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Hofer–Zehnder capacity of unit disk cotangent bundles and the loop product

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Abstract. We prove a new finiteness result for the Hofer–Zehnder capacity of certain unit disk cotangent bundles. It is proved by a computation of the pair-of-pants product on Floer homology of cotangent bundles, combined with the theory of spectral invariants. The computation of the pair-of-pants product is reduced to a simple computation of the Chas–Sullivan loop product.

Keywords. Hofer–Zehnder capacity, spectral invariant, loop product

1. Introduction

1.1. Hofer–Zehnder capacity

The study of periodic orbits of Hamiltonian systems is a fundamental problem in symplectic geometry and Hamiltonian dynamics. Hofer and Zehnder ([HZ1], [HZ2]) introduced a remarkable invariant of a symplectic manifold, which reflects behaviors of periodic orbits of Hamiltonian systems on the manifold. This invariant is called Hofer–Zehnder capacity, and we will recall its precise definition below.

If one can show that a given symplectic manifold has finite Hofer–Zehnder capacity, or succeeds in computing the capacity, it often gives strong consequences. For example, on every symplectic manifold with finite Hofer–Zehnder capacity, we have a very strong existence theorem about periodic orbits of Hamiltonian systems (see [HZ2, Chapter 4, Theorem 4]). Moreover, quantitative estimates of the Hofer–Zehnder capacity have applications in symplectic embedding problems (see [HZ2, Chapter 2]).

Thus it is an important task to understand the behavior of the capacity. However, even to prove its finiteness is hard in general. In particular, very little is known about the Hofer–Zehnder capacity of cotangent bundles, although they are the most basic examples of symplectic manifolds. In this paper, we try to add a new result on this problem, based on computations of Floer homology of cotangent bundles due to [AS1], [AS2].

First we recall the formal definition of the Hofer–Zehnder capacity. In fact we introduce a refined version, taking into account homotopy classes of periodic orbits. Let (X, ω) be a symplectic manifold without boundary. For any $H \in C^\infty(X)$, its *Hamiltonian vector*

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field X_H is defined by $\omega(X_H, \cdot) = -dH(\cdot)$. Let $\pi_1'(X)$ be the set of homotopy classes of free loops on X. For any $P \subset \pi_1'(X)$, $H \in C^\infty(X)$ is called *Hofer–Zehnder admissible* with respect to P if the following hold:

- $H(x) \le 0$ for any $x \in X$.
- Any nonconstant periodic orbit γ of X_H satisfying $[\gamma] \in P$ has a period > 1.
- There exists a nonempty open set U such that $H|_U \equiv \min H$.
- $\operatorname{supp} H$ is a compact, proper subset of X.

Then, we define the quantity

$$c_{\mathrm{HZ}}(X, \omega : P) := \sup\{-\min H \mid H \text{ is Hofer–Zehnder admissible with respect to } P\},$$

and call it the *Hofer–Zehnder capacity* with respect to P. The quantity $c_{\mathrm{HZ}}(X, \omega) := c_{\mathrm{HZ}}(X, \omega : \pi_1'(X))$ is the usual Hofer–Zehnder capacity (see e.g. [HZ2]).

Remark 1.1. *Throughout this paper, all manifolds are assumed to be connected*. Therefore for any manifold X, there is a unique element in $\pi_1'(X)$ which consists of contractible loops. We denote it as c_X.

1.2. Main result

To explain our main result, we fix some notation. Let M be a closed Riemannian manifold.

- Λ_M denotes the Hilbert manifold of free loops $S^1 (:= \mathbb{R}/\mathbb{Z}) \to M$ of Sobolev class $L^{1,2}$ (i.e. loops with square integrable derivatives).
- For any $\alpha \in \pi_1'(M)$, $\Lambda_M^\alpha := \{\gamma \in \Lambda_M \mid [\gamma] = \alpha\}$.
- For any $\gamma \in \Lambda_M$, $\bar{\gamma}$ denotes its inverse, i.e. $\bar{\gamma}(t) := \gamma(1-t)$. For any $\alpha \in \pi_1'(X)$, $\bar{\alpha} \in \pi_1'(X)$ is defined as $\bar{\alpha} := [\bar{\gamma}]$ ($[\gamma] = \alpha$).
- DT^*M denotes the unit disk cotangent bundle of M, i.e.

$$DT^*M := \{(q, p) \in T^*M \mid |p| \le 1\}.$$

- π_M denotes the natural projection map $T^*M \to M$, $(q, p) \mapsto q$. The differential forms $\lambda_M \in \Omega^1(T^*M)$ and $\omega_M \in \Omega^2(T^*M)$ are defined as

$$\begin{aligned} \lambda_M(\xi) &:= p(d\pi_M(\xi)) \qquad ((q, p) \in T^*M,\ \xi \in T_{(q,p)}(T^*M)), \\ \omega_M &:= d\lambda_M. \end{aligned}$$

Our main result is the following:

Theorem 1.2. *Let M be a closed Riemannian manifold, and $\alpha \in \pi_1'(M) \setminus \{c_M\}$. Suppose that the evaluation map* $\mathrm{ev} : \Lambda_M^\alpha \to M$, $\gamma \mapsto \gamma(0)$, *has a continuous section s (i.e. $s : M \to \Lambda_M^\alpha$ such that* $\mathrm{ev} \circ s = \mathrm{id}_M$). *Then*

$$c_{\mathrm{HZ}}(\operatorname{int} DT^*M, \omega_M : \{c_M, \alpha, \bar{\alpha}\}) < \infty.$$

The following corollary is immediate from Theorem 1.2.

Corollary 1.3. *Let M be a closed Riemannian manifold. Suppose that M admits a smooth S^1 action such that the S^1 orbit $\gamma_p : S^1 \to M$, $t \mapsto t \cdot p$, is not contractible for any $p \in M$. Then $c_{\mathrm{HZ}}(DT^*M, \omega_M) < \infty$.*

Proof. $[\gamma_p] \in \pi_1'(M) \setminus \{c_M\}$ does not depend on $p \in M$; denote it α. Now $s : p \mapsto \gamma_p$ is a continuous section of $\mathrm{ev} : \Lambda_M^\alpha \to M$, thus Theorem 1.2 completes the proof. □

Corollary 1.3 is proved in [I] in a different way.

1.3. Outline of the proof

We give an outline of the proof of Theorem 1.2, dividing it into three steps. We use several technical terms without explanations. We recall their definitions later.

Step 1: Hofer–Zehnder capacity and the pair-of-pants product. For any Liouville domain (X, λ) and $\alpha \in \pi_1'(X)$, we define a $\mathbb{Z}_2$-module $\mathrm{HF}^\alpha(X, \lambda)$, *Floer homology*. (We do not need grading. As a matter of fact, Floer homology carries only $\mathbb{Z}_2$-grading in our context.) Let us denote $\mathrm{HF}(X, \lambda) := \bigoplus_{\alpha \in \pi_1'(X)} \mathrm{HF}^\alpha(X)$. There exists a natural homomorphism $\iota : \bigoplus_i H^i(X) \to \mathrm{HF}(X)$. Moreover, $\mathrm{HF}(X)$ carries the *pair-of-pants product*, which is denoted as $*$. We prove the following result:

Theorem 1.4. *Let (X, λ) be a Liouville domain and $\alpha \in \pi_1'(X) \setminus \{c_X\}$. Suppose there exist $x \in \mathrm{HF}^\alpha(X, \lambda)$ and $y \in \mathrm{HF}^{\bar{\alpha}}(X, \lambda)$ such that $x * y = \iota(1)$. Then*

$$c_{\mathrm{HZ}}(\mathrm{int}\, X, d\lambda : \{c_X, \alpha, \bar{\alpha}\}) < \infty.$$

Remark 1.5. Suppose that X is a Liouville domain which satisfies the assumption of Theorem 1.4. Let $\tilde{X}$ be a Liouville domain which is obtained by a subcritical handle attachment to X. Then the homomorphism $\mathrm{HF}(\tilde{X}) \to \mathrm{HF}(X)$ constructed in [V1] is isomorphic by [Cie], and it is possible to show that the homomorphism preserves the pair-of-pants product. Hence $\tilde{X}$ also satisfies the assumption of Theorem 1.4, unless $\pi_1'(X) \to \pi_1'(\tilde{X})$ maps α to the trivial class.

The proof of Theorem 1.4 is based on the theory of *spectral invariants*, in particular their subadditivity with respect to the pair-of-pants product.

Step 2: Floer homology of cotangent bundles. As a next step, we make use of the following result, which enables us to compute the pair-of-pants product on Floer homology of cotangent bundles via the Chas–Sullivan loop product:

Theorem 1.6 ([V2], [SW], [AS1], [AS2]). *For any closed Riemannian manifold M, the following statements hold:*

(1) *For each $\alpha \in \pi_1'(M)$, there exists a natural isomorphism*

$$\Psi^\alpha : \mathrm{HF}^\alpha(DT^*M, \lambda_M) \to \bigoplus_i H_i(\Lambda_M^\alpha).$$

$\Psi : \mathrm{HF}(DT^*M) \to \bigoplus_{i,\alpha} H_i(\Lambda_M^\alpha)$ *denotes the direct product of* $(\Psi^\alpha)_{\alpha \in \pi_1'(M)}$.

