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Partial hyperbolicity and homoclinic tangencies

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Abstract. We show that any diffeomorphism of a compact manifold can be C^1 approximated by diffeomorphisms exhibiting a homoclinic tangency or by diffeomorphisms having a partial hyperbolic structure.

Keywords. Homoclinic tangency, heterodimensional cycle, hyperbolic diffeomorphism, generic dynamics, homoclinic class, partial hyperbolicity

1. Introduction

1.1. Characterization of partial hyperbolicity

We are interested in describing the dynamics of a large class of diffeomorphisms of a compact manifold M. This goal was achieved in a satisfactory way about 40 years ago for *hyperbolic diffeomorphisms*. These are the systems f admitting a filtration, i.e. a finite collection of open sets $\emptyset = U_0 \subset U_1 \subset \cdots \subset U_s = M$ satisfying $f(\overline{U_i}) \subset U_i$, such that the maximal invariant set Λ_i in each level $U_i \setminus U_{i-1}$ is hyperbolic: there exists a splitting of the tangent bundle into two invariant linear subbundles, $T_{\Lambda_i}M = E^{\mathrm{s}} \oplus E^{\mathrm{u}}$, such that E^{s} and E^{u} are respectively strictly contracted and expanded by a forward iterate of f. These systems have many nice properties. For instance, they possess a spectral decomposition: for a finer filtration the sets Λ_i are transitive. Moreover, they are stable under perturbations, they can be coded, etc.

We are far from well understanding the dynamics beyond hyperbolicity and one would like to extend the previous properties to weaker forms of hyperbolicity. This paper deals with the following version of partial hyperbolicity.

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Definition 1.1. An invariant set Λ of a diffeomorphism f is *partially hyperbolic* if its tangent bundle splits into invariant linear subbundles:

$$T_\Lambda M = E^{\mathrm{s}} \oplus E^{\mathrm{c}}_1 \oplus \cdots \oplus E^{\mathrm{c}}_k \oplus E^{\mathrm{u}},$$

such that each central bundle E^{c}_i is one-dimensional and considering a Riemannian metric on M there exists a forward iterate f^N which satisfies:

- $\|Df^N.v\| \leq 1/2$ for each unitary $v \in E^{\mathrm{s}}$ (one says E^{s} is (*uniformly*) *contracted*).
- $\|Df^{-N}.v\| \leq 1/2$ for each unitary $v \in E^{\mathrm{u}}$ (one says E^{u} is (*uniformly*) *expanded*).
- $\|Df^N_x.u\| \leq (1/2)\|Df^N_x.v\|$ for each $x \in \Lambda$, each $i = 0, \ldots, k$ and any unitary vectors $u \in E^{\mathrm{s}} \oplus \cdots \oplus E^{\mathrm{c}}_i$, $v \in E^{\mathrm{c}}_{i+1} \oplus \cdots \oplus E^{\mathrm{u}}$ in $T_x M$.

A diffeomorphism f is *partially hyperbolic* if there exists a filtration $\emptyset = U_0 \subset U_1 \subset \cdots \subset U_s = M$ such that the maximal invariant set Λ_i in each level $U_i \setminus U_{i-1}$ is partially hyperbolic.

The type of the decomposition can be different on each piece Λ_i. We allow $k = 0$: in this case the piece is hyperbolic. We also allow E^{s} and/or E^{u} to be trivial. Different notions of partial hyperbolicity appear in the literature; the decomposition of the central part into one-dimensional central bundles here provides a good control of the tangent dynamics. In particular, this gives a symbolic description and the existence of equilibrium states for these systems (see the discussion in Section 1.4).

The set of partially hyperbolic diffeomorphisms is open in the space $\mathrm{Diff}^1(M)$ of diffeomorphisms endowed with the C^1-topology. It is well known that there exist diffeomorphisms that cannot be approximated by hyperbolic ones (see for instance [S]). Also the partially hyperbolic diffeomorphisms are not dense in $\mathrm{Diff}^1(M)$ when $\dim(M) \geq 3$ (on surfaces this is an open problem): from [BD_1], for an open set of diffeomorphisms each filtration contains a piece Λ_i whose tangent dynamics has no non-trivial invariant continuous subbundle. The lack of partial hyperbolicity in these examples is related to the following obstruction.

Definition 1.2. A diffeomorphism $f : M \to M$ exhibits a *homoclinic tangency* if there is a hyperbolic periodic orbit whose invariant manifolds $W^{\mathrm{s}}(O)$ and $W^{\mathrm{u}}(O)$ have a non-transverse intersection.

Homoclinic tangencies generate interesting dynamical instabilities (see for instance [PT]). Our goal is to provide a dichotomy between these two notions.

Main Theorem. *Any diffeomorphism f can be approximated in* $\mathrm{Diff}^1(M)$ *by diffeomorphisms which exhibit a homoclinic tangency or by partially hyperbolic diffeomorphisms.*

This result gives a complete obstruction to partial hyperbolicity and decomposes the set of diffeomorphisms into two parts: on one of them, the global dynamics is quite well described, on the other one, the systems exhibit rich dynamical instabilities through local bifurcations. One thus obtains an example of decomposition by *phenomenon and mechanism* as discussed in [CP].

In fact, our result was motivated by a conjecture of Palis [Pa] which proposes to characterize the lack of hyperbolicity by homoclinic tangencies and heterodimensional cycles. Several previous papers are related to our work.

- [PuS] solved the Palis conjecture on surfaces and implied our result in this case.
- [W_1] and [G_2] showed that for diffeomorphisms far from tangencies, the tangent bundle splits on the closure of the hyperbolic periodic orbits of a given stable dimension.
- The previous results imply the theorem for *tame diffeomorphisms*, i.e. those which do not admit filtrations with an arbitrarily large number of non-trivial levels (see [BDPR, ABCDW]). For the same reason, the theorem holds in the conservative setting.
- [GYW] gave a local version of the theorem: far from homoclinic tangencies, any minimally non-hyperbolic set is partially hyperbolic.
- [C_3] proved the theorem above for diffeomorphisms that are not approximated by diffeomorphisms exhibiting a heterodimensional cycle; this was used in [CP] to obtain partial results on the Palis conjecture.
- [Y_2] obtained the partial hyperbolicity on aperiodic pieces of generic diffeomorphisms far from homoclinic tangencies.

In view of [W_1], one could hope to generalize our result and obtain a positive answer to the following question.

Problem 1.3. *Is it equivalent for a diffeomorphism to be non-partially hyperbolic and to be a limit in* $\mathrm{Diff}^1(M)$ *of diffeomorphisms exhibiting a homoclinic tangency?*

1.2. Precise statement

When one studies the global dynamics of a homeomorphism f of a compact metric space M, one splits the dynamics into pieces that cannot be further decomposed by filtrations. Let us define on M the following relation: $x \sim y$ if and only if for any $\varepsilon > 0$ there exists a periodic ε-pseudo-orbit which contains both x and y. Then the *chain-recurrent set* $\mathcal{R}(f)$ is the set of points $x \in M$ satisfying $x \sim x$. On this set, $\sim$ becomes an equivalence relation which defines a compact invariant decomposition of $\mathcal{R}(f)$ into its *chain-recurrence classes*. A *chain-transitive set* of f is an invariant compact set $K \subset M$ such that the restriction $f_{|K}$ has a single chain-recurrence class.

For any hyperbolic periodic orbit O of a diffeomorphism, the closure of the transverse intersections between the stable and the unstable manifolds of O is called the *homoclinic class* of O and denoted by $H(O)$. In [BC] it is proved that for a dense G_δ subset of $\mathrm{Diff}^1(M)$ the chain-recurrence classes which contain periodic points are the homoclinic classes. The other ones are called *aperiodic classes*.

In the following we denote by $\overline{\mathrm{HT}} \subset \mathrm{Diff}^1(M)$ the closure of the set of diffeomorphisms exhibiting a homoclinic tangency. Recall also that if μ is an invariant probability measure and if E is a continuous invariant one-dimensional bundle over the support of μ, then one can define the *Lyapunov exponent* of μ along E as $\int \log \|Df_{|E}\| d\mu$ by considering any Riemannian metric on M.

Theorem 1.1. *Every diffeomorphism f in a dense G_δ subset $\mathcal{G} \subset \mathrm{Diff}^1(M) \setminus \overline{\mathrm{HT}}$ has the following properties:*

1. *Any aperiodic class $\mathcal{C}$ is partially hyperbolic with a one-dimensional central bundle. Moreover, the Lyapunov exponent along E^{c} of any invariant measure supported on $\mathcal{C}$ is zero.*
2. *Any homoclinic class $H(p)$ has a partially hyperbolic structure*
$$T_{H(p)}M = E^{\mathrm{s}} \oplus E^{\mathrm{c}}_1 \oplus \cdots \oplus E^{\mathrm{c}}_k \oplus E^{\mathrm{u}}.$$
Moreover the minimal stable dimension of periodic orbits of $H(p)$ is $\dim(E^{\mathrm{s}})$ or $\dim(E^{\mathrm{s}}) + 1$. Similarly the maximal stable dimension of periodic orbits of $H(p)$ is $\dim(E^{\mathrm{s}})+k$ or $\dim(E^{\mathrm{s}})+k-1$. For every i with $1 \le i \le k$ there exist periodic points in $H(p)$ whose Lyapunov exponent along E^{c}_i is arbitrarily close to 0.

One obtains the Main Theorem from Theorem 1.1 by proving that any diffeomorphism $f \in \mathcal{G}$ is partially hyperbolic. We first recall the following properties:

- If a chain-recurrence class $\mathcal{C}$ is partially hyperbolic, then the same holds on the maximal invariant set in a neighborhood of $\mathcal{C}$.
- Any limit of a sequence of chain-recurrence classes for the Hausdorff topology is contained in a chain-recurrence class.

By compactness of the space of compact subsets of M, one deduces from Theorem 1.1 that there exists a finite family of open sets $V_1, \dots, V_\ell$ such that the maximal invariant set in each of them is partially hyperbolic and any chain-recurrence class is contained in one of the V_i. As a consequence, considering a filtration $U_1, \dots, U_s$ such that the maximal invariant set in each level set $U_i \setminus U_{i-1}$ is contained in one of the open sets V_j, one gets the partial hyperbolicity of f. This gives the Main Theorem.

Remark 1.4.
- As we said, the first item of Theorem 1.1 was already obtained in [Y_2].
- From [ABCDW], the set of stable dimensions of periodic orbits in a homoclinic class is an interval of $\{0, \dots, \dim(M)\}$.
- For a homoclinic class $H(p)$ whose minimal stable dimension of periodic points is $\dim(E^{\mathrm{s}}) + 1$, the bundle $E^{\mathrm{s}} + E^{\mathrm{c}}_1$ has some weak hyperbolicity property (see Section 8.1).
- From [C_3, Section 2.4], for aperiodic classes the extremal bundles E^{s} and E^{u} are both non-degenerate.

Theorem 1.1 answers partially [ABCDW, Conjecture 1], but other questions remains: for diffeomorphisms as in Theorem 1.1, are homoclinic classes the only possible chain-recurrence classes? does the minimal stable dimension of periodic orbits in a homoclinic class coincide with $\dim(E^{\mathrm{s}})$? One would get positive answers if a spectral decomposition holds:

Conjecture 1 (Bonatti [Bon]). *There exists a dense G_δ subset in $\mathrm{Diff}^1(M) \setminus \overline{\mathrm{HT}}$ of diffeomorphisms having only finitely many chain-recurrence classes.*

Consider now the case where the minimal stable dimension of the periodic orbits contained in the homoclinic class $H(p)$ in Theorem 1.1 is larger than $\dim(E^s)$. Since E^c_1 is not uniformly contracted, there should exist in any neighborhood of $H(p)$ some periodic orbits of stable dimension $\dim(E^s)$, contradicting the previous conjecture. We can thus formulate the following weaker conjecture.

