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Zero-temperature 2D stochastic Ising model and anisotropic curve-shortening flow

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Abstract. Let $\mathcal{D}$ be a simply connected, smooth enough domain of $\mathbb{R}^2$. For $L > 0$ consider the continuous time, zero-temperature heat bath dynamics for the nearest-neighbor Ising model on $\mathbb{Z}^2$ with initial condition such that $\sigma_x = -1$ if $x \in L\mathcal{D}$ and $\sigma_x = +1$ otherwise. It is conjectured [23] that, in the diffusive limit where space is rescaled by L, time by L^2 and $L \to \infty$, the boundary of the droplet of "$-$" spins follows a *deterministic* anisotropic curve-shortening flow, where the normal velocity at a point of its boundary is given by the local curvature times an explicit function of the local slope. The behavior should be similar at finite temperature $T < T_c$, with a different temperature-dependent anisotropy function.

We prove this conjecture (at zero temperature) when $\mathcal{D}$ is convex. Existence and regularity of the solution of the deterministic curve-shortening flow is not obvious *a priori* and is part of our result. To our knowledge, this is the first proof of mean-curvature-type droplet shrinking for a model with genuine microscopic dynamics.

Keywords. Ising model, Glauber dynamics, curve-shortening flow

Contents

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1. Introduction

Consider a thermodynamic system with two coexisting phases and imagine to prepare it in an initial condition where a droplet of one phase is immersed in the other phase. If the system undergoes a dynamics that does not conserve the order parameter, it is well understood phenomenologically [20] that the droplet will shrink in order to decrease its surface tension until it eventually disappears, and that (roughly speaking) the normal speed at a point of its boundary will be proportional to the local mean curvature. Deriving such behavior from first principles, i.e. from a microscopic model undergoing a local (stochastic) dynamics, is a much harder task and this program was started only rather recently [23]. More precisely, what one expects is that, if the initial droplet is of diameter L, it will "disappear" within a time of order L^2 (this behavior is sometimes referred to as "Lifshitz law"). Moreover, in the "diffusive limit" where $L \to \infty$ and at the same time space is rescaled by L (so that the initial droplet is of size $O(1)$) and time is accelerated by L^2, the droplet evolution should become deterministic and follow some anisotropic version of a mean curvature flow. Anisotropy (i.e. the fact that the normal velocity will also depend on the local orientation of the droplet boundary) is expected when the underlying model is defined on a lattice, as will be the case for us.

Up to now, mathematical progress on this issue has been rather modest, the main difficulty being that it is not clear how to implement the idea that the fast modes related to relaxation inside the two pure phases should decouple from slow modes related to the interface motion, which are responsible for the diffusive L^2 time scaling.

A fairly well understood situation is when the interface can be described by a height function and the bulk structure of the two phases is disregarded. This is possible (by definition) for the so-called "effective interface models" or Ginzburg–Landau $\nabla\phi$ interface models: for models with continuous heights and strictly convex potential undergoing a Langevin-type dynamics, Funaki and Spohn [10] derived the full mean curvature motion in the diffusive scaling. Another well-studied case is that of models with Kac-type potentials: in this case, mean curvature motion can be proven to emerge [5, 6, 15] in a limit where interaction range is taken to infinity at some stage, but in this limit there is no sharp interface separating the phases and the system becomes very close to mean field.

As for true lattice models, results are much more scarce. For instance, for the two-dimensional nearest-neighbor Ising model below the critical temperature, the best known upper bound on the "disappearance time" for a droplet of "$-$ phase" immersed in the "$+$ phase" is of order $L^{c(T)\log L}$ [22], very far from the expected L^2 scaling. Recently, a weak version of the Lifshitz law was proven for the three-dimensional Ising model at zero temperature: the disappearance time of a "$-$" droplet is of order L^2 (upper and lower bounds), up to multiplicative logarithmic (in L) corrections [2]. When the dimension is higher than three (always at zero temperature), an *upper bound* for the disappearance time of order $L^2(\log L)^c$, for some constant c, was proven in [18].

In this work, we concentrate on the two-dimensional nearest-neighbor Ising model on the infinite square lattice. The dynamics takes a very simple form: each spin is updated with rate one and after the update it takes the same value as the majority of its neighbors, or the value ± 1 with equal probabilities if exactly two neighbors are "$+$"

and two are "−". In this case, the disappearance time of a large "−" droplet should be asymptotically given by half its volume (number of "−" spins). Moreover, in the diffusive scaling limit the droplet boundary should be given by a deterministic curve $\gamma(t)$ whose normal speed is given by the local (signed) curvature, times a function $a(\theta)$ where θ is the angle of the local normal vector. The function $a(\cdot)$ is explicitly known (see (2.4)). In this two-dimensional setting, it is more natural to refer to such flow as "(anisotropic) curve-shortening flow" (rather than "mean curvature flow").

Our main result (Theorem 2.2) is a proof of the curve-shortening conjecture (and, as a byproduct, of the Lifshitz law) when the initial droplet is convex.

There are some previous partial results available on this problem. The scaling limit of the evolution when initially spins are "−" in the first quadrant of $\mathbb{Z}^2$ (infinite corner) and "+" elsewhere is described in [16, Section 4.2] (with the language of exclusion processes rather than spin systems). This is a simple situation because the interface motion is mapped to symmetric simple exclusion and is described by the associated height function at all times. In [23], Spohn described the scaling limit of the interface motion in a situation that more or less corresponds to the zero-temperature Ising model in an infinite vertical cylinder, with an initial condition such that the interface separating "+" from "−" spins can be globally described by a height function at all times (in particular, this cannot describe a droplet, and implicitly he has to modify the dynamics to guarantee that droplets do not appear later in the evolution). In [4], Chayes et al. proved the Lifshitz law (but not the curve-shortening conjecture) for a modified dynamics where updates which break the droplet into several droplets are forbidden. In [3], Cerf and Louhichi computed the "drift at time 0" of the droplet (for the unmodified dynamics), but their result does not yield information on the evolution for finite time $t > 0$.

An important building block of our proof of the anisotropic curve-shortening conjecture is that, as was well understood by Spohn [23], locally the interface can be (roughly speaking) described by the hydrodynamic limit of a certain zero-range process at the points where the tangent to the boundary is horizontal or vertical, and by the hydrodynamic limit of the symmetric simple exclusion process elsewhere. However, such correspondence is not exact due to updates that split the droplet into more than one connected component (see for instance Figure 4). In other words, the interface is not (even locally) the graph of a function. Also, it is a non-trivial task to patch together the various pieces of "local analysis" to control globally the evolution of the droplet. Both problems will be tamed by a sequence of monotonicity arguments, which are allowed because the dynamics conserves the stochastic ordering among configurations.

In order to prove Theorem 2.2, we also need to know that a classical solution to the anisotropic curve-shortening flow exists up to the time where the droplet disappears, and (crucially) that the solution is sufficiently regular in space and time (i.e. that the curvature is a Lipschitz function of the angle and a continuous function of time). To our surprise, we found that the existing literature on curve-shortening flows does not provide global (in time) results for the flow associated to the zero-temperature 2D Ising model. The reason is that the anisotropy function $a(\cdot)$ is not smooth (its derivative has jumps, reflecting the singularities of the surface tension at zero temperature), while the existing results assume that $a(\cdot)$ is at least C^2 (cf. [11, 12]). To prove existence, uniqueness and regularity of the solutions (cf. Theorem 2.1), we will regularize the function $a(\cdot)$ and then analyze the regularized flow following the ideas of [11, 12]. Of course, it will be crucial to guarantee that all the estimates we need are uniform in the regularization parameter, which tends to zero in the end.

The case where the initial "−" droplet is non-convex will be considered in future work. The additional difficulties are two-fold. First of all, from the analytic point of view, available global existence and regularity results for the solution of curve-shortening flows with non-convex initial condition seem to be limited to the isotropic case where $a(\cdot) \equiv 1$ [14]. Secondly, due to the fact that the droplet will move at the same time outwards and inwards at different locations according to the sign of the curvature, various monotonicity arguments we use in the rest of the paper will not work.

2. Model and results

2.1. Glauber dynamics and expected limiting evolution

Set $\mathbb{Z}^* := \mathbb{Z} + 1/2 := \{x + 1/2 : x \in \mathbb{Z}\}$. We consider the zero-temperature stochastic Ising model on $(\mathbb{Z}^*)^2$ with its usual lattice structure (x and y are linked if $|x - y| = 1$ for the l_1 norm). This is a continuous time Markov chain $(\sigma(t))_{t\geq 0}$ on the space of spin configurations on $(\mathbb{Z}^*)^2$, $\Omega := \{-1, 1\}^{(\mathbb{Z}^*)^2}$. We write $\sigma(t) = (\sigma_x(t))_{x\in(\mathbb{Z}^*)^2}$ and for simplicity we write $\sigma_x = -$ (resp. $\sigma_x = +$) instead of $\sigma_x = -1$ (resp. $\sigma_x = +1$).

The transition rules are the following: for each site $x \in (\mathbb{Z}^*)^2$, the value σ_x of the spin at x is updated independently with rate one. When the spin at a site is updated, it takes the same value as the spin of the majority of its neighbors, or the values ± 1 with equal probabilities $1/2$ if two neighbors have "+" spins and the other two have "−" spins. That these rules yield a well-defined Markov chain even in infinite volume is a standard fact (cf. [21]). In what follows (cf. (2.1)), we will consider only initial conditions where the number of "−" spins is finite. It is easy to realize that the spins outside the smallest square containing all the initial "−" spins stay "+" forever, so that in reality we have a dynamics on a finite volume and the question of existence of the process is trivial.

We are interested in the evolution of the set of "−" spins for this Markov chain when the initial condition $\sigma(0)$ is a large droplet, i.e. a finite connected set of "−" spins surrounded by "+" spins. In that case, almost surely, after a finite time τ_+, all the "−" spins have turned into "+" and the dynamics will stay forever in the all "+" configuration (which is an absorbing state). Our aim is to describe the evolution of the shape of the rescaled "−" droplet on a proper (diffusive) time scale. In the next section we make that aim more precise.

We consider a compact, simply connected subset $\mathcal{D} \subset [-1, 1]^2$ whose boundary is a closed smooth curve. Given $L \in \mathbb{N}$ we consider the Markov chain described above with initial condition

$$\sigma_x(0) = \begin{cases} -1 & \text{if } x \in (\mathbb{Z}^*)^2 \cap L\mathcal{D}, \\ +1 & \text{otherwise.} \end{cases} \tag{2.1}$$

In order to see a set of "−" spins as a subset of $\mathbb{R}^2$, each vertex $x \in (\mathbb{Z}^*)^2$ may be identified with the closed square of side-length one centered at x,

$$\mathcal{C}_x := x + [-1/2, 1/2]^2. \tag{2.2}$$

One defines

$$\mathcal{A}_L(t) := \bigcup_{\{x:\sigma_x(t)=-1\}} \mathcal{C}_x, \tag{2.3}$$

which is the "− droplet" at time t for the dynamics. The boundary of $\mathcal{A}_L(t)$ is a union of edges of $\mathbb{Z}^2$ (this is the only reason why we defined the Ising model on $(\mathbb{Z}^*)^2$).

What was conjectured by Lifshitz [20] on heuristic grounds for the low temperature Ising model is that $\mathcal{A}_L(t)$ should follow an anisotropic curve-shortening motion: after rescaling space by L and time by L^2 and letting L tend to infinity, the motion of the interface between $\mathcal{A}_L(t)$ and its complement should be deterministic and the local drift of the interface should be proportional to the curvature, with an anisotropic correction to reflect anisotropy of the underlying lattice. More precisely, one can formulate this conjecture as follows [23]: Let $\gamma(t, L)$ denote the boundary of the (random) set $(1/L)\mathcal{A}_L(L^2t)$. Then, for $L \to \infty$, $\gamma(t, L)$ should converge to a deterministic curve $\gamma(t)$ and the evolution of $(\gamma(t))_{t\geq 0}$ should be such that the normal velocity at a point $x \in \gamma(t)$ is given by the curvature at x, times an anisotropic factor $a(\theta_x)$, where θ_x is the slope of the outwards directed normal to $\gamma(t)$ at x. The velocity is directed inwards at points where $\gamma(t)$ is convex and outwards at points where it is concave. The function $a(\cdot)$ should have the explicit expression

$$a(\theta) := \frac{1}{2(|\cos(\theta)| + |\sin(\theta)|)^2}. \tag{2.4}$$

In particular, the curve $\gamma(t)$ should shrink to a point in a finite time

$$t_0 = \frac{\text{Area}(\mathcal{D})}{\int_0^{2\pi} a(\theta)\, d\theta} = \frac{\text{Area}(\mathcal{D})}{2}.$$

Note that the function $a(\cdot)$ is symmetric around 0 and $\pi/2$-periodic, which reflects the discrete symmetries of the lattice $(\mathbb{Z}^*)^2$. It is important to note that $a(\cdot)$ is C^∞ except at $\theta = j\pi/2$, $j = 0, \ldots, 3$, where it is only continuous and its first derivative has a jump: indeed, $a(\theta) \sim 1/2 - |\theta - i\pi/2|$ for θ close to $i\pi/2$, $i = 0, \ldots, 3$.

2.2. *Results*

2.2.1. Convex initial droplet. We prove the anisotropic curve-shortening conjecture in the case where the initial droplet is convex and suitably smooth. Given a strictly convex smooth domain $\mathcal{D}$ in $\mathbb{R}^2$ and letting $\gamma = \partial\mathcal{D}$ be its boundary, we parameterize it following a standard convention of convex geometry (cf. e.g. [12] and Figure 1). For $\theta \in [0, 2\pi]$ let $v(\theta)$ be the unit vector forming an anticlockwise angle θ with the horizontal axis and let

$$h(\theta) = \sup\{x \cdot v(\theta) : x \in \gamma\}$$

with $\cdot$ the usual scalar product in $\mathbb{R}^2$.

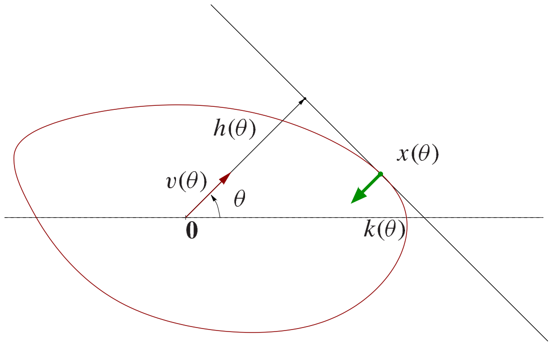

Fig. 1. A graphic description of the support function h. Given θ, consider the point $x(\theta)$ of γ that maximizes $x \cdot v(\theta)$ (it is unique if the curve is strictly convex). Then $h(\theta) = x(\theta) \cdot v(\theta)$, and $k(\theta)$ is the norm of the curvature vector of γ (bold vector) at $x(\theta)$. If the tangent to γ at x exists it is normal to $v(\theta)$ and $|h(\theta)|$ is the distance between the tangent and the origin. We emphasize that this construction works equally well when the origin is not inside γ.

Sometimes, we abusively say that γ is a convex curve if the domain $\mathcal{D}$ it encloses is convex, and we identify γ with $\mathcal{D}$.

The function $\theta \mapsto h(\theta)$ (called the *support function*) uniquely determines γ:

$$\mathcal{D} = \bigcap_{0 \le \theta \le 2\pi} \{x \in \mathbb{R}^2 : x \cdot v(\theta) \le h(\theta)\}.$$

With this parameterization, the anisotropic curve-shortening evolution reads

$$\begin{cases} \partial_t h(\theta, t) = -a(\theta) k(\theta, t), \\ h(\theta, 0) = h(\theta), \end{cases} \tag{2.5}$$

where, for a convex curve γ, $k(\theta) \ge 0$ is the curvature at the point $x(\theta) \in \gamma$ where the outward normal forms an anticlockwise angle θ with the horizontal axis and the t-derivative is taken at constant θ (see [12, Lemma 2.1] for a proof of (2.5)). Of course $h(\cdot)$ is the support function of $\partial\mathcal{D}$.

In general, even proving the existence of a solution of (2.5) with $a(\cdot)$ given in (2.4) is non-trivial, since $a(\cdot)$ has points of non-differentiability and the existing literature (e.g. [11, 12]) usually assumes that $a(\cdot)$ is at least C^2.

Our first result is

Theorem 2.1. *Let $\mathcal{D} \subset [-1, 1]^2$ be strictly convex and assume that its boundary $\gamma = \partial\mathcal{D}$ is a curve whose curvature $[0, 2\pi] \ni \theta \mapsto k(\theta)$ defines a positive, 2π-periodic, Lipschitz function. Then there exists a unique flow of convex curves $(\gamma(t))_t$ with curvature defined everywhere, such that $\gamma(0) = \gamma$ and that the corresponding support function $h(\theta, t)$ solves (2.5) for $t \ge 0$ and satisfies the correct initial condition $h(\theta, 0) = h(\theta)$. The curve $\gamma(t)$ shrinks to a point $\mathbf{x}_f \in \mathbb{R}^2$ at time $t_f = \mathrm{Area}(\mathcal{D})/2$. For $t < t_f$, $\gamma(t)$ is a smooth curve in the following sense: its curvature function $k(\cdot, t)$ is Lipschitz and bounded away from 0 and infinity on any compact subset of $[0, t_f)$.*

We let $\mathcal{D}(t)$ denote the convex closed set enclosed by $\gamma(t)$ (of course, $\mathcal{D}(0)=\mathcal{D}$). Also, we use the convention that $\mathcal{D}(t)=\{\mathbf{x}_f\}$ if $t\ge t_f$.

For $\delta>0$ let $B(x,\delta)$ denote the ball of radius δ centered at x, and for any compact set $\mathcal{C}\subset\mathbb{R}^2$ define

$$\mathcal{C}^{(\delta)}:=\bigcup_{x\in\mathcal{C}}B(x,\delta),\qquad \mathcal{C}^{(-\delta)}:=\Big(\bigcup_{x\notin\mathcal{C}}B(x,\delta)\Big)^c. \tag{2.6}$$

Note that $\mathcal{D}(t)^{(\delta)}=B(\mathbf{x}_f,\delta)$ and $\mathcal{D}(t)^{(-\delta)}=\emptyset$ if $t\ge t_f$.

An event B_L is said to occur *with high probability* (w.h.p.) if $\lim_{L\to\infty}P(B_L)=1$.

Theorem 2.2. *Under the same assumptions on $\mathcal{D}$ as in Theorem 2.1, for any $\delta>0$ one has w.h.p.*

$$\begin{aligned}&\mathcal{D}^{(-\delta)}(t)\subset\frac{1}{L}\mathcal{A}_L(L^2t)\subset\mathcal{D}^{(\delta)}(t) && \textit{for every } 0\le t\le t_f+\delta, \\ &\mathcal{A}_L(L^2t)=\emptyset && \textit{for every } t>t_f+\delta.\end{aligned} \tag{2.7}$$

In particular, one has the following convergence in probability:

$$\lim_{L\to\infty}\frac{\tau_+}{L^2\,\mathrm{Area}(\mathcal{D})}=\frac{1}{2}. \tag{2.8}$$

The reason why in Theorems 2.1 and 2.2 we do not content ourselves with, say, initial C^∞ curves is that, as we see in the next section, there is a very natural initial condition whose curvature function is only Lipschitz and not C^1 (and stays so at later times).

Theorem 2.2 does not apply directly if one considers $\mathcal{D}=[0,1]^2$ or any other non-smooth or non-strictly-convex convex set. However, approximating $\mathcal{D}$ from above and below by smooth compact sets and using monotonicity (cf. Section 2.3), one sees easily that (2.8) holds in any case. In particular, the disappearance time of an $L\times L$ square droplet is with high probability $(L^2/2)(1+o(1))$.

Theorems 2.2 and 2.3 tell us that for our choices of initial configuration, the disappearance time of the minus droplet is non-random at first order. This implies that the variation distance of our Markov chain from equilibrium (which is concentrated on the all-plus configuration) drops abruptly from 1 to 0 around time L^2t_f within a time window of width $o(L^2)\ll L^2t_f$ (we conjecture that the correct order of the window should be $O(L^{3/2})$). This is a particular instance of a phenomenon called cut-off (cf. [7] and [19]).

2.2.2. Scale-invariant droplet. A particular case of Theorem 2.2 is when the initial condition is scale invariant, i.e. the limiting evolution $(\gamma(t))_t$ is a homothetic contraction. Consider the function

$$f_0:[-1/\sqrt{2},1/\sqrt{2}]\ni x\mapsto f_0(x)=\beta\Big\{4\alpha x\int_0^x e^{2\alpha t^2}dt-e^{2\alpha x^2}\Big\}, \tag{2.9}$$

where α is the unique positive solution of

$$4\sqrt{2}\,\alpha e^{-\alpha}\int_0^{1/\sqrt{2}}e^{2\alpha t^2}dt=1$$

and

$$\beta = -\sqrt{2}\, e^{-\alpha} < 0.$$

Note that f_0 is C^∞, positive, concave, symmetric around 0 and increasing on $[-1/\sqrt{2}, 0]$. We denote by $(\mathbf{e}_1, \mathbf{e}_2)$ the canonical basis of $\mathbb{R}^2$ and $(\mathbf{f}_1, \mathbf{f}_2) = \left(\frac{\mathbf{e}_1-\mathbf{e}_2}{\sqrt{2}}, \frac{\mathbf{e}_1+\mathbf{e}_2}{\sqrt{2}}\right)$ the image of $(\mathbf{e}_1, \mathbf{e}_2)$ by the rotation of angle $-\pi/4$. We also define the curve γ_1 to be the graph of f_0 in the coordinate system $(\mathbf{f}_1, \mathbf{f}_2)$, i.e.

$$\gamma_1 := \{x\mathbf{f}_1 + f_0(x)\mathbf{f}_2 : x \in [-1/\sqrt{2}, 1/\sqrt{2}]\}.$$

If S_1 (resp. S_2) denotes the symmetry with respect to the axis $\mathbf{e}_1$ (resp. $\mathbf{e}_2$), one defines the closed curve γ by

$$\gamma = \gamma_1 \cup (S_1\gamma_1) \cup (S_2\gamma_1) \cup ((S_1 \circ S_2)\gamma_1).$$

In what follows, $\mathscr{D}$ denotes the compact, convex set enclosed by γ (see Figure 2).

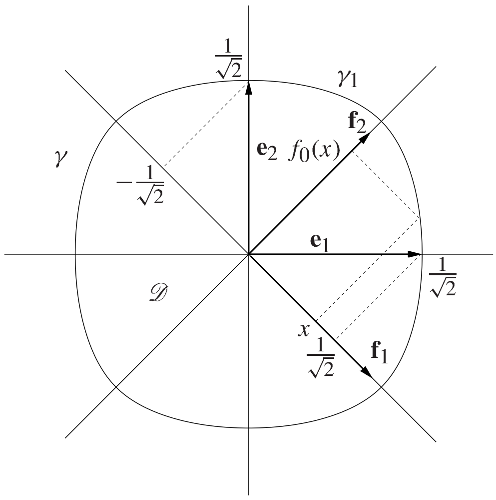

Fig. 2. The curve $\gamma = \partial\mathscr{D}$ and the coordinate systems $(\mathbf{e}_1, \mathbf{e}_2)$ and $(\mathbf{f}_1, \mathbf{f}_2)$.

One can check that the curvature function $\theta \mapsto k(\theta)$ of $\partial\mathscr{D}$ is Lipschitz and bounded away from zero, but not differentiable at $\theta = i\pi/2$, $i = 0, 1, 2, 3$. In this case, Theorem 2.2 can be formulated as follows.

Theorem 2.3. *Assume that $\mathcal{D} = \mathscr{D}$. For any $\eta > 0$, w.h.p.,*

$$(\sqrt{1-2\alpha t} - \eta)\mathscr{D} \subset \frac{1}{L}\mathcal{A}_L(tL^2) \subset (\sqrt{1-2\alpha t} + \eta)\mathscr{D} \quad \text{for every } t \geq 0 \tag{2.10}$$

where we follow the convention that $\sqrt{x} = 0$ for $x \leq 0$ and $x\mathscr{D} = \emptyset$ for $x < 0$. Moreover, one has the following convergence in probability:

$$\lim_{L\to\infty} \frac{\tau_+}{\text{Area}(L\mathscr{D})} = \alpha \lim_{L\to\infty} \frac{\tau_+}{L^2} = \frac{1}{2}. \tag{2.11}$$

It is easy to check, using Lemma 3.6 below and a couple of integrations by parts, that $\mathrm{Area}(\mathscr{D}) = 1/\alpha$, yielding the first equality in (2.11). The expression (2.9) for the invariant shape also appears, although with different notation, in the recent work [17].