(2) *Define* $c : M \to \Lambda_M$ *by* $c(p) :=$ *constant loop at* p. *Then* $\Psi(\iota(1)) = c_*[M]$, *where* $[M]$ *denotes the fundamental homology class of* M, *and* 1 *denotes the canonical element in* $H^0(DT^*M)$.
(3) *The isomorphism* Ψ *intertwines the pair-of-pants product on* $\mathrm{HF}(DT^*M, \lambda_M)$ *with the Chas–Sullivan loop product on* $\bigoplus_i H_i(\Lambda_M)$.

There are several remarks on Theorem 1.6:

Remark 1.7. • Theorem 1.6(1) is due to [V2], [SW], [AS1]. The author is not sure if one can identify the isomorphisms constructed in those papers in natural ways. In the following, we adopt the isomorphism constructed in [AS1].

- The definition of Floer homology of cotangent bundles in [AS1], [AS2] is slightly different from ours, which will be given in Section 2. We follow [V1], and consider Hamiltonians which grow linearly at ends. On the other hand, the authors of [AS1], [AS2] consider Hamiltonians which grow quadratically at ends. It is not hard to check that there exists a natural isomorphism between Floer homologies defined by these two definitions; we omit the details.
- Theorem 1.6(2) is not made explicit in [AS1]. However, it is not hard to prove it from the construction of the isomorphism in [AS1]; we omit the details.
- Theorem 1.6(3) is due to [AS2]. In [AS2], it is assumed that M is oriented. This assumption is necessary when we define the loop product over $\mathbb{Z}$. However, we do not need to assume that M is oriented, since we are working with $\mathbb{Z}_2$ coefficients.

Step 3: A key computation on the loop product. The following lemma is a key ingredient in the proof of Theorem 1.2:

Lemma 1.8. *Let* M *be a closed manifold. Suppose* $s : M \to \Lambda_M$ *satisfies* $\mathrm{ev} \circ s = \mathrm{id}_M$, *and let* $\bar{s} : M \to \Lambda_M$ *be defined by* $\bar{s}(p) := \overline{s(p)}$. *Then*

$$s_*[M] \circ \bar{s}_*[M] = c_*[M],$$

where $\circ$ *denotes the loop product on* $H_*(\Lambda_M)$.

We now deduce Theorem 1.2 from Theorem 1.4, Theorem 1.6, and Lemma 1.8. Take $s : M \to \Lambda_M$ as in the assumption in Theorem 1.2, and consider $x := \Psi^{-1}(s_*[M]) \in \mathrm{HF}^{\alpha}(DT^*M)$ and $y := \Psi^{-1}(\bar{s}_*[M]) \in \mathrm{HF}^{\bar{\alpha}}(DT^*M)$. Then we get

$$x * y = \Psi^{-1}(s_*[M] \circ \bar{s}_*[M]) = \Psi^{-1}(c_*[M]) = \iota(1)$$

from Theorem 1.6 and Lemma 1.8. Hence $c_{\mathrm{HZ}}(\mathrm{int}\, DT^*M, \omega_M : \{c_M, \alpha, \bar{\alpha}\}) < \infty$ by Theorem 1.4.

1.4. Previous work

When a domain in a symplectic manifold (with some mild conditions) is Hamiltonian displaceable, its Hofer–Zehnder capacity is finite (see e.g. [HZ2], [Schl], [U]). Hence, it is natural to study the case of Hamiltonian nondisplaceable domains. Unit disk cotangent bundles are the most basic examples here. However, as far as the author knows, very little is known concerning their Hofer–Zehnder capacity. The following proposition is an immediate consequence of the Weinstein neighborhood theorem and finiteness of Hofer–Zehnder capacity of standard symplectic balls:

Proposition 1.9. *Let M be an n-dimensional closed Riemannian manifold. If $\mathbb{C}^n$ with the standard symplectic structure admits a closed Lagrangian submanifold diffeomorphic to M, then $c_{\mathrm{HZ}}(\mathrm{int}\, DT^*M, \omega_M) < \infty$.*

M. Jiang [J] extends this observation, and proves a good upper bound of $c_{\mathrm{HZ}}(\mathrm{int}\, DT^*M, \omega_M)$ when M is the flat torus. The following remarkable result is an immediate consequence of R. Ma's [Ma, Theorem 1.3] (note that this result can be deduced from Corollary 1.3 when N is compact):

Theorem 1.10 ([Ma]). *Let N be any Riemannian manifold, and $M := N \times S^1$ with the product metric. Then $c_{\mathrm{HZ}}(\mathrm{int}\, DT^*M, \omega_M) < \infty$.*

L. Macarini [Mac, Corollary 1.4] also proved finiteness of the Hofer–Zehnder capacity of the unit disk cotangent bundle under the assumption that the base manifold admits a free circle action satisfying some conditions.

J. Weber [W] introduced the Biran–Polterovich–Salamon (BPS) capacity, based on the work of P. Biran, L. Polterovich and D. Salamon [BPS]. The BPS capacity can be considered to be a variant of the Hofer–Zehnder capacity, which detects noncontractible periodic orbits of Hamiltonian systems on cotangent bundles. Weber computed the BPS capacity completely [W, Theorem 4.3], improving results in [BPS]. For other related results concerning nonconstant periodic orbits of Hamiltonian systems, especially on noncontractible orbits, consult [W, introduction], [BPS, Section 1.2] and references therein.

1.5. Organization of the paper

In Section 2, we recall Floer theory on Liouville domains. We define truncated Floer homology of Liouville domains, and the pair-of-pants product on Floer homology. In Section 3, we prove Theorem 1.4 by using the theory of spectral invariants. In Section 4, we recall the definition of the Chas–Sullivan loop product, and prove Lemma 1.8.

2. Floer theory on Liouville domains

In this section, we recall Floer theory on Liouville domains. In Section 2.1, we recall basic objects (Liouville domains, Hamiltonians, almost complex structures) and prove a convexity result for solutions of the Floer equations (Lemma 2.2). In Section 2.2, first we define truncated Floer homology of (admissible) Hamiltonians. Then we define truncated Floer homology of Liouville domains. In Section 2.3, we define the pair-of-pants product on truncated Floer homology.

2.1. *Preliminaries*

2.1.1. Liouville domains. A *Liouville domain* is a pair (X, λ), where X is a $2n$-dimensional compact manifold with boundary and $\lambda \in \Omega^1(X)$ is such that $d\lambda$ is a symplectic form on X, and $\lambda \wedge (d\lambda)^{n-1} > 0$ on ∂X. The *Liouville vector field* Z is defined implicitly by the equation $i_Z(d\lambda) = \lambda$. It is easy to show that Z points strictly outwards on ∂X. For any Liouville domain (X, λ), $(\partial X, \lambda)$ is a contact manifold. We define

$$\mathrm{Spec}(X, \lambda) := \left\{ \int_\gamma \lambda \;\middle|\; \gamma \text{ is a periodic Reeb orbit on } (\partial X, \lambda) \right\}.$$

Obviously, $\mathrm{Spec}(X, \lambda) \subset (0, \infty)$. Moreover, it is well-known that $\mathrm{Spec}(X, \lambda)$ is closed and nowhere dense in $\mathbb{R}$.

Let $I : \partial X \times (0, 1] \to X$ be the embedding defined by

$$I(z, 1) = z, \qquad \partial_r I(z, r) = r^{-1} Z(I(z, r)).$$

It is easy to check that $I^*\lambda(z, r) = r\lambda(z)$ for any $(z, r) \in \partial X \times (0, 1]$.

Define a manifold $\hat{X}$ by

$$\hat{X} := X \cup_I \partial X \times (0, \infty),$$

and $\hat{\lambda} \in \Omega^1(\hat{X})$ by

$$\hat{\lambda}(x) := \begin{cases} \lambda(x) & (x \in X), \\ r\lambda(z) & (x = (z, r) \in \partial X \times (0, \infty)). \end{cases}$$

$(\hat{X}, \hat{\lambda})$ is called the *completion* of (X, λ); $d\hat{\lambda}$ is a symplectic form on $\hat{X}$. For each $r > 0$, $X(r)$ denotes the bounded domain in $\hat{X}$ with boundary $\partial X \times \{r\}$, i.e.

$$X(r) := \begin{cases} X \cup \partial X \times [1, r] & (r \geq 1), \\ X \setminus \partial X \times (r, 1] & (r < 1). \end{cases}$$

Example 2.1. When M is a closed Riemannian manifold, (DT^*M, λ_M) is a Liouville domain. There exists a unique diffeomorphism $\varphi : \widehat{DT^*M} \to T^*M$ such that $\varphi^*\lambda_M = \hat{\lambda}_M$ and $\varphi|_{DT^*M}$ is the inclusion $DT^*M \to T^*M$. Hence we identify (T^*M, λ_M) with the completion of (DT^*M, λ_M).

2.1.2. Hamiltonians. For $H \in C^\infty(S^1 \times \hat{X})$, $H_t \in C^\infty(\hat{X})$ is defined by $H_t(x) := H(t, x)$, and $\mathcal{P}(H)$ denotes the set of 1-periodic orbits of $(X_{H_t})_{t \in S^1}$, i.e.

$$\mathcal{P}(H) := \{x : S^1 \to \hat{X} \mid X_{H_t}(x(t)) = \partial_t x(t)\}.$$

H is *nondegenerate* when all orbits in $\mathcal{P}(H)$ are nondegenerate; H is *linear at* ∞ when there exist $a_H > 0$, $b_H \in \mathbb{R}$ and $r_0 \geq 1$ such that $H_t(z, r) = a_H r + b_H$ for any $t \in S^1$, $z \in \partial X$, $r \geq r_0$; and H is *admissible* when it is nondegenerate and linear at ∞. We denote the set of admissible Hamiltonians by $\mathcal{H}_{\mathrm{ad}}(X, \lambda)$. Notice that any $H \in \mathcal{H}_{\mathrm{ad}}(X, \lambda)$ satisfies $a_H \notin \mathrm{Spec}(X, \lambda)$, since otherwise $\mathcal{P}(H)$ contains infinitely many degenerate orbits.