Conjecture 2. *There exists a dense G_δ subset in $\mathrm{Diff}^1(M) \setminus \overline{\mathrm{HT}}$ of diffeomorphisms whose homoclinic classes have a partially hyperbolic structure $T_{H(p)}M = E^s \oplus E^c \oplus E^u$, where $\dim(E^s)$ and $\dim(E^s \oplus E^c)$ are the minimal and maximal stable dimensions of periodic orbits in $H(p)$.*

One can also wonder if a local version of Theorem 1.1 holds for homoclinic classes:

Problem 1.5. *Does there exist a dense G_δ subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that for any homoclinic class $H(O)$, either*

- *$H(O)$ is partially hyperbolic, or*
- *there exists a periodic orbit $O' \subset H(O)$ and diffeomorphisms arbitrarily close to f in $\mathrm{Diff}^1(M)$ exhibiting a homoclinic tangency associated to O'?*

1.3. Linking periodic orbits to a class

Let us explain the main difficulty in obtaining Theorem 1.1. We consider a chain-recurrence class $\mathcal{C}$ of a diffeomorphism f which belongs to a dense G_δ subset of $\mathrm{Diff}^1(M)\setminus\overline{\mathrm{HT}}$.

From [C_1], the class $\mathcal{C}$ is the Hausdorff limit of a sequence of periodic orbits. From [W_1], one gets an integer $N \geq 1$ and an invariant splitting $T_{\mathcal{C}}M = E \oplus F$ on $\mathcal{C}$ which is *N-dominated*: for each $x \in \mathcal{C}$ and any unitary vectors $u \in E_x$, $v \in F_x$ one has $\|Df^N_x.u\| \leq (1/2)\|Df^N_x.v\|$.

We thus have to analyze non-uniform bundles and try to split them with one-dimensional central bundles. From a selecting lemma of Liao, if a bundle E is not uniform, one can find in $\mathcal{C}$ an invariant compact subset K with an invariant splitting of its tangent bundle of the form $T_K M = E^s \oplus E^c \oplus E^u$, where E^c is one-dimensional and neither uniformly contracted nor expanded. By Mañé's ergodic closing lemma, one can find in any neighborhood of K some periodic orbits O^-, O^+ of stable dimensions $\dim(E^s)$ and $\dim(E^s) + 1$ respectively. If one can prove that there exist other periodic orbits $\widehat{O}^-$, $\widehat{O}^+$ with the same stable dimensions as O^-, O^+ and which approximate $\mathcal{C}$ in the Hausdorff topology, then one deduces again from [W_1] that there exists a splitting $T_{\mathcal{C}}M = E' \oplus E^c \oplus F'$ which coincides with $T_K M = E^s \oplus E^c \oplus E^u$ on K, giving a one-dimensional central bundle on $\mathcal{C}$ as required.

One way to obtain this property is to prove that the periodic orbits O^-, O^+ are *contained* in the class $\mathcal{C}$. Indeed, by [BC] one would conclude that $\mathcal{C}$ coincides with the homoclinic classes $H(O^-)$ and $H(O^+)$. Proving that O^- and O^+ are contained in $\mathcal{C}$ is the main difficulty in this subject, and so it is also for the Palis conjecture. If all the chain-recurrence classes are isolated in $\mathcal{R}(f)$ (their number is finite, and the dynamics is said to be *tame*), this problem does not appear; the same holds when the dynamics is conservative.

This difficulty is addressed by the following technical local result. When O is a periodic orbit of a diffeomorphism f and U is an open set containing O, we define the *local homoclinic class* $H(O,U)$ as the closure of the set of transverse homoclinic orbits between the stable and the unstable manifolds of O that are contained in U. It is a transitive invariant compact set contained in $\overline{U}$.

Theorem 1.2. *Let f be a diffeomorphism in a dense G_δ subset of* $\mathrm{Diff}^1(M)$ *and let Λ be a compact invariant chain-transitive set with a dominated splitting $T_\Lambda M = E^s \oplus E^c \oplus F$. Assume also that:*

- *E^s is uniformly contracted.*
- *E^c is one-dimensional and it is not uniformly contracted.*
- *There exists an ergodic measure μ supported on Λ whose Lyapunov exponent along E^c is non-zero.*

Then one of the following holds:

1. *For any neighborhood U_0 of Λ, there exists a local homoclinic class $H(O,U_0)$ containing Λ where the stable dimension of O equals* $\dim(E^s)$.
2. *For any $\theta > 0$ and any neighborhood U_0 of Λ, there exists a periodic orbit $O \subset U$ such that:*
   - *The Lyapunov exponent of O along E^c belongs to $(-\theta,\theta)$.*
   - *Λ is contained in the local homoclinic class $H(O,U_0)$.*

We point out that the above theorem does not require the diffeomorphism to be far from homoclinic tangencies.

The goal to link a set having a non-uniform central bundle to weak periodic orbits contained in a neighborhood is very similar to Liao's and Mañé's selecting lemmas (see for instance [W_2] or [M_2]). For instance, with these techniques [BGY] proved the following:

Let f be a C^1-generic diffeomorphism and let $H(O)$ be a homoclinic class with dominated splitting $E^s \oplus F$ where E^s is uniformly contracted, the stable dimension of O equals $\dim(E^s)$ *and F is not uniformly expanded. Then for any δ there are periodic orbits in H whose minimal Lyapunov exponent in F belongs to* $(0,\delta)$.

Our proof however is very different. As in [C_3], it is obtained by analyzing the dynamics along the central bundle and by using the central models introduced in [C_2]. The fact that we do not forbid heterodimensional cycles changes however the philosophy and increases the difficulty. We need to develop two new strategies presented in Sections 6 and 7.

Organization of the paper. Section 2 recalls genericity results and Section 4 presents the central models. The proof of Theorem 1.2, using results from Sections 6 and 7, is given in Section 5. The proof of Theorem 1.1 (and hence of the Main Theorem) appears in Section 3. The consequences that are presented in Section 1.4 are obtained in Section 8. Note that the first item of Theorem 1.1 about aperiodic classes can be proved in a shorter way (independently of Sections 6 and 7)—see Section 8.2.

1.4. Consequences. Dynamics far from homoclinic tangencies

Our result allows us to solve, for diffeomorphisms far from homoclinic tangencies, several problems stated for C^1-diffeomorphisms. In the following, we will say that a property holds for *C^1-generic diffeomorphisms* if it is satisfied on a dense G_δ subset of $\mathrm{Diff}^1(M)$.

Symbolic extensions. The dynamics on partially hyperbolic sets where the central bundle has a dominated splitting into one-dimensional subbundles has a nice symbolic description. It is obtained through the property of entropy expansiveness, introduced by Bowen [Bow_1].

Definition 1.6. A homeomorphism f of a compact metric space X is *entropy expansive* if there exists $\varepsilon > 0$ such that for any $x \in X$, the set

$$B(x, \varepsilon, f) = \{y \in X : \text{for any } n \in \mathbb{Z}, d(f^n(x), f^n(y)) \leq \varepsilon\}$$

has zero entropy.

Recall that a (not necessarily invariant) set $B \subset X$ *has zero entropy* when the entropy of B at any scale δ is zero: if we denote by $N(B, \delta, f, n)$ the minimal number of points $x_1, \dots, x_N$ in B such that any $x \in B$ stays δ-close to some x_i during n iterations, then $N(B, \delta, f, n)$ growths subexponentially with n for any δ.

Remark 1.7. We provide some remarks from [Bow_1].

1. This notion is weaker than expansiveness, which assumes that the dynamical ball $B(x, \varepsilon, f)$ reduces to x.
2. In the definition, one can also replace $B(x, \varepsilon, f)$ by the forward dynamical ball

$$B^+(x, \varepsilon, f) = \{y \in X : \text{for any } n \in \mathbb{N},\ d(f^n(x), f^n(y)) \leq \varepsilon\}.$$

3. One can also introduce the ball

$$B_n(x, \varepsilon, f) = \{y \in X : \text{for any } 0 \leq k \leq n,\ d(f^k(x), f^k(y)) \leq \varepsilon\}.$$

   A map is *entropy expansive* if for $\varepsilon > 0$ small the following quantity is zero for any $x \in X$:

$$h^*_f(\varepsilon) = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(B_n(x, \varepsilon, f), \delta, f, n).$$

4. If f is entropy expansive, then there exists $\varepsilon > 0$ such that the topological entropy of f coincides with the topological entropy at scale ε defined as

$$h_f(\varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log N(X, \varepsilon, f, n).$$

5. If the restriction of f to the non-wandering set $\Omega(f)$ is entropy expansive, then so is f.

From the last remark, any entropy expansive system admits *equilibrium states* (see [Bow$_2$, Proposition 2.19]): more precisely, for any continuous map $\varphi\colon X \to \mathbb{R}$, there exists a measure that realizes the following maximum over all invariant probability measures μ:

$$P_f(\varphi) = \max_{\mu}\left\{h_f(\mu) + \int \varphi\, d\mu\right\},$$

where $h_f(\mu)$ denotes the entropy of μ for f.

In [BFF] it is shown that any entropy expansive system admits a *principal symbolic extension*, i.e. there exists a compact invariant set Y of a full subshift $(\{1,\dots,k\}^{\mathbb{Z}}, \sigma)$ and a continuous semiconjugacy $\pi : Y \to X$ such that for any invariant measure μ of (Y,σ), the entropies of μ for σ and of $\pi_*(\mu)$ for f are the same.

In [CY, DFPV], it is shown that the restriction of a diffeomorphism to a compact invariant set which is partially hyperbolic and whose central bundle has a dominated splitting into one-dimensional subbundles is entropy expansive. This result together with remark 5 above and the Main Theorem implies the following.

Corollary 1.3. *Any diffeomorphism f in an open and dense subset of* $\mathrm{Diff}^1(M)\setminus\overline{\mathrm{HT}}$ *is entropy expansive. In particular, it admits principal symbolic extensions. Also, any continuous map $\varphi\colon M \to \mathbb{R}$ has an equilibrium state.*

At the time of finishing this paper, we learned that Liao, Viana and Yang [LVY] proved the entropy expansiveness for *any* diffeomorphism C^1-far from homoclinic tangencies. Their result uses the uniform dominated splitting over the support of ergodic measures for diffeomorphisms C^1-far from homoclinic tangencies. In our case, we do not have a uniform version of Theorem 1.1 that holds for general diffeomorphisms in $\mathrm{Diff}^1(M)\setminus\overline{\mathrm{HT}}$.

Lack of hyperbolicity and weak periodic points. The stability of diffeomorphisms has been related to hyperbolicity (see [M$_2$]). This goal has been achieved by considering *star diffeomorphisms*, i.e. systems whose periodic orbits do not bifurcate under small C^1-perturbation. We propose here to address a local version of this question.

For a homoclinic class $H(O)$ of f, if for any $\delta > 0$, there is a hyperbolic periodic point q such that:

- q (and its orbit) is *homoclinically related* to O, i.e. its stable and unstable manifolds intersect those of O transversally.
- q has one Lyapunov exponent in $(-\delta,\delta)$,

then one says that $H(O)$ contains *weak periodic orbits related to p*.

