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Ortho-Independent States of the C. C. R. Algebra

By

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Abstract

We determine all states of the C. C. R. C^* algebra [1] which have a certain independence property relative to a family of von Neumann sub-algebras indexed by subspaces of the test function space.

§ 1. Introduction

In [2] Kac gave a simple characterisation of the normally distributed $\boldsymbol{R}^n$-valued random variables ($n>1$) with covariance matrix $\sigma^2 I$ — namely they were those random variables $\boldsymbol{X}$ for which the components $(O\boldsymbol{X})_i$ $i=1, \dots, n$ are stochastically independent for every orthogonal matrix O (the degenerate case $\sigma=0$ being included). In this paper we generalize this result to the non-commutative context of states on the C^*-algebra of the canonical commutation relations (C. C. R.) [1], isolating the family of states which have an independence property which is pertinent to the theory of canonical Wiener processes ("quantum Brownian motion") [3].

The main result is Corollary 1 while Theorem 1 serves as a bridge between this and Kac's result (Corollary 2). The proof employs Cramer's theorem [4] (see Section 2) and the methods of Kac's proof, together with a quasi-free-type analysis of a pair (β, b) consisting of a symplectic form and a real inner product respectively which are related by an inequality. This inequality reduces to the Cauchy-Schwarz inequality in the context of the first corollary and is vacuous in the classical case — in other words it is only required for the bridging theorem.

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§ 2. Preliminaries and Notation

If β is a (possibly degenerate) symplectic form on a real vector space V, we denote by $\mathfrak{A}(V, \beta)$ the "smallest C^*-algebra of the C. C. R.' s" [1] and note that any state ω on $\mathfrak{A}(V, \beta)$ is determined by its generating functional

$$F_\omega : V \to \boldsymbol{C}$$
$$f \mapsto \omega(W_f)$$

where W_f are the unitaries, satisfying the Weyl relations: $W_f W_g = e^{-i\beta(f,g)} W_{f+g}$, which generate $\mathfrak{A}(V, \beta)$.

A state is called *regular* if the map

$$t \mapsto \omega(W_{f+tg})$$

is continuous from $\boldsymbol{R}$ to $\boldsymbol{C}$ for each f and $g \in V$ — equivalently if the map

$$f \mapsto \pi_\omega(W_f)$$

is strongly continuous when restricted to finite dimensional subspaces of V — π_ω being the G. N. S. representation of $\mathfrak{A}(V, \beta)$ corresponding to the state ω.

A map $F : V \to \boldsymbol{C}$ is the generating functional of a state on $\mathfrak{A}(V, \beta)$ exactly when

(i) $F(0) = 1$,

(ii) $\sum_{j,k=1}^{n} a_j \bar{a}_k F(f_j - f_k) e^{i\beta(f_k, f_j)} \geq 0$

$$\forall n \in \boldsymbol{N},\ (\boldsymbol{a}, \boldsymbol{f}) \in \boldsymbol{C}^n \times V^n.$$

We shall refer to maps with property (ii) as being of *β-positive type* (or of positive type when $\beta \equiv 0$). In view of Bochner's theorem the characteristic functions of n-dimensional distributions are the generating functionals of regular states on $\mathfrak{A}(\boldsymbol{R}^n)$ (i. e. $\mathfrak{A}(\boldsymbol{R}^n, 0)$).

Cramer's theorem states that if the sum of two independent $\boldsymbol{R}^n$-valued random variables is normal then each of the summands is normally distributed — in terms of characteristic functions we may state that if the product of two characteristic functions is a normal characteristic function then each is normal.