2.1.3. Almost complex structures. Let J be an almost complex structure on $\hat{X}$. It is *compatible with* $d\hat{\lambda}$ when

$$g_J : TM \otimes TM \to \mathbb{R}, \qquad v \otimes w \mapsto d\hat{\lambda}(v, Jw),$$

is a Riemannian metric (we will denote $g_J(v,v)^{1/2}$ as $|v|_J$). Let $\mathcal{J}(\hat{X}, \hat{\lambda})$ denote the set of almost complex structures on $\hat{X}$ which are compatible with $d\hat{\lambda}$.

Let $I \subset (0, \infty)$ be a nonempty interval. A family $(J_a)_{a \in A}$ of almost complex structures is *of contact type on* $\partial X \times I$ when each J_a satisfies $dr \circ J_a(z, r) = -\lambda(z)$ for any $(z, r) \in \partial X \times I$. If $(J_a)_{a\in A}$ is of contact type on $\partial X \times (r_0, \infty)$ for some r_0, then $(J_a)_{a\in A}$ is *of contact type at* ∞.

2.1.4. Convexity. The following convexity result is necessary to develop Floer theory on Liouville domains. Although it is well-known, we include its proof for the sake of completeness.

Lemma 2.2. *Let (X, λ) be a Liouville domain, $(H_{s,t})_{(s,t)\in\mathbb{R}\times S^1}$ be a family of Hamiltonians on $\hat{X}$, and $(J_{s,t})_{(s,t)\in\mathbb{R}\times S^1}$ be a family of elements in $\mathcal{J}(\hat{X}, \hat{\lambda})$. Suppose that there exists $r_0 > 0$ such that the following hold:*

- *There exist $a, b \in C^\infty(\mathbb{R})$ such that $H_{s,t}(z, r) = a(s)r + b(s)$ on $\partial X \times [r_0, \infty)$, and $a'(s) \geq 0$ for any $s \in \mathbb{R}$.*
- *$(J_{s,t})_{(s,t)\in\mathbb{R}\times S^1}$ is of contact type on $\partial X \times [r_0, \infty)$.*

Under these assumptions, if $u : \mathbb{R} \times S^1 \to \hat{X}$ satisfies the Floer equation $\partial_s u - J_{s,t}(\partial_t u - X_{H_{s,t}}(u)) = 0$ and $u^{-1}(\partial X \times (r_0, \infty))$ is bounded, then $u(\mathbb{R} \times S^1) \subset X(r_0)$.

Proof. If $u(\mathbb{R}\times S^1)$ is not contained in $X(r_0)$, then there exists $r_1 > r_0$ such that $u(\mathbb{R}\times S^1)$ is not contained in $X(r_1)$, and u is transversal to $\partial X \times \{r_1\}$. Then $D := u^{-1}(\partial X \times [r_1, \infty))$ is a compact surface with boundary, and

$$\begin{aligned} 0 < \int_D |\partial_s u|^2_{J_{s,t}} ds\, dt &= \int_D d\hat{\lambda}(\partial_t u - X_{H_{s,t}}(u), \partial_s u)\, ds\, dt \\ &= \int_D dH_{s,t}(\partial_s u)\, ds\, dt - u^*(d\hat{\lambda}). \end{aligned}$$

On the other hand, if $u(s,t) \in \partial X \times [r_0, \infty)$,

$$\begin{aligned} dH_{s,t}(\partial_s u) = a(s)\partial_s r(s,t) &\leq a(s)\partial_s r(s,t) + a'(s) r(s,t) = \partial_s(a(s) r(s,t)) \\ &= \partial_s(\hat{\lambda}(X_{H_{s,t}}(u))). \end{aligned}$$

Hence we get

$$\begin{aligned} \int_D dH_{s,t}(\partial_s u)\, ds\, dt - u^*(d\hat{\lambda}) &\leq \int_D \partial_s(\hat{\lambda}(X_{H_{s,t}}(u)))\, ds\, dt - u^*(d\hat{\lambda}) \\ &= \int_{\partial D} \hat{\lambda}(X_{H_{s,t}}(u)\, dt - du). \end{aligned}$$

We can compute the right hand side as follows (j denotes the complex structure on $\mathbb{R}\times S^1$ which is defined by $j(\partial_s)=\partial_t$):

$$\begin{aligned}\int_{\partial D}\hat{\lambda}(X_{H_{s,t}}(u)\,dt-du)&=\int_{\partial D}\hat{\lambda}(J_{s,t}\circ(X_{H_{s,t}}(u)\,dt-du))\circ j\\&=r_1\int_{\partial D}dr(X_{H_{s,t}}(u)\,dt-du)\circ j.\end{aligned}$$

The first equality follows from the Floer equation, while the second holds since $J_{s,t}$ is of contact type on $\partial X\times[r_0,\infty)$ and $u(\partial D)\subset\partial X\times\{r_1\}$. Finally,

$$\int_{\partial D}dr(X_{H_{s,t}}(u)\,dt-du)\circ j<0.$$

This is because $dr(X_{H_{s,t}})=0$ on $\partial X\times\{r_1\}$, and $dr(du(jV))>0$ when V is a vector tangent to ∂D, positive with respect to the boundary orientation. Hence we get a contradiction. $\square$

2.2. *Truncated Floer homology of Liouville domains*

In Section 2.2.1, we define truncated Floer homology of admissible Hamiltonians, and introduce *monotonicity homomorphisms*. In Section 2.2.2, we define truncated Floer homology of Liouville domains, by taking a direct limit with respect to monotonicity homomorphisms. Throughout this paper, we work in $\mathbb{Z}_2$-coefficient homology.

2.2.1. Truncated Floer homology of admissible Hamiltonians. Let (X,λ) be a Liouville domain, $H\in\mathcal{H}_{\mathrm{ad}}(X,\lambda)$, $\alpha\in\pi_1'(X)$, and $I\subset\mathbb{R}$ be a nonempty interval. $\mathrm{CF}^{I,\alpha}(H)$ denotes the free $\mathbb{Z}_2$-module generated by

$$\{x\in\mathcal{P}(H)\mid\mathcal{A}_H(x)\in I,\ [x]=\alpha\},$$

where $\mathcal{A}_H(x)$ is defined by

$$\mathcal{A}_H(x):=\int_{S^1}(x^*\hat{\lambda}-H_t(x(t)))\,dt.$$

$\mathrm{CF}^{\alpha}(H)$ abbreviates $\mathrm{CF}^{\mathbb{R},\alpha}(H)$.

Let $J=(J_t)_{t\in S^1}$ be a family of elements in $\mathcal{J}(\hat{X},\hat{\lambda})$ which is of contact type at ∞. For any distinct $x,y\in\mathcal{P}(H)$, define (with $u(s)$ denoting $S^1\to\hat{X}$, $t\mapsto u(s,t)$)

$$\hat{\mathcal{M}}(x,y):=\Big\{u:\mathbb{R}\times S^1\to\hat{X}\ \Big|\ \partial_s u-J_t(\partial_t u-X_{H_t}(u))=0,\ \lim_{s\to-\infty}u(s)=x,\ \lim_{s\to\infty}u(s)=y\Big\}.$$

Notice that one can define a natural $\mathbb{R}$ action on $\hat{\mathcal{M}}(x,y)$ by shifting trajectories in the s-variable. Let $\mathcal{M}(x,y)$ denote the quotient $\hat{\mathcal{M}}(x,y)/\mathbb{R}$. For generic $J=(J_t)_{t\in S^1}$,

$\mathcal{M}(x, y)$ is a smooth manifold. Denote by $\mathcal{M}_0(x, y)$ the 0-dimensional component of $\mathcal{M}(x, y)$. Then $\mathcal{M}_0(x, y)$ is compact (hence a finite set). Moreover,

$$\partial_{H,J} : \mathrm{CF}^\alpha(H) \to \mathrm{CF}^\alpha(H), \quad [x] \mapsto \sum_y \sharp\mathcal{M}_0(x, y) \cdot [y],$$

satisfies $\partial_{H,J}^2 = 0$. These claims are proved by the usual transversality and glueing arguments, combined with a C^0-estimate for solutions of the Floer equation (Lemma 2.2). The homology group of $(\mathrm{CF}^\alpha(H), \partial_{H,J})$ does not depend on J, and is denoted by $\mathrm{HF}^\alpha(H)$. It is called *Floer homology* of H.

