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Observability inequalities and measurable sets

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Abstract. This paper presents two observability inequalities for the heat equation over $\Omega \times (0, T)$. In the first one, the observation is from a subset of positive measure in $\Omega \times (0, T)$, while in the second, the observation is from a subset of positive surface measure on $\partial\Omega \times (0, T)$. It also proves the Lebeau–Robbiano spectral inequality when Ω is a bounded Lipschitz and locally star-shaped domain. Some applications for the above-mentioned observability inequalities are provided.

Keywords. Observability inequality, heat equation, measurable set, spectral inequality

1. Introduction

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and T be a fixed positive time. Consider the heat equation:

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

with u_0 in $L^2(\Omega)$. The solution of (1.1) will be treated as either a function from $[0, T]$ to $L^2(\Omega)$ or a function of two variables x and t. Two important a priori estimates for the above equation are as follows:

$$\|u(T)\|_{L^2(\Omega)} \le N(\Omega, T, \mathcal{D}) \int_{\mathcal{D}} |u(x, t)|\, dx\, dt \quad \text{for all } u_0 \in L^2(\Omega), \tag{1.2}$$

where $\mathcal{D}$ is a subset of $\Omega \times (0, T)$, and

$$\|u(T)\|_{L^2(\Omega)} \le N(\Omega, T, \mathcal{J}) \int_{\mathcal{J}} |\partial_\nu u(x, t)|\, d\sigma\, dt \quad \text{for all } u_0 \in L^2(\Omega), \tag{1.3}$$

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where $\mathcal{J}$ is a subset of $\partial\Omega \times (0, T)$. Such a priori estimates are called *observability inequalities*.

In the case where $\mathcal{D} = \omega\times(0, T)$ and $\mathcal{J} = \Gamma\times(0, T)$ with ω and Γ open and nonempty subsets of Ω and $\partial\Omega$ respectively, both inequalities (1.2) and (1.3) (where $\partial\Omega$ is smooth) were essentially first established, via the Lebeau–Robbiano spectral inequalities, in [28] (see also [29, 36, 16]). These two estimates were set up for linear parabolic equations (where $\partial\Omega$ is of class C^2), based on the Carleman inequality provided in [18]. In the case when $\mathcal{D} = \omega \times (0, T)$ and $\mathcal{J} = \Gamma \times (0, T)$ with ω and Γ subsets of positive measure and positive surface measure in Ω and $\partial\Omega$ respectively, both inequalities (1.2) and (1.3) were proved in [3] with the help of a propagation of smallness estimate from measurable sets for real-analytic functions, first established in [43] (see also Theorem 4). For $\mathcal{D} = \omega \times E$, with ω and E respectively an open subset of Ω and a subset of positive measure in $(0, T)$, the inequality (1.2) (with $\partial\Omega$ is smooth) was proved in [44] with the aid of the Lebeau–Robbiano spectral inequality, and it was then verified for heat equations (where Ω is convex) with lower order terms depending on the time variable, using a frequency function method, in [40]. When $\mathcal{D} = \omega\times E$, with ω and E respectively subsets of positive measure in Ω and $(0, T)$, the estimate (1.2) (with $\partial\Omega$ real-analytic) was obtained in [45].

The purpose of this study is to establish inequalities (1.2) and (1.3) when $\mathcal{D}$ and $\mathcal{J}$ are arbitrary subsets of positive measure and of positive surface measure in $\Omega \times (0, T)$ and $\partial\Omega \times (0, T)$ respectively. Such inequalities not only are mathematically interesting but also have important applications in the control theory of the heat equation, such as bang-bang control, time optimal control, null controllability over a measurable set and so on (see Section 5 for the applications).

The starting point we choose here to prove the above-mentioned two inequalities is to assume that the Lebeau–Robbiano spectral inequality holds on Ω. To introduce it, we write

$$0 < \lambda_1 \le \lambda_2 \le \cdots$$

for the eigenvalues of $-\Delta$ with the zero Dirichlet boundary condition over $\partial\Omega$, and $\{e_j : j \ge 1\}$ for the set of $L^2(\Omega)$-normalized eigenfunctions, i.e.,

$$\begin{cases} \Delta e_j + \lambda_j e_j = 0 & \text{in } \Omega, \\ e_j = 0 & \text{on } \partial\Omega. \end{cases}$$

For $\lambda > 0$ we define

$$\mathcal{E}_\lambda f = \sum_{\lambda_j \le \lambda} (f, e_j) e_j \quad \text{and} \quad \mathcal{E}_\lambda^\perp f = \sum_{\lambda_j > \lambda} (f, e_j) e_j,$$

where

$$(f, e_j) = \int_\Omega f e_j \, dx \quad \text{when } f \in L^2(\Omega),\ j \ge 1.$$

Throughout this paper the following notation is in force:

$$(f, g) = \int_\Omega f g \, dx \quad \text{and} \quad \|f\|_{L^2(\Omega)} = (f, f)^{1/2};$$

ν is the unit exterior normal vector to $\partial\Omega$; $d\sigma$ is surface measure on $\partial\Omega$; $B_R(x_0)$ stands for the ball centered at x_0 in $\mathbb{R}^n$ of radius R; $\Delta_R(x_0)$ denotes $B_R(x_0) \cap \partial\Omega$; $B_R = B_R(0)$, $\Delta_R = \Delta_R(0)$; for measurable sets $\omega \subset \mathbb{R}^n$ and $\mathcal{D} \subset \mathbb{R}^n \times (0, T)$, $|\omega|$ and $|\mathcal{D}|$ stand for the Lebesgue measures of these sets; for each measurable set $\mathcal{J}$ in $\partial\Omega \times (0, T)$, $|\mathcal{J}|$ denotes its surface measure on the lateral boundary of $\Omega \times \mathbb{R}$; $\{e^{t\Delta} : t \geq 0\}$ is the semigroup generated by Δ with zero Dirichlet boundary condition over $\partial\Omega$. Consequently, $e^{t\Delta} f$ is the solution of (1.1) with initial state f in $L^2(\Omega)$. The Lebeau–Robbiano spectral inequality is as follows:

For each $0 < R \leq 1$, *there is* $N = N(\Omega, R)$ *such that*

$$\|\mathcal{E}_\lambda f\|_{L^2(\Omega)} \leq N e^{N\sqrt{\lambda}} \|\mathcal{E}_\lambda f\|_{L^2(B_R(x_0))} \tag{1.4}$$

when $B_{4R}(x_0) \subset \Omega$, $f \in L^2(\Omega)$ *and* $\lambda > 0$.

To our best knowledge, the inequality (1.4) has been proved under the condition that $\partial\Omega$ is at least C^2 [28, 29, 30, 33]. In the current work, we obtain this inequality when Ω is a bounded Lipschitz and locally star-shaped domain in $\mathbb{R}^n$ (see Definitions 1 and 4 in Section 3). It can be observed from Section 3 that bounded C^1 domains, Lipschitz polygons in the plane, Lipschitz polyhedra in $\mathbb{R}^n$ with $n \geq 3$ and bounded convex domains in $\mathbb{R}^n$ are always bounded Lipschitz and locally star-shaped (see Remarks 4 and 6 in Section 3).

Our main results related to observability inequalities are as follows:

Theorem 1. *Suppose that a bounded domain* Ω *satisfies condition* (1.4)*, and let* $T > 0$. *Let* $x_0 \in \Omega$ *and* $R \in (0, 1]$ *be such that* $B_{4R}(x_0) \subset \Omega$. *Then, for each measurable set* $\mathcal{D} \subset B_R(x_0) \times (0, T)$ *with* $|\mathcal{D}| > 0$, *there is a positive constant* $B = B(\Omega, T, R, \mathcal{D})$ *such that*

$$\|e^{T\Delta} f\|_{L^2(\Omega)} \leq e^B \int_{\mathcal{D}} |e^{t\Delta} f(x)|\, dx\, dt \quad \text{when } f \in L^2(\Omega). \tag{1.5}$$

Theorem 2. *Suppose that a bounded Lipschitz domain* Ω *satisfies condition* (1.4)*, and let* $T > 0$. *Let* $q_0 \in \partial\Omega$ *and* $R \in (0, 1]$ *be such that* $\Delta_{4R}(q_0)$ *is real-analytic. Then, for each measurable set* $\mathcal{J} \subset \Delta_R(q_0) \times (0, T)$ *with* $|\mathcal{J}| > 0$, *there is a positive constant* $B = B(\Omega, T, R, \mathcal{J})$ *such that*

$$\|e^{T\Delta} f\|_{L^2(\Omega)} \leq e^B \int_{\mathcal{J}} |\partial_\nu e^{t\Delta} f(x)|\, d\sigma\, dt \quad \text{when } f \in L^2(\Omega). \tag{1.6}$$

The definition of real analyticity for $\Delta_{4R}(q_0)$ is given in Section 4 (see Definition 5).

Theorem 3. *Let* Ω *be a bounded Lipschitz and locally star-shaped domain in* $\mathbb{R}^n$. *Then* Ω *satisfies condition* (1.4).

It deserves mentioning that Theorem 2 also holds when Ω is a Lipschitz polyhedron in $\mathbb{R}^n$ and $\mathcal{J}$ is a measurable subset of $\partial\Omega \times (0, T)$ with positive surface measure (see Remark 11(ii)).

In Section 5 we explain some applications of Theorems 1 and 2 in the control theory of the heat equation. These include the existence of $L^\infty(\mathcal{D})$-interior and $L^\infty(\mathcal{J})$-boundary admissible controls, the uniqueness and bang-bang properties of the minimal L^∞-norm control and the uniqueness and bang-bang property for the optimal controls associated to the first type and the second type of time optimal control problems.

In this work we use the new strategy developed in [40] to prove parabolic observability inequalities: a mix of ideas from [36], the global interpolation inequality of Theorems 6 and 10 and the telescoping series method. This new strategy can also be extended to more general parabolic evolutions with variable time-dependent second order coefficients and with unbounded lower order time-dependent coefficients. To do it one must prove the global interpolation inequalities of Theorems 6 and 10 for the corresponding parabolic evolutions. These can be derived in the more general setting from the Carleman inequalities in [8, 9, 12, 15, 25] or from local versions of frequency function arguments [10, 40]. Here we choose to derive the interpolation inequalities only for the heat equation and from condition (1.4), because this is technically less involved and helps to make the presentation of the basic ideas more clear.

The rest of the paper is organized as follows: in Section 2 we prove Theorem 1; in Section 3 we show Theorem 3; in Section 4 we verify Theorem 2; Section 5 presents the applications of Theorems 1 and 2; and Section 6 is an Appendix completing some technical details.

2. Interior observability

Throughout this section, Ω denotes a bounded domain and T is a positive time. First of all, we recall the following observability estimate or propagation of smallness inequality from measurable sets:

Theorem 4. *Assume that $f : \mathbb{R}^n \supset B_{2R} \to \mathbb{R}$ is real-analytic in B_{2R} and satisfies*

$$|\partial^\alpha f(x)| \le \frac{M|\alpha|!}{(\rho R)^{|\alpha|}} \quad \text{when } x \in B_{2R},\ \alpha \in \mathbb{N}^n,$$

for some $M > 0$ and $0 < \rho \le 1$. Let $E \subset B_R$ be a measurable set with positive measure. Then there are positive constants $N = N(\rho, |E|/|B_R|)$ and $\theta = \theta(\rho, |E|/|B_R|)$ such that

$$\|f\|_{L^\infty(B_R)} \le N\left(\fint_E |f|\,dx\right)^\theta M^{1-\theta}. \tag{2.1}$$

The estimate (2.1) was first established in [43] (see also [38] and [39] for other close results). The reader can find a simpler proof of Theorem 4 in [3, §3], building on ideas from [34], [38] and [43].

Theorem 4 and condition (1.4) imply the following:

Theorem 5. *Assume that Ω satisfies (1.4), $\omega \subset B_R(x_0)$ is a subset of positive measure and $B_{4R}(x_0) \subset \Omega$, for some $R \in (0, 1]$. Then there is a positive constant $N = N(\Omega, R, |\omega|/|B_R|)$ such that*

$$\|\mathcal{E}_\lambda f\|_{L^2(\Omega)} \le N e^{N\sqrt{\lambda}} \|\mathcal{E}_\lambda f\|_{L^1(\omega)} \quad \text{when } f \in L^2(\Omega) \text{ and } \lambda > 0. \tag{2.2}$$

Proof. Without loss of generality we may assume $x_0 = 0$. Because $B_{4R} \subset \Omega$ and (1.4) holds, there is $N = N(\Omega, R)$ such that

$$\|\mathcal{E}_\lambda f\|_{L^2(\Omega)} \le N e^{N\sqrt{\lambda}} \|\mathcal{E}_\lambda f\|_{L^2(B_R)} \quad \text{when } f \in L^2(\Omega) \text{ and } \lambda > 0. \tag{2.3}$$

For any $f \in L^2(\Omega)$, define

$$u(x, y) = \sum_{\lambda_j \le \lambda} (f, e_j) e^{\sqrt{\lambda_j}\, y} e_j.$$

One can verify that $\Delta u + \partial_y^2 u = 0$ in $B_{4R}(0,0) \subset \Omega \times \mathbb{R}$. Hence, there are $N = N(n)$ and $\rho = \rho(n)$ such that

$$\|\partial_x^\alpha \partial_y^\beta u\|_{L^\infty(B_{2R}(0,0))} \le \frac{N(|\alpha| + \beta)!}{(R\rho)^{|\alpha|+\beta}} \left(\fint_{B_{4R}(0,0)} |u|^2 \, dx\, dy \right)^{1/2} \quad \text{when } \alpha \in \mathbb{N}^n,\ \beta \ge 1$$

(see [37, Chapter 5], [22, Chapter 3]). Thus, $\mathcal{E}_\lambda f$ is a real-analytic function in B_{2R} with

$$\|\partial_x^\alpha \mathcal{E}_\lambda f\|_{L^\infty(B_{2R})} \le N|\alpha|!(R\rho)^{-|\alpha|} \|u\|_{L^\infty(\Omega \times (-4,4))}, \qquad \alpha \in \mathbb{N}^n.$$

By either extending $|u|$ as zero outside of $\Omega \times \mathbb{R}$, which turns $|u|$ into a subharmonic function in $\mathbb{R}^{n+1}$, or the local properties of solutions to elliptic equations [20, Theorems 8.17, 8.25] and the orthonormality of $\{e_j : j \ge 1\}$ in Ω, there is $N = N(\Omega)$ such that

$$\|u\|_{L^\infty(\Omega\times(-4,4))} \le N\|u\|_{L^2(\Omega\times(-5,5))} \le N e^{N\sqrt{\lambda}} \|\mathcal{E}_\lambda f\|_{L^2(\Omega)}.$$

The last two inequalities show that

$$\|\partial_x^\alpha \mathcal{E}_\lambda f\|_{L^\infty(B_{2R})} \le N e^{N\sqrt{\lambda}} |\alpha|!(R\rho)^{-|\alpha|} \|\mathcal{E}_\lambda f\|_{L^2(\Omega)} \quad \text{for } \alpha \in \mathbb{N}^n,$$

with N and ρ as above. In particular, $\mathcal{E}_\lambda f$ satisfies the hypothesis in Theorem 4 with

$$M = N e^{N\sqrt{\lambda}} \|\mathcal{E}_\lambda f\|_{L^2(\Omega)},$$

and there are $N = N(\Omega, R, |\omega|/|B_R|)$ and $\theta = \theta(\Omega, R, |\omega|/|B_R|)$ with

$$\|\mathcal{E}_\lambda f\|_{L^\infty(B_R)} \le N e^{N\sqrt{\lambda}} \|\mathcal{E}_\lambda f\|_{L^1(\omega)}^\theta \|\mathcal{E}_\lambda f\|_{L^2(\Omega)}^{1-\theta}. \tag{2.4}$$

Now, the estimate (2.2) follows from (2.3) and (2.4). □

Theorem 6. *Let Ω, x_0, R and ω be as in Theorem 5. Then there are $N = N(\Omega, R, |\omega|/|B_R|)$ and $\theta = \theta(\Omega, R, |\omega|/|B_R|) \in (0, 1)$ such that*

$$\|e^{t\Delta} f\|_{L^2(\Omega)} \le \left(N e^{\frac{N}{t-s}} \|e^{t\Delta} f\|_{L^1(\omega)}\right)^\theta \|e^{s\Delta} f\|_{L^2(\Omega)}^{1-\theta} \tag{2.5}$$

when $0 \le s < t$ and $f \in L^2(\Omega)$.

Proof. Let $0 \le s < t$ and $f \in L^2(\Omega)$. Since

$$\|e^{t\Delta}\mathcal{E}_\lambda^\perp f\|_{L^2(\Omega)} \le e^{-\lambda(t-s)}\|e^{s\Delta}f\|_{L^2(\Omega)},$$

it follows from Theorem 5 that

$$\begin{aligned}\|e^{t\Delta}f\|_{L^2(\Omega)} &\le \|e^{t\Delta}\mathcal{E}_\lambda f\|_{L^2(\Omega)} + \|e^{t\Delta}\mathcal{E}_\lambda^\perp f\|_{L^2(\Omega)}\\ &\le Ne^{N\sqrt{\lambda}}\|e^{t\Delta}\mathcal{E}_\lambda f\|_{L^1(\omega)} + e^{-\lambda(t-s)}\|e^{s\Delta}f\|_{L^2(\Omega)}\\ &\le Ne^{N\sqrt{\lambda}}[\|e^{t\Delta}f\|_{L^1(\omega)} + \|e^{t\Delta}\mathcal{E}_\lambda^\perp f\|_{L^2(\omega)}] + e^{-\lambda(t-s)}\|e^{s\Delta}f\|_{L^2(\Omega)}\\ &\le Ne^{N\sqrt{\lambda}}[\|e^{t\Delta}f\|_{L^1(\omega)} + e^{-\lambda(t-s)}\|e^{s\Delta}f\|_{L^2(\Omega)}].\end{aligned}$$

Consequently,

$$\|e^{t\Delta}f\|_{L^2(\Omega)} \le Ne^{N\sqrt{\lambda}}[\|e^{t\Delta}f\|_{L^1(\omega)} + e^{-\lambda(t-s)}\|e^{s\Delta}f\|_{L^2(\Omega)}]. \tag{2.6}$$

Because

$$\max_{\lambda\ge 0} e^{A\sqrt{\lambda}-\lambda(t-s)} \le e^{\frac{N(A)}{t-s}} \quad \text{for all } A > 0,$$

it follows from (2.6) that for each $\lambda > 0$,

$$\|e^{t\Delta}f\|_{L^2(\Omega)} \le Ne^{\frac{N}{t-s}}[e^{N\lambda(t-s)}\|e^{t\Delta}f\|_{L^1(\omega)} + e^{-\lambda(t-s)/N}\|e^{s\Delta}f\|_{L^2(\Omega)}].$$

Setting $\epsilon = e^{-\lambda(t-s)}$ in the above estimate shows that

$$\|e^{t\Delta}f\|_{L^2(\Omega)} \le Ne^{\frac{N}{t-s}}[\epsilon^{-N}\|e^{t\Delta}f\|_{L^1(\omega)} + \epsilon\|e^{s\Delta}f\|_{L^2(\Omega)}] \tag{2.7}$$

for all $0 < \epsilon \le 1$. The minimization of the right hand in (2.7) over ϵ in $(0, 1)$, as well as the fact that

$$\|e^{t\Delta}f\|_{L^2(\Omega)} \le \|e^{s\Delta}f\|_{L^2(\Omega)} \quad \text{when } t > s,$$

imply Theorem 6. □

Remark 1. Theorem 6 shows that the observability or spectral elliptic inequality (2.2) implies inequality (2.5). In particular, the elliptic spectral inequality (1.4) yields

$$\|e^{t\Delta}f\|_{L^2(\Omega)} \le \left(Ne^{\frac{N}{t-s}}\|e^{t\Delta}f\|_{L^2(B_R(x_0))}\right)^\theta \|e^{s\Delta}f\|_{L^2(\Omega)}^{1-\theta} \tag{2.8}$$

when $0 \le s < t$, $B_{4R}(x_0) \subset \Omega$ and $f \in L^2(\Omega)$. In fact, both (2.2) and (2.5) or (1.4) and (2.8) are equivalent, for if (2.5) holds, take $f = \sum_{\lambda_j\le\lambda} e^{\lambda_j/\sqrt{\lambda}}a_j e_j$, $s = 0$ and $t = 1/\sqrt{\lambda}$ in (2.5) to derive that

$$\Big(\sum_{\lambda_j\le\lambda} a_j^2\Big)^{1/2} \le Ne^{N\sqrt{\lambda}}\Big\|\sum_{\lambda_j\le\lambda} a_j e_j\Big\|_{L^1(\omega)} \quad \text{when } a_j \in \mathbb{R},\ j \ge 1,\ \lambda > 0.$$

The interested reader may wish here to compare the previous claims, Theorem 3 and [40, Proposition 2.2].

