which are big with respect to the thickness of $A_m(x)$, and $J_m^{2,R}$ contains the small ones. For the study of $J_m^{1,R}$, we will basically use the fact that it does not contain too many μ-cubes. For $J_m^{2,R}$, using $\int\Delta_Q f\,d\mu=0$, we will be reduced to those μ-cubes that "intersect" the boundary of $A_m(x)$, which are not too many once again.

For $Q\in J_m^{1,R}$, we write $|(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*(\Delta_Q f\mu))(x)|\lesssim\ell(D)^{-n}\|\chi_{A_m(x)}\Delta_Q f\|_{L^1(\mu)}$. The following claim will be proved in Subsection 5.3.2 below.

Claim 5.5. *We have* $\sum_{Q\in J_m^{1,R}}\ell(Q)^{n-1/2}\lesssim\ell(D)^{n-1/2}$.

Using the fact that $V(D)$ has finitely many elements (depending only on n and the AD regularity constant of μ), the Cauchy–Schwarz inequality, Claim 5.5, and the previous estimate, we obtain

$$\begin{aligned}
&\sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}\sum_{Q\in J_m^{1,R}}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*(\Delta_Q f\mu))(x)\Big|^2\\
&\quad\lesssim\sum_{R\in V(D)}\sum_{m\in\mathcal{S}_D(x)}\Big(\sum_{Q\in J_m^{1,R}}\ell(D)^{-n}\|\chi_{A_m(x)}\Delta_Q f\|_{L^1(\mu)}\Big)^2\\
&\quad\lesssim\sum_{R\in V(D)}\sum_{m\in\mathcal{S}_D(x)}\Big(\sum_{Q\in J_m^{1,R}}\ell(Q)^{n-1/2}\Big)\Big(\sum_{Q\in J_m^{1,R}}\frac{\|\chi_{A_m(x)}\Delta_Q f\|_{L^1(\mu)}^2}{\ell(D)^{2n}\ell(Q)^{n-1/2}}\Big)\\
&\quad\lesssim\sum_{R\in V(D)}\sum_{m\in\mathcal{S}_D(x)}\sum_{Q\in\mathrm{Tr}(R)}\frac{\|\chi_{A_m(x)}\Delta_Q f\|_{L^1(\mu)}^2}{\ell(D)^{n+1/2}\ell(Q)^{n-1/2}}\\
&\quad\lesssim\sum_{R\in V(D)}\sum_{Q\in\mathrm{Tr}(R)}\Big(\frac{\ell(Q)}{\ell(D)}\Big)^{1/2}\frac{\|\Delta_Q f\|_{L^1(\mu)}^2}{\ell(D)^n\ell(Q)^n}.
\end{aligned}\tag{40}$$

We deal now with the μ-cubes $Q \in J_m^{2,R}$. Let z_Q denote the center of Q. Since $\int \Delta_Q f \, d\mu = 0$, we can decompose

$$\begin{aligned}
&(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (\Delta_Q f\mu))(x) \\
&\quad = \int \big(\chi_{A_m(x)}(y)K(x-y) - \chi_{A_m(x)}(z_Q)K(x-z_Q)\big)\Delta_Q f(y)\, d\mu(y) \\
&\quad = \int \chi_{A_m(x)}(y)(K(x-y) - K(x-z_Q))\Delta_Q f(y)\, d\mu(y) \\
&\qquad + \int (\chi_{A_m(x)}(y) - \chi_{A_m(x)}(z_Q))K(x-z_Q)\Delta_Q f(y)\, d\mu(y) \\
&\quad =: T_m^{1,\mu}(\Delta_Q f)(x) + T_m^{2,\mu}(\Delta_Q f)(x).
\end{aligned} \tag{41}$$

For the first term on the right hand side of the last equality, we have the standard estimate (by assuming C_0 small enough, so any $Q \in J_m^{2,R}$ is far from x)

$$|T_m^{1,\mu}(\Delta_Q f)(x)| \lesssim \int_{A_m(x)} \frac{|y-z_Q|}{|x-y|^{n+1}} |\Delta_Q f(y)|\, d\mu(y) \lesssim \frac{\ell(Q)}{\ell(D)^{n+1}} \|\chi_{A_m(x)}\Delta_Q f\|_{L^1(\mu)}.$$

From this estimate and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\sum_{m\in\mathcal{S}_D(x)} \Big| \sum_{R\in V(D)} \sum_{Q\in J_m^{2,R}} T_m^{1,\mu}(\Delta_Q f)(x)\Big|^2 \\
&\qquad \lesssim \sum_{R\in V(D)} \sum_{m\in\mathcal{S}_D(x)} \bigg(\sum_{Q\in J_m^{2,R}} \frac{\ell(Q)}{\ell(D)^{n+1}} \|\chi_{A_m(x)}\Delta_Q f\|_{L^1(\mu)}\bigg)^2 \\
&\qquad \lesssim \sum_{R\in V(D)} \bigg(\sum_{Q\in \mathrm{Tr}(R)} \frac{\ell(Q)}{\ell(D)^{n+1}} \sum_{m\in\mathcal{S}_D(x)} \|\chi_{A_m(x)}\Delta_Q f\|_{L^1(\mu)}\bigg)^2 \\
&\qquad \lesssim \sum_{R\in V(D)} \bigg(\sum_{Q\in \mathrm{Tr}(R)} \frac{\ell(Q)^{n+1}}{\ell(D)^{n+1}}\bigg)\bigg(\sum_{Q\in \mathrm{Tr}(R)} \frac{\|\Delta_Q f\|^2_{L^1(\mu)}}{\ell(Q)^{n-1}\ell(D)^{n+1}}\bigg).
\end{aligned}$$

Since $\ell(R) = \ell(D)$ for all $R \in V(D)$, we have $\sum_{Q\in\mathrm{Tr}(R)} (\ell(Q)/\ell(D))^{n+1} \leq \sum_{Q\in\mathcal{D}:\, Q\subset R} (\ell(Q)/\ell(R))^{n+1} \lesssim 1$. Thus, using the fact that $t \lesssim \sqrt{t}$ for all $t \lesssim 1$, we conclude

$$\begin{aligned}
&\sum_{m\in\mathcal{S}_D(x)} \Big| \sum_{R\in V(D)} \sum_{Q\in J_m^{2,R}} T_m^{1,\mu}(\Delta_Q f)(x)\Big|^2 \\
&\qquad \lesssim \sum_{R\in V(D)} \sum_{Q\in \mathrm{Tr}(R)} \bigg(\frac{\ell(Q)}{\ell(D)}\bigg)^{1/2} \frac{\|\Delta_Q f\|^2_{L^1(\mu)}}{\ell(Q)^n \ell(D)^n}.
\end{aligned} \tag{42}$$

We deal now with the second term on the right hand side of (41). Given $Q \in J_m^{2,R}$, since $\mathrm{supp}(\Delta_Q f) \subset Q$, if $Q \subset A_m(x)$ or $Q \subset (A_m(x))^c$ then we obviously have $\chi_{A_m(x)}(y) - \chi_{A_m(x)}(z_Q) = 0$ for all $y \in \mathrm{supp}(\Delta_Q f)$. Therefore, to estimate the sum of

$T_m^{2,\mu}(\Delta_Q f)(x)$ over all $Q \in J_m^{2,R}$, we can replace $J_m^{2,R}$ by

$$J_m^{3,R} := \{Q \in \text{Tr}(R) : Q \cap A_m(x) \neq \emptyset,\ Q \cap (A_m(x))^c \neq \emptyset,\ \ell(Q) \leq C_0(\epsilon_m - \epsilon_{m+1})\}.$$

For $m \in \mathcal{S}_D(x)$ and $Q \in J_m^{3,R}$, we will use the estimate $|T_m^{2,\mu}(\Delta_Q f)(x)| \lesssim \ell(D)^{-n}\|\Delta_Q f\|_{L^1(\mu)}$.

Claim 5.6. *We have* $\sum_{Q\in J_m^{3,R}} \ell(Q)^{n-1/2} \lesssim \ell(D)^{n-1}(\epsilon_m - \epsilon_{m+1})^{1/2}$.

Hence, using the fact that $V(D)$ has finitely many terms, the Cauchy–Schwarz inequality, assuming Claim 5.6 (see Subsection 5.3.2), and by the previous estimate, we deduce

$$\begin{aligned}
\sum_{m\in\mathcal{S}_D(x)} \Big|\sum_{R\in V(D)} \sum_{Q\in J_m^{2,R}} T_m^{2,\mu}(\Delta_Q f)(x)\Big|^2 &\lesssim \sum_{R\in V(D)} \sum_{m\in\mathcal{S}_D(x)} \Big(\sum_{Q\in J_m^{3,R}} \frac{\|\Delta_Q f\|_{L^1(\mu)}}{\ell(D)^n}\Big)^2 \\
&\leq \sum_{R\in V(D)} \sum_{m\in\mathcal{S}_D(x)} \Big(\sum_{Q\in J_m^{3,R}} \frac{\ell(Q)^{n-1/2}}{\ell(D)^{n-1/2}}\Big)\Big(\sum_{Q\in J_m^{3,R}} \frac{\ell(Q)^{1/2-n}}{\ell(D)^{n+1/2}}\|\Delta_Q f\|_{L^1(\mu)}^2\Big) \\
&\lesssim \sum_{R\in V(D)} \sum_{m\in\mathcal{S}_D(x)} \Big(\frac{\epsilon_m - \epsilon_{m+1}}{\ell(D)}\Big)^{1/2} \sum_{Q\in J_m^{3,R}} \frac{\ell(Q)^{1/2-n}}{\ell(D)^{n+1/2}}\|\Delta_Q f\|_{L^1(\mu)}^2 \\
&\leq \sum_{R\in V(D)} \sum_{Q\in \text{Tr}(R)} \frac{\ell(Q)^{1/2-n}}{\ell(D)^{n+1/2}}\|\Delta_Q f\|_{L^1(\mu)}^2 \sum_{\substack{m\in\mathcal{S}_D(x):\, A_m(x)\cap Q\neq\emptyset,\\ \ell(Q)\leq C_0(\epsilon_m-\epsilon_{m+1})}} \Big(\frac{\epsilon_m - \epsilon_{m+1}}{\ell(D)}\Big)^{1/2}.
\end{aligned}$$

The sum over m on the right hand side can be easily bounded by some constant depending on C_0, so we finally obtain

$$\sum_{m\in\mathcal{S}_D(x)} \Big|\sum_{R\in V(D)} \sum_{Q\in J_m^{2,R}} T_m^{2,\mu}(\Delta_Q f)(x)\Big|^2 \lesssim \sum_{R\in V(D)} \sum_{Q\in \text{Tr}(R)} \Big(\frac{\ell(Q)}{\ell(D)}\Big)^{1/2} \frac{\|\Delta_Q f\|_{L^1(\mu)}^2}{\ell(Q)^n \ell(D)^n}. \tag{43}$$

Finally, combining (40)–(43), we conclude

$$\begin{aligned}
&\sum_{m\in\mathcal{S}_D(x)} \Big|\sum_{R\in V(D)} \sum_{Q\in \text{Tr}(R)} (K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (\Delta_Q f\mu))(x)\Big|^2 \\
&\qquad \lesssim \sum_{R\in V(D)} \sum_{Q\in \text{Tr}(R)} \Big(\frac{\ell(Q)}{\ell(D)}\Big)^{1/2} \frac{\|\Delta_Q f\|_{L^1(\mu)}^2}{\ell(Q)^n \ell(D)^n},
\end{aligned} \tag{44}$$

Since $\|\Delta_Q f\|_{L^1(\mu)} \lesssim \|\Delta_Q f\|_{L^2(\mu)} \ell(Q)^{n/2}$ by Hölder's inequality, since $V(D)$ has finitely many terms, and since $\ell(R) = \ell(D)$ for all $R \in V(D)$, we get

$$\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D\sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}\sum_{Q\in\mathrm{Tr}(R)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*(\Delta_Q f\mu))(x)\Big|^2 d\mu(x)$$
$$\lesssim\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\sum_{R\in V(D)}\sum_{Q\in\mathrm{Tr}(R)}\left(\frac{\ell(Q)}{\ell(D)}\right)^{1/2}\|\Delta_Q f\|^2_{L^2(\mu)}$$
$$\le\sum_{S\in\mathrm{Trs}}\sum_{Q\in S}\sum_{R\in\mathcal{D}:\,R\supset Q}\sum_{D\in V(R)}\left(\frac{\ell(Q)}{\ell(R)}\right)^{1/2}\|\Delta_Q f\|^2_{L^2(\mu)}$$
$$\lesssim\sum_{S\in\mathrm{Trs}}\sum_{Q\in S}\|\Delta_Q f\|^2_{L^2(\mu)}\le\sum_{Q\in\mathcal{D}}\|\Delta_Q f\|^2_{L^2(\mu)}\le\|f\|^2_{L^2(\mu)}.$$

To complete the proof of Lemma 5.4, it only remains to show Claims 5.5 and 5.6. □

5.3.2. Proof of Claims 5.5 and 5.6. First of all, we need an auxiliary result whose easy proof is left to the reader.

Lemma 5.7. *Let $\Gamma := \{x\in\mathbb{R}^d : x=(y,A(y)),\ y\in\mathbb{R}^n\}$ be the graph of a Lipschitz function $A:\mathbb{R}^n\to\mathbb{R}^{d-n}$ such that* $\mathrm{Lip}(A)$ *is small enough. Then $\mathcal{H}^n_\Gamma(A^d(z,a,b))\lesssim(b-a)b^{n-1}$ for all $0<a\le b$ and $z\in\Gamma$.*

Remark 5.8. Actually, to obtain the conclusion of the lemma, one only needs $\mathrm{Lip}(A)<1$ (see [M, Lemma 4.1.9]). Let us mention that this assumption is sharp in the sense that if $\mathrm{Lip}(A)\ge 1$ then the lemma fails. However, we do not need this stronger version for our purposes.

Claims 5.5 and 5.6 follow from the next lemma, which will be proved using Lemma 5.7.

Lemma 5.9. *Let $C_0>0$ be some constant depending only on n, d, and the AD regularity constant of μ, and consider $x\in D\in\mathcal{D}_j$ for some $j\in\mathbb{Z}$. Let $\epsilon\in[2^{-j-1},2^{-j})$. Given $k\ge j$ and $R\in V(D)$, set*

$$\Lambda_k := \{Q\in\mathrm{Tr}(R)\cap\mathcal{D}_k : Q\subset A(x,\epsilon-C_0 2^{-k},\epsilon+C_0 2^{-k})\}.$$

Then $\mu(\bigcup_{Q\in\Lambda_k}Q)\lesssim 2^{-k}\ell(D)^{n-1}\approx 2^{-k-j(n-1)}$.

Proof. First of all, we can assume $k\gg j$ (otherwise, the claim follows easily using the AD regularity of μ), thus we may assume that $\mathrm{dist}(x,Q)\ge 3\epsilon/4$. For simplicity, set $S\equiv\mathrm{Tr}(R)$. By the property (f) of the corona decomposition of μ, there exists (a rotation and translation of) an n-dimensional Lipschitz graph Γ_S with $\mathrm{Lip}(\Gamma_S)\le\eta$ such that $\mathrm{dist}(y,\Gamma_S)\le\theta\,\mathrm{diam}(Q)$ whenever $y\in C_{\mathrm{cor}}Q$ and $Q\in S$, for some given constant $C_{\mathrm{cor}}\ge 2$. Since $x\in D$ and $R\in V(D)$, we have $x\in C_{\mathrm{cor}}Q$ assuming C_{cor} is large enough, and so $\mathrm{dist}(x,\Gamma_S)\le\theta\,\mathrm{diam}(Q)$. Hence, if η and θ are small enough, one can easily modify Γ_S inside $B(x,\epsilon/4)$ to obtain a Lipschitz graph Γ_S^x such that $x\in\Gamma_S^x$, and moreover

$$\mathrm{Lip}(\Gamma_S^x)\le\eta' \text{ for some } \eta' \text{ small enough,}\quad\text{and}\quad \Gamma_S^x\setminus B(x,\epsilon/4)=\Gamma_S\setminus B(x,\epsilon/4).\quad(45)$$

Using the fact that $\mathrm{dist}(x, Q) \geq 3\epsilon/4$ for all $Q \in \Lambda_k$, that $\mathrm{dist}(z_Q, \Gamma_S) \leq \theta\,\mathrm{diam}(Q)$ for the center z_Q of Q, and the last part of (45), we deduce that $\mathrm{dist}(z_Q, \Gamma_S^x) \leq \theta\,\mathrm{diam}(Q)$ for all $Q \in \Lambda_k$. So we have $B(z_Q, \theta\,\mathrm{diam}(Q)) \cap \Gamma_S^x \neq \emptyset$, which in turn yields $\mathcal{H}^n(\Gamma_S^x \cap B(z_Q, 2\theta\,\mathrm{diam}(Q))) \gtrsim (\theta\,\mathrm{diam}(Q))^n$. Therefore, since $\{B(z_Q, 2\theta\,\mathrm{diam}(Q))\}_{Q\in\Lambda_k}$ is a family with finite overlap bounded by some constant depending only on n, θ, and the AD regularity constant of μ, we have

$$\mu\Big(\bigcup_{Q\in\Lambda_k} Q\Big) \approx \sum_{Q\in\Lambda_k} \ell(Q)^n \lesssim \theta^{-n} \sum_{Q\in\Lambda_k} \mathcal{H}^n\big(\Gamma_S^x \cap B(z_Q, 2\theta\,\mathrm{diam}(Q))\big)$$
$$\lesssim \theta^{-n}\mathcal{H}^n_{\Gamma_S^x}\Big(\bigcup_{Q\in\Lambda_k} B(z_Q, 2\theta\,\mathrm{diam}(Q))\Big)$$
$$\lesssim \theta^{-n}\mathcal{H}^n_{\Gamma_S^x}\big(A(x, \epsilon - C_0 2^{-k}, \epsilon + C_0 2^{-k})\big) \lesssim \theta^{-n} 2^{-k-j(n-1)},$$

where we have used Lemma 5.7 and that $\epsilon \approx 2^{-j}$ in the last inequality. □

Proof of Claim 5.5. Recall that $J_m^{1,R} := \{Q \in \mathrm{Tr}(R) : Q \cap A_m(x) \neq \emptyset,\ \ell(Q) \geq C_0(\epsilon_m - \epsilon_{m+1})\}$, where $R \in V(D)$ and $D \in \mathcal{D}_j$. We have to check that $\sum_{Q\in J_m^{1,R}} \ell(Q)^{n-1/2} \lesssim \ell(D)^{n-1/2}$. We will split the sum into different scales and we will apply Lemma 5.9 at each scale.