2.3. *Graphical construction of the dynamics and monotonicity*

Before starting the proofs, we wish to give a construction of the Markov process (sometimes called the *graphical construction*) that yields nice monotonicity properties. We consider a family $(\tau^x)_{x\in(\mathbb{Z}^*)^2}$ of independent Poisson clock processes. More precisely, to each site $x \in (\mathbb{Z}^*)^2$ one associates a random sequence (independently from other sites) $(\tau_n^x)_{n\geq 0}$ of times such that $\tau_0^x = 0$ and $(\tau_{n+1}^x - \tau_n^x)_{n\geq 0}$ are IID exponential variables with mean one. One also defines random variables $(U_{n,x})_{n\geq 0}$, $x \in (\mathbb{Z}^*)^2$, that are IID Bernoulli variables of parameter $1/2$, with values ± 1.

Then given an initial configuration $\xi \in \{-1, 1\}^{(\mathbb{Z}^*)^2}$ one constructs the dynamics $\sigma^\xi(t)$ starting from $\sigma^\xi(0) = \xi$ as follows:

- $(\sigma_x(t))_{t\geq 0}$ is constant on all intervals of the type $[\tau_n^x, \tau_{n+1}^x)$.
- $\sigma_x(\tau_n^x)$ is chosen to be ± 1 if a strict majority of the neighbors of x satisfy $\sigma_y(\tau_n^x) = \pm 1$, and $U_{n,x}$ otherwise (this definition makes sense as, almost surely, two neighbors will not update at the same time).

This construction gives a simple way to define simultaneously the dynamics for all initial conditions (we denote by P the associated probability). Moreover this construction preserves the natural order on $\{-1, +1\}^{(\mathbb{Z}^*)^2}$, given by

$$\xi \geq \xi' \;\Leftrightarrow\; \xi_x \geq \xi'_x \text{ for every } x \in (\mathbb{Z}^*)^2$$

(this order is just the opposite of the inclusion order for the set of "−" spins, which is therefore also preserved). Indeed, if $\xi \geq \xi'$, with the above construction, one has P-a.s.

$$\forall t > 0, \qquad \sigma^\xi(t) \geq \sigma^{\xi'}(t).$$

3. Local interface dynamics

One problem one has to deal with when proving mean curvature motion for the whole droplet is that even though initially the interface between "+" and "−" (i.e. the geometric boundary of the set $\mathcal{A}_L(0)$) is a simple curve, it can later split to form several loops. In fact, as a byproduct of our results, we will see that, with large probability, only very small extra loops can be created. We will tackle this problem by introducing some auxiliary dynamics that do not allow creation of new loops and stochastically compare to the original one.

A second problem is that the interface that one has to control is not exactly the graph of a function, for which it would be easier to describe the macroscopic motion using partial differential equations. We begin by studying two dynamics for which the interface is indeed a graph, and which have locally the same large-scale behavior as the true evolution. It is more natural to introduce these dynamics as dynamics on interfaces rather than dynamics on spins. Our task will then consist in glueing together the "local results" of Theorems 3.2 and 3.4 to get Theorems 2.2 and 2.3.

3.1. Local dynamics away from the poles and the simple exclusion process

The first auxiliary dynamics is used to control the evolution of the boundary of $(1/L)\mathcal{A}_L(tL^2)$ away from the points (the *poles*) where the tangent to the deterministic curve $\gamma(t)$ is either horizontal or vertical. The evolution near the poles will be analyzed via a second auxiliary dynamics (see Section 3.2).

Given two positive natural numbers M, N consider the state space $\Omega_{M,N}$ of nearest-neighbor directed paths of length $L := M + N$ with M steps up and N steps down:

$$\Omega_{M,N} = \left\{(h_x)_{x\in\{0,\dots,M+N\}} \in \mathbb{Z}^{M+N+1} \;\middle|\; |h_{x+1} - h_x| = 1, h_0 = 0; h_{M+N} = M - N\right\}.$$

Given $h \in \Omega_{M,N}$ and $x \in \{1, \dots, L-1\}$, we denote by $h^{(x)}$ the path with a corner "flipped" at x defined by $h^{(x)}_y = h_y$ for all $y \neq x$ and

$$h^{(x)}_x := \begin{cases} h_x - 2 & \text{if } h_{x\pm 1} = h_x - 1, \\ h_x + 2 & \text{if } h_{x\pm 1} = h_x + 1, \\ h_x & \text{if } |h_{x+1} - h_{x-1}| = 2. \end{cases} \tag{3.1}$$

The dynamics on $\Omega_{M,N}$ we consider is the one that flips every corner with rate $1/2$. More precisely, it is the Markov chain whose generator $\mathcal{L}$ is defined as

$$\mathcal{L}f(h) := \frac{1}{2}\sum_{x=1}^{L-1}(f(h^{(x)}) - f(h)), \qquad \forall f : \Omega_{M,N} \to \mathbb{R}. \tag{3.2}$$

We denote by $(h(t))_{t\geq 0}$ the trajectory of the Markov chain started from the initial condition $h(0) := h^0 \in \Omega_{M,N}$.

Remark 3.1. Note that this dynamics is in one-to-one correspondence with the Ising dynamics on a rectangle $N \times M$ with "+" boundary conditions on two adjacent sides and "−" boundary conditions on the two opposite sides, provided that the initial configuration is such that the length of the $-/+$ boundary is $M + N$ (i.e. minimal possible). More precisely (see Figure 3) the correspondence is obtained by taking the graph of h, rotating

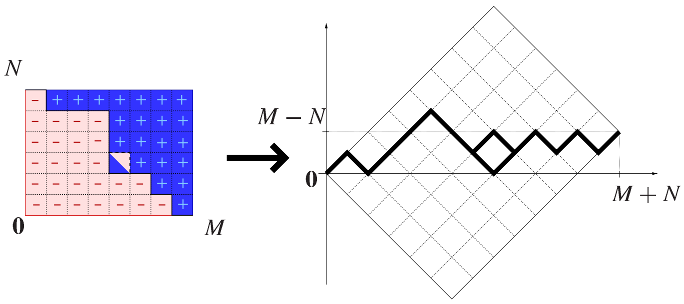

Fig. 3. One-to-one correspondence between the dynamics in a rectangle with mixed boundary conditions and the corner-flip dynamics on paths. A possible spin update together with the equivalent corner-flip are represented.

it by $\pi/4$ and rescaling space by a factor of $\sqrt{2}$ (so that squares have side-length one on the left-hand side picture). Note that we are implicitly identifying an element $h \in \Omega_{M,N}$ with a continuous function $F : [0, M+N] \to \mathbb{R}$ such that $F(x) = h_x$ for $x = 0, 1, \dots, M+N$ and $F(\cdot)$ is affine on intervals $(n, n+1)$ with integer n.

This corner-flip dynamics has been widely studied (see e.g. [24]) and can be mapped to the symmetric simple exclusion process (SSEP) on a finite interval (just say that there is a particle at $x = 0, \dots, M+N-1$ if and only if $h_{x+1} - h_x = +1$, and check that the dynamics in terms of particles coincides with that of SSEP). From hydrodynamic-limit results, it is quite clear that the rescaled version of h when M, N tends to infinity should satisfy the heat equation (see [16, Section 4.2] for an account of hydrodynamic equations for the exclusion process). However, we have not found in the literature a proof of the following precise statement we need (we give a concise proof of it in Section 7):

Theorem 3.2. *Given a* 1*-Lipschitz function* $\phi^0 : [0,1] \to \mathbb{R}$ *with* $\phi^0(0) = 0$*, let* $(h(t))_{t\geq 0}$ *be the dynamics starting from the initial condition* $h^0 \in \Omega_{M_L,N_L}$ *given by*

$$h^0_x := \begin{cases} 2\lfloor L\phi^0(x/L)/2\rfloor & \text{for } x \text{ even},\\ 2\lfloor (L\phi^0(x/L)-1)/2\rfloor + 1 & \text{for } x \text{ odd} \end{cases}$$

(M_L and N_L are implicitly fixed by L and $\phi^0(1)$). For all $T \geq 0$ and $\varepsilon > 0$, w.h.p.,

$$\sup_{t\in[0,T],\, x\in[0,1]} \frac{1}{L}|h_{\lfloor xL\rfloor}(L^2 t) - L\phi(x,t)| \leq \varepsilon$$

where $\phi : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ is the solution of the Cauchy problem

$$\begin{cases} \partial_t \phi(x,t) = \frac{1}{2}\partial_x^2 \phi(x,t), & \forall t > 0,\ \forall x \in (0,1),\\ \phi(0,t) = 0, \quad \phi(1,t) = \phi^0(1), & \forall t > 0,\\ \phi(x,0) = \phi^0(x), & \forall x \in (0,1). \end{cases} \tag{3.3}$$

Here, $\lfloor x\rfloor$ denotes the integer part of x, and the fact that h^0 does belong to Ω_{M_L,N_L} is an easy consequence of ϕ^0 being 1-Lipschitz.

3.2. Local dynamics around the poles and a zero-range process

For the definition of the second auxiliary dynamics, we use the same notation as in the previous section, but no confusion should arise as the proofs will be given in two independent sections. The state space is

$$\Omega_L := \{h : \{-L, \dots, L+1\} \to \mathbb{Z}\}. \tag{3.4}$$

For $h \in \Omega_L$ and $x \in \{-L+1, \dots, L\}$ define $h^{+,x}$ (resp. $h^{-,x}$) as the configuration such that $h^{+,x}_y = h_y$ if $y \neq x$ and $h^{+,x}_x = h_x + 1$ (resp. $h^{-,x}_x = h_x - 1$). We consider the Markov chain $(h(t))_{t\geq 0}$ started from some $h^0 \in \Omega_L$ and with generator $\mathcal{L}$ defined by

$$\mathcal{L}f(h) = \frac{1}{2}\sum_{x=-L+1}^{L} [c^{+,x}(h)(f(h^{+,x}) - f(h)) + c^{-,x}(h)(f(h^{-,x}) - f(h))] \tag{3.5}$$

where

$$c^{+,x}(h) = \mathbf{1}_{\{h_{x+1}>h_x\}} + \mathbf{1}_{\{h_{x-1}>h_x\}},$$
$$c^{-,x}(h) = \mathbf{1}_{\{h_{x+1}<h_x\}} + \mathbf{1}_{\{h_{x-1}<h_x\}}.$$

Note that the values h_{-L} and h_{L+1} are fixed in time and should be considered as boundary conditions.

Remark 3.3. This dynamics corresponds to the motion of the interface for a *modified Ising dynamics* in a vertical strip of width $2L$ with the following boundary condition: spins on the left (resp. right) boundary of the system are "+" if and only if their vertical coordinate is larger than h_{-L} (resp. h_{L+1}). The dynamics is modified in the sense that updates are discarded if after the update the boundary between the "−" and "+" domain is not a simple (open) curve (see Figure 4). It is at times more convenient to identify $h \in \Omega_L$ with a càdlàg function $H : [-L-1/2, L+3/2] \to \mathbb{Z}$ which equals identically h_n on intervals $[n-1/2, n+1/2)$ for integer n.

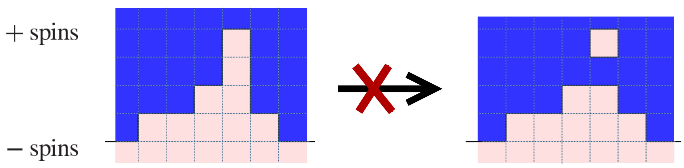

Fig. 4. An example of spin update that splits the interface into two disconnected components. The interface dynamics presented in this section does not allow this kind of move.

Another way to interpret this dynamics [23] is to look at the gradients $\eta_x = h_{x+1} - h_x$: one recognizes then a zero-range process with two type of particles (if $\eta_x = n > 0$ we say there are n particles of type A at x, if $\eta_x = -n < 0$ we say there are n type-B particles). Each particle performs a symmetric simple random walk with jump rate $1/(2n)$ (with n the occupation number of the site where the particle sits) to either left or right, and particles of different type annihilate instantaneously when they are at the same site. See Figure 5.

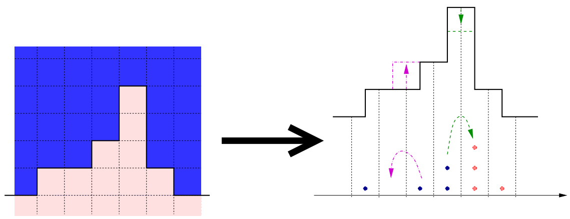

Fig. 5. Correspondence between interface dynamics and zero-range process. Arrows represent possible motions for the interface and their representation in terms of particle moves. When an A particle jumps on a B particle (green arrow), both annihilate.

In [23, Appendix A], this dynamics was considered, but in a periodized setup. A scaling limit result was given but the proof there is somewhat sketchy. Here we adapt the proof to the non-periodic case and write it in full detail.

Consider a C^2 function $\phi^0 : [-1,1] \to \mathbb{R}$ with $\phi^0(1) = \phi^0(-1) = 0$. We further assume that ϕ^0 has a finite number of changes of monotonicity. Define $\Phi_0 : \{-L, \dots, L+1\} \to \mathbb{R}$ as

$$\Phi_0(x) := L\phi^0(x/L) \tag{3.6}$$

and $h^0 : \{-L, \dots, L+1\} \to \mathbb{Z}$ by

$$h^0_x := \lfloor \Phi_0(x) \rfloor. \tag{3.7}$$

We define $\Phi : \{-L, \dots, L+1\} \times \mathbb{R}_+ \to \mathbb{R}$ as the solution of the following Cauchy problem:

$$\begin{cases} \partial_t \Phi(x,t) = \frac{1}{2}[\sigma(q_x(t)) - \sigma(q_{x-1}(t))], \\ \Phi(L+1,t) = \Phi(-L,t) = 0, \\ \Phi(x,0) = \Phi_0(x), \end{cases} \tag{3.8}$$

for every $t \geq 0$ and $x \in \{-L, \dots, L+1\}$, where $\sigma(u) = u/(1+|u|)$ and

$$q_x(t) := \Phi(x+1,t) - \Phi(x,t).$$

The result we state now is slightly weaker than Theorem 3.2 as it allows one to control the profile h only at a fixed time and not on a whole time interval.

Theorem 3.4. *Given ϕ^0 as above, consider the dynamics $(h(t))_{t\geq 0}$ described by (3.5) with initial condition h^0 as in (3.7). Then for any t, the following convergence in probability holds:*

$$\lim_{L\to\infty} \max_{x\in\{-L,\dots,L+1\}} \frac{1}{L} |h_x(L^2 t) - \Phi(x, L^2 t)| = 0. \tag{3.9}$$

It is quite intuitive that one should have $\frac{1}{L}\Phi(\lfloor Lx \rfloor, L^2 t) \to \phi(x,t)$ for any $x \in [-1,1]$, where $\phi : [-1,1] \times \mathbb{R}_+ \to \mathbb{R}$ is the solution of

$$\begin{cases} \partial_t \phi(x,t) = \dfrac{1}{2} \dfrac{\partial_x^2 \phi(x,t)}{(1+|\partial_x \phi(x,t)|)^2}, \\ \phi(1,t) = \phi(-1,t) = 0, \\ \phi(x,0) = \phi^0(x), \end{cases}$$

for $t \geq 0$ and $x \in (-1,1)$. The particular form of the non-linearity of this PDE makes the convergence question non-trivial, but fortunately Theorem 3.4 together with a comparison with the heat equation (cf. Section A.5) turns out to be sufficient for our purposes. Indeed, define $\bar\phi : [-1,1] \times \mathbb{R}_+ \to \mathbb{R}$ to be the solution of

$$\begin{cases} \partial_t \bar\phi(x,t) = \frac{1}{2}\partial_x^2 \bar\phi(x,t), \\ \bar\phi(1,t) = \bar\phi(-1,t) = 0, \\ \bar\phi(x,0) = \phi^0(x). \end{cases}$$

Corollary 3.5. *Let ϕ^0 be as above, and assume further that it is concave with $\|\partial_x \phi^0\|_\infty \leq \eta$. For every $t \geq 0$ and every $\varepsilon > 0$ the following inequality holds w.h.p.:*

$$\bar{\phi}(x/L, t) - \varepsilon \leq \frac{1}{L} h_x(L^2 t) \leq \bar{\phi}(x/L, (1+\eta)^{-2} t) + \varepsilon \quad \textit{for every } x \in \{-L, \dots, L+1\}.$$

Proof. The result follows by combining Theorem 3.4, Proposition A.9, and by taking limits of rescaled versions of Φ_1 and Φ_2 in (A.12) when L tends to infinity (cf. Lemma 7.1). □

3.3. About the scale-invariant shape

Now that we know how the interface should evolve locally (from Theorems 3.2 and 3.4), it is possible to explain why $\mathscr{D}$ should be scale invariant. By symmetries of the problem and the fact that motion is driven by curvature, the scale-invariant shape should be convex symmetric around the axes $\mathbb{R}\mathbf{e}_1$, $\mathbb{R}\mathbf{e}_2$. Therefore it is enough to consider the boundary of the intersection of $\mathscr{D}$ with the first quadrant.

From Theorem 3.2, if f is a Lipschitz function and $\partial\mathscr{D}$ is the graph of f in the coordinate system $(\mathbf{f}_1, \mathbf{f}_2)$, the initial drift in the $\mathbf{f}_2$ direction is $(1/4)\partial_x^2 f$, where the factor $1/4$ (instead of $1/2$) is due to the fact that in the correspondence between Ising dynamics and dynamics of nearest-neighboring paths, space has to be rescaled by $\sqrt{2}$, cf. Remark 3.1). One the other hand, the homothetic contraction of a shape $\mathcal{D}$ of initial velocity α gives an initial drift of the interface in the $\mathbf{f}_2$ direction

$$\alpha(-f + x\partial_x f).$$

That leads to the partial differential equation

$$\partial_x^2 f = 4\alpha(-f + x\partial_x f). \tag{3.10}$$

Next we impose the correct boundary conditions on f:

- We fix the scaling by imposing that the point $(1, 0)$ (and therefore also $(0, 1)$, $(-1, 0)$, $(0, -1)$) belongs to $\partial\mathscr{D}$. This gives

$$f(\pm 1/\sqrt{2}) = 1/\sqrt{2}. \tag{3.11}$$

- To guarantee that the curvature of $\partial\mathscr{D}$ is well defined at the point $(0, 1)$ we have to impose

$$\partial_x f(-1/\sqrt{2}) = -\partial_x f(1/\sqrt{2}) = 1. \tag{3.12}$$

We finally notice

Lemma 3.6. *The function f_0 defined in (2.9) is the unique solution of the Cauchy problem (3.10)–(3.12) for $x \in (-1/\sqrt{2}, +1/\sqrt{2})$. For other values of α the above problem has no solution.*

Proof. Uniqueness of the solution is standard from the theory of ordinary differential equations. The rest is just a matter of checking. □

3.4. *Organization of the paper*

Instead of proving directly Theorem 2.2 and then deducing Theorem 2.3 as a corollary, we decided for pedagogical reasons to give first the proof in the case of the scale-invariant droplet and then to point out what needs to be modified in the more general case of a convex droplet. The reason is that, this way, we can easily separate the question of comparing the stochastic evolution with the deterministic one (which works more or less the same in the two cases but is simpler for the invariant droplet, due to its symmetries) from the analytic, PDE-type issues which appear only in the general case.

The paper is therefore organized as follows:

- In Section 4, we show that to prove Theorem 2.3 it is sufficient to have a good control on the continuity of the interface motion (Proposition 4.2) and a result on the evolution after an "infinitesimal time" εL^2 (Proposition 4.1). Such crucial results are proven in Sections 4.3 and 4.4.
- In Section 5 we first prove Theorem 2.1 on the existence of a solution to (2.5), and then we prove Theorem 2.2 via a suitable generalization of Propositions 4.2 and 4.1.
- Finally, the hydrodynamic limit results of Theorems 3.2 and 3.4 are proven in detail in Section 7 and Appendix A respectively.

4. Proof of Theorem 2.3: evolution of the scale-invariant droplet

4.1. *Reducing to an "infinitesimal" time interval*

We decompose the proof of Theorem 2.3 into two propositions. The first (and the main one) says that after a time εL^2 the droplet looks very much the same but contracted by a factor of $1 - \alpha\varepsilon + o(\varepsilon)$.

Proposition 4.1. *For all $\delta > 0$ there exists $\varepsilon_0(\delta) > 0$ such that for all $0 < \varepsilon < \varepsilon_0(\delta)$, w.h.p.,*

$$\mathcal{A}_L(L^2\varepsilon) \subset (1 - \varepsilon(\alpha - \delta))L\mathscr{D}, \tag{4.1}$$

$$\mathcal{A}_L(L^2\varepsilon) \supset (1 - \varepsilon(\alpha + \delta))L\mathscr{D}. \tag{4.2}$$

The second proposition controls continuity in time of the rescaled motion:

Proposition 4.2. *For every $\delta > 0$, w.h.p.,*

$$\mathcal{A}_L(L^2 t) \subset (1 + \delta)L\mathscr{D} \quad \textit{for every } t \geq 0. \tag{4.3}$$

Moreover, for every $\delta > 0$ there exists $\varepsilon > 0$ such that w.h.p.,

$$\mathcal{A}_L(L^2 t) \supset (1 - \delta)L\mathscr{D} \quad \textit{for every } t \in [0, \varepsilon]. \tag{4.4}$$

Proof of Theorem 2.3 assuming Propositions 4.1 and 4.2. Given η fix δ small enough and $\varepsilon < \varepsilon_0(\delta)$. Then using (4.1) one finds that w.h.p.,

$$\mathcal{A}_L(L^2\varepsilon) \subset (1-(\alpha-\delta)\varepsilon)L\mathscr{D}. \tag{4.5}$$

Let $(\mathcal{A}_L^{(1)}(L^2t))_{t\geq 0}$ denote the evolution of the set of "−" spins for the dynamics started from initial condition "−" on $(1-\varepsilon(\alpha-\delta))L\mathscr{D}$ and "+" elsewhere. Then using the Markov property and monotonicity of the dynamics, one can couple the dynamics $(\mathcal{A}_L(L^2(\varepsilon+t)))_{t\geq 0}$ and $(\mathcal{A}_L^{(1)}(L^2t))_{t\geq 0}$ in such a way that on the event (4.5),

$$\mathcal{A}_L(L^2(\varepsilon+t)) \subset \mathcal{A}_L^{(1)}(L^2t) \quad \text{for every } t \geq 0.$$

Therefore, after conditioning on the event in (4.5) and using (4.1) for $(1-(\alpha-\delta)\varepsilon)L$ instead of L, one deduces that w.h.p.,

$$\mathcal{A}_L(L^2\varepsilon(1+(1-(\alpha-\delta)\varepsilon)^2)) \subset \mathcal{A}_L^{(1)}(L^2(1-(\alpha-\delta)\varepsilon)^2\varepsilon) \subset (1-(\alpha-\delta)\varepsilon)^2L\mathscr{D}.$$

Here we have used the fact that $\mathcal{A}_L^{(1)}(t)$ has the same law as $\mathcal{A}_{L(1-\alpha\varepsilon)}(t)$. Using this argument repeatedly one concludes that, w.h.p., for all $k \in [1, \varepsilon^{-3/2}]$,

$$\mathcal{A}_L(L^2t_k) \subset (1-(\alpha-\delta)\varepsilon)^kL\mathscr{D}$$

where t_k is defined by

$$t_k := \varepsilon\sum_{i=0}^{k-1}(1-(\alpha-\delta)\varepsilon)^{2i} = \varepsilon\frac{1-(1-(\alpha-\delta)\varepsilon)^{2k}}{1-(1-(\alpha-\delta)\varepsilon)^2}.$$

Here and below we assume $\varepsilon^{-3/2}$ to be in $\mathbb{N}$. The value $\varepsilon^{-3/2}$ could equally well be replaced by any number k_f much larger than $1/\varepsilon$; the only thing that matters is that t_{k_f} is close to $1/(2(\alpha-\delta))$. One remarks that for all values of k,

$$(1-(\alpha-\delta)\varepsilon)^k = \sqrt{1-\frac{t_k(1-(1-(\alpha-\delta)\varepsilon)^2)}{\varepsilon}} = \sqrt{1-2(\alpha-\delta)t_k+t_kO(\varepsilon)}.$$

As $(t_k)_{k\geq 0}$ is bounded above, there exists $C>0$ such that for every $k \in [1, \varepsilon^{-3/2}]$,

$$\mathcal{A}_L(L^2t_k) \subset (\sqrt{1-2(\alpha-\delta)t_k+C\varepsilon})L\mathscr{D} \subset (\sqrt{1-2\alpha t_k}+\eta/2)L\mathscr{D} \tag{4.6}$$

w.h.p., where the second inclusion holds provided that ε and δ are small enough. Combining (4.6), Proposition 4.2 and stochastic coupling, one finds that w.h.p., for every $k \in [0, \varepsilon^{-3/2}]$ and $t \in (t_k, t_{k+1})$,

$$\mathcal{A}_L(L^2t) \subset (\sqrt{1-2\alpha t_k}+3\eta/4)L\mathscr{D} \subset (\sqrt{1-2\alpha t}+\eta)L\mathscr{D},$$

and w.h.p., for every $t \geq t_{\varepsilon^{-3/2}}$,

$$\mathcal{A}_L(L^2t) \subset (\sqrt{1-2\alpha t_{\varepsilon^{-3/2}}}+3\eta/4)L\mathscr{D} \subset \eta L\mathscr{D}. \tag{4.7}$$

This ends the proof of the upper inclusion in (2.10) (note that $t_{\varepsilon^{-3/2}}$ approaches $1/(2\alpha)$ for ε, δ small). Moreover, (4.7) and stochastic domination imply that, for some constant C, w.h.p.,

$$\tau_+ \le \frac{L^2}{2\alpha}(1 + C\eta^2).$$

Indeed, it is known from [9] that a droplet of minus spins of linear size ηL disappears within a time τ_+ which w.h.p. is upper bounded by $C\eta^2 L^2$.