It is easy to check that for any $x, y \in \mathcal{P}(H)$ and $u \in \hat{\mathcal{M}}(x, y)$,

$$\mathcal{A}_H(x) - \mathcal{A}_H(y) = \int_{\mathbb{R}\times S^1} |\partial_s u(s, t)|^2_{J_t}\, ds\, dt \geq 0.$$

In particular, $\mathcal{M}(x, y) \neq \emptyset \Rightarrow \mathcal{A}_H(x) \geq \mathcal{A}_H(y)$. Hence for any nonempty interval $I \subset \mathbb{R}$, $(\mathrm{CF}^{I,\alpha}(H), \partial_{H,J})$ is a chain complex. Its homology group $\mathrm{HF}^{I,\alpha}(H)$ is called *truncated Floer homology* of H. We introduce some abbreviations:

$$\mathrm{HF}^{<a,\alpha}(H) := \mathrm{HF}^{(-\infty,a),\alpha}(H), \quad \mathrm{HF}^I(H) := \bigoplus_\alpha \mathrm{HF}^{I,\alpha}(H), \quad \mathrm{HF}(H) := \mathrm{HF}^{\mathbb{R}}(H).$$

We define $I_+, I_- \subset \mathbb{R}$ as $I_+ := (-\infty, \inf I\,] \cup I$ and $I_- := I_+ \setminus I$. The following statements are immediate from the definitions:

- For any nonempty intervals $I, I' \subset \mathbb{R}$ such that $I_\pm \subset I'_\pm$, there exists a natural homomorphism $\Phi^{II'}_H : \mathrm{HF}^{I,\alpha}(H) \to \mathrm{HF}^{I',\alpha}(H)$.
- For any $-\infty \leq a < b < c \leq \infty$, we have the exact triangle

$$\begin{tikzcd} & \mathrm{HF}^{[a,c)}(H) \arrow[dr] & \\ \mathrm{HF}^{[a,b)}(H) \arrow[ur] & & \mathrm{HF}^{[b,c)}(H) \arrow[ll, "\partial"'] \end{tikzcd}$$

Next we introduce the *monotonicity homomorphism*.

Proposition 2.3. *Let $H, H' \in \mathcal{H}_{\mathrm{ad}}(X, \lambda)$ and assume that $a_H \leq a_{H'}$. Notice that*

$$\Delta := \int_{S^1} \max(H_t - H'_t)\, dt < \infty.$$

Let $I, I' \subset \mathbb{R}$ be nonempty intervals which satisfy $I_\pm + \Delta \subset I'_\pm$, and let $\alpha \in \pi'_1(X)$. Then there exists a natural homomorphism $\Phi^{II'}_{HH'} : \mathrm{HF}^{I,\alpha}(H) \to \mathrm{HF}^{I',\alpha}(H')$. Moreover, the following properties hold:

- *When $H = H'$, $\Phi^{II'}_{HH}$ coincides with $\Phi^{II'}_H$.*

- *Suppose that $H, H', H'' \in \mathcal{H}_{\mathrm{ad}}(X)$ satisfy $a_H \le a_{H'} \le a_{H''}$, and let $I, I', I'' \subset \mathbb{R}$ be nonempty intervals. Then the following diagram commutes if the homomorphisms $\Phi^{II'}_{HH'}, \Phi^{II''}_{HH''}, \Phi^{I'I''}_{H'H''}$ are defined:*

$$
\begin{tikzcd}
 & \mathrm{HF}^{I',\alpha}(H') \arrow[dr, "\Phi^{I'I''}_{H'H''}"] & \\
\mathrm{HF}^{I,\alpha}(H) \arrow[ur, "\Phi^{II'}_{HH'}"] \arrow[rr, "\Phi^{II''}_{HH''}"] & & \mathrm{HF}^{I'',\alpha}(H'')
\end{tikzcd}
$$

Proof. The proof is almost the same as in the case of closed aspherical symplectic manifolds (see [Schw, pp. 431]). The only difference is that we need a C^0-estimate for Floer trajectories, and it follows from Lemma 2.2. □

The homomorphism $\Phi^{II'}_{HH'}$ defined in Proposition 2.3 is called the *monotonicity homomorphism*. The following corollary is immediate from Proposition 2.3.

Corollary 2.4. *Let (X, λ) be a Liouville domain, $\alpha \in \pi_1'(X)$, and $H, H' \in \mathcal{H}_{\mathrm{ad}}(X, \lambda)$.*

(1) *If $a_H = a_{H'}$, then there exists a natural isomorphism $\Phi_{HH'} : \mathrm{HF}^\alpha(H) \to \mathrm{HF}^\alpha(H')$.*
(2) *If $H_t \le H_t'$ for every $t \in S^1$, then there exists a natural homomorphism $\mathrm{HF}^{I,\alpha}(H) \to \mathrm{HF}^{I,\alpha}(H')$ for any nonempty interval I.*

2.2.2. Truncated Floer homology of Liouville domains. Let (X, λ) be a Liouville domain, $\alpha \in \pi_1'(X)$, and $I \subset \mathbb{R}$ be a nonempty interval. Setting $\mathcal{H}^{\mathrm{neg}}_{\mathrm{ad}}(X, \lambda) := \{H \in \mathcal{H}_{\mathrm{ad}}(X, \lambda) \mid H|_{S^1 \times X} < 0\}$, we define

$$
\mathrm{HF}^{I,\alpha}(X, \lambda) := \varinjlim_{H \in \mathcal{H}^{\mathrm{neg}}_{\mathrm{ad}}(X,\lambda)} \mathrm{HF}^{I,\alpha}(H),
$$

where the right hand side is a direct limit with respect to the monotonicity homomorphisms of Corollary 2.4(2). If two nonempty intervals I, I' satisfy $I_\pm \subset I'_\pm$, then there exists a natural homomorphism $\mathrm{HF}^{I,\alpha}(X, \lambda) \to \mathrm{HF}^{I',\alpha}(X, \lambda)$. We prove the following useful lemma.

Lemma 2.5. *For any $H \in \mathcal{H}_{\mathrm{ad}}(X, \lambda)$, there exists a natural isomorphism $\Psi_H : \mathrm{HF}^\alpha(H) \to \mathrm{HF}^{<a_H,\alpha}(X, \lambda)$. Moreover, if $H_-, H_+ \in \mathcal{H}_{\mathrm{ad}}(X, \lambda)$ satisfy $a_{H_-} \le a_{H_+}$, then the following diagram commutes:*

$$
\begin{tikzcd}
\mathrm{HF}^\alpha(H_-) \arrow[r, "\Psi_{H_-}"] \arrow[d, "\Phi_{H_-H_+}"'] & \mathrm{HF}^{<a_{H_-},\alpha}(X, \lambda) \arrow[d] \\
\mathrm{HF}^\alpha(H_+) \arrow[r, "\Psi_{H_+}"] & \mathrm{HF}^{<a_{H_+},\alpha}(X, \lambda)
\end{tikzcd}
$$

Proof. First we construct $\Psi_H : \mathrm{HF}^\alpha(H) \to \mathrm{HF}^{<a_H,\alpha}(X, \lambda)$. It is not hard to check that the following natural homomorphisms are all isomorphisms:

$$\begin{aligned}
\mathrm{HF}^{\alpha}(H) &\to \varinjlim_{\substack{G\in\mathcal{H}_{\mathrm{ad}}\\ a_G\le a_H}} \mathrm{HF}^{\alpha}(G),\\
\varinjlim_{\substack{G\in\mathcal{H}^{\mathrm{neg}}_{\mathrm{ad}}\\ a_G\le a_H}} \mathrm{HF}^{\alpha}(G) &\to \varinjlim_{\substack{G\in\mathcal{H}_{\mathrm{ad}}\\ a_G\le a_H}} \mathrm{HF}^{\alpha}(G),\\
\varinjlim_{\substack{G\in\mathcal{H}^{\mathrm{neg}}_{\mathrm{ad}}\\ a_G\le a_H}} \mathrm{HF}^{<a_H,\alpha}(G) &\to \varinjlim_{\substack{G\in\mathcal{H}^{\mathrm{neg}}_{\mathrm{ad}}\\ a_G\le a_H}} \mathrm{HF}^{\alpha}(G),\\
\varinjlim_{\substack{G\in\mathcal{H}^{\mathrm{neg}}_{\mathrm{ad}}\\ a_G\le a_H}} \mathrm{HF}^{<a_H,\alpha}(G) &\to \varinjlim_{G\in\mathcal{H}^{\mathrm{neg}}_{\mathrm{ad}}} \mathrm{HF}^{<a_H,\alpha}(G) = \mathrm{HF}^{<a_H,\alpha}(X,\lambda).
\end{aligned}$$

By composing the above isomorphisms and their inverses, we get an isomorphism $\Psi_H : \mathrm{HF}^{\alpha}(H) \to \mathrm{HF}^{<a_H,\alpha}(X,\lambda)$. This proves the first assertion. The second assertion follows from the above construction. ◻

It is a standard fact that for any $\delta \in (0, \min \mathrm{Spec}(X,\lambda))$, there exists a natural isomorphism (see [V1, Proposition 1.4])

$$\bigoplus_i H^i(X) \cong \mathrm{HF}^{<\delta,c_X}(X,\lambda) \cong \mathrm{HF}^{<\delta}(X,\lambda).$$

Then, for any $0 < a \le \infty$, one can define a natural homomorphism

$$\iota_a : \bigoplus_i H^i(X) \cong \mathrm{HF}^{<\delta,c_X}(X,\lambda) \to \mathrm{HF}^{<a,c_X}(X,\lambda)$$

by taking sufficiently small $\delta > 0$. The homomorphism ι_∞ coincides with ι which appears in Section 1.3 (Step 1). Using ι_a, we define an important homology class $F_a \in \mathrm{HF}^{<a,c_X}(X,\lambda)$ by $F_a := \iota_a(1)$, where 1 denotes the canonical element in $H^0(X)$.

For any $H \in \mathcal{H}_{\mathrm{ad}}(X,\lambda)$, we define $F_H \in \mathrm{HF}(H)$ by $F_H := \Psi_H^{-1}(F_{a_H})$. The second assertion in Lemma 2.5 shows that for any $H_-, H_+ \in \mathcal{H}_{\mathrm{ad}}(X,\lambda)$ with $a_{H_-} \le a_{H_+}$, we have $\Phi_{H_-H_+}(F_{H_-}) = F_{H_+}$.