Given $i \in \mathbb{Z}$ such that $2^{-i} \geq C_0(\epsilon_m - \epsilon_{m+1})$, the number of μ-cubes Q in $\mathcal{D}_i$ such that $Q \subset R$ and $Q \cap A_m(x) \neq \emptyset$ is bounded by $C\ell(R)^{n-1}2^{i(n-1)} \approx 2^{-j(n-1)+i(n-1)}$, since for all those μ-cubes, $Q \subset A(x, \epsilon_{m+1} - C2^{-i}, \epsilon_m + C2^{-i}) \subset A(x, \epsilon_m - C2^{-i+1}, \epsilon_m + C2^{-i+1})$ for some constant $C > 0$ large enough, and then by Lemma 5.9, $\mu(\bigcup_{Q\in J_m^{1,R}\cap\mathcal{D}_i} Q) \lesssim 2^{-i}\ell(D)^{n-1}$. Therefore,

$$\sum_{Q\in J_m^{1,R}} \ell(Q)^{n-1/2} = \sum_{i\in\mathbb{Z}:\, i\geq j} 2^{i/2} \sum_{Q\in J_m^{1,R}\cap\mathcal{D}_i} \ell(Q)^n \lesssim \sum_{i\in\mathbb{Z}:\, i\geq j} 2^{i/2}2^{-i}\ell(D)^{n-1}$$
$$\approx 2^{-j/2}\ell(D)^{n-1} = \ell(D)^{n-1/2}. \qquad \square$$

Proof of Claim 5.6. Recall that $J_m^{3,R} := \{Q \in \mathrm{Tr}(R) : Q \cap A_m(x) \neq \emptyset,\ Q \cap (A_m(x))^c \neq \emptyset,\ \ell(Q) \leq C_0(\epsilon_m - \epsilon_{m+1})\}$, where $R \in V(D)$ and $D \in \mathcal{D}_j$. We have to check that

$$\sum_{Q\in J_m^{3,R}} \ell(Q)^{n-1/2} \lesssim \ell(D)^{n-1}(\epsilon_m - \epsilon_{m+1})^{1/2}.$$

As before, we will split the sum into the different scales and we will apply Lemma 5.9 at each scale. Given $i \in \mathbb{Z}$ such that $2^{-i} \leq C_0(\epsilon_m - \epsilon_{m+1})$, since for any $Q \in J_m^{3,R} \cap \mathcal{D}_i$ we have $Q \subset A(x, \epsilon_{m+1} - C2^{-i}, \epsilon_{m+1} + C2^{-i}) \cup A(x, \epsilon_m - C2^{-i}, \epsilon_m + C2^{-i})$ for some constant $C > 0$ large enough, by Lemma 5.9 applied to both annuli we have $\mu(\bigcup_{Q\in J_m^{3,R}\cap\mathcal{D}_i} Q) \lesssim 2^{-i}\ell(D)^{n-1}$. Therefore,

$$\sum_{Q\in J_m^{3,R}} \ell(Q)^{n-1/2} = \sum_{i\in\mathbb{Z}:\, i\geq -\log_2(C_0(\epsilon_m-\epsilon_{m+1}))} 2^{i/2} \sum_{Q\in J_m^{3,R}\cap\mathcal{D}_i} \ell(Q)^n$$
$$\lesssim \sum_{i\in\mathbb{Z}:\, i\geq -\log_2(C_0(\epsilon_m-\epsilon_{m+1}))} 2^{-i/2}\ell(D)^{n-1} \approx (\epsilon_m - \epsilon_{m+1})^{1/2}\ell(D)^{n-1}. \qquad \square$$

5.3.3. Estimate of $\sum_{m\in\mathcal{S}_D(x)}|\sum_{R\in V(D)}\sum_{Q\in\mathrm{Stp}(R)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*(\widetilde{\Delta}_Q f\mu))(x)|^2$ *from* (39)

Lemma 5.10. *Under the notation above, we have*

$$\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D\sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}\sum_{Q\in\mathrm{Stp}(R)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*(\widetilde{\Delta}_Q f\mu))(x)\Big|^2 d\mu(x)\lesssim\|f\|_{L^2(\mu)}^2.$$

Proof. Recall the definitions of $V(D)$, $\mathrm{Tr}(R)$ and $\mathrm{Stp}(R)$ in Definition 5.3. Given R in $V(D)$, consider a μ-cube $Q\in\mathrm{Stp}(R)$. If $\mathrm{Tr}(R)\neq\emptyset$, then $Q\in\mathcal{B}\cup(\mathcal{G}\setminus\mathrm{Tr}(R))$, $Q\subset R$ and $P(Q)\in\mathrm{Tr}(R)$ (in particular, $Q\subsetneq R$). Take $S\in\mathrm{Trs}$ such that $R\in S$. By property (f) of the corona decomposition (see Subsection 2.4), we have $\mathrm{dist}(y,\Gamma_S)\le\theta\,\mathrm{diam}(P(Q))$ for all $y\in C_{\mathrm{cor}}P(Q)$. Hence, $\mathrm{dist}(y,\Gamma_S)\le C\theta\,\mathrm{diam}(Q)$ for all $y\in C_{\mathrm{cor}}Q$. On the other hand, if $\mathrm{Tr}(R)=\emptyset$ we have set $\mathrm{Stp}(R)=\{R\}$. In this case, we have $R\in\mathcal{B}$. Take S such that $D\in S$. Since $R\in V(D)$, we have $R\subset C_{\mathrm{cor}}D$ if C_{cor} is chosen large enough, and thus $\mathrm{dist}(y,\Gamma_S)\le C\theta\,\mathrm{diam}(R)$ for all $y\in C'R$, where C is as above and C' depends on C_{cor}.

Taking into account the comments above, one can prove the following claims using similar arguments to the ones in the proof of Claims 5.5 and 5.6.

Claim 5.11. *Let* $x\in D\in\mathcal{D}$, $R\in V(D)$, *and* $m\in\mathcal{S}_D(x)$. *Set* $J_m^{1,R}:=\{Q\in\mathrm{Stp}(R):Q\cap A_m(x)\neq\emptyset,\ \ell(Q)\ge C_0(\epsilon_m-\epsilon_{m+1})\}$. *Then* $\sum_{Q\in J_m^{1,R}}\ell(Q)^{n-1/2}\lesssim\ell(D)^{n-1/2}$.

Claim 5.12. *Let* $x\in D\in\mathcal{D}$, $R\in V(D)$, *and* $m\in\mathcal{S}_D(x)$. *Set* $J_m^{3,R}:=\{Q\in\mathrm{Stp}(R):Q\cap A_m(x)\neq\emptyset,\ Q\cap(A_m(x))^c\neq\emptyset,\ \ell(Q)\le C_0(\epsilon_m-\epsilon_{m+1})\}$. *Then* $\sum_{Q\in J_m^{3,R}}\ell(Q)^{n-1/2}\lesssim\ell(D)^{n-1}(\epsilon_m-\epsilon_{m+1})^{1/2}$.

The only properties of $\Delta_Q f$ that we used to obtain (44) were that $\Delta_Q f$ is supported in Q and that $\int\Delta_Q f\,d\mu=0$. The function $\widetilde{\Delta}_Q f$ is also supported in Q and has vanishing integral. Thus, if we replace $\mathrm{Tr}(R)$ by $\mathrm{Stp}(R)$, Claims 5.5 and 5.6 by Claims 5.11 and 5.12, and $\Delta_Q f$ by $\widetilde{\Delta}_Q f$, the same arguments that gave us (44) yield

$$\sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}\sum_{Q\in\mathrm{Stp}(R)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*(\widetilde{\Delta}_Q f\mu))(x)\Big|^2 \lesssim\sum_{R\in V(D)}\sum_{Q\in\mathrm{Stp}(R)}\frac{\ell(Q)^{1/2-n}}{\ell(D)^{1/2+n}}\|\widetilde{\Delta}_Q f\|_{L^1(\mu)}^2. \tag{46}$$

Below we will use the fact that $\|\widetilde{\Delta}_Q f\|_{L^1(\mu)}^2\ell(Q)^{-n}=(\int_Q|f-m_Q^\mu f|\,d\mu)^2\ell(Q)^{-n}\lesssim(m_Q^\mu|f|)^2\mu(Q)$. Notice that, by the definition of $\mathrm{Stp}(R)$ and since the corona decomposition is coherent (property (d)), any $Q\in\mathrm{Stp}(R)$ is actually a maximal μ-cube Q_S of some $S\in\mathrm{Trs}$ or $Q\in\mathcal{B}$ (and in this case $\mathrm{Tr}(R)$ is empty). Hence, if we integrate (46) in D, sum over all $D\in S\in\mathrm{Trs}$, and change the order of summation, we get

$$\begin{aligned}
&\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D\sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}\sum_{Q\in\mathrm{Stp}(R)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*(\widetilde{\Delta}_Q f\mu))(x)\Big|^2 d\mu(x)\\
&\qquad\lesssim\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\sum_{R\in V(D)}\sum_{Q\in\mathrm{Stp}(R)}\left(\frac{\ell(Q)}{\ell(D)}\right)^{1/2}\frac{\|\widetilde{\Delta}_Q f\|_{L^1(\mu)}^2}{\ell(Q)^n}\\
&\qquad\lesssim\sum_{D\in\mathcal{D}}\sum_{R\in V(D)}\sum_{S\in\mathrm{Trs}:\,Q_S\subset R}\left(\frac{\ell(Q_S)}{\ell(D)}\right)^{1/2}(m_{Q_S}^{\mu}|f|)^2\mu(Q_S)\\
&\qquad\quad+\sum_{D\in\mathcal{D}}\sum_{R\in V(D)}\sum_{Q\in\mathcal{B}:\,Q\subset R}\left(\frac{\ell(Q)}{\ell(D)}\right)^{1/2}(m_Q^{\mu}|f|)^2\mu(Q)\\
&\qquad=\sum_{S\in\mathrm{Trs}}\sum_{R\in\mathcal{D}:\,R\supset Q_S}\sum_{D\in V(R)}\left(\frac{\ell(Q_S)}{\ell(R)}\right)^{1/2}(m_{Q_S}^{\mu}|f|)^2\mu(Q_S)\\
&\qquad\quad+\sum_{Q\in\mathcal{B}}\sum_{R\in\mathcal{D}:\,R\supset Q}\sum_{D\in V(R)}\left(\frac{\ell(Q)}{\ell(R)}\right)^{1/2}(m_Q^{\mu}|f|)^2\mu(Q).
\end{aligned}$$

Finally, using the fact that $V(R)$ has finitely many elements, and that the μ-cubes Q_S with $S\in\mathrm{Trs}$ and the μ-cubes $Q\in\mathcal{B}$ satisfy the Carleson packing condition (so we can apply Carleson's embedding theorem), we deduce

$$\begin{aligned}
&\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D\sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}\sum_{Q\in\mathrm{Stp}(R)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*(\widetilde{\Delta}_Q f\mu))(x)\Big|^2 d\mu(x)\\
&\lesssim\sum_{S\in\mathrm{Trs}}(m_{Q_S}^{\mu}|f|)^2\mu(Q_S)\sum_{R\in\mathcal{D}:\,R\supset Q_S}\frac{\ell(Q_S)^{1/2}}{\ell(R)^{1/2}}+\sum_{Q\in\mathcal{B}}(m_Q^{\mu}|f|)^2\mu(Q)\sum_{R\in\mathcal{D}:\,R\supset Q}\frac{\ell(Q)^{1/2}}{\ell(R)^{1/2}}\\
&\lesssim\sum_{S\in\mathrm{Trs}}(m_{Q_S}^{\mu}|f|)^2\mu(Q_S)+\sum_{Q\in\mathcal{B}}(m_Q^{\mu}|f|)^2\mu(Q)\lesssim\|f\|_{L^2(\mu)}^2. \qquad\square
\end{aligned}$$

5.3.4. Estimate of $\sum_{m\in\mathcal{S}_D(x)}|\sum_{R\in V(D)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*((m_R^{\mu}f)\chi_R\mu))(x)|^2$ *from (39).* Recall the definitions of $V(D)$ and $\mathrm{Ch}(R)$ in Definition 5.3. We will need the following auxiliary lemma, which we prove for completeness, although we think it is known.

Lemma 5.13. *Given $D\in\mathcal{D}$ and $f\in L^2(\mu)$, set $a_D(f):=\sum_{R\in V(D)}|m_R^{\mu}f-m_D^{\mu}f|$. Then there exists $C>0$ depending only n and the AD regularity constant of μ such that*

$$\sum_{D\in\mathcal{D}}a_D(f)^2\mu(D)\le C\|f\|_{L^2(\mu)}^2.$$

Proof. By subtracting a constant if necessary, we can assume that f has mean zero. Consider the representation of f with respect to the Haar basis associated to $\mathcal{D}$, that is, $f=\sum_{Q\in\mathcal{D}}\Delta_Q f$. For $m\in\mathbb{Z}$, we define the function $u_m=\sum_{Q\in\mathcal{D}_m}\Delta_Q f$, so $f=\sum_{m\in\mathbb{Z}}u_m$ and the equality holds in $L^2(\mu)$. Given $j\in\mathbb{Z}$, define the operator

$$S_j(f):=\Big(\sum_{D\in\mathcal{D}_j}a_D(f)^2\chi_D\Big)^{1/2}.$$

We will prove that there exists a sequence $\{\sigma(k)\}_{k\in\mathbb{Z}}$ such that

$$\sum_{k\in\mathbb{Z}}\sigma(k)\le C<\infty \quad\text{and}\quad \|S_j(u_m)\|_{L^2(\mu)}\lesssim\sigma(|m-j|)\|u_m\|_{L^2(\mu)}. \tag{47}$$

Assume for the moment that (47) holds. Then, since each S_j is sublinear, by the Cauchy–Schwarz inequality and the orthogonality of the u_m's,

$$\begin{aligned}\sum_{D\in\mathcal{D}}a_D(f)^2\mu(D)&=\sum_{j\in\mathbb{Z}}\int\sum_{D\in\mathcal{D}_j}a_D(f)^2\chi_D\,d\mu=\sum_{j\in\mathbb{Z}}\|S_j(f)\|^2_{L^2(\mu)}\\&=\sum_{j\in\mathbb{Z}}\Big\|S_j\Big(\sum_{m\in\mathbb{Z}}u_m\Big)\Big\|^2_{L^2(\mu)}\le\sum_{j\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}\|S_j(u_m)\|_{L^2(\mu)}\Big)^2\\&\le\sum_{j\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}\sigma(|m-j|)\Big)\Big(\sum_{m\in\mathbb{Z}}\sigma(|m-j|)^{-1}\|S_j(u_m)\|^2_{L^2(\mu)}\Big)\\&\lesssim\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\sigma(|m-j|)\|u_m\|^2_{L^2(\mu)}=\sum_{m\in\mathbb{Z}}\|u_m\|^2_{L^2(\mu)}\sum_{j\in\mathbb{Z}}\sigma(|m-j|)\\&\lesssim\sum_{m\in\mathbb{Z}}\|u_m\|^2_{L^2(\mu)}=\|f\|^2_{L^2(\mu)},\end{aligned}$$

and the lemma follows. Let us verify (47) now. By definition,

$$\|S_j(u_m)\|^2_{L^2(\mu)}=\sum_{D\in\mathcal{D}_j}\Bigg(\sum_{R\in V(D)}\Bigg|\sum_{Q\in\mathcal{D}_m}\int\Delta_Q f\Big(\frac{\chi_R}{\mu(R)}-\frac{\chi_D}{\mu(D)}\Big)\,d\mu\Bigg|\Bigg)^2\mu(D). \tag{48}$$

Assume first that $m\ge j$. If $D\in\mathcal{D}_j$, $R\in V(D)$, and $Q\in\mathcal{D}_m$, then either $Q\cap R=\emptyset$ or $Q\subset R$. In both cases, since $\Delta_Q f$ has mean zero and is supported in Q, we have $\int\Delta_Q f\,\chi_R\,d\mu=0$. Thus, the right hand side of (48) vanishes (obviously $D\in V(D)$), and (47) follows.