The lower inclusion in (2.10) and the lower bound on τ_+ are proved in an analogous way using (4.2) instead of (4.1) and (4.4) instead of (4.3). Note that using (4.4) we have to take care to choose ε small enough, but this is possible as $t_k - t_{k-1}$ is a non-increasing function of k and $1/\varepsilon$. □

4.2. *Strategy of the proof of Proposition 4.1*

Our aim is to use Theorem 3.2 to control the motion of the interface away from the "poles", and Theorem 3.4 (or more precisely Corollary 3.5) to control the motion of the interface close to the "poles". It is therefore crucial to compare the local SSEP or the zero-range dynamics introduced in Sections 3.1 and 3.2 with the true evolution of the boundary between "+" and "−" spins.

As we have already discussed at the beginning of Section 3, however, there exists no exact mapping between the evolution of the height function associated to the two particle processes and the evolution of the $+/-$ boundary, since the original "−" droplet can break into more droplets and, strictly speaking, the interface cannot be described, even locally, as a height function. The way out is that, thanks to monotonicity arguments and to the *a priori* "continuity" information provided by Proposition 4.2, we can remove certain updates of the Markov chain, e.g. freeze certain spins to their initial value. This way, we can show that locally the interface can be stochastically compared to the height function associated to the SSEP (or to the zero-range process close to the poles). Of course, the details of the "update removal procedure" are quite different according to whether we want to prove an upper or a lower bound on the "− domain". For instance, if we want an upper bound we are allowed to freeze "−" spins or to change some "+" into "−" spins in the initial condition (this is fine thanks to monotonicity), and at the same time we can freeze the spins outside $(1+\delta)L\mathscr{D}$ to "+" (this is not allowed directly by monotonicity, but (4.3) guarantees that such spins stay "+" for all time anyway, w.h.p.). If the "update removal procedure" is performed suitably, the effect is that the various portions of the $+/-$ interface (away from and close to the poles) then become independent and evolve *exactly* like the height functions of the SSEP/zero-range process.

The approach outlined here will also be used in Section 6 in the case of a general convex initial condition (the generalization of Proposition 4.1 is Proposition 6.2).

4.3. *Upper bound: proof of (4.1) and (4.3)*

The inclusion (4.1) can be rewritten in the following manner, which is more convenient for the proof: for any positive δ and all ε small enough, w.h.p.,

$$\sigma_x(\varepsilon L^2) = + \quad \text{for every } x \in [(1 - \varepsilon(\alpha - \delta))L\mathscr{D}]^c. \tag{4.8}$$

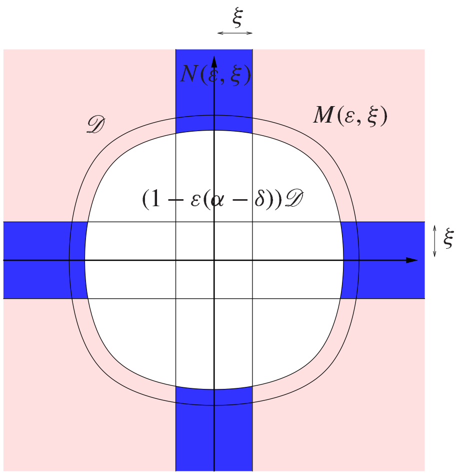

Fig. 6. The light-colored (resp. dark-colored) zones correspond to $M(\varepsilon, \xi)$ (resp. $N(\varepsilon, \xi)$) and its rotations. Together, they form a partition of the complement of $(1-\varepsilon(\alpha-\delta))\mathscr{D}$ (white central region).

Given δ, we fix a value of ξ which is small enough (depending on δ in a way that is specified in Section 4.3.2) and set (cf. Figure 6)

$$\begin{aligned} M(\varepsilon, \xi) &:= \{(x, y) \in \mathbb{R}^2 : x \geq \xi \text{ and } y \geq \xi\} \setminus [(1-\varepsilon(\alpha-\delta))\mathscr{D}], \\ N(\varepsilon, \xi) &:= \{(x, y) \in \mathbb{R}^2 : y \geq 0 \text{ and } -\xi \leq x \leq \xi\} \setminus [(1-\varepsilon(\alpha-\delta))\mathscr{D}]. \end{aligned} \tag{4.9}$$

Note that for any $\varepsilon > 0$, M, N and their successive images by rotation of angle $\pi/2$, π, $3\pi/2$ form an 8-piece cover of the complementary set $[(1-\varepsilon(\alpha-\delta))\mathscr{D}]^c$.

As the dynamics and the initial shape are invariant under these same rotations, (4.8) is proved if we can show that for ε small enough, w.h.p.,

$$\sigma_x(\varepsilon L^2) = + \quad \text{for every } x \in LM(\varepsilon, \xi), \tag{4.10}$$

$$\sigma_x(\varepsilon L^2) = + \quad \text{for every } x \in LN(\varepsilon, \xi). \tag{4.11}$$

The above new formulation of (4.1) is very convenient as it allows one to consider separately the dynamics close to the poles and away from them.

4.3.1. Proof of (4.10). For any $L > 0$, we consider the dynamics which has initial condition with "$-$" spins in $L\mathscr{D}$ and "$+$" otherwise, and the same generator as the original dynamics except that spins on the sites in $V_1 := \{\pm 1/2\} \times \{-L+1/2, \dots, L-1/2\}$ and on $V_2 := \{-L+1/2, \dots, L-1/2\} \times \{\pm 1/2\}$ are "frozen to $-$". (The construction of the dynamics is the same as in Section 2.3, except that there is no update for these sites.) We denote by $(\sigma^{(1)}_L(t))_{t\geq 0}$ the evolution of this dynamics and define

$$\mathcal{A}^{(1)}_L(t) := \bigcup_{\{x:\, \sigma^{(1)}_x(t) = -1\}} \mathcal{C}_x. \tag{4.12}$$

The graphical construction of Section 2.3 gives a natural coupling of σ and $\sigma^{(1)}$:

$$\mathcal{A}_L(t) \subset \mathcal{A}_L^{(1)}(t) \quad \text{for every } t \geq 0. \tag{4.13}$$

The advantage of the "freezing procedure" is that the evolution in the four quadrants of $(\mathbb{Z}^*)^2$ then becomes independent. The reason is that the spins on sites $(\{\pm 1/2\} \times \mathbb{Z}^*) \setminus V_1$ and $(\mathbb{Z}^* \times \{\pm 1/2\}) \setminus V_2$ are "+" for all time (recall that the spins outside the smallest square containing the initial "−" droplet stay "+" forever), so the boundary spins of all four quadrants are frozen.

The set $\mathcal{A}_L^{(1)}(t) \cap \mathbb{R}_+^2$ is a Young diagram (i.e. a collection of vertical columns of width 1 and non-negative integer heights, with heights non-increasing from left to right) for all $t \geq 0$ and we can thus consider $\partial \mathcal{A}_L^{(1)}(t) \cap \mathbb{R}_+^2$ as the graph of a (random) piecewise affine function in the coordinate system $(\mathbf{f}_1, \mathbf{f}_2)$, which we denote by $F_L(\cdot, t)$. Equation (4.10) is thus proved (for any choice of ξ) if one proves that for any $\nu < 2^{-1/2}$ and any ε small enough, w.h.p.,

$$F_L(x, \varepsilon L^2) \leq L f(x/L, (\alpha - \delta)\varepsilon) \quad \text{for every } x \in (-\nu L, \nu L), \tag{4.14}$$

where $f(\cdot, t)$ is the function whose graph in the coordinate system $(\mathbf{f}_1, \mathbf{f}_2)$ is given by the intersection of the boundary of $(1-t)\mathscr{D}$ with the half-plane $\{(x, y) \in \mathbb{R}^2 : (x, y) \cdot \mathbf{f}_2 \geq 0\}$ (the domain of definition of $f(\cdot, t)$ depends on t but includes $[-2^{-1/2}, 2^{-1/2}]$ for t small enough). By definition of $\mathscr{D}$, one has $f(\cdot, 0) = f_0(\cdot)$ (recall the definition of f_0 in (2.9)).

In practice, to prove (4.10) one has to prove (4.14) with ν such that $1/\sqrt{2} - \nu = \xi/\sqrt{2} + o(\xi)$ for ξ small (with ξ as in (4.9)). The reason is that the point of $\partial\mathscr{D}$ with horizontal coordinate ξ and positive vertical coordinate (in the coordinate system $(\mathbf{e}_1, \mathbf{e}_2)$) has horizontal coordinate $-(1-\xi)/\sqrt{2} + o(\xi)$ in the $(\mathbf{f}_1, \mathbf{f}_2)$ coordinate system.

As explained in Remark 3.1 and in Figure 3, the function $F_L(\cdot, t)$, up to space rescaling (by a factor of $\sqrt{2}$) undergoes the corner-flip dynamics of Theorem 3.2. Thus the scaling limit of F_L satisfies the heat equation, or more precisely we have the following convergence in probability for every fixed $T > 0$:

$$\lim_{L \to \infty} \sup_{x \in [-1/\sqrt{2}, 1/\sqrt{2}]} \sup_{t \leq T} \left| \frac{1}{L} F_L(xL, tL^2) - g(x, t) \right| = 0 \tag{4.15}$$

where g is the solution for $t \geq 0$ and $x \in (-1/\sqrt{2}, 1/\sqrt{2})$ of

$$\begin{cases} \partial_t g(x, t) = \frac{1}{4} \partial_x^2 g(x, t), \\ g(\cdot, 0) = f_0(\cdot), \\ g(-1/\sqrt{2}, t) = g(1/\sqrt{2}, t) = 1/\sqrt{2}. \end{cases} \tag{4.16}$$

Note that the above result plus (4.13), plus the fact that g is decreasing in t (since it stays concave through time) gives (4.3) of Proposition 4.2 for every $t \leq T < \infty$. Moreover, according to [9, Theorem 1.3], the disappearance time τ_+ of the minus-droplet is $O(L^2)$ with high probability, so that (4.3) also holds for $t > T$ provided that T was chosen large enough. As a byproduct, we have proven (4.3).

Concerning (4.10), in order to prove (4.14) it remains to show that for every $\nu \in (0, 1/\sqrt{2})$ and every $x \in (-\nu, \nu)$,

$$g(x, \varepsilon) < f(x, (\alpha - \delta)\varepsilon). \tag{4.17}$$

This is a consequence of the way f_0 was determined (see (3.10) and discussion in Section 3.3). First we notice that the time derivative of g is uniformly continuous away from the boundary points $\pm 1/\sqrt{2}$:

Lemma 4.3. *For any* $0 < \nu < 1/\sqrt{2}$*,*

$$\lim_{t\to 0} \sup\{|\partial_t g(x, s) - \partial_t g(x, 0)| : s \in [0, t] \text{ and } x \in [-\nu, \nu]\} = 0. \tag{4.18}$$

Proof. This is well known but we sketch a probabilistic proof for the sake of completeness. Let $(B_t)_{t\geq 0}$ denote a standard Brownian motion starting at $x \in [-2^{-1/2}, 2^{-1/2}]$ (with the associated expectation denoted by E_x) and let T denote the hitting time of $\{\pm 1/\sqrt{2}\}$. One has

$$\partial_x^2 g(x, t) = E_x[\partial_x^2 f_0(B_t)\mathbf{1}_{\{t<T\}}]. \tag{4.19}$$

We can thus rewrite (4.18) as

$$\lim_{t\to 0} \sup\{|E_x(\partial_x^2 f_0(B_s) - \partial_x^2 f_0(x))| : s \in [0, t] \text{ and } x \in [-\nu, \nu]\} = 0 \tag{4.20}$$

and we can conclude using the uniform continuity of $\partial_x^2 f_0$ on $[-1/\sqrt{2}, 1/\sqrt{2}]$ and well-known continuity properties of the Brownian motion. Note that (4.18) would not hold with $\nu = 1/\sqrt{2}$ because of boundary effects: for $t > 0$ one sees that $\partial_x^2 g(x, t)$ approaches zero as x approaches $\pm 1/\sqrt{2}$, since $P_x(T > t) \to 0$ when $x \to \pm 1/\sqrt{2}$. □

Therefore, for every $\eta > 0$ arbitrarily small we have for all x in $(-\nu, \nu)$, if ε is small enough,

$$g(x, \varepsilon) < f_0(x) + \varepsilon\left(\frac{1}{4}\partial_x^2 f_0(x) + \eta\right).$$

We are left with the task of proving that for ε small enough and x in $(-\nu, \nu)$,

$$f_0(x) + \varepsilon\left(\frac{1}{4}\partial_x^2 f_0(x) + \eta\right) < f(x, (\alpha - \delta)\varepsilon).$$

From the definition of $f(\cdot, t)$ as the graph in $(\mathbf{f}_1, \mathbf{f}_2)$ of the boundary of $(1 - t)\mathscr{D}$, we deduce that if ε is small enough, then uniformly for all $x \in (-\nu, \nu)$,

$$\begin{aligned} f(x, (\alpha - \delta)\varepsilon) &= [1 - (\alpha - \delta)\varepsilon] f\left(\frac{x}{1 - (\alpha - \delta)\varepsilon}, 0\right) \\ &= f_0(x) + (\alpha - \delta)\varepsilon(x\partial_x f_0(x) - f_0(x)) + O(\varepsilon^2). \end{aligned}$$

Now recall that f_0 satisfies equation (3.10), so it suffices to check that for all $x \in (-\nu, \nu)$,

$$\eta + \delta(x\partial_x f_0(x) - f_0(x)) = \eta + \frac{\delta}{4\alpha}\partial_x^2 f_0 < 0,$$

which holds provided $\eta = \eta(\delta)$ is small, since $\partial_x^2 f_0(\cdot)$ is negative and uniformly bounded away from zero. Equation (4.10) is proven. □

4.3.2. Proof of (4.11). The method is similar to the one we used for (4.10), the main difference being that, via a chain of monotonicity arguments, we analyze the evolution of the portion of interface near the "poles" by comparing it to the interface dynamics of Section 3.2 (which coincides with the height function of the zero-range process with two types of particles) instead of the "corner-flip dynamics".

Denote by $h(\cdot, t)$ the function whose graph in the coordinate system $(\mathbf{e}_1, \mathbf{e}_2)$ is given by the intersection of $(1-t)\partial\mathscr{D}$ with the upper half-plane $\mathbb{R} \times \mathbb{R}_+$, and $h_0(\cdot) = h(\cdot, 0)$. Note that h_0 is C^∞ on $(-1, 0)$ and on $(0, +1)$ by the definition of $\mathscr{D}$. The boundary condition (3.12) ensures continuity of the first derivative of h_0 at zero ($\partial_x h_0(0) = 0$); the reader can check that h_0 also has continuous second derivative and

$$\partial_x^2 h_0(0) = \frac{1}{2\sqrt{2}}\partial_x^2 f_0(-1/\sqrt{2}) = -2\alpha, \tag{4.21}$$

but the third derivative exhibits a discontinuity at 0.

Recall that ξ is the positive constant appearing in (4.9). We set $\bar{h} : [-4\xi, 4\xi] \to \mathbb{R}$ to be the function defined by the following conditions: $\bar{h} \equiv h_0$ on $[-2\xi, 2\xi]$, $\bar{h}$ is affine on $[-4\xi, -2\xi]$ and on $[2\xi, 4\xi]$ and the derivative $\partial_x \bar{h}(\cdot)$ is continuous on $(-4\xi, 4\xi)$. Since $h_0(\cdot)$ is strictly convex, we have $h_0(x) \le \bar{h}(x)$ with strict inequality outside $[-2\xi, 2\xi]$. Define also the following subsets of $\mathbb{R}^2$ (cf. Figure 7):

$$J^1 := [4\xi, \infty) \times [\bar{h}(4\xi), \infty), \qquad J^2 := (-\infty, -4\xi] \times [\bar{h}(4\xi), \infty). \tag{4.22}$$

To avoid notational complications with integer parts, we assume that $L\bar{h}(4\xi)$ and $4L\xi$ belong to $\mathbb{Z}^*$.

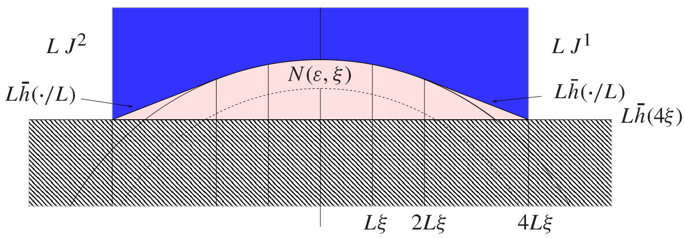

Fig. 7. In the set $L(J^1 \cup J^2)$ spins are frozen to "+" while in the dashed region they are frozen to "−". The initial condition is "+" in the dark-colored region and "−" in the light-colored one. The boundary separating the dark/light regions is determined by the function $\bar{h}(\cdot)$.

First of all, observe that, thanks to (4.3), we can freeze the spins in $L(J^1 \cup J^2)$ to their initial value "+" and, w.h.p., the dynamics will be identical for all times to the original one.

Next, we employ a chain of monotonicities, based on the graphical construction of Section 2.3. Since we are after an *upper bound* on the set of minus spins, we can freeze to "−" all spins whose vertical coordinate is below $L\bar{h}(4\xi)$. Therefore, we have just a dynamics in the set

$$Y := [-4L\xi + 1, 4L\xi - 1] \times [L\bar{h}(4\xi), \infty).$$

In principle, its initial condition is such that the spin at site $(x_1, x_2) \in Y$ is "−" if and only if $x_2 \in [L\bar{h}(4\xi), Lh_0(x_1/L)]$. The problem is however that the function $x \mapsto \max(\bar{h}(4\xi), h_0(x))$ is not concave, which precludes applying Corollary 3.5 directly later. By monotonicity, we can modify such an initial condition by adding extra "−" spins: we therefore stipulate that at time $t = 0$ the spin at site (x_1, x_2) is "−" if and only if $x_2 \in [L\bar{h}(4\xi), L\bar{h}(x_1/L)]$. Recall that $\bar{h}(x) \geq h_0(x)$, so monotonicity goes in the correct direction. With some abuse of notation, we still denote by $(\sigma(t))_{t\geq 0}$ the dynamics thus modified and by $\mathcal{A}_L(t)$ the set of minus spins. We need a final step in order to map the evolution into the zero-range process. Note that, at time $t = 0$, the boundary of $\mathcal{A}_L(t = 0)$, intersected with the strip $[-4L\xi + 1/2, 4L\xi - 1/2] \times \mathbb{R}$, can be identified with the graph of a càdlàg function

$$H_L(\cdot, 0) : [-4L\xi + 1/2, 4L\xi - 1/2] \to [L\bar{h}(4\xi) - 1/2, \infty) \cap \mathbb{Z},$$

which is constant on intervals $[n, n + 1)$ with $n \in \mathbb{Z}$ and takes boundary values $L\bar{h}(4\xi) - 1/2$ at the two endpoints ($H_L(x, 0)$ is just a discretized version of $L\bar{h}(x/L)$). However, for time $t > 0$ it is not true in general that the boundary of $\mathcal{A}_L(t)$ is still the graph of a function, simply because the set $\mathcal{A}_L(t)$ may be disconnected (see Figure 4). Let $(\sigma^{(2)}(t))_{t\geq 0}$ be the dynamics obtained by erasing all the updates that would make $\mathcal{A}_L(t)$ disconnected. It is easy to realize that since $H_L(\cdot, 0)$ has a single change of monotonicity (from non-decreasing to non-increasing, recall that $\bar{h}(\cdot)$ is concave), such erased updates can only correspond to a "−" spin turning into a "+" spin (see again Figure 4). Therefore, the set of minus spins of the dynamics $(\sigma^{(2)}(t))_{t\geq 0}$ stochastically dominates $\mathcal{A}_L(t)$; more precisely, we have shown that the coupling given by the graphical construction implies that, w.h.p. and for all $t \geq 0$,

$$\mathcal{A}_L(t) \subset \mathcal{A}_L^{(2)}(t) := \bigcup_{\{x:\, \sigma_x^{(2)}(t) = -\}} \mathcal{C}_x. \tag{4.23}$$

We let

$$H_L(\cdot, t) : [-4L\xi + 1/2, 4L\xi - 1/2] \to [L\bar{h}(4\xi) - 1/2, \infty) \cap \mathbb{Z}$$

denote the piecewise constant (random) function whose graph in the usual coordinate system $(\mathbf{e}_1, \mathbf{e}_2)$ is the intersection between $\partial\mathcal{A}_L^{(2)}(t)$ and the strip $[-4L\xi+1/2, 4L\xi-1/2] \times \mathbb{R}$. Note that $H_L(-4L\xi + 1/2, t) = H_L(4L\xi - 1/2, t) = L\bar{h}(4\xi) - 1/2$.

Equation (4.11) is proved if one shows that for any ε small enough, w.h.p.,

$$\frac{1}{L} H_L(x, \varepsilon L^2) \leq h(x/L, (\alpha - \delta)\varepsilon) \quad \text{for every } x \in (-\xi L, \xi L). \tag{4.24}$$

It is clear from Remark 3.3 that the function $H_L(\cdot,\cdot)$ follows the dynamics described in Section 3.2, with generator (3.5) (here we identify the function $H_L(\cdot,t)$ with an element of $\Omega_{4\xi L-1/2}$, see (3.4)). According to Corollary 3.5 one has for arbitrarily small $\eta > 0$, w.h.p, for all $x \in (-\xi L, \xi L)$,

$$\frac{1}{L} H_L(x, \varepsilon L^2) \le \bar{\phi}(x/L, (1 + \|\partial_x \bar{h}\|_\infty)^{-2}\varepsilon) + \eta \tag{4.25}$$

where $\|\partial_x \bar{h}\|_\infty = \sup_{[-4\xi,4\xi]} |\partial_x \bar{h}(x)|$ and $\bar{\phi}(x, L^2 t)$ is the solution of

$$\begin{cases} \partial_t \bar{\phi}(x,t) = \frac{1}{2}\partial_x^2 \bar{\phi}(x,t), \\ \bar{\phi}(-4\xi, t) = \bar{\phi}(4\xi, t) = \bar{h}(4\xi), \\ \bar{\phi}(x, 0) = \bar{h}(x) \quad \text{for every } x \in [-4\xi, 4\xi]. \end{cases} \tag{4.26}$$

Thus (4.24) will be proved if we show that

$$\bar{\phi}(x, (1 + \|\partial_x \bar{h}\|_\infty)^{-2}\varepsilon) < h(x, (\alpha - \delta)\varepsilon) \quad \text{for every } x \in [-\xi, \xi]. \tag{4.27}$$

Note that by Lemma 4.3 (which is applicable because the second derivative of $\bar{h}(\cdot) = h_0(\cdot)$ is uniformly continuous in $(-2\xi, 2\xi)$) one has, uniformly on $[-\xi, \xi]$,

$$\begin{aligned} \bar{\phi}(x, (1 + \|\partial_x \bar{h}\|_\infty)^{-2}\varepsilon) &= \bar{\phi}(x, 0) + \tfrac{1}{2}\varepsilon(1 + \|\partial_x \bar{h}\|_\infty)^{-2}\partial_x^2 \bar{\phi}(x, 0) + o(\varepsilon) \\ &= h_0(x) + \tfrac{1}{2}(1 + \|\partial_x \bar{h}\|_\infty)^{-2}(\partial_x^2 h_0(0) + r(x))\varepsilon + o(\varepsilon) \end{aligned} \tag{4.28}$$

where $r(x)$ tends to 0 for $x \to 0$. Finally, using (4.21), if ξ is chosen small enough so that both $r(x)$ and $\|\partial_x \bar{h}\|_\infty$ are sufficiently smaller than δ,

$$\bar{\phi}(x, (1 + \|\partial_x \bar{h}\|_\infty)^{-2}\varepsilon) \le h_0(x) - (\alpha - \delta/4)\varepsilon. \tag{4.29}$$

On the other hand $h(x, (\alpha - \delta)\varepsilon) \ge h_0(x) - (\alpha - \delta/2)\varepsilon$, which ends the proof of (4.11). □

4.4. *Lower bound: proof of (4.4) and (4.2)*

The proofs follow the same ideas as those of Section 4.3: we need to control the dynamics for different portions of the interface separately (around the poles and away from them) using the scaling limit results provided by Theorems 3.4 and 3.2.