Problem 1.8. *Is it true that for generic $f \in$* $\mathrm{Diff}^1(M)$*, and any homoclinic class $H(O)$, either*

- *$H(O)$ is hyperbolic, or*
- *$H(O)$ contains weak periodic orbits related to O?*

Let us discuss how to solve this problem. If $H(O)$ does not contain weak periodic orbits related to O, then by a genericity argument, one can prove that $H(O)$ admits a dominated splitting $T_{H(O)}M = E \oplus F$, where $\dim(E)$ equals the stable dimension of O, and one can get a uniform estimate for contraction and expansion at the period along the invariant bundles of periodic orbits related to O (see [GY]). By a variation of Mañé's work, one can thus answer the problem when E is uniformly contracted: [BGY] shows that F is uniformly expanded in this case.

For diffeomorphisms far from homoclinic tangencies we have:

Corollary 1.4. *For generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{HT}}$, and any homoclinic class $H(O)$, either*

- *$H(O)$ is hyperbolic, or*
- *$H(O)$ contains weak periodic orbits related to O.*

Palis conjecture. If f has two hyperbolic periodic orbits O_1 and O_2 with different stable dimension and whose invariant manifolds intersect (i.e. $W^{\mathrm{s}}(O_1) \cap W^{\mathrm{u}}(O_2)$ and $W^{\mathrm{s}}(O_2) \cap W^{\mathrm{u}}(O_1)$ are non-empty), then we say that f has a *heterodimensional cycle*. One can generalize this definition: a *robust heterodimensional cycle* is a pair of transitive hyperbolic sets K, L having different stable dimensions such that for any diffeomorphism C^1-close to f the stable and unstable sets of the continuations of K and L intersect.

Conjecture 3 (Palis). *Every robustly non-hyperbolic diffeomorphism f can be approximated by diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle.*

The following corollary of Theorem 1.1 was the main theorem of [C_3].

Corollary 1.5. *For generic $f \in \mathrm{Diff}^1(M)$ which cannot be approximated by diffeomorphisms exhibiting a homoclinic tangency or a heterodimensional cycle, any homoclinic class $H(O)$ admits a partially hyperbolic splitting*

$$T_{H(0)}M = E^{\mathrm{s}} \oplus E_1^{\mathrm{c}} \oplus E_2^{\mathrm{c}} \oplus E^{\mathrm{u}},$$

where $\dim(E_i^{\mathrm{c}}) = 0$ or 1, for $i = 1, 2$ and $\dim(E^{\mathrm{s}} \oplus E_1^{\mathrm{c}})$ coincides with the stable dimension of O.

Lyapunov stable classes. One sometimes focuses the study of a system on its attractors, since they can describe the forward behavior of most orbits. We will say that a chain-recurrence class is *Lyapunov stable* if it admits arbitrarily small neighborhoods U satisfying $f(U) \subset U$. It is known [BC] that for C^1-generic diffeomorphisms, the forward orbit of points in a dense G_δ subset of M accumulates on a Lyapunov stable class.

On restriction to these classes, the Palis conjecture holds [CP]. Combining this result with ours we get the following statement, which was suggested to us by Yi Shi.

Corollary 1.6. *For generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{HT}}$, if $\mathcal{C}$ is a Lyapunov stable chain-recurrence class, then either*

- *$\mathcal{C}$ is hyperbolic, or*
- *$\mathcal{C}$ contains a robust heterodimensional cycle.*

For generic diffeomorphisms in $\mathrm{Diff}^1(M) \setminus \overline{\mathrm{HT}}$, more is known about Lyapunov stable chain-recurrence classes. By [Y_1] they are homoclinic classes and by [BGLM] they are *residual attractors*, i.e. for each of them there exists a neighborhood U such that the forward orbit of any point in a dense G_δ subset of U accumulates on the class.

When M is connected, the bi-Lyapunov stable classes of a generic diffeomorphism far from homoclinic tangencies, i.e. the chain-recurrence classes that are Lyapunov stable both for f and f^{-1}, coincide with the whole manifold (see [Po] and previous works [ABD, PoS]). In particular, if a chain-recurrence class has non-empty interior, then M is transitive.

Bound on the number of classes. Conjecture 1 asserts that for a generic diffeomorphism $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{HT}}$ the number of chain-recurrence classes is finite. From the Main Theorem, the classes are partially hyperbolic and the number of central bundles may be large. One can bound the number of classes having a central bundle of dimension ≥ 2.

Corollary 1.7. *For generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{HT}}$, there exist only finitely many chain-recurrence classes having more than one non-uniform one-dimensional central bundle.*

Index completeness. As we explained in Section 1.3, it is sometimes important to know the stable dimensions of the periodic orbits contained in a given neighborhood of a set K or which approximate a set for the Hausdorff topology. [ABCDW] discussed this problem of *index completeness* for chain-transitive sets of generic diffeomorphisms and proved that homoclinic classes have some “inner-index-completeness” property.

For a chain-transitive set Λ of f, one defines $\mathrm{ind}(\Lambda)$ as the set of integers i such that Λ is the Hausdorff limit of a sequence of hyperbolic periodic orbits of stable dimension i. If $\mathrm{ind}(\Lambda)$ is an interval of $\mathbb{Z}$, then one says that Λ is *index complete*.

Problem 1.9. *Are chain-transitive sets of C^1-generic f index complete?*

We obtain a positive answer far from homoclinic tangencies.

Corollary 1.8. *For generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{HT}}$ and for any chain-transitive set Λ:*

1. *Λ is index complete. Moreover, Λ admits a partially hyperbolic splitting*

$$T_\Lambda M = E^{\mathrm{s}} \oplus E_1^{\mathrm{c}} \oplus \cdots \oplus E_k^{\mathrm{c}} \oplus E^{\mathrm{u}},$$

   where $\dim(E_i^{\mathrm{c}}) = 1$ *and* $\mathrm{ind}(\Lambda) = \{\dim(E^{\mathrm{s}}), \dots, \dim(E^{\mathrm{s}}) + k\}$.
2. *There exists a neighborhood U of Λ such that* $\mathrm{ind}(\Lambda)$ *coincides with the set of stable dimensions of hyperbolic periodic orbits contained in U.*

Ergodic closing lemma inside homoclinic classes. The ergodic closing lemma of Mañé [M_1] asserts that for C^1-generic diffeomorphisms any ergodic probability measure μ is the weak limit of a sequence (O_n) of periodic orbits. As mentioned in Section 1.3, it is important to know if in some cases one can choose the periodic orbits O_n in the chain-recurrence class supporting μ. This corresponds to [Bon, Conjecture 2].

We obtain a positive answer in a very particular case.

Corollary 1.9. *For any generic diffeomorphism $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{HT}}$, let $H(p)$ be a homoclinic class and let i be the minimal stable dimension of its periodic orbits. Then, for any ergodic measure μ supported on $H(p)$ and whose i^{th} Lyapunov exponent is zero, there exist periodic orbits O_n contained in $H(p)$ whose associated measures converge to μ in the weak topology.*

Note that Conjecture 2 asserts that there should not exist such μ on homoclinic classes of generic diffeomorphisms in $\mathrm{Diff}^1(M) \setminus \overline{\mathrm{HT}}$.

2. Preliminary definitions and results

For the sake of completeness, in this section we recall some definitions and results that are useful in our context. The reader may skip this section and return to it when referenced.

2.1. General definitions

Let f be a diffeomorphism of M.

The positive and negative orbits of a point, $\{f^n(x) : n \geq 0\}$ and $\{f^{-n}(x), n \geq 0\}$, are denoted by $\mathrm{Orb}^+(x)$ and $\mathrm{Orb}^-(x)$.

The *chain-stable set* $W^{\text{ch-s}}_U(K)$ of a (not necessarily f-invariant) compact set K inside a set U that contains K is the set of points $x \in M$ that can be joined to a point of K by an ε-pseudo-orbit contained in U for any $\varepsilon > 0$. When $U = M$ we denote $W^{\text{ch-s}}(K) = W^{\text{ch-s}}_U(K)$.

The *local stable set* $W^{s}_{\eta}(x)$ of size $\eta > 0$ of a point $x \in M$ is the set of points $y \in M$ such that the distance between $f^n(x)$ and $f^n(y)$ for $n \geq 0$ remains smaller than η and goes to 0 as $n \to \infty$.

The (stable) *index* of a hyperbolic periodic point is its stable dimension.

Two hyperbolic periodic orbits O_1 and O_2 contained in an open set U are *homoclinically related in* U if there exist transverse intersection points in $W^{s}(O_1) \cap W^{u}(O_2)$ and $W^{u}(O_1) \cap W^{s}(O_2)$ whose orbits are contained in U. This implies that $H(O_1, U) = H(O_2, U)$.

2.2. Genericity results

We recall some properties that hold for diffeomorphisms in a dense G_δ subset of $\mathrm{Diff}^1(M)$.

Weak periodic points. The Franks lemma [F] allows us to change the index of a periodic orbit having a weak Lyapunov exponent by a C^1-small perturbation. With a classical Baire argument, one gets the following.

Lemma 2.1. *Let f be a diffeomorphism in a dense G_δ subset of* $\mathrm{Diff}^1(M)$ *and consider a sequence (O_n) of hyperbolic periodic orbits of stable index $i > 0$ which converges in the Hausdorff topology to a set Λ. Assume also that the i^{th} Lyapunov exponent $\lambda_i(O_n)$ of O_n goes to zero as $n \to \infty$. Then there exist hyperbolic periodic orbits of index $i - 1$ arbitrarily close to Λ in the Hausdorff topology whose i^{th} Lyapunov exponent is arbitrarily close to zero.*

Connecting lemmas. We need the following lemmas for connecting orbits.

Theorem 2.2 ([H]). *For any f and any C^1 neighborhood $\mathcal{U}$ of f, there is $L = L(\mathcal{U}) \in \mathbb{N}$ such that any non-periodic point $z \in M$ has two arbitrarily small neighborhoods $B_z \subset \hat{B}_z$ with the following property.*

Let $x, y \notin U_{L,z} := \bigcup_{i=0}^{L-1} f^i(\hat{B}_z)$ have iterates $f^{n_x}(x)$ and $f^{-n_y}(y)$ in B_z with $n_x, n_y \geq 1$. Then there exists $g \in \mathcal{U}$ such that:

- *$g = f$ on $M \setminus U_{L,z}$.*
- *y belongs to the forward g-orbit of x: there is $N \leq n_x + n_y$ such that $g^N(x) = y$.*
- *The orbit segment $\{x, g(x), \dots, g^N(x)\}$ is contained in the union of the orbit segments $\{x, f(x), \dots, f^{n_x}(x)\}$, $\{f^{-n_y}(y), \dots, y\}$ and $U_{L,z}$.*

This allows one to compare local homoclinic classes (the argument is similar to [GW, Lemma 4.2]).

Corollary 2.3. *Let f be a diffeomorphism in a dense G_δ subset of $\mathrm{Diff}^1(M)$, let p, q be hyperbolic periodic points and let U, V be open sets satisfying $\overline{U} \subset V$.*

- *If $H(p, U)$ contains q and if p, q have the same index, then the orbits of p, q are homoclinically related in V.*
- *If $H(p, U)$ and $H(q, U)$ intersect, then $H(p, V)$ contains $H(q, U)$.*

The connecting lemma for pseudo-orbits proved in [BC] can be restated in the following way (see [C_1]):

Theorem 2.4 ([BC]). *Let f be a diffeomorphism in a dense G_δ subset of $\mathrm{Diff}^1(M)$, let K be a compact set, and $x, y \in K$ such that $y \in W_K^{\text{ch-u}}(x)$. Then, for any neighborhoods U_x of x, U_y of y and U_K of K, there are $z \in U_x$ and $n \in \mathbb{N}$ such that $f^n(z) \in U_y$ and $\{z, f(z), \dots, f^n(z)\} \subset U_K$.*

Using Theorem 2.4, one can improve Corollary 2.3 about local homoclinic classes and get the following (see also [BC, Section 1.2.3]).