Lemma 1. *Let $B_R(x_0) \subset \Omega$ and let $\mathcal{D} \subset B_R(x_0) \times (0, T)$ be a subset of positive measure. Set*

$$\mathcal{D}_t = \{x \in \Omega : (x, t) \in \mathcal{D}\}, \quad t \in (0, T), \quad E = \{t \in (0, T) : |\mathcal{D}_t| \geq |\mathcal{D}|/(2T)\}.$$

Then $\mathcal{D}_t$ is measurable for a.e. $t \in (0, T)$, E is measurable, $|E| \geq |\mathcal{D}|/(2|B_R|)$ and

$$\chi_E(t)\chi_{\mathcal{D}_t}(x) \leq \chi_{\mathcal{D}}(x, t) \quad \text{in } \Omega \times (0, T). \tag{2.9}$$

Proof. From Fubini's theorem,

$$|\mathcal{D}| = \int_0^T |\mathcal{D}_t|\, dt = \int_E |\mathcal{D}_t|\, dt + \int_{[0,T]\setminus E} |\mathcal{D}_t|\, dt \leq |B_R|\,|E| + |\mathcal{D}|/2. \qquad \square$$

Theorem 7. *Let $x_0 \in \Omega$ and $R \in (0, 1]$ be such that $B_{4R}(x_0) \subset \Omega$. Let $\mathcal{D} \subset B_R(x_0) \times (0, T)$ be a measurable set with $|\mathcal{D}| > 0$. Write E and $\mathcal{D}_t$ for the sets associated to $\mathcal{D}$ in Lemma 1. Then, for each $\eta \in (0, 1)$, there are $N = N(\Omega, R, |\mathcal{D}|/(T|B_R|), \eta)$ and $\theta = \theta(\Omega, R, |\mathcal{D}|/(T|B_R|), \eta) \in (0, 1)$ such that*

$$\|e^{t_2\Delta} f\|_{L^2(\Omega)} \leq \left(Ne^{N/(t_2-t_1)} \int_{t_1}^{t_2} \chi_E(s)\|e^{s\Delta} f\|_{L^1(\mathcal{D}_s)}\, ds\right)^{\theta} \|e^{t_1\Delta} f\|_{L^2(\Omega)}^{1-\theta} \tag{2.10}$$

when $0 \leq t_1 < t_2 \leq T$, $|E \cap (t_1, t_2)| \geq \eta(t_2 - t_1)$ and $f \in L^2(\Omega)$. Moreover,

$$\begin{aligned} &e^{-\frac{N+1-\theta}{t_2-t_1}} \|e^{t_2\Delta} f\|_{L^2(\Omega)} - e^{-\frac{N+1-\theta}{q(t_2-t_1)}} \|e^{t_1\Delta} f\|_{L^2(\Omega)} \\ &\qquad \leq N \int_{t_1}^{t_2} \chi_E(s)\|e^{s\Delta} f\|_{L^1(\mathcal{D}_s)}\, ds \quad \text{when } q \geq (N + 1 - \theta)/(N + 1). \end{aligned} \tag{2.11}$$

Proof. After removing from E a set with zero Lebesgue measure, we may assume that $\mathcal{D}_t$ is measurable for all t in E. From Lemma 1, $\mathcal{D}_t \subset B_R(x_0)$, $B_{4R}(x_0) \subset \Omega$ and $|\mathcal{D}_t|/|B_R| \geq |\mathcal{D}|/(2T|B_R|)$, for all $t \in E$. From Theorem 6, there are $N = N(\Omega, R, |\mathcal{D}|/(T|B_R|))$ and $\theta = \theta(\Omega, R, |\mathcal{D}|/(T|B_R|))$ such that

$$\|e^{t\Delta} f\|_{L^2(\Omega)} \leq (Ne^{\frac{N}{t-s}} \|e^{t\Delta} f\|_{L^1(\mathcal{D}_t)})^{\theta} \|e^{s\Delta} f\|_{L^2(\Omega)}^{1-\theta} \tag{2.12}$$

when $0 \leq s < t$, $t \in E$ and $f \in L^2(\Omega)$. Let $\eta \in (0, 1)$ and $0 \leq t_1 < t_2 \leq T$ satisfy $|E \cap (t_1, t_2)| \geq \eta(t_2 - t_1)$. Set $\tau = t_1 + \frac{1}{2}\eta(t_2 - t_1)$. Then

$$|E \cap (\tau, t_2)| = |E \cap (t_1, t_2)| - |E \cap (t_1, \tau)| \geq \tfrac{1}{2}\eta(t_2 - t_1). \tag{2.13}$$

From (2.12) with $s = t_1$ and the decay property of $\|e^{t\Delta} f\|_{L^2(\Omega)}$, we get

$$\|e^{t_2\Delta} f\|_{L^2(\Omega)} \leq (Ne^{\frac{N}{t_2-t_1}} \|e^{t\Delta} f\|_{L^1(\mathcal{D}_t)})^{\theta} \|e^{t_1\Delta} f\|_{L^2(\Omega)}^{1-\theta}, \quad t \in E \cap (\tau, t_2). \tag{2.14}$$

Inequality (2.10) follows from the integration of (2.14) with respect to t over $E \cap (\tau, t_2)$, Hölder's inequality with $p = 1/\theta$ and (2.13).

Inequality (2.10) and Young's inequality imply that

$$\|e^{t_2\Delta}f\|_{L^2(\Omega)} \le \epsilon\|e^{t_1\Delta}f\|_{L^2(\Omega)} + \epsilon^{-\frac{1-\theta}{\theta}} N e^{\frac{N}{t_2-t_1}} \int_{t_1}^{t_2} \chi_E(s)\|e^{s\Delta}f\|_{L^1(\mathcal{D}_s)}\,ds \quad \text{when } \epsilon > 0. \tag{2.15}$$

Multiplying first (2.15) by $\epsilon^{\frac{1-\theta}{\theta}} e^{-\frac{N}{t_2-t_1}}$ and then replacing ϵ by ϵ^{θ}, we get

$$\epsilon^{1-\theta} e^{-\frac{N}{t_2-t_1}}\|e^{t_2\Delta}f\|_{L^2(\Omega)} - \epsilon e^{-\frac{N}{t_2-t_1}}\|e^{t_1\Delta}f\|_{L^2(\Omega)} \le N\int_{t_1}^{t_2} \chi_E(s)\|e^{s\Delta}f\|_{L^1(\mathcal{D}_s)}\,ds \quad \text{when } \epsilon > 0.$$

Choosing $\epsilon = e^{-\frac{1}{t_2-t_1}}$ in the above inequality leads to

$$e^{-\frac{N+1-\theta}{t_2-t_1}}\|e^{t_2\Delta}f\|_{L^2(\Omega)} - e^{-\frac{N+1}{t_2-t_1}}\|e^{t_1\Delta}f\|_{L^2(\Omega)} \le N\int_{t_1}^{t_2} \chi_E(s)\|e^{s\Delta}f\|_{L^1(\mathcal{D}_s)}\,ds.$$

This implies (2.11) for $q \ge (N+1-\theta)/(N+1)$. □

The reader can find the proof of the lemma below in either [32, pp. 256–257] or [40, Proposition 2.1].

Lemma 2. *Let E be a subset of positive measure in $(0,T)$. Let l be a density point of E. Then, for each $z > 1$, there is $l_1 = l_1(z,E)$ in (l,T) such that the sequence $\{l_m\}$ defined as*

$$l_{m+1} = l + z^{-m}(l_1 - l), \quad m = 1, 2, \ldots,$$

satisfies

$$|E \cap (l_{m+1}, l_m)| \ge \tfrac{1}{3}(l_m - l_{m+1}) \quad \textit{when } m \ge 1. \tag{2.16}$$

Proof of Theorem 1. Let E and $\mathcal{D}_t$ be the sets associated to $\mathcal{D}$ in Lemma 1, and l be a density point in E. For $z > 1$ to be fixed later, $\{l_m\}$ denotes the sequence associated to l and z in Lemma 2. Because (2.16) holds, we may apply Theorem 7 with $\eta = 1/3$, $t_1 = l_{m+1}$ and $t_2 = l_m$, for each $m \ge 1$, to deduce that there are $N = N(\Omega, R, |\mathcal{D}|/(T|B_R|)) > 0$ and $\theta = \theta(\Omega, R, |\mathcal{D}|/(T|B_R|)) \in (0,1)$ such that

$$e^{-\frac{N+1-\theta}{l_m-l_{m+1}}}\|e^{l_m\Delta}f\|_{L^2(\Omega)} - e^{-\frac{N+1-\theta}{q(l_m-l_{m+1})}}\|e^{l_{m+1}\Delta}f\|_{L^2(\Omega)} \le N\int_{l_{m+1}}^{l_m} \chi_E(s)\|e^{s\Delta}f\|_{L^1(\mathcal{D}_s)}\,ds \quad \text{when } q \ge \frac{N+1-\theta}{N+1} \text{ and } m \ge 1. \tag{2.17}$$

Setting $z = 1/q$ in (2.17) (which leads to $1 < z \le \frac{N+1}{N+1-\theta}$) and

$$\gamma_z(t) = e^{-\frac{N+1-\theta}{(z-1)(l_1-l)t}}, \quad t > 0,$$

recalling that

$$l_m - l_{m+1} = z^{-m}\,(z-1)\,(l_1 - l) \qquad \text{for } m \geq 1,$$

we have

$$\begin{aligned}\gamma_z(z^{-m})\|e^{l_m\Delta} f\|_{L^2(\Omega)} &- \gamma_z(z^{-m-1})\|e^{l_{m+1}\Delta} f\|_{L^2(\Omega)}\\ &\leq N \int_{l_{m+1}}^{l_m} \chi_E(s)\|e^{s\Delta} f\|_{L^1(\mathcal{D}_s)}\,ds \quad \text{when } m \geq 1. \qquad (2.18)\end{aligned}$$

Choose now

$$z = \frac{1}{2}\left(1 + \frac{N+1}{N+1-\theta}\right).$$

The choice of z and Lemma 2 determines l_1 in (l, T), and from (2.18),

$$\begin{aligned}\gamma(z^{-m})\|e^{l_m\Delta} f\|_{L^2(\Omega)} &- \gamma(z^{-m-1})\|e^{l_{m+1}\Delta} f\|_{L^2(\Omega)}\\ &\leq N \int_{l_{m+1}}^{l_m} \chi_E(s)\|e^{s\Delta} f\|_{L^1(\mathcal{D}_s)}\,ds \quad \text{when } m \geq 1 \qquad (2.19)\end{aligned}$$

with

$$\gamma(t) = e^{-A/t} \quad \text{and} \quad A = A(\Omega, R, E, |\mathcal{D}|/\,(T|B_R|)) = \frac{2\,(N+1-\theta)^2}{\theta\,(l_1 - l)}.$$

Finally, because

$$\|e^{T\Delta} f\|_{L^2(\Omega)} \leq \|e^{l_1\Delta} f\|_{L^2(\Omega)}, \qquad \sup_{t\geq 0}\|e^{t\Delta} f\|_{L^2(\Omega)} < \infty, \qquad \lim_{t\to 0+}\gamma(t) = 0,$$

and from (2.9), the addition of the telescoping series in (2.19) gives

$$\|e^{T\Delta} f\|_{L^2(\Omega)} \leq N e^{zA} \int_{\mathcal{D}\cap(\Omega\times[l,l_1])} |e^{t\Delta} f(x)|\,dx\,dt \qquad \text{for } f \in L^2(\Omega)$$

which proves (1.5) with $B = zA + \log N$. □

Remark 2. The constant B in Theorem 1 depends on E because the choice of $l_1 = l_1(z, E)$ in Lemma 2 depends on the possible complex structure of the measurable set E (see the proof of Lemma 2 in [40, Proposition 2.1]). When $\mathcal{D} = \omega \times (0, T)$, one may take $l = T/2$, $l_1 = T$, $z = 2$, and then

$$B = A(\Omega, R, |\omega|/|B_R|)/T.$$

Remark 3. The proof of Theorem 1 also implies the following observability estimate:

$$\sup_{m\geq 0}\ \sup_{l_{m+1}\leq t\leq l_m} e^{-z^{m+1}A}\|e^{t\Delta} f\|_{L^2(\Omega)} \leq N \int_{\mathcal{D}\cap(\Omega\times[l,l_1])} |e^{t\Delta} f(x)|\,dx\,dt$$

for f in $L^2(\Omega)$, and with z, N and A as defined along the proof of Theorem 1. Here, $l_0 = T$.

3. Spectral inequalities

Throughout this section, Ω is a bounded domain in $\mathbb{R}^n$ and ν_q is the unit exterior normal vector at $q \in \partial\Omega$.

Definition 1. Ω is a *Lipschitz domain* (sometimes called strongly Lipschitz or a Lipschitz graph domain) with constants m and ϱ when for each point p on the boundary of Ω there is a rectangular coordinate system $x = (x', x_n)$ and a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying

$$\phi(0') = 0, \quad |\phi(x_1') - \phi(x_2')| \le m|x_1' - x_2'| \quad \text{for all } x_1', x_2' \in \mathbb{R}^{n-1}, \tag{3.1}$$

$p = (0', 0)$ in this coordinate system and

$$\begin{aligned} Z_{m,\varrho} \cap \Omega &= \{(x', x_n) : |x'| < \varrho,\ \phi(x') < x_n < 2m\varrho\}, \\ Z_{m,\varrho} \cap \partial\Omega &= \{(x', \phi(x')) : |x'| < \varrho\}, \end{aligned} \tag{3.2}$$

where $Z_{m,\varrho} = B_\varrho' \times (-2m\varrho, 2m\varrho)$.

Definition 2. Ω is a C^1 *domain* when it is a Lipschitz domain and the functions φ associated to points p in $\partial\Omega$ satisfying (3.1) and (3.2) are in $C^1(\mathbb{R}^{n-1})$. Then there is

$$\theta : [0, \infty) \to [0, \infty), \quad \text{nondecreasing}, \quad \lim_{t\to 0^+} \theta(t) = 0,$$

with

$$|(q - p) \cdot \nu_q| \le |p - q|\theta(|p - q|) \quad \text{when } p, q \in \partial\Omega. \tag{3.3}$$

Definition 3. Ω is a *lower* C^1 *domain* when it is a Lipschitz domain and

$$\liminf_{q\in\partial\Omega,\, q\to p} \frac{(q - p) \cdot \nu_q}{|q - p|} \ge 0 \quad \text{for each } p \in \partial\Omega. \tag{3.4}$$

Remark 4. Lipschitz polygons in the plane, convex domains in $\mathbb{R}^n$ (see [37, p. 72, Lemma 3.4.1] for a proof that convex domains in $\mathbb{R}^n$ are Lipschitz domains (or strongly Lipschitz domains, to keep pace with Morrey's definition)) or Lipschitz polyhedra in $\mathbb{R}^n$, $n \ge 3$, are lower C^1 domains. In fact, in all these cases, for $p \in \partial\Omega$, there is $r_p > 0$ such that

$$(q - p) \cdot \nu_q \ge 0 \quad \text{for a.e. } q \in B_{r_p}(p) \cap \partial\Omega.$$

When Ω is convex, $r_p = \infty$. In general, a Lipschitz domain Ω is a lower C^1 domain when the Lipschitz functions ϕ describing its boundary can be decomposed as the sum of two Lipschitz functions, $\phi = \phi_1 + \phi_2$, with ϕ_1 convex over $\mathbb{R}^{n-1}$ and ϕ_2 satisfying

$$\lim_{\epsilon\to 0^+} \sup_{|x_1'-x_2'|<\epsilon} |\nabla\phi_2(x_1') - \nabla\phi_2(x_2')| = 0.$$

In particular, C^1 domains are lower C^1 domains (see (3.3)).

Definition 4. A Lipschitz domain Ω in $\mathbb{R}^n$ is *locally star-shaped* when for each $p \in \partial\Omega$ there are x_p in Ω and $r_p > 0$ such that

$$|p - x_p| < r_p \quad \text{and} \quad B_{r_p}(x_p) \cap \Omega \text{ is star-shaped with center } x_p. \tag{3.5}$$

In particular,

$$(q - x_p) \cdot \nu_q \geq 0 \quad \text{for a.e. } q \in B_{r_p}(x_p) \cap \partial\Omega.$$

Remark 5. The compactness of $\partial\Omega$ shows that when Ω is locally star-shaped, there are a finite set $\mathcal{A} \subset \Omega$, $0 < \epsilon, \rho \leq 1$ and a family of positive numbers $0 < r_x \leq 1$, $x \in \mathcal{A}$, such that

$$\begin{aligned} &\partial\Omega \subset \bigcup_{x \in \mathcal{A}} B_{r_x}(x), \qquad B_{(1+\epsilon)r_x}(x) \cap \Omega \text{ is star-shaped,} \\ &\mathcal{A} \subset \Omega^{4\rho}, \qquad \overline{\Omega} \setminus \Omega^{4\rho} \subset \bigcup_{x \in \mathcal{A}} B_{r_x}(x). \end{aligned} \tag{3.6}$$

Here,

$$\Omega^\eta = \{x \in \Omega : d(x, \partial\Omega) > \eta\} \quad \text{when } \eta > 0.$$

Remark 6. Theorem 8 below shows that Lipschitz polygons in the plane, C^1 domains, convex domains and Lipschitz polyhedra in $\mathbb{R}^n$, $n \geq 3$, are locally star-shaped. Lipschitz domains in $\mathbb{R}^n$ with Lipschitz constant $m < 1/2$ are also locally star-shaped. Recall that not all polygons in the plane or polyhedra in $\mathbb{R}^3$ are Lipschitz domains (see [42, p. 496, pp. 508–509] or the two-brick domain of [26, p. 303]).