Assume now that $m<j$. Set $\widetilde{D}:=\bigcup_{R\in V(D)}R$. Recall that $\Delta_Q f:=\sum_{U\in\mathrm{Ch}(Q)}\chi_U(m^\mu_U f-m^\mu_Q f)$, so $\Delta_Q f$ is constant in each $U\in\mathrm{Ch}(Q)$. Hence, if for some $U\in\mathrm{Ch}(Q)$ we have $\widetilde{D}\subset U$ or $\widetilde{D}\subset\operatorname{supp}\mu\setminus U$, then $(R\cup D)\subset U$ or $(R\cup D)\cap U=\emptyset$ for all $R\in V(D)$, and so

$$\begin{aligned}&\int\chi_U(m^\mu_U f-m^\mu_Q f)\Big(\frac{\chi_R}{\mu(R)}-\frac{\chi_D}{\mu(D)}\Big)\,d\mu\\&\qquad=(m^\mu_U f-m^\mu_Q f)\int_U\Big(\frac{\chi_R}{\mu(R)}-\frac{\chi_D}{\mu(D)}\Big)\,d\mu=0\end{aligned}$$

for all $R\in V(D)$. Therefore, if we set $m^\mu_{U,Q}f:=m^\mu_U f-m^\mu_Q f$, using the fact that $V(D)$ has finitely many elements and that $\int|\mu(R)^{-1}\chi_R-\mu(D)^{-1}\chi_D|\,d\mu\le 2$ for all $R\in V(D)$, we deduce from (48) that

$$\|S_j(u_m)\|^2_{L^2(\mu)}$$
$$= \sum_{D\in\mathcal{D}_j}\Bigg(\sum_{R\in V(D)}\bigg|\sum_{Q\in\mathcal{D}_m}\int \sum_{\substack{U\in\mathrm{Ch}(Q):\\ \widetilde{D}\cap U\neq\emptyset,\ \widetilde{D}\cap U^c\neq\emptyset}} \chi_U\, m^\mu_{U,Q} f\left(\frac{\chi_R}{\mu(R)}-\frac{\chi_D}{\mu(D)}\right)d\mu\bigg|\Bigg)^2\mu(D)$$
$$\lesssim \sum_{D\in\mathcal{D}_j}\Bigg(\sum_{Q\in\mathcal{D}_m}\sum_{\substack{U\in\mathrm{Ch}(Q):\,\widetilde{D}\cap U\neq\emptyset,\\ \widetilde{D}\cap U^c\neq\emptyset}}|m^\mu_{U,Q}f|\Bigg)^2\mu(D)$$
$$= \sum_{D\in\mathcal{D}_j}\Bigg(\sum_{\substack{U\in\mathcal{D}_{m+1}:\,\widetilde{D}\cap U\neq\emptyset,\\ \widetilde{D}\cap U^c\neq\emptyset}}|m^\mu_{U,P(U)}f|\Bigg)^2\mu(D). \tag{49}$$

It is not hard to show that, since $m<j$ and $D\in\mathcal{D}_j$, the number of μ-cubes $U\in\mathcal{D}_{m+1}$ such that $\widetilde{D}\cap U\neq\emptyset$ and $\widetilde{D}\cap U^c\neq\emptyset$ is bounded by some constant depending only on n and the AD regularity constant of μ (but not on the precise value of m). Hence,

$$\sum_{D\in\mathcal{D}_j}\Bigg(\sum_{\substack{U\in\mathcal{D}_{m+1}:\,\widetilde{D}\cap U\neq\emptyset,\\ \widetilde{D}\cap U^c\neq\emptyset}}|m^\mu_{U,P(U)}f|\Bigg)^2\mu(D)\lesssim \sum_{D\in\mathcal{D}_j}\sum_{\substack{U\in\mathcal{D}_{m+1}:\,\widetilde{D}\cap U\neq\emptyset,\\ \widetilde{D}\cap U^c\neq\emptyset}}|m^\mu_{U,P(U)}f|^2\mu(D)$$
$$=\sum_{U\in\mathcal{D}_{m+1}}|m^\mu_{U,P(U)}f|^2\mu\Bigg(\bigcup_{\substack{D\in\mathcal{D}_j:\,\widetilde{D}\cap U\neq\emptyset,\\ \widetilde{D}\cap U^c\neq\emptyset}}D\Bigg). \tag{50}$$

Fix $U\in\mathcal{D}_{m+1}$. Recall that $\widetilde{D}:=\bigcup_{R\in V(D)}R$, so $\mathrm{diam}(\widetilde{D})\approx\mathrm{diam}(D)$. Thus, there exists a constant $\tau_0>0$ such that

$$\bigcup_{D\in\mathcal{D}_j:\,\widetilde{D}\cap U\neq\emptyset,\ \widetilde{D}\cap U^c\neq\emptyset} D\subset\{x\in U:\mathrm{dist}(x,\operatorname{supp}\mu\setminus U)\le\tau_0\ell(D)\}$$
$$\cup\{x\in\operatorname{supp}\mu\setminus U:\mathrm{dist}(x,U)\le\tau_0\ell(D)\}$$
$$=\{x\in U:\mathrm{dist}(x,\operatorname{supp}\mu\setminus U)\le\tau_0 2^{m-j+1}\ell(U)\}$$
$$\cup\{x\in\operatorname{supp}\mu\setminus U:\mathrm{dist}(x,U)\le\tau_0 2^{m-j+1}\ell(U)\}.$$

If $m\ll j$, then $\tau:=\tau_0 2^{m-j+1}<1$, so we can apply the *small boundaries condition* (9) of Subsection 2.3 to obtain $\mu(\bigcup_{D\in\mathcal{D}_j:\,\widetilde{D}\cap U\neq\emptyset,\ \widetilde{D}\cap U^c\neq\emptyset}D)\le C\tau^{1/C}2^{-mn}$. On the contrary, if $|m-j|\lesssim 1$, then $\tau^{1/C}\approx 1$, so $\mu(\bigcup_{D\in\mathcal{D}_j:\,\widetilde{D}\cap U\neq\emptyset,\ \widetilde{D}\cap U^c\neq\emptyset}D)\le\mu(C_1U)\lesssim 2^{-mn}\approx\tau^{1/C}2^{-mn}$, for some large constant $C_1>0$. Thus, in any case, $\mu(\bigcup_{D\in\mathcal{D}_j:\,\widetilde{D}\cap U\neq\emptyset,\ \widetilde{D}\cap U^c\neq\emptyset}D)\lesssim 2^{(m-j)/C}\ell(U)^n$, and combining this with (50) and (49) we conclude that, for $m<j$,

$$\|S_j(u_m)\|^2_{L^2(\mu)}\lesssim 2^{(m-j)/C}\sum_{U\in\mathcal{D}_{m+1}}|m^\mu_U f-m^\mu_{P(U)}f|^2\ell(U)^n$$
$$\approx 2^{(m-j)/C}\int\sum_{U\in\mathcal{D}_{m+1}}\chi_U|m^\mu_U f-m^\mu_{P(U)}f|^2\,d\mu=2^{-|m-j|/C}\|u_m\|^2_{L^2(\mu)},$$

which gives (47) with $\sigma(k)=2^{-|k|/(2C)}$ and finishes the proof of the lemma. □

Lemma 5.14. *Under the notation above, we have*

$$\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D\sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*((m_R^\mu f)\chi_R\mu))(x)\Big|^2\,d\mu(x)\lesssim\|f\|_{L^2(\mu)}^2.$$

Proof. Recall that, given $D\in\mathcal{D}$, we have set $\widetilde{D}:=\bigcup_{R\in V(D)}R$. For $x\in D$, we have

$$\begin{aligned}\sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}&(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*((m_R^\mu f)\chi_R\mu))(x)\Big|^2\\&\lesssim\sum_{m\in\mathcal{S}_D(x)}|(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*((m_D^\mu f)\chi_{\widetilde{D}}\mu))(x)|^2\\&\quad+\sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*((m_R^\mu f-m_D^\mu f)\chi_R\mu))(x)\Big|^2.\end{aligned}\tag{51}$$

We are going to estimate the two terms on the right hand side of (51) separately. For the second one, recall also that, given $m\in\mathcal{S}_D(x)$, we have set $A_m(x):=A(x,\epsilon_{m+1},\epsilon_m)$. We write

$$\begin{aligned}&|(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*((m_R^\mu f-m_D^\mu f)\chi_R\mu))(x)|\\&\leq|m_R^\mu f-m_D^\mu f|\int_{A_m(x)}|K(x-y)|\chi_R(y)\,d\mu(y)\lesssim|m_R^\mu f-m_D^\mu f|\,\mu(A_m(x)\cap R)\ell(D)^{-n}.\end{aligned}$$

Therefore, interchanging the order of summation,

$$\begin{aligned}\sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}&(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*((m_R^\mu f-m_D^\mu f)\chi_R\mu))(x)\Big|^2\\&\lesssim\Big(\sum_{m\in\mathcal{S}_D(x)}\sum_{R\in V(D)}|m_R^\mu f-m_D^\mu f|\mu(A_m(x)\cap R)\ell(D)^{-n}\Big)^2\\&\leq\Big(\sum_{R\in V(D)}|m_R^\mu f-m_D^\mu f|\frac{\mu(R)}{\ell(D)^n}\Big)^2\approx\Big(\sum_{R\in V(D)}|m_R^\mu f-m_D^\mu f|\Big)^2=a_D(f)^2,\end{aligned}$$

where $a_D(f)$ are the coefficients introduced in Lemma 5.13. If we integrate on D and sum over all $D\in S$ and $S\in\mathrm{Trs}$, we can apply Lemma 5.13, and we finally obtain

$$\begin{aligned}\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D\sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}&(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*((m_R^\mu f-m_D^\mu f)\chi_R\mu))(x)\Big|^2d\mu(x)\\&\lesssim\sum_{D\in\mathcal{D}}a_D(f)^2\mu(D)\lesssim\|f\|_{L^2(\mu)}^2.\end{aligned}\tag{52}$$

Let us now estimate the first term on the right hand side of (51). Let L_D be a minimizing n-plane for $\alpha_\mu(D)$, which is defined in (10), and let L_D^x be the n-plane parallel to L_D which contains x. Given $z\in\mathbb{R}^d$, let p_0^x denote the orthogonal projection onto L_D^x. Let $g_1,g_2:\mathbb{R}\to[0,1]$ be such that $\operatorname{supp}g_1\subset(-2\varepsilon\ell(D),2\varepsilon\ell(D))$,

$\operatorname{supp} g_2 \subset (-\ell(D)\varepsilon, \ell(D)\varepsilon)^c$, and $g_1+g_2=1$, where $\varepsilon>0$ is some fixed constant small enough. For $z\in\mathbb{R}^d$, consider the projection onto L_D^x given by

$$p^x(z) := \left(x+(p_0^x(z)-x)\frac{|z-x|}{|p_0^x(z)-x|}\right)g_2(|p_0^x(z)-x|)+p_0^x(z)g_1(|p_0^x(z)-x|). \tag{53}$$

Since $\operatorname{supp} g_2$ does not contain the origin, p^x is well defined. Moreover, if $z\in\mathbb{R}^d$ is such that $g_2(|p_0^x(z)-x|)=1$, then $|z-x|=|p^x(z)-x|$.

Let $C_*>0$ be a small constant which will be fixed below. Assume that $\alpha_\mu(10D)\geq C_*$. Then we can easily estimate

$$\begin{aligned}\sum_{m\in\mathcal{S}_D(x)} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*((m_D^\mu f)\chi_{\widetilde{D}}\mu))(x)|^2 &= |m_D^\mu f|^2 \sum_{m\in\mathcal{S}_{\mathcal{D}}(x)}\left|\int_{A_m(x)\cap\widetilde{D}} K(x-y)\,d\mu(y)\right|^2\\ &\lesssim |m_D^\mu f|^2\left(\sum_{m\in\mathcal{S}_{\mathcal{D}}(x)}\int_{A_m(x)\cap\widetilde{D}} |K(x-y)|\,d\mu(y)\right)^2\\ &\lesssim |m_D^\mu f|^2\left(\int_{\widetilde{D}}\ell(D)^{-n}\,d\mu(y)\right)^2 \lesssim |m_D^\mu f|^2 \lesssim |m_D^\mu f|^2\alpha_\mu(10D)^2. \end{aligned}\tag{54}$$

From now on, we assume that $\alpha_\mu(10D)<C_*$. By assuming C_* small enough, it is not difficult to show that then the distance between $\widetilde{D}$ and L_D^x is smaller than $\ell(D)/1000$. Moreover, p^x restricted to $\{y\in A_m(x) : \operatorname{dist}(y,L_D^x)\leq\ell(D)/1000\}$ is a Lipschitz function with Lipschitz constant depending only n, d, and the AD regularity constant of μ. Furthermore, by taking ε small enough, we have

$$p^x(z)=x+(p_0^x(z)-x)\frac{|z-x|}{|p_0^x(z)-x|} \tag{55}$$

for all $z\in\{y\in\widetilde{D}\cap A_m(x) : \operatorname{dist}(y,L_D^x)\leq\ell(D)/1000\}\subset\operatorname{supp}\mu$.

Recall that $D\in S$ for some $S\in\mathrm{Trs}$. Let Q_S be the maximal μ-cube of S, and set

$$\nu_x := p_\sharp^x(\chi_{40Q_S}\mu), \tag{56}$$

where $p_\sharp^x$ denotes measure transport by p^x. Then, since $\operatorname{supp}\mu\cap A_m(x)\subset\widetilde{D}$ by the construction of $\widetilde{D}$,

$$\begin{aligned}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*((m_D^\mu f)\chi_{\widetilde{D}}\mu))(x) &= (m_D^\mu f)\int_{A_m(x)} K(x-y)\,d\mu(y)\\ &= (m_D^\mu f)\int_{A_m(x)} K(x-y)\,d(\mu-\nu_x)(y)+(m_D^\mu f)\int_{A_m(x)} K(x-y)\,d\nu_x(y)\\ &=: U1_m(x)+U2_m(x).\end{aligned}\tag{57}$$

Claim 5.15. *Under the notation above, we have*

$$\sum_{m\in\mathcal{S}_D(x)} |U1_m(x)|^2 \lesssim |m_D^\mu f|^2 \Bigg(\beta_{1,\mu}(D)^2 + \alpha_\mu(D)^2 + \left(\frac{\operatorname{dist}(x, L_D)}{\ell(D)}\right)^2\Bigg).$$

Proof of Claim 5.15. By (55), $y \in A_m(x)$ if and only if $p^x(y) \in A_m(x)$ in the integral defining $U1_m(x)$. Since $|y - p^x(y)| \lesssim \operatorname{dist}(y, L_D^x) \le \operatorname{dist}(y, L_D) + \operatorname{dist}(x, L_D)$ for all $y \in \operatorname{sup}\mu \cap A_m(x)$,

$$\begin{aligned}
|U1_m(x)| &\le |m_D^\mu f| \int_{A_m(x)} |K(x-y) - K(x - p^x(y))|\, d\mu(y)\\
&\lesssim \frac{|m_D^\mu f|}{\ell(D)^{n+1}} \int_{A_m(x)} |y - p^x(y)|\, d\mu(y)\\
&\lesssim \frac{|m_D^\mu f|}{\ell(D)^{n+1}} \int_{A_m(x)} (\operatorname{dist}(y, L_D) + \operatorname{dist}(x, L_D))\, d\mu(y).
\end{aligned}$$

If L_D^1 denotes a minimizing n-plane for $\beta_1(D)$, then $\operatorname{dist}_{\mathcal{H}}(L_D \cap B_D, L_D^1 \cap B_D) \lesssim \alpha_\mu(D)\ell(D)$, so $\operatorname{dist}(y, L_D) \lesssim \operatorname{dist}(y, L_D^1) + \alpha_\mu(D)\ell(D)$ for $y \in CD$ (see [To4]). Therefore,

$$\begin{aligned}
\sum_{m\in\mathcal{S}_D(x)} |U1_m(x)|^2 &\lesssim \left(\frac{|m_D^\mu f|}{\ell(D)^{n+1}} \sum_{m\in\mathcal{S}_D(x)} \int_{A_m(x)} (\operatorname{dist}(y, L_D) + \operatorname{dist}(x, L_D))\, d\mu(y)\right)^2\\
&\lesssim |m_D^\mu f|^2 \Bigg(\ell(D)^{-n-1} \int_{CD} (\operatorname{dist}(y, L_D) + \operatorname{dist}(x, L_D))\, d\mu(y)\Bigg)^2\\
&\lesssim |m_D^\mu f|^2 \Bigg(\beta_{1,\mu}(D)^2 + \alpha_\mu(D)^2 + \left(\frac{\operatorname{dist}(x, L_D)}{\ell(D)}\right)^2\Bigg). \qquad \square
\end{aligned}$$

Let us consider $U2_m(x)$ now. We can assume that ν_x is absolutely continuous with respect to $\mathcal{H}^n_{L_D^x}$ (for example, by convolving it with an approximation of the identity and making a limiting argument). Let h_x be the corresponding density, so

$$\nu_x = h_x \mathcal{H}^n_{L_D^x}. \tag{58}$$

We may also assume that $h_x \in L^2(\mathcal{H}^n_{L_D^x})$. So,

$$U2_m(x) = (m_D^\mu f) \int_{A_m(x)} K(x-y)\, d\nu_x(y) = (m_D^\mu f) \int_{A_m(x)} K(x-y) h_x(y)\, d\mathcal{H}^n_{L_D^x}(y).$$

Roughly speaking, we are going to estimate $U2_m(x)$ in terms of some coefficients derived from a decomposition of h_x in a suitable basis. Later on, we will need to relate these coefficients to the α_μ's but, in order to do this, we need the elements of the basis which decompose h_x to be at least Lipschitz. Actually, since ν_x is a transport measure of μ (and h_x is the density function corresponding to ν_x), we can easily estimate integrals of the type $\int g\, d(\mu - \nu_x)$ whenever g is Lipschitz with compact support. This is the main reason to use a wavelet basis instead of a Haar basis in the study of $U2_m(x)$.

Let us now introduce a suitable wavelet basis.

Definition 5.16. Let $\mathcal{D}^n$ be the standard dyadic lattice of $\mathbb{R}^n$. Let $\{\psi_Q^k\}_{Q\in\mathcal{D}^n,\,k=1,\dots,2^n-1}$ be an orthonormal basis of $\mathcal{C}^1$ wavelets on $\mathbb{R}^n$ such that (see [Da, Part I]):

(a) $\psi_Q^k:\mathbb{R}^n\to\mathbb{R}$ is a $\mathcal{C}^1$ function for all $Q\in\mathcal{D}^n$ and $k=1,\dots,2^n-1$.
(b) There exist $C>1$ and $\psi_0:[0,C]^n\to\mathbb{R}$ with $\|\psi_0\|_2=1$, $\|\psi_0\|_\infty\lesssim 1$, and such that, for any $Q\in\mathcal{D}^n$ and $k=1,\dots,2^n-1$, there exists $l\in\mathbb{Z}^n$ such that $\psi_Q^k(y)=\psi_0(y/\ell(Q)-l)\ell(Q)^{-n/2}$ for all $y\in\mathbb{R}^n$.
(c) $\|\psi_Q^k\|_2=1$, $\int\psi_Q^k\,d\mathcal{L}^n=0$ and $\int\psi_Q^k\psi_R^l\,d\mathcal{L}^n=0$, for all $Q,R\in\mathcal{D}^n$ and $k,l=1,\dots,2^n-1$ such that $(Q,k)\neq(R,l)$, where $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$.
(d) $\operatorname{supp}\psi_Q^k\subset C_wQ$ for all $Q\in\mathcal{D}^n$ and $k=1,\dots,2^n-1$, where $C_w>1$ is some fixed constant (which depends on n). In particular, for any $j\in\mathbb{Z}$ the supports of the functions in $\bigcup_{Q\in\mathcal{D}^n:\,\ell(Q)=2^{-j}}\{\psi_Q^k\}_{k=1,\dots,2^n-1}$ have finite overlap.
(e) $\|\psi_Q^k\|_\infty\lesssim\ell(Q)^{-n/2}$ and $\|\nabla\psi_Q^k\|_\infty\lesssim\ell(Q)^{-n/2-1}$ for all $Q\in\mathcal{D}^n$, $k=1,\dots,2^n-1$.
(f) If $h\in L^2(\mathcal{L}^n)$, then $h=\sum_{Q\in\mathcal{D}^n,\,k=1,\dots,2^n-1}\Delta_Q^k h$, where $\Delta_Q^k h:=(\int h\psi_Q^k\,d\mathcal{L}^n)\psi_Q^k$.