2.3. *Product structure*

First we define the pair-of-pants Riemann surface Π. The following definition is taken from [AS2, pp. 1602–1603]. In the disjoint union $\mathbb{R}\times[-1,0] \sqcup \mathbb{R}\times[0,1]$, we consider the identifications

$$\begin{aligned}
(s,-1) &\sim (s,0^-), & (s,0^+) &\sim (s,1) & (s\le 0),\\
(s,0^-) &\sim (s,0^+), & (s,-1) &\sim (s,1) & (s\ge 0),
\end{aligned}$$

and define Π to be the quotient. We take the standard complex structure at every point of Π other than $P := (0,0) \sim (0,-1) \sim (0,1)$. On a neighborhood of P, we define a complex structure by the following holomorphic coordinate:

$$\{\zeta \in \mathbb{C} \mid |\zeta| < 1/\sqrt{2}\} \to \Pi,$$
$$\zeta = \sigma + \tau i \mapsto \begin{cases} (\sigma^2 - \tau^2, 2\sigma\tau) & (\sigma \geq 0), \\ (\sigma^2 - \tau^2, 2\sigma\tau + 1) & (\sigma \leq 0, \tau \geq 0), \\ (\sigma^2 - \tau^2, 2\sigma\tau - 1) & (\sigma \leq 0, \tau \leq 0). \end{cases} \qquad (\star)$$

j_Π denotes the complex structure on Π. We need the following convexity result:

Lemma 2.6. *Let $(H_{s,t})_{(s,t)\in\Pi}$ be a family of Hamiltonians on $\hat{X}$, and $(J_{s,t})_{(s,t)\in\Pi}$ be a family of elements in $\mathcal{J}(\hat{X}, \hat{\lambda})$. Suppose that there exists $r_0 > 0$ such that the following hold:*

- *There exist $a, b \in C^\infty(\mathbb{R})$ such that $H_{s,t}(z, r) = a(s)r + b(s)$ on $\partial X \times [r_0, \infty)$, and $a'(s) \geq 0$ for any $s \in \mathbb{R}$.*
- *$(J_{s,t})_{(s,t)\in\Pi}$ is of contact type on $\partial X \times [r_0, \infty)$.*

If $u : \Pi \to \hat{X}$ satisfies the Floer equation

$$\partial_s u - J_{s,t}(\partial_t u - X_{H_{s,t}}(u)) = 0 \quad (\text{at } (s,t) \neq P),$$
$$J_P \circ du \circ j_\Pi - du = 0 \quad (\text{at } P),$$

and $u^{-1}(\partial X \times (r_0, \infty))$ is bounded, then $u(\Pi) \subset X(r_0)$.

Remark 2.7. The Floer equation in Lemma 2.6 may look strange at first. In a neighborhood of P, it is written as $\partial_\sigma u - J\partial_\tau u + (2\sigma J - 2\tau)X_H(u) = 0$, by using the holomorphic coordinate $(\star)$ (we omit subscripts for J and H).

Proof. If $u(\Pi)$ is not contained in $X(r_0)$, then there exists $r_1 > r_0$ such that $u(\Pi)$ is not contained in $X(r_1)$, u is transversal to $\partial X \times \{r_1\}$ and $u(P) \notin \partial X \times \{r_1\}$. Then $D := u^{-1}(\partial X \times [r_1, \infty))$ is a compact surface with boundary, and $P \notin \partial D$.

The rest of the proof is almost the same as that of Lemma 2.2, after replacing D with $D \setminus \{P\}$. The only delicate point is that we have to check

$$\int_{D\setminus\{P\}} \partial_s(\hat{\lambda}(X_{H_{s,t}}(u)))\, ds\, dt - u^*(d\hat{\lambda}) = \int_{\partial D} \hat{\lambda}(X_{H_{s,t}}(u)\, dt - du),$$

where we cannot apply Stokes's theorem (when $P \in \operatorname{int} D$, $D \setminus \{P\}$ is not compact). It is enough to consider the case $P \in \operatorname{int} D$. Take a complex chart $(\star)$ near P, and set $D_\varepsilon := \{\zeta \in \mathbb{C} \mid 0 \leq |\zeta| \leq \varepsilon\}$. Then the above identity is proved as follows:

$$\int_{D\setminus\{P\}} \partial_s(\hat{\lambda}(X_{H_{s,t}}(u)))\, ds\, dt - u^*(d\hat{\lambda}) = \lim_{\varepsilon\to 0} \int_{D\setminus D_\varepsilon} \partial_s(\hat{\lambda}(X_{H_{s,t}}(u)))\, ds\, dt - u^*(d\hat{\lambda})$$
$$= \lim_{\varepsilon\to 0} \int_{\partial D_\varepsilon \cup \partial D} \hat{\lambda}(X_{H_{s,t}}(u)\, dt - du) = \int_{\partial D} \hat{\lambda}(X_{H_{s,t}}(u)\, dt - du).$$

The last equality holds since the norm of $\hat{\lambda}(X_{H_{s,t}}(u)\, dt - du)$ is bounded on any compact neighborhood of P, and the length of ∂D_ε goes to 0 as $\varepsilon \to 0$. □

Let $H, K \in \mathcal{H}_{\mathrm{ad}}(X, \lambda)$. Suppose that the following holds:

(P0) $\partial_t^r H|_{t=0} = \partial_t^r K|_{t=0}$ for any integer $r \geq 0$. In particular, $a_H = a_K$.

We define $H * K \in C^\infty(S^1 \times \hat{X})$ by

$$(H * K)_t := \begin{cases} 2H_{2t} & (0 \le t \le 1/2), \\ 2K_{2t-1} & (1/2 \le t \le 1). \end{cases}$$

Suppose also that:

(P1) $H * K \in \mathcal{H}_{\mathrm{ad}}(X)$,
(P2) $x(0) \neq y(0)$ for any $x \in \mathcal{P}(H)$ and $y \in \mathcal{P}(K)$.

Let $(J_t)_{-1 \le t \le 1}$ be a family of elements in $\mathcal{J}(\hat{X}, \hat{\lambda})$ which is of contact type at ∞ and $\partial_t^r J_t|_{t=-1} = \partial_t^r J_t|_{t=0} = \partial_t^r J_t|_{t=1}$ for any integer $r \ge 0$. For any $x \in \mathcal{P}(H)$, $y \in \mathcal{P}(K)$ and $z \in \mathcal{P}(H * K)$, let $\mathcal{M}(x, y : z)$ denote the set of $u : \Pi \to \hat{X}$ which satisfy

$$\begin{aligned} \partial_s u - J_t(\partial_t u - \tfrac{1}{2} X_{(H*K)_{(t+1)/2}}(u)) = 0 \quad & (\text{at } (s,t) \neq P), \\ J_P \circ du \circ j - du = 0 \quad & (\text{at } P), \end{aligned}$$

with boundary conditions

$$\begin{aligned} \lim_{s \to -\infty} u(s,t) &= y(t) && (0 \le t \le 1), \\ \lim_{s \to -\infty} u(s,t) &= x(t+1) && (-1 \le t \le 0), \\ \lim_{s \to \infty} u(s,t) &= z((t+1)/2) && (-1 \le t \le 1). \end{aligned}$$

For generic $(J_t)_{-1 \le t \le 1}$, $\mathcal{M}(x, y : z)$ is a smooth manifold. Its 0-dimensional component $\mathcal{M}_0(x, y : z)$ is compact (hence a finite set). Moreover,

$$\mathrm{CF}(H) \otimes \mathrm{CF}(K) \to \mathrm{CF}(H * K), \qquad [x] \otimes [y] \mapsto \sum_z \sharp \mathcal{M}_0(x, y : z) \cdot [z],$$

is a chain map. These claims are proved by the usual transversality and glueing arguments, combined with a C^0-estimate for Floer trajectories, which follows from Lemma 2.6. Hence we can define the *pair-of-pants product* on Floer homology of Hamiltonians:

$$\mathrm{HF}(H) \otimes \mathrm{HF}(K) \to \mathrm{HF}(H * K), \qquad \alpha \otimes \beta \mapsto \alpha * \beta.$$

Simple computations show that for any $x \in \mathcal{P}(H)$, $y \in \mathcal{P}(K)$, $z \in \mathcal{P}(H * K)$ and $u \in \mathcal{M}(x, y : z)$,

$$\mathcal{A}_H(x) + \mathcal{A}_K(y) - \mathcal{A}_{H*K}(z) = \int_{\Pi \setminus \{P\}} |\partial_s u|_{J_t}^2 \, ds\, dt \ge 0.$$

In particular, $\mathcal{M}(x, y : z) \neq \emptyset \Rightarrow \mathcal{A}_H(x) + \mathcal{A}_K(y) \ge \mathcal{A}_{H*K}(z)$. Hence for any $-\infty \le a, b \le \infty$, one can define the pair-of-pants product on truncated Floer homology of Hamiltonians:

$$\mathrm{HF}^{<a}(H) \otimes \mathrm{HF}^{<b}(K) \to \mathrm{HF}^{<a+b}(H * K).$$

By using Lemma 2.6, it is easy to show that it commutes with monotonicity homomorphisms:

Lemma 2.8. *Suppose that $H, K, \bar{H}, \bar{K} \in \mathcal{H}_{\mathrm{ad}}(X, \lambda)$ satisfy the following:*

(1) *H and K satisfy* (P0), (P1), (P2).
(2) *$\bar{H}$ and $\bar{K}$ satisfy* (P0), (P1), (P2).
(3) *$H_t \le \bar{H}_t$, $K_t \le \bar{K}_t$ for every $t \in S^1$.*

Then the following diagram commutes for any $-\infty \le a, b \le \infty$:

$$\begin{tikzcd}
\mathrm{HF}^{<a}(H) \otimes \mathrm{HF}^{<b}(K) \arrow[r] \arrow[d, "\Phi_{H\bar{H}} \otimes \Phi_{K\bar{K}}"'] & \mathrm{HF}^{<a+b}(H * K) \arrow[d, "\Phi_{H*K, \bar{H}*\bar{K}}"] \\
\mathrm{HF}^{<a}(\bar{H}) \otimes \mathrm{HF}^{<b}(\bar{K}) \arrow[r] & \mathrm{HF}^{<a+b}(\bar{H} * \bar{K})
\end{tikzcd}$$

By Lemma 2.8, one can define the pair-of-pants product on truncated Floer homology of Liouville domains: $\mathrm{HF}^{<a}(X, \lambda) \otimes \mathrm{HF}^{<b}(X, \lambda) \to \mathrm{HF}^{<a+b}(X, \lambda)$.