4.4.1. Proof of (4.4). The inclusion (4.4) is absolutely crucial to start the proof of (4.2) and quite independent of the rest. The proof is very similar to that of [2, Theorem 2], so we only sketch the main steps. Set

$$D := \{x \in (\mathbb{Z}^*)^2 : d(x, (1-\delta)L\mathscr{D}) \le 1\}, \qquad D' := (\mathbb{Z}^*)^2 \cap ((1+\delta^3)L\mathscr{D})^c$$

and consider a modified dynamics $(\tilde\sigma(t))_{t\ge 0}$ (whose law is denoted $\tilde{\mathbf{P}}$), with the same initial condition as $(\sigma(t))_{t\ge 0}$ and the rules that: (i) after each update, any "−" spin which has more than two "+" neighbors is turned into "+", and the operation is repeated as long as such spins exist; (ii) the dynamics stops at $\tilde\tau_{D,D'}$, the first time when there is either a "+" spin in D or a "−" spin in D'. We also define $\tilde\tau_D$ to be the first time when there is a "+" spin in D, and $\tau_{D,D'}$, τ_D to be the analogous random times for the original dynamics.

Note that, by (4.3), w.h.p. $\tau_{D,D'} = \tau_D$. Note also that the two dynamics can be coupled in such a way that $\tau_{D,D'} = \tau_D$ implies $\tilde\tau_{D,D'} = \tilde\tau_D \le \tau_D$ (thanks to (i) above, since before $\tilde\tau_{D,D'}$ the modified dynamics has fewer "−" spins than the original one). Therefore,

$$\mathbf{P}(\tau_D \le \varepsilon L^2) = \mathbf{P}(\tau_D \le \varepsilon L^2;\ \tau_{D,D'} = \tau_D) + o(1) \le \tilde{\mathbf{P}}(\tilde\tau_D \le \varepsilon L^2;\ \tilde\tau_{D,D'} = \tilde\tau_D) + o(1)$$

and it suffices to prove for instance that

$$\tilde{\mathbf{P}}(\tilde\tau_{D,D'} \le \varepsilon L^2;\ \tilde\tau_D = \tilde\tau_{D,D'}) \le \exp(-\gamma L)$$

for some $\varepsilon = \varepsilon(\delta)$, $\gamma > 0$. For this, one first observes (as in [2, (8.6)]) that when $\tilde\tau_{D,D'} = \tilde\tau_D$ the difference between the number of "+" spins at time $\tilde\tau_D$ and the number of "+" spins at time 0 is at least $c\delta^2L^2$ deterministically, for some $c > 0$.

Finally, (as in [2, (8.10)]) one proves that

$$\tilde{\mathbf{P}}\big(|\{x : \tilde\sigma_x(\varepsilon L^2) = +\}| - |\{x : \tilde\sigma_x(0) = +\}| \ge c\delta^2L^2\big) \le \exp(-\gamma L)$$

if $\varepsilon = \varepsilon(\delta)$ is small enough. This is based on the fact (cf. [2, Lemma 8.5]) that, for times smaller than $\tilde\tau_{D,D'}$, the rate of increase of the number of "+" spins is uniformly bounded by a constant.

4.4.2. Scheme of the proof of (4.2). Given some fixed $\delta > 0$, we want to prove that for $\varepsilon > 0$ small enough, w.h.p.,

$$(1 - (\alpha + \delta)\varepsilon)L\mathscr{D} \subset \mathcal{A}_L(\varepsilon L^2), \tag{4.30}$$

or equivalently

$$\sigma_x(\varepsilon L^2) = - \quad \text{for every } x \in (1 - (\alpha + \delta)\varepsilon)L\mathscr{D}. \tag{4.31}$$

Given ξ small enough (depending on δ) and ν small enough (depending on ξ), we define (cf. Figure 8)

$$\begin{aligned} U &:= (1-\nu)\mathscr{D}, \\ A_1(\varepsilon) &:= [((1 - (\alpha + \delta)\varepsilon)\mathscr{D}) \setminus U] \cap [\xi, +\infty)^2, \\ B_1(\varepsilon) &:= [((1 - (\alpha + \delta)\varepsilon)\mathscr{D}) \setminus U] \cap ([-\xi, \xi] \times \mathbb{R}^+), \end{aligned} \tag{4.32}$$

and $A_i, B_i, i = 2, 3, 4$, as the images of $A_1(\varepsilon)$, $B_1(\varepsilon)$ by the rotation of angle $(i-1)\pi/2$.

One has

$$(1 - (\alpha + \delta)\varepsilon)\mathscr{D} = U \cup \Big(\bigcup_{i=1}^4 A_i\Big) \cup \Big(\bigcup_{i=1}^4 B_i\Big),$$

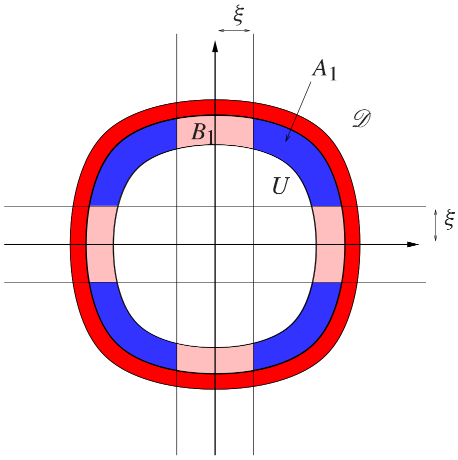

Fig. 8. The large droplet is $\mathscr{D}$ and $(1-\varepsilon(\alpha+\delta))\mathscr{D}$ is obtained by removing the external dark layer. The white central region U together with A_1, B_1 and its rotations (deformed rectangular regions) form a partition of $(1-\varepsilon(\alpha+\delta))\mathscr{D}$.

and hence (using rotational symmetries), to prove (4.30), it is sufficient to prove that for ε small enough, w.h.p.,

$$LU \subset \mathcal{A}_L(\varepsilon L^2), \tag{4.33}$$

$$LA_1(\varepsilon) \subset \mathcal{A}_L(\varepsilon L^2), \tag{4.34}$$

$$LB_1(\varepsilon) \subset \mathcal{A}_L(\varepsilon L^2). \tag{4.35}$$

The first line is a direct consequence of (4.4) provided that ε is chosen small enough (how small depends on ν). Actually, one has the following stronger statement that will be useful for what follows: if ε is small then w.h.p.,

$$LU \subset \mathcal{A}_L(tL^2) \quad \text{for every } t \le \varepsilon. \tag{4.36}$$

The main work is thus to prove (4.34) and (4.35).

4.4.3. Proof of (4.35). This is similar to the proof of (4.11), except that monotonicities will be needed in the opposite direction.

Let $\bar{h} : [-2\xi, 2\xi] \to \mathbb{R}$ be a concave, twice differentiable, even function such that

$$\begin{aligned} \bar{h}(x) &= h_0(x), \quad \forall x \in [-\xi, \xi], \\ \bar{h}(x) &< h_0(x), \quad \forall x \in [-2\xi, -\xi) \cup (\xi, 2\xi], \end{aligned}$$

where $h_0(\cdot)$ was defined in Section 4.3.2 to be the graph of $\partial\mathscr{D} \cap (\mathbb{R} \times \mathbb{R}^+)$ in the $(\mathbf{e}_1, \mathbf{e}_2)$ coordinate system. Once ξ is fixed, we choose ν and $\bar{h}$ such that the point $(2\xi, \bar{h}(2\xi))$ lies in the interior of U.

Using (4.36), we can freeze the spins with vertical coordinate $L\bar{h}(2\xi)$ and horizontal coordinate in $(-2L\xi, 2L\xi)$ (we assume for notational convenience that $2L\xi$ and $L\bar{h}(2\xi)$

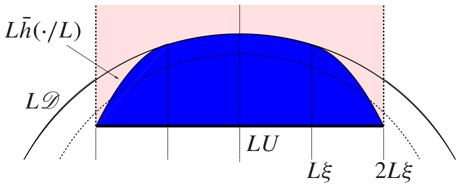

Fig. 9. The sites in the dashed vertical lines are frozen to "+" and those of the horizontal bold segment to "−" so that the dynamics in the colored infinite rectangle is independent of the rest of the system. At time $t = 0$ the sites in the dark-colored region (whose upper boundary is determined by $\bar{h}(\cdot)$) are "−", while those of the light-colored one are "+". The function $\bar{h}(\cdot)$ is such that the base of the dark-colored region is in LU.

are in $\mathbb{Z}^*$) to their initial value "−", and w.h.p., the dynamics we obtain is identical to the original one up to time εL^2.

Next we use a chain of monotonicities based on the graphical construction of Section 2.3. Since we are after a *lower bound* on the set of minuses, we can freeze to "+" all the spins with horizontal coordinate $\pm 2L\xi$ and vertical coordinate larger than $L\bar{h}(2\xi)$. Once this is done, we are reduced to considering the dynamics restricted to the set

$$Y_2 := [-2L\xi + 1, 2L\xi - 1] \times [L\bar{h}(2\xi) + 1, \infty),$$

as spins on its boundary are fixed. In principle, the initial condition one should consider is such that $(x_1, x_2) \in Y_2$ has spin "−" iff $x_2 \in [L\bar{h}(2\xi) + 1, Lh_0(x_1/L)]$, but again by monotonicity, we can add extra "+" spins: we stipulate that, at time $t = 0$, (x_1, x_2) has spin "−" iff $x_2 \in [L\bar{h}(2\xi) + 1, L\bar{h}(x_1/L)]$. With some abuse of notation, the dynamics thus modified is still called $(\sigma(t))_{t \geq 0}$.

As for the proof of (4.11), we need a final step to map the dynamics onto the interface dynamics of Theorem 3.4, the problem being exactly the same as then: it is not true that the boundary of $\mathcal{A}_L(t)$ stays connected for all t. The solution adopted in the previous section (leading to the dynamics $(\sigma^{(2)}(t))_t$, see discussion before (4.23)) does not work here as we are now looking for a *lower bound*.

Let $(\sigma^{(3)}(t))_t$ be the dynamics that evolves like $(\sigma(t))_t$ except that any spin that has three "+" neighbors is turned instantaneously into "+" (see Figure 10). The coupling given by the graphical construction implies that

$$\bigcup_{\{x:\, \sigma^{(3)}_x(t) = -\}} \mathcal{C}_x =: \mathcal{A}^{(3)}_L(t) \subset \mathcal{A}_L(t). \tag{4.37}$$

Moreover our choice of initial condition guarantees that $\mathcal{A}^{(3)}_L(t)$ stays connected for all time, since the set $\mathscr{D}$ is convex.

We denote by $H_L(\cdot, t)$ the càdlàg function $[-2\xi, 2\xi] \to \mathbb{R}$ whose graph corresponds to the intersection between $\partial\mathcal{A}^{(3)}_L(t)$ and the vertical strip $[-2L\xi + 1/2, 2L\xi - 1/2] \times \mathbb{R}$. Note that $H_L(\cdot, t)$ can be visualized as a collection of columns of width 1 and integer height. With this notation and (4.37), the inclusion (4.35) is proved if one has, w.h.p.,

$$\frac{1}{L} H_L(x, L^2\varepsilon) \geq h(x/L, (\alpha + \delta)\varepsilon) \quad \text{for every } x \in (-\xi L, \xi L). \tag{4.38}$$

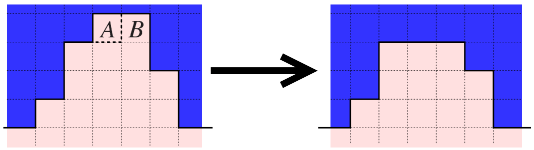

Fig. 10. Light-colored (resp. dark-colored) squares denote "−" (resp. "+") spins. In our modified dynamics $\sigma^{(3)}$, when a spin has three "+" neighbors, it is instantaneously turned into "+". On the figure, if spin at A is updated and turns into "+", then the spin B has three "+" neighbors and therefore also turns instantaneously to "+".

Now we want to relate the dynamics of H_L to that of Theorem 3.4. The relation is almost identical to that discussed in Remark 3.3, except for a slight difference in the way particles of types A and B annihilate in the zero-range process. Given $\mathbb{Z} \ni x = -2L\xi + 1/2, \dots, 2L\xi - 1/2$, we say again that there are $n > 0$ particles of type A at time t at site x if $\lim_{y\to x^+} H_L(y,t) - \lim_{y\to x^-} H_L(y,t) = n$, and that there are $n > 0$ particles of type B if the same difference equals $-n$. Then it is easy to realize that, under the dynamics $(\sigma^{(3)}(t))_{t\ge 0}$, each particle performs a symmetric simple random walk with jump rate $1/(2n)$ both to the right and to the left (with n the occupation number of the site where the particle is), and that particles of different type annihilate immediately if they are *at sites of distance* 1 (and not *at the same site*): this is the effect of flipping instantaneously "−" spins with more than two "+" neighbors. Note also that, due to convexity of $\bar h(\cdot)$, particles of type A are always to the left of particles of type B. Therefore, if we take $H_L(\cdot,t)$ and we eliminate one of the columns of maximal height (see Figure 11) (note that there are always at least two such), the modified height function thus obtained follows exactly the evolution of Theorem 3.4.

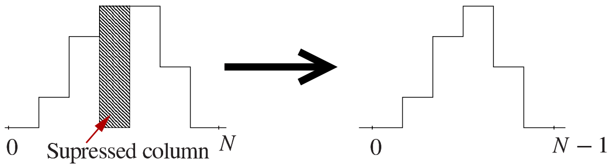

Fig. 11. Left: the height function associated to the "+/−" boundary for the dynamics $\sigma^{(3)}(t)$. Right: the same height function, with one of the highest columns removed; this follows the same evolution as in Theorem 3.4. The fact that the new interface is one step shorter makes no difference in the macroscopic limit.

Of course, the erased column does not change the scaling limit so that one can apply Theorem 3.4 and Corollary 3.5 and get for any t and $\eta > 0$, w.h.p.,

$$\frac{1}{L} H_L(x, L^2 t) \ge \bar\phi(x/L, t) - \eta \qquad \text{for every } x \in \{-2\xi L, \dots, 2\xi L\},$$

where

$$\begin{cases} \partial_t \bar\phi(x,t) = \frac{1}{2}\partial_x^2 \bar\phi(x,t), \\ \bar\phi(2\xi,t) = \bar\phi(-2\xi,t) = \bar h(2\xi), \\ \bar\phi(x,0) = \bar h(x) \quad \text{for every } x \in [-2\xi, 2\xi]. \end{cases}$$

Therefore, (4.38) is proved if one can check that $\bar{\phi}(x,\varepsilon) > h(x,(\alpha+\delta)\varepsilon)$ for every $x \in [-\xi,\xi]$. This is proved is the same manner as (4.27): one just needs to choose ξ small enough. $\square$

4.4.4. Proof of (4.34). First of all, one freezes to "$-$" all the spins on the cross-shaped region of sites in LU (cf. (4.32)) such that at least one of their coordinates is $\pm 1/2$. The inclusion (4.36) guarantees that if ε is chosen small enough, w.h.p. the dynamics so obtained coincides with the original one up to time εL^2 if ε is small enough.

Then one defines $(\sigma^{(4)}(t))_{t\geq 0}$ as the dynamics obtained by changing the initial condition in the following manner: all spins $(x,y) \in L\mathscr{D}$ with either $|x| \geq L(1-\nu)$ or $|y| \geq L(1-\nu)$ are changed from "$-$" to "$+$" (recall that ν is the constant that enters the definition of U in (4.32)) and therefore they stay "$+$" forever, since they have at least three "$+$" neighbors. Note that, this way, the evolution in each quadrant of $(\mathbb{Z}^*)^2$ is independent. By monotonicity, we get, w.h.p., for every $t \leq \varepsilon L^2$,

$$\bigcup_{\{x:\,\sigma_x^{(4)}(t)=-\}} \mathcal{C}_x =: \mathcal{A}_L^{(4)}(t) \subset \mathcal{A}_L(t),$$

and therefore (4.34) is proved if one can show that

$$LA_1(\varepsilon) \subset \mathcal{A}_L^{(4)}(\varepsilon L^2). \tag{4.39}$$

Next, note that $\partial\mathcal{A}_L^{(4)}(t)\cap\mathbb{R}_+^2$ in the coordinate system $(\mathbf{f}_1,\mathbf{f}_2)$ is the graph of a random piecewise affine function

$$F_L : [-(1-\nu)L/\sqrt{2}, (1-\nu)L/\sqrt{2}] \to \mathbb{R}$$

which follows the corner-flip dynamics described in Theorem 3.2 (apart from space rescaling by a factor of $\sqrt{2}$). For this reason one gets, w.h.p.,

$$\lim_{L\to\infty} \sup_{x\in[-(1-\nu)/\sqrt{2},(1-\nu)/\sqrt{2}]} \sup_{t\leq\varepsilon} \left| \frac{1}{L} F_L(xL, tL^2) - g(x,t) \right| = 0 \tag{4.40}$$

where

$$\begin{cases} \partial_t g(x,t) = \frac{1}{4}\partial_x^2 g(x,t), \\ g(-(1-\nu)/\sqrt{2}, t) = g((1-\nu)/\sqrt{2}, t) = (1-\nu)/\sqrt{2}, \\ g(x,0) = \bar{f}(x) \quad \text{for every } x \in [-(1-\nu)/\sqrt{2}, (1-\nu)/\sqrt{2}], \end{cases}$$

and $\bar{f}$ is the profile of the initial condition, i.e.

$$\bar{f}(x) := \min(f_0(x), (1-\nu)\sqrt{2} - |x|)). \tag{4.41}$$

Let P (resp. P_1) be the point on $\partial\mathscr{D}$ whose coordinates (x,y) (resp. (x_1,y_1)) in the coordinate system $(\mathbf{e}_1,\mathbf{e}_2)$ satisfy $x>0$, $y=1-\nu$ (resp. $x_1>0$, $y_1=h_0(\xi)$). Denote by $-d < 0$ (resp. $-d_1 < 0$) the horizontal coordinate of P (resp. of P_1) in the coordinate system $(\mathbf{f}_1, \mathbf{f}_2)$.

In view of (4.40) and of the definition of $A_1(\varepsilon)$ in (4.32), the inclusion (4.39) is satisfied if

$$g(x, \varepsilon) > f(x, (\alpha + \delta)\varepsilon) \quad \text{for every } x \in (-d_1, d_1).$$

The proof of this is very similar to that of (4.17) provided that $\bar{f}$ coincides with f_0 in a domain strictly containing $(-d_1, d_1)$ (this guarantees for instance that $\partial_x^2 \bar{f}(\cdot)$ is uniformly continuous in a domain containing $(-d_1, d_1)$, so that the drift $\partial_t g$ is continuous in time, cf. Lemma 4.3). For this to hold, it is enough to assume that $d > d_1$, i.e. that ν in (4.32) has been chosen sufficiently small as a function of ξ so that $1 - \nu > h_0(\xi)$. □

5. Proof of Theorem 2.1: existence of anisotropic curve-shortening flow with convex initial condition

Let us first recall some properties of the support function $h(\cdot)$ of a convex curve γ. First of all, if γ is contained in the convex set bounded by γ' then $h(\theta) \le h'(\theta)$ for every θ. Next, the support function is related to the curvature and to the length $L(\gamma)$ of γ by (cf. [12, Lemma 1.1])

$$\partial_\theta^2 h(\theta) + h(\theta) = \frac{1}{k(\theta)}, \tag{5.1}$$

$$L(\gamma) = \int_0^{2\pi} h(\theta)\, d\theta = \int_0^{2\pi} \frac{1}{k(\theta)}\, d\theta. \tag{5.2}$$

Also (cf. [13, Lemma 4.1.1], with the warning that what they call θ is $\theta - \pi/2$ for us), the Cartesian coordinates $(x(\theta), y(\theta))$ of the point of γ where the outward directed normal forms an anticlockwise angle θ with the positive horizontal axis can be expressed as

$$x(\theta) = h(0) - \int_0^\theta \frac{\sin(s)}{k(s)}\, ds, \tag{5.3}$$

$$y(\theta) = h(\pi/2) + \int_{\pi/2}^\theta \frac{\cos(s)}{k(s)}\, ds. \tag{5.4}$$

Under the flow (2.5), the time derivatives of area and length are (cf. [12, Lemma 2.1])

$$\frac{d}{dt} \operatorname{Area}(\gamma(t)) = -\int_0^{2\pi} a(\theta)\, d\theta, \tag{5.5}$$

$$\frac{d}{dt} L(\gamma(t)) = -\int_0^{2\pi} a(\theta) k(\theta, t)\, d\theta. \tag{5.6}$$

For the moment these are formal statements since we do not know yet that the flow exists.

5.1. *Proof of Theorem 2.1*

Uniqueness of the flow is trivial, so we concentrate on existence. First of all, we need to regularize the functions $a(\cdot)$ and $k(\cdot)$. Given $0 < w < 1$ we define $a^{(w)}(\cdot)$ to be a family of smooth approximations of the anisotropy function $a(\cdot)$. More precisely:

Assumption 5.1.

(1) $a^{(w)}(\cdot)$ is 2π-periodic and C^∞;
(2) $a^{(w)}(\theta) \overset{w\to 0}{\longrightarrow} a(\theta)$ uniformly in θ;
(3) for fixed θ, the function $w \mapsto a^{(w)}(\theta)$ is non-increasing;
(4) the function $a^{(w)}(\cdot)$ is Lipschitz, uniformly in $w > 0$ (this is possible because the function $a(\cdot)$ itself is 1-Lipschitz);
(5) the functions $w \mapsto \|\partial_\theta^2 a^{(w)}\|_\infty := \max_\theta |\partial_\theta^2 a^{(w)}(\theta)|$ and $w \mapsto \|\partial_\theta^3 a^{(w)}\|_\infty$ are bounded, uniformly for w in any compact subset of $(0, 1)$.

A possible choice is

$$a^{(w)}(\theta) = (a * g^{(w)})(\theta) + \varepsilon_w$$

where $g^{(w)}$ is a centered Gaussian of variance w^2. In the convolution it is understood that $a(\cdot)$ is seen as a 2π-periodic function on $\mathbb{R}$ and ε_w is chosen so that $a^{(w)}(\cdot)$ satisfies the monotonicity condition with respect to w. It is easy to check that one can choose $\varepsilon_w = -Cw$ for some suitably large C. Indeed, monotonicity in w is guaranteed if for $w' < w$ one has

$$\varepsilon_{w'} - \varepsilon_w \geq \|a * (g^{(w)} - g^{(w')})\|_\infty.$$

On the other hand, since $a(\cdot)$ is Lipschitz, one sees easily that $\|a * (g^{(w)} - g^{(w')})\|_\infty = O(w - w')$.

Also, we approximate γ with a sequence of convex curves $(\gamma^{(w)})_{0<w<1}$ that have the following properties:

Assumption 5.2.

