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Twistor transforms of quaternionic functions and orthogonal complex structures

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Abstract. The theory of slice-regular functions of a quaternion variable is applied to the study of orthogonal complex structures on domains Ω of $\mathbb{R}^4$. When Ω is a symmetric slice domain, the twistor transform of such a function is a holomorphic curve in the Klein quadric. The case in which Ω is the complement of a parabola is studied in detail and described by a rational quartic surface in the twistor space $\mathbb{CP}^3$.

Keywords. Orthogonal complex structures, quaternions, regular quaternionic functions, twistor methods

1. Introduction

It is well known that quaternions can be used to describe orthogonal complex structures on 4-dimensional Euclidean space, since their imaginary units parametrise isomorphisms $\mathbb{R}^4 \cong \mathbb{C}^2$. The resulting theory is invariant by the group $SO(5,1)$ of conformal automorphisms of $S^4 = \mathbb{HP}^1$, locally isomorphic to the group $SL(2,\mathbb{H})$ of linear fractional transformations (see Section 4). Moreover, it crops up naturally in the description of certain dense open subsets of $\mathbb{R}^4 = \mathbb{H}$, which possess complex structures that are induced from algebraic surfaces in the twistor space $\mathbb{CP}^3$ that fibres over S^4.

We begin with the general definition of this class of structures.

Definition 1.1. Let (M^4, g) be a 4-dimensional oriented Riemannian manifold. An *almost complex structure* is an endomorphism $J : TM \to TM$ satisfying $J^2 = -\mathrm{id}$. It is said to be *orthogonal* if it is an orthogonal transformation, that is, $g(J\mathbf{v}, J\mathbf{w}) = g(\mathbf{v}, \mathbf{w})$ for every $\mathbf{v}, \mathbf{w} \in T_pM$. In addition, we shall assume that such a J preserves the orientation of M. An orthogonal almost complex structure is said to be an *orthogonal complex structure*, abbreviated to *OCS*, if J is integrable.

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Every element I in the 2-sphere

$$\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\} \tag{1.1}$$

of imaginary unit quaternions can be regarded as a complex structure on $\mathbb{H}$ in a tautological fashion as follows. After identifying each tangent space $T_p\mathbb{H}$ with $\mathbb{H}$ itself, we define the complex structure by left multiplication by I, i.e. $J_p\mathbf{v} = I\mathbf{v}$ for all $\mathbf{v} \in T_p\mathbb{H} \cong \mathbb{H}$. Such a complex structure is called *constant* and it is clearly orthogonal: if we choose $I' \in \mathbb{S}$ such that $I \perp I'$ then J_p is associated to the matrix

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

with respect to the basis $1, I, I', II'$.

Any OCS defined globally on $\mathbb{H}$ is known to be constant (see [30]). More generally,

Theorem 1.2 ([24]). *Let J be an OCS of class C^1 on $\mathbb{R}^4 \setminus \Lambda$, where Λ is a closed set having zero 1-dimensional Hausdorff measure: $\mathcal{H}^1(\Lambda) = 0$. Then either J is constant or J can be maximally extended to the complement $\mathbb{R}^4 \setminus \{p\}$ of a point. In both cases, J is the push-forward of the standard OCS on $\mathbb{R}^4$ under a conformal transformation.*

If the real axis $\mathbb{R}$ is removed from $\mathbb{H}$ then a totally different structure $\mathbb{J}$ can be constructed, thanks to the property of $\mathbb{H}$ that we are about to describe. If, for all $I \in \mathbb{S}$, we set $L_I = \mathbb{R} + I\mathbb{R}$, then $L_I \cong \mathbb{C}$ and

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}} L_I, \qquad \bigcap_{I \in \mathbb{S}} L_I = \mathbb{R}.$$

In fact, every $q \in \mathbb{H} \setminus \mathbb{R}$ can be uniquely expressed in terms of its real part $\mathrm{Re}(q)$ and imaginary part $\mathrm{Im}(q) = q - \mathrm{Re}(q)$ as $q = \mathrm{Re}(q) + I_q|\mathrm{Im}(q)|$ where

$$I_q = \frac{\mathrm{Im}(q)}{|\mathrm{Im}(q)|} \in \mathbb{S}.$$

For all $q \in \mathbb{H} \setminus \mathbb{R}$ and for all $\mathbf{v} \in T_q(\mathbb{H} \setminus \mathbb{R}) \cong \mathbb{H}$, we now define $\mathbb{J}_q\mathbf{v} = I_q\mathbf{v}$.

Observe that $\mathbb{J}$ is integrable. Indeed, when we express

$$\mathbb{H} \setminus \mathbb{R} = \mathbb{R} + \mathbb{S} \cdot \mathbb{R}^+ \;\cong\; \mathbb{CP}^1 \times \mathbb{C}^+, \tag{1.2}$$

as the product of the Riemann sphere and the upper half-plane $\mathbb{C}^+ = \{x + iy \in \mathbb{C} : y > 0\}$, then $\mathbb{J}$ is none other than the product complex structure on the two factors. Furthermore, $\mathbb{J}$ is orthogonal: this fact can be proven reasoning pointwise.

In general, an orthogonal almost complex structure J defined on an open subset Ω of $\mathbb{R}^4$ is simply a map $\Omega \to \mathbb{S}$. It is well known that J is integrable if and only if this map is holomorphic as a mapping from the almost complex manifold (Ω, J) to the Riemann sphere (provided the orientation of the latter is chosen correctly, see [11]). It follows that J is integrable if and only if the associated section over Ω of the trivial bundle $\mathbb{R}^4 \times \mathbb{S}$ is holomorphic for the natural twisted complex structure on this product. This complex manifold can be compactified (by adding a projective line $\mathbb{CP}^1$) to the "twistor space" $\mathbb{CP}^3$, which fibres over $S^4 = \mathbb{R}^4 \cup \{\infty\}$. In practice then, an OCS J on an open subset Ω of S^4 arises from a *holomorphic* submanifold of $\mathbb{CP}^3$ [22, 8]. This tautological principle underlies the proof of Theorem 1.2, as well as our study of the properties of $\mathbb{J}$. In Section 4, we shall write down complex coordinates for the OCS $\mathbb{J}$, and use these in Section 5 to show that $\mathbb{J}$ arises from a quadric in the twistor space $\mathbb{CP}^3$ (see Proposition 5.1). This fact was the basis of the next result.

Theorem 1.3 ([24]). *Let J be an OCS of class C^1 on $\mathbb{R}^4 \setminus \Lambda$ where Λ is a round circle or a straight line, and assume that J is not conformally equivalent to a constant OCS. Then J is unique up to sign, and $\mathbb{R}^4 \setminus \Lambda$ is a maximal domain of definition for J.*

Thus $\mathbb{J}$ and $-\mathbb{J}$ are the only non-constant OCS's on $\mathbb{H} \setminus \mathbb{R}$. One might hope to achieve similar characterisations of more complicated complex structures.

For a closed subset Λ of $\mathbb{H}$ other than those described in the previous theorems, it is in general an open question as to whether OCS's exist on $\mathbb{H} \setminus \Lambda$, let alone the task of attempting their classification. No systematic methods were known for addressing this problem, other than attempting to identify graphs in the twistor space $\mathbb{CP}^3$ manufactured from some obvious algebraic subsets.

In the present work, we solve such a problem working directly on $\mathbb{H}$ by exhibiting a rather surprising link with the class of quaternionic functions recently introduced in [15]. This theory is based on the following notion of regularity.

Definition 1.4. Let Ω be a domain in $\mathbb{H}$ and let $f : \Omega \to \mathbb{H}$. For all $I \in \mathbb{S}$, let us denote $\Omega_I = \Omega \cap L_I$ and $f_I = f_{|\Omega_I}$. The function f is called *slice regular* if, for all $I \in \mathbb{S}$, the restriction f_I is holomorphic, i.e. $\bar{\partial}_I f : \Omega_I \to \mathbb{H}$ defined by

$$\bar{\partial}_I f(x + Iy) = \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy)$$

vanishes identically.

Throughout this paper we use "regular" to stand for "slice regular" (see Remark 2.5). Provided Ω satisfies the additional properties of Definition 2.1, regular functions defined on Ω are necessarily real analytic, and they have some nice properties in common with holomorphic functions of one complex variable. We shall explain this in Section 2.

Section 3 is devoted to a detailed study of points where the differential of a regular function f is not invertible, and takes account of a phenomenon whereby f can be constant on isolated 2-spheres. Theorem 3.9 fully describes the behaviour of f in such cases. The link between regular functions and OCS's, central to this paper, is established in Section 4. We discuss various automorphisms of $\mathbb{J}$ and the domain $\mathbb{H} \setminus \mathbb{R}$, and enumerate more general classes of maps that preserve orthogonality.

Section 5 contains a twistorial interpretation of the quaternionic analysis, in which one passes from $\mathbb{HP}^1$ to $\mathbb{CP}^3$ in order to render the theory complex algebraic. Theorem 5.3 asserts that a regular function f lifts to a holomorphic mapping from an open set of a quadric (the graph of $\mathbb{J}$) in $\mathbb{CP}^3$ onto another complex surface in $\mathbb{CP}^3$. This enables us to associate to f a holomorphic curve $\mathcal{F}$ in the Klein quadric in $\mathbb{CP}^5$, satisfying a reality condition; moreover, the construction can be inverted (see Theorem 5.7). The curve $\mathcal{F}$ is the transform of the title, and it encodes the action of f on the family of spheres $x + y\mathbb{S}$ symmetric with respect to the real axis. Properties of regular functions can then be interpreted in natural holomorphic terms.

We illustrate our approach by describing OCS's induced by mapping the real axis of $\mathbb{H}$ onto a parabola γ. Because of the importance of this example, it effectively occupies the last two sections. Section 6 exploits the theory of the quadratic polynomial $q^2 + qi + c$ over $\mathbb{H}$ and properties of its roots (generically, there are two). In Section 7, we show that the induced complex structures arise from a quartic surface $\mathscr{K}$ in $\mathbb{CP}^3$, whose singular points form three lines. It is one of a number of quartic scrolls whose classification (described in [9] and references therein) might be relevant for the treatment of analogous orthogonal complex structures. The map f determines a resolution of $\mathscr{K}$ via Theorem 7.1, and the "twistor geometry of conics" could be further developed using Lie sphere techniques [25].

In [24, Theorem 14.4 of v1], it was shown that a (non-singular) algebraic subvariety of $\mathbb{CP}^3$ of degree $d > 2$ cannot define a single-valued OCS unless one removes a set of real dimension at least 3. By applying similar methods, we prove the non-existence of an OCS on the complement $\mathbb{H}\setminus\gamma$ of the parabola (see Theorem 7.5). Indeed, in the geometry of $\mathbb{H} \setminus \gamma$ that we describe in this paper, one needs to remove a solid paraboloid $\mathfrak{G}$ as well as γ in order to define an OCS.

2. Factorisation of regular functions

Definition 1.4 has its origin in work of Cullen [7]. The class of regular functions includes every polynomial or power series of the form

$$f(q) = \sum_{n\in\mathbb{N}} q^n a_n$$

on its ball of convergence $B(0, R) = \{q \in \mathbb{H} : |q| < R\}$. Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers $0, 1, 2, \ldots$.

Conversely, a regular function on a ball $B(0, R)$ is the sum of a power series of the same type. Furthermore, the set of power series converging in $B(0, R)$ is a ring with the usual addition operation $+$ and the multiplicative operation $*$ defined by

$$\Big(\sum_{n\in\mathbb{N}} q^n a_n\Big) * \Big(\sum_{n\in\mathbb{N}} q^n b_n\Big) = \sum_{n\in\mathbb{N}} q^n \sum_{k=0}^{n} a_k b_{n-k}. \tag{2.1}$$

Recently, [4] identified a larger class of domains on which the theory of regular functions is natural and not limited to quaternionic power series.

Definition 2.1. A domain $\Omega \subseteq \mathbb{H}$ is called a *slice domain* if $\Omega_I = \Omega \cap L_I$ is an open connected subset of $L_I \cong \mathbb{C}$ for all $I \in \mathbb{S}$, and the intersection of Ω with the real axis is non-empty. A slice domain Ω is a *symmetric slice domain* if it is symmetric with respect to the real axis.

The word *symmetric* will refer to axial symmetry with respect to $\mathbb{R}$ throughout the paper. Recall the 2-sphere $\mathbb{S}$ defined in (1.1). If Ω is a symmetric slice domain then $\Omega \setminus \mathbb{R}$ can be expressed as

$$\Omega \setminus \mathbb{R} = \bigcup_{x+iy \in V} (x + y\mathbb{S}) \;\cong\; \mathbb{CP}^1 \times V \tag{2.2}$$

relative to (1.2), where $x + y\mathbb{S} = \{x + Iy : I \in \mathbb{S}\}$ and V is an open connected subset of $\mathbb{C}^+$.

As explained in [4], regular functions on a symmetric slice domain admit an identity principle, and they are C^∞. These properties are not granted for other types of domains (see e.g. [4]). Additionally, it was proven in the same paper that the multiplicative operation $*$ can be extended so that the set of regular functions on a symmetric slice domain Ω is endowed with a ring structure. It turns out that this algebraic structure is strictly related to the distribution of the zeros (see [12, 14, 16]). In fact, the zero set of a regular function on a symmetric slice domain consists of isolated points or isolated 2-spheres of type $x + y\mathbb{S}$, and can be studied as follows.