In order to reduce the notation, we may think that a cube of $\mathcal{D}^n$ is not only a subset of $\mathbb{R}^n$, but a couple (Q,k), where Q is a subset of $\mathbb{R}^n$ and $k=1,\dots,2^n-1$. In particular, there exist 2^n-1 cubes in $\mathcal{D}^n$ such that the subsets they represent in $\mathbb{R}^n$ coincide. We make this abuse of notation to avoid using the superscript k in the previous definition. Then we can rewrite the wavelet basis as $\{\psi_Q\}_{Q\in\mathcal{D}^n}$, with the evident adjustments of the properties (a), ..., (f) in Definition 5.16.

Let $\mathcal{D}_x^{n,0}$ be a fixed dyadic lattice of the n-plane L_D^x, and let $\{\psi_Q\}_{Q\in\mathcal{D}_x^{n,0}}$ be a wavelet basis as the one introduced in Definition 5.16 but defined on L_D^x. Denote by E_D^x the n-dimensional vector space which defines L_D^x, and let $\{Q_k^0\}_{k\in\mathbb{Z}}$ be a fixed sequence of nested dyadic cubes in E_D^x having the origin as a common vertex and such that $\ell(Q_k^0)=2^{-k}$ for all $k\in\mathbb{Z}$. Given $s\in E_D^x$, set $\mathcal{D}_x^{n,s}:=\{s+Q:Q\in\mathcal{D}_x^{n,0}\}$ (notice that, for any $k\in\mathbb{Z}$, the family $\{Q\in\mathcal{D}_x^{n,s}:\ell(Q)=2^{-k}\}$ is periodic in the parameter s). For any $Q\in\mathcal{D}_x^{n,0}$ and $y\in L_D^x$, if $Q'=s+Q\in\mathcal{D}_x^{n,s}$, we define $\psi_{Q'}(y)\equiv\psi_{s+Q}(y):=\psi_Q(y-s)$. Then $\{\psi_{Q'}\}_{Q'\in\mathcal{D}_x^{n,s}}$ is also a wavelet basis defined on L_D^x. Consider the decomposition of h_x in (58) with respect to this basis,

$$h_x=\sum_{Q\in\mathcal{D}_x^{n,s}}\Delta_Q^\psi h_x=\sum_{Q\in\mathcal{D}_x^{n,0}}\Delta_{Q,s}^\psi h_x,\tag{59}$$

where $\Delta_{Q,s}^\psi h_x(z):=(\int h_x(y)\psi_Q(y-s)\,d\mu(y))\psi_Q(z-s)$ (recall that, for any Q in $\mathcal{D}_x^{n,s}$, $\int\psi_Q\,d\mathcal{H}^n_{L_D^x}=0$). We set $Y(Q_S):=-\log_2(\ell(Q_S))$, and given $Q\in\mathcal{D}_x^{n,s}$, we set $Y(Q):=-\log_2(\ell(Q))$ and $Y'(Q):=\max\{Y(Q_S),Y(Q)\}$. Given $\Omega\subset E_D^x$, denote by $m_{s\in\Omega}g$ the average of a function $g:E_D^x\to\mathbb{R}$ over all $s\in\Omega$ and with respect to $\mathcal{H}^n_{E_D^x}$. Then by the periodicity of $\{\psi_Q\}_{Q\in\mathcal{D}_x^{n,s}}$ in the parameter s (recall Definition 5.16(b)) and (59), we can write

$$h_x=m_{s\in Q^0_{Y(Q_S)}}(h_x)=\sum_{Q\in\mathcal{D}_x^{n,0}}m_{s\in Q^0_{Y(Q_S)}}(\Delta_{Q,s}^\psi h_x)=\sum_{Q\in\mathcal{D}_x^{n,0}}m_{s\in Q^0_{Y'(Q)}}(\Delta_{Q,s}^\psi h_x).$$

It could seem strange to introduce an averaging (with respect to s) at this point. However, it will be necessary in order to obtain the estimate in Lemma 5.20(d). More precisely, we cannot ensure that the estimate in (92), which is used in the proof of Lemma 5.20(d), holds for a particular s because we do not have such a level of control on ν_x, but to take an average overcomes the problem. That is the only point where the averaging with respect to s is used.

Set

$$J := \{Q \in \mathcal{D}_x^{n,0} : \operatorname{supp} \psi_Q(\cdot - s) \cap \operatorname{supp} \chi_{2^{-j-1}}^{2^{-j}}(x - \cdot) \neq \emptyset \text{ for some } s \in Q^0_{Y'(Q)}\}. \tag{60}$$

Then

$$U2_m(x) = (m_D^\mu f) \int_{A_m(x)} K(x-y) \sum_{Q \in J} m_{s \in Q^0_{Y'(Q)}} (\Delta^\psi_{Q,s} h_x(y))\, d\mathcal{H}^n_{L_D^x}(y). \tag{61}$$

Recall that $D \in \mathcal{D}_j$ and $m \in \mathcal{S}_D(x)$. Since $x \in D$ and $\ell(D) = 2^{-j}$, if $Q \in J$, then $D \subset B(x, C_a \ell(Q))$ or $Q \subset B(x, C_a \ell(D))$ for some constant $C_a > 0$ large enough. In particular, if $\ell(Q) \gtrsim \ell(D)$ then $D \subset B(z_Q, C_a \ell(Q))$, and if $\ell(Q) \leq C\ell(D)$ with $C > 0$ small enough then $Q \subset B(z_D, C_a \ell(D))$, where z_Q denotes the center of $Q \subset L_D^x$ and z_D denotes the center of $D \in \mathcal{D}$. From (60), we define

$$\begin{aligned} J_1 &:= \{Q \in J : \ell(Q) \leq C\ell(D)\} \subset \{Q \in \mathcal{D}_x^{n,0} : Q \subset B(z_D, C_a \ell(D))\}, \\ J_2 &:= J \setminus J_1 \subset \{Q \in \mathcal{D}_x^{n,0} : D \subset B(z_Q, C_a \ell(Q))\}. \end{aligned} \tag{62}$$

Since $\int_{A_m(x)} K(x-y)\, d\mathcal{H}^n_{L_D^x}(y) = 0$ by antisymmetry, if x' denotes some fixed point in $A(x, 2^{-j-1}, 2^{-j}) \cap L_D^x$, we have

$$\begin{aligned} &\int_{A_m(x)} K(x-y) \sum_{Q \in J_2} m_{s \in Q^0_{Y'(Q)}} (\Delta^\psi_{Q,s} h_x(y))\, d\mathcal{H}^n_{L_D^x}(y) \\ &\quad = \int_{A_m(x)} K(x-y) \sum_{Q \in J_2} m_{s \in Q^0_{Y'(Q)}} (\Delta^\psi_{Q,s} h_x(y) - \Delta^\psi_{Q,s} h_x(x'))\, d\mathcal{H}^n_{L_D^x}(y). \end{aligned} \tag{63}$$

Then, using (61), (62), the fact that $Y'(Q) = Y(Q)$ for all $Q \in J_1$ (because $D \subset Q_S$), and (63),

$$\begin{aligned} U2_m(x) &= (m_D^\mu f) \int_{A_m(x)} K(x-y) \sum_{Q \in J_1} m_{s \in Q^0_{Y(Q)}} (\Delta^\psi_{Q,s} h_x(y))\, d\mathcal{H}^n_{L_D^x}(y) \\ &\quad + (m_D^\mu f) \int_{A_m(x)} K(x-y) \sum_{Q \in J_2} m_{s \in Q^0_{Y'(Q)}} (\Delta^\psi_{Q,s} h_x(y) - \Delta^\psi_{Q,s} h_x(x'))\, d\mathcal{H}^n_{L_D^x}(y) \\ &=: U3_m(x) + U4_m(x). \end{aligned} \tag{64}$$

Claim 5.17. *Under the notation above, we have*

$$\sum_{m \in \mathcal{S}_D(x)} |U4_m(x)|^2 \lesssim |m_D^\mu f|^2 \sum_{Q \in J_2} \left(\frac{\ell(D)}{\ell(Q)}\right)^{1/2} \ell(Q)^{-n} (m_{s \in Q^0_{Y'(Q)}} \|\Delta^\psi_{Q,s} h_x\|_2)^2.$$

Proof of Claim 5.17. By property (e) of the wavelet basis in Definition 5.16, we have $|\Delta^{\psi}_{Q,s}h_x(y) - \Delta^{\psi}_{Q,s}h_x(x')| \le \|\nabla(\Delta^{\psi}_{Q,s}h_x)\|_\infty |x'-y| \lesssim \|\Delta^{\psi}_{Q,s}h_x\|_2 |x'-y| \ell(Q)^{-n/2-1}$. Moreover, if $y \in A_m(x)$, then $|x'-y| \lesssim \ell(D)$. Therefore,

$$
\begin{aligned}
&|U4_m(x)|\\
&\le \sum_{Q\in J_2} |m^\mu_D f| \int_{A_m(x)} |K(x-y)| m_{s\in Q^0_{Y'(Q)}}(|\Delta^{\psi}_{Q,s}h_x(y) - \Delta^{\psi}_{Q,s}h_x(x')|)\, d\mathcal{H}^n_{L^x_D}(y)\\
&\lesssim \sum_{Q\in J_2} |m^\mu_D f| m_{s\in Q^0_{Y'(Q)}}(\|\Delta^{\psi}_{Q,s}h_x\|_2)\ell(D)^{1-n}\ell(Q)^{-n/2-1}\mathcal{H}^n_{L^x_D}(A_m(x)).
\end{aligned}
$$

Then, by the Cauchy–Schwarz inequality and as $J_2 \subset \{Q \in \mathcal{D}^{n,0}_x : D \subset B(z_Q, C_a\ell(Q))\}$ (in particular, $\ell(D)/\ell(Q) \lesssim (\ell(D)/\ell(Q))^{1/2}$),

$$
\begin{aligned}
&\sum_{m\in\mathcal{S}_D(x)} |U4_m(x)|^2\\
&\quad\lesssim \Bigg(\sum_{m\in\mathcal{S}_D(x)}\sum_{Q\in J_2} |m^\mu_D f|\, m_{s\in Q^0_{Y'(Q)}}(\|\Delta^{\psi}_{Q,s}h_x\|_2)\frac{\ell(Q)^{n/2+1}}{\ell(D)^{n-1}}\,\mathcal{H}^n_{L^x_D}(A_m(x))\Bigg)^2\\
&\quad\le \Bigg(\sum_{Q\in J_2} |m^\mu_D f|\, m_{s\in Q^0_{Y'(Q)}}(\|\Delta^{\psi}_{Q,s}h_x\|_2)\ell(D)\ell(Q)^{n/2+1}\Bigg)^2\\
&\quad\le \Bigg(\sum_{Q\in J_2}\frac{\ell(D)}{\ell(Q)}\Bigg)\Bigg(\sum_{Q\in J_2} |m^\mu_D f|^2 (m_{s\in Q^0_{Y'(Q)}}\|\Delta^{\psi}_{Q,s}h_x\|_2)^2\frac{\ell(D)}{\ell(Q)^{n+1}}\Bigg)\\
&\quad\lesssim |m^\mu_D f|^2 \sum_{Q\in J_2}\Bigg(\frac{\ell(D)}{\ell(Q)}\Bigg)^{1/2}\ell(Q)^{-n}(m_{s\in Q^0_{Y'(Q)}}\|\Delta^{\psi}_{Q,s}h_x\|_2)^2. \qquad\square
\end{aligned}
$$

We are going to estimate $U3_m(x)$ with techniques very similar to the ones used in Subsections 5.3.1 and 5.3.3. First of all, let $b_* > 0$ be a small constant which will be fixed later on, and consider the family $\mathcal{P} := \{Q \in \mathcal{D}^{n,0}_x : \ell(Q) \le \ell(D)\}$. Let Stp denote the set of cubes $Q \in \mathcal{P}$ such that there exists $R_Q \in \mathcal{D}$ with $\ell(R_Q) = \ell(Q)$, $10R_Q \cap (p^x)^{-1}(\operatorname{supp}\psi_Q) \neq \emptyset$, and

$$
\sum_{R\in\mathcal{D}:\, R_Q\subset R,\, \ell(R)\le\ell(D)} \alpha_\mu(10R) \ge b_* \quad\text{but}\quad \sum_{R\in\mathcal{D}:\, P(R_Q)\subset R,\, \ell(R)\le\ell(D)} \alpha_\mu(10R) < b_*. \tag{65}
$$

Observe that if Q and Q' are different and belong to Stp, then $Q \cap Q' = \emptyset$. Notice also that $D \notin$ Stp because we assumed $\alpha_\mu(10D) < C_*$. Finally, denote by Tr the set of cubes $Q \in \mathcal{P} \setminus$ Stp such that $R \notin$ Stp for all $R \in \mathcal{P}$ with $R \supset Q$. Then $\mathcal{P} = \mathrm{Tr} \cup \bigcup_{Q\in\mathrm{Stp}}\{R \in \mathcal{P} : R \subset Q\}$. By taking C_* small enough we can assume that, if $R \in J_1 \cap \mathcal{P}$ and $R \subset Q$ for some $Q \in$ Stp, then $Q \in J_1$. So we write

$$
\begin{aligned}
&\sum_{Q\in J_1} m_{s\in Q^0_{Y(Q)}}(\Delta^{\psi}_{Q,s}h_x)\\
&\qquad= \sum_{Q\in J_1\cap\mathrm{Tr}} m_{s\in Q^0_{Y(Q)}}(\Delta^{\psi}_{Q,s}h_x) + \sum_{Q\in J_1\cap\mathrm{Stp}}\ \sum_{R\in J_1\cap\mathcal{P}:\, R\subset Q} m_{s\in Q^0_{Y(Q)}}(\Delta^{\psi}_{R,s}h_x)
\end{aligned}
$$

Set $\widetilde{\Delta}^{\psi}_{Q,s} h_x := \sum_{R\in\mathcal{P}:R\subset Q} \Delta^{\psi}_{R,s} h_x$. Then using the definition of J_1 and J, we can split

$$\begin{aligned} U3_m(x) &= (m^{\mu}_D f) \int_{A_m(x)} K(x-y) \sum_{Q\in J_1\cap \mathrm{Tr}} m_{s\in Q^0_{Y(Q)}} (\Delta^{\psi}_{Q,s} h_x(y))\, d\mathcal{H}^n_{L^x_D}(y) \\ &\quad + (m^{\mu}_D f) \int_{A_m(x)} K(x-y) \sum_{Q\in J_1\cap \mathrm{Stp}} m_{s\in Q^0_{Y(Q)}} (\widetilde{\Delta}^{\psi}_{Q,s} h_x(y))\, d\mathcal{H}^n_{L^x_D}(y) \\ &=: U3^a_m(x) + U3^b_m(x). \end{aligned} \tag{66}$$

Claim 5.18. *Under the notation above, we have*

$$\sum_{m\in\mathcal{S}_D(x)} |U3^a_m(x)|^2 \lesssim |m^{\mu}_D f|^2 \sum_{Q\in J_1\cap \mathrm{Tr}} \left(\frac{\ell(Q)}{\ell(D)}\right)^{1/2} \|m_{s\in Q^0_{Y(Q)}} (\Delta^{\psi}_{Q,s} h_x)\|_2^2\, \ell(D)^{-n}.$$

For simplicity of notation, we have set $\|\cdot\|_p := \|\cdot\|_{L^p(\mathcal{H}^n_{L^x_D})}$.

Proof of Claim 5.18. Notice that $\mathcal{H}^n_{L^x_D}(A_m(x)) \lesssim (\epsilon_m - \epsilon_{m+1})\ell(D)^{n-1}$. Moreover, the function $m_{s\in Q^0_{Y(Q)}} (\Delta^{\psi}_{Q,s} h_x)$ is supported in CQ and has vanishing integral, because the same holds for each $\Delta^{\psi}_{Q,s} h_x$ with $s \in Q^0_{Y(Q)}$. Hence, the sum $\sum_{m\in\mathcal{S}_D(x)} |U3^a_m(x)|^2$ can be estimated using arguments very similar to the ones in Subsection 5.3.1 (see (44)), and the analogues of Lemma 5.7 and Claims 5.5 and 5.6 for $\mathcal{H}^n_{L^x_D}$ follow easily. One obtains the expected estimate. □

Claim 5.19. *Under the notation above, we have*

$$\sum_{m\in\mathcal{S}_D(x)} |U3^b_m(x)|^2 \lesssim |m^{\mu}_D f|^2 \sum_{Q\in J_1\cap \mathrm{Stp}} \left(\frac{\ell(Q)}{\ell(D)}\right)^{1/2} \frac{\|m_{s\in Q^0_{Y(Q)}} (\widetilde{\Delta}^{\psi}_{Q,s} h_x)\|_1^2}{\ell(D)^n \ell(Q)^n}.$$

Proof of Claim 5.19. Since $m_{s\in Q^0_{Y(Q)}} (\widetilde{\Delta}^{\psi}_{Q,s} h_x)$ has vanishing integral and it is supported in a neighborhood of Q, the term $U3^b_m(x)$ can be estimated in the same manner (but now we do not use the estimate $\|m_{s\in Q^0_{Y(Q)}} (\widetilde{\Delta}^{\psi}_{Q,s} h_x)\|_1^2 \lesssim \ell(Q)^n \|m_{s\in Q^0_{Y(Q)}} (\widetilde{\Delta}^{\psi}_{Q,s} h_x)\|_2^2$), and one obtains the expected estimate (compare with (46)). □

Recall that we have fixed $x \in D \in S \in \mathrm{Trs}$, and we denote by Q_S the maximal μ-cube in S from the corona decomposition, so $D \subset Q_S$. The following lemma, whose proof is given in Subsection 5.3.5, yields the suitable estimates for $m_{s\in Q^0_{Y'(Q)}} (\Delta^{\psi}_{Q,s} h_x)$ and $m_{s\in Q^0_{Y'(Q)}} (\widetilde{\Delta}^{\psi}_{Q,s} h_x)$ (recall that h_x is given by (58)).