(1) $\gamma^{(w)} \supset \gamma^{(w')} \supset \gamma$ or equivalently $h^{(w)}(\cdot) \geq h^{(w')}(\cdot) \geq h(\cdot)$ if $0 < w' < w$;
(2) $\lim_{w\to 0} h^{(w)}(\cdot) = h(\cdot)$ uniformly in θ, so that γ is the limit of $\gamma^{(w)}$ in the Hausdorff distance;
(3) the Lipschitz constant $\mathbb{L}(k^{(w)})$ of the curvature function $k^{(w)}(\cdot)$ is finite uniformly in w, $k^{(w)}(\cdot) \to k(\cdot)$ uniformly and $\limsup_{w\to 0} \mathbb{L}(k^{(w)}) \leq \mathbb{L}(k)$;
(4) the first three derivatives with respect to θ of $k^{(w)}(\theta)$ are uniformly bounded for w in any compact subset of $(0, 1)$.

(Just as for the regularization of $a(\cdot)$ into $a^{(w)}(\cdot)$, a possible construction of $h^{(w)}(\cdot)$ is obtained by convolving $h(\cdot)$ with a Gaussian of variance w^2 and adding a suitable constant ε_w.)

For the regularized mean curvature motion, it follows from [12] that the equation

$$\begin{cases} \partial_t h^{(w)}(\theta, t) = -a^{(w)}(\theta) k^{(w)}(\theta, t), \\ h^{(w)}(\theta, 0) = h^{(w)}(\theta), \end{cases} \tag{5.7}$$

admits a solution corresponding to a flow of curves $(\gamma^{(w)}(t))_{t\geq 0}$ which remain convex and shrink to a point in a finite time

$$\tilde{t}_f := t_f^{(w)} = \mathrm{Area}(\gamma^{(w)}(0))/\int_0^{2\pi} a^{(w)}(\theta)\,d\theta$$

(cf. (5.5) with $a(\cdot)$ replaced by $a^{(w)}(\cdot)$). To lighten notation, we will often write $\tilde{h}(\cdot,\cdot)$, $\tilde{\gamma}(t), \tilde{a}(\cdot)$, etc. for the regularized quantities $h^{(w)}(\cdot,\cdot), \gamma^{(w)}(t), a^{(w)}(\cdot)$, etc. Thanks to Assumption 5.1, we have $\int_0^{2\pi} a^{(w)}(\theta)\,d\theta \to \int_0^{2\pi} a(\theta)\,d\theta = 2$ as $w \to 0$ and therefore $t_f^{(w)} = t_f(1+o(1))$ when $w \to 0$, with t_f defined in Theorem 2.1.

From (5.1) and (5.7) one can check that the curvature satisfies the parabolic equation

$$\begin{cases} \partial_t \tilde{k} = \tilde{k}^2 \partial_\theta^2(\tilde{a}\tilde{k}) + \tilde{a}\tilde{k}^3, \\ \tilde{k}(\theta,0) = \tilde{k}(\theta). \end{cases} \tag{5.8}$$

Also, following [13] it is possible to see that the curvature function stays C^∞ until $\tilde{t}_f$ (since $\tilde{a}$ is C^∞). However, estimates on the regularity will *not* be necessarily uniform in the regularization parameter w and we will need to be very careful on this point.

For fixed t, set

$$\gamma(t) := \lim_{w\to 0} \gamma^{(w)}(t) \tag{5.9}$$

where convergence is in the Hausdorff metric. *A posteriori*, since we will see that $(\gamma(t))_t$ provides the (unique) solution to our curve-shortening equation, it follows that the limit (5.9) does not depend on the choice of regularization. Existence of the limit (in the Hausdorff metric) along subsequences is guaranteed by the Blaschke selection theorem [8, Th. 32] which says that a family of convex subsets of a bounded subset of $\mathbb{R}^n$ admits a subsequence converging to a non-empty convex set. Uniqueness of the limit follows from the fact that $\gamma^{(w')}(t) \subset \gamma^{(w)}(t)$ if $w' < w$ and $t < t_f^{(w')}$ (because $a^{(w)}(\theta)$ is decreasing in w and the curve is smooth at all times). One has to use the fact that convergence in the Hausdorff distance also holds for the boundary curves.

Since volume is continuous in the topology induced by the Hausdorff metric [8, Ch. 4] we also see that $\mathrm{Area}(\gamma(t)) = \mathrm{Area}(\gamma) - t\int_0^{2\pi} a(\theta)\,d\theta = \mathrm{Area}(\gamma) - 2t$; for $t \to t_f$ the curve $\gamma(t)$ shrinks to a point (its diameter shrinks to zero). We will prove

Theorem 5.1. *The flow of curves $(\gamma(t))_{t<t_f}$ defined in (5.9) is a classical solution of the anisotropic curve-shortening flow (2.5) for $0 \leq t < t_f$.*

Definition 5.2. For $t < \tilde{t}_f$ let $\tilde{k}_{\max}(t)$ (resp. $\tilde{k}_{\min}(t)$) be the maximal (resp. minimal) curvature of $\tilde{\gamma}(t)$. We let $\tilde{k}_{\max} := \tilde{k}_{\max}(0)$ and $\tilde{a}_{\max} := \max_\theta \tilde{a}(\theta)$, and similarly for $\tilde{k}_{\min}$ and $\tilde{a}_{\min}$. Also, $k_{\min(\max)}$ and $a_{\min(\max)}$ are defined similarly to $\tilde{k}_{\min(\max)}, \tilde{a}_{\min(\max)}$ but with $\tilde{k}(\cdot), \tilde{a}(\cdot)$ replaced by $k(\cdot), a(\cdot)$.

It is crucial that $\tilde{k}_{\max}(t)$ stays bounded, uniformly for w small, as long as the disappearance time is not approached:

Proposition 5.3 (Regularity estimate). *Assume that the curvature function $k(\cdot)$ is Lipschitz. There exists $w_0 > 0$ such that, for all $b > 0$, $t < t_f(1-b)$, and $0 < w \le w_0$,*

$$\tilde{k}_{\max}(t) \le C_1, \tag{5.10}$$

$$\max_\theta |\partial_\theta(\tilde{a}(\theta)\tilde{k}(\theta,t))| \le C_2(\mathbb{L}(k)+1), \tag{5.11}$$

where $\mathbb{L}(k)$ is the Lipschitz constant of $k(\cdot)$. The constants C_1 and C_2 depend only b and on $k_{\max}$.

Proof. The proof is based on ideas of [12]. However, it is important to make sure that estimates are uniform in $w \le w_0$ (in [12] the anisotropy function $a(\cdot)$ is assumed to be C^2, so there was no need to regularize it).

Fix $w > 0$. First we get a lower bound on $\tilde{k}_{\min}(t)$. Note first of all that at time zero the minimal curvature is bounded away from zero (uniformly in w): indeed, using (5.2) and the fact that the curvature function is $\mathbb{L}(k)$-Lipschitz, one has

$$L(\gamma(0)) = \int_0^{2\pi} \frac{1}{k(\theta)}\,d\theta \ge 2\int_0^{\pi} \frac{1}{k_{\min}+\mathbb{L}(k)\theta}\,d\theta = \frac{2}{\mathbb{L}(k)}\log\frac{\mathbb{L}(k)\pi + k_{\min}}{k_{\min}}. \tag{5.12}$$

Then, since the length of $\gamma(0)$ is finite, $k_{\min}$ must be positive.

Set for simplicity

$$g = g(\theta,t) = \tilde{a}(\theta)\tilde{k}(\theta,t).$$

Formula (5.8) gives

$$\partial_t g = \frac{1}{\tilde{a}}(g^2\partial_\theta^2 g + g^3) =: g(\theta,t)u(\theta,t). \tag{5.13}$$

This, together with the fact that $\tilde{a}(\cdot)$ and $\tilde{k}(\cdot,t)$ are smooth, implies that

$$\frac{d}{dt}\min_\theta g \ge \frac{\min_\theta g^3}{\tilde{a}_{\max}} \ge 0$$

(at the minimum point the second derivative is positive) so that

$$\tilde{k}_{\min}(t) \ge \frac{\tilde{a}_{\min}}{\tilde{a}_{\max}}\tilde{k}_{\min} \ge Ck_{\min} > 0 \tag{5.14}$$

with C independent of w (say for $w \le w_0$) thanks to the uniform convergence $a^{(w)}(\cdot) \to a(\cdot)$ and $k^{(w)}(\cdot) \to k(\cdot)$.

Next the real work: bounding $\tilde{k}_{\max}(t)$ uniformly in w. From (5.13) one sees that, since $\tilde{a}(\cdot)$ and $\tilde{k}(\cdot,t)$ are smooth,

$$\frac{d}{dt}\max_\theta g \le \frac{1}{\tilde{a}_{\min}}(\max_\theta g)^3. \tag{5.15}$$

From this one immediately sees that $\tilde{k}_{\max}(t)$ is upper bounded uniformly in $w \le w_0$, up to some time t_1 depending only on $k_{\max}$. However the solution of $\dot{x} = x^3$ explodes in finite time, and certainly before the time $\tilde t_f$ when the curve shrinks to a point, so we need to do better.

For this, we define $z(t) = \min_\theta u(\theta,t)$ (cf. (5.13)). Then, taking the derivative of u with respect to t shows (cf. [12, Lemma 4.2] for details) that

$$\frac{d}{dt}z(t) \ge 2z(t)^2,$$

so that if $z(0) \ge 0$ we get $z(t) \ge 0$, while if $z(0) \le 0$ we get $z(t) \ge -(1/|z(0)| + 2t)$. Altogether,

$$u(\theta,t) \ge -\frac{1}{2t}$$

uniformly in θ and $w \le w_0$. Now we use this to get a uniform bound on $\|\partial_\theta g\|_\infty$ in terms of $\tilde k_{\max}(t)$. Without loss of generality suppose that there exists θ_1 such that $\partial_\theta g(\theta_1,t) = \|\partial_\theta g\|_\infty$ (if this is not the case one can still find θ_1 such that $\partial_\theta g(\theta_1,t) = -\|\partial_\theta g\|_\infty$ and apply the same method). Let also $\theta_2 > \theta_1$ be such that $\partial_\theta g(\theta_2,t) = 0$ (such an angle exists since g is periodic). Then, from the definition (5.13) of u,

$$\begin{aligned}\|\partial_\theta g\|_\infty &= -\int_{\theta_1}^{\theta_2} \partial_\theta^2 g\, d\theta = -\int_{\theta_1}^{\theta_2}\left(\frac{u(\theta,t)}{\tilde k(\theta,t)} - \tilde a(\theta,t)\tilde k(\theta,t)\right) d\theta\\ &\le \frac{1}{2t}\int_{\theta_1}^{\theta_2}\frac{d\theta}{\tilde k(\theta,t)} + (\theta_2-\theta_1)\tilde a_{\max}\tilde k_{\max}(t) \le \frac{L(\gamma(0))}{t} + C_4\tilde k_{\max}(t).\end{aligned}$$

In the last inequality we used (5.2) and then (5.6), which says that $L(\tilde\gamma(t)) \le L(\tilde\gamma(0)) \le 2L(\gamma(0))$. Since $g = \tilde a\tilde k$ and by assumption $\tilde a$ is C^∞ and Lipschitz uniformly in w, one deduces that

$$\|\partial_\theta \tilde k\|_\infty \le L(\gamma(0))/t + C_5\tilde k_{\max}(t) \le C_6(t)\tilde k_{\max}(t) \tag{5.16}$$

and C_6 can be chosen to be decreasing in t. From this it is trivial to see that, if θ_0 is such that $\tilde k(\theta_0,t) = \tilde k_{\max}(t)$, one has

$$\tilde k(\theta,t) \ge \tilde k_{\max}(t)/2 \quad \text{whenever} \quad |\theta-\theta_0| \le \alpha(t) \tag{5.17}$$

for some $\alpha(t)$ increasing in t (it could vanish for $t \to 0$). Next, one proves that for $t < (1-b)t_f$,

$$E(t) := \int_0^{2\pi} \tilde a(\theta)\log(g(\theta,t))\, d\theta \le C_7 \tag{5.18}$$

where C_7 depends only on $a_{\max}$ and on b and on the maximal curvature $k_{\max}$ of the initial curve $\gamma(0)$. Indeed, (5.18) is obvious for $t = 0$, since the initial curvature is bounded by assumption. To get the control for $t > 0$, one observes (cf. [12, Propositions 5.3 and 5.4]) that

$$\frac{d}{dt}E(t) \le 2\tilde a_{\max}\frac{L(\tilde\gamma(0))}{\mathrm{Area}(\tilde\gamma((1-b)t_f))}\left(-\frac{d}{dt}L(\tilde\gamma(t))\right).$$

The prefactor is bounded since $b > 0$ and the time integral of the time derivative of the length gives at most $L(\tilde{\gamma}(0))$. At this point we are almost done: by (5.17),

$$\begin{aligned} C_7 &\geq \int_0^{2\pi} \tilde{a}(\theta) \log(g(\theta, t))\, d\theta \\ &\geq 2\alpha(t)\tilde{a}_{\min} \log(\tilde{a}_{\min}\tilde{k}_{\max}(t)/2) + 2\pi\tilde{a}_{\max} \log[\min(1, \tilde{a}_{\min}\tilde{k}_{\min}(t))], \end{aligned} \tag{5.19}$$

and this (recall that $\tilde{k}_{\min}(t) \geq Ck_{\min} > 0$, cf. (5.14)) gives us an upper bound on $\tilde{k}_{\max}(t)$ uniformly in $w \leq w_0$ and $t < (1-b)t_f$: up to t_1 one uses the upper bound which comes from (5.15), and after t_1 the one from (5.19); inequality (5.10) is proven. When t approaches the disappearance time $\tilde{t}_f$ (i.e. when b approaches zero), the upper bound diverges (because C_7 diverges), as it should.

Now (5.16) says that the curvature function is Lipschitz with a Lipschitz constant C that depends on t, b and $L(0)$ but not on w. This is not yet the desired (5.11) because the bound diverges for $t \to 0$. To prove (5.11) remark that, by (5.13),

$$\begin{aligned} \partial_t\partial_\theta g &= \partial_\theta\left(\frac{g^2}{\tilde{a}}\partial_\theta^2 g + \frac{g^3}{\tilde{a}}\right) \\ &= -\frac{\partial_\theta\tilde{a}}{\tilde{a}^2}(g^2\partial_\theta^2 g + g^3) + \frac{1}{\tilde{a}}(2g\partial_\theta g\partial_\theta^2 g + g^2\partial_\theta^3 g + 3g^2\partial_\theta g). \end{aligned}$$

At the point where $\partial_\theta g$ is maximized, $\partial_\theta^2 g$ cancels and $\partial_\theta^3 g$ is non-positive. This, together with the boundedness of g uniformly in $w \leq w_0, \theta \in [0, 2\pi]$ and $t < (1-b)t_f$, implies

$$\partial_t \max_\theta \partial_\theta g(\theta, t) \leq C_8\Big(1 + \max_\theta \partial_\theta g(\theta, t)\Big).$$

where C_8 just depends on $k_{\max}$ and b. Integrating with respect to time, one gets

$$\max_\theta \partial_\theta g(\theta, t) \leq C_9\Big[\max_\theta \partial_\theta(a^{(w)}(\theta)k^{(w)}(\theta)) + 1\Big]$$

with C_9 depending only on C_8. Also, observe that

$$\partial_\theta(a^{(w)}(\theta)k^{(w)}(\theta)) \leq \tfrac{3}{4}|\partial_\theta k^{(w)}(\theta)| + C_{10} \leq \tfrac{3}{4}\mathbb{L}(k^{(w)}) + C_{10}$$

with C_{10} a constant depending on $k_{\max}$, since for w small $a_{\max}^{(w)} < 3/4$ and $a^{(w)}$ is uniformly Lipschitz. Finally, from Assumption 5.2(3), we can conclude that $\partial_\theta(a^{(w)}(\theta)k^{(w)}(\theta)) \leq C_{10} + \mathbb{L}(k)$ for w small. An analogous lower bound can be found on $\partial_t \min_\theta \partial_\theta g(\theta, t)$, and this gives (5.11). □

Following [13] it is possible to prove that, once we have bounds on the curvature and on $\|\partial_\theta g(\cdot, t)\|_\infty$, for every $n \geq 2$ and $t < t_f(1-b)$ the derivatives $\partial_\theta^n g(\theta, t)$ are also bounded. The bounds we get are in general *not uniform* in w but this is not very important for our purposes. Indeed, we will need only:

Proposition 5.4. *Fix $b > 0$. There exists a function $c(w)$, which is non-increasing with respect to $w \in (0, w_0]$, such that for $t < (1-b)t_f$,*

$$\max_\theta |\partial_t^2 \tilde{h}(\theta, t)| \le c(w). \tag{5.20}$$

Proof. Recall (5.7) and (5.13):

$$\partial_t^2 \tilde{h} = -\frac{1}{\tilde{a}}(g^2 \partial_\theta^2 g + g^3). \tag{5.21}$$

Thus we just have to bound $\partial_\theta^2 g$, since we have already proved that g itself is bounded. For this, we adapt the method used by Gage and Hamilton [13] for the special case of the isotropic curve-shortening flow where $a \equiv 1$. What they observed [13, Lemma 4.4.2] is that, if the curvature and its θ-derivative are bounded (which we proved in Proposition 5.3), the t-derivative of $\Phi(t) := \int_0^{2\pi} [\partial_\theta^2 g(\theta, t)]^4 \, d\theta$ can be upper bounded by a constant times $\Phi(t)$ itself, and then one can integrate the inequality with respect to t to get a bound on $\Phi(t)$ in terms of $\Phi(0)$. In our case, with a similar computation, we find that $(d/dt)\Phi(t)$ is upper bounded by $\Phi(t)$ times a constant depending on $\|\partial_\theta a^{(w)}\|_\infty$, which is finite uniformly for $w \le 1$. Since $\Phi(0)$ is also bounded for w in compact subsets of $(0, 1)$ (cf. Assumptions 5.1(5) and 5.2(4)), we get $\Phi(t) \le c_1(w)$ for $w \in (0, 1)$ and $t < (1-b)t_f$, and we can choose c_1 to be decreasing. In general, c_1 will diverge when w approaches zero.

A similar computation (cf. [13, Lemma 4.4.3] when $a(\theta) \equiv 1$) shows that

$$\Psi(t) := \int_0^{2\pi} [\partial_\theta^3 g(\theta, t)]^2 \, d\theta \le c_2(w)$$

with $c_2(\cdot)$ decreasing in $(0, 1)$. Then one uses the fact that for a smooth, 2π-periodic function f one has (cf. [13, Corollary 4.4.4])

$$\|f\|_\infty^2 \le C \int_0^{2\pi} (f^2 + (\partial_\theta f)^2) \, d\theta$$

for some universal constant C, applied with $f(\cdot) = \partial_\theta^2 g(\cdot, t)$, to get $\|\partial_\theta^2 g\|_\infty \le c_3(w)$ as we wished. □

Proof of Theorem 5.1. We are now ready to prove that $(\gamma(t))_t$ provides a classical solution of (2.5). This is based on the following easy consequence of the Arzelà–Ascoli Theorem:

Lemma 5.5. *Let $f^{(n)}$ be a sequence of periodic C^1 functions on $[0, 2\pi]$ such that both sequences $f^{(n)}$ and $\partial_x f^{(n)}$ are uniformly bounded and equicontinuous. If $f^{(n)} \to f$ as $n \to \infty$, then f is C^1 and $\partial_x f = \lim_n \partial_x f^{(n)}$, where the convergence is uniform and does not require subsequences.*

First of all, we note that $\tilde{h}(\cdot, t)$ does converge (for $w \to 0$) to $h(\cdot, t)$ for every fixed $t < t_f$. This just follows from the fact that $\tilde{\gamma}(t)$ converges to $\gamma(t)$ in the Hausdorff distance. Furthermore, convergence is uniform in $t < t_f(1-b)$ for every fixed b. This is true because the area difference between $\tilde{\gamma}(t)$ and $\gamma(t) \subset \tilde{\gamma}(t)$ tends to zero when w does (uniformly in t) and the curvature is uniformly bounded: then, if $\tilde{h}(\theta, t) - h(\theta, t)$ were larger than some δ independent of w for some (θ, t), necessarily the area difference would be larger than some $c(\delta)$ at that time.

Applying Lemma 5.5 and recalling (5.1), we find that, for t fixed, $\partial_\theta \tilde{h}(\theta, t)$ and $\tilde{k}(\theta, t)$ converge to $\partial_\theta h(\theta, t)$ and $k(t, \theta)$ respectively and this convergence is uniform in θ (knowing that the curvature is Lipschitz is important here). Note by the way that $k(\cdot, t)$ is Lipschitz, since $\|\partial_\theta \tilde{k}(\cdot, t)\|_\infty$ is uniformly bounded.

Then applying dominated convergence (which is allowed in view of Proposition 5.3), one gets

$$h(\theta, t) - h(\theta, s) = -\int_s^t a(\theta) k(\theta, u)\, du,$$

which is an integrated version of (2.5). To get the stronger statement (2.5), we need to prove that $k(\theta, t)$ is continuous as a function of t.

First of all, we prove that one can find a function $\varepsilon : (0, 1) \ni w \mapsto \varepsilon(w) \in \mathbb{R}_+$, increasing and going to zero as $w \to 0$ such that for all θ and $t \leq (1 - b)t_f$,

$$|\tilde{k}(\theta, t) - k(\theta, t)| \leq \varepsilon(w). \tag{5.22}$$

If this were not the case then, thanks to $\tilde{k}(\cdot, t)$ and $k(\cdot, t)$ being uniformly Lipschitz, we would have, say, for arbitrarily small w and for some $\varepsilon > 0$ and $t < t_f(1 - b)$,

$$\tilde{k}(\theta, t) - k(\theta, t) \geq \varepsilon$$

for $\theta \in [\bar{\theta}, \bar{\theta} + \varepsilon]$ for some $\bar{\theta} \in [0, 2\pi]$. But since (cf. (5.1))

$$(\partial_\theta^2 + 1)(h(\theta, t) - \tilde{h}(\theta, t)) = \frac{1}{k(\theta, t)} - \frac{1}{\tilde{k}(\theta, t)},$$

this would contradict the uniform convergence of $\tilde{h}(\cdot, \cdot)$ to $h(\cdot, \cdot)$.

On the other hand, from Proposition 5.4, for all θ and $t, s \leq (1 - b)t_f$,

$$|\tilde{k}(\theta, t) - \tilde{k}(\theta, s)| \leq c(w)|t - s|.$$

Together with (5.22) this implies that

$$|k(\theta, t) - k(\theta, s)| \leq \inf_w (2\varepsilon(w) + c(w)|t - s|). \tag{5.23}$$

The right-hand side clearly tends to zero with $|t - s|$ (choose a sequence $\{w_k\}$ tending to zero; if $c(w_k)$ does not diverge we are done; otherwise, compute the right-hand side for $w = w_k$ with the largest value of k such that $c(w_k) \leq |t - s|^{-1/2}$). This shows that $t \mapsto k(\theta, t)$ is continuous away from t_f, and the proof is complete. □

6. Proof of Theorem 2.2: evolution of a convex droplet

The proof is very similar to that of Theorem 2.3 in the scale-invariant case (Section 4), and therefore it will only be sketched. We will also try to use the same notation as in Section 4 as much as possible.

First we present two statements that are analogous to Propositions 4.2 and 4.1:

Proposition 6.1. *Let $\mathcal{D}$ be convex with a bounded curvature function. For every $\alpha > 0$, w.h.p.,*

$$\mathcal{A}_L(L^2 t) \subset L\mathcal{D}^{(\alpha)} \quad \textit{for every } t \geq 0 \tag{6.1}$$

(*recall definition* (2.6)). *Moreover, for every $\alpha > 0$ there exists $\varepsilon_1(\alpha, k_{\max}) > 0$ such that w.h.p.,*

$$\mathcal{A}_L(L^2 t) \supset L\,\mathcal{D}^{(-\alpha)} \quad \textit{for every } t \in [0, \varepsilon_1]. \tag{6.2}$$

Proof. The proof of (6.1) is essentially identical to that of (4.3), so we give no details. As for (6.2), given α it is possible to find a finite collection $\{\mathscr{D}_i\}_i$ such that:

- each $\mathscr{D}_i$ is an open convex subset of $\mathbb{R}^2$, obtained from (the interior of) the invariant shape $\mathscr{D}$ via a suitable translation and shrinking;
- $\mathscr{D}_i \subset \mathcal{D}$ for every i;
- $\bigcup_i \mathscr{D}_i \supset \mathcal{D}^{(-\alpha/2)}$.