Theorem 8. *Let Ω be a lower C^1 domain. Then Ω is locally star-shaped.*

Proof. Let (x', x_n) and ϕ be the rectangular coordinate system and the Lipschitz function associated to $p \in \partial\Omega$, satisfying (3.1) and (3.2). Let $x_p = p + \delta e$, $e = (0', 1)$, and $r_p = 2\delta$, where $\delta > 0$ will be chosen later. Clearly x_p is in Ω, $|p - x_p| < r_p$ and $B_{r_p}(x_p) \subset Z_{m,\varrho}$, when $0 < \delta < \min\{\varrho/2, 2m\varrho/3\}$. Moreover, for almost every q in $B_{r_p}(x_p) \cap \partial\Omega$, we have $q = (x', \phi(x'))$ for some x' in B'_ρ,

$$\nu_q = \frac{(\nabla\phi(x'), -1)}{\sqrt{1 + |\nabla\phi(x')|^2}}, \qquad \text{and} \quad |q - p| \leq r_p + \delta = 3\delta$$

and

$$(q - x_p) \cdot \nu_q = (q - p) \cdot \nu_q - \delta e \cdot \nu_q \geq (q - p) \cdot \nu_q + \frac{\delta}{\sqrt{1 + m^2}}.$$

From (3.4) there is $s_p > 0$ such that

$$(q - p) \cdot \nu_q \geq -\frac{|q - p|}{3\sqrt{1 + m^2}} \quad \text{when } q \in \partial\Omega \text{ and } |q - p| < s_p. \tag{3.7}$$

Thus, (3.5) holds for the choices we made of x_p, r_p and s_p, provided that δ is chosen with $0 < \delta < \min\{\varrho/2, 2m\varrho/3, s_p/3\}$. $\square$

Theorem 3 will follow from Lemmas 3 and 4 below.

Lemma 3. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, let $R > 0$ and assume that $B_R(x_0) \cap \Omega$ is star-shaped with center at some $x_0 \in \overline{\Omega}$. Then*

$$\|u\|_{L^2(B_{r_2}(x_0)\cap\Omega)} \leq \|u\|^{\theta}_{L^2(B_{r_1}(x_0)\cap\Omega)} \|u\|^{1-\theta}_{L^2(B_{r_3}(x_0)\cap\Omega)} \quad \text{with } \theta = \frac{\log \frac{r_3}{r_2}}{\log \frac{r_3}{r_1}} \tag{3.8}$$

when $\Delta u = 0$ in $B_R(x_0) \cap \Omega$, $u = 0$ on $\Delta_R(x_0)$ and $0 < r_1 < r_2 < r_3 \leq R$.

Remark 7. See [27, Lemma 3.1] for a proof of Lemma 3, and [2] and [1] for related results but with spheres replacing balls. The relevance of the assumption that $B_R(x_0) \cap \Omega$ is star-shaped in the proof of [27, Lemma 3.1] is that

$$(q - x_0) \cdot \nu_q \geq 0 \quad \text{for a.e. } q \text{ in } B_R(x_0) \cap \partial\Omega$$

and certain terms arising in the arguments can be dropped because of their nonnegative sign. In fact, (3.8) is the logarithmic convexity of the L^2-norm of u over $B_r(x_0) \cap \Omega$ for $0 < r \leq R$ with respect to the variable $\log r$. Lemma 3 extends up to the boundary the classical interior three-spheres inequality for harmonic functions, first established for complex analytic functions by Hadamard [21] and extended to harmonic functions by several authors [19]:

$$\|u\|_{L^2(B_{r_2}(x_0))} \leq \|u\|^{\theta}_{L^2(B_{r_1}(x_0))} \|u\|^{1-\theta}_{L^2(B_{r_3}(x_0))} \quad \text{with } \theta = \frac{\log \frac{r_3}{r_2}}{\log \frac{r_3}{r_1}} \tag{3.9}$$

when $\Delta u = 0$ in Ω, $B_R(x_0) \subset \Omega$ and $0 < r_1 < r_2 < r_3 \leq R$.

When $B_R(x_0) \subset \Omega$, their intersection is just $B_R(x_0)$, and a proof for (3.9) is within the one for [27, Lemma 3.1].

Lemma 4. *Let Ω be a Lipschitz domain with constants m and ϱ, let $0 < r < \varrho/10$ and let q be in $\partial\Omega$. Then there are $N = N(n, m)$ and $\theta = \theta(n, m) \in (0, 1)$ such that*

$$\|u\|_{L^2(B_r(q)\cap\Omega)} \leq N r^{3\theta/2} \|\partial_\nu u\|^{\theta}_{L^2(\Delta_{6r}(q))} \|u\|^{1-\theta}_{L^2(B_{8r}(q)\cap\Omega)}$$

for all harmonic functions u in $B_{8r}(q) \cap \Omega$ satisfying $u = 0$ on $B_{8r}(q) \cap \partial\Omega$.

To prove Lemma 4 we use the Carleman inequality of Lemma 5 below. As far as the authors know, the first L^2-type Carleman inequality with a radial weight and whose proof was worked out in Cartesian coordinates appeared in [24, Lemma 1]. Lemma 5 borrows ideas from [24, Lemma 1] but the proof here is somewhat simpler. In [2, p. 518] there also appears an interpolation inequality similar to the one in Lemma 4 but with the L^2-norms replaced by L^1-norms. The inequality in [2, p. 518] does hold, though its proof in [2] is not correct; it follows from Lemma 4 and properties of harmonic functions.

Lemma 5. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ with $\Omega \subset B_R$ and let $\tau > 0$. Assume that $0 \notin \overline{\Omega}$. Then*

$$\int_\Omega |x|^{-2\tau} u^2 \, dx \leq \frac{R^2}{4\tau^2} \int_\Omega |x|^{-2\tau+2} (\Delta u)^2 \, dx - \frac{R^2}{2\tau} \int_{\partial\Omega} q \cdot \nu \, |q|^{-2\tau} (\partial_\nu u)^2 \, d\sigma$$

for all u in $C^2(\overline{\Omega})$ satisfying $u = 0$ on $\partial\Omega$.

Proof. Let u be in $C^2(\overline{\Omega})$, $u = 0$ on $\partial\Omega$ and define $f = |x|^{-\tau} u$. Then

$$|x|^{1-\tau}\Delta u = |x|\left(\Delta f + \frac{\tau^2}{|x|^2} f\right) + \frac{2\tau}{|x|}\left(x \cdot \nabla f + \frac{n-2}{2} f\right). \tag{3.10}$$

Square both sides of (3.10) to get

$$\begin{aligned} |x|^{2-2\tau}(\Delta u)^2 = 4\tau^2 |x|^{-2}\left(x \cdot \nabla f + \frac{n-2}{2} f\right)^2 + |x|^2\left(\Delta f + \frac{\tau^2}{|x|^2} f\right)^2 \\ + 4\tau\left(x \cdot \nabla f + \frac{n-2}{2} f\right)\left(\Delta f + \frac{\tau^2}{|x|^2} f\right). \end{aligned} \tag{3.11}$$

Observe that

$$\int_\Omega |x|^{-2}\left(x \cdot \nabla f + \frac{n-2}{2} f\right) f\, dx = \frac{1}{2}\int_{\partial\Omega} \frac{q \cdot \nu}{|q|^2} f^2\, d\sigma = 0,$$

and integrate (3.11) over Ω. Then we get the identity

$$\begin{aligned} \int_\Omega |x|^{2-2\tau}(\Delta u)^2\, dx = 4\tau^2 \int_\Omega |x|^{-2}\left(x \cdot \nabla f + \frac{n-2}{2} f\right)^2 dx \\ + \int_\Omega |x|^2\left(\Delta f + \frac{\tau^2}{|x|^2} f\right)^2 dx + 4\tau \int_\Omega \left(x \cdot \nabla f + \frac{n-2}{2} f\right)\Delta f\, dx. \end{aligned} \tag{3.12}$$

The Rellich–Nečas or Pohozaev identity

$$\nabla \cdot [x|\nabla f|^2 - 2(x \cdot \nabla f)\nabla f] + 2(x \cdot \nabla)\Delta f = (n-2)|\nabla f|^2$$

and the identity

$$\Delta(f^2) = 2f\Delta f + 2|\nabla f|^2$$

give the formula

$$\nabla \cdot [x|\nabla f|^2 - 2(x \cdot \nabla f)\nabla f] + 2\left(x \cdot \nabla f + \frac{n-2}{2} f\right)\Delta f = \frac{n-2}{2}\Delta(f^2).$$

The integration of this identity over Ω implies the formula

$$4\int_\Omega \left(x \cdot \nabla f + \frac{n-2}{2} f\right)\Delta f\, dx = 2\int_{\partial\Omega} q \cdot \nu (\partial_\nu f)^2 d\sigma, \tag{3.13}$$

and plugging (3.13) into (3.12) gives the identity

$$\begin{aligned} \int_\Omega |x|^{2-2\tau}(\Delta u)^2\, dx = 4\tau^2 \int_\Omega |x|^{-2}\left(x \cdot \nabla f + \frac{n-2}{2} f\right)^2 dx \\ + \int_\Omega |x|^2\left(\Delta f + \frac{\tau^2}{|x|^2} f\right)^2 dx + 2\tau \int_{\partial\Omega} q \cdot \nu (\partial_\nu f)^2\, d\sigma. \end{aligned} \tag{3.14}$$

Next, the identity

$$-\int_\Omega \left(x\cdot\nabla f + \frac{n-2}{2} f\right) f\,dx = \int_\Omega f^2\,dx,$$

the Cauchy–Schwarz inequality and $\Omega \subset B_R$ show that

$$R^{-2}\int_\Omega f^2\,dx \le \int_\Omega |x|^{-2}\left(x\cdot\nabla f + \frac{n-2}{2} f\right)^2 dx. \tag{3.15}$$

Now, Lemma 5 follows from (3.14) and (3.15). □

Proof of Lemma 4. After rescaling and translation we may assume that $q = 0$, $r = 1$ and that there is a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying

$$\begin{aligned}
&\phi(0') = 0, \quad |\phi(x_1') - \phi(x_2')| \le m|x_1' - x_2'| \quad \text{for all } x_1', x_2' \in \mathbb{R}^{n-1},\\
&Z_{m,10} \cap \Omega = \{(x', x_n) : |x'| < 10,\ \phi(x') < x_n < 20m\},\\
&Z_{m,10} \cap \partial\Omega = \{(x', \phi(x')) : |x'| < 10\},
\end{aligned} \tag{3.16}$$

where $Z_{m,10} = B'_{10} \times (-20m, 20m)$. Recall that $e = (0', 1)$. Thus, $-e \notin \overline{\Omega}$ and

$$B_1 \subset B_2(-e) \subset B_5(-e) \subset B_6. \tag{3.17}$$

Choose ψ in $C_0^\infty(B_5(-e))$ with $\psi \equiv 1$ in $B_3(-e)$ and $\psi \equiv 0$ outside $B_4(-e)$. From (3.16), there is $\beta = \beta(m) \in (0, 1)$ such that

$$B_{2\beta}(-e) \cap \Omega = \emptyset. \tag{3.18}$$

By translating the inequality in Lemma 5 from 0 to $-e$ we find that

$$\begin{aligned}
\left\| |x+e|^{-\tau} f \right\|_{L^2(B_5(-e)\cap\Omega)} &\le \left\| |x+e|^{-\tau+1} \Delta f \right\|_{L^2(B_5(-e)\cap\Omega)}\\
&\quad + \left\| |q+e|^{1/2-\tau} \partial_\nu f \right\|_{L^2(B_5(-e)\cap\partial\Omega)}
\end{aligned} \tag{3.19}$$

when f is in $C_0^2(B_5(-e) \cap \overline{\Omega})$, $f = 0$ on $B_5(-e) \cap \partial\Omega$ and $\tau \ge 20$. Let then u be harmonic in $B_8 \cap \Omega$ with $u = 0$ on $B_8 \cap \partial\Omega$ and take $f = u\psi$ in (3.19). We get

$$\begin{aligned}
\left\| |x+e|^{-\tau} u\psi \right\|_{L^2(B_5(-e)\cap\Omega)} &\le \left\| |x+e|^{1-\tau} (u\Delta\psi + 2\nabla\psi\cdot\nabla u) \right\|_{L^2(B_5(-e)\cap\Omega)}\\
&\quad + \left\| |q+e|^{1/2-\tau} \psi \partial_\nu u \right\|_{L^2(B_5(-e)\cap\partial\Omega)} \quad \text{when } \tau \ge 20.
\end{aligned} \tag{3.20}$$

Next, from $\psi \equiv 1$ on $B_3(-e)$ and (3.17), we have

$$2^{-\tau}\|u\|_{L^2(B_1\cap\Omega)} \le \left\| |x+e|^{-\tau} f \right\|_{L^2(B_5(-e)\cap\Omega)} \quad \text{when } \tau \ge 1. \tag{3.21}$$

Also, it follows from (3.18) that

$$\left\| |q+e|^{1/2-\tau} \psi \partial_\nu u \right\|_{L^2(B_5(-e)\cap\partial\Omega)} \le \beta^{-\tau} \|\partial_\nu u\|_{L^2(\Delta_6)} \quad \text{when } \tau \ge 1. \tag{3.22}$$

Now, $|\Delta\psi| + |\nabla\psi|$ is supported in $B_4(-e) \setminus B_3(-e)$, where $|x+e| \geq 3$ and

$$\begin{aligned} \left\| |x+e|^{1-\tau}(u\Delta\psi + 2\nabla\psi\cdot\nabla u)\right\|_{L^2(B_5(-e)\cap\Omega)} \\ &\leq N3^{-\tau}\left\| |u| + |\nabla u|\right\|_{L^2(B_4(-e)\setminus B_3(-e)\cap\Omega)} \\ &\leq N3^{-\tau}\|u\|_{L^2(B_6\cap\Omega)} \quad \text{when } \tau \geq 1, \text{ with } N = N(n). \end{aligned} \tag{3.23}$$

Putting together (3.20)–(3.23), we get

$$\|u\|_{L^2(B_1\cap\Omega)} \leq N[(2/3)^\tau \|u\|_{L^2(B_6\cap\Omega)} + (2/\beta)^\tau \|\partial_\nu u\|_{L^2(\Delta_6)}] \quad \text{when } \tau \geq 20.$$

This inequality shows that there is $N = N(n, m) \geq 1$ such that

$$\|u\|_{L^2(B_1\cap\Omega)} \leq N[\epsilon\|u\|_{L^2(B_8\cap\Omega)} + \epsilon^{-N}\|\partial_\nu u\|_{L^2(\Delta_6)}] \quad \text{when } 0 < \epsilon \leq (2/3)^{20}. \tag{3.24}$$

Finally, the minimization of (3.24) shows that the conclusion of Lemma 4 holds for $q = 0$ and $r = 1$, which completes the proof. □

Proof of Theorem 3. Without loss of generality we may assume that $x_0 = 0$ and $B_{4R} \subset \Omega$ for some fixed $0 < R \leq 1$. To prove that (1.4) holds with $x_0 = 0$, we first show that under the hypothesis of Theorem 3, there are ρ and θ in $(0, 1)$ and $N \geq 1$ such that

$$\|u\|_{L^2(\Omega\times[-1,1])} \leq N\|u\|^\theta_{L^2(\Omega^{2\rho}\times[-2,2])}\|u\|^{1-\theta}_{L^2(\Omega\times[-4,4])} \tag{3.25}$$

when

$$\begin{cases} \Delta u + \partial_y^2 u = 0 & \text{in } \Omega\times\mathbb{R}, \\ u = 0 & \text{on } \partial\Omega\times\mathbb{R}. \end{cases} \tag{3.26}$$

To prove (3.25), we recall that (3.6) holds and $\mathcal{A} = \{x_1, \ldots, x_l\}$ for some x_i in $\Omega^{4\rho}$, $i = 1, \ldots, l$, for some $l \geq 1$. Because $B_{(1+\epsilon)r_{x_i}}(x_i) \cap \Omega$ is star-shaped with center x_i, the same holds for $B_{(1+\epsilon)r_{x_i}}(x_i, \tau) \cap \Omega \times \mathbb{R}$ and with center (x_i, τ), $i = 1, \ldots, l$, for all $\tau \in \mathbb{R}$. Then, from Lemma 3,

$$\|u\|_{L^2(B_{r_{x_i}}(x_i,\tau)\cap\Omega\times\mathbb{R})} \leq \|u\|^\theta_{L^2(B_\rho(x_i,\tau)\cap\Omega\times\mathbb{R})}\|u\|^{1-\theta}_{L^2(B_{(1+\epsilon)r_{x_i}}(x_i,\tau)\cap\Omega\times\mathbb{R})} \tag{3.27}$$

for $i = 1, \ldots, l$ with

$$\theta = \min\{\log(1+\epsilon)/\log((1+\epsilon)r_{x_i}/\rho) : i = 1, \ldots, l\}. \tag{3.28}$$

Also, from (3.9),

$$\|u\|_{L^2(B_\rho(x,\tau))} \leq \|u\|^\theta_{L^2(B_{\rho/4}(x,\tau))}\|u\|^{1-\theta}_{L^2(B_{2\rho}(x,\tau))} \quad \text{with } \theta = \log 2/\log 8 \tag{3.29}$$

when x is in the closure of $\Omega^{4\rho}$ and $\tau \in \mathbb{R}$. Because (3.6), (3.27), (3.29) hold and there are $z_1, \ldots, z_m$ in the closure of $\Omega^{4\rho}$ with

$$\overline{\Omega^{4\rho}} \subset \bigcup_{i=1}^m B_\rho(z_i),$$

it is now clear that there are $N = N(l, m, \rho, \mathcal{A}) \geq 1$ and a new $\theta = \theta(l, m, \rho, \mathcal{A})$ in $(0, 1)$ such that (3.25) holds.