Theorem 2.2. *Let $f \not\equiv 0$ be a regular function on a symmetric slice domain Ω, and let $x + y\mathbb{S} \subset \Omega$. There exist $m \in \mathbb{N}$ and a regular function $\tilde{f} : \Omega \to \mathbb{H}$ not identically zero in $x + y\mathbb{S}$ such that*

$$f(q) = [(q - x)^2 + y^2]^m \tilde{f}(q).$$

If $\tilde{f}$ has a zero $p_1 \in x + y\mathbb{S}$ then such a zero is unique and there exist $n \in \mathbb{N}$, $p_2, \dots, p_n \in x + y\mathbb{S}$ (with $p_l \neq \bar{p}_{l+1}$ for all $l \in \{1, \dots, n-1\}$) such that

$$\tilde{f}(q) = (q - p_1) * \cdots * (q - p_n) * g(q)$$

for some regular function $g : \Omega \to \mathbb{H}$ that does not have zeros in $x + y\mathbb{S}$.

Definition 2.3. In the situation of Theorem 2.2, f is said to have *spherical multiplicity* $2m$ at $x + y\mathbb{S}$ and *isolated multiplicity* n at p_1. Finally, the *total multiplicity* of $x + y\mathbb{S}$ for f is defined as the sum $2m + n$.

The meaning of these notions is clarified by the case of quadratic polynomials, which will also prove useful in the final sections of this paper.

Example 2.4. Let $\alpha, \beta \in \mathbb{H}$ and set $P(q) = (q - \alpha) * (q - \beta)$.

(1) If β is not in the same sphere $x + y\mathbb{S}$ as α then P has two distinct roots, α and $(\alpha - \bar{\beta})\beta(\alpha - \bar{\beta})^{-1}$, each having isolated multiplicity 1.
(2) If α, β lie in the same sphere $x + y\mathbb{S}$ but $\alpha \neq \bar{\beta}$ then $\alpha = (\alpha - \bar{\beta})\beta(\alpha - \bar{\beta})^{-1}$ is the unique root and it has isolated multiplicity 2.

(3) Finally, if $\alpha = \bar{\beta} \in x + y\mathbb{S}$ then the zero set of P is $x + y\mathbb{S}$, which has spherical (and total) multiplicity 2.

Observe that, in (1), the second root $(\alpha - \bar{\beta})\beta(\alpha - \bar{\beta})^{-1}$ does not necessarily equal β but it does lie in the same sphere as β.

Remark 2.5. The class characterised by Definition 1.4 differs significantly from other classes of quaternionic functions introduced during the last century, in particular that arising from solutions of the Fueter equation, relevant to the study of maps between hypercomplex manifolds. (See [3, 5, 29] and references therein.) The class of Fueter regular functions has the advantage that it has an interpretation on any quaternionic manifold [23]. It has a subtle but deep algebraic structure that was identified by Joyce [18], whose work was interpreted by Quillen [21]. But the latter is less relevant to the theory of quaternionic polynomials and power series, application of which is really the subject of this paper.

3. The differential of a regular function

In this section, we will prove some new results concerning the (real) differential f_* of a regular function f. We first recall two theorems proven in [27].

Theorem 3.1. *Let f be a regular function on a symmetric slice domain Ω, and let $x_0, y_0 \in \mathbb{R}$ be such that $x_0 + y_0\mathbb{S} \subset \Omega$. For all $q_0 \in x_0 + y_0\mathbb{S}$, there exists a sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathbb{H}$ such that*

$$f(q) = \sum_{n\in\mathbb{N}} [(q - x_0)^2 + y_0^2]^n [A_{2n} + (q - q_0)A_{2n+1}] \tag{3.1}$$

in every $U(x_0 + y_0\mathbb{S}, R) = \{q \in \mathbb{H} : |(q - x_0)^2 + y_0^2| < R^2\} \subseteq \Omega$.

Let $q_0 = x_0 + Iy_0$. We now see that regular functions are not merely holomorphic in the plane L_I, but that they have an analogous derivative perpendicular to L_I. The resulting formulae involve the first two coefficients (after A_0).

Theorem 3.2. *Let f be a regular function on a symmetric slice domain Ω, let $q_0 \in \Omega$ and let A_n be the coefficients of the expansion (3.1). Then for all $v \in \mathbb{H}$ with $|v| = 1$ the derivative of f along v can be computed at q_0 as*

$$\lim_{t\to 0} \frac{f(q_0 + tv) - f(q_0)}{t} = vA_1 + (q_0 v - v\bar{q}_0)A_2. \tag{3.2}$$

For more details, we refer the reader to [27].

If we identify $T_{q_0}\mathbb{H}$ with $\mathbb{H} = L_I \oplus L_I^\perp$ then for all $u \in L_I$ and $w \in L_I^\perp$,

$$(f_*)_{q_0}(u + w) = u(A_1 + 2\,\mathrm{Im}(q_0)A_2) + wA_1. \tag{3.3}$$

In other words, *the differential f_* at q_0 acts by right multiplication by $A_1 + 2\,\mathrm{Im}(q_0)A_2$ on L_I and by right multiplication by A_1 on $L_I^\perp$.*

We will now make use of (3.3) to investigate the rank of f_*.

Proposition 3.3. *Let f be a regular function on a symmetric slice domain Ω, let $q_0 = x_0 + Iy_0 \in \Omega \setminus \mathbb{R}$ and let A_n be the coefficients of the expansion* (3.1)*.*

- *If $A_1 = 0$ then f_* has rank 2 at q_0 if $A_2 \neq 0$, and rank 0 if $A_2 = 0$.*
- *If $A_1 \neq 0$ then either f_* is invertible at q_0 or it has rank 2 at q_0; the latter happens if and only if $1 + 2\,\mathrm{Im}(q_0)A_2A_1^{-1} \in L_I^{\perp}$.*

Finally, for all $x_0 \in \Omega \cap \mathbb{R}$, f_* is invertible at x_0 if $A_1 \neq 0$; it has rank 0 at x_0 if $A_1 = 0$.

Proof. If $A_1 = 0$ then $(f_*)_{q_0}(u + w) = u\,2\,\mathrm{Im}(q_0)A_2$ for all $u \in L_I$, $w \in L_I^{\perp}$. Hence the kernel of $(f_*)_{q_0}$ is $L_I^{\perp}$ if $A_2 \neq 0$, the whole space $\mathbb{H}$ if $A_2 = 0$.

Let us now turn to the case $A_1 \neq 0$ and observe that for all $u \in L_I$ and $w \in L_I^{\perp}$,

$$(f_*)_{q_0}(u + w) = [u(1 + 2\,\mathrm{Im}(q_0)A_2A_1^{-1}) + w]A_1.$$

The differential $(f_*)_{q_0}$ is not invertible if and only if $1 + 2\,\mathrm{Im}(q_0)A_2A_1^{-1} = p \in L_I^{\perp}$. In this case, if $p = 0$ then the kernel of $(f_*)_{q_0}$ is L_I; if $p \neq 0$ then the kernel is the 2-plane of vectors $-wp^{-1} + w$ for $w \in L_I^{\perp}$.

The last statement is proved by observing that if $x_0 \in \Omega \cap \mathbb{R}$ then $(f_*)_{x_0}v = vA_1$ for all $v \in \mathbb{H}$. □

Now let us study in detail the set

$$N_f = \{q \in \Omega : f_* \text{ is not invertible at } q\},$$

which we may call the *singular set* of f. In this study, we will make use of the following specific subset of N_f (see [13, 14]).

Definition 3.4. Let Ω be a symmetric slice domain and let $f : \Omega \to \mathbb{H}$ be a regular function. The *degenerate set* of f is the union D_f of the 2-spheres $x + y\mathbb{S}$ (with $y \neq 0$) such that $f_{|x+y\mathbb{S}}$ is constant.

The following observation will also prove useful (see [2]).

Remark 3.5. For all $q_0 = x_0 + Iy_0 \in \mathbb{H} \setminus \mathbb{R}$, setting $\Psi(q) = (q - q_0)(q - \bar{q}_0)^{-1}$ defines a stereographic projection of $x_0 + y_0\mathbb{S}$ onto the plane $L_I^{\perp}$ from the point $\bar{q}_0$.

Indeed, if we choose $J \in \mathbb{S}$ with $J \perp I$ and set $K = IJ$ then for all $q = x_0 + Ly_0$ with $L = tI + uJ + vK \in \mathbb{S}$ we have $\Psi(q) = (L - I)(L + I)^{-1} = \frac{u+Iv}{1+t}K$ and $(\mathbb{R} + I\mathbb{R})K = L_I^{\perp}$.

We are now in a position to characterise the points of the singular set in algebraic terms.

Proposition 3.6. *Let f be a regular function on a symmetric slice domain Ω, and let $q_0 = x_0 + Iy_0 \in \Omega$. Then f_* is not invertible at q_0 if, and only if, there exist $\widetilde{q}_0 \in x_0 + y_0\mathbb{S}$ and a regular function $g : \Omega \to \mathbb{H}$ such that*

$$f(q) = f(q_0) + (q - q_0) * (q - \widetilde{q}_0) * g(q). \tag{3.4}$$

Equivalently, f_* is not invertible at q_0 if, and only if, $f - f(q_0)$ has total multiplicity $n \geq 2$ at $x_0 + y_0\mathbb{S}$. Moreover, q_0 belongs to the degenerate set D_f if, and only if, it belongs to the singular set N_f and there exists a regular $g : \Omega \to \mathbb{H}$ such that equation (3.4) holds for $\widetilde{q}_0 = \bar{q}_0$. The latter is equivalent to saying that $f - f(q_0)$ has spherical multiplicity $n \geq 2$ at $x_0 + y_0\mathbb{S}$.

Proof. If $q_0 \in \Omega \setminus \mathbb{R}$ then it belongs to D_f if and only if, f is constant on the 2-sphere $x_0 + y_0\mathbb{S}$, i.e. there exists a regular function $g : \Omega \to \mathbb{H}$ such that

$$f(q) = f(q_0) + [(q - x_0)^2 + y_0^2] * g(q) = f(q_0) + (q - q_0) * (q - \bar{q}_0) * g(q).$$

This happens if and only if the coefficient A_1 in the expansion (3.1) vanishes.

Now let us turn to the case $q_0 \in \Omega \setminus \mathbb{R}$, $q_0 \notin D_f$. By Proposition 3.3, $q_0 \in N_f$ if and only if $1 + 2\,\mathrm{Im}(q_0)A_2A_1^{-1} = p \in L_I^{\perp}$. Thanks to the previous remark, $p \in L_I^{\perp}$ if, and only if, there exists $\widetilde{q}_0 \in (x_0 + y_0\mathbb{S}) \setminus \{\bar{q}_0\}$ such that $p = (\widetilde{q}_0 - q_0)(\widetilde{q}_0 - \bar{q}_0)^{-1}$. The last formula is equivalent to

$$2\,\mathrm{Im}(q_0)A_2A_1^{-1} = (\widetilde{q}_0 - q_0 - \widetilde{q}_0 + \bar{q}_0)(\widetilde{q}_0 - \bar{q}_0)^{-1} = -2\,\mathrm{Im}(q_0)(\widetilde{q}_0 - \bar{q}_0)^{-1},$$

i.e. $A_1 = (\bar{q}_0 - \widetilde{q}_0)A_2$. Finally, the last equality is equivalent to

$$\begin{aligned} f(q) &= A_0 + (q - q_0)(\bar{q}_0 - \widetilde{q}_0)A_2 + [(q - x_0)^2 + y_0^2]A_2 + [(q - x_0)^2 + y_0^2](q - q_0) * h(q) \\ &= f(q_0) + (q - q_0) * [\bar{q}_0 - \widetilde{q}_0 + q - \bar{q}_0]A_2 + (q - q_0) * [(q - x_0)^2 + y_0^2] * h(q) \\ &= f(q_0) + (q - q_0) * (q - \widetilde{q}_0)A_2 + (q - q_0) * (q - \widetilde{q}_0) * (q - \overline{\widetilde{q}}_0) * h(q) \\ &= f(q_0) + (q - q_0) * (q - \widetilde{q}_0) * [A_2 + (q - \overline{\widetilde{q}}_0) * h(q)], \end{aligned}$$

for some regular $h : \Omega \to \mathbb{H}$.

To conclude, we observe that if $x_0 \in \Omega \cap \mathbb{R}$ then $A_1 = 0$ if and only if

$$f(q) = f(q_0) + (q - x_0)^2 * g(q) = f(q_0) + (q - x_0) * (q - x_0) * g(q)$$

for some regular function $g : \Omega \to \mathbb{H}$. □

We end this section with a complete characterisation of the singular set of f, proving in particular that it is empty when f is an injective function. The following two results, from [13, 14], will prove useful to this end.

Theorem 3.7. *Let Ω be a symmetric slice domain and let $f : \Omega \to \mathbb{H}$ be a non-constant regular function. The degenerate set D_f is closed in $\Omega \setminus \mathbb{R}$ and it has empty interior.*

Theorem 3.8 (Open Mapping Theorem). *Let f be a regular function on a symmetric slice domain Ω and let D_f be its degenerate set. Then $f : \Omega \setminus \bar{D}_f \to \mathbb{H}$ is open.*

We are now in a position to fully characterise the singular set of f.

Theorem 3.9. *Let Ω be a symmetric slice domain and let $f : \Omega \to \mathbb{H}$ be a non-constant regular function. Then its singular set N_f has empty interior. Moreover, for a fixed $q_0 = x_0 + Iy_0 \in N_f$, let $n > 1$ be the total multiplicity of $f - f(q_0)$ at $x_0 + y_0\mathbb{S}$. Then there exist a neighbourhood U of q_0 and a neighbourhood T of $x_0 + y_0\mathbb{S}$ such that, for all $q_1 \in U$, the sum of the total multiplicities of the zeros of $f - f(q_1)$ in T equals n; in particular, for all $q_1 \in U \setminus N_f$ the preimage of $f(q_1)$ includes at least two distinct points of T.*

Proof. Since f is not constant, the singular set N_f has empty interior if and only if $N_f \setminus D_f$ does. By Proposition 3.3, the rank of f_* equals 2 at each point of $N_f \setminus D_f$. If $N_f \setminus D_f$ contained a non-empty open set A, its image $f(A)$ could not be open by the constant rank theorem, and the open mapping theorem 3.8 would be contradicted.