If we have a von Neumann algebra N and a state ω on N, then a family of von Neumann subalgebras $\{N_\lambda : \lambda \in \Lambda\}$ is said to be *stochastically independent* in the state ω if

$$\omega(\prod_{i=1}^{n} A_i) = \prod_{i=1}^{n} \omega(A_i) \quad \forall n \in \boldsymbol{N},\ A_i \in N_{\lambda_i},\ \{\lambda_1, \ldots, \lambda_n\} \subseteq \Lambda.$$

Each state ω on $\mathfrak{A}(V, \beta)$ gives rise to a state on the von Neumann algebra $\{\pi_\omega(W_f) : f \in V\}''$ through the cyclic vector Ω_ω of the G. N. S. representation. When V has a complex inner product space structure and β is the symplectic form given by the imaginary part of the inner product we write $\mathfrak{A}(V, \mathrm{Im}\langle .\,, .\rangle)$. We now introduce a new term: a state ω on $\mathfrak{A}(V, \mathrm{Im}\langle .\,, .\rangle)$, or its generating functional F_ω, will be called *ortho-independent* if it satisfies

$$F_\omega(f+g) = F_\omega(f) F_\omega(g) \text{ whenever } \langle f, g\rangle = 0.$$

An example is the Fock state which has generating functional $f \mapsto \exp\left[-\frac{1}{2}\|f\|^2\right]$.

We shall be considering a triple (V, β, b) consisting of a real vector space, a symplectic form and an inner product (positive definite, symmetric, bilinear form) for which

$$b(f, f) b(g, g) \geq \beta(f, g)^2 \quad \forall f, g \in V.$$

$\bar{V}$ will denote the completion of V in the inner product norm given by b. In view of the above inequality there is a unique continuous extension β' of β to $\bar{V}$ and, by the Riesz representation theorem, there is an operator M on $\bar{V}$ such that

$$\beta'(f, g) = b(Mf, g) \quad \forall f, g \in \bar{V}.$$

We now drop the primes. Letting $\bar{V} = V_0 \oplus V_1$, where $V_0 = \{f \in \bar{V} : \beta(f, g) = 0 \quad \forall g \in \bar{V}\}$ and $V_1 = V_0^{\perp}$, we have the following properties [5]:

(i) $M = 0 \oplus N$, $\|N\| \leqq 1$.

(ii) $N = JP = PJ$ with P positive, J orthogonal and symplectic on V_1.

(iii) J gives a β-allowed pre-Hilbert structure to V_1 [6].

$U = V_0 \oplus V_0 \oplus V_1$ may be made into a complex pre-Hilbert space by defining:

$$i(f, g, h) = (-g, f, Jh),$$
$$B((f_1, g_1, h_1), (f_2, g_2, h_2)) = b(f_1, f_2) + b(g_1, g_2) + b(Ph_1, h_2) + i(b(f_1, g_2) - b(g_1, f_2) + \beta(h_1, h_2))$$

and then $\bar{V}$ may be identified with the real-linear subset $V_0 \oplus \{0\} \oplus V_1$ of U. Thus for $f_i = f_i^0 + f_i^1 \in \bar{V}$,

$$B((f_1^0, f_1^1), (f_2^0, f_2^1)) = b(f_1^0, f_2^0) + b(Pf_1^1, f_2^1) + i\beta(f_1^1, f_2^1).$$

Finally, *quasi-free states* on $\mathfrak{A}(V, \beta)$ are precisely those states whose generating functionals take the form

$$\mu_\omega : f \mapsto \exp\left[im(f) - \frac{1}{2}t(f, f)\right],$$

where $m \in V^*$ — the algebraic dual of V — and t is a non-negative definite, symmetric bilinear form on V satisfying the inequality

$$t(f, f)t(g, g) \geq \beta(f, g)^2 \quad \forall f, g \in V.$$

m is the first truncated functional of the state ω and t is the symmetric bilinear form determined by the quadratic form of the second truncated functional of the state ω.

§ 3. Statement of Results

Theorem 1. *Let (V, β) be a symplectic space, b a real inner product on V for which*

(a) $\quad b(f, f)b(g, g) \geq \beta(f, g)^2 \quad \forall f, g \in V,$

(b) $\quad \dim V_0 \neq 1$ *where* $V_0 = \ker \beta'$,

and F the generating functional of a regular state ω on $\mathfrak{A}(V, \beta)$. Then, provided ***either*** *the continuous extension of β to $\bar{V}$ is non-degenerate (i. e. $V_0 = \{0\}$) or ω extends to a state on $\mathfrak{A}(\bar{V}, \beta')$, the following are equivalent*:

(i) $\quad F(f+g) = F(f)F(g)$ *whenever* $B(f, g) = 0$,

(ii) $\quad F(f) = \exp\left[im(f) - \frac{1}{2}(\sigma_0^2 b(f_0, f_0) + \sigma_1^2 b(Pf_1, f_1))\right]$

for $f = f_0 + f_1$ *where* $m \in V^*$, $\sigma_0 \in [0, \infty)$, $\sigma_1 \in [1, \infty)$,

(iii) $\quad F(f+g) = F(f)F(g)$ *whenever* $\mathrm{Re}\, B(f, g) = 0$.