Corollary 2.5. *Let f be a diffeomorphism in a dense G_δ subset of $\mathrm{Diff}^1(M)$. Then, for any hyperbolic periodic point p, any compact sets Λ, Δ such that Λ is contained in the local chain-stable and chain-unstable sets of p inside Δ, and any neighborhood U of Δ, the local homoclinic class $H(p, U)$ contains Λ.*

In particular, the homoclinic classes of a C^1-generic diffeomorphism are chain-recurrence classes. Also considering a hyperbolic periodic orbit O and two chain-transitive compact sets $K \subset \Lambda$ contained in an open set U such that $K \subset H(O, U)$, the class $H(O, V)$ contains Λ for any open neighborhood V of $\overline{U}$.

The following global connecting lemma allows one to control the support of orbit segments.

Theorem 2.6 ([C$_1$]). *For any diffeomorphism f in a dense G_δ subset of $\mathrm{Diff}^1(M)$ and for any $\delta > 0$, there exists $\varepsilon > 0$ such that for any ε-pseudo-orbit segment $X = \{z_0, z_1, \dots, z_n\}$, one can find an orbit segment $O = \{x, f(x), \dots, f^m(x)\}$ which is δ-close to X for the Hausdorff distance.*

Moreover if X is a periodic pseudo-orbit (i.e. $z_n = z_0$), then x can be chosen m-periodic. In particular chain-transitive sets are Hausdorff limits of periodic orbits.

Dominated splitting far from homoclinic tangencies. The existence of dominated splitting far from homoclinic tangencies was obtained in [PuS] in the two-dimensional setting and in [W$_1$] in the general case. The basic idea is that periodic orbits may have at most one Lyapunov exponent close to zero.

Theorem 2.7 ([W$_1$, W$_2$]). *Let $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{HT}}$ and let $i \in \{1, \dots, \dim(M) - 1\}$. Then the tangent bundle above the set of hyperbolic periodic points with index i has a dominated splitting $E \oplus F$ such that $\dim(E) = i$.*

Moreover, there exist $\delta, C > 0$, $\sigma \in (0, 1)$ and an integer $N \geq 1$ such that if $\mathcal{O}$ is a hyperbolic periodic orbit of period τ and if $E \oplus E^c \oplus F$ is the splitting over $T_{\mathcal{O}}M$ into the characteristic spaces whose Lyapunov exponents belong to $(-\infty, \delta]$, $(-\delta, \delta)$, $[\delta, \infty)$, then:

- *E^c has dimension at most one.*
- *The splitting is N-dominated.*
- *For any $x \in \mathcal{O}$ we have*

$$\prod_{k=0}^{[\tau/N]-1} \|Df_{|E}(f^{kN}(x))\| \leq C\sigma^\tau, \qquad \prod_{k=0}^{[\tau/N]-1} \|Df^{-1}_{|F}(f^{-kN}(x))\| \leq C\sigma^\tau. \tag{2.1}$$

The last result induces dominated splitting over homoclinic classes.

Corollary 2.8. *For any f in a dense G_δ subset of $\mathrm{Diff}^1(M) \setminus \overline{\mathrm{HT}}$ there exists δ such that any homoclinic class $H(O)$ satisfies the following.*

If O has no Lyapunov exponent in $(-\delta, \delta)$, then $H(O)$ has a dominated splitting $E \oplus F$ where $\dim(E)$ coincides with the stable index of O.

If O has a Lyapunov exponent in $(-\delta, \delta)$, then $H(O)$ has a dominated splitting $E \oplus E^c \oplus F$ where E^c is one-dimensional and the Lyapunov exponent of O along E^c belongs to $(-\delta, \delta)$.

Proof. The set of hyperbolic periodic points with Lyapunov exponents close to those of O is dense in $H(O)$ by [BCDG, Lemma 4.1]. By Theorem 2.7 the result follows. □

The following is a consequence of Theorem 2.7 and Liao's selecting lemma. The proof is exactly the same as in [C$_3$, Theorem 1]; the statement we give now is more local.

Theorem 2.9 ([C$_3$, Theorem 1]). *Let $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{HT}}$ and let K_0 be a compact invariant set having a dominated splitting $E \oplus F$. If E is not uniformly contracted, then for any neighborhood U of K_0 one of the following occurs:*

1. *K_0 intersects a local homoclinic class $H(O,U)$ associated to a periodic orbit O whose stable index is strictly less than* $\dim(E)$.
2. *K_0 intersects local homoclinic classes $H(O_n,U)$ associated to periodic orbits O_n whose stable index is equal to* $\dim(E)$ *and which contain weak periodic orbits: for any $\delta>0$ there exists a sequence of hyperbolic periodic orbits $\mathcal{O}_n$ that are homoclinically related in U, converge in the Hausdorff topology to a compact subset K of K_0, have stable index equal to* $\dim(E)$*, and have maximal Lyapunov exponent along E belonging to* $(-\delta,0)$.
3. *There exists an invariant compact set $K\subset K_0$ which has partially hyperbolic structure $E^{\mathrm{s}}\oplus E^{\mathrm{c}}\oplus E^{\mathrm{u}}$. Moreover,* $\dim(E^{\mathrm{s}})<\dim(E)$*, the central bundle E^{c} is one-dimensional and any invariant measure supported on K has Lyapunov exponent along E^{c} equal to* 0.

Theorems 2.7 and 2.4 allow one to extend the dominated splitting over a chain-transitive set to larger sets.

Proposition 2.10 ([C3, Proposition 1.10]). *Let f be a diffeomorphism in a dense G_δ set $\mathcal{G}_{\mathrm{Ext}}\subset \mathrm{Diff}^1(M)\setminus\overline{\mathrm{HT}}$ and let K be a chain-transitive set which has a partially hyperbolic splitting $E^{\mathrm{s}}\oplus E^{\mathrm{c}}\oplus E^{\mathrm{u}}$ such that E^{c} is one-dimensional and such that any invariant measure supported on K has Lyapunov exponent along E^{c} equal to* 0. *Then, for any chain-transitive set A that strictly contains K there exists a chain-transitive set $A'\subset A$ that strictly contains K and there exists a dominated decomposition $E_1\oplus E^{\mathrm{c}}\oplus E_3$ on A' that extends the partially hyperbolic structure of K. Moreover, the set A' is the Hausdorff limit of periodic orbits of stable index* $\dim(E^{\mathrm{s}})$ *whose Lyapunov exponent along E^{c} is arbitrarily close to zero.*

Indices of a homoclinic class. We now state a result from [ABCDW, Corollaries 2 and 3]:

Theorem 2.11 ([ABCDW, Corollaries 2 and 3]). *Let f be a diffeomorphism in a dense G_δ subset of* $\mathrm{Diff}^1(M)\setminus\overline{\mathrm{HT}}$ *and let H be a homoclinic class having hyperbolic saddles of stable indices α and β with $\alpha<\beta$. Then:*

1. *H contains periodic points of stable index j for every $\alpha\le j\le\beta$.*
2. *For every $j\in\{\alpha+1,\dots,\beta\}$ there exist periodic orbits in H of stable index* $j-1$ *and j whose j^{th} Lyapunov exponent is arbitrarily close to* 0.

Remark 2.1. The last theorem is also valid for local homoclinic classes (with the same proof): if $H=H(p,U)$ is a local homoclinic class containing hyperbolic periodic points of indices $\alpha<\beta$, then for any neighborhood V of $\overline{U}$, the local homoclinic class $H(p,V)$ contains hyperbolic periodic points of any index $i\in\{\alpha,\dots,\beta\}$.

Heterodimensional cycles. Robust heterodimensional cycles can be obtained from periodic points with different indices contained in the same chain-recurrence class.

Theorem 2.12 ([BDK]). *For any diffeomorphism f in a dense G_δ subset of* $\mathrm{Diff}^1(M)$*, if $H(p)$ is a homoclinic class containing hyperbolic periodic points of different indices, then $H(p)$ contains a robust heterodimensional cycle.*

2.3. Dominated splitting, hyperbolic times

In this section we consider an invariant set Λ with a dominated splitting $T_\Lambda M = E \oplus F$.

Extension to the closure. The following property is classical.

Lemma 2.13. *An N-dominated splitting $T_\Lambda M = E \oplus F$ over an invariant set Λ extends uniquely onto the closure of Λ as an invariant N-dominated splitting.*

Adapted norm. Using [G_1], one can replace the Riemannian norm by another one which is adapted to the domination: for this norm, there exists $\lambda \in (0,1)$ such that for any point $x \in \Lambda$ and any unitary vectors $v^E \in E_x$ and $v^F \in F_x$ we have

$$\|Df.v^E\| \leq \lambda \|Df.v^F\|.$$

Generalization to forward invariant sets. Consider a small neighborhood U of Λ and consider the set Λ^+ of points whose forward f-orbit is contained in $\overline{U}$. Similarly Λ^- denotes the set of points whose backward f-orbit is contained in $\overline{U}$. The bundle E on Λ extends uniquely as a forward invariant continuous bundle that is tangent to a center-stable cone. The bundle E is also characterized in the following way: for any $x \in \Lambda^+$ and any vectors $u \in E_x \setminus \{0\}$ and $v \in T_x M \setminus E_x$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \|Df^n(x).u\| < \liminf_{n\to\infty} \frac{1}{n} \log \|Df^n(x).v\|.$$

Hyperbolic points. Let Λ be a compact invariant set and E be a continuous invariant bundle over Λ. For any $C > 0$ and $\sigma \in (0,1)$, we say that a point $x \in \Lambda$ is (C,σ,E)-*hyperbolic for* f if for any $n \in \mathbb{N}$,

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i(x))}\| \leq C\sigma^n.$$

For any $x \in \Lambda$, if $f^n(x)$ is (C,σ,E)-hyperbolic for f, then n is called a (C,σ,E)-*hyperbolic time* of x for f.

Note that the definition of (C,σ,E)-hyperbolic points extends to the set Λ^+ introduced above. One defines similarly the (C,σ,E)-hyperbolic times for f^{-1}. When there is no ambiguity between f and f^{-1}, we just say that a point is (C,σ,E)-hyperbolic.

The next lemma gives the existence of $(1,\sigma,E)$-hyperbolic points, which is a weak version of the well-known Pliss lemma.

Lemma 2.14. *Let Λ be an invariant set, E be a continuous invariant bundle on Λ and $\sigma \in (0,1)$. Then for any $x \in \Lambda$ with*

$$\limsup_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{n} \log \|Df|_{E(f^i(x))}\| < \log \sigma,$$

there is a $(1,\sigma,E)$-hyperbolic point y for f in the forward orbit of x.

Proof. Let $\varphi\colon \Lambda \to \mathbb{R}$ be the function defined by $\varphi(x) = \log\|Df|E(x)\| - \log\sigma$ so that $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x)) < 0$. If $\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x)) < 0$ for all $n \geq 1$ then x is $(1, \sigma, E)$-hyperbolic. Otherwise, there exists $n_0 \geq 1$ such that:

- $\frac{1}{n_0}\sum_{i=0}^{n_0-1}\varphi(f^i(x)) \geq 0$.
- $\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x)) < 0$ for all $n > n_0$.

This implies that $y = f^{n_0}(x)$ is $(1, \sigma, E)$-hyperbolic. Indeed, for any $n \geq 1$,

$$\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(y)) = \frac{1}{n}\Big(\sum_{i=0}^{n+n_0-1}\varphi(f^i(x)) - \sum_{i=0}^{n_0-1}\varphi(f^i(x))\Big).$$

By the above equalities, the first sum on the right hand side is less than 0 and the second is not. So the difference is negative and y is $(1, \sigma, E)$-hyperbolic. $\square$

We are interested by finding (C, σ, E)-hyperbolic points for f and (C, σ, F)-hyperbolic points for f^{-1} that are close.