Lemma 5.20. *Assume that $\alpha_\mu(D) < C_*$ for some constant $C_* > 0$ small enough. Given $Q \in \mathcal{D}_x^{n,0}$, there exist constants $C_1, C_2 > 1$ depending on C_* and b_* (see (65)) such that:*

(a) *if $Q \in J_2$ and $\ell(Q) > \ell(Q_S)$, then $m_{s\in Q^0_{Y'(Q)}}(\|\Delta^\psi_{Q,s} h_x\|_2) \lesssim \ell(Q_S)^n \ell(Q)^{-n/2}$,*

(b) *if $Q \in J_2$ and $\ell(Q) \le \ell(Q_S)$, then*

$$m_{s\in Q^0_{Y'(Q)}}(\|\Delta^\psi_{Q,s} h_x\|_2) \lesssim \Bigg(\sum_{R\in\mathcal{D}:\, D\subset R\subset B(z_Q, C_1\ell(Q))} \alpha_\mu(C_1 R) + \frac{\operatorname{dist}(x, L_D)}{\ell(D)}\Bigg)\ell(Q)^{n/2},$$

(c) *if $Q \in J_1 \cap \mathrm{Tr}$, then there exists $Q_0 \equiv Q_0(x, Q) \in \mathcal{D}$ depending on x and $Q \in \mathcal{D}_x^{n,0}$ such that $Q_0 \subset C_2 D$, $\ell(Q_0) \approx \ell(Q)$, $Q_0 \cap (p^x)^{-1}(\operatorname{supp}\psi_Q) \neq \emptyset$ and*

$$\|m_{s\in Q^0_{Y(Q)}}(\Delta^\psi_{Q,s} h_x)\|_2 \lesssim \Bigg(\sum_{R\in\mathcal{D}:\, Q_0\subset R\subset C_2 D} \alpha_\mu(C_2 R) + \frac{\operatorname{dist}(x, L_D)}{\ell(D)}\Bigg)\ell(Q)^{n/2},$$

(d) *if $Q \in J_1 \cap \mathrm{Stp}$, then $\|m_{s\in Q^0_{Y(Q)}}(\widetilde{\Delta}^\psi_{Q,s} h_x)\|_1 \lesssim \ell(Q)^n$.*

We are ready to put all the estimates together to bound the first term on the right hand side of (51). From (54), (57), (64), and (66) we have

$$\begin{aligned}\sum_{m\in\mathcal{S}_D(x)} |(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * ((m^\mu_D f)\chi_{\widetilde{D}}\mu))(x)|^2 &\lesssim |m^\mu_D f|^2 \alpha_\mu(10D)^2 \\ &+ \sum_{m\in\mathcal{S}_D(x)} (|U1_m(x)|^2 + |U3^a_m(x)|^2 + |U3^b_m(x)|^2 + |U4_m(x)|^2). \end{aligned} \tag{67}$$

Let us deal with $U1_m(x)$ (the term $|m^\mu_D f|^2\alpha_\mu(10D)^2$ above is handled in the same manner). If L^1_D and L^2_D denote minimizing n-planes for $\beta_{1,\mu}(D)$ and $\beta_{2,\mu}(D)$, respectively, one can show that $\operatorname{dist}_{\mathcal{H}}(L_D \cap B_D, L^1_D \cap B_D) \lesssim \alpha_\mu(D)\ell(D)$ and $\operatorname{dist}_{\mathcal{H}}(L^1_D \cap B_D, L^2_D \cap B_D) \lesssim \beta_{2,\mu}(D)\ell(D)$, so we have $\operatorname{dist}(x, L_D) \lesssim \operatorname{dist}(x, L^2_D) + \beta_{2,\mu}(D)\ell(D) + \alpha_\mu(D)\ell(D)$ for $x \in D$. Then by Claim 5.15 and Carleson's embedding theorem,

$$\begin{aligned}\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D \sum_{m\in\mathcal{S}_D(x)} |U1_m|^2\, d\mu &\lesssim \sum_{D\in\mathcal{D}} \int_D |m^\mu_D f|^2 \Bigg(\beta_{1,\mu}(D)^2 + \alpha_\mu(D)^2 + \bigg(\frac{\operatorname{dist}(x, L_D)}{\ell(D)}\bigg)^2\Bigg)\, d\mu(x) \\ &\lesssim \sum_{D\in\mathcal{D}} |m^\mu_D f|^2 \ell(D)^n (\beta_{1,\mu}(D)^2 + \alpha_\mu(D)^2 + \beta_{2,\mu}(D)^2) \lesssim \|f\|^2_{L^2(\mu)}. \end{aligned} \tag{68}$$

For the case of $U3^a_m(x)$, by Claim 5.18 and Lemma 5.20(c) applied to the μ-cubes in $J_1 \cap \mathrm{Tr}$, we have

$$
\begin{aligned}
&\sum_{S\in \mathrm{Trs}}\sum_{D\in S}\int_D \sum_{m\in \mathcal{S}_D(x)} |U3_m^a|^2\,d\mu \\
&\quad\lesssim \sum_{D\in\mathcal{D}} |m_D^\mu f|^2 \int_D \sum_{Q\in J_1\cap \mathrm{Tr}} \left(\frac{\ell(Q)}{\ell(D)}\right)^{1/2} \frac{\|m_{s\in Q^0_{Y(Q)}}(\Delta^\psi_{Q,s}h_x)\|_2^2}{\ell(D)^n}\,d\mu(x)\\
&\quad\lesssim \sum_{D\in\mathcal{D}} |m_D^\mu f|^2 \int_D \sum_{Q\in J_1\cap \mathrm{Tr}} \left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2} \Big(\sum_{\substack{R\in\mathcal{D}:\\ Q_0(x,Q)\subset R\subset C_2D}} \alpha_\mu(C_2R)\Big)^2\,d\mu(x)\\
&\quad\quad+ \sum_{D\in\mathcal{D}} |m_D^\mu f|^2 \int_D \sum_{Q\in J_1\cap \mathrm{Tr}} \left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2} \left(\frac{\mathrm{dist}(x,L_D)}{\ell(D)}\right)^2 d\mu(x) =: S_1+S_2.
\end{aligned}
$$

Recall that $J_1\subset\{Q\in\mathcal{D}_x^{n,0}: Q\subset B(z_D, C_a\ell(D))\}$. Then $\sum_{Q\in J_1}(\ell(Q)/\ell(D))^{n+1/2}\lesssim 1$, and since $\mathrm{dist}(x,L_D)\lesssim \mathrm{dist}(x,L_D^2)+\beta_{2,\mu}(D)\ell(D)+\alpha_\mu(D)\ell(D)$ for $x\in D$, we have $S_2\lesssim\sum_{D\in\mathcal{D}}|m_D^\mu f|^2(\beta_{2,\mu}(D)^2+\alpha_\mu(D)^2)\ell(D)^n$, and hence $S_2\le C\|f\|^2_{L^2(\mu)}$, by Carleson's embedding theorem. For S_1, since $\ell(Q)\approx\ell(Q_0(x,Q))$ (recall the definition of $Q_0\equiv Q_0(x,Q)$ in Lemma 5.20(c)), $Q_0(x,Q)\subset C_2D$, and every $Q_0\in\mathcal{D}$ intersects $(p^x)^{-1}(\mathrm{supp}\,\psi_Q)$ for finitely many cubes $Q\in\mathcal{D}_x^{n,0}$ (with a bound for the number of such cubes Q independent of x and Q_0), we have

$$
\begin{aligned}
&\sum_{Q\in J_1\cap\mathrm{Tr}}\left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2}\Big(\sum_{R\in\mathcal{D}:\,Q_0(x,Q)\subset R\subset C_2D}\alpha_\mu(C_2R)\Big)^2\\
&\quad=\sum_{P\in\mathcal{D}:\,P\subset C_2D}\ \sum_{\substack{Q\in\mathcal{D}_x^{n,0}:\,Q\subset B(z_D,C_a\ell(D)),\\ Q_0(x,Q)=P}}\left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2}\Big(\sum_{R\in\mathcal{D}:\,P\subset R\subset C_2D}\alpha_\mu(C_2R)\Big)^2\\
&\quad\lesssim\sum_{P\in\mathcal{D}:\,P\subset C_2D}\left(\frac{\ell(P)}{\ell(D)}\right)^{n+1/2}\Big(\sum_{R\in\mathcal{D}:\,P\subset R\subset C_2D}\alpha_\mu(C_2R)\Big)^2.
\end{aligned}
$$

By the Cauchy–Schwarz inequality,

$$
\begin{aligned}
&\sum_{P\in\mathcal{D}:\,P\subset C_2D}\left(\frac{\ell(P)}{\ell(D)}\right)^{n+1/2}\Big(\sum_{R\in\mathcal{D}:\,P\subset R\subset C_2D}\alpha_\mu(C_2R)\Big)^2\\
&\qquad\lesssim\sum_{P\in\mathcal{D}:\,P\subset C_2D}\left(\frac{\ell(P)}{\ell(D)}\right)^{n+1/2}\log_2\left(\frac{\ell(D)}{\ell(P)}\right)\sum_{R\in\mathcal{D}:\,P\subset R\subset C_2D}\alpha_\mu(C_2R)^2\\
&\qquad\lesssim\sum_{R\in\mathcal{D}:\,R\subset C_2D}\alpha_\mu(C_2R)^2\sum_{P\in\mathcal{D}:\,P\subset R}\left(\frac{\ell(P)}{\ell(D)}\right)^{n+1/4}\\
&\qquad\lesssim\sum_{R\in\mathcal{D}:\,R\subset C_2D}\alpha_\mu(C_2R)^2\left(\frac{\ell(R)}{\ell(D)}\right)^{n+1/4}=:\lambda_1(D)^2. \qquad (69)
\end{aligned}
$$

By standard arguments one can easily show that these λ_1 coefficients satisfy the Carleson packing condition, so by (69) and Carleson's embedding theorem we obtain $S_1 \lesssim \sum_{D\in\mathcal{D}} |m_D^\mu f|^2 \ell(D)^n \lambda_1(D)^2 \lesssim \|f\|_{L^2(\mu)}^2$, which combined with $S_2 \lesssim \|f\|_{L^2(\mu)}^2$ yields

$$\sum_{S\in\mathrm{Trs}} \sum_{D\in S} \int_D \sum_{m\in\mathcal{S}_D(x)} |U3_m^a|^2\, d\mu \lesssim \|f\|_{L^2(\mu)}^2. \tag{70}$$

Let us deal now with $U3_m^b$. By Claim 5.19 and Lemma 5.20(d) applied to the μ-cubes in $J_1 \cap \mathrm{Stp}$, we have

$$\begin{aligned}\sum_{S\in\mathrm{Trs}} \sum_{D\in S} \int_D \sum_{m\in\mathcal{S}_D(x)} |U3_m^b|^2\, d\mu \\ &\lesssim \sum_{D\in\mathcal{D}} |m_D^\mu f|^2 \int_D \sum_{Q\in J_1\cap\mathrm{Stp}} \left(\frac{\ell(Q)}{\ell(D)}\right)^{1/2} \frac{\|m_{s\in Q^0_{Y(Q)}}(\widetilde{\Delta}^\psi_{Q,s} h_x)\|_1^2}{\ell(D)^n \ell(Q)^n}\, d\mu(x) \\ &\lesssim \sum_{D\in\mathcal{D}} |m_D^\mu f|^2 \int_D \sum_{Q\in J_1\cap\mathrm{Stp}} \left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2} d\mu.\end{aligned}$$

Given $D \in \mathcal{D}$, consider the family $\Lambda_D := \{R \in \mathcal{D} : R = R_Q$ for some $x \in D$ and some $Q \in J_1 \cap \mathrm{Stp}\}$ (see the definition of R_Q in (65)). Observe that every $R \in \mathcal{D}$ intersects $(p^x)^{-1}(Q \cap L_D^x)$ for finitely many μ-cubes $Q \in \mathcal{D}_x^{n,0}$ such that $\ell(Q) = \ell(R)$. Thus, similarly to what we did for $Q \in J_1 \cap \mathrm{Tr}$ in the case of $U3_m^a$, we have

$$\begin{aligned}\sum_{D\in\mathcal{D}} |m_D^\mu f|^2 \int_D \sum_{Q\in J_1\cap\mathrm{Stp}} \left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2} d\mu &\lesssim \sum_{D\in\mathcal{D}} |m_D^\mu f|^2 \int_D \sum_{R\in\Lambda_D} \left(\frac{\ell(R)}{\ell(D)}\right)^{n+1/2} d\mu \\ &\lesssim \sum_{D\in\mathcal{D}} |m_D^\mu f|^2 \sum_{R\in\Lambda_D} \left(\frac{\ell(R)}{\ell(D)}\right)^{n+1/2} \mu(D) = \sum_{D\in\mathcal{D}} |m_D^\mu f|^2 \lambda_2(D)^2 \mu(D),\end{aligned}$$

where we have set $\lambda_2(D)^2 := \sum_{R\in\Lambda_D} (\ell(R)/\ell(D))^{n+1/2}$. Since the α_μ's satisfy the Carleson packing condition, it is not hard to show that the same holds for the λ_2's. Indeed, since for any $R \in \Lambda_D$ we have $\sum_{R'\in\mathcal{D}:\, R\subset R',\, \ell(R)\le\ell(D)} \alpha_\mu(10R') \ge b_*$ by (65), then

$$\lambda_2(D)^2 \le b_*^{-2} \sum_{R\in\Lambda_D} \left(\frac{\ell(R)}{\ell(D)}\right)^{n+1/2} \Big(\sum_{R'\in\mathcal{D}:\, R\subset R',\, \ell(R)\le\ell(D)} \alpha_\mu(10R') \Big)^2,$$

and we can proceed as in (69). Hence, putting these estimates together and using Carleson's embedding theorem for the λ_2's, we obtain

$$\sum_{S\in\mathrm{Trs}} \sum_{D\in S} \int_D \sum_{m\in\mathcal{S}_D(x)} |U3_m^b|^2\, d\mu \lesssim \|f\|_{L^2(\mu)}^2. \tag{71}$$

We deal now with $U4_m(x)$. By Claim 5.17 and Lemma 5.20(a)&(b) applied to the cubes in J_2,

$$\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D\sum_{m\in\mathcal{S}_D(x)}|U4_m|^2\,d\mu$$

$$\lesssim \sum_{S\in\mathrm{Trs}}\sum_{D\in S}|m^\mu_D f|^2\int_D\sum_{Q\in J_2}\left(\frac{\ell(D)}{\ell(Q)}\right)^{1/2}\frac{m_{s\in Q^0_{Y'(Q)}}(\|\Delta^\psi_{Q,s}h_x\|_2)^2}{\ell(Q)^n}\,d\mu$$

$$\lesssim \sum_{S\in\mathrm{Trs}}\sum_{D\in S}|m^\mu_D f|^2\int_D\sum_{Q\in J_2:\,\ell(Q)\le\ell(Q_S)}\left(\frac{\ell(D)}{\ell(Q)}\right)^{1/2}$$

$$\times\left[\Bigg(\sum_{R\in\mathcal{D}:\,D\subset R\subset B(z_Q,C_1\ell(Q))}\alpha_\mu(C_1R)\Bigg)^2+\left(\frac{\mathrm{dist}(x,L_D)}{\ell(D)}\right)^2\right]d\mu$$

$$+\sum_{S\in\mathrm{Trs}}\sum_{D\in S}|m^\mu_D f|^2\int_D\sum_{Q\in J_2:\,\ell(Q)>\ell(Q_S)}\left(\frac{\ell(D)}{\ell(Q)}\right)^{1/2}\frac{\ell(Q_S)^{2n}}{\ell(Q)^{2n}}\,d\mu=:S_3+S_4. \quad (72)$$

Regarding S_3, since $\mathrm{dist}(x,L_D)\lesssim\mathrm{dist}(x,L^2_D)+\beta_{2,\mu}(D)\ell(D)+\alpha_\mu(D)\ell(D)$ for $x\in D$ and $\sum_{Q\in J_2}(\ell(D)/\ell(Q))^{1/2}\lesssim 1$, the second term in the definition of S_3 is bounded by $\sum_{D\in\mathcal{D}}|m^\mu_D f|^2(\beta_{2,\mu}(D)^2+\alpha_\mu(D)^2)\ell(D)^n$, and hence by $C\|f\|^2_{L^2(\mu)}$, by Carleson's embedding theorem. For the first term in S_3, by the Cauchy–Schwarz inequality,

$$\sum_{S\in\mathrm{Trs}}\sum_{D\in S}|m^\mu_D f|^2\int_D\sum_{Q\in J_2:\,\ell(Q)\le\ell(Q_S)}\left(\frac{\ell(D)}{\ell(Q)}\right)^{1/2}\Bigg(\sum_{\substack{R\in\mathcal{D}:\\ D\subset R\subset B(z_Q,C_1\ell(Q))}}\alpha_\mu(C_1R)\Bigg)^2d\mu$$

$$\lesssim\sum_{S\in\mathrm{Trs}}\sum_{D\in S}|m^\mu_D f|^2$$

$$\times\int_D\sum_{\substack{Q\in J_2:\\ \ell(Q)\le\ell(Q_S)}}\left(\frac{\ell(D)}{\ell(Q)}\right)^{1/2}\log_2\left(\frac{\ell(Q)}{\ell(D)}\right)\sum_{\substack{R\in\mathcal{D}:\\ D\subset R\subset B(z_Q,C_1\ell(Q))}}\alpha_\mu(C_1R)^2\,d\mu$$

$$\lesssim\sum_{D\in\mathcal{D}}|m^\mu_D f|^2\int_D\sum_{\substack{R\in\mathcal{D}:\\ D\subset R}}\alpha_\mu(C_1R)^2\sum_{\substack{Q\in\mathcal{D}^{n,0}_x:\\ R\subset B(z_Q,C_1\ell(Q))}}\left(\frac{\ell(D)}{\ell(Q)}\right)^{1/4}d\mu.$$

Notice that $\sum_{Q\in\mathcal{D}^{n,0}_x:\,R\subset B(z_Q,C_1\ell(Q))}(\ell(D)/\ell(Q))^{1/4}\lesssim(\ell(D)/\ell(R))^{1/4}$, thus the right side of the preceding inequality is bounded above by

$$\sum_{D\in\mathcal{D}}|m^\mu_D f|^2\ell(D)^n\sum_{R\in\mathcal{D}:\,D\subset R}\alpha_\mu(C_1R)^2\left(\frac{\ell(D)}{\ell(R)}\right)^{1/4}=:\sum_{D\in\mathcal{D}}|m^\mu_D f|^2\ell(D)^n\lambda_3(D)^2. \quad (73)$$

By standard arguments one can show that the λ_3's satisfy the Carleson packing condition, so by Carleson's embedding theorem again, the last term in (73) is bounded by $C\|f\|^2_{L^2(\mu)}$. Thus we obtain $S_3 \lesssim \|f\|^2_{L^2(\mu)}$.