From (3.9) with $r_1 = r/4$, $r_2 = r$, $r_3 = 2r$, we see that

$$\|u\|_{L^2(B_r(x_1,y_1))} \leq \|u\|^{\theta}_{L^2(B_{r/4}(x_1,y_1))} \|u\|^{1-\theta}_{L^2(B_{2r}(x_1,y_1))}, \qquad \theta = \log 2/\log 8, \tag{3.30}$$

when $B_{2r}(x_1, y_1) \subset \Omega \times [-4, 4]$. From (3.30) and the fact that

$$B_{r/4}(x_1, y_1) \subset B_r(x_2, y_2) \quad \text{when } (x_2, y_2) \in B_{r/4}(x_1, y_1),$$

we find that

$$\|u\|_{L^2(B_r(x_1,y_1))} \leq \|u\|^{\theta}_{L^2(B_r(x_2,y_2))} \|u\|^{1-\theta}_{L^2(\Omega\times[-4,4])}, \qquad \theta = \log 2/\log 8, \tag{3.31}$$

when $B_{2r}(x_1, y_1) \subset \Omega \times [-4, 4]$, (x_2, y_2) is in $B_{r/4}(x_1, y_1)$ and $r > 0$. Because ρ is now fixed and $\overline{\Omega}$ is compact, there are $0 < r \leq 1$ and $k \geq 2$, which depend on ρ, the geometry of Ω, and R, such that for all (x_0, y_0) in $\Omega^{2\rho} \times [-2, 2]$ there are k points $(x_1, y_1), \ldots, (x_k, y_k)$ in $\Omega^{2\rho} \times [-2, 2]$ with

$$B_{2r}(x_i, y_i) \subset \Omega \times [-4, 4], \quad (x_{i+1}, y_{i+1}) \in B_{r/4}(x_i, y_i) \quad \text{for } i = 0, \ldots, k-1,$$
$$(x_k, y_k) = (0, R/16) \quad \text{and} \quad B_r(0, R/16) \subset B^+_{R/8}(0, 0).$$

Together with (3.31) this shows that

$$\|u\|_{L^2(B_r(x_i,y_i))} \leq \|u\|^{\theta}_{L^2(B_r(x_{i+1},y_{i+1}))} \|u\|^{1-\theta}_{L^2(\Omega\times[-4,4])} \quad \text{for } i = 0, \ldots, k-1, \tag{3.32}$$

while the compactness of Ω and iteration of (3.32) imply that for some new $N = N(l, m, k, \rho, \mathcal{A}, R) \geq 1$ and $0 < \theta = \theta(l, m, k, \rho, \mathcal{A}, R) < 1$,

$$\|u\|_{L^2(\Omega^{2\rho}\times[-2,2])} \leq N \|u\|^{\theta}_{L^2(B^+_{R/8}(0,0))} \|u\|^{1-\theta}_{L^2(\Omega\times[-4,4])}. \tag{3.33}$$

Putting together (3.25) and (3.33), it follows that when u satisfies (3.26),

$$\|u\|_{L^2(\Omega\times[-1,1])} \leq N \|u\|^{\theta}_{L^2(B^+_{R/8}(0,0))} \|u\|^{1-\theta}_{L^2(\Omega\times[-4,4])}, \tag{3.34}$$

with N and θ as before. Finally, proceeding as in [28], for $\lambda > 0$ we take

$$u(x, y) = \sum_{\lambda_j \leq \lambda} a_j \frac{\sinh(\sqrt{\lambda_j}\, y)}{\sqrt{\lambda_j}} e_j(x), \tag{3.35}$$

with $\{a_j\}$ a sequence of real numbers. Then u satisfies (3.26) and also $u(x, 0) \equiv 0$ in $\overline{\Omega}$. Now, we may assume that $R/8 \leq \varrho/10$ and Lemma 4 implies

$$\|u\|_{L^2(B^+_{R/8}(0,0))} \leq N \|\partial_y u(\cdot, 0)\|^{\theta}_{L^2(B_R)} \|u\|^{1-\theta}_{L^2(\Omega\times[-4,4])}. \tag{3.36}$$

Combining (3.34) with (3.36) leads to

$$\|u\|_{L^2(\Omega\times[-1,1])} \leq N \|\partial_y u(\cdot, 0)\|^{\theta}_{L^2(B_R)} \|u\|^{1-\theta}_{L^2(\Omega\times[-4,4])}.$$

This, along with (3.35) and standard arguments in [29], shows that Ω satisfies (1.4). $\square$

Remark 8. The reader can now easily derive, with arguments based on Lemmas 3, 4 and Theorem 3, the following spectral inequality:

Theorem 9. *Suppose that Ω is Lipschitz and satisfies (1.4). Let $q_0 \in \partial\Omega$, $\tau \in \mathbb{R}$ and $R \in (0, 1]$. Then there is $N = N(\Omega, R, \tau)$ such that*

$$\Big(\sum_{\lambda_j \le \lambda} (a_j^2 + b_j^2)\Big)^{1/2} \le e^{N\sqrt{\lambda}} \Big\| \sum_{\lambda_j \le \lambda} (a_j e^{\sqrt{\lambda_j}\, y} + b_j e^{-\sqrt{\lambda_j}\, y}) \partial_\nu e_j \Big\|_{L^2(B_R(q_0,\tau)\cap\partial\Omega\times\mathbb{R})}$$

for all sequences $\{a_j\}$ and $\{b_j\}$ in $\mathbb{R}$. In particular, the above inequality holds when Ω is a bounded Lipschitz and locally star-shaped domain.

4. Boundary observability

Throughout this section, Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and T is a positive time. We first study quantitative estimates of real analyticity with respect to the space-time variables for caloric functions in $\Omega \times (0, T)$ with zero lateral Dirichlet boundary conditions. Let

$$G(x, y, t) = \sum_{j \ge 1} e^{-\lambda_j t} e_j(x) e_j(y) \tag{4.1}$$

be the Green function for $\Delta - \partial_t$ on Ω with zero lateral Dirichlet boundary condition. By the maximum principle,

$$0 \le G(x, y, t) \le (4\pi t)^{-n/2} e^{-|x-y|^2/4t},$$

and

$$G(x, x, t) = \sum_{j \ge 1} e^{-\lambda_j t} e_j(x)^2 \le (4\pi t)^{-n/2}. \tag{4.2}$$

Definition 5. Let $q_0 \in \partial\Omega$ and $0 < R \le 1$. We say that $\triangle_{4R}(q_0)$ is *real-analytic* with constants ϱ and δ if for each $q \in \triangle_{4R}(q_0)$, there are a new rectangular coordinate system where $q = 0$, and a real-analytic function $\phi : \mathbb{R}^{n-1} \supset B'_\varrho \to \mathbb{R}$ satisfying

$$\begin{aligned} &\phi(0') = 0, \quad |\partial^\alpha \phi(x')| \le |\alpha|!\delta^{-|\alpha|-1} \quad \text{when } x' \in B'_\varrho,\ \alpha \in \mathbb{N}^{n-1},\\ &B_\varrho \cap \Omega = B_\varrho \cap \{(x', x_n) : x' \in B'_\varrho,\ x_n > \phi(x')\},\\ &B_\varrho \cap \partial\Omega = B_\varrho \cap \{(x', x_n) : x' \in B'_\varrho,\ x_n = \phi(x')\}. \end{aligned}$$

Here, B'_ϱ denotes the open ball of radius ϱ and with center at $0'$ in $\mathbb{R}^{n-1}$.

Lemma 6. *Let $q_0 \in \partial\Omega$ and $R \in (0, 1]$. Assume that $\triangle_{4R}(q_0)$ is real-analytic with constants ϱ and δ. Then there are $N = N(\varrho, \delta)$ and $\rho = \rho(\varrho, \delta) \in (0, 1]$ such that*

$$|\partial_x^\alpha \partial_t^\beta e^{t\Delta} f(x)| \le \frac{N(t-s)^{-n/4} e^{8R^2/(t-s)} |\alpha|!\,\beta!}{(R\rho)^{|\alpha|}((t-s)/4)^\beta} \|e^{s\Delta} f\|_{L^2(\Omega)} \tag{4.3}$$

when $x \in B_{2R}(q_0) \cap \overline{\Omega}$, $0 \le s < t$, $\alpha \in \mathbb{N}^n$ and $\beta \ge 0$.

Proof. It suffices to prove (4.3) for $s = 0$. Let $f = \sum_{j\geq 1} a_j e_j$. Set

$$u(x,t,y) = \sum_{j=1}^{\infty} a_j e^{-\lambda_j t + \sqrt{\lambda_j}\, y} e_j(x), \quad x \in \overline{\Omega},\ t > 0,\ y \in \mathbb{R},$$

$$u(x,t) = u(x,t,0) = \sum_{j=1}^{\infty} a_j e^{-\lambda_j t} e_j(x) = e^{t\Delta} f(x), \quad x \in \overline{\Omega},\ t > 0.$$

We have

$$\begin{cases} \partial_y^2 \partial_t^\beta u + \Delta \partial_t^\beta u = 0 & \text{in } \Omega \times \mathbb{R}, \\ \partial_t^\beta u = 0 & \text{on } \partial\Omega \times \mathbb{R}, \end{cases}$$

$$\partial_t^\beta u(x,t,y) = (-1)^\beta \sum_{j\geq 1} a_j \lambda_j^\beta e^{-\lambda_j t + \sqrt{\lambda_j}\, y} e_j(x). \tag{4.4}$$

Because $\triangle_{4R}(q_0)$ is real-analytic, there are $N = N(\varrho, \delta)$ and $\rho = \rho(\varrho, \delta)$ such that

$$\begin{aligned} &\|\partial_x^\alpha \partial_t^\beta u(\cdot, t, \cdot)\|_{L^\infty(B_{2R}(q_0,0)\cap\Omega\times\mathbb{R})} \\ &\quad \leq \frac{N|\alpha|!}{(R\rho)^{|\alpha|}} \left(\fint_{B_{4R}(q_0,0)\cap\Omega\times\mathbb{R}} |\partial_t^\beta u(x,t,y)|^2\, dx\, dy \right)^{1/2} \quad \text{when } \alpha \in \mathbb{N}^n,\ \beta \geq 0 \end{aligned} \tag{4.5}$$

(see [37, Chapter 5] and [22, Chapter 3]). Now, it follows from (4.4) that

$$|\partial_t^\beta u(x,t,y)| \leq \Big(\sum_{j\geq 1} a_j^2 \Big)^{1/2} \Big(\sum_{j\geq 1} \lambda_j^{2\beta} e^{-2\lambda_j t + 2\sqrt{\lambda_j}\, y} e_j(x)^2 \Big)^{1/2}.$$

Also,

$$\begin{aligned} \sum_{j\geq 1} \lambda_j^{2\beta} e^{-2\lambda_j t + 2\sqrt{\lambda_j}\, y} e_j(x)^2 &= t^{-2\beta} e^{y^2/t} \sum_{j\geq 1} (\lambda_j t)^{2\beta} e^{-\lambda_j t} e_j(x)^2 e^{-(\sqrt{\lambda_j t} - y/\sqrt{t})^2} \\ &\leq t^{-2\beta} e^{y^2/t} \sum_{j\geq 1} (\lambda_j t)^{2\beta} e^{-\lambda_j t} e_j(x)^2 = t^{-2\beta} e^{y^2/t} \sum_{j\geq 1} (\lambda_j t)^{2\beta} e^{-\lambda_j t/2} e^{-\lambda_j t/2} e_j(x)^2. \end{aligned}$$

Next Stirling's formula shows that

$$\max_{x\geq 0} x^{2\beta} e^{-x/2} = 4^{4\beta} \beta^{2\beta} e^{-2\beta} \lesssim 4^{4\beta} (\beta!)^2 \quad \text{when } \beta \geq 0.$$

Finally, the above three inequalities, as well as (4.2), imply that

$$|\partial_t^\beta u(x,t,y)| \leq (2\pi t)^{-n/4} e^{y^2/2t} \Big(\sum_{j\geq 1} a_j^2 \Big)^{1/2} (t/4)^{-\beta} \beta! \tag{4.6}$$

for $t > 0$, $x \in \overline{\Omega}$, $y \in \mathbb{R}$ and $\beta \geq 0$. The latter inequality and (4.5) imply that

$$|\partial_x^\alpha \partial_t^\beta u(x,t)| \leq \frac{N t^{-n/4} e^{8R^2/t} |\alpha|!\, \beta!}{(R\rho)^{|\alpha|} (t/4)^\beta} \Big(\sum_{j\geq 1} a_j^2 \Big)^{1/2}$$

when $x \in B_{2R}(q_0) \cap \overline{\Omega}$, $t > 0$, $\alpha \in \mathbb{N}^n$ and $\beta \geq 0$. $\square$

The next caloric interpolation inequality plays the same role for the boundary case as inequality (2.8) for the interior case.

Theorem 10. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with constants (m, ϱ) and satisfying condition (1.4). Then, given $0 < R \le 1$, $0 \le t_1 < t_2 \le T \le 1$ and $q \in \partial\Omega$, there are $N = N(\Omega, R, m, \varrho) \ge 1$ and $\theta = \theta(m, \varrho) \in (0, 1)$ such that*

$$\|e^{t_2\Delta} f\|_{L^2(\Omega)} \le \left(N e^{\frac{N}{t_2-t_1}} \|\partial_\nu e^{t\Delta} f\|_{L^2(\Delta_R(q)\times(t_1,t_2))}\right)^\theta \|e^{t_1\Delta} f\|^{1-\theta}_{L^2(\Omega)}, \qquad f \in L^2(\Omega).$$

To prove Theorem 10, we need some lemmas. We begin with the following Carleman inequality (see [9] and [11]).

Lemma 7. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with constants (m, ϱ) and with $0 \notin \Omega$, and let $\sigma(t) = te^{-Mt}$, $M > 0$. Then*

$$\begin{aligned}\sqrt{\tau M}\|\sigma(t)^{-\tau} e^{-|x|^2/8t} h\|_{L^2(\Omega\times(0,\infty))} &\le \|t^{1/2}\sigma(t)^{-\tau} e^{-|x|^2/8t}(\Delta + \partial_t)h\|_{L^2(\Omega\times(0,\infty))} \\ &\quad + \left\| |q|^{1/2}\sigma(t)^{-\tau} e^{-|q|^2/8t}\partial_\nu h\right\|_{L^2(\partial\Omega\times(0,\infty))}\end{aligned}$$

when $\tau \ge 1$ and $h \in C_0^\infty(\overline{\Omega} \times [0, \infty))$ with $h = 0$ on $\partial\Omega \times [0, \infty)$.

Proof. First, let $f = \sigma(t)^{-\tau} e^{-|x^2|/8t} h$. Then

$$\begin{aligned}\sigma(t)^{-\tau} e^{-|x|^2/8t}(\Delta + \partial_t)h &= \sigma(t)^{-\tau} e^{-|x|^2/8t}(\Delta + \partial_t)(\sigma(t)^\tau e^{|x|^2/8t} f) \\ &= \Delta f - \frac{|x|^2}{16t^2} f + \tau\partial_t(\log\sigma) f + \partial_t f + \frac{x}{2t}\cdot\nabla f + \frac{n}{4t} f.\end{aligned}$$

Thus

$$\begin{aligned}&\|t^{1/2}\sigma(t)^{-\tau} e^{-|x|^2/8t}(\Delta + \partial_t)h\|^2_{L^2(\Omega\times(0,\infty))} \\ &\quad = \left\| t^{1/2}\left(\Delta f - \frac{|x|^2}{16t^2} f + \tau\partial_t(\log\sigma) f\right)\right\|^2_{L^2(\Omega\times(0,\infty))} \\ &\qquad + \left\| t^{1/2}\left(\partial_t f + \frac{x}{2t}\cdot\nabla f + \frac{n}{4t} f\right)\right\|^2_{L^2(\Omega\times(0,\infty))} \\ &\qquad + \int_{\Omega\times(0,\infty)} 2t\left(\Delta f - \frac{|x|^2}{16t^2} f + \tau\partial_t(\log\sigma) f\right)\left(\partial_t f + \frac{x}{2t}\cdot\nabla f + \frac{n}{4t} f\right) dx\,dt. \qquad (4.7)\end{aligned}$$

Next, integrating by parts we have the following two identities:

$$\begin{aligned}&\int_{\Omega\times(0,\infty)} 2t\partial_t f\,\Delta f\,dx\,dt \\ &\quad = \int_{\Omega\times(0,\infty)} (-t\partial_t|\nabla f|^2 + 2t\nabla\cdot(\partial_t f\nabla f))\,dx\,dt = \int_{\Omega\times(0,\infty)} |\nabla f|^2\,dx\,dt, \qquad (4.8)\end{aligned}$$

$$
\begin{aligned}
\int_{\Omega\times(0,\infty)} 2t\partial_t f\left(-\frac{|x|^2}{16t^2}(f+\tau\partial_t(\log\sigma)f)\right)dx\,dt
&= \int_{\Omega\times(0,\infty)}\left(-\frac{|x|^2}{16t}\,\partial_t f^2+\tau t\partial_t(\log\sigma)\partial_t f^2\right)dx\,dt\\
&= \int_{\Omega\times(0,\infty)}\left(-\frac{|x|^2}{16t^2}f^2-\tau\partial_t(t\partial_t(\log\sigma))f^2\right)dx\,dt. \qquad (4.9)
\end{aligned}
$$

The Rellich–Nečas or Pohozaev identity

$$
\begin{aligned}
\nabla\cdot[(x|\nabla f|^2)-2(x\cdot\nabla f)\nabla f] &= (n-2)|\nabla f|^2-2(x\cdot\nabla f)\Delta f\\
&= (n-2)|\nabla f|^2-2\left(x\cdot\nabla f+\frac{n}{2}f\right)\Delta f+nf\Delta f\\
&= \frac{n}{2}\Delta(f^2)-2|\nabla f|^2-2\left(x\cdot\nabla f+\frac{n}{2}f\right)\Delta f
\end{aligned}
$$

gives the formula

$$
\left(x\cdot\nabla f+\frac{n}{2}f\right)\Delta f=\nabla\cdot\left[(x\cdot\nabla f)\nabla f-\frac{x}{2}|\nabla f|^2\right]+\frac{n}{4}\,\Delta(f^2)-|\nabla f|^2.
$$

Integrating the above identity in Ω, we find that for each $t>0$,

$$
\int_\Omega\left(x\cdot\nabla f+\frac{n}{4}f\right)\Delta f\,dx=\frac{1}{2}\int_{\partial\Omega}q\cdot\nu(\partial_\nu f)^2\,d\sigma-\int_\Omega|\nabla f|^2\,dx. \qquad (4.10)
$$

On the other hand,

$$
\begin{aligned}
\int_{\Omega\times(0,\infty)}&\left(x\cdot\nabla f+\frac{n}{2}f\right)\left(-\frac{|x|^2}{16t^2}f+\tau\partial_t(\log\sigma)f\right)dx\,dt\\
&= \int_{\Omega\times(0,\infty)}\frac{x}{2}\cdot\nabla(f^2)\left(-\frac{|x|^2}{16t^2}+\tau\partial_t(\log\sigma)\right)dx\,dt\\
&\quad+\frac{n}{2}\int_{\Omega\times(0,\infty)}\left(\tau\partial_t(\log\sigma)-\frac{|x|^2}{16t^2}\right)f^2\,dx\,dt\\
&= \int_{\Omega\times(0,\infty)}\frac{|x|^2}{16t^2}f^2\,dx\,dt. \qquad (4.11)
\end{aligned}
$$

Combining (4.7) and the identities (4.8)–(4.11), we get

$$
\begin{aligned}
&\|t^{1/2}\sigma(t)^{-\tau}e^{-|x|^2/8t}(\Delta+\partial_t)h\|^2_{L^2(\Omega\times(0,\infty))}\\
&\qquad\geq -\tau\int_{\Omega\times(0,\infty)}\partial_t(t\partial_t(\log\sigma))f^2\,dx\,dt+\int_{\partial\Omega\times(0,\infty)}\frac{1}{2}q\cdot\nu(\partial_\nu f)^2\,d\sigma\,dt.
\end{aligned}
$$

Choosing then $\sigma(t)=te^{-Mt}$, $M>0$, $\partial_t(t\partial_t(\log\sigma))=-M$, leads to the desired estimate. □

In Lemmas 8–11 below, we assume that Ω is a Lipschitz domain with constants (m, ϱ), and $0 \in \partial\Omega$. In Lemmas 8 and 11 we also assume that Ω is, near the origin, the region above the graph $x_n = \phi(x')$, with ϕ as in (3.1) and (3.2), so that $-\rho e$ is not in Ω when $0 < \rho \le m\varrho$, $e = (0', 1)$.

Lemma 8. *Let $\sigma(t) = te^{-t}$. Then, for $0 < \rho \le m\varrho$,*

$$\begin{aligned}\sqrt{\tau}\|\sigma(t)^{-\tau}e^{-|x+\rho e|^2/8t}h\|_{L^2(\Omega\times(0,\infty))}&\\ \le \|t^{1/2}\sigma(t)^{-\tau}e^{-|x+\rho e|^2/8t}(\Delta+\partial_t)h\|_{L^2(\Omega\times(0,\infty))}&\\ +\left\||q+\rho e|^{1/2}\sigma(t)^{-\tau}e^{-|q+\rho e|^2/8t}\partial_\nu h\right\|_{L^2(\partial\Omega\times(0,\infty))}&\end{aligned}$$

when $\tau \ge 1$ and $h \in C_0^\infty(\overline{\Omega}\times[0,\infty))$ with $h = 0$ on $\partial\Omega\times[0,\infty)$.

Proof. This follows from Lemma 7 by translation. □

Lemma 9. *There is $N = N(n)$ such that*

$$\|\nabla u\|_{L^2(B_R\cap\Omega\times(0,T/2))} \le N\sqrt{R^{-2}+T^{-1}}\,\|u\|_{L^2(B_{4R/3}\cap\Omega\times(0,T))}$$

when u satisfies $\partial_t u + \Delta u = 0$ in $B_{4R/3}\cap\Omega\times[0,T]$, and $u = 0$ on $\Delta_{4R/3}\times[0,T]$, for some $R, T > 0$.