Given $\eta > 0$, thanks to Proposition 4.2 there exists $\varepsilon > 0$ such that, w.h.p., for every $t < \varepsilon$,

$$\mathcal{A}_L(L^2 t) \supset \bigcup_i L\mathscr{D}_i^{(-\eta)}.$$

Here we use monotonicity (because $\mathscr{D}_i \subset \mathcal{D}$) and the fact that the union of a finite number of events which occur w.h.p. still has probability tending to 1. Note that the choice of ε depends on η but also on the diameter of the smallest set in the collection $\{\mathscr{D}_i\}_i$ and consequently on $k_{\max}$. Then, if η is small enough (depending on α) it is clear that $\bigcup_i \mathscr{D}_i^{(-\eta)} \supset \mathcal{D}^{(-\alpha)}$ (recall that the $\mathscr{D}_i$ are open sets, so every $x \in \mathcal{D}^{(-\alpha)}$ is contained in the interior of at least one $\mathscr{D}_i$). □

Proposition 6.2. *Let $\mathcal{D}$ be a convex set whose curvature function is $\mathbb{L}(k)$-Lipschitz and is bounded away from zero and infinity. For all $\delta > 0$ there exists $\varepsilon_0(\delta, k_{\min}, k_{\max}, \mathbb{L}(k)) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, w.h.p.,*

$$\mathcal{A}_L(L^2\varepsilon) \subset L\mathcal{D}(\varepsilon(1-\delta)), \tag{6.3}$$

$$\mathcal{A}_L(L^2\varepsilon) \supset L\mathcal{D}(\varepsilon(1+\delta)), \tag{6.4}$$

where we recall that $\mathcal{D}(t)$ is the set enclosed by the curve $\gamma(t)$.

Proof of Theorem 2.2 assuming Propositions 6.2 and 6.1. It is enough to prove (2.7) for $t < (1-b)t_f$ and arbitrary $b > 0$. Then the statement for $t \geq (1-b)t_f$ and also (2.8) follow from the fact that the disappearance time of a droplet of diameter ℓ is w.h.p. $O(\ell^2)$ (recall that $\gamma(t)$ shrinks to a point when $t \to t_f$ in the sense that its diameter converges to zero). Define $k^*_{\min} > 0$ (resp. $k^*_{\max}, \mathbb{L}^*_k < \infty$) to be the infimum (resp. maximum) of $k_{\min}(s)$ (resp. $k_{\max}(s), \mathbb{L}(k(s))$) on $[0, (1-b)t_f]$. Fix δ' small and let $\varepsilon < \varepsilon_0(\delta', k^*_{\min}, k^*_{\max}, \mathbb{L}^*_k)$ and $\varepsilon < \varepsilon_1(\delta/2, k^*_{\max})$ with $\varepsilon_0, \varepsilon_1$ defined in Propositions 6.1 and 6.2. Using the Markov property and the monotonicity of our process we get, w.h.p., for any k such that $\varepsilon k < (1-b)t_f$,

$$\mathcal{A}_L(L^2 k\varepsilon) \subset L\mathcal{D}(k\varepsilon(1-\delta')). \tag{6.5}$$

From (6.5) and Proposition 6.1 we deduce that w.h.p., for every $t \le (1-b)t_f$,

$$\mathcal{A}_L(L^2 t) \subset L[\mathcal{D}(\lfloor t/\varepsilon \rfloor \varepsilon(1-\delta'))]^{(\delta/2)} \subset L[\mathcal{D}((t-\varepsilon)(1-\delta'))]^{(\delta/2)}.$$

Setting $\varepsilon' = t_f\delta' + \varepsilon$ this implies that w.h.p.,

$$\mathcal{A}_L(L^2 t) \subset L[\mathcal{D}(t-\varepsilon')]^{(\delta/2)} \quad \text{for every } t \le (1-b)t_f.$$

Finally observe (this follows from (2.5)) that the Hausdorff distance between $\mathcal{D}(t-\varepsilon')$ and $\mathcal{D}(t)$ is at most $\varepsilon' k^*_{\max} \max_\theta |a(\theta)|$, so that if ε' is chosen such that $\varepsilon' k^*_{\max} \max_\theta |a(\theta)| < \delta/2$ we get (2.7).

The lower bound is proven similarly; here one has to use the assumption $\varepsilon < \varepsilon_1(\delta/2, k^*_{\max})$. □

6.1. Upper bound: proof of (6.3)

Definition 6.3. Define $(P_i(t))_{i=1}^4$ to be the four "poles" of $\mathcal{D}(t)$, where the tangent vector is either horizontal or vertical (recall that $\mathcal{D}(t)$ is strictly convex at all times under our assumptions, cf. discussion after (5.12), so that the four poles are distinct and uniquely defined). $P_1(t)$ denotes the "north pole" and the others are numbered in clockwise order. Denote by $(x(P_i(t)), y(P_i(t)))$ (resp. $(u(P_i(t)), v(P_i(t)))$) the coordinates of $P_i(t)$ in the coordinate system $(\mathbf{f}_1, \mathbf{f}_2)$ (resp. $(\mathbf{e}_1, \mathbf{e}_2)$). When $t = 0$ we omit the time coordinate.

An equivalent formulation of (6.3) is: for all $\delta > 0$ and ε small enough, w.h.p.,

$$\sigma_x(\varepsilon L^2) = + \quad \text{for every } x \in L[\mathcal{D}(\varepsilon(1-\delta))]^c. \tag{6.6}$$

Given some small ξ we divide $[\mathcal{D}(\varepsilon(1-\delta))]^c$ into eight pieces $(M_i)_{i=1}^4$ and $(N_i)_{i=1}^4$ as follows (this is analogous to the definition (4.9) in the scale-invariant case, cf. Figure 6):

$$M_1(\varepsilon, \xi) := ([u(P_1) + \xi, \infty) \times [v(P_2) + \xi, \infty)) \setminus \mathcal{D}(\varepsilon(1-\delta)),$$

while $N_1(\varepsilon, \xi)$ is the infinite component of $([u(P_1) - \xi, u(P_1) + \xi] \times \mathbb{R}) \setminus \mathcal{D}(\varepsilon(1-\delta))$ which contains P_1. The sets M_i, N_i are defined analogously for $i = 2, 3, 4$, so that $[\mathcal{D}(\varepsilon(1-\delta))]^c = \bigcup_{i=1}^4 (M_i \cup N_i)$. Equation (6.6) is proved if one can prove that for every i, and ε small enough, w.h.p.,

$$\sigma_x(\varepsilon L^2) = + \quad \text{for every } x \in LM_i(\varepsilon, \xi), \tag{6.7}$$

$$\sigma_x(\varepsilon L^2) = + \quad \text{for every } x \in LN_i(\varepsilon, \xi). \tag{6.8}$$

Of course one can focus on $i = 1$, the other cases being obtained by a permutation of coordinates.

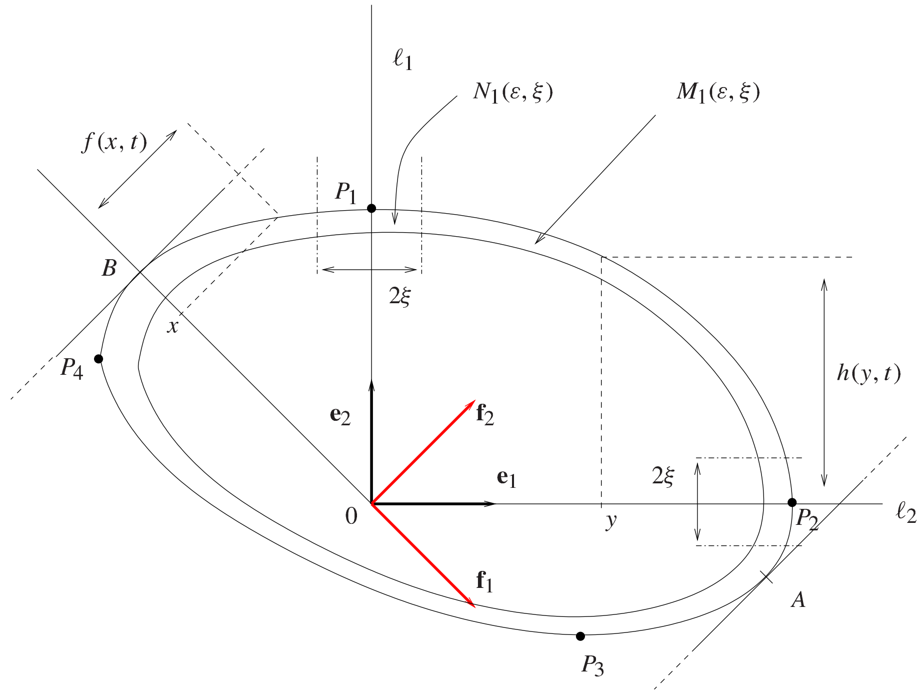

Fig. 12. The larger convex set is $\mathcal{D}$ and the smaller one is $\mathcal{D}(\varepsilon(1-\delta))$. The poles P_i of $\mathcal{D}$ are marked with black dots (for convenience we have chosen P_1 on the vertical axis and P_2 on the horizontal one). The graph in $(\mathbf{f}_1, \mathbf{f}_2)$ of the anticlockwise portion of $\partial\mathcal{D}$ between A and B is $f(\cdot, 0)$ and the graph in $(\mathbf{e}_1, \mathbf{e}_2)$ of the portion of $\partial\mathcal{D}$ between P_4 and P_2 is $h(\cdot, 0)$. For the proof of (6.7), boundary spins to the left of ℓ_1 are set to "$-$" below P_1 and to "$+$" above; boundary spins below ℓ_2 are set to "$-$" to the left of P_2 and to "$+$" to the right.

6.1.1. Proof of (6.7). We use the notation $f(\cdot, t)$ for the function whose graph in the coordinate system $(\mathbf{f}_1, \mathbf{f}_2)$ is the portion of $\partial\mathcal{D}(t)$ which goes in the anticlockwise direction from point A where the tangent forms an angle $\pi/4$ with the horizontal axis (cf. Figure 12) to point B where the angle is $(5/4)\pi$. The domain of definition of $f(\cdot, t)$ decreases with time (because $\mathcal{D}(t)$ shrinks) but for t small enough it includes $[x(P_1), x(P_2)]$. Let $\mathcal{D}_1$ be the "triangular-shaped" region bounded by $\partial\mathcal{D}$, by the vertical line ℓ_1 passing through P_1 and by the horizontal line ℓ_2 passing through P_2 (note that $\mathcal{D}_1$ may not be included in $\mathcal{D}$).

We consider a modified dynamics in the north-east quadrant $[Lu(P_1), \infty) \times [Lv(P_2), \infty)$ bounded by the lines $L\ell_1, L\ell_2$. All the spins are initially "$-$" in $L\mathcal{D}_1$ and "$+$" otherwise. As for boundary spins, the spins at distance at most 1 to the left of $L\ell_1$ are frozen to "$-$" if they are below LP_1 and to "$+$" if they are above. The spins at distance at most 1 below $L\ell_2$ are frozen to "$-$" if they are to the left of LP_2 and to "$+$" otherwise (see Figure 12). In the quadrant under consideration, this dynamics dominates the original one (for the inclusion order of the set of "$-$" spins). Let $F_L(\cdot, t)$ denote the function whose graph in $(\mathbf{f}_1, \mathbf{f}_2)$ is the interface between "$-$" and "$+$" spins for this dynamics.

Using exactly the same argument as in (4.15) we get

$$\lim_{L\to\infty} \sup_{x\in[x(P_1),x(P_2)]} \sup_{t\le T} \left| \frac{1}{L} F_L(xL, tL^2) - g(x,t) \right| = 0$$

in probability, where g is the solution for $t \ge 0$ and $x \in (x(P_1), x(P_2))$ of

$$\begin{cases} \partial_t g(x,t) = \frac{1}{4}\partial_x^2 g(x,t), \\ g(\cdot,t) = f(\cdot,0), \\ g(x(P_1),t) = y(P_1) \text{ and } g(x(P_2),t) = y(P_2). \end{cases}$$

Thus it remains to prove that for every $\tilde{x}_1, \tilde{x}_2$ satisfying $x(P_1) < \tilde{x}_1 < \tilde{x}_2 < x(P_2)$ and every $x \in (\tilde{x}_1, \tilde{x}_2)$,

$$g(x,\varepsilon) < f(x,(1-\delta)\varepsilon). \tag{6.9}$$

Lemma 4.3 (which is valid also in this case, since the curvature is Lipschitz and therefore $\partial_x^2 f(\cdot,0)$ is uniformly continuous) allows us to write that for any fixed η, and ε small enough,

$$g(x,\varepsilon) \le f(x,0) + \frac{\varepsilon}{4}(\partial_x^2 f(x,0) + \eta). \tag{6.10}$$

We are left with the task of estimating the right-hand side of (6.9). For any $\theta \in (0,\pi/2)$ and $s > 0$ define $x(\theta,s)$ to be the $\mathbf{f}_1$ coordinate, in the $(\mathbf{f}_1,\mathbf{f}_2)$ coordinate system, of the point of $\gamma(s)$ where the outward normal vector forms an anticlockwise angle θ with the horizontal vector $\mathbf{e}_1$. Note that for $s \ge 0$, $x(\cdot,s)$ defines a bijective function. We denote by $\theta(\cdot,s)$ its inverse.

It is more practical for the purposes of this section to rewrite the curve-shortening flow in the $(\mathbf{f}_1,\mathbf{f}_2)$ coordinate system. Using the explicit expression (2.4) of $a(\theta)$, some trigonometry and the expression $|f''(x)|/(1+(f'(x))^2)^{3/2}$ for the absolute value of the curvature at the point $(x, f(x))$ of the curve given by the graph of a function $x \mapsto f(x)$, one finds that for $\theta \in (0,\pi/2)$,

$$a(\theta)k(\theta,s) = -\frac{1}{4}\partial_x^2 f(x(\theta,s),s)\cos(\theta-\pi/4)$$

and

$$\partial_t f(x,s) = -\frac{a(\theta(x,s))k(\theta(x,s),s)}{\cos(\theta(x,s)-\pi/4)} = \frac{1}{4}\partial_x^2 f(x,s), \tag{6.11}$$

so that

$$f(x,(1-\delta)\varepsilon) = f(x,0) + \int_0^{(1-\delta)\varepsilon} \frac{1}{4}\partial_x^2 f(x,s)\,ds.$$

We need therefore to prove time regularity of $\partial_x^2 f(\cdot,s)$:

Lemma 6.4. *One has*

$$\sup\{|\partial_t f(x,s) - \partial_t f(x,0)| : s \in [0,t] \text{ and } x \in [x(P_1),x(P_2)]\} \le \Psi(t, k_{\max}, k_{\min}, \mathbb{L}(k)) \tag{6.12}$$

where Ψ tends to zero with the first argument.

Proof. Recall from Section 5 that the curvature function $k(\theta, s)$ is jointly continuous in (θ, s) and that its modulus of continuity depends only on $k_{\max}, k_{\min}, \mathbb{L}(k)$. Thus using (6.11) it is sufficient to prove that $\theta(x, s)$ is a continuous function in s uniformly in x:

$$\sup\{|\theta(x,s) - \theta(x,0)| : s \in [0,t] \text{ and } x \in [x(P_1), x(P_2)]\} \le \Psi_2(t, k_{\max}, k_{\min}, \mathbb{L}(k))$$

where again Ψ_2 tends to zero as $t \to 0$. This comes from the continuity of $x(\theta, \cdot)$:

$$\sup\{|x(\theta,s) - x(\theta,0)| : s \in [0,t] \text{ and } \theta \in [0,\pi/2]\} \le \Psi_3(t, k_{\max}, k_{\min}, \mathbb{L}(k)),$$

and from the fact that $x(\cdot, s)$ is strictly monotone: for $t \ge 0$,

$$\inf\{|\partial_\theta x(\theta,s)| : s \le t, \theta \in [0,\pi/2]\} > c(k_{\min}) > 0.$$

Both properties are consequences of

$$x(\theta,t) = x(\pi/4,t) - \int_{\pi/4}^{\theta} \frac{\cos(\theta' - \pi/4)\, d\theta'}{k(\theta',t)},$$

which is easily derived from (5.3)–(5.4). □

We finally see that for $x \in (x(P_1), x(P_2))$ and ε small enough (as a function of $k_{\min}, k_{\max}, \mathbb{L}(k)$),

$$f(x,(1-\delta)\varepsilon) \ge f(x,0) + (1-\delta)\frac{\varepsilon}{4}(\partial_x^2 f(x,0) - \eta).$$

Thus, combining this with (6.10), we see that (6.9) is proved if one has

$$\partial_x^2 f(x,0) + \eta < (1-\delta)(\partial_x^2 f(x,0) - \eta),$$

i.e. $2\eta + \delta\partial_x^2 f(x,0) \le 0$. For this it is sufficient to have η small enough, since (cf. (6.11)) $\sup\{\partial_x^2 f(x,0) : x \in [x(P_1), x(P_2)]\}$ can be upper bounded by a negative constant times the minimal curvature $k_{\min}$, which is strictly positive. □

6.1.2. Proof of (6.8). Set $h(\cdot, t)$ to be the continuous concave function whose graph in the $(\mathbf{e}_1, \mathbf{e}_2)$ coordinate system is the portion of $\gamma(t)$ which goes from $P_2(t)$ to $P_4(t)$ with the anticlockwise orientation. Given a small η choose ξ small enough so that $\sup\{|\partial_x h(x,0)| : u(P_1) - \xi \le x \le u(P_1) + \xi\} \le \eta$.

Consider the C^1 function $\bar{h}(\cdot)$ equal to $h(\cdot, 0)$ on $[u(P_1) - 2\xi, u(P_1) + 2\xi]$ and affine outside. Assume for definiteness that $\bar{h}(u(P_1) - 4\xi) \le \bar{h}(u(P_1) + 4\xi)$. Define $\xi^- = u(P_1) - 4\xi$ and $\xi^+ = \inf\{x > u(P_1) : \bar{h}(x) = \bar{h}(\xi^-)\}$. We consider the restriction of $\bar{h}$ to $[\xi^-, \xi^+]$ and still call it $\bar{h}$. Define

$$J^1 := [\xi^+, \infty) \times [\bar{h}(\xi^+), \infty), \qquad J^2 := (-\infty, \xi^-] \times [\bar{h}(\xi^+), \infty).$$

We consider the same chain of monotonicities as in the scale-invariant case (Section 4.3.2) and we end up with a dynamics in the half-strip $[L\xi^-, L\xi^+] \times [L\bar{h}(\xi^+), \infty)$ with boundary spins frozen to "+" in $L(J^1 \cup J^2)$ and to "−" in $\mathbb{Z}^* \times (-\infty, L\bar{h}(\xi^+)]$ and an initial condition with "−" spins under the graph of $L\bar{h}(\cdot/L)$. Also, the dynamics thus obtained does not allow moves that make the interface disconnected. Calling this dynamics $(\sigma_2(t))_{t\geq 0}$, we see that (4.23) is satisfied.

Define $H_L : [L\xi^-, L\xi^+] \to \mathbb{Z}$ to be the function whose graph in $(\mathbf{e}_1, \mathbf{e}_2)$ is the interface between "+" and "−" spins. We have to prove

$$\frac{1}{L} H_L(Lx, \varepsilon L^2) \leq h(x, (1-\delta)\varepsilon) \quad \text{for every } x \in (u(P_1) - \xi, u(P_1) + \xi). \tag{6.13}$$

Following the same steps as in (4.25) to (4.28) (recall that $\partial_x^2 h(\cdot, 0)$ is uniformly continuous by the Lipschitz curvature assumption) one finds that the left-hand side of (6.13) is upper bounded w.h.p. by

$$h(x, 0) + \frac{\varepsilon}{2}(1+\eta)^{-2}\big(\partial_x^2 h(u(P_1), 0) + r(x, \mathbb{L}(k))\big) + o(\varepsilon), \tag{6.14}$$

where $r(x, \mathbb{L}(k))$ tends to 0 when $x \to u(P_1)$.

To estimate the r.h.s. of (6.13), one remarks that, in analogy with (6.11),

$$\partial_s h(x, s) = -a(\theta(x, s))k(\theta(x, s), s)/\sin(\theta(x, s)), \tag{6.15}$$

so that $\partial_t h(x, t)$ is continuous in x and t (since θ is around $\pi/2$, $\sin(\theta(x, s))$ is bounded away from zero). Moreover

$$\partial_t h(u(P_1), 0) = \frac{1}{2}\partial_x^2 h(u(P_1), 0), \tag{6.16}$$

which can be obtained directly from $a(0) = 1/2$ and from the fact that the curvature of $\mathcal{D}$ at the north pole P_1 equals minus the second derivative of $h(x, 0)$ computed at $x = u(P_1)$. Thus for every $x \in (u(P_1) - \xi, u(P_1) + \xi)$,

$$h(x, (1-\delta)\varepsilon) \geq h(x, 0) + (1-\delta)\frac{\varepsilon}{2}(1+\eta)\partial_x^2 h(u(P_1), 0), \tag{6.17}$$

and (6.13) is proven (by combining (6.14) and (6.17)) if one chooses η and ξ small enough.

6.2. Lower bound: proof of (6.4)

We are confident that the reader is by now convinced that the proof of Theorem 2.2 is essentially identical to that in the scale-invariant case, modulo the fact that the definitions of the various subsets of $\mathbb{R}^2$ needed to define the regions where spins are frozen to "−" or "+" (U, J^1, J^2, etc.) have to be adapted in the obvious way due to the lack of discrete-rotation symmetry of the general initial droplet $\mathcal{D}$. We will therefore skip the proof of (6.4) altogether and limit ourselves to indicating the only point where some (minor) care has to be taken.

The definition of the set U in (4.32) is replaced by $U := \mathcal{D}^{(-\nu)}$ (cf. (2.6)). Let s_1 be the vertical segment obtained by moving downwards from the "north pole of U" up to the point c where s_1 meets s_2, the horizontal segment obtained by moving to the left from the "east pole" of U until c is reached. To prove the analog of (4.34), mimicking

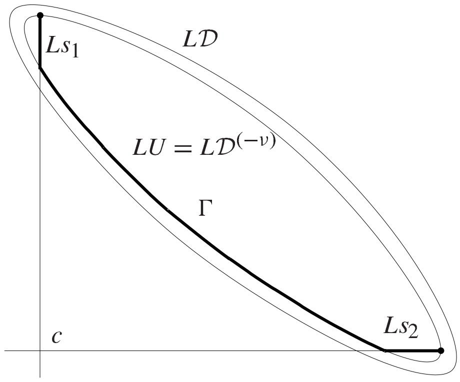

Fig. 13

the proof given in Section 4.4.4, one would like to apply (6.2) in order to freeze to "−" all the spins along the two rescaled segments Ls_1, Ls_2. This is however not allowed in general, because nothing guarantees that they are entirely contained in LU, i.e., that $c \in U$ (this problem does not occur for the invariant shape $\mathscr{D}$, where c is the origin). The solution however is simple (cf. Figure 13): one just freezes to "−" all the spins along the portions of Ls_1, Ls_2 which are inside LU, and along the shorter portion of $L\partial U$ which connects them (call this portion Γ). The point is that in this situation the $+/-$ interface between north and east poles follows again the corner dynamics and Theorem 3.2 is applicable. The freezing of "−" spins along Γ is equivalent to putting a hard-wall constraint in the corner dynamics (the interface is not allowed to cross a zig-zag path which approximates Γ), but this is irrelevant: since Γ is at distance of order L away from the linear profile the corner dynamics approaches for long times, the probability that the interface even feels the hard-wall constraint within the diffusive times of order L^2 we are interested in goes to zero with L (this again can be seen via Theorem 3.2). Other than that, the proof of (6.4) is identical to that in the $\mathcal{D} = \mathscr{D}$ case.