In order to prove our second statement, let us introduce the notation $f_{q_1}(q) = f(q) - f(q_1)$ for $q_1 \in \Omega$. By Proposition 3.6, if $q_0 = x_0 + Iy_0 \in N_f$ then the total multiplicity n of f_{q_0} at $x_0 + y_0\mathbb{S}$ is strictly greater than 1. We will now make use of the operation of symmetrisation defined in [4] by setting $f^s = f * f^c = f^c * f$ where f^c is defined by

$$\Big(\sum_{n\in\mathbb{N}} q^n a_n\Big)^c = \sum_{n\in\mathbb{N}} q^n \bar{a}_n$$

on power series, and appropriately extended to regular functions on symmetric slice domains. By direct computation, the total multiplicity of each 2-sphere $x + y\mathbb{S}$ for f^s is twice its total multiplicity for f. Hence, in our case $x_0 + y_0\mathbb{S}$ has total multiplicity $2n$ for $f^s_{q_0}$, i.e. there exists a regular $h : \Omega \to \mathbb{H}$ having no zeros in $x_0 + y_0\mathbb{S}$ such that $f^s_{q_0}(q) = [(q - x_0)^2 + y_0^2]^n h(q)$ and

$$f^s_{q_0}(z) = [(z - x_0)^2 + y_0^2]^n\, h(z) = (z - q_0)^n (z - \bar{q}_0)^n h(z)$$

for all $z \in \Omega_I$. Furthermore, $f^s_{q_0}(\Omega_I) \subseteq L_I$ as explained in [14], so that the restriction of $f^s_{q_0}$ to Ω_I can be viewed as a holomorphic complex function. Let us choose an open 2-disc Δ centred at q_0 such that $f^s_{q_0}$ has no zeros in $\bar{\Delta} \setminus \{q_0\}$, with Δ strictly included both in Ω_I and in the half-plane of L_I that contains q_0. If we denote by F_{q_1} the restriction of $f^s_{q_1}$ to Δ then $q_1 \mapsto F_{q_1}$ is continuous in the topology of compact uniform convergence and F_{q_0} has a zero of multiplicity n at q_0 (and no other zero). We claim that there exists a neighbourhood U of q_0 such that for all $q_1 \in U$ the sum of the multiplicities of the zeros of F_{q_1} in Δ is n: if this were not the case, it would be possible to construct a sequence $\{q_k\}_{k\in\mathbb{N}}$ converging to q_0 such that $\{F_{q_k}\}_{k\in\mathbb{N}}$ contradicted Hurwitz's theorem (in the version of [6]). Now let $R > 0$ be the radius of Δ and let

$$T = T(x_0 + y_0\mathbb{S}, R) = \{x + Jy : |x - x_0|^2 + |y - y_0|^2 < R^2,\ J \in \mathbb{S}\}$$

be its symmetric completion. Then for all $q_1 \in U$, $2n$ equals the sum of the spherical multiplicities of the zeros of $f^s_{q_1}$ in T. Hence, there exist points $q_2, \ldots, q_n \in T$ and a function $h : \Omega \to \mathbb{H}$ having no zeros in T such that

$$f_{q_1}(q) = (q - q_1) * \cdots * (q - q_n) * h(q).$$

Now let us suppose $q_1 = x_1 + Jy_1 \in U \setminus N_f$. Then f_* is invertible at q_1 and, by Proposition 3.6, the point q_2 cannot belong to the sphere $x_1 + y_1\mathbb{S}$. Hence, $f_{q_1}(q)$ has at least two zeros in T and the preimage of $f(q_1)$ via f intersects T at least twice. □

Corollary 3.10. *Let Ω be a symmetric slice domain and let $f : \Omega \to \mathbb{H}$ be a regular function. If f is injective then its singular set N_f is empty.*

4. Induced complex structures

This section starts by explaining the link between the orthogonal complex structure $\mathbb{J}$ and the theory of regular functions. The idea is simple, namely that a regular function maps $\mathbb{J}$ locally to another *orthogonal* complex structure. The main formula, (4.1) below, expresses this fact even more simply, but its consequences are significant.

The real differential of an injective regular function is invertible at all points, thanks to Corollary 3.10. This allows us to push forward the complex structure $\mathbb{J}$ we defined on $\mathbb{H} \setminus \mathbb{R}$; we recall that for all $q \in \mathbb{H} \setminus \mathbb{R}$ and for all $\mathbf{v} \in T_q\mathbb{H} \cong \mathbb{H}$,

$$\mathbb{J}_q\mathbf{v} = I_q\mathbf{v} \quad \text{where} \quad I_q = \frac{\operatorname{Im}(q)}{|\operatorname{Im}(q)|} \in \mathbb{S}.$$

Definition 4.1. Let Ω be a symmetric slice domain, and let $f : \Omega \to \mathbb{H}$ be an injective regular function. The *induced structure* on $f(\Omega \setminus \mathbb{R})$ is the push-forward

$$\mathbb{J}^f = f_*\,\mathbb{J}\,(f_*)^{-1}.$$

We now find an explicit expression for this induced structure, thereby proving that it is orthogonal.

Proposition 4.2. *Let Ω be a symmetric slice domain, and let $f : \Omega \to \mathbb{H}$ be an injective regular function. Then*

$$\mathbb{J}^f_{f(q)}\mathbf{v} = I_q\mathbf{v} \tag{4.1}$$

for all $q \in \Omega \setminus \mathbb{R}$ and $\mathbf{v} \in T_{f(q)}f(\Omega \setminus \mathbb{R}) \cong \mathbb{H}$. As a consequence, $\mathbb{J}^f$ is an OCS on $f(\Omega \setminus \mathbb{R})$.

Proof. If $\mathbf{w} = (f_*)^{-1}_{f(q)}\mathbf{v}$ then by direct computation

$$\mathbb{J}^f_{f(q)}\mathbf{v} = (f_*)_q\mathbb{J}_q(f_*)^{-1}_{f(q)}\mathbf{v} = (f_*)_q\mathbb{J}_q\mathbf{w} = (f_*)_q I_q\mathbf{w}.$$

Thanks to formula (3.2) and to the fact that q and I_q commute,

$$(f_*)_q I_q\mathbf{w} = I_q\mathbf{w}A_1 + (qI_q\mathbf{w} - I_q\mathbf{w}\bar{q})A_2 = I_q(\mathbf{w}A_1 + (q\mathbf{w} - \mathbf{w}\bar{q})A_2) = I_q(f_*)_q\mathbf{w} = I_q\mathbf{v},$$

as desired.

Now, f is an injective holomorphic map from the complex manifold $(\Omega \setminus \mathbb{R}, \mathbb{J})$ to the almost complex manifold $(f(\Omega \setminus \mathbb{R}), \mathbb{J}^f)$. Hence, $\mathbb{J}^f$ can be viewed as a holomorphic map from the almost complex manifold $(f(\Omega \setminus \mathbb{R}), \mathbb{J}^f)$ to $\mathbb{S}$, a fact which implies that $\mathbb{J}^f$ itself is an OCS (see the remarks preceding Theorem 1.3). □

It is important to note that (4.1) asserts that f is not in general $\mathbb{J}$-holomorphic. This is because I_q does not in general coincide with $I_{f(q)}$ and, in other words, f does not preserve the subspace $\mathbb{C} \cong L_{I_q} \subset \mathbb{H}$ defined at every point q of its domain. Next, we shall however discuss examples in which f *is* $\mathbb{J}$-holomorphic.

At this juncture, we need to introduce the quaternionic projective line $\mathbb{HP}^1$. We define this to be the set of equivalence classes $[q_1, q_2]$, where

$$[q_1, q_2] = [pq_1, pq_2], \qquad \forall p \in \mathbb{H}^*. \tag{4.2}$$

The choice of *left* multiplication is dictated by the choice in Definition 1.4. We choose to embed $\mathbb{H}$ as the affine line in $\mathbb{HP}^1$ by mapping $q \in \mathbb{H}$ to $[1, q]$, so that $[0, 1]$ is the point of infinity.

Once one identifies $S^4 = \mathbb{H} \cup \{\infty\}$ with $\mathbb{HP}^1$, it is well known that the group of conformal transformations corresponds to the group of invertible transformations

$$[q_1, q_2] \mapsto [q_1 d + q_2 c, q_1 b + q_2 a], \qquad a, b, c, d \in \mathbb{H} \tag{4.3}$$

(with ordering chosen to match [26]). The invertibility condition is

$$|a|^2|d|^2 + |b|^2|c|^2 - 2\,\mathrm{Re}(\bar{b}d\bar{c}a) \neq 0,$$

the left-hand side being the real determinant when $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ is embedded in $\mathfrak{gl}(16, \mathbb{R})$. When restricted to the affine line, (4.3) becomes the linear fractional transformation

$$q \mapsto (qc + d)^{-1}(qa + b). \tag{4.4}$$

The group generated by these transformations is double covered by $SL(2, \mathbb{H})$. Since the condition that a complex structure be orthogonal depends only on the underlying conformally flat structure of $\mathbb{R}^4$, any element of $SL(2, \mathbb{H})$ certainly maps $\mathbb{J}$ to another *orthogonal* complex structure.

The maps (4.4) are not in general regular, in part because regularity is not conserved under composition. However an analogous class of regular Möbius transformations was defined by the third author in [26]. These coincide with (4.4) when c and d are real, in which case we see that (4.4) is regular because $qc + d$ takes values in the complex plane L_I where $I = I_q$ and its reciprocal can be expanded as a power series in q with real coefficients.

Proposition 4.3. *The subgroup of* $SL(2, \mathbb{H})$ *that maps* $\mathbb{J}$ *to itself is*

$$SO(2, \mathbb{H}) \cong Sp(1) \times_{\mathbb{Z}_2} SL(2, \mathbb{R}),$$

consisting of elements of (4.4) *such that* a, b, c, d *are all real multiples of the same unit quaternion* $\varepsilon \in Sp(1) \cong SU(2)$.

Proof. Suppose that the bijection f maps $\mathbb{J}$ to itself. Inherent in this condition is that f maps the real axis (and so its complement) to itself. Thus

$$(x\bar{c}+\bar{d})(xa+b) = x^2\bar{c}a + x(\bar{d}a+\bar{c}b) + \bar{d}b \in \mathbb{R}, \qquad \forall x \in \mathbb{R}.$$

This obviously holds given the hypothesis. Conversely, if $\bar{c}a$, $\bar{d}a+\bar{c}b$, $\bar{d}b$ are all real then we can write $a=\alpha c$ and $b=\beta d$ with $\alpha, \beta \in \mathbb{R}$. This implies that $\alpha\bar{d}c+\beta\bar{c}d \in \mathbb{R}$, which (since $\alpha \neq \beta$) forces $\bar{d}c \in \mathbb{R}$ and so $c=\gamma d$ with $\gamma \in \mathbb{R}$. We conclude by setting $d=\delta\varepsilon$ with $\delta \in \mathbb{R}$, $\varepsilon \in \mathbb{H}$ and $|\varepsilon|=1$.

Any element of $SO(2,\mathbb{H})$ is the composition of a real Möbius transformation $f \in SL(2,\mathbb{R})$ and the mapping

$$f_\varepsilon\colon\ q \mapsto \varepsilon^{-1}q\,\varepsilon, \qquad \varepsilon \in Sp(1). \tag{4.5}$$

As remarked above, the former type preserves $\mathbb{J}$. The latter fixes the real axis, so by the identity principle cannot be regular unless $\varepsilon = \pm 1$ and f_ε is the identity. On the other hand, it *is* $\mathbb{J}$-holomorphic. For the differential of (4.5) maps $\mathbf{v} \in \mathbb{H}$ to $\varepsilon^{-1}\mathbf{v}\varepsilon$ and

$$\mathbb{J}((f_\varepsilon)_*\mathbf{v}) = I_{\varepsilon^{-1}q\varepsilon}(\varepsilon^{-1}\mathbf{v}\varepsilon) = \varepsilon^{-1}I_q\varepsilon(\varepsilon^{-1}\mathbf{v}\varepsilon) = \varepsilon^{-1}(I_q\mathbf{v})\varepsilon = (f_\varepsilon)_*(I_q\mathbf{v}) = (f_\varepsilon)_*(\mathbb{J}\mathbf{v}),$$

so the induced complex structure at $\varepsilon^{-1}q\,\varepsilon$ coincides with $\mathbb{J}$. □

The group $SO(2,\mathbb{H})$, sometimes denoted $SO^*(4)$, has the same complexification as the orthogonal group $SO(4) = Sp(1) \times_{\mathbb{Z}_2} Sp(1)$. It double covers the *isometry* group of $\mathbb{H}\setminus\mathbb{R} \cong \mathbb{CP}^1 \times \mathbb{C}^+$, endowed with the product of metrics with constant curvature ± 1 (recall (1.2)). The representation of $SO(2,\mathbb{H})$ on the space of real symmetric 3×3 matrices was the object of study in [24, Section 4].

Remark 4.4. The following classes of functions all map the orthogonal complex structure $\mathbb{J}$ to another *orthogonal* complex structure:

(1) $SO(2,\mathbb{H})/\mathbb{Z}_2$, the group of $\mathbb{J}$-holomorphic conformal transformations;
(2) $SL(2,\mathbb{H})/\mathbb{Z}_2$, the group of all conformal transformations;
(3) the pseudo-group of all $\mathbb{J}$-holomorphic regular maps;
(4) the pseudo-group of all $\mathbb{J}$-holomorphic maps;
(5) the set of all injective regular maps.