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\alpha \in C^\infty(\mathbb{R})$ be such that $\psi = 1$ in B_R, $\psi = 0$ outside $B_{7R/6}$, $0 \le \psi \le 1$, $|\Delta\psi| \le NR^{-2}$, $\alpha = 1$ in $(-\infty, T/2]$, $\alpha = 0$ in $[3T/4, \infty)$, $0 \le \alpha \le 1$ and $|\partial_t\alpha| \le N/T$. Then

$$\begin{aligned}\int_{B_R\cap\Omega\times(0,T/2))} 2|\nabla u|^2\,dx\,dt &\le \int_{\Omega\times(0,\infty)} \psi(x)\alpha(t)(\Delta+\partial_t)(u^2)\,dx\,dt\\ &= \int_{\Omega\times(0,\infty)} (\Delta-\partial_t)(\psi(x)\alpha(t))u(x,t)^2\,dx\,dt - \int_\Omega \psi(x)u(x,0)^2\,dx\\ &\le N(R^{-2}+T^{-1})\int_{B_{4R/3}\cap\Omega\times(0,T)} u^2\,dx\,dt.\end{aligned}$$ □

Lemma 10. *Let $0 < \rho \le 1$ and $T > 0$. Then there is $N = N(n)$ such that*

$$\|u(t)\|_{L^2(B_{3\rho}\cap\Omega)} \ge \tfrac{1}{2}\|u(0)\|_{L^2(B_\rho\cap\Omega)} \tag{4.12}$$

when

$$0 < t \le \min\left\{\frac{T}{2}, \frac{\rho^2}{N\log^+\left(\frac{N\|u\|_{L^2(B_4\cap\Omega\times[0,T])}}{\rho\|u(0)\|_{L^2(B_\rho\cap\Omega)}}\right)}\right\} \tag{4.13}$$

and u satisfies $\partial_t u + \Delta u = 0$ in $B_4\cap\Omega\times[0,T]$ and $u = 0$ on $\Delta_4\times[0,T]$.

Here, $\log^+ x = \max\{\log x, 0\}$ and $\frac{1}{0} = \infty$.

Proof. Let $\psi \in C_0^\infty(B_{3\rho})$ satisfy $\psi = 1$ in $B_{2\rho}$, $0 \le \psi \le 1$ and $\rho|\nabla\psi| + \rho^2|D^2\psi| \le N$. Set $f(x,t) = u(x,t)\psi(x)$. Then

$$|\Delta f + \partial_t f| \le N(\rho^{-2}|u| + \rho^{-1}|\nabla u|)\chi_{B_{3\rho}\setminus B_{2\rho}} \quad \text{in } \Omega \times [0,T].$$

Define

$$H(t) = \int_\Omega f(x,t)^2 G(x-y,t)\,dx \quad \text{when } t>0,\ y \in B_\rho \cap \Omega,$$

with $G(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$. We have

$$\begin{aligned}\frac{d}{dt}H(t) &= 2\int_\Omega (f(\Delta f + \partial_t f) + |\nabla f|^2) G(x-y,t)\,dx\\ &\ge -N\int_{(B_{3\rho}\setminus B_{2\rho})\cap\Omega} (\rho^{-2}|u|^2 + |\nabla u|^2) G(x-y,t)\,dx\\ &\ge -N t^{-n/2} e^{-\rho^2/4t}\int_{B_{3\rho}\cap\Omega} (\rho^{-2}|u|^2 + |\nabla u|^2)\,dx\\ &\ge -N\rho^{-n} e^{-\rho^2/8t}\int_{B_{3\rho}\cap\Omega} (\rho^{-2}|u|^2 + |\nabla u|^2)\,dx\end{aligned}$$

for $0 < t < T$. Integrating the above inequality in $(0,t)$ yields

$$\int_\Omega f(x,t)^2 G(x-y,t)\,dx - u(y,0)^2 \ge -N\rho^{-n} e^{-\rho^2/8t}\int_{B_{3\rho}\cap\Omega}\int_0^t (\rho^{-2}|u|^2 + |\nabla u|^2)\,d\tau\,dx.$$

This, along with Lemma 9 with $R = 3\rho$, $T/2 = t$, shows that

$$\begin{aligned}&\int_{B_{3\rho}\cap\Omega} u(x,t)^2 G(x-y,t)\,dx - u(y,0)^2\\ &\qquad \ge -N\rho^{-n-2}(1+\rho^2/t) e^{-\rho^2/8t}\int_{B_{4\rho}\cap\Omega}\int_0^{2t} u(x,\tau)^2\,dx\,d\tau\\ &\qquad \ge -N\rho^{-n-2} e^{-\rho^2/16t}\int_{B_{4\rho}\cap\Omega}\int_0^{2t} u(x,\tau)^2\,dx\,d\tau\end{aligned}$$

when $y \in B_\rho \cap \Omega$ and $0 < 2t \le T$. Integrating the above inequality over $y \in B_\rho \cap \Omega$ and recalling that

$$\int_{\mathbb{R}^n} G(x-y,t)\,dy = 1 \quad \text{for all } x \in \mathbb{R}^n,\ t > 0,$$

we get

$$\int_{B_{3\rho}\cap\Omega} u(x,t)^2\,dx \ge \int_{B_\rho\cap\Omega} u(y,0)^2\,dy - N\rho^{-2} e^{-\rho^2/16t}\|u\|^2_{L^2(B_{4\rho}\cap\Omega\times[0,2t])}$$

when $0 < 2t \le T$. The last inequality shows that (4.12) holds when t satisfies (4.13) with $N = N(n)$. □

Lemma 11. *There are $\rho \in (0,1)$, $N = N(m,\varrho) \geq 1$ and $\theta = \theta(m,\varrho) \in (0,1)$ such that*

$$\|u(0)\|_{L^2(B_\rho\cap\Omega)} \leq \left(Ne^{N/T}\|\partial_\nu u\|_{L^2(\Delta_8\times[0,T])}\right)^\theta \|u\|^{1-\theta}_{L^2(B_8\cap\Omega\times[0,T])} \tag{4.14}$$

when u satisfies $\partial_t u + \Delta u = 0$ in $B_8 \cap \Omega \times [0,T]$, and $u = 0$ on $\Delta_8 \times [0,T]$.

The readers can find a similar interpolation inequality to (4.14) in [6, Theorem 4.6] though not with optimal T dependency, so that its application to observability boundary inequalities does not imply optimal cost constants.

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\alpha \in C^\infty(\mathbb{R})$ be such that $\psi = 1$ in B_4, $\psi = 0$ outside B_5, $0 \leq \psi \leq 1$, $|\nabla\psi| + |D^2\psi| \leq N$, $\alpha = 1$ in $(-\infty, 1]$, $\alpha = 0$ in $[2,\infty)$, $0 \leq \alpha(t) \leq 1$ and $|\partial_t\alpha(t)| \leq 1$. Let τ be a positive number satisfying $4/\tau \leq \min\{T,1\}$. Take $h(x,t) = u(x,t)\psi(x+\rho e)\alpha(\tau t)$ in Lemma 8, with $\rho \in (0,1)$ to be fixed later. Then

$$|\Delta h + \partial_t h| \leq N[(1+\tau)|u| + |\nabla u|]\chi_E(x,t)$$

with

$$E = B_5(-\rho e) \cap \Omega \times [0, 2/\tau] \setminus B_4(-\rho e) \cap \Omega \times [0, 1/\tau].$$

Since $t/e \leq \sigma(t) \leq t$ in $(0,1)$, where σ is as in Lemma 8, we have

$$t^{1/2}\sigma(t)^{-\tau}e^{-|x+\rho e|^2/8t} \leq e^\tau t^{1/2-\tau}e^{-|x+\rho e|^2/8t} \leq e^\tau \tau^{\tau-1/2} \quad \text{when } (x,t) \in E.$$

This shows that

$$\begin{aligned}\|t^{1/2}\sigma(t)^{-\tau}(\Delta h + \partial_t h)e^{-|x+\rho e|^2/8t}\|_{L^2(\Omega\times(0,\infty))}&\\ \leq Ne^\tau\tau^{\tau-1/2}\big\|(1+\tau)|u| + |\nabla u|\big\|_{L^2(B_6\cap\Omega\times[0,2/\tau])}&.\end{aligned} \tag{4.15}$$

From Lemma 9 with $R = 6$ and $T = 2/\tau$, we get

$$\big\|(1+\tau)|u| + |\nabla u|\big\|_{L^2(B_6\cap\Omega\times[0,2/\tau])} \leq N\tau\|u\|_{L^2(B_8\cap\Omega\times[0,T])}. \tag{4.16}$$

From (4.15) and (4.16), it follows that

$$\begin{aligned}\|t^{1/2}\sigma(t)^{-\tau}(\Delta h + \partial_t h)e^{-|x+\rho e|^2/8t}\|_{L^2(\Omega\times(0,\infty))}&\\ \leq e^\tau\tau^{\tau+1/2}N\|u\|_{L^2(B_8\cap\Omega\times[0,T])}, \quad N = N(n)&.\end{aligned} \tag{4.17}$$

Next, because $\partial\Omega$ is Lipschitz, there is a positive number $\beta = \beta(m,\varrho)$ such that $\Omega \cap B_{2\beta\rho}(-\rho e) = \emptyset$. Then

$$\begin{aligned}|q+\rho e|^{1/2}\sigma(t)^{-\tau}e^{-|q+\rho e|^2/8t} &\leq \sqrt{8}\,e^\tau t^{-\tau}e^{-\beta^2\rho^2/2t}\\ &\leq \sqrt{8}(\beta^2\rho^2/2)^{-\tau}\tau^\tau \quad \text{on } B_5(-\rho e)\cap\partial\Omega\times(0,1).\end{aligned}$$

Thus

$$\left\| |q+\rho e|^{1/2}\sigma(t)^{-\tau}e^{-|q+\rho e|^2/8t}\partial_\nu h\right\|_{L^2(\partial\Omega\times(0,\infty))} \leq \sqrt{8}\left(\frac{2}{\beta^2\rho^2}\right)^{\tau}\tau^{\tau}\|\partial_\nu u\|_{L^2(\Delta_8\times(0,T))}. \tag{4.18}$$

From (4.17) and (4.18) and Lemma 8, it follows that

$$\tau^{1/2}\|t^{-\tau}e^{-|x+\rho e|^2/8t}u\|_{L^2(B_4(-\rho e)\cap\Omega\times[0,1/\tau])} \leq N\left[e^{\tau}\tau^{\tau+1/2}\|u\|_{L^2(B_8\cap\Omega\times(0,T))}+\left(\frac{2}{\beta^2\rho^2}\right)^{\tau}\tau^{\tau}\|\partial_\nu u\|_{L^2(\Delta_8\times(0,T))}\right] \tag{4.19}$$

for $4/\tau \leq \min\{1,T\}$ and with $N = N(m,\varrho,n)$. Because

$$B_{3\rho}\cap\Omega\times[\rho^2/2\tau,\rho^2/\tau]\subset B_{4\rho}(-\rho e)\cap\Omega\times[\rho^2/2\tau,\rho^2/\tau]\subset B_{4\rho}(-\rho e)\cap\Omega\times[0,1/\tau],$$

we have

$$\tau^{1/2}t^{-\tau}e^{-|x+\rho e|^2/8t}\geq\tau^{\tau+1/2}\rho^{-2\tau}e^{-4\tau}\quad \text{for } (x,t)\in B_{4\rho}(-\rho e)\cap\Omega\times[\rho^2/2\tau,\rho^2/\tau].$$

Hence, the left hand side of (4.19) satisfies

$$\tau^{1/2}\|t^{-\tau}e^{-|x+\rho e|^2/8t}u\|_{L^2(B_4(-\rho e)\cap\Omega\times[0,1/\tau])} \geq \tau^{\tau+1/2}\rho^{-2\tau}e^{-4\tau}\|u\|_{L^2(B_{3\rho}\cap\Omega\times[\rho^2/2\tau,\rho^2/\tau])}. \tag{4.20}$$

Also, from Lemma 10,

$$\|u\|_{L^2(B_{3\rho}\cap\Omega\times[\rho^2/2\tau,\rho^2/\tau])}\geq\frac{\rho}{\sqrt{8}}\tau^{-1/2}\|u(0)\|_{L^2(B_\rho\cap\Omega)}$$

when

$$\frac{1}{\tau}\leq\min\left\{T,\frac{1}{N\log^+\left(\frac{N\|u\|_{L^2(B_8\cap\Omega\times[0,T])}}{\rho\|u(0)\|_{L^2(B_\rho\cap\Omega)}}\right)}\right\}.$$

This, along with (4.19) and (4.20), shows that

$$\tau^{\tau}\rho^{-2\tau}e^{-4\tau}\rho\|u(0)\|_{L^2(B_\rho\cap\Omega)} \leq N\left[e^{\tau}\tau^{\tau+1/2}\|u\|_{L^2(B_8\cap\Omega\times[0,T])}+\left(\frac{2}{\beta^2\rho^2}\right)^{\tau}\tau^{\tau}\|\partial_\nu u\|_{L^2(\Delta_8\times(0,T))}\right]$$

when

$$\frac{1}{\tau}\leq\min\left\{\frac{1}{4},\frac{T}{4},\frac{1}{N\log^+\left(\frac{N\|u\|_{L^2(B_8\cap\Omega\times[0,T])}}{\rho\|u(0)\|_{L^2(B_\rho\cap\Omega)}}\right)}\right\}. \tag{4.21}$$

Thus, there is $N = N(m, \varrho, n)$ such that

$$\rho \|u(0)\|_{L^2(B_\rho \cap \Omega)} \le (e^7 \rho^2)^\tau \frac{N}{2} \|u\|_{L^2(B_8 \cap \Omega \times [0,T])} + N^\tau \|\partial_\nu u\|_{L^2(\Delta_8 \times [0,T])} \tag{4.22}$$

when τ satisfies (4.21). Fix now ρ so that $e^7\rho^2 = e^{-1}$ and choose

$$\tau = 4 + \frac{4}{T} + N \log^+ \left(\frac{N \|u\|_{L^2(B_8 \cap \Omega \times [0,T])}}{\rho \|u(0)\|_{L^2(B_\rho \cap \Omega)}} \right). \tag{4.23}$$

Clearly, τ satisfies (4.21). Moreover,

$$\begin{aligned}(e^7\rho^2)^\tau \frac{N}{2} \|u\|_{L^2(B_8 \cap \Omega \times [0,T])} &= e^{-\tau} \left(\frac{\frac{N}{2} \|u\|_{L^2(B_8 \cap \Omega \times [0,T])}}{\rho \|u(0)\|_{L^2(B_\rho \cap \Omega)}} \right) (\rho \|u(0)\|_{L^2(B_\rho \cap \Omega)}) \\ &\le \frac{\rho}{2} \|u(0)\|_{L^2(B_\rho \cap \Omega)}.\end{aligned}$$

This together with (4.22) shows that

$$\rho \|u(0)\|_{L^2(B_\rho \cap \Omega)} \le 2N^\tau \|\partial_\nu u\|_{L^2(\Delta_8 \times [0,T])}.$$

From (4.23), it follows that

$$N^\tau = e^{\tau \log N} = e^{(4+4/T)\log N} \left(\frac{N \|u\|_{L^2(B_8 \cap \Omega \times [0,T])}}{\rho \|u(0)\|_{L^2(B_\rho \cap \Omega)}} \right)^{N \log N},$$

and then we get

$$\begin{aligned}&[\rho \|u(0)\|_{L^2(B_\rho \cap \Omega)}]^{1 + N \log N} \\ &\qquad \le e^{(4+4/T)\log N} (N \|u\|_{L^2(B_8 \cap \Omega \times [0,T])})^{N \log N} \|\partial_\nu u\|_{L^2(\Delta_8 \times [0,T])} \\ &\qquad \le (N e^{N/T} \|u\|_{L^2(B_8 \cap \Omega \times [0,T])})^{N \log N} \|\partial_\nu u\|_{L^2(\Delta_8 \times [0,T])}.\end{aligned}$$

In particular,

$$\|u(0)\|_{L^2(B_\rho \cap \Omega)} \le (N e^{N/T} \|\partial_\nu u\|_{L^2(\Delta_8 \times [0,T])})^\theta \|u\|^{1-\theta}_{L^2(B_8 \cap \Omega \times [0,T])}, \ \theta = \frac{1}{1 + N \log N}.$$

□

By translation and rescaling, Lemma 11 is equivalent to the following:

Lemma 12. *Let Ω be a Lipschitz domain with constants (m, ϱ), and let $0 < R \le 1$, $q \in \partial\Omega$, and $T > 0$. Then there are $\rho \in (0, 1]$, $N = N(m, \varrho, n) \ge 1$ and $\theta = \theta(m, \varrho, n)$ in $(0, 1)$ such that*

$$\begin{aligned}&\|u(0)\|_{L^2(B_{\rho R}(q) \cap \Omega)} \\ &\qquad \le (N e^{NR^2/T} R^{1/2} \|\partial_\nu u\|_{L^2(\Delta_{8R}(q) \times [0,T])})^\theta \, (R^{-1} \|u\|_{L^2(B_{8R}(q) \cap \Omega \times [0,T])})^{1-\theta}\end{aligned}$$

when u satisfies $\partial_t u + \Delta u = 0$ in $B_{8R}(q) \cap \Omega \times [0, T]$ and $u = 0$ on $\Delta_{8R}(q) \times [0, T]$.

Proof of Theorem 10. It suffices to prove Theorem 10 when $t_2 = T$, $t_1 = 0$ and $q = 0$. From Theorem 6 or (2.8), there are $N = N(\Omega, R)$ and $0 < \theta_1 < 1$, $\theta_1 = \theta_1(\Omega, R)$, such that

$$\|e^{T\Delta} f\|_{L^2(\Omega)} \le (Ne^{N/T}\|e^{T\Delta} f\|_{L^2(B_{\rho R/20}(0',\rho R/2))})^{\theta_1}\|f\|_{L^2(\Omega)}^{1-\theta_1}, \quad f \in L^2(\Omega), \tag{4.24}$$

with $\rho \in (0, 1)$ as in Lemma 12. On the other hand, it follows from Lemma 12 that

$$\|e^{T\Delta} f\|_{L^2(B_{\rho R}\cap\Omega)} \le (Ne^{N/T}\|\partial_\nu e^{(T-t)\Delta} f\|_{L^2(B_{8R}\cap\partial\Omega\times[0,T])})^{\theta_2}\|e^{(T-t)\Delta} f\|_{L^2(B_{8R}\cap\Omega\times[0,T])}^{1-\theta_2}.$$

In particular,

$$\|e^{T\Delta} f\|_{L^2(B_{\rho R}\cap\Omega)} \le (Ne^{N/T}\|\partial_\nu e^{t\Delta} f\|_{L^2(\Delta_{8R}\times[0,T])})^{\theta_2}\|f\|_{L^2(\Omega)}^{1-\theta_2}.$$

This, together with (4.24) and $B_{\rho R/20}(0', \rho R/2) \subset B_{\rho R}\cap\Omega$, leads to the desired estimate. □

Remark 9. It follows from Theorem 10 that there are constants $N = N(\Omega, R, n)$ and $\theta = \theta(\Omega, n) \in (0, 1)$ such that

$$\|e^{T\Delta} f\|_{L^2(\Omega)} \le (Ne^{N/[(\epsilon_2-\epsilon_1)T]}\|\partial_\nu e^{t\Delta} f\|_{L^2(\Delta_R(q)\times[\epsilon_1 T,\epsilon_2 T])})^{\theta}\|f\|_{L^2(\Omega)}^{1-\theta} \tag{4.25}$$

when $f \in L^2(\Omega)$ and $0 < \epsilon_1 < \epsilon_2 < 1$.