Lemma 2.15. *Consider a set Λ and a constant $\lambda \in (0, 1)$ as above. Consider also $\sigma, \rho \in (0, 1)$ such that $\lambda < \rho\sigma$ and a sequence (y_k) in Λ such that one of the following two conditions holds:*

$$\limsup_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{n}\log\|Df^{-1}|_{E(f^{-i}(y_k))}\| < \log(\rho^{-1}),$$
$$\limsup_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{n}\log\|Df^{-1}|_{F(f^{-i}(y_k))}\| < \log\sigma.$$

Then one of the following properties holds:

- *There exists $C > 0$ such that infinitely many y_k are (C, σ, F)-hyperbolic for f^{-1}.*
- *For each $k \geq 0$, there exists a backward iterate z_k of y_k which is $(1, \sigma, F)$-hyperbolic for f^{-1} and any accumulation point of (z_k) is $(1, \rho, E)$-hyperbolic for f.*

Proof. Note that under the first condition, the domination between E and F gives, for each k,

$$\limsup_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{n}\log\|Df^{-1}|_{F(f^{-i}(y_k))}\| < \log\lambda + \log(\rho^{-1}) < \log\sigma,$$

so that the second condition holds. Thus, by Lemma 2.14, one can get a $(1, \sigma, F)$-hyperbolic point $f^{-\ell_k}(y_k)$ for f^{-1} in the backward orbit of y_k. Let z_k be the last one (corresponding to the smallest $\ell_k \geq 0$). After taking a subsequence if necessary, two cases are possible:

(1) There is $N_0 \in \mathbb{N}$ such that $\ell_k \leq N_0$.
(2) $\lim_{k\to\infty}\ell_k = \infty$.

For case (1), we take

$$C = \max_{x\in\Lambda,\, 1\le n\le N_0}\left\{\prod_{i=0}^{n-1}\|Df^{-1}|_{F(f^{-i}(x))}\|\right\}.$$

It is clear that any y_k is (C, σ, F)-hyperbolic and the first case of the lemma occurs.

For case (2), we claim that for each k and for any $m \in [0, \ell_k] \cap \mathbb{N}$,

$$\prod_{i=1}^{m}\|Df^{-1}|_{F(f^i(z_k))}\| \ge \sigma^m.$$

This is proved by induction (the case $m = 0$ being clear). Assume that it holds up to some integer $m \in [0, \ell_k - 1] \cap \mathbb{N}$ and that it is not satisfied for $m + 1$. For each $0 \le n \le m$ we have

$$\prod_{i=n+1}^{m+1}\|Df^{-1}|_{F(f^i(f^{-\ell_k}(y_k)))}\| < \sigma^{m+1-n}.$$

Since $f^{-\ell_k}(y_k)$ is $(1, \sigma, F)$-hyperbolic, this shows that $f^{m+1-\ell_k}(y_k)$ is also $(1, \sigma, F)$-hyperbolic, contradicting the maximality of $-\ell_k$. Hence the claim holds.

Any limit point z of the sequence (z_k) thus satisfies, for each $m \ge 0$,

$$\prod_{i=1}^{m}\|Df^{-1}|_{F(f^i(z))}\| \ge \sigma^m.$$

The domination between E and F implies

$$\prod_{i=0}^{m-1}\|Df|_{E(f^i(z))}\| \le \sigma^{-m}\lambda^m \le \rho^m,$$

proving that z is $(1, \rho, E)$-hyperbolic. □

Local manifolds. A (*locally invariant*) *plaque family* tangent to E is a continuous map $\mathcal{D}$ from the linear bundle E over Λ into M satisfying:

- For each $x \in \Lambda$ the induced map $\mathcal{D}_x : E_x \to M$ is a C^1-embedding that is tangent to E_x at the point $\mathcal{D}_x(0) = x$.
- The family $(\mathcal{D}_x)_{x\in\Lambda}$ of C^1-embeddings is continuous.
- The plaque family is locally forward invariant, i.e., there exists a neighborhood U of the zero section in E such that for each $x \in \Lambda$, the image of $\mathcal{D}_x(E_x \cap U)$ by f is contained in $\mathcal{D}_{f(x)}(E_{f(x)})$.

This definition extends to locally forward invariant plaque families tangent to E over the set Λ^+. In a symmetric way, one can consider plaque families tangent to F over Λ^-, which are locally backward invariant. In the following, $\mathcal{D}_x$ will also denote the image of the embedding $\mathcal{D}_x$.

Plaque families tangent to E over an invariant set Λ always exist by [HPS]; the same proof gives locally forward invariant plaque families over Λ^+.

The following lemma ensures the existence of a *local stable manifold* at (C, σ, E)-hyperbolic points. The proof is classical (see for instance [ABC, Section 8.2]).

Lemma 2.16. *Let $\mathcal{D}$ be a locally forward invariant plaque family tangent to E over Λ^+. Then, for any constants $C > 0$, $\sigma < \sigma' < 1$, there exist $C', r > 0$ such that for any (C, σ, E)-hyperbolic point $x \in \Lambda^+$, the ball $B^E(x, r)$ in $\mathcal{D}_x$ is contracted by forward iterations. More precisely, for each $r' \in (0, r)$ and $n \geq 0$, the image $f^n(B^E(x, r'))$ is contained in $\mathcal{D}_{f^n(x)}$ and has diameter smaller than $C'(\sigma')^n r'$.*

The stable and unstable manifolds of E-hyperbolic and F-hyperbolic points that are close intersect:

Lemma 2.17. *Consider plaque families $\mathcal{D}^E, \mathcal{D}^F$ tangent to E, F over the sets Λ^+, Λ^- respectively. For any $C > 0$ and $\sigma \in (0, 1)$, there exists $r > 0$ such that if $y \in \Lambda^-$ is (C, σ, F)-hyperbolic for f^{-1} and $x \in \Lambda^+$ is (C, σ, E)-hyperbolic for f, and if the distance between x, y is smaller than r, then the local stable manifold at x and the local unstable manifold at y intersect.*

Proof. For $x \in \Lambda^+$ and $y \in \Lambda^-$ that are close, the plaques $\mathcal{D}^E_x$ and D^F_y intersect at a point z close to x, y. By the previous lemma, this point belongs to the local stable manifold at x and the local unstable manifold at y. □

If Λ has a splitting $T_\Lambda M = E' \oplus E^c \oplus F$, the following result allows one to get intersections between stable manifolds tangent to E' and unstable manifolds tangent to F. The argument is the same as for [BGW, Theorem 13] and [C_2, Proposition 3.7].

Lemma 2.18. *Assume that Λ has a dominated splitting $T_\Lambda M = E \oplus F = (E' \oplus E^c) \oplus F$ where E^c is one-dimensional, and consider plaque families $\mathcal{D}^{E'}, \mathcal{D}^F$ tangent to E', F over Λ^+, Λ^- respectively. Assume also that Λ^+ contains a curve I tangent to E and transverse to E'. For any interior point x of I there exists $r > 0$ with the following property.*

For any $y \in \Lambda^-$ that is r-close to x, the plaque $\mathcal{D}^F_y$ intersects $\mathcal{D}^{E'}_{x'}$ for some $x' \in I$. In particular for any $C > 0$ and $\sigma \in (0, 1)$, if r is small enough, Lemma 2.16 implies that if y is (C, σ, F)-hyperbolic for f^{-1} and all points of I are (C, σ, E')-hyperbolic for f, then the local unstable manifold of y and the local stable manifold of x' intersect.

Remark 2.2. In the setting of the previous lemma, if $E' = E^s$ is uniformly contracted, then any point in I is (C, σ, E')-hyperbolic for f.

A weak shadowing lemma for measures. We now state a result regarding hyperbolic measures whose Oseledets splitting is dominated. The argument is classical [C_3, Proposition 1.4]. The version we state now is local but the proof is the same.

Proposition 2.19. *Let f be a C^1 diffeomorphism and μ be an ergodic measure which has no zero Lyapunov exponent and whose Oseledets splitting $E^s \oplus E^u$ is dominated. Then μ is supported on a homoclinic class. Indeed, for any neighborhood U of the support of μ, there is a sequence of hyperbolic periodic orbits O_n of index $\dim(E^s)$ that are homoclinically related in U, converge to the support of μ in the Hausdorff topology and the invariant measures supported on the O_n converge to μ in the weak-$*$ topology.*

3. Theorem 1.2 implies Theorem 1.1

In this section we prove Theorem 1.1 assuming that Theorem 1.2 holds. The proof will be based on several results stated in Section 2. Before we go into details, we give a general idea of the argument.

Since we are far from homoclinic tangencies, we know from the results in Section 2 that any chain-recurrence class $\mathcal{C}$ has a dominated splitting. We distinguish the cases when $\mathcal{C}$ is an aperiodic or a homoclinic class. In the first case we know that $\mathcal{C}$ is the Hausdorff limit of periodic orbits and that any ergodic measure on $\mathcal{C}$ has vanishing Lyapunov exponent, so we manage to get a dominated splitting with a one-dimensional central bundle. If $\mathcal{C}$ is a homoclinic class and if a bundle E is not uniformly contracted, then E decomposes over a subset $\Lambda \subset \mathcal{C}$ and Λ satisfies the conditions of Theorem 1.2, so E can be decomposed over the whole class $\mathcal{C}$.

Now we give the precise proof of Theorem 1.1. It uses the following lemma. The generic set in $\mathrm{Diff}^1(M) \setminus \overline{\mathrm{HT}}$ that appears in the statements of both Lemma 3.1 and Theorem 1.1 is the set of diffeomorphisms satisfying Theorem 1.2 and all the generic results of Section 2.

Lemma 3.1. *Let f be a diffeomorphism which belongs to a dense G_δ subset of* $\mathrm{Diff}^1(M) \setminus \overline{\mathrm{HT}}$ *and let $\mathcal{C}$ be a chain-recurrent class of f such that:*

- *$\mathcal{C}$ has a dominated splitting $T_{\mathcal{C}}M = E \oplus F$.*
- *$\mathcal{C}$ is the Hausdorff limit of periodic orbits of index* $\dim(E)$.
- *$\mathcal{C}$ does not contain any periodic point whose index is less than* $\dim(E)$.
- *E is not uniformly contracted.*

Then there exists a dominated splitting $T_{\mathcal{C}}M = E' \oplus E^{\mathrm{c}} \oplus F'$ where $\dim(E^{\mathrm{c}}) = 1$ *and* $\dim(E') < \dim(E)$ *and $\mathcal{C}$ is the Hausdorff limit of periodic orbit of index* $\dim(E')$.

Proof. Since E is not uniformly contracted, Theorem 2.9 applies. By our assumptions, the first case in the conclusion does not occur.

If the second case of Theorem 2.9 occurs, then by Corollary 2.5, for any $\delta > 0$ small, $\mathcal{C}$ is the homoclinic class of a periodic orbit whose maximal Lyapunov exponent along E belongs to $(-\delta, 0)$. Then $\mathcal{C}$ is the Hausdorff limit of periodic orbits of index $\dim(E) - 1$, and by Corollary 2.8, the class has a dominated splitting $T_{\mathcal{C}}M = E' \oplus E^{\mathrm{c}} \oplus F$, as required.

We thus assume that $\mathcal{C}$ contains a minimal set K with a partially hyperbolic splitting $T_K M = E^{\mathrm{s}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{u}}$ satisfying the third case of Theorem 2.9. We can take such K with least $\dim(E^{\mathrm{s}})$: for any other minimal set $K' \subset \mathcal{C}$ having a dominated splitting $T_{K'}M = F^{\mathrm{s}} \oplus F^{\mathrm{c}} \oplus F^{\mathrm{u}}$ and such that any invariant probability measure supported on K' has zero Lyapunov exponent along F^{c}, we have $\dim(E^{\mathrm{s}}) \leq \dim(F^{\mathrm{s}})$.