7. Proof of Theorem 3.2: scaling limit for SSEP

The first step is to discretize (3.3), so that instead of working with $\phi(\cdot,\cdot)$ we get the solution $\Phi(\cdot,\cdot)$ of the analogous discrete Cauchy problem:

$$\begin{cases} \partial_t \Phi(x,t) = \frac{1}{2}\Delta\Phi(x,t), \\ \Phi^L(0,t) = h_0^0 = 0, \\ \Phi^L(L,t) = h_L^0, \\ \Phi^L(x,0) = h_x^0, \end{cases}$$

for every $t \geq 0$ and $x \in \{1,\dots,L-1\}$. Here Δ is the discrete Laplacian operator:

$$(\Delta f)(x) := f(x+1) + f(x-1) - 2f(x), \qquad \forall x \in \{1,\dots,L-1\}. \tag{7.1}$$

Note that $\Phi(x,t) = \mathbb{E}[h_x(t)]$, with $(h(t))_{t\geq 0}$ the process with generator (3.2), and that $\Phi(0,t) - h_0(t) = \Phi(L,t) - h_L(t) = 0$. It is a standard result that Φ, the solution of the discrete space heat equation, converges to ϕ in all reasonable norms when $L \to \infty$ in the diffusive limit. We record this result here:

Lemma 7.1.

$$\lim_{L\to\infty} \max_{t\in[0,T]} \max_{x\in[0,1]} \frac{1}{L} |\Phi(\lfloor xL \rfloor, tL^2) - L\phi(x,t)| = 0.$$

By Lemma 7.1, it remains to prove

$$\lim_{L\to\infty} \mathbb{P}\Big[\max_{t\in[0,TL^2]} \max_{x\in\{1,\dots,L-1\}} |h_x(t) - \Phi(x,t)| < \varepsilon L \Big] = 1. \tag{7.2}$$

Both $h_\cdot(t)$ and $\Phi(\cdot,t)$ are 1-Lipschitz functions (for all t) so that $|h_\cdot(t) - \Phi(\cdot,t)|$ is 2-Lipschitz and

$$\Big\{ \max_{x\in\{1,\dots,L-1\}} |h_x(t) - \Phi(x,t)| \geq a \Big\} \Rightarrow \Big\{ \sum_{x=1}^{L-1} [h_x(t) - \Phi(x,t)]^2 \geq a^3/3 \Big\}.$$

As a consequence, (7.2) is equivalent to the following $\mathbb{L}_2$ convergence statement:

Proposition 7.2. *The following convergence in probability holds:*

$$\lim_{L\to\infty} \sup_{t\in[0,L^2T]} \frac{1}{L^3} \sum_{x=1}^{L-1} [h_x(t) - \Phi(x,t)]^2 = 0. \tag{7.3}$$

Proof. The restriction of the operator Δ to

$$\Lambda_L = \big\{ g : \{0,\dots,L\} \to \mathbb{R} : g(0) = g(L) = 0 \big\}$$

is self-adjoint (for the canonical scalar product on $\mathbb{R}^{L-1}$ denoted henceforth by $\langle\cdot,\cdot\rangle$), and the family of functions

$$f_k : \{0,\dots,L\} \ni x \mapsto \sqrt{2/L}\,\sin(k\pi x/L), \qquad k = 1,\dots,L-1,$$

forms an orthonormal basis of Λ_L of Δ-eigenfunctions, with respective eigenvalues

$$-\lambda_k := 2\cos(\pi k/L) - 2 < 0.$$

As the function $x \mapsto h_x(t) - \Phi(x,t)$ is in Λ_L, it can be decomposed with respect to this basis. We use the notation H_t^k for its k-th coordinate (multiplied by $\sqrt{L/2}$ for convenience):

$$H_t^k := \sum_{x=0}^{L} [h_x(t) - \Phi(x,t)] \sin(k\pi x/L).$$

The quantity one wants to estimate in (7.3) is

$$\sup_{t\in[0,L^2T]} \frac{2}{L^4} \sum_{k=1}^{L-1} (H_t^k)^2 \leq \frac{2}{L^4} \sum_{k=1}^{L-1} \sup_{t\in[0,L^2T]} (H_t^k)^2. \tag{7.4}$$

We control the right-hand side by controlling each H_t^k separately.

Lemma 7.3. *For every L, $k \in \{1, \dots, L-1\}$ and $t > 0$ one has deterministically*

$$|H_t^k| \le 4L^2/k. \tag{7.5}$$

Moreover for any given T, w.h.p.,

$$|H_t^k| \le L^{7/4} \quad \text{for all } k \le (\log L)^{1/3} \text{ and } t \le L^2 T. \tag{7.6}$$

Proof. The first statement is easy. By summation by parts,

$$H_t^k = \sum_{x=1}^{L} \big([h_x(t) - \Phi(x,t)] - [h_{x-1}(t) - \Phi(x-1,t)] \big) \sum_{y=x}^{L} \sin(k\pi y/L).$$

Then one can check that for every x and k,

$$|[h_x(t) - \Phi(x,t)] - [h_{x-1}(t) - \Phi(x-1,t)]| \le 2,$$

$$\Big| \sum_{y=x}^{L} \sin(k\pi y/L) \Big| \le 2L/k,$$

so that (7.5) follows.

For the second statement, first, one notices that for all $k \in \{1, \dots, L-1\}$, the functions

$$F_k : \Omega_{M_L,N_L} \ni h \mapsto \sum_{x=0}^{L} \sin(\pi k x/L) \left[h_x - \frac{h_L - h_0}{L} x \right]$$

are eigenfunctions of $\mathcal{L}$ with respective eigenvalues $-\lambda_k/2$. Indeed, F_k is just a linear combination of the coordinate function $A_x : h \mapsto h_x$ (plus a constant), and it can be seen from the very definition (3.2) of the generator $\mathcal{L}$ that

$$\mathcal{L}(A_x)(h) = \tfrac{1}{2}(\Delta h)(x) = \tfrac{1}{2}(\Delta \tilde{h})(x)$$

with the notation $\tilde{h}_x = h_x - \frac{h_L - h_0}{L} x$. Hence (note that $\tilde{h} \in \Lambda_L$)

$$2\mathcal{L}F_k(h) = \sqrt{L/2}\,\langle f_k, \Delta\tilde{h} \rangle = \sqrt{L/2}\,\langle \Delta f_k, \tilde{h} \rangle = -\lambda_k \sqrt{L/2}\,\langle f_k, \tilde{h} \rangle = -\lambda_k F_k(h).$$

As a consequence one can rewrite

$$H_t^k = \sum_{x=0}^{L} \sin(k\pi x/L) \tilde{h}_x(t) - e^{-\lambda_k t/2} \sum_{x=0}^{L} \sin(k\pi x/L) \tilde{h}_x(0)$$

and notice that $M_t^k := e^{\lambda_k t/2} H_t^k$ is a martingale. Therefore one can get the result by computing the second moment of M_t^k and using Doob's inequality.

It is not difficult to bound the quadratic variation of M^k. Notice that

$$\mathbb{E}[(M_t^k)^2] = \mathbb{E}\left[\int_0^t d\langle M^k \rangle_s \right]$$

and that

$$d\langle M^k\rangle_s = e^{\lambda_k s} d\langle H^k\rangle_s = e^{\lambda_k s}\sum_{k=1}^{L-1} \sin^2(k\pi x/L)\frac{(\Delta(h(t))(x))^2}{4}\, ds \le L e^{\lambda_k s}\, ds$$

so that $\mathbb{E}[(M_t^k)^2] \le L\int_0^t e^{\lambda_k s}\, ds$. Therefore (using $\lambda_k = \pi^2k^2/L^2(1+o(1))$ uniformly for all $k \le (\log L)^{1/3}$),

$$\mathbb{P}\Big[\sup_{t\in[0,L^2T]} |H_t^k| \ge a\Big] \le \mathbb{P}\Big[\sup_{t\in[0,L^2T]} |M_t^k| \ge a\Big] \le C\frac{L^3 e^{\lambda_k L^2 T}}{a^2k^2}.$$

Using this inequality for $a := L^{7/4}$ and all $k \le (\log L)^{1/3}$ one gets

$$\mathbb{P}\big[\exists t \in [0, L^2T],\ \exists k \le (\log L)^{1/3},\ |H_t^k| \ge L^{7/4}\big] \le \sum_{k\le(\log L)^{1/3}} \frac{C}{k^2\sqrt{L}} e^{k^2\pi^2 T}.$$

One can check that the right-hand side above tends to zero when L goes to infinity, which finishes the proof of Lemma 7.3. □

We now turn to (7.4):

$$\frac{2}{L^4}\sum_{k=1}^{L-1} \sup_{t\in[0,L^2T]} (H_t^k)^2 \le \frac{2}{L^4}\sum_{k\le(\log L)^{1/3}} \sup_{t\in[0,L^2T]} (H_t^k)^2 + 32\sum_{k=\lceil(\log L)^{1/3}\rceil}^{L} k^{-2}.$$

The second term tends to zero (it is roughly $(\log L)^{-1/3}$). The first one is w.h.p. less than

$$\frac{2}{L^4}\sum_{k\le(\log L)^{1/3}} L^{7/2} \le \frac{\log L}{\sqrt{L}}.$$

This completes the proof of Proposition 7.2 and thus also the one of Theorem 3.2. □

Appendix A. Proof of Theorem 3.4: scaling limit for the zero-range process

This section follows quite closely the computations in Appendix A of [23].

A.1. Particle system and monotonicity

For $x \in \{-L, \dots, L\}$ we denote by $\eta_x := h_{x+1} - h_x$ the discrete gradient of h in x. A configuration $h \in \Omega_L$ can be alternatively given by $\eta \in \Theta_L := \{\eta : \{-L, \dots, L\} \to \mathbb{Z}\}$. It turns out that the zero-range process description of the dynamics (cf. Section 3.2) is easier to work with.

For a more formal description of the dynamics we write its generator explicitly. For $\eta \in \Theta_L$ and $x \in \{-L, \dots, L-1\}$, we define the configuration $\vec{\eta}^{(x)}$ as

$$\begin{aligned} \vec{\eta}^{(x)}(x) &:= \eta_x - \mathrm{sg}(\eta_x), \\ \vec{\eta}^{(x)}(x+1) &:= \eta_{x+1} + \mathrm{sg}(\eta_x), \\ \vec{\eta}^{(x)}(y) &:= \eta_y, \quad \forall y \notin \{x, x+1\}. \end{aligned}$$

We define $\overleftarrow{\eta}^{(x)}$ analogously for $x \in \{-L+1, \dots, L\}$ replacing $x+1$ in the second and third lines by $x-1$. The sign function sg is given by

$$\mathrm{sg}(a) := \begin{cases} 1 & \text{if } a > 0, \\ -1 & \text{if } a < 0, \\ 0 & \text{if } a = 0. \end{cases}$$

The generator of the chain seen in the state space Θ_L is given by

$$\mathcal{L}f := \frac{1}{2} \sum_{x=-L}^{L-1} [f(\vec{\eta}^{(x)}) + f(\overleftarrow{\eta}^{(x+1)}) - 2f(\eta)]. \tag{A.1}$$

Note that the dynamics conserves the sum of the η's, i.e. the value of h_{L+1}.

Before going to the core of the proof, we need to change the initial condition slightly. In order to compare with the original one, one needs the following monotonicity statement:

Proposition A.1 (Coupling).

(i) *There is a canonical way of constructing simultaneously the dynamics with generator (3.5) from all possible initial configurations h^0. It has the following monotonicity property: given h^0 and $\bar{h}^0$ with $h_x^0 \geq \bar{h}_x^0$ for all x, the dynamics h and $\bar{h}$ starting from h^0 and $\bar{h}^0$ respectively satisfy $h_x(t) \geq \bar{h}_x(t)$ for every t and x. Moreover, the dynamics started from $h^0 + a$, $a \in \mathbb{Z}$, (a vertically translated version of h^0, including the boundary conditions h_0 and h_{L+1}) is simply $(h(t) + a)_{t \geq 0}$.*

(ii) *There is a canonical way of constructing the dynamics with generator (A.1) from all possible initial configurations η^0. It has the following monotonicity property: given η^0 and $\bar{\eta}^0$ with $\eta_x^0 \geq \bar{\eta}_x^0$ for all x, the dynamics η and $\bar{\eta}$ starting from η^0 and $\bar{\eta}^0$ respectively satisfy $\eta_x(t) \geq \bar{\eta}_x(t)$ for every t and x.*

Proof. The idea of the proof is a canonical construction of the process, similarly to what is done in Section 2.3. It is quite classic but we perfom it here for the sake of completeness.

- For $x \in \{-L+1, L\}$ we define $(\tau_{n,x})_{n \geq 0}$ and $(\tau'_{n,x})_{n \geq 0}$ to be two IID clock processes, with $\tau_{0,x} = 0$ and $\tau_{n+1,x} - \tau_{n,x}$ IID exponential variables of mean 2.
- The process $h(\cdot)$ is càdlàg and constant in time except at the ringing times of the clock processes. At time $\tau_{n,x}$ only h_x is modified, as follows: $h_x(\tau_{n,x}) = h_x(\tau_{n,x}^-) + \mathrm{sg}(h_{x-1}(\tau_{n,x}^-) - h_x(\tau_{n,x}^-))$, the other coordinates being left unchanged. At time $\tau'_{n,x}$ only h_x is modified, as follows: $h_x(\tau'_{n,x}) = h_x((\tau'_{n,x})^-) + \mathrm{sg}(h_{x+1}((\tau'_{n,x})^-) - h_x((\tau'_{n,x})^-))$, the other coordinates being left unchanged.

The reader can check that this allows one to couple the dynamics from all possible initial conditions and that the coupling has the desired properties. This coupling induces a coupling on η that also has the right properties. $\square$

A.2. *Changing the initial condition*

We prove (3.9) working with an initial condition which is not the one, h^0, described in (3.7), which is random and for which the number of particle at a site is given by a geometric variable. The reason for this change of initial condition will appear in the proof of (iii) in Lemma A.3. We explain in this section why this implies the result starting from h^0.

Given a continuous function $\phi^0 : [-1, 1] \to \mathbb{R}$ with $\phi^0(\pm 1) = 0$ and with a finite number of changes of monotonicity, set $(\hat{\eta}_x)_{x\in\{-L,\dots,L\}}$ to be a family of independent variables with the following distribution: if $\phi^0((x+1)/L) - \phi^0(x/L) \geq 0$ then $\hat{\eta}_x$ is a geometric variable of mean $L(\phi^0((x+1)/L) - \phi^0(x/L))$, and if $(\phi^0((x+1)/L) - \phi^0(x/L)) < 0$ then $-\hat{\eta}_x$ is a geometric variable of mean $L(\phi^0(x/L) - \phi^0((x+1)/L))$ (with the convention that $\phi^0(1 + 1/L) = 0$). One sets

$$\hat{h}^0_x = \sum_{y=-L}^{x-1} \hat{\eta}_y.$$

Note that for every $\varepsilon > 0$, w.h.p.,

$$\hat{h}^0_x - L^{1/2+\varepsilon} \leq h^0_x \leq \hat{h}^0_x + L^{1/2+\varepsilon} \quad \text{for every } x \in \{-L, \dots, L+1\}.$$

Let $(h(t))_{t\geq 0}$, $(\hat{h}(t))_{t\geq 0}$ be the dynamics with generator (3.5) started with initial condition h^0, $\hat{h}^0$ respectively, constructed using the canonical way of Proposition A.1(i). Then with high probability, for every $t > 0$ and $x \in \{-L, \dots, L\}$,

$$\hat{h}_x(t) - L^{1/2+\varepsilon} \leq h_x(t) \leq \hat{h}_x(t) + L^{1/2+\varepsilon}.$$

Therefore in order to prove (3.9) for $h(\cdot)$, it is sufficient to prove it for $\hat{h}(\cdot)$. We let $\hat{\eta}_x(t) = \hat{h}_{x+1}(t) - \hat{h}_x(t)$ denote the gradient of $\hat{h}$.

A.3. *Proof of an $\mathbb{L}_2$ statement*

For $(\hat{h}(t))_{t\geq 0}$ defined above one has

Proposition A.2. *For any $t \geq 0$,*

$$\lim_{L\to\infty} \mathbb{E}\left[\frac{1}{L^3} \sum_{x=-L}^{L+1} (\Phi(x, L^2 t) - \hat{h}_x(L^2 t))^2\right] = 0.$$

This result does not directly imply (3.9) ($\hat{h}$ may have *a priori* unbounded gradients), but it is not to difficult to deduce it from Proposition A.2 (see Section A.4). In the rest of the section, for ease of notation we write h, η instead of $\hat{h}, \hat{\eta}$.

Before starting the proof we need some technical statements. First note, recalling the definition of the generator (3.5), that for every $x \in \{-L+1, \dots, L\}$,

$$\begin{aligned} 2\partial_t \mathbb{E}[h_x(t)] &= \mathbb{E}[\mathrm{sg}(\eta_x(t)) - \mathrm{sg}(\eta_{x-1}(t))], \\ 2\partial_t \mathbb{E}[h_x^2(t)] &= \mathbb{E}\big[2h_x(t)(\mathrm{sg}(\eta_x(t)) - \mathrm{sg}(\eta_{x-1}(t))) + (|\mathrm{sg}(\eta_x(t))| + |\mathrm{sg}(\eta_{x-1}(t))|)\big]. \end{aligned} \tag{A.2}$$

Now some remarks:

Lemma A.3. *The following properties hold* (*recall notation in* (3.2))*:*

(i) $\max_x |q_x(t)|$ *is a non-increasing function of* t. *As a consequence,*

$$\forall t > 0,\ \forall x \in \{-L, \dots, L\}, \qquad |q_x(t)| \le \|\partial_x \phi^0\|_\infty.$$

(ii) $\max_x |\sigma(q_{x+1}(t)) - \sigma(q_x(t))|$ *is a non-increasing function of* t (*recall that* $\sigma(u) = u/(1+|u|)$). *Then, using also* (i), *for some* $C(\phi^0) = C(\|\partial_x\phi^0\|_\infty, \|\partial_x^2\phi^0\|_\infty) < \infty$ *one has*

$$\forall t > 0,\ \forall x \in \{-L, \dots, L\}, \qquad |q_{x+1}(t) - q_x(t)| \le C(\phi^0)/L. \tag{A.3}$$

(iii) *For any* t, *the random vectors* $(\eta_x(t))_{x\in\{-L,\dots,L\}}$ *and* $(-\eta_x(t))_{x\in\{-L,\dots,L\}}$ *are stochastically dominated by* $2L+1$ *IID geometric variables with mean* $\|\partial_x\phi^0\|_\infty$.

Proof. For (i) it is sufficient to show that $Q(t) = \max_x q_x(t)$ is non-increasing (by a similar argument one shows that $\min q_x(t)$ is non-decreasing). As the maximum over finitely many differentiable functions, $\max_x q_x(t)$ possesses a right and a left derivative everywhere, and the right derivative is equal to

$$\partial_t^+ Q(t) = \max_{x\in \operatorname{argmax} q_\cdot(t)} \partial_t q_x(t).$$

For any x in $\max_{x\in\operatorname{argmax} q_\cdot(t)}$, one has

$$2\partial_t q_x(t) = \sigma(q_{x+1}(t)) + \sigma(q_{x-1}(t)) - 2\sigma(q_x(t)) \le 0$$

(as $\sigma(q_x(t))$ is maximal), and therefore $Q(t)$ is decreasing.

For (ii): Using the same argument as for (i), we have to note that for any fixed time T and x_0 where $\max_x[\sigma(q_{x+1}) - \sigma(q_x)](T)$ is attained one has

$$\begin{aligned} 2[\partial_t\{\sigma(q_{x_0+1}) - \sigma(q_{x_0})\}](T) &= \sigma'(q_{x_0+1}(T))[\sigma(q_{x_0+2}(T)) - \sigma(q_{x_0+1}(T))] \\ &\quad + \sigma'(q_{x_0}(T))[\sigma(q_{x_0}(T)) - \sigma(q_{x_0-1}(T))] \\ &\quad - [\sigma'(q_{x_0+1}(T)) + \sigma'(q_{x_0}(T))][\sigma(q_{x_0+1}(T)) - \sigma(q_{x_0}(T))] \le 0. \end{aligned}$$

Therefore,

$$|\sigma(q_{x+1}(t)) - \sigma(q_x(t))| \le C(\phi^0)/L.$$

In order to deduce (A.3), write

$$\sigma(q_{x+1}(t)) - \sigma(q_x(t)) = \sigma'(y)[q_{x+1}(t) - q_x(t)]$$

for some $q_{x+1}(t) \leq y \leq q_x(t)$. Since the q_x are bounded (point (i)) and $\sigma(\cdot)$ has uniformly positive derivative on bounded intervals, (A.3) follows.

For (iii): One has $L(\phi^0((x+1)/L) - \phi^0(x/L)) \leq \|\partial_x \phi^0\|_\infty$ so that the initial configuration η^0 is stochastically dominated by $\tilde{\eta}^0$, the configuration given by $2L+1$ IID geometric variables with mean $\|\partial_x \phi^0\|_\infty$. According to Proposition A.1(ii), one can couple the two dynamics η and $\tilde{\eta}$ starting from η^0 and $\tilde{\eta}^0$ so that $\eta(t) \leq \tilde{\eta}(t)$ for all $t \geq 0$. For fixed t the law of $\tilde{\eta}(t)$ is the same as the one of $\tilde{\eta}^0$, as this distribution is stationary for the dynamics. The other domination is proved in the same way. □

Proof of Proposition A.2. We estimate the difference between $\mathbb{E}[L^{-3}\sum_{x=-L}^{L+1}(\Phi(x, L^2t) - h_x(L^2t))^2]$ and the same quantity at time zero, by considering it as the integral of its time derivative:

$$
\begin{aligned}
&\mathbb{E}\bigg[\frac{1}{L^3}\sum_{x=-L}^{L+1}(\Phi(x,L^2t)-h_x(L^2t))^2\bigg] - \mathbb{E}\bigg[\frac{1}{L^3}\sum_{x=-L}^{L+1}(\Phi(x,0)-h_x(0))^2\bigg]\\
&= \frac{1}{L^3}\int_0^{L^2t}\sum_{x=-L+1}^{L}\partial_s\mathbb{E}[(\Phi(x,s)-h_x(s))^2]\,ds\\
&= \frac{1}{L^3}\sum_{x=-L+1}^{L}\int_0^{L^2t}\mathbb{E}\big\{(\Phi(x,s)-h_x(s))(\sigma(q_x(s))-\sigma(q_{x-1}(s)))\\
&\qquad\qquad - \Phi(x,s)(\mathrm{sg}(\eta_x(s))-\mathrm{sg}(\eta_{x-1}(s)))\\
&\quad + h_x(s)(\mathrm{sg}(\eta_x(s))-\mathrm{sg}(\eta_{x-1}(s))) + \tfrac{1}{2}(|\mathrm{sg}(\eta_x(s))| + |\mathrm{sg}(\eta_{x-1}(s))|)\big\}\,ds\\
&= \frac{1}{L^3}\sum_{x=-L}^{L}\int_0^{L^2t}\mathbb{E}\big[-q_x(s)\sigma(q_x(s)) + \eta_x(s)\sigma(q_x(s)) + q_x(s)\,\mathrm{sg}(\eta_x(s))\\
&\qquad\qquad - (|\eta_x(s)| - |\mathrm{sg}(\eta_x(s))|)\big]\,ds\\
&\quad - \frac{1}{L^3}\int_0^{L^2t}\mathbb{E}\big[h_{L+1}(s)(\mathrm{sg}(\eta_L(s)) - \sigma(q_L(s))) + \tfrac{1}{2}(|\mathrm{sg}(\eta_{-L}(s))| + |\mathrm{sg}(\eta_L(s))|)\big]\,ds.
\end{aligned}
$$

The second equality is obtained by expanding the product and using (A.2) and (3.8) to estimate the derivative of each term in the expansion of the square. The third equality is obtained via summation by parts; it gives a term that is due to the boundary effect (the second one), which can be bounded as follows:

$$
\begin{aligned}
&L^{-3}\bigg|\int_0^{L^2t}\mathbb{E}\big[h_{L+1}(s)(\mathrm{sg}(\eta_L(s)) - \sigma(q_L(s))) + \tfrac{1}{2}(|\mathrm{sg}(\eta_{-L}(s))| + |\mathrm{sg}(\eta_L(s))|)\big]\,ds\bigg|\\
&\qquad \leq CL^{-1}(1 + \mathbb{E}\,|h_{L+1}|) = O(L^{-1/2}).
\end{aligned}
$$

Indeed, $h_{L+1}(t)$ is constant through time and is the sum of $2L+1$ independent variables. The mean of this sum is 0 and the variance of each term is bounded as we have supposed ϕ^0 to be smooth. The variance of h_{L+1} is thus $O(L)$. We can also neglect the second term in the first line, as

$$\mathbb{E}\left[\frac{1}{L^3}\sum_{x=-L}^{L+1}(\Phi(x,0)-h_x(0))^2\right]=\frac{1}{L^3}\sum_{x=-L}^{L+1}\mathrm{Var}(h_x(0))=O(L^{-1}),$$

where the last equality is easy to obtain once we notice that h_x is the sum of $L+x$ independent geometric variables with bounded variance.