Lemma 13 below is a rescaled and translated version of [3, Lemma 2].

Lemma 13. *Let f be analytic in $[a, a+L]$ with $a \in \mathbb{R}$ and $L > 0$, and F be a measurable set in $[a, a+L]$. Assume that there are positive constants M and ρ such that*

$$|f^{(k)}(x)| \le Mk!(2\rho L)^{-k} \quad \text{for } k \ge 0,\ x \in [a, a+L]. \tag{4.26}$$

Then there are $N = N(\rho, |F|/L)$ and $\gamma = \gamma(\rho, |F|/L) \in (0, 1)$ such that

$$\|f\|_{L^\infty(a,a+L)} \le N\left(\fint_F |f|\, d\tau\right)^{\gamma} M^{1-\gamma}.$$

Lemma 14. *Let $q_0 \in \partial\Omega$ and $\mathfrak{J} \subset \Delta_R(q_0) \times (0, T)$ with $|\mathfrak{J}| > 0$. Set*

$$\mathfrak{J}_t = \{x \in \partial\Omega : (x, t) \in \mathfrak{J}\}, \quad t \in (0, T), \qquad E = \{t \in (0, T) : |\mathfrak{J}_t| \ge |\mathfrak{J}|/(2T)\}.$$

Then $\mathfrak{J}_t \subset \Delta_R(q_0)$ is measurable for a.e. $t \in (0, T)$, E is measurable in $(0, T)$, $|E| \ge |\mathfrak{J}|/(2|\Delta_R(q_0)|)$ and $\chi_E(t)\chi_{\mathfrak{J}_t}(x) \le \chi_{\mathfrak{J}}(x, t)$ on $\partial\Omega \times (0, T)$.

Proof. From Fubini's theorem,

$$|\mathfrak{J}| = \int_0^T |\mathfrak{J}_t|\, dt = \int_E |\mathfrak{J}_t|\, dt + \int_{[0,T]\setminus E} |\mathfrak{J}_t|\, dt \le |\Delta_R(x_0)|\,|E| + |\mathfrak{J}|/2.$$ □

Theorem 11. *Suppose that Ω satisfies condition (1.4). Assume that $q_0 \in \partial\Omega$ and $R \in (0, 1]$ are such that $\triangle_{4R}(q_0)$ is real-analytic. Let $\mathfrak{J}$ be a subset in $\Delta_R(q_0) \times (0, T)$ of positive surface measure on $\partial\Omega \times (0, T)$, and E and $\mathfrak{J}_t$ be the measurable sets associated to $\mathfrak{J}$ in Lemma 14. Then, for each $\eta \in (0, 1)$, there are $N = N(\Omega, R, |\mathfrak{J}|/(T|\Delta_R(q_0)|), \eta)$ and $\theta = \theta(\Omega, R, |\mathfrak{J}|/(T|\Delta_R(q_0)|), \eta) \in (0, 1)$ such that*

$$\|e^{t_2\Delta} f\|_{L^2(\Omega)} \le \left(N e^{N/(t_2-t_1)} \int_{t_1}^{t_2} \chi_E(t) \|\partial_\nu e^{t\Delta} f\|_{L^1(\mathfrak{J}_t)}\, dt \right)^{\theta} \|e^{t_1\Delta} f\|_{L^2(\Omega)}^{1-\theta} \tag{4.27}$$

when $0 \le t_1 < t_2 \le T$ with $t_2 - t_1 < 1$, $|E \cap (t_1, t_2)| \ge \eta(t_2 - t_1)$ and $f \in L^2(\Omega)$. Moreover,

$$\begin{aligned} e^{-\frac{N+1-\theta}{t_2-t_1}} \|e^{t_2\Delta} f\|_{L^2(\Omega)} &- e^{-\frac{N+1-\theta}{q(t_2-t_1)}} \|e^{t_1\Delta} f\|_{L^2(\Omega)} \\ &\le N \int_{t_1}^{t_2} \chi_E(t) \|\partial_\nu e^{t\Delta} f\|_{L^1(\mathfrak{J}_t)}\, dt \quad \text{when } q \ge \frac{N+1-\theta}{N+1}. \end{aligned} \tag{4.28}$$

Proof. Because $\triangle_{4R}(x_0)$ is real-analytic, according to Lemma 6, there are $N = N(\varrho, \delta)$ and $\rho = \rho(\varrho, \delta)$, $0 < \rho \le 1$, such that

$$|\partial_x^\alpha \partial_t^\beta e^{t\Delta} f(x)| \le \frac{N(t-s)^{-n/4} e^{8R^2/(t-s)} |\alpha|!\, \beta!}{(R\rho)^{|\alpha|}((t-s)/4)^\beta} \|e^{s\Delta} f\|_{L^2(\Omega)} \tag{4.29}$$

when $x \in \Delta_{2R}(x_0)$, $0 \le s < t$, $\alpha \in \mathbb{N}^n$ and $\beta \ge 0$.

Let $0 \le t_1 < t_2 \le T$ with $t_2 - t_1 < 1$, $\mathfrak{J} \subset \Delta_R(x_0) \times (0, T)$ with positive surface measure in $\partial\Omega \times (0, T)$. Let $\mathfrak{J}_t$ and E be the measurable sets associated to $\mathfrak{J}$ in Lemma 14. Assume that $\eta \in (0, 1)$ satisfies

$$|E \cap (t_1, t_2)| \ge \eta(t_2 - t_1).$$

Define

$$\begin{aligned} \tau &= t_1 + \tfrac{1}{20}\eta(t_2 - t_1), & \tilde{t}_1 &= t_1 + \tfrac{1}{8}\eta(t_2 - t_1), \\ \tilde{\tau} &= t_2 - \tfrac{1}{20}\eta(t_2 - t_1), & \tilde{t}_2 &= t_2 - \tfrac{1}{8}\eta(t_2 - t_1). \end{aligned}$$

Then

$$t_1 < \tau < \tilde{t}_1 < \tilde{t}_2 < \tilde{\tau} < t_2 \quad \text{and} \quad |E \cap (\tilde{t}_1, \tilde{t}_2)| \ge \tfrac{3}{4}\eta(t_2 - t_1).$$

Taking $T = t_2 - t_1$ (different from the T in Theorem 11 and only used here), $\epsilon_1 = \eta/20$, $\epsilon_2 = 1 - \eta/20$ and replacing f by $e^{t_1\Delta} f$ in (4.25), we get

$$\|e^{t_2\Delta} f\|_{L^2(\Omega)} \le N e^{N/(t_2-t_1)} \|\partial_\nu e^{t\Delta} f\|_{L^2(\Delta_R(x_0)\times(\tau,\tilde{\tau}))}^{\theta} \|e^{t_1\Delta} f\|_{L^2(\Omega)}^{1-\theta} \tag{4.30}$$

with $N = N(\Omega, R, \eta)$ and $\theta = \theta(\Omega) \in (0, 1)$. From (4.29) with $\beta = 0$, $|\alpha| = 1$ and $s = t_1$, there is $N = N(\Omega, R, \eta)$ such that

$$\|\partial_\nu e^{t\Delta} f\|_{L^\infty(\Delta_{2R}(x_0)\times(\tau,\tilde{\tau}))} \le N e^{N/(t_2-t_1)} \|e^{t_1\Delta} f\|_{L^2(\Omega)}. \tag{4.31}$$

Next, (4.30), the interpolation inequality

$$\|\partial_\nu e^{t\Delta} f\|_{L^2(\Delta_R(x_0)\times(\tau,\tilde{\tau}))} \le \|\partial_\nu e^{t\Delta} f\|^{1/2}_{L^1(\Delta_R(x_0)\times(\tau,\tilde{\tau}))}\|\partial_\nu e^{t\Delta} f\|^{1/2}_{L^\infty(\Delta_R(x_0)\times(\tau,\tilde{\tau}))}$$

and (4.31) yield

$$\|e^{t_2\Delta} f\|_{L^2(\Omega)} \le N e^{N/(t_2-t_1)}\|e^{t_1\Delta} f\|^{1-\theta/2}_{L^2(\Omega)}\|\partial_\nu e^{t\Delta} f\|^{\theta/2}_{L^1(\Delta_R(x_0)\times(\tau,\tilde{\tau}))} \tag{4.32}$$

with $N = N(\Omega, R, \eta)$ and $\theta = \theta(\Omega) \in (0, 1)$. Next, setting

$$v(x,t) = \partial_\nu e^{t\Delta} f(x) \quad \text{ for } x \in \Delta_{2R}(x_0),\ t > 0,$$

we have

$$\|v\|_{L^1(\Delta_R(x_0)\times(\tau,\tilde{\tau}))} \le (\tilde{\tau}-\tau)\int_{\Delta_R(x_0)} \|v(x,\cdot)\|_{L^\infty(\tau,\tilde{\tau})}\, d\sigma. \tag{4.33}$$

Write $[\tau, \tilde{\tau}] = [a, a+L]$, with $a = \tau$ and $L = \tilde{\tau} - \tau = (1-\eta/10)(t_2 - t_1)$. Also, (4.29) with $|\alpha| = 1$ and $s = t_1$ shows that there is $N = N(\Omega, R, \eta)$ such that for each fixed $x \in \Delta_{2R}(x_0)$, $t \in [\tau, \tilde{\tau}]$ and $\beta \ge 0$,

$$\begin{aligned} |\partial_t^\beta v(x,t)| &\le \frac{N e^{N/(t-t_1)}\beta!}{((t-t_1)/4)^\beta}\|e^{t_1\Delta} f\|_{L^2(\Omega)} \le \frac{N e^{N/(t_2-t_1)}\beta!}{(\eta(t_2-t_1)/80)^\beta}\|e^{t_1\Delta} f\|_{L^2(\Omega)} \\ &= \frac{M\beta!}{(2\rho L)^\beta} \end{aligned} \tag{4.34}$$

with

$$M = N e^{N/(t_2-t_1)}\|e^{t_1\Delta} f\|_{L^2(\Omega)} \quad \text{and} \quad \rho = \frac{\eta}{16(10-\eta)}.$$

From (4.34), Lemma 13 with $F = E \cap (\tilde{t}_1, \tilde{t}_2)$, and observing that

$$\frac{15\eta}{2(10-\eta)} \le \frac{|F|}{L} \le \frac{5(4-\eta)}{2(10-\eta)},$$

we find that for each x in $\Delta_{2R}(x_0)$,

$$\|v(x,\cdot)\|_{L^\infty(\tau,\tilde{\tau})} \le \left(\fint_{E\cap(\tilde{t}_1,\tilde{t}_2)} |v(x,t)|\, dt\right)^\gamma (N e^{N/(t_2-t_1)}\|e^{t_1\Delta} f\|_{L^2(\Omega)})^{1-\gamma} \tag{4.35}$$

with $N = N(\Omega, R, \eta)$ and $\gamma = \gamma(\eta) \in (0, 1)$. Thus, (4.33) and (4.35) yield

$$\begin{aligned} &\|v\|_{L^1(\Delta_R(x_0)\times(\tau,\tilde{\tau}))} \\ &\quad \le \int_{\Delta_R(x_0)} \left(\int_{E\cap(\tilde{t}_1,\tilde{t}_2)} |v(x,t)|\, dt\right)^\gamma d\sigma\, (N e^{N/(t_2-t_1)}\, \|e^{t_1\Delta} f\|_{L^2(\Omega)})^{1-\gamma} \\ &\quad \le \left(\int_{E\cap(\tilde{t}_1,\tilde{t}_2)}\int_{\Delta_R(x_0)} |v(x,t)|\, d\sigma\, dt\right)^\gamma N e^{N/(t_2-t_1)}\|e^{t_1\Delta} f\|^{1-\gamma}_{L^2(\Omega)} \end{aligned} \tag{4.36}$$

with N and γ as above. In the second inequality in (4.36) we have used Hölder's inequality.

Next, from (4.29) with $\beta = 0$ and $s = t_1$, we see that for $t \in (\tilde{t}_1, \tilde{t}_2)$ we have

$$t - t_1 \geq \tilde{t}_1 - t_1 = \tfrac{1}{8}\eta(t_2 - t_1)$$

and

$$\|\partial_x^\alpha v(\cdot, t)\|_{L^\infty(\Delta_{2R}(x_0))} \leq \frac{N e^{N/(t_2-t_1)}|\alpha|!}{(R\rho)^{|\alpha|}} \|e^{t_1\Delta} f\|_{L^2(\Omega)}$$

for all $\alpha \in \mathbb{N}^n$ and with $N = N(\Omega, R, \eta)$. Also, $|\mathcal{J}_t| \geq |\mathcal{J}|/(2T)$ when $t \in E$. By the obvious generalization of Theorem 4 to real-analytic hypersurfaces, there are $N = N(\Omega, R, \eta, |\mathcal{J}|/(T|\Delta_R(x_0)|))$ and $\vartheta = \vartheta(\Omega, |\mathcal{J}|/(T|\Delta_R(x_0)|)) \in (0, 1)$ such that

$$\int_{\Delta_R(x_0)} |v(x,t)|\, d\sigma \leq \left(\int_{\mathcal{J}_t} |v(x,t)|\, d\sigma\right)^{\vartheta} \left(N e^{N/(t_2-t_1)} \|e^{t_1\Delta} f\|_{L^2(\Omega)}\right)^{1-\vartheta} \tag{4.37}$$

when $t \in E \cap (\tilde{t}_1, \tilde{t}_2)$. Now (4.36) and (4.37), together with Hölder's inequality, imply that

$$\|v\|_{L^1(\Delta_R(x_0)\times(\tau,\tilde{\tau}))} \leq \left(N e^{N/(t_2-t_1)} \int_{E\cap(\tilde{t}_1,\tilde{t}_2)} \int_{\mathcal{J}_t} |v(x,t)|\, d\sigma\, dt\right)^{\vartheta\gamma} \|e^{t_1\Delta} f\|_{L^2(\Omega)}^{1-\vartheta\gamma}.$$

This, along with (4.32) and the definition of v, leads to the first estimate in the theorem.

The second estimate can be proved with the method used in the proof of the second part of Theorem 7. □

Proof of Theorem 2. Let E and $\mathcal{J}_t$ be the sets associated to $\mathcal{J}$ in Lemma 14, and l be a density point in E. For $z > 1$ to be fixed later, $\{l_m\}$ denotes the sequence associated to l and z in Lemma 2. Because of (2.16) and from Theorem 11 with $\eta = 1/3$, $t_1 = l_{m+1}$ and $t_2 = l_m$, with $m \geq 1$, there are $N = N(\Omega, R, |\mathcal{J}|/(T|\Delta_R(q_0)|)) > 0$ and $\theta = \theta(\Omega, R, |\mathcal{J}|/(T|\Delta_R(q_0)|)) \in (0, 1)$ such that

$$\begin{aligned} &e^{-\frac{N+1-\theta}{l_m - l_{m+1}}} \|e^{l_m\Delta} f\|_{L^2(\Omega)} - e^{-\frac{N+1-\theta}{q(l_m - l_{m+1})}} \|e^{l_{m+1}\Delta} f\|_{L^2(\Omega)} \\ &\quad \leq N \int_{l_{m+1}}^{l_m} \chi_E(s) \|\partial_\nu e^{s\Delta} f\|_{L^1(\mathcal{J}_s)}\, ds \quad \text{when } q \geq \frac{N+1-\theta}{N+1} \text{ and } m \geq 1. \end{aligned}$$

Let

$$z = \frac{1}{2}\left(1 + \frac{N+1}{N+1-\theta}\right).$$

Then we can use the same arguments as those in the proof of Theorem 1 to verify the conclusion of Theorem 2. □

Remark 10. The proof of Theorem 2 also implies the following observability estimate:

$$\sup_{m\geq 0} \sup_{l_{m+1}\leq t\leq l_m} e^{-z^{m+1}A} \|e^{t\Delta} f\|_{L^2(\Omega)} \leq N \int_{\mathcal{J}\cap(\partial\Omega\times[l,l_1])} |\partial_\nu e^{t\Delta} f(x)|\, d\sigma\, dt$$

for f in $L^2(\Omega)$, with $A = 2(N+1-\theta)^2/[\theta(l_1 - l)]$ and with z, N and θ as given along the proof of Theorem 2. Here, $l_0 = T$.

Remark 11. (i) In Theorem 2, one may relax the hypothesis on Ω and allow $\triangle_{4R}(q_0)$ to be piecewise analytic or simply require that

$$|\{q \in \triangle_{4R}(q_0) : \Delta_{4r}(q) \text{ is not real-analytic for some } r \leq R\}| = 0.$$

(ii) In particular, Theorem 2 holds when Ω is a Lipschitz polyhedron in $\mathbb{R}^n$ and $\mathcal{J}$ is a measurable subset with positive surface measure in $\partial\Omega \times (0, T)$. For if Ω is a Lipschitz polyhedron, Theorem 8 shows that Ω satisfies (1.4). Also, $\mathcal{J}$ must have a boundary density point (q, τ), $q \in \partial\Omega$, $\tau \in (0, T)$, with q in the interior of one the open flat faces of $\partial\Omega$. Thus, we can find $R > 0$ such that

$$\frac{|\Delta_R(q) \times (\tau - R, \tau + R) \cap \mathcal{J}|}{|\Delta_R(q) \times (\tau - R, \tau + R)|} \geq \frac{1}{2},$$

with $\triangle_{4R}(q)$ contained in a flat face of $\partial\Omega$. Then we replace the original set $\mathcal{J}$ by $\Delta_R(q) \times (\tau - R, \tau + R) \cap \mathcal{J} \subset \Delta_R(q) \times (0, T)$, set $q_0 = q$ and apply Theorem 2 as stated.

(iii) Theorem 2 improves the work in [35].

Remark 12. When $\mathcal{J} = \Gamma \times (0, T)$ and $\Gamma \subset \Delta_R(q_0)$ is a measurable set, one may take $l = T/2$, $l_1 = T$, $z = 2$, and the constant B in Theorem 2 becomes

$$B = A(\Omega, R, |\Gamma|/|\Delta_R(q_0)|)/T.$$

Remark 13. Theorem 10 also implies the following: if Ω is only a bounded Lipschitz domain and satisfies (1.4), the heat equation is null controllable with $L^\infty(\Gamma \times (0, T))$ controls when Γ is an open subset of $\partial\Omega$. This seems to be a new result and we give its proof in the Appendix (Section 6).

5. Applications

Throughout this section, we assume that $T > 0$ and Ω is a bounded Lipschitz domain satisfying condition (1.4), and we show several applications of Theorems 1 and 2 to control problems for the heat equation.

First of all, we will show that Theorems 1 and 2 imply null controllability with controls restricted over measurable subsets in $\Omega \times (0, T)$ and $\partial\Omega \times (0, T)$ respectively. Let $\mathcal{D}$ be a measurable subset with positive measure in $B_R(x_0) \times (0, T)$ with $B_{4R}(x_0) \subset \Omega$. Let $\mathcal{J}$ be a measurable subset with positive surface measure in $\Delta_R(q_0) \times (0, T)$, where $q_0 \in \partial\Omega$, $R \in (0, 1]$ and $\triangle_{4R}(q_0)$ is real-analytic. Consider the following controlled heat equations:

$$\begin{cases} \partial_t u - \Delta u = \chi_{\mathcal{D}} v & \text{in } \Omega \times (0, T], \\ u = 0 & \text{on } \partial\Omega \times [0, T], \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{5.1}$$

and

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0, T], \\ u = g\chi_{\mathcal{J}} & \text{on } \partial\Omega \times [0, T], \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{5.2}$$

where $u_0 \in L^2(\Omega)$, $v \in L^\infty(\Omega \times (0,T))$ and $g \in L^\infty(\partial\Omega \times (0,T))$ are controls. We say that u is a solution to (5.2) if $v \equiv u - e^{t\Delta}u_0$ is the unique solution defined in [14, Theorem 3.2] (see also [4, Theorems 8.1 and 8.3]) to

$$\begin{cases} \partial_t v - \Delta v = 0 & \text{in } \Omega \times (0,T), \\ v = g\chi_{\mathcal{J}} & \text{on } \partial\Omega \times (0,T), \\ v(0) = 0 & \text{in } \Omega, \end{cases}$$

with g in $L^p(\partial\Omega \times (0,T))$ for some $2 \le p \le \infty$. From now on, we always denote by $u(\cdot\,; u_0, v)$ and $u(\cdot\,; u_0, g)$ the solutions of (5.1) and (5.2) corresponding to v and g respectively.