Recall that $\mathbb{J}$ is merely the product complex structure relative to (2.2). Note that the group $PGL(2,\mathbb{C})$ of automorphisms of $\mathbb{CP}^1$ (acting as the identity on $\mathbb{C}^+$) lies in (4) but not (3). This is because an element of $PGL(2,\mathbb{C})$ is only regular if it is the identity (by the remark following (4.5)). We shall investigate a mapping properly in class (5) in Section 6.

In preparation for the next section, we next describe a pair u, v of holomorphic coordinates for the complex structure $\mathbb{J}$.

It is well known that conjugation by any quaternion (as in (4.5)) acts on $\mathbb{H} = \mathbb{R}\oplus\mathbb{R}^3$ as a rotation on the $\mathbb{R}^3$ summand (and obviously fixes the real part of any element $q \in \mathbb{H}$). With this in mind, define

$$Q_u = 1 + uj, \qquad u \in \mathbb{C},$$

and consider the mapping

$$\phi\colon \mathbb{C}\times\mathbb{C}^+ \to \mathbb{H}, \qquad (u,v)\mapsto Q_u^{-1} v\, Q_u.$$

We shall write $v = x + iy$, so $y > 0$ and the complex number v represents an element in the upper half-plane $\mathbb{C}^+$.

It follows that $\phi(u,i)$ belongs to the sphere $\mathbb{S}$ of unit quaternions, and (up to a factor $-i$) it is in fact the inverse of the stereographic projection $\mathbb{S}\to\mathbb{C}$ from the point $-i$ (cf. (3.5)). To see this, suppose that

$$I = ai + bj + ck = Q_u^{-1} i\, Q_u \in \mathbb{S},$$

so that

$$(1+uj)(ai + (b+ic)j) = i + iuj,$$

whence

$$u = -i\frac{b+ic}{1+a}. \tag{4.6}$$

We can write any $I \in \mathbb{S}\setminus\{-i\}$ as $\phi(u,i)$ for a unique $u\in\mathbb{C}$, but of course if we extend the domain of u to $\mathbb{CP}^1$ by allowing u to assume the value ∞, then ϕ exhibits the isomorphism (1.2).

An element $q\in\mathbb{H}\setminus\mathbb{R}$ can now be represented by means of ϕ:

$$q = Q_u^{-1} v\, Q_u = (1+uj)^{-1} v(1+uj), \tag{4.7}$$

and this makes it easier to analyse maps with $\mathbb{H}\setminus\mathbb{R}$ as domain. For example, to apply an element of $PGL(2,\mathbb{C})$ (mentioned in Remark 4.4) one merely replaces u in (4.7) by $(cu+d)/(au+b)$ with $a,b,c,d\in\mathbb{C}$. The relevance of (4.7) in the study of power series can be seen immediately by computing

$$q^n = (Q_u^{-1} v\, Q_u)^n = Q_u^{-1} v^n Q_u.$$

In the next section, we shall "eliminate" the inverse term Q_u^{-1} by passing to projective coordinates, and interpreting the above facts twistorially.

5. Twistor lifts

Recall the "leftish" definition (4.2) of the quaternionic projective line $\mathbb{HP}^1$. It enables us to define the *twistor projection*

$$\pi\colon \mathbb{CP}^3\to\mathbb{HP}^1, \qquad [Z_0,Z_1,Z_2,Z_3]\mapsto [Z_0+Z_1 j, Z_2+Z_3 j]. \tag{5.1}$$

Observe that, with our choices, changing the complex representative $(Z_0,Z_1,Z_2,Z_3)\in\mathbb{C}^4\setminus\{0\}$ will not affect the image in $\mathbb{HP}^1$.

Embed $\mathbb{H}$ as the affine line in $\mathbb{HP}^1$ by mapping $q\in\mathbb{H}$ to $[1,q]$. Note that $[0,1]$ then corresponds to the point $\infty\in S^4$, and that

$$\pi^{-1}(\infty) = \{[0,0,Z_2,Z_3] : [Z_2,Z_3]\in\mathbb{CP}^1\}. \tag{5.2}$$

With the notation of the previous section,

$$[1, q] = [Q_u, vQ_u] = [1 + uj, v(1 + uj)] = \pi[1, u, v, uv].$$

We therefore obtain

Proposition 5.1. *The complex manifold* $(\mathbb{H} \setminus \mathbb{R}, \mathbb{J})$ *is biholomorphic to the open subset* $\mathcal{Q}^+$ *of the quadric*

$$\mathcal{Q} = \{[Z_0, Z_1, Z_2, Z_3] \in \mathbb{CP}^3 : Z_0Z_3 = Z_1Z_2\}$$

whose elements satisfy at least one of the following conditions:

- $Z_0 \neq 0$ *and* $Z_2/Z_0 \in \mathbb{C}^+$,
- $Z_1 \neq 0$ *and* $Z_3/Z_1 \in \mathbb{C}^+$.

Consider a quaternionic entire function

$$f(q) = \sum_{n\in\mathbb{N}} q^n a_n = \sum_{n\in\mathbb{N}} q^n (b_n + c_n j), \tag{5.3}$$

where $b_n, c_n \in \mathbb{C}$. Repeating the calculation above with $q = Q_u^{-1} v\, Q_u$, we have

$$\begin{aligned}
[1, f(q)] = \Big[Q_u, \sum_{n\in\mathbb{N}} v^n Q_u a_n\Big] &= \Big[1 + uj, \sum_{n\in\mathbb{N}} v^n (1 + uj)(b_n + c_n j)\Big] \\
&= \Big[1 + uj, \sum_{n\in\mathbb{N}} v^n (b_n - u\bar{c}_n) + \sum_{n\in\mathbb{N}} v^n (c_n + u\bar{b}_n) j\Big] \\
&= [1 + uj,\ g(v) - u\,\hat{h}(v) + (h(v) + u\,\hat{g}(v))j], \qquad (5.4)
\end{aligned}$$

where

$$g(v) = \sum_{n\in\mathbb{N}} b_n v^n, \qquad h(v) = \sum_{n\in\mathbb{N}} c_n v^n \tag{5.5}$$

are holomorphic functions, and $\hat{g}(z) = \overline{g(\bar{z})}$, $\hat{h}(z) = \overline{h(\bar{z})}$ are obtained from g, h by Schwarz reflection.

This shows that the holomorphic mapping $F : \mathcal{Q}^+ \to \mathbb{CP}^3$ defined by

$$F[1, u, v, uv] = [1, u, g(v) - u\hat{h}(v), h(v) + u\hat{g}(v)] \tag{5.6}$$

satisfies $\pi \circ F = f \circ \pi$, and will therefore be called the *twistor lift* of f.

Remark 5.2. The conversion of f into the two holomorphic functions g, h can be regarded as an application of the splitting lemma of [15]: for all v in the plane $L_i = \mathbb{C}$ we have $f(v) = g(v) + h(v)j$. The fact that F preserves u in the second slot is another way of expressing the formula (4.1) in Proposition 4.2, and the linearity in u reflects the fact that F is really transforming lines on $\mathcal{Q}^+$; we explain this below. If the coefficients a_n are all real (so $h = 0$), then F maps $\mathcal{Q}^+$ into the quadric $\mathcal{Q}$. In this very special case, f will in fact be $\mathbb{J}$-holomorphic.

The technique used in (5.4) can be generalised to other domains in $\mathbb{H} \setminus \mathbb{R}$. This is accomplished by the next result that also places the argument on a more rigorous footing.

Theorem 5.3. *Let Ω be a symmetric slice domain, and let $f : \Omega \to \mathbb{H}$ be a regular function. Then f admits a twistor lift to $\mathscr{O} = \pi^{-1}(\Omega \setminus \mathbb{R}) \cap \mathscr{Q}^+$; in other words, there exists a holomorphic mapping $F \colon \mathscr{O} \to \mathbb{CP}^3$ of the form (5.6) such that $\pi \circ F = f \circ \pi$.*

Proof. The theorem will be proved if we show that for all

$$U = U(x_0 + y_0\mathbb{S}, R) = \{q \in \mathbb{H} : |(q - x_0)^2 + y_0^2| < R^2\} \subseteq \Omega \setminus \mathbb{R}$$

the restriction $f_{|U}$ can be lifted to a holomorphic $F : \pi^{-1}(U) \cap \mathscr{Q}^+ \to \mathbb{CP}^3$. By Theorem 3.1, for each $q_0 \in x_0 + y_0\mathbb{S}$ there exists $\{A_n\}_{n\in\mathbb{N}} \subset \mathbb{H}$ such that for all $q \in U$,

$$f(q) = \sum_{n\in\mathbb{N}} P_n(q)A_n$$

where $P_{2n}(q) = [(q-x_0)^2+y_0^2]^n$ and $P_{2n+1}(q) = P_{2n}(q)(q-q_0)$. Now, if $q = Q_u^{-1} v\, Q_u$ then $P_{2n}(q) = Q_u^{-1} P_{2n}(v) Q_u$ and

$$P_{2n+1}(q) = Q_u^{-1} P_{2n}(v) Q_u (Q_u^{-1} v\, Q_u - q_0) = Q_u^{-1} P_{2n}(v)(v - Q_u q_0 Q_u^{-1}) Q_u$$

so that

$$f(q) = Q_u^{-1} \sum_{n\in\mathbb{N}} \widetilde{P}_n(v) Q_u A_n = Q_u^{-1} \Big(\sum_{n\in\mathbb{N}} \widetilde{P}_n(v) \widetilde{A}_n \Big) Q_u$$

with $\widetilde{A}_n = Q_u A_n Q_u^{-1}$, $\widetilde{P}_{2n}(v) = [(v - x_0)^2 + y_0^2]^n$ and $\widetilde{P}_{2n+1}(v) = \widetilde{P}_{2n}(v)(v - Q_u q_0 Q_u^{-1})$. Let $V \subseteq \mathbb{C}^+$ be such that $U = \{x + Iy : I \in \mathbb{S}, x + iy \in V\}$. Then by the same estimates used in [27] to prove Theorem 3.1, $\sum_{n\in\mathbb{N}} \widetilde{P}_n(v)\widetilde{A}_n$ converges absolutely and uniformly on compact sets in $\mathbb{C} \times V$.

Furthermore,

$$\begin{aligned}
[1, f(q)] &= \Big[Q_u, \sum_{n\in\mathbb{N}} \widetilde{P}_n(v) Q_u A_n \Big] \\
&= \Big[Q_u, \sum_{m\in\mathbb{N}} \widetilde{P}_{2m}(v) \big(Q_u A_{2m} + (v - Q_u q_0 Q_u^{-1}) Q_u A_{2m+1} \big) \Big] \\
&= \Big[Q_u, \sum_{m\in\mathbb{N}} \widetilde{P}_{2m}(v) \big(Q_u A_{2m} + v Q_u A_{2m+1} - Q_u q_0 A_{2m+1} \big) \Big] \\
&= \Big[1 + uj, \sum_{m\in\mathbb{N}} \widetilde{P}_{2m}(v) \big((1 + uj)(A_{2m} - q_0 A_{2m+1}) + v(1 + uj) A_{2m+1} \big) \Big] \\
&= \Big[1 + uj, \sum_{m\in\mathbb{N}} \widetilde{P}_{2m}(v) \big\{ \big(A_{2m} - q_0 A_{2m+1} + v A_{2m+1} \big) \\
&\qquad\qquad + u\big(jA_{2m} - jq_0 A_{2m+1} + vjA_{2m+1} \big) \big\} \Big]
\end{aligned}$$

Hence, there exist holomorphic $g, h \colon V \to \mathbb{C}$ such that $[1, f(q)]$ equals

$$\big[1 + uj, g(v) + h(v)j + u(\hat{g}(v)j - \hat{h}(v)) \big] = \pi \big[1, u, g(v) - u\,\hat{h}(v), h(v) + u\hat{g}(v) \big],$$

as required. □

Remark 5.4. Let f_1, f_2 be regular functions on a symmetric slice domain Ω. If g_1, h_1 and g_2, h_2 are the couples of functions appearing in their twistor lifts according to formula (5.6) then $f_l(v) = g_l(v) + h_l(v)j$ for all $v \in \Omega_i = \Omega \cap \mathbb{C}$ and $l = 1, 2$. By the definition of regular (or star) product in (2.1), $f_1 * f_2$ splits as

$$f_1 * f_2 = (g_1 g_2 - h_1 \hat{h}_2) + (g_1 h_2 + h_1 \hat{g}_2)j \tag{5.7}$$

in $L_i = \mathbb{C}$. Hence, the twistor lift of $f_1 * f_2$ is immediately determined by the lifts of f_1 and f_2.

If $\Omega \subset \mathbb{H}$ is a symmetric slice domain that is properly included in $\mathbb{H}$, the lifting F of a regular function $f : \Omega \to \mathbb{H}$ does not, in general, extend to all of $\mathscr{Q}^+$.

Example 5.5. If $f : B(0, 1) \to \mathbb{H}$ is defined as $f(q) = \sum_{n\in\mathbb{N}} q^{2^n}$ then for $q = Q_u^{-1} v Q_u$,

$$[1, f(q)] = \pi\Big[1, u, \sum_{n\in\mathbb{N}} v^{2^n}, u \sum_{n\in\mathbb{N}} v^{2^n}\Big],$$

where $\sum_{n\in\mathbb{N}} v^{2^n}$ cannot be holomorphically extended near any point v with $|v| = 1$.