Moreover, the isomorphism in Lemma 2.5 commutes with products. More precisely, if $H, K \in \mathcal{H}_{\mathrm{ad}}(X, \lambda)$ satisfy (P0), (P1), (P2), then the following diagram commutes ($a := a_H = a_K$):

$$\begin{tikzcd}
\mathrm{HF}(H) \otimes \mathrm{HF}(K) \arrow[r] \arrow[d, "\Psi_H \otimes \Psi_K"'] & \mathrm{HF}(H * K) \arrow[d, "\Psi_{H*K}"] \\
\mathrm{HF}^{<a}(X, \lambda) \otimes \mathrm{HF}^{<a}(X, \lambda) \arrow[r] & \mathrm{HF}^{<2a}(X, \lambda)
\end{tikzcd}$$

3. Spectral invariants and Hofer–Zehnder capacity

The goal of this section is to prove Theorem 1.4. The proof depends on the theory of *spectral invariants*, which has been developed by several authors (see [FGS], [MS, Section 12.4] and references therein). In Section 3.1, we define the spectral invariants and summarize their basic properties. In Section 3.2, we prove Theorem 1.4.

3.1. *Spectral invariants*

Let (X, λ) be a Liouville domain, and $H \in \mathcal{H}_{\mathrm{ad}}(X, \lambda)$. For any $a \in \mathbb{R}$, there exists an exact triangle

$$\begin{tikzcd}
 & \mathrm{HF}(H) \arrow[dr, "j^a"] & \\
\mathrm{HF}^{<a}(H) \arrow[ur, "i^a"] & & \mathrm{HF}^{\ge a}(H) \arrow[ll, "\partial^a"]
\end{tikzcd}$$

Recall that there exists a natural isomorphism $\Psi_H : \mathrm{HF}(H) \to \mathrm{HF}^{<a_H}(X, \lambda)$. Then, for any $x \in \mathrm{HF}^{<a_H}(X, \lambda)$, we define the *spectral invariant* $\rho(H : x)$ by

$$\rho(H : x) := \inf\{a \in \mathbb{R} \mid \Psi_H^{-1}(x) \in \operatorname{Im} i^a\} = \inf\{a \in \mathbb{R} \mid j^a(\Psi_H^{-1}(x)) = 0\}.$$

Notice that $\rho(H : 0) = -\infty$.

In Lemma 3.1 below, we summarize basic properties of the spectral invariant. First we introduce some notation:

- For $H \in C^\infty(S^1 \times \hat{X})$ and $\alpha \in \pi_1'(X)$, we define

$$\mathcal{P}^\alpha(H) := \{x \in \mathcal{P}(H) \mid [x] = \alpha\}, \qquad \mathrm{Spec}^\alpha(H) := \{\mathcal{A}_H(x) \mid x \in \mathcal{P}^\alpha(H)\},$$
$$\mathrm{Spec}(H) := \{\mathcal{A}_H(x) \mid x \in \mathcal{P}(H)\}.$$

  Suppose that $H \in C^\infty(S^1 \times \hat{X})$ is linear at ∞ and $a_H \notin \mathrm{Spec}(X, \lambda)$. Then it is easy to see that $\mathrm{Spec}^\alpha(H), \mathrm{Spec}(H) \subset \mathbb{R}$ are closed, nowhere dense sets.
- For $H \in C_0^\infty(S^1 \times \hat{X})$ (C_0^∞ denotes the set of compactly supported smooth functions), its *Hofer norm* is defined as

$$\|H\| := \int_{S^1} (\max H_t - \min H_t)\, dt.$$

Lemma 3.1. (1) *For any* $H \in \mathcal{H}_{\mathrm{ad}}(X, \lambda)$ *and* $x \in \mathrm{HF}^{<a_H,\alpha}(X, \lambda) \setminus \{0\}$,

$$\rho(H : x) \in \mathrm{Spec}^\alpha(H).$$

(2) *Suppose* $H, K \in \mathcal{H}_{\mathrm{ad}}(X, \lambda)$ *are such that* $\mathrm{supp}(H - K)$ *is compact. Then, for any* $x \in \mathrm{HF}^{<a_H}(X, \lambda) \setminus \{0\}$,

$$|\rho(H : x) - \rho(K : x)| \le \|H - K\|.$$

(3) *Suppose* $H, K \in \mathcal{H}_{\mathrm{ad}}(X, \lambda)$ *satisfy* (P0), (P1), (P2) *of Section* 2.3. *Then, for any* $x, y \in \mathrm{HF}^{<a_H}(X, \lambda)$,

$$\rho(H * K : x * y) \le \rho(H : x) + \rho(K : y).$$

Proof. (1) Suppose $\rho(H : x) \notin \mathrm{Spec}^\alpha(H)$. We abbreviate $\rho(H : x)$ as ρ. Since $x \ne 0$, we have $\rho \ne -\infty$. Since $\mathrm{Spec}^\alpha(H)$ is closed, there exists $\varepsilon > 0$ such that $[\rho-\varepsilon, \rho+\varepsilon] \cap \mathrm{Spec}^\alpha(H) = \emptyset$. Hence $\mathrm{HF}^{[\rho-\varepsilon,\rho+\varepsilon),\alpha}(H) = 0$, and so $\mathrm{HF}^{<\rho-\varepsilon,\alpha}(H) \to \mathrm{HF}^{<\rho+\varepsilon,\alpha}(H)$ is an isomorphism. Hence we get

$$\mathrm{Im}\big(\mathrm{HF}^{<\rho-\varepsilon,\alpha}(H) \to \mathrm{HF}^\alpha(H)\big) = \mathrm{Im}\big(\mathrm{HF}^{<\rho+\varepsilon,\alpha}(H) \to \mathrm{HF}^\alpha(H)\big).$$

However, the definition of the spectral invariant implies that $\Psi_H^{-1}(x) \notin \mathrm{Im}(\mathrm{HF}^{<\rho-\varepsilon,\alpha}(H) \to \mathrm{HF}^\alpha(H))$ and $\Psi_H^{-1}(x) \in \mathrm{Im}(\mathrm{HF}^{<\rho+\varepsilon,\alpha}(H) \to \mathrm{HF}^\alpha(H))$, hence a contradiction.

(2) By Proposition 2.3, for any $a \in \mathbb{R}$ there exists a monotonicity homomorphism $\mathrm{HF}^{<a}(H) \to \mathrm{HF}^{<a+\|H-K\|}(K)$. The commutativity of the diagram (horizontal arrows are monotonicity homomorphisms)

$$\begin{tikzcd}
\mathrm{HF}^{<a}(H) \arrow[r] \arrow[d, "i^a"'] & \mathrm{HF}^{<a+\|H-K\|}(K) \arrow[d, "i^{a+\|H-K\|}"] \\
\mathrm{HF}(H) \arrow[r] & \mathrm{HF}(K)
\end{tikzcd}$$

shows that $\rho(K : x) \le \rho(H : x) + \|H - K\|$. A similar argument shows that $\rho(H : x) \le \rho(K : x) + \|H - K\|$, hence (2) is proved.

(3) follows from the fact that the isomorphism in Lemma 2.5 commutes with products (see the last paragraph of Section 2.3). ◻

Using Lemma 3.1(2), we can define spectral invariants for a larger class of Hamiltonians. Suppose $H \in C^\infty(S^1 \times \hat{X})$ is linear at ∞ and $a_H \notin \mathrm{Spec}(X, \lambda)$, but $\mathcal{P}(H)$ may contain degenerate orbits. Take a sequence $(H_j)_{j=1,2,\ldots}$ of admissible Hamiltonians such that $\mathrm{supp}(H_j - H)$ is compact for any j and $\lim_{j\to\infty} \|H_j - H\| = 0$. Define

$$\rho(H : x) := \lim_{j\to\infty} \rho(H_j : x).$$

By Lemma 3.1(2), the right hand side exists and does not depend on the choices of $(H_j)_j$. The following lemma is immediate from Lemma 3.1 and the above definition.