Corollary 1. *For each $u_0 \in L^2(\Omega)$, there are bounded control functions v and g with*

$$\|v\|_{L^\infty(\Omega\times(0,T))} \le C_1\|u_0\|_{L^2(\Omega)}, \qquad \|g\|_{L^\infty(\partial\Omega\times(0,T))} \le C_2\|u_0\|_{L^2(\Omega)},$$

such that $u(T; u_0, v) = 0$ and $u(T; u_0, g) = 0$. Here $C_1 = C(\Omega, T, R, \mathcal{D})$ and $C_2 = C(\Omega, T, R, \mathcal{J})$.

Proof. We only prove boundary controllability. Let E be the measurable set associated to $\mathcal{J}$ in Lemma 14. Write

$$\widetilde{\mathcal{J}} = \{(x,t) : (x, T-t) \in \mathcal{J}\} \quad \text{and} \quad \widetilde{E} = \{t : T - t \in E\}.$$

Let $l > 0$ be a density point of $\widetilde{E}$ (hence $T - l$ is a density point of E). We choose z, l_1 and a sequence $\{l_m\}$ as in the proof of Theorem 2 but with $\mathcal{J}$ and E replaced by $\widetilde{\mathcal{J}}$ and $\widetilde{E}$. It is clear that

$$0 < l < \cdots < l_{m+1} < l_m < \cdots < l_1 < l_0 = T, \qquad \lim_{m\to\infty} l_m = l.$$

We set

$$\mathcal{M} = \mathcal{J} \cap (\partial\Omega \times [T - l_1, T - l]) \subset \mathcal{J}.$$

It is clear that $|\mathcal{M}| > 0$. The proof of Theorem 2, the change of variables $t = T - \tau$ and Remark 10 show that the observability inequality

$$\|\varphi(0)\|_{L^2(\Omega)} \le e^B \int_{\mathcal{M}} |\partial_\nu \varphi(p,t)|\, d\sigma\, dt \tag{5.3}$$

holds when φ is the unique solution in $L^\infty([0,T], L^2(\Omega)) \cap L^2([0,T], H_0^1(\Omega))$ to

$$\begin{cases} \partial_t \varphi + \Delta\varphi = 0 & \text{in } \Omega \times [0,T), \\ \varphi = 0 & \text{on } \partial\Omega \times [0,T), \\ \varphi(T) = \varphi_T & \text{in } \partial\Omega, \end{cases} \tag{5.4}$$

for some φ_T in $L^2(\Omega)$. Set

$$X = \{\partial_\nu \varphi|_{\mathcal{M}} : \varphi(t) = e^{(T-t)\Delta}\varphi_T \text{ for } 0 \le t \le T, \text{ for some } \varphi_T \in L^2(\Omega)\}.$$

Since $\mathcal{M} \subset \partial\Omega \times [T - l_1, T - l]$, X is a subspace of $L^1(\mathcal{M})$ (see (6.4) and (6.5)), and from (5.3), the linear mapping $\Lambda : X \to \mathbb{R}$ defined by

$$\Lambda(\partial_\nu \varphi|_{\mathcal{M}}) = (u_0, \varphi(0))$$

satisfies

$$|\Lambda(\partial_\nu \varphi|_{\mathcal{M}})| \le e^B \|u_0\|_{L^2(\Omega)} \int_{\mathcal{M}} |\partial_\nu \varphi(p, t)|\, d\sigma\, dt \quad \text{when } \partial_\nu \varphi|_{\mathcal{M}} \in X.$$

From the Hahn–Banach theorem, there is a linear extension $T : L^1(\mathcal{M}) \to \mathbb{R}$ of Λ with

$$\begin{aligned} T(\partial_\nu \varphi|_{\mathcal{M}}) &= (u_0, \varphi(0)) && \text{when } \partial_\nu \varphi|_{\mathcal{M}} \in X, \\ |T(f)| &\le e^B \|u_0\|\, \|f\|_{L^1(\mathcal{M})} && \text{for all } f \in L^1(\mathcal{M}). \end{aligned}$$

Thus, T is in $L^1(\mathcal{M})^* = L^\infty(\mathcal{M})$ and there is g in $L^\infty(\mathcal{M})$ satisfying

$$T(f) = \int_{\mathcal{M}} fg\, d\sigma\, dt \quad \text{for all } f \in L^1(\mathcal{M}), \quad \text{and} \quad \|g\|_{L^\infty(\mathcal{M})} \le e^B \|u_0\|.$$

We extend g over $\partial\Omega \times (0, T)$ by setting it to be zero outside $\mathcal{M}$ and denote the extended function by g again. Then $u(T; u_0, g) = 0$ provided that we know that

$$\int_\Omega u(T; u_0, g)\varphi_T\, dx = \int_\Omega u_0 \varphi(0)\, dx - \int_{\mathcal{M}} g\, \partial_\nu \varphi\, d\sigma\, dt \quad \text{for all } \varphi_T \in L^2(\Omega). \tag{5.5}$$

To prove (5.5), we first use the unique solvability for the problem

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0, T], \\ u = \gamma & \text{on } \partial\Omega \times [0, T], \\ u(0) = 0 & \text{in } \Omega, \end{cases}$$

with lateral Dirichlet data γ in $L^p(\partial\Omega \times (0, T))$, $2 \le p \le \infty$, established in [14, Theorem 3.2] (see also [4, Theorems 8.1 and 8.3]). Then, because $g\chi_{\mathcal{M}}$ is bounded and supported in $\partial\Omega \times [T - l_1, T - l] \subset \partial\Omega \times (2\eta, T - 2\eta)$ for some $\eta > 0$, the calculations leading to (5.5) can be justified via the regularization of $g\chi_{\mathcal{M}}$ and the approximation of Ω by smooth domains $\{\Omega_j : j \ge 1\}$ as in [4, Lemma 2.2]. For the sake of completeness we provide the detailed proof of this identity in the Appendix. □

Now we apply Theorems 1 and 2 to get the bang-bang property for the minimal time control problems usually called the first type of time optimal control problems. They are stated as follows: Let ω be a measurable subset with positive measure in $B_R(x_0)$, with $B_{4R}(x_0) \subset \Omega$. Suppose that $\Delta_{4R}(q_0)$ is real-analytic for some $q_0 \in \partial\Omega$ and $R \in (0, 1]$, and let Γ be a measurable subset of $\Delta_R(x_0)$ with positive surface measure. For each $M > 0$, define the following control constraint sets:

$$\begin{aligned} \mathcal{U}_M^1 &= \{v \text{ measurable on } \Omega \times \mathbb{R}^+ : |v(x, t)| \le M \text{ for a.e. } (x, t) \in \Omega \times \mathbb{R}^+\}, \\ \mathcal{U}_M^2 &= \{g \text{ measurable on } \partial\Omega \times \mathbb{R}^+ : |g(x, t)| \le M \text{ for a.e. } (x, t) \in \partial\Omega \times \mathbb{R}^+\}. \end{aligned}$$

Let $u_0 \in L^2(\Omega) \setminus \{0\}$. Consider the minimal time control problems

$$(TP)^1_M : T^1_M \equiv \min_{v \in \mathcal{U}^1_M} \left\{ t > 0 : e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta}(\chi_\omega v)\, ds = 0 \right\},$$

$$(TP)^2_M : T^2_M \equiv \min_{g \in \mathcal{U}^2_M} \{ t > 0 : u(x, t; g) = 0 \text{ for a.e. } x \in \Omega \},$$

where $u(\cdot, \cdot; g)$ is the solution to

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ u = g\chi_\Gamma & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{5.6}$$

Any solution of $(TP)^i_M$, $i = 1, 2$, is called a *minimal time control* for this problem. According to Theorem 1 and [41, Theorem 3.3], Problem $(TP)^1_M$ has solutions. By Theorem 2, using the same arguments as in [41, proof of Theorem 3.3], we can verify that there is $g \in \mathcal{U}^2_M$ such that for some $t > 0$, $u(x, t; g) = 0$ for a.e. $x \in \Omega$.

Lemma 15. *Problem $(TP)^2_M$ has solutions.*

Proof. Let $\{t_n\}_{n\geq 1}$ with $t_n \searrow T^2_M$ and $g_n \in \mathcal{U}^2_M$ be such that $u(x, t_n; g_n) = 0$ over Ω. Hence, for a subsequence,

$$g_n \to g^* \quad \text{weakly star in } L^\infty(\partial\Omega \times (0, t_1)). \tag{5.7}$$

It suffices to show that

$$u_n(x, t_n) \equiv u(x, t_n; g_n) \to u^*(x, T^2_M) \equiv u(x, T^2_M; g^*) \quad \text{for all } x \in \Omega. \tag{5.8}$$

For this purpose, let $G(x, y, t)$ be the Green function for $\Delta - \partial_t$ in $\Omega \times \mathbb{R}$ with zero lateral Dirichlet boundary condition. [14, Theorems 1.3 and 1.4, and p. 643] show that for $g \in \mathcal{U}^2_M$ and $(x, t) \in \Omega \times (0, T)$,

$$u(x, t; g) = e^{t\Delta} u_0 - \int_0^t \int_{\partial\Omega} \partial_{\nu_q} G(x, q, t - s)\chi_\Gamma(q, s) g(q, s)\, d\sigma_q\, ds \tag{5.9}$$

and

$$\int_0^T \int_{\partial\Omega} |\partial_{\nu_q} G(x, q, \tau)|^2\, d\sigma_q\, d\tau < \infty \quad \text{when } x \in \Omega,\ T > 0. \tag{5.10}$$

Also, by standard interior parabolic regularity there is $N = N(n, \epsilon)$ with

$$|u(x, t; g) - u(x, s; g)| \leq N|t - s|(\|g\|_{L^\infty(\partial\Omega\times(0,T))} + \|u_0\|_{L^2(\Omega)}) \tag{5.11}$$

when $d(x, \partial\Omega) > \sqrt{\epsilon}$ and $t > s \geq \epsilon$. Now, when $x \in \Omega$ with $d(x, \partial\Omega) > \sqrt{\epsilon}$,

$$|u_n(x, t_n) - u^*(x, T^2_M)| \leq |u_n(x, t_n) - u_n(x, T^2_M)| + |u_n(x, T^2_M) - u^*(x, T^2_M)|.$$

This along with (5.7), (5.9)–(5.11) indicates that (5.8) holds for all $x \in \Omega$ with $d(x, \partial\Omega) > \sqrt{\epsilon}$. Since $\epsilon > 0$ is arbitrary, (5.8) follows at once. ◻

Now, one can use the same methods as in [44], as well as in Lemma 15, to get the following consequences of Theorems 1 and 2 respectively:

Corollary 2. *Problem $(TP)^1_M$ has the bang-bang property: any minimal time control v satisfies $|v(x,t)| = M$ for a.e. $(x,t) \in \omega \times (0, T^1_M)$. Consequently, this problem has a unique minimal time control.*

Corollary 3. *Problem $(TP)^2_M$ has the bang-bang property: any minimal time boundary control g satisfies $|g(x,t)| = M$ for a.e. $(x,t) \in \Gamma \times (0, T^2_M)$. Consequently, this problem has a unique minimal time control.*

Next, we make use of Theorems 1 and 2 to study the bang-bang property for time optimal control problems where the interest is in retarding the initial time of the action of a control with bounded L^∞-norm. These problems are usually called the second type of time optimal control problems and are stated as follows: Let $T > 0$ and $M > 0$. Write ω and Γ for the sets given in Problems $(TP)^1_M$ and $(TP)^2_M$ respectively. Consider the controlled heat equations

$$\begin{cases} \partial_t u - \Delta u = \chi_\omega \chi_{(\tau,T)} v & \text{in } \Omega \times (0,T], \\ u = 0 & \text{on } \partial\Omega \times [0,T], \\ u(0) = u_0 & \text{in } \Omega \end{cases} \tag{5.12}$$

and

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,T], \\ u = \chi_\Gamma \chi_{(\tau,T)} g & \text{on } \partial\Omega \times [0,T], \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{5.13}$$

where $u_0 \in L^2(\Omega)$. Write $u(\cdot\,; \chi_{(\tau,T)} v)$ and $u(\cdot\,; \chi_{(\tau,T)} g)$ for the solutions to (5.12) corresponding to $\chi_{(\tau,T)} v$, and to (5.13) corresponding to $\chi_{(\tau,T)} g$. Define the following control constraint sets:

$$\mathcal{U}^1_{M,T} = \{v \text{ measurable on } \Omega \times (0,T) : |v(x,t)| \le M \text{ for a.e. } (x,t) \in \Omega \times (0,T)\},$$
$$\mathcal{U}^2_{M,T} = \{g \text{ measurable on } \partial\Omega \times (0,T) : |g(x,t)| \le M \text{ for a.e. } (x,t) \in \partial\Omega \times (0,T)\}.$$

Consider the time optimal control problems

$$(TP)^1_{M,T} : \ \tau^1_{M,T} \equiv \sup_{v \in \mathcal{U}^1_{M,T}} \{\tau \in [0,T) : u(T; \chi_{(\tau,T)} v) = 0\},$$
$$(TP)^2_{M,T} : \ \tau^2_{M,T} \equiv \sup_{g \in \mathcal{U}^2_{M,T}} \{\tau \in [0,T) : u(T; \chi_{(\tau,T)} g) = 0\}.$$

Any solution of $(TP)^i_{T,M}$, $i = 1, 2$, is called an *optimal control* for the corresponding problem.

Now, we can use the same arguments as in [40, proof of Theorem 3.4] to get the following consequences of Theorems 1 and 2 respectively:

Corollary 4. *Any optimal control v^* to Problem $(TP)^1_{M,T}$, if it exists, has the bang-bang property: $|v^*(x,t)| = M$ for a.e. $(x,t) \in \omega \times (\tau^1_{M,T}, T)$.*

Corollary 5. *Any optimal control g^* to Problem $(TP)^2_{M,T}$, if it exists, has the bang-bang property: $|g^*(x,t)| = M$ for a.e. $(x,t) \in \Gamma \times (\tau^2_{M,T}, T)$.*

Remark 14. By Theorem 1 (see also Remark 2) and the energy decay property for the heat equation, one can easily prove the following: for a fixed $M > 0$, there is $v \in \mathcal{U}^1_{M,T}$ such that $u(T; \chi_{(0,T)}v) = 0$ when T is large enough (such a control v is called an *admissible control*); while for a fixed $T > 0$, the same holds when M is large enough. The same can be said about Problem $(TP)^2_{M,T}$ because of Theorem 2 (see also Remark 12). In the case when Problem $(TP)^1_{M,T}$ has admissible controls, one can easily prove the existence of time optimal controls for this problem. If Problem $(TP)^2_{M,T}$ has admissible controls, one can make use of a similar method to the proof of Lemma 15 to verify the existence of time optimal controls for this problem.

Finally, we utilize Theorems 1 and 2 to study the bang-bang property for the minimal norm control problems stated as follows: Let $\mathcal{D}$ and $\mathcal{J}$ be the subsets given at the beginning of this section. Let $u_0 \in L^2(\Omega)$. Define two control constraint sets as follows:

$$\mathcal{V}_{\mathcal{D}} = \{v \in L^\infty(\Omega \times (0,T)) : u(T; u_0, v) = 0\},$$
$$\mathcal{V}_{\mathcal{J}} = \{g \in L^\infty(\partial\Omega \times (0,T)) : u(T; u_0, g) = 0\}.$$

Consider the minimal norm control problems:

$$(NP)_{\mathcal{D}} : \ M_{\mathcal{D}} \equiv \min\{\|v\|_{L^\infty(\Omega\times(0,T))} : v \in \mathcal{V}_{\mathcal{D}}\},$$
$$(NP)_{\mathcal{J}} : \ M_{\mathcal{J}} \equiv \min\{\|g\|_{L^\infty(\partial\Omega\times(0,T))} : g \in \mathcal{V}_{\mathcal{J}}\}.$$

Any solution of $(NP)_{\mathcal{D}}$ (or $(NP)_{\mathcal{J}}$) is called a *minimal norm control* for this problem. According to Corollary 1, the sets $\mathcal{V}_{\mathcal{D}}$ and $\mathcal{V}_{\mathcal{J}}$ are not empty. Since $\mathcal{V}_{\mathcal{D}}$ is not empty, it follows from the standard arguments that Problem $(NP)_{\mathcal{D}}$ has solutions. Because $\mathcal{V}_{\mathcal{J}}$ is not empty, by using similar arguments to those in the proof of Lemma 15, we can show that Problem $(NP)_{\mathcal{J}}$ has solutions. Now, one can use the same methods as in [40] to get the following consequences of Theorems 1 and 2 respectively:

Corollary 6. *Problem $(NP)_{\mathcal{D}}$ has the bang-bang property: any minimal norm control v satisfies $|v(x,t)| = M_{\mathcal{D}}$ for a.e. $(x,t) \in \mathcal{D}$. Consequently, this problem has a unique minimal norm control.*

Corollary 7. *Problem $(NP)_{\mathcal{J}}$ has the bang-bang property: any minimal norm boundary control g satisfies $|g(x,t)| = M_{\mathcal{J}}$ for a.e. $(x,t) \in \mathcal{J}$. Consequently, this problem has a unique minimal norm control.*

6. Appendix

Proof of (5.5). For each $(p,\tau) \in \partial\Omega \times \mathbb{R}$ and fixed $\xi > 0$, we define

$$\Gamma(p) = \{x \in \Omega : |x - p| \le (1+\xi)d(x,\partial\Omega)\},$$
$$\Gamma(p,\tau) = \{(x,t) \in \Omega \times (0,T) : |x-p| + \sqrt{|t-\tau|} \le (1+\xi)d(x,\partial\Omega)\}.$$

These are called respectively *elliptic* and *parabolic nontangential approach regions* from the interior of $\Omega \times (0, T)$ to (p, τ). In particular,

$$\Gamma(p) \times \{\tau\} \subset \Gamma(p, \tau) \quad \text{for all } (p, \tau) \in \partial\Omega \times (0, T).$$

When $u : \Omega \to \mathbb{R}$ or $u : \Omega \times (0, T) \to \mathbb{R}$ (or $\mathbb{R}^n$), define the *elliptic* and *parabolic nontangential maximal function* of u in $\partial\Omega \times (0, T)$ as

$$u^*(p) = \sup_{x \in \Gamma(p)} |u(x)|, \quad u^\sharp(p, \tau) = \sup_{(x,t) \in \Gamma(p,\tau)} |u(x, t)|, \quad p \in \partial\Omega,\ \tau \in (0, T).$$

Let $\eta > 0$ be fixed such that $[T - l_1, T - l] \subset [2\eta, T - 2\eta]$, with l and l_1 as defined in Corollary 1. Denote by u the solution to

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0, T), \\ u = g\chi_{\mathcal{M}} \equiv \gamma & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega \end{cases}$$

(see the beginning of Section 5 for the definition of the solution).