The quadric $\mathscr{Q}$ of Proposition 5.1 is of course biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$; the rulings are parametrised by u and v. An axially symmetric sphere $x + y\mathbb{S}$ can be identified with the line

$$\ell_v = \big\{[1, u, x + iy, (x + iy)u] : u \in \mathbb{C} \cup \{\infty\}\big\}$$

in $\mathbb{CP}^3$ defined by fixing $v = x + iy$. We may regard ℓ_v as a point in the Grassmannian $\mathbb{G}r_2(\mathbb{C}^4)$ that parametrises lines in $\mathbb{CP}^3$, or equivalently (by means of the Plücker embedding) as a point in the Klein quadric in $\mathbb{P}(\bigwedge^2\mathbb{C}^4) = \mathbb{CP}^5$.

An important feature of F is that it maps each line ℓ_v into another *line* in $\mathbb{CP}^3$, since the expression in (5.6) is linear in u (when the right-hand side is homogenised). This is also a consequence of the fact, shown simply in [4, 13], that f maps $x + y\mathbb{S}$ to another sphere $a + \mathbb{S}b$ (with $a, b \in \mathbb{H}$) in an affine manner. Indeed, the projection π establishes a bijective correspondence between lines $\mathbb{CP}^1$ in $\mathbb{CP}^3$ and "2-spheres" in $\mathbb{R}^4$, where a "2-sphere" might be an S^2 of finite radius (in some 3-space), or a 2-plane, or a point. A detailed study of this correspondence is given in [25].

The action of the unit imaginary quaternion j on $\mathbb{H}^2$, or equivalently

$$j\colon [Z_0, Z_1, Z_2, Z_3] \mapsto [-\bar{Z}_1, \bar{Z}_0, -\bar{Z}_3, \bar{Z}_2] \tag{5.8}$$

on $\mathbb{C}^4$, induces a real structure

$$\sigma\colon [\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6] \mapsto [\bar{\zeta}_1, \bar{\zeta}_5, -\bar{\zeta}_4, -\bar{\zeta}_3, \bar{\zeta}_2, \bar{\zeta}_6] \tag{5.9}$$

on $\mathbb{CP}^5$, expressed here relative to the basis $\{e^{01}, e^{02}, e^{03}, e^{12}, e^{13}, e^{23}\}$ of $\bigwedge^2 \mathbb{C}^4$ (where $e^{lm} = e^l \wedge e^m$). A fixed point of σ corresponds to a j-invariant line in $\mathbb{CP}^3$, i.e. a fibre of π or a "2-sphere" of zero radius. The fixed point set must therefore by parametrised by S^4, and one can see this directly as follows.

In the above coordinates, the Klein quadric is given by

$$\zeta_1\zeta_6 - \zeta_2\zeta_5 + \zeta_3\zeta_4 = 0. \tag{5.10}$$

Splitting the homogeneous coordinates ζ_l into their real and imaginary parts, the equation of the fixed point set is

$$x_1x_6 - x_2^2 - x_3^2 - x_4^2 - x_5^2 = 0,$$

and defines the real quadric $\mathscr{N}$ of null directions in the projectivised Lorentz space $\mathbb{P}(\mathbb{R}^{1,5})$. The situation is summarised by the diagram

$$\begin{array}{ccc} \mathbb{G}\mathrm{r}_2(\mathbb{C}^4) & \subset & \mathbb{CP}^5 \\ \cup & & \cup \\ \mathbb{HP}^1 & = S^4 = \mathscr{N} \subset & \mathbb{P}(\mathbb{R}^{1,5}) \end{array} \tag{5.11}$$

that reflects the usual $SO(1,5)$-invariant representation of the conformal 4-sphere.

Now let $f\colon \Omega \to \mathbb{H}$ be a regular function, where Ω is a symmetric slice domain and $\Omega \setminus \mathbb{R} \cong \mathbb{CP}^1 \times V$ as in (2.2), and let F be its twistor lift. We may consider the mapping

$$\mathcal{F}\colon V \to \mathbb{G}\mathrm{r}_2(\mathbb{C}^4), \qquad v \mapsto F(\ell_v). \tag{5.12}$$

Definition 5.6. We shall call $\mathcal{F}$ the *twistor transform* of f.

The idea is simple, and exploits the philosophy of Shapiro's paper [25] that twistor geometry is the complexification of Lie sphere geometry. The domain $\mathbb{H} \setminus \mathbb{R}$ is a disjoint union of the sets $x + y\mathbb{S}$, which are mapped by f to other spheres in $\mathbb{H}$, and $\mathcal{F}$ merely encodes the induced map of spheres.

Theorem 5.7. *Let Ω be a symmetric slice domain, and let $V = \Omega \cap \mathbb{C}^+$ so that $\Omega \setminus \mathbb{R} \cong \mathbb{CP}^1 \times V$. If $f\colon \Omega \to \mathbb{H}$ is a regular function and $\mathcal{F}$ is its twistor transform (5.12), then $\mathcal{F}$ extends to $\overline{V}$ by setting $\mathcal{F}(\bar{v}) = \sigma(\mathcal{F}(v))$, and consequently it defines a holomorphic curve on $\Omega_i = \Omega \cap \mathbb{C}$. Conversely, any such "real" holomorphic curve $G\colon V \to \mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$ equals the transform $\mathcal{F}$ of a unique regular map $f\colon \Omega \to \mathbb{H}$, provided $\zeta_6 \circ G$ is never zero.*

Remark 5.8. The condition $\zeta_6(G(v)) = 0$ asserts that $G(v)$ intersects the fibre (5.2) over ∞. If the intersection is proper, then the projection $\pi(G(v))$ is a *plane* in $\mathbb{H}$ rather than a 2-sphere. But a regular function f must transform $x + \mathbb{S}y$ into a genuine 2-sphere for all $v = x + iy \in \mathbb{C}^+$, and it is this 2-sphere that is the projection of $\mathcal{F}(v)$. When one allows f to have poles, it is possible for $\mathcal{F}(v)$ to properly intersect, or to equal, $\pi^{-1}(\infty)$. The former happens if and only if f has a 2-sphere of non-removable poles, one of which has order 0. The latter happens if and only if f has a 2-sphere of non-removable poles, all having positive order. (For the definition and characterisation of poles, see [28].) For a given "real" holomorphic curve $G\colon V \to \mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$, removing those $v = x + iy \in V$ such that $\zeta_6 \circ G(v) = 0$, one can reconstruct a regular f that will have poles on the 2-spheres $x + \mathbb{S}y$; examples are presented after the proof.

Proof of Theorem 5.7. It is an easy matter to determine $\mathcal{F}$ explicitly when f is given by (5.3) and g, h by (5.5). It follows that $F(\ell_v)$ has linear equations

$$Z_2 = gZ_0 - \hat{h}Z_1, \qquad Z_3 = hZ_0 + \hat{g}Z_1, \tag{5.13}$$

whose coefficients determine the vectors

$$[g, -\hat{h}, -1, 0], \qquad [h, \hat{g}, 0, -1],$$

that we can wedge together. Relative to our basis $\{e^{lm}\}_{l,m}$,

$$\mathcal{F}(v) = [\zeta_1, \ldots, \zeta_6] = [g\hat{g} + \hat{h}h, h, -g, \hat{g}, \hat{h}, 1]. \tag{5.14}$$

This confirms that $\mathcal{F}$ is holomorphic. The final "1" fixes the projective class, and we can ignore the quadratic term because $\mathcal{F}(v)$ must satisfy (5.10), expressing the fact that the 4-form $\mathcal{F}(v) \wedge \mathcal{F}(v)$ vanishes. Comparing (5.14) with (5.9), and recalling that $\hat{g}(v) = \overline{g(\bar{v})}$, $\hat{h}(v) = \overline{h(\bar{v})}$ shows that $\mathcal{F}$ intertwines complex conjugation with σ.

Conversely, given a mapping $G\colon V \to \mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$ such that $G(\bar{v}) = \sigma(G(v))$ and with $\zeta_6 \neq 0$, we may assume that $\zeta_6 \equiv 1$ and recover $g = -\zeta_3$ and $h = \zeta_2$, and (for all $v \in V$ and $u \in \mathbb{C}$) we set

$$\begin{aligned}\big[1, f((1+uj)^{-1}v(1+uj))\big] &= \pi\big[1, u, g(v) - u\,\hat{h}(v), h(v) + u\,\hat{g}(v)\big]\\ &= \big[1 + uj, g(v) + h(v)j + u(\hat{g}(v)j - \hat{h}(v))\big].\end{aligned}$$

In order to verify that f is regular, we compute:

$$\begin{aligned}&\left(\frac{\partial}{\partial x} + (1+uj)^{-1}i(1+uj)\frac{\partial}{\partial y}\right) f\big(x + (1+uj)^{-1}i(1+uj)y\big)\\ &= \left(\frac{\partial}{\partial x} + (1+uj)^{-1}i(1+uj)\frac{\partial}{\partial y}\right)(1+uj)^{-1}\big\{g(x+iy) + h(x+iy)j\\ &\qquad\qquad + u[\hat{g}(x+iy)j - \hat{h}(x+iy)]\big\}\\ &= (1+uj)^{-1}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\big\{g(x+iy) + h(x+iy)j + u[\hat{g}(x+iy)j - \hat{h}(x+iy)]\big\},\end{aligned}$$

which is identically zero. □

Examples 5.9. An example of the phenomenon mentioned in Remark 5.8 is provided by the curve

$$G(v) = [1, 0, i - v, v + i, 0, v^2 + 1] = \left[\frac{1}{v^2+1}, 0, -\frac{1}{v+i}, \frac{1}{v-i}, 0, 1\right].$$

This satisfies the reality condition, and

$$G(-i) = [1, 0, 2i, 0, 0, 0], \qquad G(i) = [1, 0, 0, 2i, 0, 0]$$

are lines that properly intersect $\pi^{-1}(\infty)$. Retracing our steps back as in the previous proof, we recover $g(v) = 1/(v+i)$ and $h \equiv 0$, and

$$f(q) = Q_u^{-1}\left(\frac{1}{v+i} + \frac{u}{v-i}j\right) = (q^2+1)^{-1}(q-i),$$

in the notation (4.7). This is the so-called *regular reciprocal* of $q+i$, also denoted by $(q+i)^{-*}$. It has a non-removable pole of order 0 at $q = i$, and poles of order 1 at all other points of $\mathbb{S}$ (see [13, 28]).

Let us consider a second example: a "real" curve G giving rise to the regular function $f(q) = (q^2+1)^{-2}(q-i) = (q^2+1)^{-1}(q+i)^{-*}$, which has a sphere of non-removable poles of strictly positive order. It is

$$\begin{aligned} G(v) &= [1, 0, (v^2+1)(i-v), (v^2+1)(v+i), 0, (v^2+1)^3] \\ &= \left[\frac{1}{(v^2+1)^3}, 0, -\frac{1}{(v^2+1)(v+i)}, \frac{1}{(v^2+1)(v-i)}, 0, 1\right], \end{aligned}$$

which equals $[1, 0, 0, 0, 0, 0]$, i.e. $\pi^{-1}(\infty)$, when $v = \pm i$.

If $f_r\colon \mathbb{R} \to \mathbb{H}$ is real analytic, we can extend it to a regular function $f\colon \Omega \to \mathbb{H}$ for some symmetric slice domain Ω neighbouring $\mathbb{R}$. The twistor transform of f is none other than the *complexification* of f_r, obtained using (5.11). If f_r is algebraic then $\mathcal{F}$ will be a rational curve.

Examples 5.9 fit into this scheme, and in Section 7, we shall see an example in which $\mathcal{F}$ is a rational quartic curve (see (7.8)). However, an even simpler case arises when f_r is the identity, for then

$$\mathcal{F}(v) = \ell_v = [v^2, 0, -v, v, 0, 1]$$

is defined for all $v \in \mathbb{CP}^1$, and the image of $\mathcal{F}$ is a conic in some $\mathbb{CP}^2$. Indeed, $\bigwedge^2(\mathbb{C}^4)$ is the complexification of the Lie algebra

$$\mathfrak{so}(2, \mathbb{H}) \cong \mathfrak{sp}(1) \oplus \mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(1, 2),$$

and the $\mathbb{CP}^2$ arises by projectivising the final summand.

To conclude, our focus on axially symmetric spheres and the complex structure $\mathbb{J}$ has reduced the conformal group to $SO(2, \mathbb{H})$. This provides the extra structure on the Klein quadric with which to study regular functions. We point out that not only do the second and third slot in the twistor transform (5.14) determine the twistor lift, but they also enable one to compute the transform of a star product, thanks to formula (5.7). In this way, (2.1) is converted into a multiplication within the class of holomorphic curves under consideration in Theorem 5.7.

Remark 5.10. Let us consider a domain $V \subseteq \mathbb{C}^+$ and a holomorphic curve $G\colon V \to \mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$ which is not necessarily real (that is, which does not necessarily extend to a domain in $\mathbb{C}$ which is symmetric with respect to the real axis). Then, provided $\zeta_6 \circ G \neq 0$, the curve G will still define a mapping $F\colon \mathscr{O} \to \mathbb{CP}^3$ on an open subset $\mathscr{O} \cong \mathbb{CP}^1 \times V$ of $\mathscr{Q}^+$ as in (5.6), except that $g, \hat{g}$ and $h, \hat{h}$ are now unrelated. We can therefore define

$f : \pi(\mathscr{O}) \to \mathbb{H}$ so that $f \circ \pi = \pi \circ F$. This f will be a regular function that does not necessarily extend to a symmetric slice domain, but that still maps $\mathbb{J}$ to the orthogonal complex structure transforming each $\mathbf{v} \in T_{f(q)} f(\pi(\mathscr{O})) \cong \mathbb{H}$ to $I_q \mathbf{v}$.