Lemma 3.2. *Suppose $H \in C^\infty(S^1 \times \hat{X})$ is linear at ∞ and $a_H \notin \mathrm{Spec}(X, \lambda)$.*

(1) *For any $x \in \mathrm{HF}^{<a_H,\alpha}(X, \lambda) \setminus \{0\}$, $\rho(H : x) \in \mathrm{Spec}^\alpha(H)$.*
(2) *For any $K \in C^\infty(S^1 \times \hat{X})$ such that $\mathrm{supp}(H - K)$ is compact and for any $x \in \mathrm{HF}^{<a_H}(X, \lambda) \setminus \{0\}$,*

$$|\rho(H : x) - \rho(K : x)| \le \|H - K\|.$$

(3) *Suppose that $2a_H \notin \mathrm{Spec}(X, \lambda)$. If $K \in C^\infty(S^1 \times \hat{X})$ is linear at ∞ and satisfies $\partial_t^r H|_{t=0} = \partial_t^r K|_{t=0}$ for any integer $r \ge 0$, then*

$$\rho(H * K : x * y) \le \rho(H : x) + \rho(K : y) \qquad \text{for any } x, y \in \mathrm{HF}^{<a_H}(X, \lambda).$$

3.2. Proof of Theorem 1.4

In this subsection, we use the following notation:

- For any $H \in C_0^\infty(\mathrm{int}\, X)$, $a \in \mathbb{R}$ and $\nu \in C^\infty([1,\infty))$, we define $H_{a,\nu} : S^1 \times \hat{X} \to \mathbb{R}$ by
$$H_{a,\nu}(t, x) := \begin{cases} aH(x) & (x \in \mathrm{int}\, X), \\ \nu(r) & (x = (z, r) \in \partial X \times [1, \infty)). \end{cases}$$
- For any $K \in C^\infty(S^1 \times \hat{X})$ which is linear at ∞ and $a_K \notin \mathrm{Spec}(X, \lambda)$, we abbreviate $\rho(K : F_{a_K})$ as $\rho(K)$.

The proof of Theorem 1.4 is based on the following proposition:

Proposition 3.3. *Let (X, λ) be a Liouville domain, $H \in C_0^\infty(\mathrm{int}\, X)$, $\nu \in C^\infty([1, \infty))$. Suppose that:*

(1) *There exists $r_0 > 1$ such that $\nu(r) \equiv 0$ on $[1, r_0]$.*
(2) *There exist $r_1 > 1$ and $a_\nu \in (0, -\min H) \setminus \mathrm{Spec}(X, \lambda)$ such that $\nu'(r) \equiv a_\nu$ on $[r_1, \infty)$.*
(3) *$S(\nu) := \sup_{r\ge 1}(r\nu'(r) - \nu(r)) < -\min H$.*

Under these assumptions, if H is Hofer–Zehnder admissible with respect to c_X, then $\rho(H_{1,\nu}) = -\min H$.

To prove Proposition 3.3, we need the following lemma:

Lemma 3.4. *Fix $r > 1$. Suppose that $K \in \mathcal{H}_{\mathrm{ad}}(X, \lambda)$ is time independent, and linear on $\partial X \times [r, \infty)$. If the C^2-norm of $K|_{X(r)}$ is sufficiently small, then $\rho(K) = -\min K$.*

Proof. Since K is time independent, we can define $k \in C^\infty(\hat{X})$ by $k(x) := K(t, x)$. If the C^2-norm of $K|_{X(r)}$ is sufficiently small, the Floer complex of K is identified with the Morse complex of k, and it induces an isomorphism $\mathrm{HF}(K) \cong \bigoplus_i H^i(X)$. Since $F_K \in \mathrm{HF}(K)$ corresponds to $1 \in H^0(X)$ in this isomorphism, $\sum_{q \in \mathrm{CrP}_0(K)} a_q[q]$ represents F_K if and only if $a_q = 1$ for any $q \in \mathrm{CrP}_0(k) := \{\text{critical points of } k \text{ with Morse index } 0\}$. Hence $\rho(K) = \max_{q \in \mathrm{CrP}_0(k)} -k(q) = -\min K$. □

Proof of Proposition 3.3. Suppose $H \in C_0^\infty(\mathrm{int}\, X)$ is Hofer–Zehnder admissible with respect to c_X. For any $c > 0$, there exists $K \in C_0^\infty(\mathrm{int}\, X)$ which satisfies $|H - K|_{C^0} < c$ and has the following properties:

- Any nonconstant contractible periodic orbit of X_K has period larger than 1.
- $\min K < \min H$.
- $\min K$ is isolated in the set of critical values of K.
- $\min K$ is attained by a unique point $p_K \in X$. Moreover, the constant loop at p_K is nondegenerate as an element of $\mathcal{P}(K)$.

Therefore it is enough to show $\rho(K_{1,\nu}) = -\min K$ for any $K \in C_0^\infty(\mathrm{int}\, X)$ which satisfies the above conditions. We prove this in three steps.

Step 1. There exist $0 < \varepsilon_0 < 1$ and $0 < \delta_0 < (\min \mathrm{Spec}(X, \lambda))/a_\nu$ such that $\rho(K_{\varepsilon,\delta\nu}) = -\varepsilon \min K$ for any $\varepsilon \in (0, \varepsilon_0]$ and $\delta \in (0, \delta_0]$.

When ε and δ are sufficiently small, the C^2-norm of $K_{\varepsilon,\delta\nu}|_{X(r_1)}$ is sufficiently small. Hence the claim follows at once from Lemma 3.4, by approximating $K_{\varepsilon,\delta\nu}$ by admissible time independent Hamiltonians.

Step 2. $\rho(K_{1,\delta\nu}) = -\min K$ for any $0 < \delta < \min\{\delta_0, (-\varepsilon_0 \min K)/S(\nu)\}$.

For any $\varepsilon \in (0, 1]$, $\mathcal{P}^{c_X}(\varepsilon K)$ consists of only constant loops at critical points of K, since every nonconstant contractible periodic orbit of X_K has period larger than 1. On the other hand, $\mathcal{A}_{K_{\varepsilon,\delta\nu}}(x) \le \delta S(\nu)$ for any $x \in \mathcal{P}(K_{\varepsilon,\delta\nu})$ which is not contained in X. Hence $\mathrm{Spec}^{c_X}(K_{\varepsilon,\delta\nu}) \subset (-\infty, \delta S(\nu)] \cup -\varepsilon\, \mathrm{CrV}(K)$, where $\mathrm{CrV}(K)$ denotes the set of critical values of K. Since $\delta a_\nu < \min \mathrm{Spec}(X, \lambda)$, we have $F_{\delta a_\nu} \neq 0$. Hence Lemma 3.2(1) shows that $\rho(K_{\varepsilon,\delta\nu}) \in (-\infty, \delta S(\nu)] \cup -\varepsilon\, \mathrm{CrV}(K)$.

Let $I := \{\varepsilon \in [\varepsilon_0, 1] \mid \rho(K_{\varepsilon,\delta\nu}) = -\varepsilon \min K\}$. Step 1 shows $\varepsilon_0 \in I$. Lemma 3.2(2) shows that $\rho(K_{\varepsilon,\delta\nu})$ depends continuously on ε, hence I is closed. Moreover, since $\delta S(\nu) < -\varepsilon_0 \min K$ and $\mathrm{CrV}(K)$ is nowhere dense, I is open. Hence $I = [\varepsilon_0, 1]$. In particular, $\rho(K_{1,\delta\nu}) = -\min K$.

Step 3. $\rho(K_{1,\nu}) = -\min K$.

Since $S(\nu) < -\min H < -\min K$, we have $\mathrm{Spec}(K_{1,\nu}) \subset (-\infty, -\min K]$. Hence $\rho(K_{1,\nu}) \le -\min K$ is clear. Therefore it is enough to prove $\rho(K_{1,\nu}) \ge -\min K$. Take δ so that $0 < \delta < \min\{\delta_0, (-\varepsilon_0 \min K)/S(\nu)\}$. Take $c > 0$ so that $\mathrm{CrV}(K) \cap (-\infty, \min K + c] = \{\min K\}$ (this is possible since $\min K$ is isolated in $\mathrm{CrV}(K)$) and

$S(\nu) < -\min K - c$. Consider the following commutative diagram, where vertical arrows are monotonicity homomorphisms (to be precise, we have to perturb $K_{1,\delta\nu}$ and $K_{1,\nu}$ to define $\mathrm{HF}(K_{1,\delta\nu})$ and $\mathrm{HF}(K_{1,\nu})$; we omit this point) :

$$\begin{tikzcd} \mathrm{HF}(K_{1,\delta\nu}) \arrow[r] \arrow[d] & \mathrm{HF}^{\geq -\min K - c}(K_{1,\delta\nu}) \arrow[d] \\ \mathrm{HF}(K_{1,\nu}) \arrow[r] & \mathrm{HF}^{\geq -\min K - c}(K_{1,\nu}) \end{tikzcd}$$

Since $\min K$ is attained at a unique point p_K, it follows that $\mathrm{CF}^{\geq -\min K - c}(K_{1,\delta\nu})$ and $\mathrm{CF}^{\geq -\min K - c}(K_{1,\nu})$ are generated by p_K. Moreover, $K_{1,\delta\nu}|_{S^1\times X} = K_{1,\nu}|_{S^1\times X}$. In particular, $K_{1,\delta\nu}$ and $K_{1,\nu}$ coincide near p_K. Therefore the right arrow is an isomorphism.

By Step 2, $\rho(K_{1,\delta\nu}) = -\min K$. Hence $F_{K_{1,\delta\nu}}$ does not vanish under the top arrow. Since the right arrow is an isomorphism, $F_{K_{1,\nu}}$ does not vanish under the bottom arrow, i.e. $\rho(K_{1,\nu}) \geq -\min K - c$. Since $c > 0$ is arbitrarily small, $\rho(K_{1,\nu}) \geq -\min K$. $\square$

Before going to the proof of Theorem 1.4, let us mention the following corollary of Proposition 3.3.