Let γ^ε in $C_0^1(\partial\Omega \times (0, T))$ be a regularization of γ in $\partial\Omega \times [0, T]$ such that

$$\|\gamma^\varepsilon\|_{L^\infty(\partial\Omega\times[0,T])} + \epsilon\, \|\gamma^\varepsilon\|_{C^1(\partial\Omega\times[0,T])} \le \|\gamma\|_{L^\infty(\partial\Omega\times[0,T])},$$

$$\operatorname{supp}(\gamma^\varepsilon) \subset \partial\Omega \times [\eta, T - \eta],$$

and let v^ε be the solution to

$$\begin{cases} \partial_t v^\varepsilon - \Delta v^\varepsilon = 0 & \text{in } \Omega \times (0, T), \\ v^\varepsilon = \gamma^\varepsilon & \text{on } \partial\Omega \times (0, T), \\ v^\varepsilon(0) = 0 & \text{in } \Omega. \end{cases}$$

From [14, Theorem 3.2] and either [4, Theorem 6.1] or [5, Theorem 2.9],

$$\|v^\varepsilon\|_{L^\infty(\partial\Omega\times[0,T])} + \epsilon\, \|(\nabla v^\epsilon)^\sharp\|_{L^2(\partial\Omega\times[0,T])} \le \|\gamma\|_{L^\infty(\partial\Omega\times[0,T])}, \tag{6.1}$$

and the limits

$$\lim_{\substack{(x,t)\in\Gamma(p,\tau) \\ (x,t)\to(p,\tau)}} \nabla v^\epsilon(x, t) = \nabla v^\epsilon(p, \tau)$$

exist and are finite for a.e. (p, τ) in $\partial\Omega \times (0, T)$. Also, $v_\epsilon \in C(\overline{\Omega} \times [0, T]) \cap C^\infty(\Omega \times [0, T])$, $v^\varepsilon = 0$ for $t \le \eta$, and $v^\varepsilon = 0$ on $\partial\Omega \times (T - \eta, T]$. Moreover, the Hölder regularity up to the boundary for bounded solutions to parabolic equations with zero local lateral Dirichlet data shows that there are positive constants $N = N(m, \varrho, \eta)$ and $\alpha = \alpha(m, \varrho) \in (0, 1)$ such that

$$|v^\varepsilon(x_1, t_1) - v^\varepsilon(x_2, t_2)| \le N[|x_1 - x_2|^2 + |t_1 - t_2|]^{\alpha/2} \|\gamma\|_{L^\infty(\partial\Omega\times[0,T])} \tag{6.2}$$

when $x_1, x_2 \in \overline{\Omega}$, $T - \eta/2 \le t_1, t_2 \le T$ [31, Theorems 6.28 and 6.32].

Let $\varphi(t) = e^{(T-t)\Delta}\varphi_T$, $t \in (0, T)$, where φ_T is in $L^2(\Omega)$. From the regularity of caloric functions [13, Theorem 1.7],

$$\varphi \in C([0, T]; L^2(\Omega)) \cap C^\infty(\Omega \times [0, T)) \cap C(\overline{\Omega} \times [0, T)), \tag{6.3}$$

and from [14, Theorems 1.3 and 1.4] or the proof of (6.4) and (6.5) later in this appendix, there are $N = N(m, \varrho)$ and $\epsilon = \epsilon(m, \varrho, n) > 0$ such that

$$\|(\nabla\varphi)^*\|_{L^\infty(0,T-\delta;\, L^{2+\epsilon}(\partial\Omega))} \leq N e^{1/\delta} \|\varphi_T\|_{L^2(\Omega)} \tag{6.4}$$

when $0 < \delta < T$ and the limit

$$\lim_{\substack{x\in\Gamma(p)\\ x\to p}} \nabla\varphi(x, \tau) = \nabla\varphi(p, \tau) \tag{6.5}$$

exists and is finite for a.e. $p \in \partial\Omega$ and for all $\tau \in (0, T)$. Now, let $\Omega_j \subset \overline{\Omega}_{j+1} \subset \Omega$, $j \geq 1$, be a sequence of C^∞ domains approximating Ω as in [4, Lemma 2.2]. Set $u^\varepsilon = v^\varepsilon + e^{t\Delta}u_0$. By Green's formula,

$$\frac{d}{dt}\int_{\Omega_j} u^\varepsilon(t)\varphi(t)\, dx = \int_{\partial\Omega_j} (\partial_{\nu_j} u^\varepsilon \varphi - \partial_{\nu_j}\varphi u^\varepsilon)\, d\sigma_j.$$

Integrating the above identity over $[\delta, T-\delta]$ for a fixed $\delta \in (0, \eta/2)$, we get

$$\begin{aligned}\int_{\Omega_j} u^\varepsilon(T-\delta)\varphi(T-\delta)\, dx - \int_{\Omega_j} u^\varepsilon(\delta)\varphi(\delta)\, dx \\ = \int_{\partial\Omega_j\times(\delta,T-\delta)} (\partial_{\nu_j} u^\varepsilon\varphi - \partial_{\nu_j}\varphi u^\varepsilon)\, d\sigma_j\, dt.\end{aligned} \tag{6.6}$$

Recall that $u^\varepsilon(\delta) = e^{\delta\Delta}u_0$ and let $j \to \infty$ in (6.6) with ϵ and δ held fixed. Then (6.1), (6.3), (6.5) and the dominated convergence theorem show that

$$\int_\Omega u^\varepsilon(T-\delta)\varphi(T-\delta)\, dx = \int_\Omega (e^{\delta\Delta}u_0)\varphi(\delta)\, dx - \int_{\partial\Omega\times(\delta,T-\delta)} \gamma^\varepsilon\partial_\nu\varphi\, d\sigma\, dt.$$

Because γ^ε is supported in $[\eta, T-\eta]$, this is the same as

$$\int_\Omega u^\varepsilon(T-\delta)\varphi(T-\delta)\, dx = \int_\Omega (e^{\delta\Delta}u_0)\varphi(\delta)\, dx - \int_{\partial\Omega\times(\eta,T-\eta)} \gamma^\varepsilon\partial_\nu\varphi\, d\sigma\, dt \tag{6.7}$$

when $0 < \delta < \eta/8$. Next, from (6.2),

$$u^\varepsilon(T-\delta) = v^\varepsilon(T-\delta) + e^{(T-\delta)\Delta}u_0 = v^\varepsilon(T) + e^{T\Delta}u_0 + O(\delta^{\alpha/2}),$$

uniformly for $x \in \overline{\Omega}$, when $0 < \delta < \eta/8$. Hence, after letting $\delta \to 0$ in (6.7), we get

$$\int_\Omega (v^\varepsilon(T) + e^{T\Delta}u_0)\varphi(T)\, dx = \int_\Omega u_0\varphi(0)\, dx - \int_{\partial\Omega\times(\eta,T-\eta)} \gamma^\varepsilon\partial_\nu\varphi\, d\sigma\, dt.$$

Also, from (6.1) and (6.2), v^ε converges uniformly over $\overline{\Omega} \times [T-\eta/2, T]$ to some continuous function $\widetilde{v}$ as $\varepsilon \to 0$. We claim that $\widetilde{v} = v$. If this is the case, after letting $\varepsilon \to 0$ in the last equality, we get

$$\int_\Omega u(T)\varphi(T)\,dx = \int_\Omega u_0\varphi(0)\,dx - \int_{\partial\Omega\times(\eta,T-\eta)} \gamma\,\partial_\nu\varphi\,d\sigma\,dt,$$

since $\gamma^\varepsilon(p,\tau) \to \gamma(p,\tau)$ for a.e. $(p,\tau) \in \partial\Omega \times (0,T)$, and since (6.3) holds and

$$\operatorname{supp}(\gamma^\varepsilon) \cup \operatorname{supp}(\gamma) \subset \partial\Omega \times [\eta, T-\eta].$$

Recalling that $\gamma = g\chi_{\mathcal{M}}$, we get

$$\int_\Omega u(T)\varphi(T)\,dx = \int_\Omega u_0\varphi(0)\,dx - \int_{\partial\Omega\times(0,T)} g\chi_{\mathcal{M}}\partial_\nu\varphi\,d\sigma\,dt.$$

Hence, (5.5) is proved.

To verify that $\widetilde{v} = v$ over $\overline{\Omega}\times[0,T]$, observe that because $v^\epsilon - v$ is the unique solution to

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega\times(0,T),\\ u = \gamma^\epsilon - \gamma & \text{on } \partial\Omega\times(0,T),\\ u(0) = 0 & \text{in } \Omega, \end{cases}$$

whose parabolic nontangential maximal function is in $L^2(\partial\Omega\times(0,T))$ (see [14, Theorem 3.2]), we have

$$\|(v^\varepsilon - v)^\sharp\|_{L^2(\partial\Omega\times(0,T))} \le N\|\gamma^\varepsilon - \gamma\|_{L^2(\partial\Omega\times(0,T))}. \tag{6.8}$$

For fixed p in $\partial\Omega$, we may assume that $p = (0',0)$ and that near p,

$$\Omega \cap Z_{m,\varrho} = \{(x',x_n) : \phi(x') < x_n < 2m\varrho,\ |x'| \le \varrho\},$$

with ϕ as in (3.1) and (3.2). Then

$$\begin{aligned} &\int_0^T \int_{B'_\varrho} \int_{\phi(y')}^{\phi(y')+m\varrho} |F(y',y_n,t)|^2\,dy'\,dy_n\,dt \\ &\qquad \le m\varrho \int_0^T \int_{B'_\varrho} F^\sharp(y',y_n,t)^2\,dy'\,dt \le m\varrho \int_{\partial\Omega\times(0,T)} F^\sharp(p,t)^2\,d\sigma\,dt \end{aligned}$$

for all functions F. The above estimate, a covering argument and (6.8) show that

$$\|v^\epsilon - v\|_{L^2(\Omega_{m\varrho}\times(0,T))} \le N\|\gamma^\epsilon - \gamma\|_{L^2(\partial\Omega\times(0,T))} \tag{6.9}$$

with $\Omega_\eta = \{x \in \Omega : d(x,\partial\Omega) \le \eta\}$. Recalling that $v^\epsilon = v = 0$ for $t \le \eta$, the local boundedness properties of solutions to parabolic equations [31, Theorem 6.17] show that

$$|(v^\varepsilon - v)(x,\tau)| \le \left(\fint_{B_{R/20}(x)\times[\tau-R^2/20^2,\tau]} |v^\varepsilon - v|^2\,dy\,ds\right)^{1/2}$$

when $x \in \partial\Omega_R$, $0 \le \tau \le T$, and taking $R < m\varrho/20$ above, we find from (6.9) that

$$\|v^\varepsilon - v\|_{L^\infty(\Omega^R\times\{0\}\cup\partial\Omega^R\times[0,T])} \le N_R\|\gamma^\varepsilon - \gamma\|_{L^2(\partial\Omega\times(0,T))}.$$

By the maximum principle and the above estimate

$$\|v^\varepsilon - v\|_{L^\infty(\Omega^R\times[0,T])} \le N_R\|\gamma^\varepsilon - \gamma\|_{L^2(\partial\Omega\times(0,T))} \to 0 \quad \text{as } \varepsilon \to 0,$$

which shows that $\widetilde{v} = v$ in $\overline{\Omega} \times [0, T]$. □

Next we give the proof of (6.4) and (6.5). For this purpose, we first need to recall the following known result which follows from [7, Theorem 3, Lemmas 1 and 8] (see also [23, Theorem 5.19]):

Lemma 16. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $p \in \partial\Omega$ and suppose u in $C(\overline{\Omega}\cap B_{2R}(p))$ satisfies $\Delta u = 0$ in $\Omega\cap B_{2R}(p)$ and $u = 0$ on $\partial\Omega\cap B_{2R}(p)$. Then there are $N = N(m,\varrho)$ and $\epsilon = \epsilon(m,\varrho,n) \in (0,1)$ such that*

$$\|(\nabla u)^*\|_{L^{2+\epsilon}(\Delta_R(p))} \le NR^{-1-\frac{n}{2}+\frac{n-1}{2+\epsilon}}\|u\|_{L^2(B_{2R}(p)\cap\Omega)}.$$

Moreover, the limit

$$\nabla u(q) = \lim_{\substack{x\in\Gamma(q)\\ x\to q}} \nabla u(x) \text{ exists and is finite for a.e. } q \in \Delta_R(p).$$

Proof of (6.4) and (6.5). A covering of the lateral boundary of $\Omega \times (-1/2, 1/2)$ and the application of Lemma 16 to the harmonic functions $u_j(x, y) = e^{\sqrt{\lambda_j}\,y}e_j(x)$ with $R = 1/2$, $j \ge 1$, show that

$$\|(\nabla e_j)^*\|_{L^{2+\epsilon}(\partial\Omega)} \le Ne^{\sqrt{\lambda_j}} \quad \text{and} \quad \lim_{\substack{x\in\Gamma(p)\\ x\to p}} \nabla e_j(x) = \nabla e_j(p) \tag{6.10}$$

exists and is finite for a.e. $p \in \partial\Omega$ and for all $j \ge 1$. Recall that

$$\varphi(t) = e^{(T-t)\Delta}\varphi_T = \sum_{j\ge1} e^{-\lambda_j(T-t)}(\varphi_T, e_j)e_j, \qquad t \in [0, T],$$

when φ_T is in $L^2(\Omega)$. Then, for $(p,\tau) \in \partial\Omega \times [0, T-\delta]$, $\delta > 0$ and $x \in \Gamma(p)$ with $t \le T - \delta$, we have

$$|\nabla\varphi(x,t)| \le \sum_{j\ge1} e^{-\lambda_j\delta}|(\varphi_T, e_j)|(\nabla e_j)^*(p).$$

Thus,

$$(\nabla\varphi)^*(p,\tau) \le \sum_{j\ge1} e^{-\lambda_j\delta}|(\varphi_T, e_j)|(\nabla e_j)^*(p),$$

and from (6.10),

$$
\begin{aligned}
&\|(\nabla\varphi)^*(\cdot,\tau)\|_{L^{2+\epsilon}(\partial\Omega)} \le \sum_{j\ge1} e^{-\lambda_j\delta}|(\varphi_T,e_j)|\,\|(\nabla e_j)^*\|_{L^{2+\epsilon}(\partial\Omega)}\\
&\le N\sum_{j\ge1} e^{-\lambda_j\delta+\sqrt{\lambda_j}}|(\varphi_T,e_j)| \le Ne^{1/\delta}\sum_{j\ge1} e^{-\lambda_j\delta/2}|(\varphi_T,e_j)|\\
&\le Ne^{1/\delta}\Big(\sum_{j\ge1}|(\varphi_T,e_j)|^2\Big)^{1/2}\Big(\sum_{j\ge1}e^{-\lambda_j\delta}\Big)^{1/2} = Ne^{1/\delta}\|\varphi_T\|_{L^2(\Omega)}\Big(\sum_{j\ge1}e^{-\lambda_j\delta}\Big)^{1/2}.
\end{aligned}
$$

Now, integrate (4.2) over Ω to find that

$$
\sum_{j\ge1} e^{-\lambda_j\delta} \le (4\pi\delta)^{-n/2}|\Omega|
$$

and get (6.4). Next, for $i=1,\dots,n$, and $\varphi_k=\sum_{j\le k} e^{-(T-t)\lambda_j}(\varphi_T,e_j)e_j$, we have

$$
\begin{aligned}
&\Big|\Big\{p\in\partial\Omega : \limsup_{\substack{x\in\Gamma(p)\\ x\to p}} \partial_i\varphi(x,t) - \liminf_{\substack{x\in\Gamma(p)\\ x\to p}} \partial_i\varphi(x,t) > \lambda\Big\}\Big|\\
&\quad= \Big|\Big\{p\in\partial\Omega : \limsup_{\substack{x\in\Gamma(p)\\ x\to p}} (\partial_i\varphi-\partial_i\varphi_k)(x,t) - \liminf_{\substack{x\in\Gamma(p)\\ x\to p}} (\partial_i\varphi-\partial_i\varphi_k)(x,t) > \lambda\Big\}\Big|\\
&\quad\le |\{p\in\partial\Omega : (\nabla\varphi-\nabla\varphi_k)^*(p,t) > \lambda/2\}|\\
&\quad\le \frac{4N^2e^{2/\delta}}{\lambda^2}\sum_{j>k}(\varphi_T,e_j)^2 \quad \text{when } t\le T-\delta,
\end{aligned}
$$

which shows that (6.5) holds after letting k tend to infinity. □

Proof of Remark 13. From the estimate in Theorem 10 with $\Delta_R(q)\subset\Gamma$,

$$
\|e^{t_2\Delta}f\|_{L^2(\Omega)} \le \big(Ne^{\frac{N}{t_2-t_1}}\|\partial_\nu e^{t\Delta}f\|_{L^2(\Gamma\times(t_1,t_2))}\big)^{\theta}\|e^{t_1\Delta}f\|^{1-\theta}_{L^2(\Omega)}, \qquad f\in L^2(\Omega),
$$

using the telescoping series method, we get the following L^2-observability inequality:

$$
\|e^{L\Delta}f\|_{L^2(\Omega)} \le Ne^{N/L}\|\partial_\nu e^{t\Delta}f\|_{L^2(\Gamma\times(L/2,L))}, \qquad L\in(0,T). \tag{6.11}
$$

Next, recall the L^p-interpolation inequality

$$
\|\partial_\nu e^{t\Delta}f\|_{L^2(\Gamma\times(L/2,L))} \le \|\partial_\nu e^{t\Delta}f\|^{\frac{\epsilon}{2(1+\epsilon)}}_{L^1(\Gamma\times(L/2,L))}\|\partial_\nu e^{t\Delta}f\|^{\frac{2+\epsilon}{2(1+\epsilon)}}_{L^{2+\varepsilon}(\Gamma\times(L/2,L))}, \tag{6.12}
$$

and the bound

$$
\|\partial_\nu e^{t\Delta}f\|_{L^{2+\epsilon}(\Gamma\times(L/2,L))} \le Ne^{N/L}\|f\|_{L^2(\Omega)}, \tag{6.13}
$$

which follows from (6.4) with $T = L$ and $\delta = L/2$. Then, from (6.11)–(6.13),

$$\begin{aligned}\|e^{L\Delta} f\|_{L^2(\Omega)} &\le (Ne^{N/L}\|\partial_\nu e^{t\Delta} f\|_{L^1(\Gamma\times(L/2,L))})^\rho \|f\|_{L^2(\Omega)}^{1-\rho}\\ &\le (Ne^{N/L}\|\partial_\nu e^{t\Delta} f\|_{L^1(\Gamma\times(0,L))})^\rho \|f\|_{L^2(\Omega)}^{1-\rho}\end{aligned}$$

with $\rho = \frac{\epsilon}{2(1+\epsilon)}$. In particular,

$$\|e^{t_2\Delta} f\|_{L^2(\Omega)} \le (Ne^{N/(t_2-t_1)}\|\partial_\nu e^{t\Delta} f\|_{L^1(\Gamma\times(t_1,t_2))})^\rho \|e^{t_1\Delta} f\|_{L^2(\Omega)}^{1-\rho}$$

when $0 \le t_1 < t_2 \le T \le 1$. Finally, making use of the telescoping series arguments, we get

$$\|e^{T\Delta} f\|_{L^2(\Omega)} \le Ne^{N/T}\|\partial_\nu e^{t\Delta} f\|_{L^1(\Gamma\times(0,T))}, \qquad f \in L^2(\Omega).$$

This, together with Corollary 1, yields the statement in Remark 13. □

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