The estimate of S_4 from (72) is easier:

$$S_4 \lesssim \sum_{S\in\mathrm{Trs}} \sum_{D\in S} |m^\mu_D f|^2 \int_D \sum_{\substack{Q\in\mathcal{D}^{n,0}_x:\,\ell(Q)>\ell(Q_S),\\ D\subset B(z_Q,C_1\ell(Q))}} \frac{\ell(D)^{1/2}\ell(Q_S)^{2n}}{\ell(Q)^{2n+1/2}}\,d\mu(x).$$

As before, $\sum_{Q\in\mathcal{D}^{n,0}_x:\,\ell(Q)>\ell(Q_S),\,D\subset B(z_Q,C_1\ell(Q))} \ell(Q)^{-2n-1/2} \lesssim \ell(Q_S)^{-2n-1/2}$, thus

$$\begin{aligned} S_4 &\lesssim \sum_{S\in\mathrm{Trs}} \sum_{D\in S} |m^\mu_D f|^2 \ell(D)^n \left(\frac{\ell(D)}{\ell(Q_S)}\right)^{1/2} \lesssim \sum_{D\in\mathcal{D}} |m^\mu_D f|^2 \ell(D)^n \sum_{S\in\mathrm{Trs}:\,S\ni D} \left(\frac{\ell(D)}{\ell(Q_S)}\right)^{1/2} \\ &=: \sum_{D\in\mathcal{D}} |m^\mu_D f|^2 \ell(D)^n \lambda_4(D)^2. \end{aligned}$$

Similarly to the case of the λ_3 coefficients, one can show that the λ_4's also satisfy the Carleson packing condition, thus $S_4 \lesssim \|f\|^2_{L^2(\mu)}$ by Carleson's embedding theorem. Actually, if one defines $\widehat{\alpha}_\mu(Q) = 1$ if $Q = Q_S$ for some $S \in \mathrm{Trs}$ and $\widehat{\alpha}_\mu(Q) = 0$ otherwise, using the packing condition for the μ-cubes Q_S with $S \in \mathrm{Trs}$, one can easily verify that the $\widehat{\alpha}_\mu$'s satisfy the Carleson packing condition. Then

$$\lambda_4(D)^2 = \sum_{S\in\mathrm{Trs}:\,D\subset Q_S} \left(\frac{\ell(D)}{\ell(Q_S)}\right)^{1/2} \widehat{\alpha}_\mu(Q_S)^2 = \sum_{Q\in\mathcal{D}:\,D\subset Q} \left(\frac{\ell(D)}{\ell(Q)}\right)^{1/2} \widehat{\alpha}_\mu(Q)^2,$$

and we can argue as in the case of the λ_3's in (73).

By the estimates of S_3 and S_4, we obtain

$$\sum_{S\in\mathrm{Trs}} \sum_{D\in S} \int_D \sum_{m\in\mathcal{S}_D(x)} |U4_m|^2\,d\mu \lesssim \|f\|^2_{L^2(\mu)}. \tag{74}$$

Finally, plugging (68), (70), (71), and (74) in (67), and combining the result with (51) and (52), we conclude that

$$\sum_{S\in\mathrm{Trs}} \sum_{D\in S} \int_D \sum_{m\in\mathcal{S}_D(x)} \Big| \sum_{R\in V(D)} (K\chi^{\epsilon_m}_{\epsilon_{m+1}} * ((m^\mu_R f)\chi_R\mu))(x)\Big|^2\,d\mu(x) \lesssim \|f\|^2_{L^2(\mu)},$$

and Lemma 5.14 is finally proved, modulo Lemma 5.20. □

5.3.5. Proof of Lemma 5.20. See (56) and (58) for the definitions of ν_x and h_x.

Proof of Lemma 5.20(a). By Definition 5.16(e), for any $s \in Q^0_{Y'(Q)}$ we have

$$\|\Delta^{\psi}_{Q,s} h_x\|_\infty \lesssim |\langle h_x, \psi_{s+Q}\rangle| \ell(Q)^{-n/2} \lesssim \ell(Q)^{-n} \int h_x \, d\mathcal{H}^n_{L^x_D} = \ell(Q)^{-n} \int d\nu_x$$
$$= \ell(Q)^{-n} \int d(p^x_\sharp(\chi_{40Q_S}\mu)) = \ell(Q)^{-n} \int_{40Q_S} d\mu \lesssim \frac{\ell(Q_S)^n}{\ell(Q)^n}.$$

Hence, $\|\Delta^{\psi}_{Q,s} h_x\|_2 \leq \|\Delta^{\psi}_{Q,s} h_x\|_\infty \mathcal{L}^n(\operatorname{supp}\psi_{s+Q})^{1/2} \lesssim \ell(Q_S)^n \ell(Q)^{-n/2}$ for all $s \in Q^0_{Y'(Q)}$, and Lemma 5.20(a) follows by taking the average over $s \in Q^0_{Y'(Q)}$. □

Proof of Lemma 5.20(b). Since $D \subset B(z_Q, C_a\ell(Q))$, $D \in S$, and $\ell(D) \lesssim \ell(Q) \leq \ell(Q_S)$, by taking C_{cor} large enough (see property (f) in Subsection 2.4), we can assume that μ is well approximated by Γ_S in a neighborhood of Q. We are going to show that, for each $s \in Q^0_{Y'(Q)}$,

$$\|\Delta^{\psi}_{Q,s} h_x\|_2 \lesssim \Bigg(\sum_{R\in\mathcal{D}:\, D\subset R\subset B(z_Q, C_1\ell(Q))} \alpha_\mu(C_1 R) + \frac{\operatorname{dist}(x, L_D)}{\ell(D)} \Bigg) \ell(Q)^{n/2}, \tag{75}$$

and Lemma 5.20(b) will follow by taking the average over $s \in Q^0_{Y'(Q)}$.

Fix $Q \in J_2$, so $D \subset B(z_Q, C_a\ell(Q))$ with $\ell(Q) \leq \ell(Q_S)$, and $s \in Q^0_{Y'(Q)}$. Take $Q' \in \mathcal{D}$ such that $\ell(Q) = \ell(Q')$ and $Q \subset B(z_{Q'}, 3\ell(Q))$. Recall that $\operatorname{supp}\psi_{s+Q} \subset CQ$ and $|\nabla\psi_{s+Q}| \lesssim \ell(Q)^{-n/2-1}$. Let $\phi_{s+Q'}$ be an extension of ψ_{s+Q}, i.e., let $\phi_{s+Q'} : \mathbb{R}^d \to \mathbb{R}$ be such that $\operatorname{supp}\phi_{s+Q'} \subset B_{Q'} \subset \mathbb{R}^d$, $|\nabla\phi_{s+Q'}| \lesssim \ell(Q')^{-n/2-1}$ and $\phi_{s+Q'} = \psi_{s+Q}$ in L^x_D.

Let $L_{Q'}$ be a minimizing n-plane for $\alpha_\mu(C_1 Q')$, where $C_1 > 1$ is some large constant to be fixed below, and let $L^x_{Q'}$ be the n-plane parallel to $L_{Q'}$ which contains x. Let $\sigma_{Q'} := c_{Q'}\mathcal{H}^n_{L_{Q'}}$ be a minimizing measure for $\alpha_\mu(C_1 Q')$ and define $\sigma^x_{Q'} := c_{Q'}\mathcal{H}^n_{L^x_{Q'}}$. Finally, set $\sigma := c_{Q'}\mathcal{H}^n_{L^x_D}$. Since ψ_{s+Q} has vanishing integral in L^x_D, we also have $\int \phi_{s+Q'} \, d\mathcal{H}^n_{L^x_D} = 0$. Hence,

$$\|\Delta^{\psi}_{Q,s} h_x\|_2 = \|\langle h_x, \psi_{s+Q}\rangle \psi_{s+Q}\|_2 = |\langle h_x, \psi_{s+Q}\rangle| = \left| \int_{L^x_D} \phi_{s+Q}(y) \, d\nu_x(y) \right|$$
$$= \left| \int \phi_{s+Q'}(y) \, d(\nu_x - \sigma)(y) \right| \lesssim \ell(Q)^{-n/2-1} \operatorname{dist}_{B_{Q'}}(\nu_x, \sigma). \tag{76}$$

We can assume that

$$\sum_{R\in\mathcal{D}:\, D\subset R\subset B(z_Q, C_1\ell(Q))} \alpha_\mu(C_1 R) \leq b_*, \tag{77}$$

otherwise Lemma 5.20(b) follows easily. By assuming (77) one can show that the angle between L^x_D and $L^x_{Q'}$ is small. By the triangle inequality, we have

$$\operatorname{dist}_{B_{Q'}}(\nu_x, \sigma) \leq \operatorname{dist}_{B_{Q'}}(\nu_x, p^x_\sharp \sigma^x_{Q'}) + \operatorname{dist}_{B_{Q'}}(p^x_\sharp \sigma^x_{Q'}, \sigma). \tag{78}$$

To deal with the first term on the right hand side of (78), let h be a Lipschitz function such that $\operatorname{supp} h \subset B_{Q'}$ and $\operatorname{Lip}(h) \le 1$. Then, since $\operatorname{supp} \mu$ is well approximated in CQ' by a Lipschitz graph Γ_S with small slope, the function $h \circ p^x$ restricted to $\operatorname{supp} \mu \cup L_Q^x$ can be extended to a Lipschitz function supported in $B_{C_1 Q'}$ (if C_1 is large enough) with $\operatorname{Lip}(h \circ p^x)$ bounded by a constant which only depends on n, d, and $\operatorname{Lip}(\Gamma_S)$. Therefore,

$$\begin{aligned}\left|\int_{B_{Q'}} h\, d(\nu_x - p^x_\sharp \sigma^x_{Q'})\right| &= \left|\int_{B_{C_1 Q'}} h \circ p^x\, d(\mu - \sigma^x_{Q'})\right| \lesssim \operatorname{dist}_{B_{C_1 Q'}}(\mu, \sigma^x_{Q'}) \\ &\le \operatorname{dist}_{B_{C_1 Q'}}(\mu, \sigma_{Q'}) + \operatorname{dist}_{B_{C_1 Q'}}(\sigma_{Q'}, \sigma^x_{Q'}) \lesssim \alpha_\mu(C_1 Q')\ell(Q)^{n+1} + \operatorname{dist}(x, L_{Q'})\ell(Q)^n. \end{aligned} \tag{79}$$

Since $x \in D$ and $D \subset C_1 Q'$ (if $C_1 > C_b$), by [To4, Remark 5.3] we have

$$\operatorname{dist}(x, L_{Q'}) \lesssim \sum_{R \in \mathcal{D}:\, D \subset R \subset C_1 Q'} \alpha_\mu(R)\ell(R) + \operatorname{dist}(x, L_D). \tag{80}$$

Taking the supremum over all possible Lipschitz functions h in (79) and using $\ell(D) \le \ell(R) \lesssim \ell(Q)$ in the sum above, we get

$$\operatorname{dist}_{B_{Q'}}(\nu_x, p^x_\sharp \sigma^x_{Q'}) \lesssim \sum_{R \in \mathcal{D}:\, D \subset R \subset C_1 Q'} \alpha_\mu(C_1 R)\ell(Q)^{n+1} + \frac{\operatorname{dist}(x, L_D)}{\ell(D)}\, \ell(Q)^{n+1}. \tag{81}$$

To estimate the second term on the right hand side of (78), notice that $p^x_\sharp \sigma = \sigma$ because $p^x|_{L^x_D} = \mathrm{Id}$. Hence, as in (79),

$$\begin{aligned}\operatorname{dist}_{B_{Q'}}(p^x_\sharp \sigma^x_{Q'}, \sigma) &= \operatorname{dist}_{B_{Q'}}(p^x_\sharp \sigma^x_{Q'}, p^x_\sharp \sigma) \lesssim \operatorname{dist}_{B_{C_1 Q'}}(\sigma^x_{Q'}, \sigma) \\ &\le \operatorname{dist}_{B_{C_1 Q'}}(\sigma^x_{Q'}, \sigma_{Q'}) + \operatorname{dist}_{B_{C_1 Q'}}(\sigma_{Q'}, \sigma) \\ &\lesssim \operatorname{dist}_{B_{C_1 Q'}}(\mathcal{H}^n_{L^x_{Q'}}, \mathcal{H}^n_{L_{Q'}}) + \operatorname{dist}_{B_{C_1 Q'}}(\mathcal{H}^n_{L_{Q'}}, \mathcal{H}^n_{L_D}) + \operatorname{dist}_{B_{C_1 Q'}}(\mathcal{H}^n_{L_D}, \mathcal{H}^n_{L^x_D}) \\ &\lesssim \operatorname{dist}(x, L_{Q'})\ell(Q)^n + \operatorname{dist}_{B_{C_1 Q'}}(\mathcal{H}^n_{L_{Q'}}, \mathcal{H}^n_{L_D}) + \operatorname{dist}(x, L_D)\ell(Q)^n. \end{aligned}$$

The term $\operatorname{dist}_{B_{C_1 Q'}}(\mathcal{H}^n_{L_{Q'}}, \mathcal{H}^n_{L_D})$ can be estimated using the intermediate μ-cubes between D and $C_1 Q'$ (similarly to (81)), and we obtain

$$\operatorname{dist}_{B_{C_1 Q}}(\mathcal{H}^n_{L_Q}, \mathcal{H}^n_{L_D}) \lesssim \sum_{R \in \mathcal{D}:\, D \subset R \subset C_1 Q} \alpha_\mu(C_1 R)\ell(Q)^{n+1}.$$

Thus, by (80) and since $\ell(D) \lesssim \ell(Q)$,

$$\operatorname{dist}_{B_{Q'}}(p^x_\sharp \sigma^x_{Q'}, \sigma) \lesssim \sum_{R \in \mathcal{D}:\, D \subset R \subset C_1 Q'} \alpha_\mu(C_1 R)\ell(Q)^{n+1} + \frac{\operatorname{dist}(x, L_D)}{\ell(D)}\ell(Q)^{n+1}.$$

Then (75) follows by plugging this last inequality and (81) in (78) combined with (76), and recalling that $\ell(Q) \approx \ell(Q')$. □

Proof of Lemma 5.20(c). Given $Q \in J_1 \cap \mathrm{Tr}$, using (65) we have

$$\sum_{R' \in \mathcal{D}: R \subset R',\, \ell(R') \leq \ell(D)} \alpha_\mu(10R') < b_*$$

for all $R \in \mathcal{D}$ with $\ell(R) = \ell(Q)$ and such that $R \cap (p^x)^{-1}(\operatorname{supp} \psi_{s+Q}) \neq \emptyset$ for all $s \in Q^0_{Y(Q)}$. By assuming b_* small enough, we are going to show that for some $Q_0(x, Q) \in \mathcal{D}$ as in statement (c) and all $s \in Q^0_{Y(Q)}$ we have

$$\|\Delta^\psi_{Q,s} h_x\|_2 \lesssim \left(\sum_{R \in \mathcal{D}:\, Q_0 \subset R \subset C_2 D} \alpha_\mu(C_2 R) + \frac{\operatorname{dist}(x, L_D)}{\ell(D)} \right) \ell(Q)^{n/2}. \tag{82}$$

As before, Lemma 5.20(c) will follow by averaging over $s \in Q^0_{Y(Q)}$, and noting that $\|m_{s \in Q^0_{Y(Q)}} (\Delta^\psi_{Q,s} h_x)\|_2 \leq m_{s \in Q^0_{Y(Q)}} \|\Delta^\psi_{Q,s} h_x\|_2$ by Minkowski's integral inequality.

Take $Q \in J_1 \cap \mathrm{Tr}$. Let C_2 be some large constant which will be fixed later on, and let $Q_0 \in \mathcal{D}$ be a minimal μ-cube such that $C_2 Q_0$ contains $\operatorname{supp} \mu \cap (p^x)^{-1}(\operatorname{supp} \psi_{s+Q} \cap L^x_D)$ for all $s \in Y(Q)$. We can assume that $Q_0 \subset C_2 D$ if C_2 is large enough and, by (65), we may also suppose that $\sum_{R \in \mathcal{D}:\, Q_0 \subset R \subset C_2 D} \alpha_\mu(C_2 R)$ is small enough. Hence, if L_{Q_0} is a minimizing n-plane for $\beta_{\infty,\mu}(C_2 Q_0)$, the angle between L_{Q_0} and L^x_D is also small enough, since it is bounded by $\sum_{R \in \mathcal{D}:\, Q_0 \subset R \subset C_2 D} \alpha_\mu(C_2 R)$ (see [To4, Lemma 5.2] for a related argument). It is not hard to show that then

$$\operatorname{diam}(\Gamma \cap (p^x)^{-1}(Q \cap L^x_D)) \lesssim \ell(Q). \tag{83}$$

Let L_{Q_0} and $\sigma_{Q_0} := c_{Q_0} \mathcal{H}^n_{L_{Q_0}}$ be a minimizing n-plane and measure for $\alpha_\mu(C_2 Q_0)$, respectively. Fix $z_{Q_0} \in L_{Q_0} \cap B_{C_2 Q_0}$ and let L_r be an n-plane parallel to L^x_D which contains z_{Q_0}. Finally, define the measures $\sigma_r := c_{Q_0} \mathcal{H}^n_{L_r}$ and $\sigma' := c_{Q_0} \mathcal{H}^n_{L^x_D}$.

Since σ' is a multiple of $\mathcal{H}^n_{L^x_D}$, similarly to (76) and using the triangle inequality,

$$\begin{aligned} \|\Delta^\psi_{Q,s} h_x\|_2 \ell(Q)^{n/2+1} &\lesssim \operatorname{dist}_{B_Q}(\nu_x, \sigma') \\ &\leq \operatorname{dist}_{B_Q}(\nu_x, p^x_\sharp \sigma_{Q_0}) + \operatorname{dist}_{B_Q}(p^x_\sharp \sigma_{Q_0}, p^x_\sharp \sigma_r) + \operatorname{dist}_{B_Q}(p^x_\sharp \sigma_r, \sigma'), \end{aligned} \tag{84}$$

where we have set $B_Q := B(z_Q, 3\ell(Q)) \subset \mathbb{R}^d$ (for these computations, we may also assume that $\ell(Q)$ is small enough in comparison with $\ell(D)$).

Arguing as in (79), if C_2 is large enough, we have

$$\operatorname{dist}_{B_Q}(\nu_x, p^x_\sharp \sigma_{Q_0}) = \operatorname{dist}_{B_Q}(p^x_\sharp \mu, p^x_\sharp \sigma_{Q_0}) \lesssim \alpha_\mu(C_2 Q_0) \ell(Q)^{n+1}, \tag{85}$$

and

$$\operatorname{dist}_{B_Q}(p^x_\sharp \sigma_{Q_0}, p^x_\sharp \sigma_r) \lesssim \operatorname{dist}_{B_{C_2 Q_0}}(\sigma_{Q_0}, \sigma_r) \lesssim \operatorname{dist}_{\mathcal{H}}(L_{Q_0} \cap B_{C_2 Q_0}, L_r \cap B_{C_2 Q_0}) \ell(Q)^n.$$

Let γ be the angle between L_r and L_{Q_0} (which is the same as the one between L_D and L_{Q_0}). Since $z_{Q_0} \in L_{Q_0} \cap L_r \cap B_{C_2 Q_0}$, we have $\operatorname{dist}_{\mathcal{H}}(L_{Q_0} \cap B_{C_2 Q_0}, L_r \cap B_{C_2 Q_0}) \lesssim \sin(\gamma) \ell(Q)$, and it is not difficult to show that $\sin(\gamma) \lesssim \sum_{R \in \mathcal{D}:\, Q_0 \subset R \subset C_2 D} \alpha_\mu(C_2 R)$.