Consider the following family $\mathcal{F}$ of compact invariant chain-transitive subsets of $\mathcal{C}$, say Λ, such that:

- Λ contains K.
- $T_\Lambda M = E' \oplus E^{\mathrm{c}} \oplus F'$ dominated with $\dim(E') = \dim(E^{\mathrm{s}})$ and $\dim(E^{\mathrm{c}}) = 1$.

- Λ is the Hausdorff limit of periodic orbits $\mathcal{O}$ whose Lyapunov exponent along E^{c} is arbitrarily weak (i.e., for any $\delta > 0$ there exists a sequence $\mathcal{O}_n$ of periodic orbits converging in the Hausdorff topology to Λ such that the Lyapunov exponent of each $\mathcal{O}_n$ along E^{c} belongs to $(-\delta, \delta)$).

Notice that the last condition makes sense since every periodic orbit near Λ has a splitting $E' \oplus E^{\mathrm{c}} \oplus F'$. Note also that K belongs to $\mathcal{F}$ by Theorem 2.6. Order the family $\mathcal{F}$ by inclusion, that is, $\Lambda_1 \leq \Lambda_2$ if $\Lambda_1 \subset \Lambda_2$. We will apply Zorn's Lemma to get a maximal set within $\mathcal{F}$. Let $\{\Lambda_\alpha : \alpha \in \Gamma\}$ be a totally ordered chain and consider

$$\Lambda_\infty = \overline{\bigcup_{\alpha\in\Gamma} \Lambda_\alpha}.$$

It follows that Λ_∞ is the Hausdorff limit of periodic orbits whose $(\dim E'+1)^{\mathrm{th}}$ Lyapunov exponent is arbitrarily weak. Thus, by Theorem 2.7, the splitting along these periodic orbits extends to the closure and so Λ_∞ has a dominated splitting $E' \oplus E^{\mathrm{c}} \oplus F'$. Thus, Λ_∞ is an upper bound set of $\{\Lambda_\alpha : \alpha \in \Gamma\}$. Now, Zorn's Lemma yields a maximal set $\Lambda \in \mathcal{F}$ with respect to the order.

In the case $\Lambda = \mathcal{C}$ we immediately obtain the conclusion of the lemma. Now assume that $\Lambda \neq \mathcal{C}$. By the maximality of Λ, Proposition 2.10 implies that there exists a measure supported on Λ whose Lyapunov exponent along E^{c} is not zero. Note also that the bundle E' is uniformly contracted since otherwise Theorem 2.9 would contradict the minimality of $\dim(E') = \dim(E^{\mathrm{s}})$ for K. Theorem 1.2 can thus be applied.

By our assumptions, the first property of Theorem 1.2 does not happen. The second property holds and implies that $\mathcal{C}$ is the Hausdorff limit of periodic orbits whose $(\dim(E') + 1)^{\mathrm{th}}$ Lyapunov exponent is arbitrarily close to zero. By Theorem 2.7 and Lemma 2.13, the splitting $T_\Lambda M = E' \oplus E^{\mathrm{c}} \oplus F'$ on Λ extends as a dominated splitting on $\mathcal{C}$. By Lemma 2.1, $\mathcal{C}$ is the Hausdorff limit of periodic orbit of index $\dim(E')$, ending the proof. $\square$

Let $\mathcal{C}$ be a chain-recurrent class of f. Assume first that $\mathcal{C}$ is aperiodic. By Theorem 2.6 it is the Hausdorff limit of a sequence of periodic orbits. By Theorem 2.7, we get a dominated splitting $T_{\mathcal{C}}M = E \oplus F$ such that $\dim(E)$ coincides with the index of the periodic orbits. Since $\mathcal{C}$ is aperiodic, it is not a hyperbolic set, hence either E is not uniformly contracted or F is not uniformly expanded. Assume for instance that the former holds. One can then apply Lemma 3.1 above. This gives another dominated splitting $T_{\mathcal{C}}M = E' \oplus E^{\mathrm{c}} \oplus F'$ such that $\mathcal{C}$ is the Hausdorff limit of periodic orbits of index $\dim(E') < \dim(E)$. If E' is not uniformly contracted, the lemma applies again. Applying the lemma inductively, one can assume that $\mathcal{C}$ has a splitting $T_{\mathcal{C}}M = E^{\mathrm{s}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{cu}}$ where E^{s} is uniformly contracted and E^{c} has dimension 1 and is not uniformly contracted. Since $\mathcal{C}$ is aperiodic, the conclusion of Theorem 1.2 does not occur. Consequently, for any invariant probability measure supported on $\mathcal{C}$, the Lyapunov exponent along E^{c} is zero. By domination, this implies that E^{cu} is uniformly expanded, as in Theorem 1.1.

We assume now that $\mathcal{C}$ is a homoclinic class $H(p)$. By Theorems 2.11 and 2.7, there exists a dominated splitting

$$T_{H(p)}M = F^{\mathrm{cs}} \oplus F^{\mathrm{c}}_1 \oplus \cdots \oplus F^{\mathrm{c}}_s \oplus F^{\mathrm{cu}}$$

such that each bundle F_i^c is one-dimensional, $\dim(F^{cs})$ is equal to the minimal index of periodic orbits of $H(p)$, and $\dim(M) - \dim(F^{cu})$ is equal to the maximal index of periodic orbits of $H(p)$. Moreover for each i, there exist periodic orbits in $H(p)$ whose Lyapunov exponent along F_i^c is arbitrarily close to zero.

If the bundle F^{cs} is not uniformly contracted, then Lemma 3.1 applies. As before, by induction we get a dominated splitting $T_{H(p)}M = E^s \oplus E^c \oplus E^{cu}$ such that $\dim(E^s) < \dim(F^{cs})$, the bundle E^c is one-dimensional and not uniformly contracted, and E^s is uniformly contracted. Since $H(p)$ contains hyperbolic periodic points, there exists an invariant measure supported on $H(p)$ whose Lyapunov exponent along E^c is not zero; hence Theorem 1.2 can be used. The first case of the conclusion does not happen since $\dim(E^s)$ is smaller than the minimal index of periodic points in $H(p)$. The second case is thus satisfied: the minimal index $\dim(F^{cs})$ coincides with $\dim(E^s) + 1$ so that $F^{cs} = E^s \oplus E^c$. Moreover there exist periodic points whose Lyapunov exponent along E^c is arbitrarily close to zero.

The same holds for the bundle F^{cu}, giving the conclusion of Theorem 1.1.

4. Central models and some consequences

When dealing with sets having a dominated splitting where one bundle is one-dimensional, the first author developed a tool, known as "Central Models", which proved to be useful in this context (see [C$_2$], [C$_3$]). We will recall the main classification result regarding these central models.

4.1. Definition of central models

Let K be a compact invariant set with a dominated splitting of the form $T_K M = E_1 \oplus E^c \oplus E_3$ where E^c is one-dimensional. Consider a locally invariant *plaque family* $\mathcal{D}^c$ tangent to E^c. It always exists: using a result of [HPS] recalled in Section 2.3, one can consider center-stable and center-unstable plaque families $\mathcal{D}^{cs}$, $\mathcal{D}^{cu}$ tangent to the bundles $E_1 \oplus E^c$ and $E^c \oplus E_3$ respectively and choose $\mathcal{D}^c$ such that for each $x \in K$, the image of $\mathcal{D}^c_x$ is contained in the intersection of the images of $\mathcal{D}^{cs}_x$ and $\mathcal{D}^{cu}_x$.

Plaque families tangent to E^c is a way to introduce central manifolds for points $x \in K$. When such a family $\mathcal{D}^c$ is fixed and $\eta > 0$ small is given, one defines the *central manifold of size* η at $x \in K$ as

$$W^c_\eta(x) = \mathcal{D}^c_x(-\eta, \eta).$$

The above properties of the central plaque family allow one to lift the dynamics of f as a fibered dynamics $\hat{f}$ on the bundle E^c that is locally defined in a neighborhood of the (invariant) zero section of E^c. (See [C$_2$] and [C$_3$] for details.) First, one notices that the action induced by f on the unitary bundle associated to E^c is the union of one or two chain-recurrence classes. We denote by $\hat{K}$ one of these classes: it can be viewed as a collection of central half-plaques of $\mathcal{D}^c$. These half-plaques can be parameterized, giving a projection map $\pi : \hat{K} \times [0, \infty) \to M$ such that $\pi(\{\hat{x}\} \times [0, \infty)) \subset \mathcal{D}^c_x$ and $\pi(\hat{x}, 0) = x$ for each $\hat{x} \in \hat{K}$ above a point $x \in K$. Since the plaque family $\mathcal{D}^c$ is locally invariant, the map f can be lifted as a map $\hat{f}$ defined on a neighborhood of $\hat{K} \times \{0\}$ which is fiber preserving and locally invertible. The set $\hat{K} \times [0, \infty)$ endowed with the map $\hat{f}$ is called a *central model* of (K, f) for the bundle E^{c}.

Sometimes it is useful to consider central models associated to smaller plaques. When $\eta > 0$ is given we say that the central model is *associated to the plaques* W^{c}_η if for each $\hat{x} \in \hat{K}$ the projection $\pi(\{\hat{x}\} \times [0, \infty))$ is contained in $W^{\mathrm{c}}_\eta(x)$ where $x = \pi(\hat{x}, 0)$.

Two cases can occur when K is chain-transitive:

- Either the action induced by f on the unitary bundle associated to E^{c} has only one chain-recurrence class. Equivalently, there is no continuous orientation of E^{c} preserved by the action of Df. In this case $\hat{K}$ is a two-fold covering of K. Moreover if $\hat{x}^-, \hat{x}^+$ in $\hat{K}$ are the two lifts of $x \in K$, then $\pi(\{\hat{x}^-\} \times [0, \infty)) \cup \pi(\{\hat{x}^+\} \times [0, \infty))$ contains a uniform neighborhood of x in $\mathcal{D}^{\mathrm{c}}_x$.
- Or the unitary bundle associated to E^{c} is the union of two chain-recurrence classes $\hat{K}^-, \hat{K}^+$. In this case, $\hat{K}^-, \hat{K}^+$ are copies of K. Consider any two central models $\pi^\pm \colon \hat{K}^\pm \times [0, \infty) \to M$. Then for any lifts $\hat{x}^\pm \in \hat{K}^\pm$ of $x \in K$, the union $\pi(\{\hat{x}^-\} \times [0, \infty)) \cup \pi(\{\hat{x}^+\} \times [0, \infty))$ contains a uniform neighborhood of x in $\mathcal{D}^{\mathrm{c}}_x$.

4.2. Classification of central dynamics

Let us consider a central model $\hat{f} \colon \hat{K} \times [0, 1] \to \hat{K} \times [0, \infty)$. We introduce some definitions used to describe its dynamics.

Definition 4.1. (a) When $\hat{Z}$ is a subset of $\hat{K}$, we define the *chain-unstable set* of $\hat{Z}$ for the dynamics of the central model, $\hat{W}^{\text{ch-u}}(\hat{Z})$, as the set of points $y \in \hat{Z} \times [0, \infty)$ such that for any $\varepsilon > 0$, there exists a sequence $\{x_0, \dots, x_n\}$ in $\hat{Z}$ and a sequence $\{y_0, \dots, y_n\}$ in $\hat{Z} \times [0, 1]$ satisfying:

- $y_k \in \{x_k\} \times [0, 1]$ for each $k \in \{0, \dots, n\}$.
- $y_0 = (x_0, 0)$ and $y_n = y$.
- y_k and $\hat{f}(y_{k-1})$ are ε-close for each $k \in \{1, \dots, n\}$.