Corollary 3.5. *Let (X, λ) be a Liouville domain, and $a \in (0,\infty)\setminus\mathrm{Spec}(X, \lambda)$. If $F_a = 0$, then $c_{\mathrm{HZ}}(\mathrm{int}\, X, d\lambda : \{c_X\}) \leq a$.*

Proof. If $c_{\mathrm{HZ}}(\mathrm{int}\, X, d\lambda : \{c_X\}) > a$, then there exists $H \in C^\infty_0(\mathrm{int}\, X)$ which is Hofer–Zehnder admissible with respect to c_X, and $-\min H > a$. Take $\nu \in C^\infty([1,\infty))$ which satisfies (1)–(3) of Proposition 3.3 and $a_\nu = a$. Then Proposition 3.3 states that $\rho(H_{1,\nu}) = -\min H$. On the other hand, $\rho(H_{1,\nu}) = -\infty$ as $F_a = 0$; a contradiction. $\square$

Corollary 3.5 implies that any subcritical Weinstein manifold has a finite Hofer–Zehnder capacity, since Floer homology of such a manifold vanishes (see [Cie]). However, this result itself is an immediate consequence of the energy-capacity inequality in [Schl]. If X is the unit disk cotangent bundle of a closed manifold, then one can see immediately from Theorem 1.6 that $F_a \neq 0$ for any $a > 0$. Hence we cannot apply Corollary 3.5 to prove Theorem 1.2. This is the reason why we have to use the product structure on Floer homology to prove Theorem 1.2.

Now we prove Theorem 1.4. In fact, we prove the following quantitative result:

Theorem 3.6. *Let (X, λ) be a Liouville domain, let $a > 0$ be such that $a, 2a \notin \mathrm{Spec}(X, \lambda)$, and let $\alpha \in \pi_1'(X) \setminus \{c_X\}$. Suppose there exist $x \in \mathrm{HF}^{<a,\alpha}(X, \lambda)$ and $y \in \mathrm{HF}^{<a,\bar{\alpha}}(X, \lambda)$ such that $x * y = F_{2a}$. Then $c_{\mathrm{HZ}}(\mathrm{int}\, X, d\lambda : \{c_X, \alpha, \bar{\alpha}\}) \leq 2a$.*

Proof. Suppose that there exists $H \in C^\infty_0(\mathrm{int}\, X)$ which is Hofer–Zehnder admissible with respect to $\{c_X, \alpha, \bar{\alpha}\}$ and $-\min H > 2a$. Take $\nu \in C^\infty([1,\infty))$ which satisfies (1)–(3) of Proposition 3.3 and $a_\nu = 2a$. Then Proposition 3.3 yields $\rho(H_{1,\nu}) = -\min H$.

Since H is Hofer–Zehnder admissible with respect to α, and α is noncontractible, we have $\mathcal{P}^\alpha(H/2) = \emptyset$. On the other hand, for any $\gamma \in \mathcal{P}(H_{1/2,\nu/2})$ which is not contained in X, $\mathcal{A}_{H_{1/2,\nu/2}}(\gamma) \leq S(\nu)/2 < -\min H/2$. Hence $\mathrm{Spec}^\alpha(H_{1/2,\nu/2}) \subset (-\infty, -\min H/2)$.

By Lemma 3.1(1), $\rho(H_{1/2,\nu/2} : x) < -\min H/2$. By the same arguments, we get $\rho(H_{1/2,\nu/2} : y) < -\min H/2$. Hence we deduce a contradiction:

$$-\min H = \rho(H_{1,\nu} : F_{2a}) \leq \rho(H_{1/2,\nu/2} : x) + \rho(H_{1/2,\nu/2} : y) < -\min H,$$

where the first inequality follows from Lemma 3.1(3). □

Proof of Theorem 1.4. Suppose there exist $x \in \mathrm{HF}^{\alpha}(X,\lambda)$ and $y \in \mathrm{HF}^{\bar{\alpha}}(X,\lambda)$ such that $x * y = \iota(1) = F_\infty$. Since $\mathrm{HF}(X,\lambda) = \lim_{a\to\infty} \mathrm{HF}^{<a}(X,\lambda)$, for sufficiently large $a > 0$ there exist $x' \in \mathrm{HF}^{<a,\alpha}(X,\lambda)$ and $y' \in \mathrm{HF}^{<a,\bar{\alpha}}(X,\lambda)$ such that $x' * y' = F_{2a}$. Now apply Theorem 3.6. □

4. The loop product

4.1. *Definition of the loop product*

First we recall the definition of the loop product, which was introduced in [CS]. The following exposition is taken from Section 1.2 in [AS2] (although the authors of [AS2] work on $C^0(S^1, M)$ rather than $L^{1,2}(S^1, M)$).

First we recall the definition of the *Umkehr map*. Let X be a Hilbert manifold, Y be its closed submanifold of codimension n, and $i : Y \to X$ be the inclusion map. Let NY denote the normal bundle of Y. The tubular neighborhood theorem (see [L, IV, Sections 5-6]) states that there exists a unique (up to isotopy) open embedding $u : NY \to X$ such that $u(y,0) = i(y)$ for any $y \in Y$. Setting $U := u(NY)$, the *Umkehr map* $i_! : H_*(X) \to H_{*-n}(Y)$ is defined as

$$H_*(X) \to H_*(X, X\setminus Y) \xrightarrow{\cong} H_*(U, U\setminus Y) \xrightarrow{(u_*)^{-1}} H_*(NY, NY\setminus Y) \xrightarrow{\tau} H_{*-n}(Y).$$

The second arrow is the isomorphism given by excision, the last one is the Thom isomorphism associated to the vector bundle $NY \to Y$ (although it is not oriented, we can consider the Thom isomorphism since we are working with $\mathbb{Z}_2$-coefficient homology). The following lemma is immediate from the above definition:

Lemma 4.1. *Let X be a Hilbert manifold, Y be its closed submanifold of codimension n, and $i : Y \to X$ be the inclusion map. Let Z be a k-dimensional closed manifold, and $f : Z \to X$ be a smooth map which is transversal to Y. Then $i_!(f_*[Z]) = f_*[f^{-1}(Y)]$ in $H_{k-n}(Y)$.*

Now we define the loop product. Let M be an n-dimensional manifold. Let us consider the evaluation map $\mathrm{ev}\times\mathrm{ev} : \Lambda_M \times \Lambda_M \to M \times M$, $(\gamma,\gamma') \mapsto (\gamma(0), \gamma'(0))$, and the diagonal $\Delta_M := \{(x,y) \in M\times M \mid x = y\} \subset M \times M$. Then $\Theta_M := (\mathrm{ev}\times\mathrm{ev})^{-1}(\Delta_M)$ is an n-codimensional submanifold of $\Lambda_M \times \Lambda_M$. Let us denote the embedding map $\Theta_M \to \Lambda_M \times \Lambda_M$ by e. Moreover, $\Gamma : \Theta_M \to \Lambda_M$ denotes the concatenation map, i.e.

$$\Gamma(\gamma,\gamma')(t) := \begin{cases} \gamma(2t) & (0 \leq t \leq 1/2), \\ \gamma'(2t-1) & (1/2 \leq t \leq 1). \end{cases}$$

Then the loop product

$$\circ : H_i(\Lambda_M) \otimes H_j(\Lambda_M) \to H_{i+j-n}(\Lambda_M)$$

is defined as the composition of the following three homomorphisms:

$$H_i(\Lambda_M) \otimes H_j(\Lambda_M) \xrightarrow{\times} H_{i+j}(\Lambda_M \times \Lambda_M) \xrightarrow{e_!} H_{i+j-n}(\Theta_M) \xrightarrow{\Gamma_*} H_{i+j-n}(\Lambda_M).$$

The first arrow is the usual cross-product in singular homology. The second one is the Umkehr map associated to the embedding $e : \Theta_M \to \Lambda_M \times \Lambda_M$, and the last one is induced by the concatenation map Γ.

4.2. *Proof of Lemma 1.8*

Let $s : M \to \Lambda_M^\alpha$ be a continuous map satisfying $\mathrm{ev} \circ s = \mathrm{id}_M$. It is easy to see that s can be approximated by a *smooth* map $s' : M \to \Lambda_M^\alpha$ which satisfies $\mathrm{ev} \circ s' = \mathrm{id}_M$.

Hence, we may assume s is smooth. It is clear that $s_*[M] \times \bar{s}_*[M] = (s \times \bar{s})_*[M \times M]$. Moreover, since $(\mathrm{ev} \times \mathrm{ev}) \circ (s \times \bar{s}) = \mathrm{id}_{M\times M}$ and $\Theta_M = (\mathrm{ev} \times \mathrm{ev})^{-1}(\Delta_M)$, it follows that $s \times \bar{s} : M \times M \to \Lambda_M \times \Lambda_M$ is transversal to Θ_M. Then Lemma 4.1 shows that $s_*[M] \circ \bar{s}_*[M] = \Gamma(s, \bar{s})_*[M]$.

Hence it is enough to show that two continuous maps $\Gamma(s, \bar{s}), c : M \to \Lambda_M$ are homotopic. Define $K : M \times [0, 1] \to \Lambda_M$ by

$$K(p,t)(\tau) := \begin{cases} s(p)(2t\tau) & (0 \le \tau \le 1/2), \\ s(p)(2t - 2t\tau) & (1/2 \le \tau \le 1). \end{cases}$$

Then K is a homotopy between $\Gamma(s, \bar{s})$ and c. □

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