Thus,

$$\operatorname{dist}_{B_Q}(p^x_\sharp \sigma_{Q_0}, p^x_\sharp \sigma_r) \lesssim \sum_{R\in\mathcal{D}:\, Q_0\subset R\subset C_2 D} \alpha_\mu(C_2 R)\ell(Q)^{n+1}. \tag{86}$$

Let us estimate the last term on the right hand side of (84). Since $c_{Q_0} \lesssim 1$, we have $\operatorname{dist}_{B_Q}(p^x_\sharp \sigma_r, \sigma') \lesssim \operatorname{dist}_{B_Q}(p^x_\sharp \mathcal{H}^n_{L_r}, \mathcal{H}^n_{L^x_D})$. Let h be a 1-Lipschitz function supported in B_Q and such that

$$\operatorname{dist}_{B_Q}(p^x_\sharp \mathcal{H}^n_{L_r}, \mathcal{H}^n_{L^x_D}) \approx \left| \int h\, d(p^x_\sharp \mathcal{H}^n_{L_r} - \mathcal{H}^n_{L^x_D}) \right|.$$

Set $d := \operatorname{dist}(z_{Q_0}, L^x_D)$. Since $Q \in J_1 \subset J$ and $\ell(Q) \leq C\ell(D)$, if C is small enough then $\operatorname{dist}(x, B_Q) \gtrsim \ell(D)$. Without loss of generality, we may assume that $x = 0$ and that $L^x_D = \mathbb{R}^n \times \{0\}^{d-n}$, so $L_r = z_{Q_0} + \mathbb{R}^n \times \{0\}^{d-n}$. Thus, if we set $z'_{Q_0} := (z^{n+1}_{Q_0}, \dots, z^d_{Q_0})$, we see that $d = |z'_{Q_0}|$ and p^x restricted to $L_r \cap B_Q$ can be written in the following manner: $p^x : y = (y^1, \dots, y^n, z'_{Q_0}) \mapsto (F(y^1, \dots, y^n), 0)$, where $F : \mathbb{R}^n \setminus \{0\}^n \to \mathbb{R}^n$ is defined by

$$F(y) = y\frac{\sqrt{|y|^2 + d^2}}{|y|} = y\sqrt{1 + \frac{d^2}{|y|^2}}.$$

Therefore, $\int h\, d(p^x_\sharp \mathcal{H}^n_{L_r}) = \int h \circ p^x\, d\mathcal{H}^n_{L_r} = \int_{\mathbb{R}^n} (h \circ p^x)(y, z'_{Q_0})\, dy = \int_{\mathbb{R}^n} h(F(y), 0)\, dy$, and we also have $\int h\, d\mathcal{H}^n_{L^x_D} = \int_{\mathbb{R}^n} h((y,0))\, dy = \int_{\mathbb{R}^n} h(F(y), 0) J(F)(y)\, dy$ by a change of variables, where $J(F)$ denotes the Jacobian of F. Hence

$$\left| \int h\, d(p^x_\sharp \mathcal{H}^n_{L_r} - \mathcal{H}^n_{L^x_D}) \right| \lesssim \int_{\mathbb{R}^n} |h(F(y), 0)|\, |1 - J(F)(y)|\, dy. \tag{87}$$

Notice that, because of the assumptions on $\operatorname{supp} h(F(\cdot), 0)$ and since $z_{Q_0} \in B_{C_2 Q_0}$ and $Q_0 \subset C_2 D$, we have $d \lesssim |y|$ for all $y \in \operatorname{supp} h(F(\cdot), 0)$. If F_i denotes the ith coordinate of F, it is straightforward to check that $\partial_{y^j} F_i(y) = -d^2 y^i y^j |y|^{-3}(|y|^2 + d^2)^{-1/2}$ if $i \neq j$ and $\partial_{y^i} F_i(y) = (1 + d^2/|y|^2)^{1/2} - d^2 (y^i)^2 |y|^{-3}(|y|^2 + d^2)^{-1/2}$. Thus, we easily obtain

$$|1 - J(F)(y)| \lesssim d/|y| \lesssim d/\ell(D) \tag{88}$$

for all $y \in \operatorname{supp} h(F(\cdot), 0)$. Since $\operatorname{diam}(\operatorname{supp} h(F(\cdot), 0)) \lesssim \ell(Q)$ and $h((F(\cdot), 0))$ is Lipschitz, using (88) and taking the supremum in (87) over all such functions h, we have $\operatorname{dist}_{B_Q}(p^x_\sharp \mathcal{H}^n_{L_r}, \mathcal{H}^n_{L^x_D}) \lesssim \ell(Q)^{n+1} d/\ell(D)$. Finally, by [To4, Remark 5.3] and since $z_{Q_0} \in L_{Q_0}$,

$$d \lesssim \operatorname{dist}(z_{Q_0}, L_D) + \operatorname{dist}(L_D, L^x_D) \lesssim \sum_{R\in\mathcal{D}:\, Q_0\subset R\subset C_2 D} \alpha_\mu(C_2 R)\ell(R) + \operatorname{dist}(x, L_D),$$

and thus

$$\begin{aligned}\operatorname{dist}_{B_Q}&(p^x_\sharp \mathcal{H}^n_{L_r}, \mathcal{H}^n_{L^x_D}) \\ &\lesssim \sum_{R\in\mathcal{D}:\, Q_0\subset R\subset C_2 D} \alpha_\mu(C_2 R)\ell(Q)^{n+1} + \frac{\operatorname{dist}(x, L_D)}{\ell(D)}\, \ell(Q)^{n+1}.\end{aligned} \tag{89}$$

Finally, (82) follows by applying (85), (86), and (89) to (84). □

Proof of Lemma 5.20(d). As remarked just before (60), this is the key point where taking averages of dyadic lattices with respect to the parameter s is necessary. Given Q in $J_1 \cap \mathrm{Stp}$, we have to show that $\|m_{s\in Q^0_{Y(Q)}}(\widetilde{\Delta}^{\psi}_{Q,s} h_x)\|_1 \lesssim \ell(Q)^n$. Unlike (a)–(c), the estimate in (d) does not hold for a particular choice of s in general but, as we will see, it holds on average. Recall that, for a fixed $s \in Q^0_{Y(Q)}$,

$$\begin{aligned}\widetilde{\Delta}^{\psi}_{Q,s} h_x &= \sum_{R\in\mathcal{P}:\, R\subset Q} \Delta^{\psi}_{R,s} h_x \\ &= \sum_{\substack{R\in\mathcal{P}:\,\mathrm{supp}\,\psi_R\cap Q\neq\emptyset\\ \ell(R)\leq\ell(Q)}} \chi_{s+Q}\, \Delta^{\psi}_{R,s} h_x - \sum_{\substack{R\in\mathcal{P}:\,\mathrm{supp}\,\psi_R\cap Q\neq\emptyset\\ \ell(R)\leq\ell(Q),\, R\not\subset Q}} \chi_{s+Q}\, \Delta^{\psi}_{R,s} h_x \\ &\quad + \sum_{\substack{R\in\mathcal{P}:\\ R\subset Q}} \chi_{(s+Q)^c}\, \Delta^{\psi}_{R,s} h_x =: I_s + II_s + III_s.\end{aligned}$$

We are going to estimate I_s, II_s, and III_s separately. For the case of I_s, we have

$$\begin{aligned}\chi_{s+Q}\, h_x &= \chi_{s+Q} \sum_{R\in\mathcal{D}^{n,0}_x:\,\ell(R)>\ell(Q)} \Delta^{\psi}_{R,s} h_x + \chi_{s+Q} \sum_{R\in\mathcal{D}^{n,0}_x:\,\ell(R)\leq\ell(Q)} \Delta^{\psi}_{R,s} h_x \\ &= \chi_{s+Q}\, I'_s + I_s,\end{aligned}$$

where we have set $I'_s := \sum_{R\in\mathcal{D}^{n,0}_x:\,\ell(R)>\ell(Q)} \Delta^{\psi}_{R,s} h_x$. On the one hand, since $Q \in J_1\cap\mathrm{Stp}$, (65) holds. Thus, using $\sum_{R\in\mathcal{D}:\,P(R_Q)\subset R,\,\ell(R)\leq\ell(D)} \alpha_\mu(10R) < b_*$, one can show that

$$\|\chi_{s+Q}\, h_x\|_1 \lesssim \ell(Q)^n \tag{90}$$

(see above (83) for a related argument). On the other hand, since $\|\chi_{s+Q}\, h_x\|_1 \lesssim \ell(Q)^n$, it is known that then $\|\chi_{s+Q} I'_s\|_1 \lesssim \ell(Q)^n$ (see [Da, Part I], in particular pay attention to the last sum in equation (46) of Part I). Combining these estimates, we conclude that $\|I_s\|_1 \lesssim \ell(Q)^n$.

Let us now deal with II_s. First of all, we split II_s into different scales, that is,

$$\sum_{\substack{R\in\mathcal{P}:\,\mathrm{supp}\,\psi_R\cap Q\neq\emptyset\\ \ell(R)\leq\ell(Q),\, R\not\subset Q}} \chi_{s+Q}\, \Delta^{\psi}_{R,s} h_x = \sum_{k\geq Y(Q)} \sum_{\substack{R\in\mathcal{P}:\,\mathrm{supp}\,\psi_R\cap Q\neq\emptyset\\ \ell(R)=2^{-k},\, R\not\subset Q}} \chi_{s+Q}\, \Delta^{\psi}_{R,s} h_x.$$

Observe that if $k \geq Y(Q)$, $\mathrm{supp}\,\psi_R \cap Q \neq \emptyset$, $\ell(R) = 2^{-k}$, and $R \not\subset Q$, then $s + R \subset U_{C2^{-k}}(s + \partial Q)$, where $C > 1$ is some fixed constant and $U_{C2^{-k}}(s + \partial Q) := \{z \in L^x_D : \mathrm{dist}(z, s + \partial Q) < C2^{-k}\}$. Hence, using Definition 5.16(e) and the definition of h_x, we get

$$\|II_s\|_1 \leq \sum_{k\geq Y(Q)} \sum_{\substack{R\in\mathcal{P}:\,\mathrm{supp}\,\psi_R\cap Q\neq\emptyset\\ \ell(R)=2^{-k},\, R\not\subset Q}} \|\Delta^{\psi}_{R,s} h_x\|_1 \lesssim \sum_{k\geq Y(Q)} \nu_x(U_{C2^{-k}}(s + \partial Q)).$$

The case of III_s can be dealt with using very similar techniques, and one obtains the same estimate. Therefore,

$$\|m_{s\in Q^0_{Y(Q)}}(\widetilde{\Delta}^{\psi}_{Q,s}h_x)\|_1 = \|m_{s\in Q^0_{Y(Q)}}(I_s+II_s+III_s)\|_1 \le m_{s\in Q^0_{Y(Q)}}\|I_s+II_s+III_s\|_1$$
$$\lesssim \ell(Q)^n + m_{s\in Q^0_{Y(Q)}}\Big(\sum_{k\ge Y(Q)} \nu_x(U_{C2^{-k}}(s+\partial Q))\Big). \tag{91}$$

Using Fubini's theorem, it is not difficult to show that

$$m_{s\in Q^0_{Y(Q)}}\nu_x(U_{C2^{-k}}(s+\partial Q))) \lesssim 2^{-k}\ell(Q)^{-1}\nu_x(CQ) \tag{92}$$

for all $k \ge Y(Q)$ (see [To2, Lemma 7.5] for example, for a related argument). Since $Q\in \mathrm{Stp}$, (65) holds, and so, as in (90), we have $\nu_x(CQ)\lesssim \ell(Q)^n$, thus

$$m_{s\in Q^0_{Y(Q)}}\Big(\sum_{k\ge Y(Q)}\nu_x(U_{C2^{-k}}(s+\partial Q))\Big) \lesssim \ell(Q)^n.$$

If we combine this last estimate with (91), we are done. □

5.3.6. Final estimates. From Lemmas 5.4, 5.10, and 5.14, we obtain

$$\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D \sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}\sum_{Q\in\mathrm{Tr}(R)}(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (\Delta_Q f)\mu)(x)\Big|^2 d\mu(x)$$
$$+\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D \sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}\sum_{Q\in\mathrm{Stp}(R)}(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (\widetilde{\Delta}_Q f)\mu)(x)\Big|^2 d\mu(x)$$
$$+\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D \sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (m^{\mu}_R f)\chi_R\mu)(x)\Big|^2 d\mu(x) \lesssim \|f\|^2_{L^2(\mu)}.$$

Combining this estimate with (39), we deduce

$$\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D \sum_{m\in\mathcal{S}_D(x)}|(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (f\mu))(x)|^2 d\mu(x) \lesssim \|f\|^2_{L^2(\mu)}.$$

Finally, using (35) and (36), we conclude that

$$\|(\mathcal{V}^{\mathcal{S}}_\rho\circ\mathcal{T}^\mu)f\|^2_{L^2(\mu)} \lesssim \sum_{D\in\mathcal{D}}\int_D\sum_{m\in\mathcal{S}_D(x)}|(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (f\mu))(x)|^2 d\mu(x) \lesssim \|f\|^2_{L^2(\mu)}.$$

This finishes the proof of Theorem 5.1.

6. If $\mathcal{V}_\rho \circ \mathcal{R}^\mu : L^2(\mu) \to L^2(\mu)$ is a bounded operator, then μ is a uniformly n-rectifiable measure

Let $C_\mu > 0$ be the AD regularity constant of an AD regular measure μ, that is, $C_\mu^{-1} r^n \leq \mu(B(x,r)) \leq C_\mu r^n$ for all $x \in \operatorname{supp}\mu$ and $0 \leq r < \operatorname{diam}(\operatorname{supp}\mu)$. For simplicity of notation, we may assume that $\operatorname{diam}(\operatorname{supp}\mu) = \infty$ (the general case follows by minor modifications in our arguments). As before, we denote by $\mathcal{D}$ the dyadic lattice of μ-cubes introduced in Subsection 2.3.

In this section, we set $K(x) = x|x|^{-n-1}$ for $x \neq 0$. Recall that, given $\epsilon > 0$, a Radon measure μ, and $f \in L^1(\mu)$, we have set $\mathcal{R}^\mu f := \{R_\epsilon^\mu f\}_{\epsilon>0}$, where

$$R_\epsilon^\mu f(x) = \int_{|x-y|>\epsilon} K(x-y) f(y)\, d\mu(y).$$

In order to prove the main theorem of this section (Theorem 6.8), we need first to introduce some notation and state some preliminary results.

Definition 6.1 (Special truncation of the Riesz transform). For $\epsilon > 0$, let φ_ϵ be as in Definition 2.1. Given $m \in \mathbb{Z}$ and a Radon measure μ in $\mathbb{R}^d$, we set

$$S_m\mu(x) := \int (\varphi_{2^{-m-1}}(x-y) - \varphi_{2^{-m}}(x-y)) K(x-y)\, d\mu(y).$$

Lemma 6.2 ([DS1, Lemma 5.8]). *Given $Q \in \mathcal{D}$, there exist $n+1$ points $x_0, \ldots, x_n$ in Q (and thus in $\operatorname{supp}\mu$) such that $\operatorname{dist}(x_j, L_{j-1}) \geq C\ell(Q)$, where L_k denotes the k-plane passing through $x_0, \ldots, x_k$, and where C depends only on n and C_μ.*

Lemma 6.3 ([To4, Lemma 7.4 and Remark 7.5]). *Let $Q \in \mathcal{D}$ and $x_0, \ldots, x_n \in Q$ be as in Lemma 6.2. Denote $r = \operatorname{diam}(Q)$, and let $m, p \in \mathbb{Z}$ be such that $t \geq s > 4r$ for $t = 2^{-p}$ and $s = 2^{-m}$. Suppose that $A(x_0, 2^{-m-1/2}, 2^{-m+1/2}) \cap \operatorname{supp}\mu \neq \emptyset$. Then any point $x_{n+1} \in 3Q$ satisfies*

$$\operatorname{dist}(x_{n+1}, L_0) \lesssim s \sum_{j=1}^{n+1} \sum_{k=p}^{m} |S_k\mu(x_j) - S_k\mu(x_0)| + \frac{r^2}{s} + \frac{rs}{t}, \tag{93}$$

where L_0 is the n-plane passing through $x_0, \ldots, x_n$.

The following proposition is a direct consequence of the techniques used in the last section of [To4]. We give the proof for completeness.