We define the *chain-stable set* of $\hat{Z}$ similarly. Note that we do not assume that $\hat{Z}$ is compact or invariant.

(b) A point $\hat{x} \in \hat{K}$ has a *chain-recurrent central segment* if there exists $a > 0$ such that $\{\hat{x}\} \times [0, a]$ is contained in the chain-stable and chain-unstable sets of $\hat{K} \times \{0\}$.

(c) We say that the dynamics of $\hat{f}$ is *thin trapped* if there exists arbitrarily small open neighborhoods U of $\hat{K} \times \{0\}$ such that $\hat{f}(\overline{U}) \subset U$ and $U \cap (\{\hat{x}\} \times [0, \infty))$ is an interval for each $\hat{x} \in \hat{K}$.

The next classification results are the basis of this theory. They restate [C_2, Section 2] and [C_3, Proposition 2.2].

Theorem 4.1. *Let $(\hat{K} \times [0, \infty), \hat{f})$ be a central model and assume that K is chain-transitive. Then:*

1. *The dynamics of $\hat{f}$ is not thin trapped if and only if the chain-unstable set of $\hat{K}$ contains a non-trivial interval $\{\hat{y}\} \times [0, a]$, $a > 0$.*

2. *The dynamics of $\hat{f}$ and $\hat{f}^{-1}$ are not thin trapped if and only if there is a chain-recurrent central segment.*
3. *If the dynamics of $\hat{f}$ is thin trapped and the dynamics of f^{-1} is not thin trapped then there is a neighborhood of $\hat{K} \times \{0\}$ contained in the chain-stable set of $\hat{K} \times \{0\}$.*

Proof. Let us explain the first item:

- If the chain-unstable set of $\hat{K} \times \{0\}$ is reduced to $\hat{K} \times \{0\}$, we apply [C_2, Lemma 2.7] (the ε-chain-unstable sets of $\hat{K} \times \{0\}$ are arbitrarily small trapped strips): the dynamics is thin trapped.
- If the chain-unstable set of $\hat{K} \times \{0\}$ contains a point $(\hat{y}, a)$ with $a > 0$, it also contains the interval $\{\hat{y}\} \times [0, a]$. Clearly, the dynamics cannot be thin trapped.

The second item is [C_2, Proposition 2.5]. For the third item: if the dynamics of $\hat{f}$ is thin trapped and the dynamics of f^{-1} is not, then there is no chain-recurrent central segment and by the first item the chain-stable set of $\hat{K} \times \{0\}$ is not trivial. By [C_2, Lemma 2.8], there is a neighborhood of $\hat{K} \times \{0\}$ contained in the chain-stable set of $\hat{K} \times \{0\}$. □

We now come back to the manifold M and discuss the central dynamics of K.

Definition 4.2. (a) K has a *chain-recurrent central segment* if in some central model of (K, f), there exists a chain-recurrent central segment. More precisely, a point $x \in K$ has a chain-recurrent central segment with respect to a central model of (K, f) if in this central model there exists a chain-recurrent central segment $\{\hat{x}\} \times [0, a]$, $a > 0$, where $\hat{x}$ lifts x.

(b) The central dynamics is *thin trapped* if there exist one or two central models $(\hat{K} \times [0, \infty), \hat{f})$ or $(\hat{K}^{\pm} \times [0, \infty), \hat{f}^{\pm})$ that are thin trapped and $\hat{K}$ or $\hat{K}^{-} \cup \hat{K}^{+}$ contain the whole unitary bundle associated to E^{c}.

From the classification result, if K is chain-transitive and has no central model with non-trivial chain-unstable set, then the central dynamics of K is thin trapped. Moreover, this does not depend on the choice of the central models:

Lemma 4.2. *Assume that K is chain-transitive and that its central dynamics is thin trapped. Then, for any central model $(\hat{K} \times [0, \infty), \hat{f})$ of K, the dynamics is thin trapped.*

Proof. This is contained in [C_3, proof of Lemma 2.5]. □

4.3. Dynamics with a chain-recurrent central segment

For the next result we need a preliminary lemma.

Lemma 4.3. *Let f be a diffeomorphism, Λ be a compact invariant chain-transitive set with a dominated splitting $E \oplus E^{c} \oplus F$ where E^{c} is one-dimensional, and $(W^{c}(x))$ be a locally invariant plaque family tangent to E^{c} over Λ. There is $\theta_0 > 0$ such that for any small neighborhood U_Λ of Λ, any $\eta > 0$ small, and any arc I satisfying:*

- *there exists $x \in \Lambda$ satisfying $f^k(I) \subset W^{c}_{\eta}(f^k(x))$ for all $k \geq 0$,*
- *$I \subset W^{\text{ch-s}}_{U_\lambda}(\Lambda)$,*

- *there exists a sequence of periodic points p_n whose orbit $\mathcal{O}(p_n)$ is contained in U_Λ, whose central exponent belongs to $(-\theta_0, \theta_0)$ and such that p_n converge to an interior point y of I,*

then for some n large enough and any neighborhood U'_Λ of $\overline{U_\Lambda}$ we have

$$W^{\mathrm{u}}_\eta(\mathcal{O}(p_n)) \cap W^{\text{ch-s}}_{U'_\Lambda}(\Lambda) \neq \emptyset.$$

Proof. Assume that the Riemannian norm is adapted to the domination $(E \oplus E^{\mathrm{c}}) \oplus F$ and that $\lambda \in (0, 1)$ is a constant as in Section 2.3.

Let $\varepsilon > 0$ be small. Since all the curves $f^k(I)$, $k \geq 0$, have length smaller than η, choosing η small enough we have, for each $z \in I$,

$$\|Df^k_{|E^{\mathrm{c}}}(z)\| \leq \frac{\mathrm{length}(f^k(I))}{\mathrm{length}(I)} e^{\varepsilon k}.$$

Withe the domination, this implies that there exist $C_E > 0$ and $\sigma_E \in (0, 1)$ such that any point $z \in I$ is (C_E, σ_E, E)-hyperbolic for f.

We choose $\rho, \sigma_F \in (0, 1)$ such that $\lambda < \rho\sigma_F$ and set $\theta_0 = \log(\rho^{-1})$. We then apply Lemma 2.15 to the sequence (p_n) and discuss the two cases of the conclusion.

In the first case, taking a subsequence, the points (p_n) are (C_F, σ_F, F)-hyperbolic for f^{-1} and some constant $C_F > 0$. We then set $C = \sup(C_E, C_F)$ and $\sigma = \sup(\sigma_E, \sigma_F, \rho)$. From Lemma 2.18, for n large the local unstable manifold of p_n in the plaque tangent to F at p_n intersects the local stable manifold of some point $z_n \in I$ close to y in the plaque tangent to E at y_n. The result follows in this case.

In the second case, there exist points q_n in the orbit of p_n that are $(1, \sigma_F, F)$-hyperbolic for f^{-1} and which converge to some point z such that:

- z belongs to the maximal invariant set in $\overline{U_\Lambda}$.
- z is $(1, \rho, E^{\mathrm{c}})$-hyperbolic for f.
- $z \in W^{\text{ch-s}}_{\overline{U_\Lambda}}(\Lambda)$.

By Lemma 2.17, for n large the points z and p_n are close, hence the local stable manifold at z and the local unstable manifold at p_n intersect. Since $z \in W^{\text{ch-s}}_{\overline{U_\Lambda}}(\Lambda)$, one deduces that $p_n \in W^{\text{ch-s}}_{U'_\Lambda}(\Lambda)$. □

Applying Lemma 4.3 to f and f^{-1}, one gets the following corollary.

Corollary 4.4. *Let f be a diffeomorphism, Λ be a compact invariant chain-transitive set with a dominated splitting $E \oplus E^{\mathrm{c}} \oplus F$ where E^{c} is one-dimensional and $\mathcal{D}$ be a plaque family tangent to E^{c}. There is $\theta_0 > 0$ such that for any small neighborhood U_Λ of Λ and any $\eta > 0$ small and considering:*

- *a chain-recurrent central segment I of K associated to some central model and to the plaques W^{c}_η,*
- *a sequence (p_n) of periodic points, which converge to an interior point y of I, whose orbit is contained in U_Λ, and whose central Lyapunov exponent belongs to $(-\theta_0, \theta_0)$,*

then for any neighborhood U'_Λ of $\overline{U_\Lambda}$ and some n large, the point p_n belongs to the chain-stable and chain-unstable sets $W^{\text{ch-s}}_{U'_\Lambda}(\Lambda)$ and $W^{\text{ch-u}}_{U'_\Lambda}(\Lambda)$.

4.4. Thin trapped central dynamics

When K is hyperbolic, it is contained in a local homoclinic class. The following result is analogous in the case the central dynamics is thin trapped.

Proposition 4.5. *Let f be a diffeomorphism and Λ be a compact invariant chain-transitive set. Assume that:*

- *There exists a compact invariant set $K \subset \Lambda$ having a partial hyperbolic structure $T_K M = E^{\mathrm{s}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{u}}$ with E^{c} one-dimensional.*
- *The central dynamics of K is thin trapped.*
- *There are a neighborhood U_Λ of Λ, a locally invariant plaque family $\mathcal{D}^{\mathrm{c}}$ tangent to E^{c} over K and a plaque $\mathcal{D}^{\mathrm{c}}_x$, $x \in K$, contained in the chain-stable set $W^{\text{ch-s}}_{U_\Lambda}(\Lambda)$.*

Then, for any neighborhoods U'_Λ, U_K of $\overline{U_\Lambda}$ and K there exists a periodic point p such that:

- *The whole orbit of p is contained in U_K.*
- *p is contained in both the local chain-stable and chain-unstable sets of Λ in U'_Λ.*

Proof. Note first that one can assume that *any* plaque $\mathcal{D}^{\mathrm{c}}_y$ is contained in the local chain-stable set $W^{\text{ch-s}}_{U_\Lambda}(\Lambda)$. Indeed, if $(z_0, \dots, z_n)$ is an ε-pseudo-orbit between $z_0 = y$ and $z_n = x$ with $\varepsilon > 0$ small enough, then, since the plaque family $\mathcal{D}$ is trapped, $f(\mathcal{D}_{z_k})$ is contained in a small neighborhood of $\mathcal{D}_{z_{k+1}}$ for each k. This implies that any point of $\mathcal{D}_y$ can be joined to $\mathcal{D}_x$ by a pseudo-orbit.

Note also that we can always replace K by a minimal subset, so that one can assume that K is chain-transitive. Let U be a small neighborhood of K, so that the partially hyperbolic structure $E^{\mathrm{s}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{u}}$ on K extends to the maximal invariant set in U. This allows us to introduce some locally invariant plaque families $(W^{\mathrm{cs}}(x))$ and $(W^{\mathrm{c}}(x))$ tangent to $E^{\mathrm{s}} \oplus E^{\mathrm{c}}$ and E^{c} over this maximal invariant set. We may assume that $\overline{W^{\mathrm{c}}(x)} \subset W^{\mathrm{cs}}(x)$ for each x.