6. Removing a parabola

In this section, we will investigate the possibility of defining a non-constant orthogonal complex structure on $\mathbb{H} \setminus \gamma$, where γ is the parabola

$$\gamma = \{t^2 + it : t \in \mathbb{R}\},$$

via the regular function $f(q) = q^2 + qi$ (which maps $\mathbb{R}$ onto γ).

Theorem 6.1. *Let $f(q) = q^2 + qi$ on $\mathbb{H}$. Its degenerate set D_f is empty and its singular set N_f is the 2-plane $-i/2 + j\mathbb{R} + k\mathbb{R}$, whose image is the paraboloid of revolution*

$$\Gamma = \{x_0 + jx_2 + kx_3 : x_0, x_2, x_3 \in \mathbb{R},\ x_0 = 1/4 - (x_2^2 + x_3^2)\}. \tag{6.1}$$

The restriction $f : N_f \to \Gamma$ is a bijection. Furthermore, $f(\mathbb{H} \setminus N_f) = \mathbb{H} \setminus \Gamma$ and the inverse image of every $c \in \mathbb{H} \setminus \Gamma$ consists of two points. The mapping $f : \mathbb{H} \to \mathbb{H}$ is therefore a double covering branched over Γ and $f(\mathbb{H} \setminus \mathbb{R}) = \mathbb{H} \setminus \gamma$.

Proof. For any given $c \in \mathbb{H}$, the polynomial $f(q) - c = q^2 + qi - c$ has at least one root by the quaternionic fundamental theorem of algebra (see, for instance, [17]). Hence $f^{-1}(c)$ contains at least a point α. Therefore, $q^2 + qi - c$ can be factored as $(q - \alpha) * (q - \beta)$ for some $\beta \in \mathbb{H}$. According to Example 2.4, either

$$f^{-1}(c) = \{\alpha, (\alpha - \bar{\beta})\beta(\alpha - \bar{\beta})^{-1}\},$$

or α is a point of the degenerate set D_f and $\beta = \bar{\alpha}$. But the latter is excluded because here $\alpha + \beta = -i$. Hence, $D_f = \emptyset$.

Moreover,

$$(\alpha - \bar{\beta})\beta(\alpha - \bar{\beta})^{-1} = (\alpha + \bar{\alpha} - i)(-\alpha - i)(\alpha + \bar{\alpha} - i)^{-1}.$$

We split $\alpha = z + wj$ with $z = x_0 + ix_1$ and $w = x_2 + ix_3$ in $\mathbb{C}$. Thus, $\bar{\alpha} = \bar{z} - wj$ and

$$\begin{aligned}(\alpha + \bar{\alpha} - i)(-\alpha - i)(\alpha + \bar{\alpha} - i)^{-1} &= (z + \bar{z} - i)(-z - i - wj)(z + \bar{z} - i)^{-1} \\ &= -z - i - (z + \bar{z} - i)(z + \bar{z} + i)^{-1} wj \\ &= -z - i + \frac{1 - (z + \bar{z})^2 + 2(z + \bar{z})i}{1 + (z + \bar{z})^2} wj \\ &= -(z + i/2) - i/2 + e^{2\theta i} wj,\end{aligned}$$

where $\tan\theta = z + \bar{z}$. This corresponds to rotating z by 180° about the point $-i/2$, and rotating w by an angle 2θ about the origin. Thus,

$$f^{-1}(c) = \left\{z + wj, -\left(z + \tfrac{1}{2}i\right) - \tfrac{1}{2}i + e^{2i\theta} wj\right\} \tag{6.2}$$

consists of two distinct points except when $z + \bar{z} = 0$ and $z = -(z + i/2) - i/2$, which means $z = -i/2$. Hence, the singular set N_f is the 2-plane $-i/2 + \mathbb{R}j + \mathbb{R}k$.

We complete the proof observing that

$$f\big(-\tfrac{1}{2}i + wj\big) = \tfrac{1}{4} - |w|^2 - wk,$$

so that f maps N_f bijectively into the paraboloid Γ defined by (6.1). □

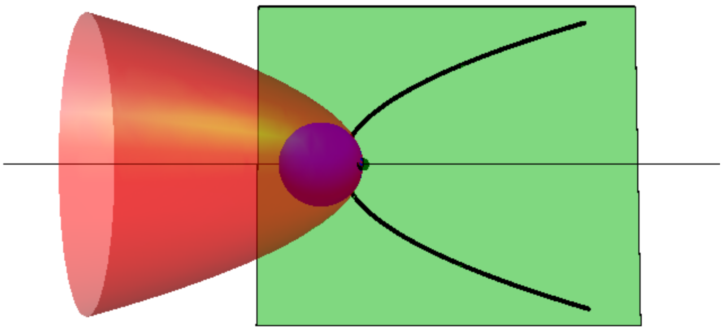

Fig. 1. In $\mathbb{R}^4$, the paraboloid Γ intersects the plane L_i containing γ in the parabola's focus $(1/4, 0, 0, 0)$. It intersects any 3-space containing γ in a parabola with vertex at this focus. The significance of the osculating sphere $(x_0 + 1/4)^2 + x_2^2 + x_3^2 = 1/4$ to Γ is explained in Section 7 (see (7.10)).

Inspecting the proof of the previous lemma, we find that f is injective when restricted either to the open right half-space

$$\mathbb{H}^+ = \{x_0 + ix_1 + jx_2 + kx_3 : x_0 > 0\}$$

or to the open left half-space $\mathbb{H}^-$. It is easily seen that their boundary $i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$ maps onto the (3-dimensional) solid paraboloid

$$\mathfrak{G} = \{x_0 + jx_2 + kx_3 : x_0, x_2, x_3 \in \mathbb{R}, x_0 \le 1/4 - (x_2^2 + x_3^2)\}$$

bounded by Γ in the 3-space $\mathbb{R} + j\mathbb{R} + k\mathbb{R}$, and that $f(\mathbb{H}^+) = \mathbb{H} \setminus \mathfrak{G} = f(\mathbb{H}^-)$. Note that only Γ is of fundamental significance; distinguishing the half-spaces $\mathbb{H}^+, \mathbb{H}^-$ and boundary $i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$ determines the "inside" of the paraboloid, but this choice is not unique.

We next use our choices to define single-valued complex structures on a dense open set of $\mathbb{H}$:

Proposition 6.2. *Let $f(q) = q^2 + qi$. Let $\mathbb{J}^+$ denote the complex structure induced by the restriction $f : \mathbb{H}^+ \to \mathbb{H}$ on $\mathbb{H} \setminus (\gamma \cup \mathfrak{G})$ and let $\mathbb{J}^-$ be induced by $f : \mathbb{H}^- \to \mathbb{H}$. Then $\mathbb{H} \setminus (\gamma \cup \mathfrak{G})$ is the maximal open domain of definition for both $\mathbb{J}^+$ and $\mathbb{J}^-$. Indeed, both $\mathbb{J}^+$ and $\mathbb{J}^-$ extend continuously to Γ but neither of them extends continuously to any point of $\mathfrak{G} \setminus \Gamma$.*

Proof. The function f is injective in $\mathbb{H}^+ \cup N_f$. Its inverse

$$f^{-1} : (\mathbb{H} \setminus \mathfrak{G}) \cup \Gamma \to \mathbb{H}$$

is continuous by the open mapping theorem, since the degenerate set of f is empty. Hence, $p \mapsto I_{f^{-1}(p)}$ is continuous and the structure $\mathbb{J}^+$ can be extended continuously to Γ by setting $\mathbb{J}^+_p \mathbf{v} = I_{f^{-1}(p)}\mathbf{v}$ for all $p \in \Gamma$.

Now, let $p \in \mathfrak{G} \setminus \Gamma$: since f^{-1} is defined and continuous on $(\mathbb{H} \setminus \mathfrak{G}) \cup \Gamma$, the structure $\mathbb{J}^+$ admits a continuous extension to p if and only if for $a \in \mathbb{H} \setminus (\mathfrak{G} \cup \gamma)$,

$$\lim_{a \to p} I_{f^{-1}(a)}$$

exists. We saw in the proof of the previous lemma that $f^{-1}(p) = \{q, q'\}$ with $q = ti + wj$ and $q' = -(t+1)i + wj$ for some $t > -1/2$ and $w \in \mathbb{C}$. Since q and q' are both in $\overline{\mathbb{H}}^+$, the aforementioned limit exists if and only if $I_q = I_{q'}$. However, by direct computation

$$I_q = \frac{ti + wj}{|ti + wj|} \neq \frac{-(t+1)i + wj}{|-(t+1)i + wj|} = I_{q'},$$

and the proof is complete. □

Finally, let us compare the two structures that we introduced.

Theorem 6.3. *The structures $\mathbb{J}^+$ and $\mathbb{J}^-$ have the following properties on $\mathbb{H} \setminus (\gamma \cup \mathfrak{G})$:*

(1) *The four structures $\mathbb{J}^+, \mathbb{J}^-, -\mathbb{J}^+, -\mathbb{J}^-$ are distinct outside $L_i = \mathbb{C}$.*
(2) *At every point of $\{x_0 + ix_1 : x_0 > x_1^2\}$, the structures $\mathbb{J}^+$ and $\mathbb{J}^-$ both coincide with left multiplication by $-i$.*
(3) *At every point of $\{x_0 + ix_1 : x_0 < x_1^2\}$, the structures $\mathbb{J}^+$ and $-\mathbb{J}^-$ coincide; in $\{x_0 + ix_1 : x_0 < x_1^2,\ x_1 > 0\}$, they both coincide with left multiplication by i; in $\{x_0 + ix_1 : x_0 < x_1^2,\ x_1 < 0\}$, they coincide with left multiplication by $-i$.*

At every point of the paraboloid Γ, the extended structures $\mathbb{J}^+$ and $\mathbb{J}^-$ coincide.

Proof. For every point $c \in \mathbb{C} \setminus (\gamma \cup \mathfrak{G})$,

$$f^{-1}(c) = \left\{z, -\left(z + \tfrac{1}{2}i\right) - \tfrac{1}{2}i\right\}$$

for some $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. If $c \in \{x_0 + ix_1 : x_0 > x_1^2\}$ then z and $-(z + i/2) - i/2$ both belong to $\{x_0 + ix_1 : -1 < x_1 < 0\}$ so that $\mathbb{J}^+_c$ and $\mathbb{J}^-_c$ both coincide with left multiplication by $-i$. If $c \in \{x_0 + ix_1 : x_0 < x_1^2,\ x_1 > 0\}$ then $z \in \{x_0 + ix_1 : x_1 > 0\}$ and $-(z + i/2) - i/2$ belongs to $\{x_0 + ix_1 : x_1 < -1\}$ so that $\mathbb{J}^+_c$ coincides with left multiplication by i and $\mathbb{J}^-_c$ coincides with left multiplication by $-i$. Moreover, if $c \in \{x_0 + ix_1 : x_0 < x_1^2,\ x_1 < 0\}$ then $z \in \{x_0 + ix_1 : x_1 < -1\}$ and $-(z + i/2) - i/2 \in \{x_0 + ix_1 : x_1 > 0\}$ so that $\mathbb{J}^+_c$ coincides with left multiplication by $-i$ and $\mathbb{J}^-_c$ coincides with left multiplication by i.

Every point $c \in \mathbb{H} \setminus (\gamma \cup \mathfrak{G} \cup \mathbb{C})$ has preimage (6.2) for some $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ and non-zero $w \in \mathbb{C}$; by direct inspection, $w \neq \pm e^{2\theta i} w$ so that $\mathbb{J}^+_c \neq \pm \mathbb{J}^-_c$.

The final assertion follows from the fact that the preimage of each $c \in \Gamma$ consists of one point. □

7. A rational quartic scroll

We apply the theory of Section 5 to the example of Section 6, again with $f(q) = q^2 + qi$.

From the point of view of complex structures, f maps the plane L_i containing the parabola $\gamma = \{t^2 + ti : t \in \mathbb{R}\}$ onto itself. It maps $-i/2$ to the focus $1/4$, but is $2:1$ elsewhere in L_i. It follows that, pointwise for $q \in L_i$, the induced complex structure $\mathbb{J}^f_{f(q)}$ is determined by the action of f on L_i. This will be a key feature of the following interpretation, in which the multi-valued structure $\mathbb{J}^f$ is encoded in the twistor lift of f. We shall see that the action of f on L_i gives rise to double points in the "graph" of $\mathbb{J}^f$.

The twistor lift is given by (5.6), where

$$g(v) = v^2 + iv, \quad \hat{g}(v) = v^2 - iv, \quad h(v) = 0. \tag{7.1}$$

This extends to a holomorphic mapping $F : \mathscr{Q} \to \mathbb{CP}^3$ by allowing v to assume values in $\mathbb{C}$ rather than just $\mathbb{C}^+$. To be more precise, we shall adopt homogeneous coordinates for $\mathbb{CP}^3$ throughout this final section. Then

$$F[st, su, tv, uv] = [s^2t, s^2u, t(v^2 + isv), u(v^2 - ivs)], \tag{7.2}$$

with $([s, v], [t, u]) \in \mathbb{CP}^1 \times \mathbb{CP}^1$, is well defined and consistent with (5.6).

If we set $Z_0 = s^2t$, $Z_1 = s^2u$ etc., then

$$Z_1Z_2 - Z_0Z_3 = 2is^3tuv, \qquad Z_1Z_2 + Z_0Z_3 = 2s^2tuv^2.$$

It follows that the image of F lies in the quartic surface $\mathscr{K}$ defined by the equation $K = 0$, where

$$K(Z_0, Z_1, Z_2, Z_3) = (Z_1Z_2 - Z_0Z_3)^2 + 2Z_1Z_0(Z_1Z_2 + Z_0Z_3). \tag{7.3}$$

Observe that $\mathscr{K}$ is a *real* subvariety of the twistor space, invariant by the antilinear involution (5.8).