Proposition 6.4. *Given $\epsilon_0 > 0$, there exist $\delta_0 > 0$ and $m_0, k_0 \in \mathbb{N}$ depending on ϵ_0, n, and C_μ such that, for all $i \in \mathbb{Z}$ and all $Q \in \mathcal{D}_i$ with $\beta_{1,\mu}(Q) > \epsilon_0$, there exist $k \in \mathbb{Z}$ with $|k| \leq k_0$ and $P \in \mathcal{D}_{i+k+m_0}$ such that $P \subset 4Q$ and $|S_{i+k}\mu(x)| \geq \delta_0$ for all $x \in P$.*

Proof. Fix $\epsilon_0 > 0$. Let $Q \in \mathcal{D}_i$ be such that $\beta_{1,\mu}(Q) > \epsilon_0$. Take $x_0, \dots, x_n$ in Q as in Lemma 6.2, denote $r = \operatorname{diam} Q$, and let $m \in \mathbb{Z}$ (to be fixed below) be such that $4r < 2^{-m} =: s$ and $A(x_0, 2^{-m-1/2}, 2^{-m+1/2}) \cap \operatorname{supp}\mu \neq \emptyset$ (we assume $\operatorname{diam}(\operatorname{supp}\mu) = \infty$). By Lemma 6.3, for $t := 2^{-p} \geq s$ to be fixed below and all $x_{n+1} \in 3Q$,

$$\begin{aligned}\operatorname{dist}(x_{n+1}, L_0) &\lesssim s \sum_{j=1}^{n+1} \sum_{k=p}^{m} |S_k\mu(x_j) - S_k\mu(x_0)| + \frac{r^2}{s} + \frac{rs}{t} \\ &\lesssim s \sum_{k=p}^{m} \sum_{j=0}^{n+1} |S_k\mu(x_j)| + \frac{r^2}{s} + \frac{rs}{t}.\end{aligned}$$

Then by integrating over $x_{n+1} \in 3Q$, for some constant $C_1 > 0$ depending only on n and C_μ we have

$$\begin{aligned}\epsilon_0 < \beta_{1,\mu}(Q) &\leq \frac{1}{\ell(Q)^n} \int_{3Q} \frac{\operatorname{dist}(x_{n+1}, L_0)}{\ell(Q)}\, d\mu(x_{n+1}) \\ &\leq C_1 \left(\frac{s}{r} \sum_{k=p}^{m} \left(\frac{1}{\ell(Q)^n} \int_{3Q} |S_k\mu(x_{n+1})|\, d\mu(x_{n+1}) + \sum_{j=0}^{n} |S_k\mu(x_j)| \right) + \frac{r}{s} + \frac{s}{t} \right).\end{aligned}$$

Thus,

$$\frac{r}{s}\left(\frac{\epsilon_0}{C_1} - \frac{r}{s} - \frac{s}{t} \right) \leq \sum_{k=p}^{m} \left(\int_{3Q} \frac{|S_k\mu(x_{n+1})|}{\ell(Q)^n}\, d\mu(x_{n+1}) + \sum_{j=0}^{n} |S_k\mu(x_j)| \right).$$

We can easily choose s and t large enough (depending on r, ϵ_0, and C_1) such that, for some constant $\epsilon_1 > 0$ depending only on ϵ_0, n and C_μ,

$$0 < \epsilon_1 \leq \sum_{k=p}^{m} \left(\int_{3Q} \frac{|S_k\mu(x_{n+1})|}{\ell(Q)^n}\, d\mu(x_{n+1}) + \sum_{j=0}^{n} |S_k\mu(x_j)| \right). \tag{94}$$

Notice that, since $t = 2^{-p}$ and $s = 2^{-m}$ were chosen depending on $r \approx 2^{-i}$, the sum on the right hand side of (94) has a finite number of terms which only depends on ϵ_0, n and C_μ. Therefore, there exist $k_0 \in \mathbb{N}$ and $C_2 > 0$ depending only on ϵ_0, n and C_μ such that, for some negative integer k with $|k| \leq k_0$ and some $j = 0, \dots, n$,

$$\epsilon_1 \leq C_2 \left(\frac{1}{\ell(Q)^n} \int_{3Q} |S_{i+k}\mu|\, d\mu + |S_{i+k}\mu(x_j)| \right),$$

which implies that there exist C_3 (depending on C_2) and $z \in 3Q$ such that $\epsilon_1 \leq C_3 |S_{i+k}\mu(z)|$.

Given $x \in \operatorname{supp}\mu$, if $|x - z| \leq 2^{-i-k}$, then

$$\begin{aligned}|S_{i+k}\mu(x) - S_{i+k}\mu(z)| &\leq \int_{|y-z| \lesssim 2^{-i-k}} \|\nabla(\varphi_{i+k} K)\|_\infty |x - z|\, d\mu(y) \\ &\lesssim 2^{(i+k)(n+1)} |x - z| \int_{|y-z| \lesssim 2^{-i-k}} d\mu(y) \lesssim 2^{i+k} |x - z|.\end{aligned}$$

Hence if $|x-z| \leq C_4 2^{-i-k}$ with $C_4 > 0$ small enough, we have $C_3|S_{i+k}\mu(x)-S_{i+k}\mu(z)| \leq \epsilon_1/2$, so $\epsilon_1/2 \leq C_3|S_{i+k}\mu(x)|$. Therefore, there exist $m_0 \in \mathbb{N}$ depending on C_4 (and thus on ϵ_0, n, and C_μ) and $P \in \mathcal{D}_{i+k+m_0}$ such that $\epsilon_1/2 \leq C_3|S_{i+k}\mu(x)|$ for all $x \in P$. We can also assume that $P \subset 4Q$ by taking C_4 small enough, and since $|k| \leq k_0$ we have $\ell(P) \approx \ell(Q)$. The proposition follows by setting $\delta_0 := \epsilon_1/(2C_3) > 0$. □

Definition 6.5. Given $\epsilon_0 > 0$, let $\delta_0, m_0 > 0$ be as in Proposition 6.4. Set

$$\mathcal{B} := \{Q \in \mathcal{D} : \beta_{1,\mu}(Q) > \epsilon_0\}, \qquad \widetilde{\mathcal{B}} := \bigcup_{k\in\mathbb{Z}} \{Q \in \mathcal{D}_{k+m_0} : |S_k\mu(x)| \geq \delta_0 \text{ for all } x \in Q\}.$$

Given $P, R \in \mathcal{D}$ with $P \subset R$, we set $F_P^R = \sum_{Q\in\widetilde{\mathcal{B}}:\, P\subset Q\subset R} \chi_Q$ and $F^R = \sum_{Q\in\widetilde{\mathcal{B}}:\, Q\subset R} \chi_Q$.

Lemma 6.6. *Let $\rho > 0$. Assume that there exists $C_0 > 0$ such that, for all $R \in \mathcal{D}$,*

$$\int_R (F^R)^{2/\rho}\, d\mu \leq C_0\mu(R). \tag{95}$$

Then there exists $C > 0$ such that $\sum_{Q\in\widetilde{\mathcal{B}}:\, Q\subset R} \mu(Q) \leq C\mu(R)$ for all $R \in \mathcal{D}$.

Proof. Let $M > 1$ be large enough (it will be fixed below). For $R \in \mathcal{D}$, set

$$\begin{aligned}
\mathrm{Tree}(R) &:= \{Q \in \widetilde{\mathcal{B}} : Q \subset R,\ \chi_Q F_Q^R \leq M\chi_Q\},\\
\mathrm{Top}_0(R) &:= \{P \in \widetilde{\mathcal{B}} : P \subset R,\ \chi_P F_P^R > M\chi_P, \text{ and } \chi_Q F_Q^R \leq M\chi_Q\\
&\qquad\qquad \text{for all } Q \in \widetilde{\mathcal{B}} \text{ such that } P \subsetneq Q \subset R\}.
\end{aligned}$$

For $m \geq 1$, set $\mathrm{Top}_m(R) := \bigcup_{P\in\mathrm{Top}_{m-1}(R)} \mathrm{Top}_0(P)$, and $\mathrm{Top}(R) := \bigcup_{m\geq 0} \mathrm{Top}_m(P)$. Notice that if $R \in \widetilde{\mathcal{B}}$ then $R \in \mathrm{Tree}(R)$, because $M > 1$. Notice also that

$$\{Q \in \widetilde{\mathcal{B}} : Q \subset R\} = \mathrm{Tree}(R) \cup \bigcup_{P\in\mathrm{Top}(R)} \mathrm{Tree}(P), \tag{96}$$

and the union is disjoint.

Fix $R \in \mathcal{D}$. Then by (96),

$$\begin{aligned}
\sum_{Q\in\widetilde{\mathcal{B}}:\, Q\subset R} \mu(Q) &= \sum_{Q\in\mathrm{Tree}(R)} \mu(Q) + \sum_{P\in\mathrm{Top}(R)} \sum_{Q\in\mathrm{Tree}(P)} \mu(Q)\\
&= \int_R \sum_{Q\in\mathrm{Tree}(R)} \chi_Q\, d\mu + \int_R \sum_{P\in\mathrm{Top}(R)} \sum_{Q\in\mathrm{Tree}(P)} \chi_Q\, d\mu.
\end{aligned} \tag{97}$$

Given $x \in R$ and $P \in \mathcal{D}$ such that $P \subset R$, by the definition of $\mathrm{Tree}(P)$, we have

$$\sum_{Q\in\mathrm{Tree}(P)} \chi_Q(x) \leq M\chi_P(x).$$

Therefore, by (97),

$$\sum_{Q\in\widetilde{\mathcal{B}}:\,Q\subset R}\mu(Q)\leq M\mu(R)+\int_R\sum_{P\in\mathrm{Top}(R)}M\chi_P\,d\mu=M\Big(\mu(R)+\sum_{m\geq 0}\sum_{P\in\mathrm{Top}_m(R)}\mu(P)\Big).\tag{98}$$

We are going to prove that, if M is large enough,

$$\sum_{P\in\mathrm{Top}_m(R)}\mu(P)\leq 2^{-m}\mu(R)\tag{99}$$

for all $m\geq 0$, and then, by (98), we will finally obtain

$$\sum_{Q\in\widetilde{\mathcal{B}}:\,Q\subset R}\mu(Q)\leq M\mu(R)+M\sum_{m\geq 0}2^{-m}\mu(R)\leq 3M\mu(R),$$

and the lemma will be proven.

Notice that, if $P, P'\in\mathrm{Top}_0(R)$ are different, then $P\cap P'=\emptyset$ because of the last condition in the definition of $\mathrm{Top}_0(R)$. So, to verify (99), it is enough to show that, for all $m\geq 0$,

$$\sum_{P\in\mathrm{Top}_{m+1}(R)}\mu(P)<\frac{1}{2}\sum_{P\in\mathrm{Top}_m(R)}\mu(P).\tag{100}$$

We have

$$\sum_{P\in\mathrm{Top}_{m+1}(R)}\mu(P)=\sum_{P\in\mathrm{Top}_m(R)}\sum_{Q\in\mathrm{Top}_0(P)}\mu(Q)\tag{101}$$

and $\sum_{Q\in\mathrm{Top}_0(P)}\chi_Q=\chi_U$, where $U:=\bigcup_{Q\in\mathrm{Top}_0(P)}Q\subset P$. If $x\in U$, there exists $Q\in\mathrm{Top}_0(P)$ such that $x\in Q$, so $1=\chi_Q(x)<M^{-2/\rho}(F_Q^P(x))^{2/\rho}\leq M^{-2/\rho}(F^P(x))^{2/\rho}$, and then using (95) we have

$$\begin{aligned}\sum_{Q\in\mathrm{Top}_0(P)}\mu(Q)&=\int_P\sum_{Q\in\mathrm{Top}_0(P)}\chi_Q\,d\mu\\&=\int_U 1\,d\mu<M^{-2/\rho}\int_P(F^P)^{2/\rho}\,d\mu\leq\frac{C_0}{M^{2/\rho}}\mu(P),\end{aligned}$$

which, in combination with (101), yields (100) by taking $M>(2C_0)^{\rho/2}$. □

Lemma 6.7. *Assume that, for some $C_1>0$, $\sum_{Q\in\widetilde{\mathcal{B}}:\,Q\subset R}\mu(Q)\leq C_1\mu(R)$ for all $R\in\mathcal{D}$. Then there exists $C_2>0$ such that $\sum_{Q\in\mathcal{B}:\,Q\subset R}\mu(Q)\leq C_2\mu(R)$ for all $R\in\mathcal{D}$.*

Proof. Given $Q\in\mathcal{B}$, by Proposition 6.4, there exists $P_Q\in\mathcal{D}_{k+m_0}$ for some $k\in\mathbb{Z}$ such that $P_Q\subset 4Q$, $\mu(P_Q)\geq C_0\mu(Q)$, and $|S_k\mu(x)|\geq\delta_0$ for all $x\in P_Q$, where $C_0>0$ is some small constant. Thus, in particular, $P_Q\in\widetilde{\mathcal{B}}$ for all $Q\in\mathcal{B}$. Since $P_Q\subset 4Q$ and $\mu(P_Q)\geq C_0\mu(Q)$ for all $Q\in\mathcal{B}$, given $P\in\widetilde{\mathcal{B}}$ there are finitely many μ-cubes $Q\in\mathcal{B}$ such that $P_Q=P$, and the number of such μ-cubes is bounded above by a constant depending only on n, C_0, and C_μ. Hence, since $4R$ is contained in the union of a bounded number of μ-cubes with side length $\ell(R)$, we have

$$\sum_{Q\in\mathcal{B}:\,Q\subset R}\mu(Q)\leq C_0^{-1}\sum_{Q\in\mathcal{B}:\,Q\subset R}\mu(P_Q)\lesssim\sum_{P\in\widetilde{\mathcal{B}}:\,P\subset 4R}\mu(P)\leq C_1\mu(R)$$

for all $R\in\mathcal{D}$, as desired. □

Theorem 6.8. *Let $\rho>0$. Given an n-dimensional AD regular measure μ, if $\mathcal{V}_\rho\circ\mathcal{R}^\mu$ is a bounded operator in $L^2(\mu)$, then μ is uniformly n-rectifiable.*

Proof. It is easy to see that, if $\mathcal{V}_\rho\circ\mathcal{R}^\mu$ is a bounded operator in $L^2(\mu)$, then R_*^μ is also bounded in $L^2(\mu)$. By Theorem 1.2 in [DS2, Part III, Chapter 1], in order to show that μ is uniformly n-rectifiable, it is enough to show that μ satisfies the Weak Geometric Lemma, i.e., for any $\epsilon_0>0$, the set $\mathcal{B}$ is a Carleson set. In other words, it suffices to show that there exists a constant $C>0$ depending on ϵ_0 such that $\sum_{Q\in\mathcal{B}:\,Q\subset R}\mu(Q)\leq C\mu(R)$ for all $R\in\mathcal{D}$. By Lemmas 6.7 and 6.6, this holds if, for some $\rho>0$, there exists $C>0$ depending on ϵ_0 such that, for all $R\in\mathcal{D}$,

$$\int_R (F^R)^{2/\rho}\,d\mu\leq C\mu(R). \tag{102}$$

Notice that, for $m\in\mathbb{Z}$ and $f\in L^1(\mu)$, $S_m(f\mu)=T^\mu_{\varphi_{2^{-m-1}}}f-T^\mu_{\varphi_{2^{-m}}}f$, where S_m is introduced in Definition 6.1 and $T^\mu_{\varphi_\epsilon}$ is as in Definition 2.1 (remember that now K denotes the Riesz kernel), thus

$$\sum_{k\in\mathbb{Z}}|S_k(f\mu)(x)|^\rho\leq((\mathcal{V}_\rho\circ\mathcal{T}^\mu_\varphi)f(x))^\rho. \tag{103}$$

We may assume that $\rho\geq 1$, since $(\mathcal{V}_{\widetilde{\rho}}\circ\mathcal{R}^\mu)f(x)\leq(\mathcal{V}_\rho\circ\mathcal{R}^\mu)f(x)$ for $\widetilde{\rho}\geq\rho$, and then the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho\circ\mathcal{R}^\mu$ for some $\rho>0$ implies the $L^2(\mu)$ boundedness of $\mathcal{V}_{\widetilde{\rho}}\circ\mathcal{R}^\mu$ for all $\widetilde{\rho}\geq\rho$. Since $\varphi_{\mathbb{R}}(2^{2m}t^2)$ is a convex combination of the functions $\chi_{\{s\in\mathbb{R}:\,s>\epsilon\}}(t)$ for $\epsilon>0$, using $\rho\geq 1$ and Minkowski's integral inequality it is not hard to show that the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho\circ\mathcal{R}^\mu$ implies the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho\circ\mathcal{T}^\mu_\varphi$ (see Subsection 5.2, or [CJRW1, Lemma 2.4], for a similar argument). Therefore, for any $M>0$, we have

$$\|(\mathcal{V}_\rho\circ\mathcal{T}^\mu_\varphi)\chi_{MR}\|^2_{L^2(\mu)}\leq C\mu(MR)\leq C\mu(R)\quad\text{ for all } R\in\mathcal{D}. \tag{104}$$

Fix $\epsilon_0>0$, let $\delta_0, m_0>0$ be as in Proposition 6.4, and let $R\in\mathcal{D}$. Given $x\in R$ and $k\in\mathbb{Z}$, for any $Q\in\mathcal{D}_{k+m_0}\cap\widetilde{\mathcal{B}}$ such that $x\in Q\subset R$ we have $|S_k\mu(x)|\geq\delta_0$. Notice that, since $Q\in\mathcal{D}_{k+m_0}$ and $Q\subset R$, there exists $M>1$ depending only on n and m_0 such that $\delta_0\leq|S_k\mu(x)|=|S_k(\chi_{MR}\mu)(x)|$. Therefore, using (103) and the fact that for each $k\in\mathbb{Z}$ there is at most one μ-cube $Q\in\mathcal{D}_{k+m_0}$ such that $x\in Q\subset R$, we obtain

$$\begin{aligned}F^R(x)&=\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_{k+m_0}\cap\widetilde{\mathcal{B}}:\,x\in Q\subset R}\chi_Q(x)\leq\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_{k+m_0}\cap\widetilde{\mathcal{B}}:\,x\in Q\subset R}\delta_0^{-\rho}|S_k(\chi_{MR}\mu)(x)|^\rho\\&\leq\delta_0^{-\rho}\sum_{k\in\mathbb{Z}}|S_k(\chi_{MR}\mu)(x)|^\rho\leq\delta_0^{-\rho}((\mathcal{V}_\rho\circ\mathcal{T}^\mu_\varphi)\chi_{MR}(x))^\rho\end{aligned} \tag{105}$$

and then, by (104),

$$\int_R (F^R)^{2/\rho}\,d\mu \le \delta_0^{-2}\int_R \big((\mathcal{V}_\rho\circ\mathcal{T}_\varphi^\mu)\chi_{MR}\big)^2\,d\mu \le \delta_0^{-2}\|(\mathcal{V}_\rho\circ\mathcal{T}_\varphi^\mu)\chi_{MR}\|_{L^2(\mu)}^2 \le C\mu(R)$$

for all $R\in\mathcal{D}$. This yields (102), and the theorem follows. □

Remark 6.9. Let $\{r_m\}_{m\in\mathbb{Z}}\subset(0,\infty)$ be a fixed decreasing sequence defining $\mathcal{O}$. If there exists $C>0$ such that $C^{-1}r_m\le r_m-r_{m+1}\le Cr_m$ for all $m\in\mathbb{Z}$, then the last inequality in (105) still holds if we replace $\mathcal{V}_\rho$ by $\mathcal{O}$ (by taking $\rho=2$ from the beginning). Hence, Theorem 6.8 still holds after replacing $\mathcal{V}_\rho$ by $\mathcal{O}$ for this particular sequence $\{r_m\}_{m\in\mathbb{Z}}$. However, we do not know if it holds for any $\{r_m\}_{m\in\mathbb{Z}}\subset(0,\infty)$.

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