((i) *must be dropped when* $\dim V_1 = 2$.)

Note: 1. (ii) may be written

$$F(f) = \exp\left[im(f) - \frac{1}{2}B(\sigma_0 f_0 + \sigma_1 f_1, \sigma_0 f_0 + \sigma_1 f_1)\right].$$

2. These states are thus the quasi-free states whose bilinear form is given by

$$(f, g) \mapsto b([\sigma_0^2 Q_{V_0} + \sigma_1^2 P Q_{V_1}]f, g)$$

where Q_{V_0}, Q_{V_1} are the orthogonal projections onto the subspaces V_0, V_1 of $\bar{V}$.

Corollary 1. *Let h be a complex pre-Hilbert space of dimension greater than or equal to 2 and F the generating functional of a regular state on $\mathfrak{A}(h, \mathrm{Im}\langle .,. \rangle)$ then the following conditions are equivalent:*

(i) *F is ortho-independent*

(ii) $F(f) = \exp\left[im(f) - \frac{\sigma^2}{2}\|f\|^2\right]$ $\sigma \in [1, \infty)$, $m : h \to \boldsymbol{R}$ *real-linear*

(iii) $F(f+g) = F(f)F(g)$ *whenever* $\mathrm{Re}\langle f, g \rangle = 0$;

when $\dim h = 1$ *the result remains true if* (i) *is dropped.*

Corollary 2. *Let V be a finite dimensional real inner product space. If X is a V-valued random variable whose co-ordinates, with respect to each orthogonal basis, are stochastically independent then X is normally distributed with covariance $\sigma^2 I$, i.e.*

$$\hat{\mu}_X(v) := \boldsymbol{E}(\exp[i(X, v)]) = \exp\left[i(m, v) - \frac{1}{2}\sigma^2(v, v)\right] \quad m \in V, \sigma \in [0, \infty),$$

the degenerate distributions where $\sigma = 0$ are possible.

Proof of Corollary 2. $\hat{\mu}_X$ is a generating functional on $\mathfrak{A}(V)$ satisfying the hypothesis of our theorem, thus

$$\hat{\mu}_X(v) = \exp\left[im(v) - \frac{1}{2}\sigma^2 b(v, v)\right], \text{ since } V = V_0 = \bar{V},$$

$$= \exp\left[i(m, v) - \frac{1}{2}\sigma^2(v, v)\right], \text{ since } V^* \cong V \text{ for } \dim V < \infty.$$

Proof of Corollary 1. In this case the hypothesis of the theorem is satisfied since the continuous extension of the symplectic form is obviously non-degenerate, moreover this means that $V \subseteq V_1 = \bar{V}$ and, since $\mathrm{Im}\langle f, g \rangle = \mathrm{Re}\langle if, g \rangle$, we have $P = I$ so that

$$F(f) = \exp\left[im(f) - \frac{1}{2}\sigma^2\|f\|^2\right].$$

Thus (i)$\Rightarrow$(ii). Clearly (ii)$\Rightarrow$(iii)$\Rightarrow$(i) since

$$\exp\left[im(f+g)-\frac{\sigma^2}{2}\|f+g\|^2\right]=\exp\left[im(f)-\frac{\sigma^2}{2}\|f\|^2\right]\exp\left[im(g)-\frac{\sigma^2}{2}\|g\|^2\right]$$
$$\times\exp[-\sigma^2\,\mathrm{Re}\langle f,\ g\rangle].$$

§ 4. Proof of the Theorem

The following well-known properties of functions F, G of β-positive type will be used:

(βi) $F(-f)=\overline{F(f)}$,

(βii) F is also of $\tilde{\beta}$-positive type where $\tilde{\beta}=-\beta$,

(βiii) FG is of positive type.