We already know from Theorem 6.1 that f is generically $2:1$. If $f(q_1) = f(q_2)$ then the two pairs of points in the quadric $\mathscr{Q}$ of Proposition 5.1 lying over q_1, q_2 must map to the typically four points of $\mathscr{K}$ lying over the common image point. This shows that F is generically $1:1$. In order to describe the action of F more explicitly, consider the intersection of $\mathscr{K}$ with the quadric $Z_1Z_2 - Z_0Z_3 = 0$, which we can in fact identify with $\mathscr{Q}$. This intersection also satisfies $Z_0Z_1Z_2 = 0$. With the notation

$$m_{ln} = \{[Z_0, Z_1, Z_2, Z_3] : Z_l = 0 = Z_n\}, \tag{7.4}$$

it must therefore consist of the four lines $m_{01}, m_{02}, m_{13}, m_{23}$ forming a "square".

Theorem 7.1. *The mapping F is a birational equivalence between $\mathscr{Q}$ and $\mathscr{K}$. In fact, F maps $\mathscr{Q}$ onto $\mathscr{K}$ and is injective on the complement of $m_{02} \cup m_{13}$ over which it is a $2:1$ branched covering. The singular locus of $\mathscr{K}$ is the union $m_{02} \cup m_{13} \cup m_{01}$, and the set of cusp points is $m_{01} \cup \{[0, 1, 0, 1/4], [1, 0, 1/4, 0]\}$.*

Proof. To prove that F is onto, first suppose that $Z_0 \neq 0$ and $Z_1 \neq 0$. Then we can define

$$u/t = Z_1/Z_0, \qquad 2iv/s = (Z_1Z_2 - Z_0Z_3)/(Z_0Z_1),$$

and $[Z_l]$ has a unique inverse image.

If $Z_0 = 0$ then $Z_1Z_2 = 0$, whereas if $Z_1 = 0$ then $Z_0Z_3 = 0$. These equations define the lines m_{02}, m_{13}, m_{01}, so we need to consider the three possibilities in turn:

- If $Z_0 = Z_2 = 0$ then $t = 0$ and F maps $[s, v] \in \mathbb{CP}^1$ to $[0, s^2, 0, v^2 - isv] \in m_{02}$.
- If $Z_1 = Z_3 = 0$ then $u = 0$ and F maps $[s, v] \in \mathbb{CP}^1$ to $[s^2, 0, v^2 + isv, 0] \in m_{13}$.
- If $Z_0 = 0 = Z_1$ then $s = 0$ and F maps $[t, u] \in \mathbb{CP}^1$ to $[0, 0, t, u]$, so F is the identity on m_{01}.

It follows that F is injective unless $[t, u]$ equals $[1, 0]$ or $[0, 1]$ *and* $[s, v]$ does not equal $[1, 0]$ or $[0, 1]$. Moreover, restricted to $m_{02} \cup m_{13}$, the map F is $2:1$ except over the vertices

$$[1, 0, 0, 0], \quad [0, 1, 0, 0], \quad [0, 0, 1, 0], \quad [0, 0, 0, 1] \tag{7.5}$$

of the square, where it acts as the identity.

Any singular point $[Z_l]$ of $\mathscr{K}$ must satisfy

$$\begin{aligned}
0 &= \frac{\partial K}{\partial Z_0} = -2Z_3(Z_1Z_2 - Z_0Z_3) + 2Z_1(Z_1Z_2 + 2Z_0Z_3),\\
0 &= \frac{\partial K}{\partial Z_1} = 2Z_2(Z_1Z_2 - Z_0Z_3) + 2Z_0(2Z_1Z_2 + Z_0Z_3),\\
0 &= \frac{\partial K}{\partial Z_2} = 2Z_1(Z_1Z_2 - Z_0Z_3) + 2Z_0Z_1^2,\\
0 &= \frac{\partial K}{\partial Z_3} = -2Z_0(Z_1Z_2 - Z_0Z_3) + 2Z_0^2Z_1.
\end{aligned}$$

The last two equations imply that $Z_0 = 0$ or $Z_1 = 0$. Since $[Z_l]$ satisfies (7.3), we know that it must lie in $m_{02} \cup m_{13} \cup m_{01}$. Conversely, any point in the union of the three lines satisfies all four equations above, and is singular.

To find the cusp points, one can compute the 2×2 minors (determinants) of the Hessian $\left(\frac{\partial^2 K}{\partial Z_l \partial Z_n}\right)_{l,n}$ at the singular points. When $Z_0 = Z_1 = 0$, all minors are identically zero, and every point of m_{01} is a cusp. Over m_{02} the relevant determinant equals $4Z_1^3(4Z_3 - Z_1)$, and over m_{13} it is $4Z_0^3(4Z_2 - Z_0)$, indicating additional cusps at $[0, 1, 0, 1/4]$ and $[1, 0, 1/4, 0]$. □

Recall from (5.2) that $m_{01} = \pi^{-1}(\infty)$. To make the twistor projection π more explicit, set

$$\tilde{q} = x_0 + ix_1 + (x_2 + ix_3)j = w_1 + w_2 j, \qquad w_1, w_2 \in \mathbb{C}, \tag{7.6}$$

so that $[Z_l] \in \pi^{-1}(\tilde{q})$ if and only if $Z_2 + Z_3 j = (Z_0 + Z_1 j)(w_1 + w_2 j)$. (In practice, we will want to consider points $\tilde{q} = f(q)$, so the tilde is to avoid confusion). This implies that $\pi^{-1}(\tilde{q})$ has equations

$$Z_2 = w_1 Z_0 - \bar{w}_2 Z_1, \qquad Z_3 = w_2 Z_0 + \bar{w}_1 Z_1, \tag{7.7}$$

and coordinates

$$[\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6] = [|w_1|^2 + |w_2|^2, w_2, -w_1, \bar{w}_1, \bar{w}_2, 1]$$

in $\mathbb{CP}^5$ (cf. (5.13) and (5.14)), so that it is fixed by the action of σ. We obtain

- $Z_2 = 0 = Z_3 \Rightarrow w_1 = 0 = w_2 \Rightarrow \tilde{q} = 0$,
- $Z_0 = 0 = Z_2$ or $Z_1 = 0 = Z_3 \Rightarrow w_2 = 0 \Rightarrow \tilde{q} \in L_i$,
- $Z_0 = 0 = Z_3$ or $Z_1 = 0 = Z_2 \Rightarrow w_1 = 0 \Rightarrow \tilde{q} \in L_i^{\perp}$.

Therefore $m_{23} = \pi^{-1}(0)$ and m_{02}, m_{13} both project to $L_i \cup \infty$. It follows from Theorem 7.1 that topologically, *$\mathscr{K}$ is obtained from $\mathscr{Q}$ by carrying out a $\mathbb{Z}_2$ identification on each of the two punctured planes lying over $L_i \setminus \{0\}$*. The main features of $\mathscr{K}$ are represented by Figure 2.

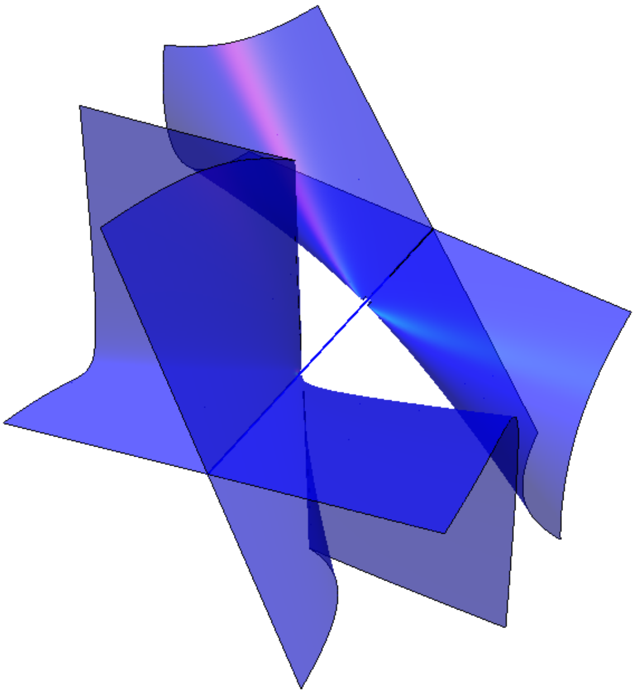

Fig. 2. The real affine slice of $\mathscr{K}$ defined by $K(x, y, z, 1/4) = 0$. The vertical line is m_{01}, and the diagonal one containing an extra isolated cusp corresponds to m_{02}.

From (5.14) and (7.1), the twistor transform of f is the mapping $\mathcal{F} : \mathbb{CP}^1 \to \mathbb{CP}^5$ given by

$$\mathcal{F}(v) = [v^4 + v^2, 0, -v^2 - iv, v^2 - iv, 0, 1]. \tag{7.8}$$

Its image lies in a 2-quadric, the intersection of the Klein quadric with two of its tangent hyperplanes, those at the points [0, 1, 0, 0, 0, 0] and [0, 0, 0, 0, 1, 0] of $\mathbb{CP}^5$ that correspond to m_{02} and m_{13} respectively. It has a cusp at [1, 0, 0, 0, 0, 0] that corresponds to m_{01}. By construction, $\mathscr{K}$ is the ruled surface or *scroll* containing, for each $v \in \mathbb{C}$, the line $F(\ell_v) = \mathcal{F}(v)$, intersection of the planes

$$Z_2 = (v^2 + iv)Z_0 \quad \text{and} \quad Z_3 = (v^2 - iv)Z_1. \tag{7.9}$$

This line is a twistor fibre if and only if it satisfies the reality condition $\sigma(\mathcal{F}(v)) = \mathcal{F}(v)$, equivalently, $v = \bar{v} = x \in \mathbb{R}$. Indeed,

$$F(\ell_x) = \{[Z_0, Z_1, (x^2 + ix)Z_0, (x^2 - ix)Z_1] : [Z_0, Z_1] \in \mathbb{CP}^1\} = \pi^{-1}(x^2 + ix)$$

for each $x \in \mathbb{R}$. Of the six lines (7.4), only $m_{01} = \ell_\infty$ and $m_{23} = \ell_0$ belong to the ruling parametrised by the twistor transform $\mathcal{F}$.

Remark 7.2. It follows from the above descriptions that the rational ruled quartic surface $\mathscr{K}$ is of type 1(iii) in Dolgachev's list [9, Theorem 10.4.15] and, as explained in that book, of type IV(B) in Edge's list [10], these classifications building on work of Cayley and Cremona. In classical terminology, the singular locus of $\mathscr{K}$ is the *double curve*, and the singular points (7.5) are the *pinch points* of $\mathscr{K}$ [9]. The singular locus is effectively determined by the properties of $\mathcal{F}$ mentioned immediately after (7.8), and the equation (7.3) is a variant of Rohn's normal form (see [20, Lemma 1.10 and §3.2.8]).

Substituting $Z_2 = w_1 Z_0$ and $Z_3 = \bar{w}_1 Z_1$ in (7.3), we see that points in $\pi^{-1}(L_i) \cap \mathscr{K}$ are characterised by the equation

$$Z_0^2 Z_1^2 (x_1^2 - x_0) = 0,$$

confirming that each point of L_i is covered by exactly two double points of $\mathscr{K}$. These affine singular points arise naturally as intersection points of lines in the ruling. Indeed, $F(\ell_v)$, $F(\ell_w)$ meet where

$$Z_2 = (v^2 + iv)Z_0 = (w^2 + iw)Z_0, \qquad Z_3 = (v^2 - iv)Z_1 = (w^2 - iw)Z_1,$$

yielding $Z_0 = 0 = Z_2$ and $v + w = i$, or $Z_1 = 0 = Z_3$ and $v + w = -i$.

The lines $F(\ell_v)$, $F(\ell_{\pm i - v})$ are distinct unless $v = \pm i/2$. In the former case, they meet m_{02} or m_{13} (depending on the sign) transversely at a node of $\mathscr{K}$. Moreover, if they meet over γ, one of the two lines is a twistor fibre. On the other hand, each of the two double lines $\ell_{i/2}, \ell_{-i/2}$ is tangent to $\mathscr{K}$ at a cusp lying over the focus of γ, i.e., at a point of $\mathscr{K} \cap \pi^{-1}(1/4)$. One of these two pinch points is shown in Figure 2, and we can establish a direct link with Figure 1 as follows. If we use (7.2), we obtain

$$F\left[1, u, \tfrac{i}{2}, \tfrac{i}{2}u\right] = \left[1, u, -\tfrac{3}{4}, \tfrac{1}{4}u\right]$$

for $u \in \mathbb{C}$. We therefore see that

$$F(\ell_{i/2}) = \left\{\left[1, u, -\tfrac{3}{4}, \tfrac{1}{4}u\right] : u \in \mathbb{C} \cup \{\infty\}\right\}$$

and that the projection $\pi(F(\ell_{i/2})) = \pi(F(\ell_{-i/2}))$ is parametrised stereographically in $\mathbb{H} = \mathbb{R}^4$ by

$$\frac{1}{4(1+|u|^2)}(|u|^2 - 3 + 4uj) = \frac{1}{4(1+|u|^2)}(|u|^2 - 3, 0, 4\,\mathrm{Re}\,u, 4\,\mathrm{Im}\,u) \qquad (7.10)$$

(here Im denotes the complex imaginary part, $\mathrm{Im} : \mathbb{C} \to \mathbb{R}$). This is the sphere

$$f\left(\tfrac{1}{2}\mathbb{S}\right) = -\tfrac{1}{4} + \tfrac{i}{2}\mathbb{S} = \{x_0 + x_2 j + x_3 k : (x_0 + 1/4)^2 + x_2^2 + x_3^2 = 1/4\}$$

visible in Figure 1, containing the focus of γ.