Set

$$A(x,s):=-q_x(s)\sigma(q_x(s))+\eta_x(s)\sigma(q_x(s))+q_x(s)\,\mathrm{sg}(\eta_x(s))-(|\eta_x(s)|-|\mathrm{sg}(\eta_x(s))|).$$

From the previous equations one gets

$$\mathbb{E}\left[\frac{1}{L^3}\sum_{x=-L+1}^{L}(\Phi(x,L^2t)-h_x(L^2t))^2\right]=\frac{1}{L^3}\int_0^{L^2t}\sum_{x=-L+1}^{L}\mathbb{E}[A(s,x)]\,ds+o(1). \tag{A.4}$$

To understand better the rest of the proof, the reader should notice that if $(\eta_x(s))_{x\in\{-L,\dots,L\}}$ were distributed like geometric variables it would be possible to factorize $\mathbb{E}[A(x,s)]$ into a product of negative sign and from equation (A.4) the proof would be over. Indeed, for $q>0$ and η distributed like a geometric variable of mean $u>0$ (or $-\eta$ distributed like a geometric variable of mean $-u>0$),

$$\mathbb{E}\big[-q\sigma(q)+\eta\sigma(q)+q\,\mathrm{sg}(\eta)-(|\eta|-|\mathrm{sg}(\eta)|)\big]=-(q-u)(\sigma(q)-\sigma(u))\le 0$$

(recall that $\sigma(\cdot)$ is an increasing function). It is not true in general that the $\eta_x(s)$ are geometrically distributed for $s>0$ but it is reasonable to think that their distribution is close to geometric: as the system mixes locally in finite time, what one should observe on finite but large windows is close to an equilibrium measure, and from [1] it is known that the only (infinite-volume translation invariant) equilibrium measures for the zero-range process are convex combinations of products of geometric variables. Most of our efforts will therefore be focused on proving convergence to the infinite volume measure for a space-time averaged version of the probability distribution of the $\eta_x(s)$ (using this space-time average is somehow crucial for the proof to work).

As the limiting object is an infinite volume measure, it is somewhat more convenient to consider $\eta(s)$ as an element of $\mathbb{Z}^{\mathbb{Z}}$ by periodizing it: for the system of size $2L+1$ one sets $\eta_{x+k(2L+1)}=\eta_x$ for every $k\in\mathbb{Z}$ and $x\in\{-L,\dots,L\}$. For $y\in\mathbb{Z}$ one defines θ_y to be the shift operator $\eta\mapsto\theta_x\eta$ defined by

$$\forall x\in\mathbb{Z},\qquad(\theta_y\eta)_x:=\eta_{x+y}. \tag{A.5}$$

We define for each $L>0$ the measure μ_t^L on $\mathbb{Z}^{\mathbb{Z}}$ our space-time averaged measures by its action on local functions (for $K\in\mathbb{N}$ we call $f(\eta)$ a *K-local function* if f is bounded and can be written as a function of $\eta_{|[-K,K]}$; f is a *local function* if there exists a K such that f is K-local):

$$\mu_t^L(f):=\mathbb{E}\left[\frac{1}{tL^2}\frac{1}{2L+1}\int_0^{L^2t}\sum_{y=-L}^{L}f(\theta_y(\eta(s)))\,ds\right]. \tag{A.6}$$

We want to prove that any limit point (when $L \to \infty$) of μ_t^L is an equilibrium and use this information to bound the right-hand side of (A.4).

We introduce some notation to describe the limiting measure. For $u \in \mathbb{R}$ define ρ^u to be a measure on $\eta = (\eta_x)_{x\in\mathbb{Z}}$ such that the η_x are IID geometric variables of mean u if $u \geq 0$, while the $-\eta_x$ are IID geometric variables of mean $-u$ if $u < 0$. If ν is a probability measure on $\mathbb{R}$, define

$$\rho^\nu := \int \rho^u \, \nu(du).$$

Proposition A.4. *Fix $t > 0$. For any subsequence of $(\mu_t^{L_n})_{n\geq 0}$, it is possible to find a subsubsequence $(\mu_t^{L'_n})_{n\geq 0}$ that converges locally to ρ^ν with ν a probability measure on $\mathbb{R}$ with support in $[-\|\partial_x\phi^0\|_\infty, \|\partial_x\phi^0\|_\infty]$, in the sense that for any local function f,*

$$\lim_{n\to\infty} \mu_t^{L'_n}(f) = \rho^\nu(f). \tag{A.7}$$

As a consequence, for any local function f,

$$\limsup_{L\to\infty} \mu_t^L(f) \leq \max_{u\in[-\|\partial_x\phi^0\|_\infty, \|\partial_x\phi^0\|_\infty]} \rho^u(f). \tag{A.8}$$

Remark A.5. Note that the convergence does not hold in the total variation distance: indeed, the μ_t^L give mass one to L-periodic η whereas these configurations have mass zero for the limiting measure.

Proof of Proposition A.4. For any fixed $K > 0$, the sequence of laws of $(\eta_x)_{x\in[-K,K]}$ under $\mu_t^{L_n}$ is tight by Lemma A.3(iii) and hence we can extract a converging subsequence. By a diagonal procedure it is possible to extract a subsequence L'_n of L_n and a family of measures $(\mu_K)_{K\geq 0}$ on $\mathbb{Z}^{[-K,K]}$ such that the law of $(\eta_x)_{x\in[-K,K]}$ under $\mu_t^{L'_n}$ converges to μ^K for all K. By construction for $H \geq K$, μ_H projected on $\mathbb{Z}^{[-K,K]}$ is equal to μ_K, and by the Kolmogorov extension theorem there exists a measure μ on $\mathbb{Z}^{\mathbb{Z}}$ such that μ projected on $\mathbb{Z}^{[-K,K]}$ equals μ_K for all K. Therefore, for every local function f,

$$\lim_{n\to\infty} \mu_t^{L'_n}(f) = \mu(f).$$

We have to show that μ can be written as ρ^ν. First one remarks that $\mu_t^{L_n}$ is translation invariant, so that μ is too. A second point to make is that μ-almost surely all the η_x (that are not equal to zero) have the same sign. Indeed,

$$\begin{aligned}\mu(\exists x, x' \in \mathbb{Z},\ \eta_x\eta'_x < 0) &= \lim_{K\to\infty} \mu(\exists x, x' \in [-K, K],\ \eta_x\eta'_x < 0)\\ &= \lim_{K\to\infty}\lim_{n\to\infty} \mu_t^{L'_n}(\exists x, x' \in [-K, K],\ \eta_x\eta'_x < 0)\end{aligned}$$

and

$$\mu_t^L(\exists x, x' \in [-K, K],\ \eta_x \eta'_x < 0)$$
$$= \frac{1}{tL^2(2L+1)} \int_0^{L^2 t} \sum_{y=-L}^{L} \mathbb{P}\big[\exists x, x' \in [-K+y, K+y],\ \eta_x(s)\eta_{x'}(s) < 0\big]\, ds.$$

One realizes easily that

$$\sum_{y=-L}^{L} \mathbf{1}_{\{\exists x, x' \in [-K+y, K+y],\ \eta_x \eta_{x'} < 0\}}$$

is upper bounded by $2K+1$ times the number of sign changes in $(\eta_x)_{x \in [-L, L+1]}$. From the definition of the dynamics, a transition can only lower the number of sign changes. Its initial value is smaller than the number of changes of monotonicity of ϕ^0 (which is assumed to be finite) plus one (the "plus one" can come from periodizing). Therefore

$$\sum_{y=-L}^{L} \mathbb{P}\big[\exists x, x' \in [-K+y, K+y],\ \eta_x(s)\eta_{x'}(s) < 0\big] \le 2KC(\phi^0).$$

A third point is to show that μ is an invariant measure for the infinite volume dynamics (the infinite volume version of (A.1); call its generator $\mathcal{L}^\infty$). For f a K-local function, one has (for $L \ge K$ large enough)

$$\mu_t^L(\mathcal{L}^\infty f) = \frac{1}{tL^2} \frac{1}{2L+1} \int_0^{L^2 t} \sum_{y=-L}^{L} \mathbb{E}\mathcal{L}^\infty (f \circ \theta_y)(\eta(s))\, ds.$$

For $y \in [-L+K, L-K]$ the infinite volume generator applied to f has the same effect as the finite volume generator so that

$$\int_0^t \mathbb{E}[\mathcal{L}^\infty (f \circ \theta_y)(\eta(s))]\, ds = \int_0^t \partial_s \mathbb{E}[(f \circ \theta_y)(\eta(s))]\, ds$$
$$= \mathbb{E}[(f \circ \theta_y)(\eta(t)) - (f \circ \theta_y)(\eta(0))].$$

Therefore

$$\mu_t^L(\mathcal{L}^\infty f) = \frac{1}{tL^2} \frac{1}{2L+1} \sum_{y=-L+K}^{L-K} \mathbb{E}[(f \circ \theta_y)(\eta(tL^2)) - (f \circ \theta_y)(\eta(0))]$$
$$+ \frac{1}{tL^2} \frac{1}{2L+1} \int_0^{L^2 t} \Big(\sum_{y=-L}^{-L+K-1} + \sum_{y=L-K+1}^{L} \Big) \mathbb{E}[\mathcal{L}^\infty (f \circ \theta_y)(\eta(s))]\, ds = O(1/L).$$

As a consequence, for any local function f,

$$\mu(\mathcal{L}^\infty f) = \lim_{n \to \infty} \mu_t^{L'_n}(\mathcal{L}^\infty f) = 0.$$

Restricted on the event that the η_x all have the same sign, $\mathcal{L}^\infty$ is the generator of the zero-range process with one type of particle and therefore μ is a translation invariant measure for the zero-range process. From [1, Theorem 1.9] one can write $\mu = \rho^\nu$ for some ν. By Lemma A.3(iii), under μ, at time zero, η is dominated by an IID family of geometric variables of mean $\|\partial_x \phi^0\|_\infty$ and so is $-\eta$. This implies the claim on the support of ν.

The second statement of Proposition A.4 is standard; we include its proof for completeness. Given a local f one can extract a subsequence L_n such that

$$\lim_{n\to\infty} \mu_t^{L_n}(f) = \limsup_{L\to\infty} \mu_t^L(f).$$

From L_n one can extract a subsequence L'_n such that $\mu_t^{L'_n}$ converges to ρ^ν so that

$$\lim_{n\to\infty} \mu_t^{L_n}(f) = \lim_{n\to\infty} \mu_t^{L'_n}(f) = \int \rho^u(f)\, \nu(du),$$

which ends the proof. □

Fix a large integer l. For $y \in \mathbb{Z}$ set

$$B_y := \{1+y, \dots, l+y\}. \tag{A.9}$$

For notational convenience, similarly to η in (A.5), one now considers periodized versions $(q_x(s))_{s\in\mathbb{Z}}$ of $q(s)$ and $(A(x,s))_{x\in\mathbb{Z}}$ of $A(\cdot, s)$.

Now, one uses Proposition A.4 to control each term in $\mathbb{E}\sum A(s,x)$.

Lemma A.6.

$$\lim_{l\to\infty} \limsup_{L\to\infty} \frac{1}{L^3}\mathbb{E}\int_0^{L^2 t} \sum_{y=-L}^{L} \left| \frac{1}{l}\sum_{x\in B_y} q_x(s)\,\mathrm{sg}(\eta_x(s)) - q_y(s)\sigma\left(\frac{1}{l}\sum_{x\in B_y}\eta_x(s)\right) ds\right| = 0. \tag{A.10}$$

Proof. Fix $l > 0$. For L large enough, and all $y \in \{-L, \dots, L-l\}$,

$$\begin{aligned}&\left| \frac{1}{l}\sum_{x\in B_y} q_x(s)\,\mathrm{sg}(\eta_x(s)) - q_y(s)\sigma\left(\frac{1}{l}\sum_{x\in B_y}\eta_x(s)\right)\right| \\ &\qquad \le |q_y(s)|\left|\frac{1}{l}\sum_{x\in B_y}\mathrm{sg}(\eta_x(s)) - \sigma\left(\frac{1}{l}\sum_{x\in B_y}\eta_x(s)\right)\right| + \max_{x\in B_y}|q_x(s) - q_y(s)|.\end{aligned}$$

Moreover, uniformly in $y \in \{-L, \dots, L-l\}$, as a consequence of Lemma A.3(ii),

$$\max_{y\in\{-L,\dots,L-l\},\, x\in B_y,\, s\ge 0} |q_x(s) - q_y(s)| = O(l/L).$$

The contribution of $y \in \{L-l+1, L\}$ to the sum over y in (A.10) is $O(l)$. Therefore summing over $y \in \{-L, \dots, L\}$, integrating over s and taking expectation one gets

$$\begin{aligned}
&\left| \int_0^{tL^2} \mathbb{E}\Big[\sum_{y=-L}^{L} \Big(\frac{1}{l} \sum_{x \in B_y} q_x(s)\,\mathrm{sg}(\eta_x(s)) \Big) - q_y(s) \sigma\Big(\frac{1}{l} \sum_{x \in B_y} \eta_x(s) \Big) \Big] ds \right| \\
&\le \Big(\max_{y,s} |q_y(s)| \Big) \left| \int_0^{tL^2} \mathbb{E}\Big[\sum_{y=-L}^{L} \Big(\frac{1}{l} \sum_{x \in B_y} \mathrm{sg}(\eta_x(s)) \Big) - \sigma\Big(\frac{1}{l} \sum_{x \in B_y} \eta_x(s) \Big) \Big] ds \right| + O(lL^2) \\
&= \Big(\max_{y,s} |q_y(s)| \Big) tL^2(2L+1) \mu_t^L \left(\left| \frac{1}{l} \sum_{x \in B_0} \mathrm{sg}(\eta_x) - \sigma\Big(\frac{1}{l} \sum_{x \in B_0} \eta_x \Big) \right| \right) + O(lL^2)
\end{aligned}$$

where μ_t^L is defined in (A.6). Therefore, the proof will be finished provided we show

$$\lim_{l \to \infty} \limsup_{L \to \infty} \mu_t^L \left(\left| \frac{1}{l} \sum_{x \in B_0} \mathrm{sg}(\eta_x) - \sigma\Big(\frac{1}{l} \sum_{x \in B_0} \eta_x \Big) \right| \right) = 0.$$

From Proposition A.4 one has

$$\begin{aligned}
&\limsup_{L \to \infty} \mu_t^L \left(\left| \frac{1}{l} \sum_{x \in B_0} \mathrm{sg}(\eta_x) - \sigma\Big(\frac{1}{l} \sum_{x \in B_0} \eta_x \Big) \right| \right) \\
&\qquad \le \sup_{0 \le u \le \|\partial_x \phi^0\|_\infty} \rho^u \left(\left| \frac{1}{l} \sum_{x \in B_0} \mathrm{sg}(\eta_x) - \sigma\Big(\frac{1}{l} \sum_{x \in B_0} \eta_x \Big) \right| \right)
\end{aligned}$$

and one can check that the right-hand side tends to zero as $l \to \infty$: we note that for every x one has $\rho^u(\mathrm{sg}(\eta_x)) = \sigma(u)$, and the law of large numbers tells us that the two terms $\frac{1}{l} \sum_{x \in B_y} \mathrm{sg}(\eta_x)$ and $\sigma\big(\frac{1}{l} \sum_{x \in B_y} \eta_x\big)$ have the same limit as $l \to \infty$. However, because of the sup over u one needs more quantitative estimates than the law of large numbers to conclude. For instance we can get them by using the second moment method; we leave the details to the reader. □

Similarly to Lemma A.6 one shows

Lemma A.7.

$$\lim_{l \to \infty} \limsup_{L \to \infty} \frac{1}{L^3} \int_0^{L^2 t} \sum_{y=-L}^{L} \mathbb{E}(G(\eta(s))) = \lim_{l \to \infty} \limsup_{L \to \infty} \frac{t(2L+1)}{L} \mu_t^L(G(\eta)) = 0$$

where

$$G(\eta) = \left| \frac{1}{l} \sum_{x \in B_y} \big(|\eta_x(s)| - |\mathrm{sg}(\eta_x(s))| \big) - \Big(\frac{1}{l} \sum_{x \in B_y} \eta_x(s) \Big) \sigma\Big(\frac{1}{l} \sum_{x \in B_y} \eta_x(s) \Big) \right|.$$

Proof. The proof is very similar to that of Lemma A.6, the only additional technical point being that the function $G(\eta)$ is not bounded so that one cannot use Proposition A.4 directly. However stochastic domination given by Lemma A.3(iii) allows us to get the same conclusion by considering the function $\eta \mapsto \min(G(\eta), K)$, and letting K tend to infinity afterwards. Altogether one gets

$$\limsup_{L\to\infty} \mu_t^L(G) \leq \sup_{0\leq u\leq \|\partial_x \phi^0\|_\infty} \rho^u(G).$$

We end the proof in the same way as for the previous lemma, remarking that

$$\rho^u(|\eta_x| - |\mathrm{sg}(\eta_x)|) = u\sigma(u). \qquad \square$$

Now we are ready to conclude:

$$\begin{aligned}
\sum_{y=-L}^{L} A(y,s) &= \sum_{y=-L}^{L} \Bigg\{ -q_y(s)\sigma(q_y(s)) + \frac{1}{l}\sum_{x\in B_y} \big(\eta_x(s)\sigma(q_x(s)) \\
&\qquad + q_x(s)\,\mathrm{sg}(\eta_x(s)) - (|\eta_x(s)| - |\mathrm{sg}(\eta_x(s))|)\big) \Bigg\} \\
&\leq \sum_{y=-L}^{L} -q_y(s)\sigma(q_y(s) + \left(\frac{1}{l}\sum_{x\in B_y}\eta_x(s)\right)\sigma(q_y(s)) + q_y(s)\sigma\left(\frac{1}{l}\sum_{x\in B_y}\eta_x(s)\right) \\
&\qquad - \left(\frac{1}{l}\sum_{x\in B_y}\eta_x(s)\right)\sigma\left(\frac{1}{l}\sum_{x\in B_y}\eta_x(s)\right) + R(s,l,L) \\
&= R(s,l,L) - \sum_{y=-L}^{L}\left[q_y(s) - \frac{1}{l}\sum_{x\in B_y}\eta_x(s)\right]\left[\sigma(q_y(s)) - \sigma\left(\frac{1}{l}\sum_{x\in B_y}\eta_x(s)\right)\right]
\end{aligned}$$

where

$$\begin{aligned}
R(s,l,L) = -\sum_{y=-L}^{L} \frac{1}{l}&\left(\sum_{x\in B_y}\eta_x(s)\big(\sigma(q_y(s)) - \sigma(q_x(s))\big)\right) \\
&+ \left|\frac{1}{l}\sum_{x\in B_y} q_x(s)\,\mathrm{sg}(\eta_x(s)) - q_y(s)\sigma\left(\frac{1}{l}\sum_{x\in B_y}\eta_x(s)\right)\right| \\
&+ \left|\frac{1}{l}\sum_{x\in B_y}\big(|\eta_x(s)| - |\mathrm{sg}(\eta_x(s))|\big) - \left(\frac{1}{l}\sum_{x\in B_y}\eta_x(s)\right)\sigma\left(\frac{1}{l}\sum_{x\in B_y}\eta_x(s)\right)\right|
\end{aligned}$$

and the second term is non-positive ($a-b$ and $\sigma(a)-\sigma(b)$ have the same sign).

According to (ii)–(iii) in Lemma A.3 (to control the first term) and Lemmata A.6 and A.7,

$$\lim_{l\to\infty}\limsup_{L\to 0} \frac{1}{L^3}\int_0^{L^2 t} \mathbb{E}R(s,l,L)\,ds = 0.$$

This implies

$$\limsup_{L\to\infty} \frac{1}{L^3} \int_0^{L^2 t} \sum_{y=-L}^{L} \mathbb{E}\, A(x,s)\, ds \le 0$$

and therefore the result that we want to prove, from (A.4). $\square$

A.4. *Concluding the proof of Theorem 3.4*

It is not hard to transform the $\mathbb{L}_2$ statement of Proposition A.2 into the desired "almost sure" statement:

Proposition A.8. *For any $\varepsilon > 0$ and $t \ge 0$, w.h.p.,*

$$\max_{x\in\{-L,\dots,L+1\}} \frac{1}{L} |\Phi(x, L^2 t) - \hat{h}_x(L^2 t)| \le \varepsilon.$$

Proof. Also here we write h for $\hat{h}$. Note that from Lemma A.3(iii) the random vector $(|\eta_x(t)|)_{x\in\{-L,\dots,L\}}$ is stochastically dominated for every t by a vector of IID time-independent geometric variables. This implies that there exists a constant C such that for any $t \ge 0$, w.h.p.,

$$|h_x(t) - h_y(t)| \le C|x-y| \quad \text{if } x, y \in \{-L,\dots,L\},\ |x-y| \ge \log L \tag{A.11}$$

(this can be proved by using large deviation estimates and a union bound on $x, y \in \{-L,\dots,L\}$). Moreover Lemma A.3(i) ensures that $\Phi(\cdot, t)$ is always Lipschitz so that (A.11) also holds for $\Phi(\cdot,t) - h_\cdot(t)$.

With (A.11) and L large enough, one has

$$\left\{\max_{x\in\{-L,\dots,L+1\}} |\Phi(x, L^2 t) - h_x(L^2 t)| \ge \varepsilon L\right\}$$
$$\subset \left\{\sum_{x\in\{-L,\dots,L+1\}} |\Phi(x, L^2 t) - h_x(L^2 t)|^2 \ge \frac{\varepsilon^3 L^3}{10C}\right\},$$

so that the left-hand side event has small probability when L is large, otherwise Proposition A.2 would be false. $\square$

A.5. *Laplacian bounds*

Recall that $\Phi(x,t)$ is the solution of the Cauchy problem (3.8). We want to bound $\Phi(x,t)$ above and below by the solution of a suitable heat equation. For this, we will suppose that the function ϕ^0, through which the initial condition Φ_0 for $\Phi(x,t)$ is defined, is concave on $[-1,1]$ (in addition to the assumptions required for Theorem 3.4). One defines the evolution $\Phi_1(x,t)$ as the solution of

$$\begin{cases} \partial_t \Phi_1(x,t) = \frac{1}{2}\Delta\Phi_1(x,t), \\ \Phi_1(-L,t) = \Phi(L+1,t) = 0, \\ \Phi_1(x,0) = \Phi_0(x), \end{cases}$$

for $t \geq 0, x \in \{-L+1, L\}$. Also we define $\Phi_2(x,t)$ as the solution of the analogous equation (with the same boundary values) where the discrete Laplacian is multiplied by $(1/2)\sigma'(\|\partial_x \phi^0\|_\infty) = 1/(1+\|\partial_x \phi^0\|_\infty)^2$.

Proposition A.9. *For all $t \geq 0$ and $x \in \{-L, \dots, L+1\}$ one has*

$$\Phi_1(x,t) \leq \Phi(x,t) \leq \Phi_2(x,t). \tag{A.12}$$

Proof. We prove the upper bound, the lower one being very similar. Suppose that the result does not hold and set

$$T := \max\big\{t : \Phi(x,s) \leq \Phi_2(x,s) \text{ for every } s \leq t \text{ and } x \in \{-L, \dots, L+1\}\big\}.$$

Note that by the property of the heat equation, $\Phi_2(x,t)$ is a strictly concave function of x for all positive t (except in the case where one starts from the flat initial condition, but in that case the statement is trivial). Let x_0 be such that

$$\Phi(x_0, T) = \Phi_2(x_0, T).$$

Then one remarks that $q_{x_0}(T) - q_{x_0-1}(T) < 0$ (by strict concavity of $\Phi_2(\cdot, T)$) and that by Lemma A.3, $\max_x |q_x(t)| \leq \|\partial_x \phi^0\|_\infty$, so that

$$\sigma(q_{x_0}(T)) - \sigma(q_{x_0-1}(T)) < (q_{x_0}(T) - q_{x_0-1}(T))\sigma'(\|\partial_x \phi_0\|_\infty)$$

(since $\sigma'(\cdot)$ is decreasing on $\mathbb{R}^+$) and hence

$$\begin{aligned} 2\partial_t[\Phi_2 - \Phi](x_0, T) &= \sigma'(\|\partial_x \phi_0\|_\infty)\Delta\Phi_2(x,t) - \sigma(q_{x_0}(T)) + \sigma(q_{x_0-1}(T)) \\ &> \sigma'(\|\partial_x \phi_0\|_\infty)\big[(\Phi_2(x+1,t) + \Phi_2(x-1,t)) - (\Phi(x+1,t) + \Phi(x-1,t))\big]. \end{aligned}$$

Since the last expression is non-negative, one has $\Phi(x,t) < \Phi_2(x,t)$ on an interval $[T, T+\varepsilon(x)]$ for some $\varepsilon(x) > 0$ and every $x \in \{-L, \dots, L+1\}$, and that concludes the proof since the only possibility is that $T = \infty$. ◻

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