(βi) and (βii) are simply verified and (βiii) follows from (βii) and the fact that the component-wise product of positive semi-definite matrices is positive semi-definite.

We shall consider $F_0=F|_{V_0}$ and $F_1=F|_{V_1}$ separately (or simply F in case $V_0=\{0\}$).

Now in view of the argument deducing Corollary 1 from the theorem, it is only necessary to prove (i)→(ii) when $\dim V_1\neq 2$ and (iii)→(ii) when $\dim V_1=2$—the remaining implications being immediate.

A. Assume first that V_1 is four-dimensional as a real vector space (complex dimension two) and let $\{f,\ g\}$ be a pair of its elements orthonormal with respect to B.

(i) Define $\lambda,\ \mu:\boldsymbol{C}\to\boldsymbol{C}$ by
$\lambda(u)=F_1(uf),\ \mu(u)=F_1(ug)$.

(ii) Since $B(u(f+e^{i\theta}g),\ v(f-e^{i\theta}g))=0\ \ \forall u,\ v\in\boldsymbol{C},\ \theta\in\boldsymbol{R}$ we have $\lambda(u+v)\mu(e^{i\theta}(u-v))=\lambda(u)\lambda(v)\mu(e^{i\theta}u)\mu(-e^{i\theta}v)$.

(iii) Letting $\theta=0$, $u=v$: $\lambda(u)=\lambda\left(\frac{u}{2}\right)^2\mu\left(\frac{u}{2}\right)\mu\left(-\frac{u}{2}\right)$

$$u=-v:\ \mu(u)=\mu\left(\frac{u}{2}\right)^2\lambda\left(\frac{u}{2}\right)\lambda\left(-\frac{u}{2}\right)$$

in particular λ and μ are in fact of positive type by (βiii).

(iv) Considering λ and μ as maps from $\boldsymbol{R}^2$ to $\boldsymbol{C}$, they are characteristic functions.

(v) Now assume, for the moment, that λ and μ are even functions, and thus real:

(a) $\lambda(u)=\mu(u)=\lambda\left(\frac{u}{2}\right)^4=\lambda\left(\frac{u}{2^k}\right)^{4^k}\quad \forall u\in \boldsymbol{C},\ k\in \boldsymbol{N}$

so, by the continuity of λ, and the fact that $\lambda(0)=1$,

$$0<\lambda(u)\leq 1,\quad \forall u\in \boldsymbol{C}.$$

(b) Putting $u=v$ in (ii) and using the previous inequality we also have

$$\lambda(e^{i\theta}u)=\lambda(u),\quad \forall u\in \boldsymbol{C},\ \theta\in \boldsymbol{R}.$$

(c) Since λ is non-vanishing and continuous we may apply Cauchy's method to the functional equation

$$\lambda(u+v)\lambda(u-v)=\lambda(u)^2\lambda(v)^2,\quad u,\ v\in \boldsymbol{R}$$

to obtain

$\lambda(t)=e^{kt^2}$ where $k\leq 0$, by (a).

(d) In view of (b), $\lambda(u)=\exp\left[-\frac{1}{2}\sigma^2|u|^2\right],\ u\in \boldsymbol{C}$

so that if

$$h=uf+vg,$$
$$F(h)=\lambda(u)\mu(v)=\exp\left[-\frac{1}{2}\sigma^2 B(h,h)\right].$$

(vi) Now in order to eliminate the assumption that λ and μ are even consider the functions $\lambda\bar{\lambda}$, $\mu\bar{\mu}$—these are even characteristic functions on $\boldsymbol{R}^2$ and so the above analysis applies to them, moreover Cramer's theorem tells us that

$$\lambda(x+iy)=\exp\left[i(m_1x+n_1y)-\frac{1}{2}(\boldsymbol{x},\ C_1\boldsymbol{x})\right],$$
$$\mu(x+iy)=\exp\left[i(m_2x+n_2y)-\frac{1}{2}(\boldsymbol{x},\ C_2\boldsymbol{x})\right],\ \boldsymbol{x}=\begin{pmatrix}x\\ y\end{pmatrix}\in \boldsymbol{R}^2$$