The next result asserts that the surface $\mathscr{K}$ is in effect determined by the parabola $\gamma = f(\mathbb{R})$.

Proposition 7.3. *Any algebraic surface in $\mathbb{CP}^3$ containing $\pi^{-1}(\gamma)$ contains $\mathscr{K}$.*

Proof. Let $p = p(Z_0, Z_1, Z_2, Z_3)$ be a polynomial that vanishes on $F(\ell_x)$ for all $x \in \mathbb{R}$. We shall actually show that K divides p.

Referring to (7.2), we see that $P = p \circ F$ vanishes on all ℓ_x, so

$$P(Z_0, Z_1, xZ_0, xZ_1) = 0, \qquad \forall\, [Z_0, Z_1] \in \mathbb{CP}^1,\ x \in \mathbb{R}.$$

Since P is a polynomial, we can replace $x \in \mathbb{R}$ by $x \in \mathbb{C}$, and so P vanishes on the quadric $\mathscr{Q}$. It follows from Theorem 7.1 that p vanishes on $\mathscr{K}$.

The polynomial K must be irreducible because no line or quadric in $\mathbb{CP}^3$ could contain $\pi^{-1}(\gamma)$. It follows from the Nullstellensatz that K divides p. □

In view of the proposition, properties of $\mathscr{K}$ are naturally associated to the 4-dimensional conformal geometry of the parabola. We are now in a position to consolidate the analysis that we are carrying out in parallel to Section 6.

Theorem 7.4. *Let $\tilde{q} = f(q) \in \mathbb{H}$. The cardinality of the fibre $\pi^{-1}(\tilde{q}) \cap \mathscr{K}$ is different from 4 in the following cases:*

(1) *$\tilde{q} \in \gamma$ iff $\pi^{-1}(\tilde{q}) \subset \mathscr{K}$;*
(2) *$\tilde{q} \in L_i \setminus \gamma$ iff $\pi^{-1}(\tilde{q})$ contains exactly two singular points of $\mathscr{K}$;*
(3) *$\tilde{q} \in \Gamma \setminus \left\{\frac{1}{4}\right\}$ iff $\pi^{-1}(\tilde{q})$ is tangent to $\mathscr{K}$ at two smooth points.*

Proof. We tackle this by investigating the intersection of a given fibre $\pi^{-1}(\tilde{q})$ with the lines $F(\ell_v)$ as v varies. Referring to (7.7) and (7.9), we need to solve the equations

$$(v^2 + iv)Z_0 = w_1 Z_0 - \bar{w}_2 Z_1, \qquad (v^2 - iv)Z_1 = w_2 Z_0 + \bar{w}_1 Z_1,$$

which yield

$$\left((v^2 - iv - \bar{w}_1)(v^2 + iv - w_1) + |w_2|^2\right) Z_0 Z_1 = 0.$$

If $Z_0 = 0$ or $Z_1 = 0$ (two antipodal points on the fibre) then $w_2 = 0$ and

$$\bar{w}_1 = v^2 - iv \quad \text{or} \quad w_1 = v^2 + iv. \qquad (7.11)$$

For fixed $\tilde{q} = w_1 \in L_i$, there are a total of four solutions ℓ_v unless $\tilde{q} = 1/4$ is the focus of γ. We already know that these solutions are associated to two distinct singular points of $\mathscr{K}$.

The line $F(\ell_v)$ can only equal a fibre $\pi^{-1}(\tilde{q})$ if it hits both points $Z_0 = 0$ and $Z_1 = 0$ of that fibre. This forces the "or" in (7.11) to be an "and", whence $v = x \in \mathbb{R}$ and $\tilde{q} \in \gamma$. This is statement (1), and (2) also follows.

If $Z_0 Z_1 \neq 0$ then it is the vanishing of

$$R(v) = v^4 + (1 - 2x_0)v^2 - 2x_1 v + C,$$

where $C = |\tilde{q}|^2 = \sum_{l=0}^{3} x_l^2$, that determines the points of $\pi^{-1}(\tilde{q}) \cap \mathscr{K}$. Since the coefficients are real, the roots of $R(v)$ must occur in conjugate pairs.

A computation of the discriminant of $R(v)$ shows that it equals 16 times

$$\begin{aligned} D = C &- 8C^2 + 16C^3 - 8Cx_0 + 32C^2x_0 + 24Cx_0^2 - 32C^2x_0^2 - 32Cx_0^3 \\ &+ 16Cx_0^4 - x_1^2 + 36Cx_1^2 + 6x_0x_1^2 - 72Cx_0x_1^2 - 12x_0^2x_1^2 + 8x_0^3x_1^2 - 27x_1^4. \end{aligned}$$

When $x_1 = 0$, this expression reduces to $C(-1 + 4C + 4x_0 - 4x_0^2)^2$, so the zero set of D contains

$$x_0 = 1/4 - x_2^2 - x_3^2, \qquad x_1 = 0, \tag{7.12}$$

which is the equation of Γ. We need to show that there are no other zeros.

When we substitute $x_l = \pm t$, the factor D becomes a polynomial of degree 6 in t with leading term $9216t^6$ irrespective of the choice of signs. Hence it is positive on the boundary of a sufficiently large cube, and must attain a minimum inside the cube. It suffices to show that the only critical points of D occur at (7.12). It turns out that $\partial D/\partial x_l$ is divisible by x_l for $l = 1, 2, 3$, so we set

$$\delta_l = \begin{cases} \dfrac{\partial D}{\partial x_0}, & l = 0, \\[2mm] \dfrac{1}{x_l}\dfrac{\partial D}{\partial x_l}, & l = 1, 2, 3. \end{cases}$$

A computation shows that the system $\delta_l = 0$ for $l = 0, 1, 2, 3$ only has real solutions with $x_1 = 0$, and there are no further solutions with $x_2 = 0$ or $x_3 = 0$.

We already know that the quartic has no singular points over $\mathbb{H} \setminus L_i$. If two of the four points coincide at a smooth point of $\mathscr{K}$ then $\mathscr{K}$ is tangent to the fibre at this point. Statement (3) follows. □

Theorem 7.4 provides essentially the same information as Theorem 6.3. In a neighbourhood U of a generic point of $\mathbb{H}$, the quartic $\mathscr{K}$ is a $4:1$ covering of U whose leaves are holomorphic sections of the twistor space $\mathbb{CP}^3$. In particular, $\mathscr{K}$ contains the graphs of the complex structures $\pm\mathbb{J}^+, \pm\mathbb{J}^-$ defined by Proposition 6.2 on the complement $\mathbb{H} \setminus (\gamma \cup \mathfrak{G})$ of the parabola γ and the solid paraboloid $\mathfrak{G}$. These two pairs coincide over L_i, but the two leaves cross transversely (so both $\mathbb{J}^+$ and $\mathbb{J}^-$ are well defined) over $L_i \setminus (\gamma \cup \{1/4\})$. On the other hand, along directions approaching Γ the graphs become vertical, so the covariant derivatives $\nabla\mathbb{J}^\pm$ (computed in any conformally flat metric) are unbounded.

The non-trivial computations required in the proof of statement (3) highlight the difficulty in computing the discriminant locus of some quite simple algebraic surfaces relative to the twistor projection π. In this paper, the quartic $\mathscr{K}$ was "reverse engineered" via Proposition 7.3. The problem of understanding the discriminant locus of generic quartics (or even cubics) remains open. The last proof also serves to demonstrate the relative effectiveness of the approach in Section 6, which achieved the same result using quaternions.

Having identified the surface $\mathscr{K}$, we can proceed with the proof of the next result by mimicking that of [24, Theorem 3.10].

Theorem 7.5.

(1) *There is no orthogonal complex structure whose maximal domain of definition is* $\mathbb{H} \setminus \gamma$.
(2) *Let J be an OCS defined on $\mathbb{H} \setminus (\gamma \cup \mathfrak{G})$. If, for each $p \in \mathfrak{G}$, the limit points of J at p are among the limit points of $\mathbb{J}^+, \mathbb{J}^-, -\mathbb{J}^+, -\mathbb{J}^-$ at p, then J is itself one of these four complex structures.*

Proof. (1) First suppose that J is an OCS defined on $\mathbb{H} \setminus \gamma$. Let $\mathscr{S}$ be the graph of J, which is a complex submanifold of $\pi^{-1}(\mathbb{H} \setminus \gamma)$ by [22, 8]. If the holomorphic function K vanishes on an open subset of $\mathscr{S}$ then it vanishes identically, and $\mathscr{S} \subset \mathscr{K}$. This is not possible because $\mathscr{K}$ is not transversal to the twistor fibres over Γ, while J is smooth over Γ. It follows that $\mathscr{S} \cap \mathscr{K}$ has complex dimension at most 1 at each point.

Now consider the set

$$A = \mathscr{S} \cap (\mathbb{CP}^3 \setminus \mathscr{K}) = \mathscr{S} \setminus (\mathscr{S} \cap \mathscr{K}). \tag{7.13}$$

Since $\mathscr{S}$ is a complex analytic subset of $\pi^{-1}(\mathbb{H} \setminus \gamma)$, A is a complex analytic subset of the smaller open subset $\mathbb{CP}^3 \setminus \mathscr{K}$. We can therefore apply Bishop's lemma [1, Lemma 9] in an attempt to extend A across the algebraic variety $B = \mathscr{K}$ of complex dimension k with $k = 2$. In order to do this we need to know that the $2k$-dimensional Hausdorff measure of $\overline{A} \cap \mathscr{K}$ is zero, where $\overline{A}$ is the closure of A in $\mathbb{CP}^3$. But $\overline{A} \cap \mathscr{K}$ is a subset of

$$(\mathscr{S} \cap \mathscr{K}) \cup \pi^{-1}(\gamma) \cup \pi^{-1}(\infty),$$

which is a union of real submanifolds of dimension less than and equal to 3.

The extension is therefore possible, and we conclude that $\overline{A}$ is complex analytic. Moreover, $\overline{A} = \overline{\mathscr{S}}$. Indeed, for all $p \in \overline{\mathscr{S}}$ and for all neighbourhoods U of p in $\mathbb{CP}^3$, the fact that $U \cap \mathscr{S} \neq \emptyset$ implies that $U \cap A = (U \cap \mathscr{S}) \setminus (\mathscr{S} \cap \mathscr{K}) \neq \emptyset$ because $\mathscr{S} \cap \mathscr{K}$ has complex dimension at most 1. Since $\overline{\mathscr{S}}$ is complex analytic, it follows from standard results (such as Mumford's treatment [19]) that $\overline{\mathscr{S}}$ is algebraic. However, it then has degree one since it meets a generic line in just one point, so J must be conformally constant and extend at least to $\mathbb{H}$ minus a point.

(2) Now suppose that J is an OCS defined on $O = \mathbb{H} \setminus (\gamma \cup \mathfrak{G})$, with the stated limiting property over $\mathfrak{G}$. The graph $\mathscr{S}$ of J is a complex submanifold of $\pi^{-1}(O)$. If $\mathscr{S} \cap \mathscr{K}$ has complex dimension 2 at some point then $\mathscr{S} \subset \mathscr{K}$, and $\mathscr{S}$ must coincide with one of the smooth branches of $\mathscr{K}$ over O, namely the graph of one of $\mathbb{J}^+, \mathbb{J}^-, -\mathbb{J}^+, -\mathbb{J}^-$. We can therefore assume that $\mathscr{S} \cap \mathscr{K}$ has complex dimension at most 1 at each point.

Define A as in (7.13). Then $\overline{A} \cap \mathscr{K}$, a subset of

$$(\pi^{-1}(\mathfrak{G}) \cap \mathscr{K}) \cup (\mathscr{S} \cap \mathscr{K}) \cup \pi^{-1}(\gamma) \cup \pi^{-1}(\infty),$$

again has 4-dimensional Hausdorff measure equal to zero. Indeed, $\pi^{-1}(\mathfrak{G}) \cap \mathscr{K}$ has real dimension 3 in view of Theorem 7.4. The fact that A is an analytic subset of $\mathbb{CP}^3 \setminus \mathscr{K}$ follows because $\mathscr{S}$ is analytic in

$$\mathbb{CP}^3 \setminus (\pi^{-1}(\mathfrak{G}) \cup \pi^{-1}(\gamma) \cup \pi^{-1}(\infty))$$

and $A = \mathscr{S} \setminus \mathscr{K}$ intersects neither $\pi^{-1}(\gamma) \cup \pi^{-1}(\infty)$ (as before) nor $\pi^{-1}(\mathfrak{G})$ (thanks to the hypothesis on the limit points). Applying Bishop's lemma, we deduce that $\overline{A} = \overline{\mathscr{S}}$ is analytic. Arguing as above, J must be conformally constant, but such a J cannot coincide with $\pm\mathbb{J}^{\pm}$ over $\mathfrak{G}$. □

To sum up, Proposition 7.3 and Theorem 7.4 imply that the paraboloid Γ is determined algebraically by the parabola γ. The vertex of Γ is the focus of γ, and the two points of $\mathscr{K}$ lying above this vertex are cusps. On the other hand, $\pi^{-1}(\Gamma)$ is not a complex submanifold of $\mathscr{K}$ because the 2-plane $N_f = f^{-1}(\Gamma)$ is not $\mathbb{J}$-holomorphic (see (6.1)).

The subsets γ, L_i and Γ of $\mathbb{H}$ (together with the point at infinity) constitute the discriminant locus of $\mathscr{K}$, and no OCS whose graph lies in $\mathscr{K}$ can be smoothly extended over γ and Γ. Theorem 7.5 tells us that such an OCS is characterised analytically by its limiting values over a 3-dimensional region, such as $\mathfrak{G}$, with boundary Γ.

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