where

$$2C_1=2C_2=kI,$$

thus if $h=uf+vg$,

$$F_1(h)=\lambda(u)\mu(v)=\exp\left[im(h)-\frac{1}{2}\sigma^2 B(h,h)\right],$$

where

$$m:V_1\to \boldsymbol{R}$$

is the map defined by

$$m((x_1+iy_1)f+(x_2+iy_2)g)=m_1x_1+n_1y_1+m_2x_2+n_2y_2$$

B. We now extend the result to $\dim V_1 \geq 4$. For any (complex) two-dimensional subspace U of V_1, $F_1|_U$ determines a pair (m_U, σ_U) by the above method. Define (m, σ) by $m(f) = m_U(f)$, $\sigma = \sigma_U$ where U is any two (complex)-dimensional subspace of V_1 containing f. We must now check that these are well-defined—if they are, then clearly

$$F_1(f) = \exp\left[im(f) - \frac{1}{2}\sigma^2 B(f, f)\right], \quad f \in V_1.$$

Let U and W be two (complex) two-dimensional subspaces of V_1 containing a particular non-zero vector f, then

$$\exp\left[im_U(f) - \frac{1}{2}\sigma_U^2 B(f, f)\right] = F_1(f) = \exp\left[im_W(f) - \frac{1}{2}\sigma_W^2 B(f, f)\right]$$

thus $\sigma_U = \sigma_W$ and, in view of the continuity of the maps $t \to m_W(tf)$, $t \to m_U(tf)$ we have $m_U(f) = m_W(f)$. Thus σ and m are well-defined.

C. Now consider F_0, and suppose that V_0 is two-dimensional. Let $\{f, g\}$ be an orthonormal pair in V_0.

(i)′ Define $\lambda, \mu : \boldsymbol{R} \to \boldsymbol{C}$ as in (i). (iii)-(vi) may be repeated with $\boldsymbol{R}^2$ replaced by $\boldsymbol{R}$ (and (v) b omitted) we then obtain, for $h = xf + yg$

$$F_0(h) = \lambda(x)\mu(y) = \exp\left[im(h) - \frac{1}{2}\sigma^2 B(h, h)\right].$$

D. The argument in **B** may be used to extend the result to $\dim V_0 \geq 2$.

E. It remains to deal with the case of V_1 being one (complex)-dimensional. Let f be a vector in V_1 normalised with respect to B.

(i)″ Define $\lambda, \mu : \boldsymbol{R} \to \boldsymbol{C}$ by $\lambda(t) = F_1(tf)$; $\mu(t) = F_1(itf)$.

(ii)″ Since $\operatorname{Re} B((s+it)f, (t-is)f) = 0$ $\forall s, t \in \boldsymbol{R}$ we have

$$\lambda(s+t)\lambda(t-s) = \lambda(s)\lambda(t)\mu(t)\mu(-s)$$

and we may proceed as in **C** to obtain, for $h = (s+it)f$

$$F_1(h) = \lambda(s)\mu(t) = \exp\left[im(h) - \frac{1}{2}\sigma^2 B(h, h)\right].$$

The proof is now complete.

§ 5. Conclusion

The importance of the result seems to lie essentially in the fact that it singles out those regular states of the C. C. R. algebra $\mathfrak{A}(h, \mathrm{Im}\langle\cdot,\cdot\rangle)$ for which the von Neumann algebras generated by the Weyl operators with argument ranging over orthogonal subspaces of h are stochastically independent. To elucidate we state

Corollary 3. *For a regular state ω on $\mathfrak{A}(h, \mathrm{Im}\langle\cdot,\cdot\rangle)$ the following are equivalent:*

(i) *F_ω is ortho-independent*

(ii) *$N_\lambda = \{\pi_\omega(W_f) : f \in h_\lambda\}''$ are stochastically independent in the state determined by Ω_ω, for any family $\{h_\lambda : \lambda \in \Lambda\}$ of* ***orthogonal*** *subspaces of h.*

The proof is simply a matter of taking strong limits of uniformly bounded sequences of linear combinations of $\pi_\omega(W_f)s$.

Finally we remark that the statement of Kac's result in [2] is incorrect — the independence conditions do not force the distribution to have zero mean.

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