By assumption and Lemma 4.2, for any central model $(\hat{K} \times [0, \infty), \hat{f})$ over K associated to the plaque family $(W^{\mathrm{c}}(x))$, the dynamics is thin trapped. We claim that there exists a continuous map $\delta \colon \hat{K} \to (0, \infty)$ such that for each $\hat{x} \in \hat{K}$, we have

$$\hat{f}(\{\hat{x}\} \times [0, \delta(\hat{x})]) \subset \{\hat{f}(\hat{x})\} \times [0, \delta(\hat{f}(\hat{x}))). \tag{4.1}$$

Indeed, since the dynamics is thin trapped, there exists an open neighborhood V of $\hat{K} \times \{0\}$ in the central model such that $\hat{f}(\overline{V}) \subset V$. Let S be the union of the intervals $\{\hat{x}\} \times [0, a]$ contained in V. Since V is open, this is an open set of the form $\{\{\hat{x}\} \times [0, a(\hat{x}))\}$, where $a \colon \hat{K} \to (0, \infty)$ is lower semicontinuous. The image of $\overline{S}$ by $\hat{f}$ is contained in U. Since it is a union of segments $\{\hat{x}\} \times [0, a]$, it is contained in S. One deduces that $\hat{f}(\overline{S})$ is a compact set of the form $\{\{\hat{x}\} \times [0, b(\hat{x})]\}$, where $b \colon \hat{K} \to (0, \infty)$ is upper semicontinuous. Hence, there exists a continuous map $\delta \colon \hat{K} \to (0, \infty)$ such that $b(x) < \delta(x) < a(x)$ at every point (such a map exists locally and can be obtained globally by a partition of unity). The claim follows.

The property (4.1) above shows that one can reduce the plaques $W^{\mathrm{c}}(x)$ so that for each $x \in K$ we have

$$f(\overline{W^{\mathrm{c}}(x)}) \subset W^{\mathrm{c}}(f(x)).$$

Note that this property extends to any point in the maximal invariant set in a small neighborhood U' of K. We will use the following properties.

1. By [C_2, Lemma 3.11], there exists $\varepsilon > 0$ such that for any periodic orbit $\mathcal{O} \subset U'$, any point in the ε-neighborhood of $W^{\mathrm{c}}(q)$ inside the plaque $W^{\mathrm{cs}}(q)$ for $q \in \mathcal{O}$ belongs to the stable manifold of a periodic point $p \in W^{\mathrm{c}}(q)$. Having chosen U' and the plaques $W^{\mathrm{c}}(x)$ small enough, such a periodic point p is arbitrarily close to some point $z \in K$.
2. Since the central dynamics of K is thin trapped, for any $z \in K$, any point inside the plaque $\mathcal{D}^{\mathrm{c}}_z$ belongs to K^+. When p is close enough to $z \in K$ one can apply Lemma 2.18 and Remark 2.2: the strong unstable manifold of p meets the strong stable manifold of a point $z' \in \mathcal{D}^{\mathrm{c}}_z$ close to z. Since by assumption z' belongs to the chain-stable set of Λ inside U_Λ, the periodic point p belongs to the chain-stable set of Λ inside U'_Λ.

By [C_3, Lemma 2.9] any partially hyperbolic set K whose central bundle is one-dimensional and thin trapped satisfies the shadowing lemma:[1] for any $\delta > 0$, there exists $\varepsilon > 0$ such that any ε-pseudo-orbit in K is δ-shadowed by an orbit in M. This implies that there exists a periodic orbit $\mathcal{O}$ contained in an arbitrarily small neighborhood of K. In particular, there exists $y \in K$ such that the local strong unstable manifold of y meets the ε-neighborhood of $W^{\mathrm{c}}(q)$ inside the plaque $W^{\mathrm{cs}}(q)$ for some $q \in \mathcal{O}$. By the first property above, this implies that there is a periodic point $p \in W^{\mathrm{c}}(q)$ which belongs to the local chain-unstable set of Λ. By the second property, p belongs to the local chain-stable set of Λ inside U'_Λ, as required.

Note that when the bundle E^{s} (or E^{u}) is trivial the local strong stable (or unstable) manifold of any point x is reduced to x but the proof is unchanged. When E^{u} is trivial, we can also apply [C_3, Proposition 2.7], which shows that K contains a periodic orbit, hence gives the conclusion of Proposition 4.5. □

Remark 4.3. The following stronger statement does not require the uniform expansion along E^{u}.

Let Λ be a compact invariant chain-transitive set. Assume that:

- *There is a compact invariant set $K \subset \Lambda$ and a dominated splitting $T_K M = E^{\mathrm{s}} \oplus E^{\mathrm{c}} \oplus F$ with E^{c} one-dimensional and E^{s} uniformly contracted.*
- *The central dynamics of K is thin trapped.*
- *There exists a (C, σ, F)-hyperbolic point for f^{-1} in K, for some constants C, σ.*
- *There are a neighborhood U_Λ of Λ, a locally invariant plaque family $\mathcal{D}^{\mathrm{c}}$ tangent to E^{c} over K and a plaque $\mathcal{D}^{\mathrm{c}}_x$ contained in the chain-stable set $W^{\mathrm{ch\text{-}s}}_{U_\Lambda}(\Lambda)$.*

[1] We stated Proposition 4.5 in a general setting. In case f belongs to a dense G_δ subset of $\mathrm{Diff}^1(M)$, the argument could be simplified since any chain-recurrence set is the limit in the Hausdorff topology of a sequence of periodic orbits.

Then, for any neighborhoods U'_Λ, U_K of $\overline{U_\Lambda}$ and K there exists a periodic point p such that:

- The whole orbit of p is contained in U_K.
- p is contained in both the local chain-stable and chain-unstable sets of Λ in U'_Λ.

We will use this result only in the alternative proof of Proposition 8.2, in Section 8.2. For this reason, we only sketch the proof. Assume that the Riemannian norm is adapted to the domination $(E^s \oplus E^c) \oplus F$ and consider $\lambda \in (0, 1)$ as in Section 2.3. As in the proof of Proposition 4.5, any plaque of the family $\mathcal{D}^c$ is contained in $W^{\text{ch-s}}_{U_\Lambda}(\Lambda)$.

If K supports an ergodic measure which is hyperbolic (i.e. all its Lyapunov exponents are non-zero) and whose stable spaces have dimension $\dim(E^s)$ or $\dim(E^s \oplus E^c)$, then the conclusion follows. Indeed, the Oseledets splitting is dominated and by Proposition 2.19, there exists a sequence of hyperbolic periodic orbits that converge to a subset of K and are homoclinically related in a small neighborhood of K.

If E^c is uniformly contracted, since there exists a (C, σ, F)-hyperbolic point for f^{-1}, there exists an ergodic measure μ_0 supported on K such that the integral of $\log \|Df^{-1}_{|F}\|$ is smaller than σ. Such a measure is hyperbolic, has stable dimension $\dim(E^s \oplus E^c)$ and we are done.

If K supports an ergodic measure μ whose central Lyapunov exponent is non-zero and larger than $\log \lambda$, then by domination, all the exponents of μ along F are positive, the measure is hyperbolic, has stable dimension $\dim(E^s)$ or $\dim(E^s \oplus E^c)$ and we are done too.

If for some invariant compact set $K' \subset K$, any invariant probability measure on K' has central Lyapunov exponent zero, then Proposition 4.5 can be applied and the conclusion follows. One deduces that E^c is not uniformly contracted on K and for any compact invariant set $K' \subset K$, there exists a measure whose central Lyapunov exponent is smaller than $\log \lambda$.

Liao's selecting argument thus applies (see [W_2, Lemma 3.8]): again, there exists a sequence of hyperbolic periodic orbits that converge to a subset of K and are homoclinically related together in a small neighborhood of K, concluding the proof.

As for the proof of Proposition 4.5, the argument is simpler in the generic setting: since there exists a (C, σ, F)-hyperbolic point for f^{-1}, one may find an invariant measure μ supported on K such that all the Lyapunov exponents along the bundle F are positive. By Mañé's ergodic closing lemma [M_1], there exists a sequence of periodic orbits whose associated invariant measures converge to μ in the weak-* topology. The rest of the argument is similar to the end of the proof of Proposition 4.5.

5. Proof of Theorem 1.2

In this section we reduce the proof to some technical propositions (Propositions 5.4 and 5.5) that will be proved in later sections. Let Λ be as in the hypothesis of Theorem 1.2 with a dominated splitting $E^s \oplus E^c \oplus F$ with E^c one-dimensional and E^s uniformly contracted. We consider a small neighborhood U_0 of Λ and a small constant θ, and we have to show that the conclusion of Theorem 1.2 is satisfied by U_0 and θ. We also fix a locally invariant family of central plaques $\mathcal{D}^c$ over Λ. All the central manifolds $W^c_\eta(x)$ and all central models that we will consider are taken from this plaque family.

Let us give the idea of the arguments contained in this section. Recall that E^c is not uniformly contracted but there exists an ergodic measure supported on Λ with non-vanishing central Lyapunov exponent. If there exist ergodic hyperbolic measures on Λ with arbitrarily weak central Lyapunov exponent, we are done. In the other case any measure has a zero Lyapunov exponent or all its Lyapunov exponents are bounded away from zero. From that we deduce that the set X of points with good central contraction is non-empty and there exist compact invariant sets K supporting only invariant measures with vanishing central Lyapunov exponent. Combining these with the analysis of the central models we split the proof into two cases (Propositions 5.4 and 5.5) discussed in the next two sections.

5.1. *Generic assumption*

Let U, V be open sets in M and $\theta > 0$. We define $\mathcal{O}(U, V, \theta)$ as the set of diffeomorphisms of M having a hyperbolic periodic orbit O contained in U which meets V and whose $(\dim(E^s) + 1)^{\text{th}}$ Lyapunov exponent belongs to $(-\theta, \theta)$.

Let $\mathcal{B}_0$ be a countable basis of open sets in M and $\mathcal{B}$ the open sets in M that are finite union of elements of $\mathcal{B}_0$. The set $\mathcal{B}$ has the following properties: it is countable and for each compact set $K \subset M$ and each open set $V \subset M$ containing K, there exists $U \in \mathcal{B}$ such that $K \subset U \subset V$.

We then define the dense G_δ set

$$\mathcal{G} = \bigcap_{(U,V,\theta)\in\mathcal{B}\times\mathcal{B}\times\mathbb{Q}^+} \big(\mathcal{O}(U, V, \theta) \cup (\mathrm{Diff}^1(M) \setminus \overline{\mathcal{O}(U, V, \theta)})\big). \tag{5.1}$$

The dense G_δ subset of $\mathrm{Diff}^1(M)$ given by Theorem 1.2 is the set of diffeomorphisms whose periodic orbits are hyperbolic and contained in the intersection of the dense G_δ sets given by Theorem 2.4 (consequences of the connecting lemma for pseudo-orbits), Theorem 2.6 (approximation of pseudo-orbits by orbits in the Hausdorff topology), Corollary 2.5 (coincidence between chain-recurrence classes with periodic points and homoclinic classes and local version of it) and of $\mathcal{G}$ defined above.

5.2. *Measures supported on* Λ

For an ergodic measure μ supported on Λ we denote by $L^c(\mu)$ the Lyapunov exponent of μ along E^c. Note that this also defines the Lyapunov exponent $L^c(O)$ along E^c of a periodic orbit $O \subset U_0$.

We know that there exists an ergodic measure μ such that $L^c(\mu) \neq 0$. If $L^c(\mu) > 0$ then by the domination on F and since E^s is uniformly contracted we see that μ is a hyperbolic measure whose Oseledets splitting $E^s_\mu \oplus E^u_\mu$ coincides a.e. with $E^s \oplus (E^c \oplus F)$. Now, Proposition 2.19 yields the first option of Theorem 1.2.

Thus, assume from now on that for any ergodic measure μ in Λ we have $L^c(\mu) \leq 0$. The next lemma says that if there are ergodic measures with negative central Lyapunov exponent arbitrarily close to zero we can conclude the proof.

Lemma 5.1. *Assume that for every $\epsilon > 0$ there exists an ergodic measure μ in Λ such that $L^c(\mu) \in (-\epsilon, 0)$. Then the second option of Theorem 1.2 holds.*

Proof. Let μ be an ergodic measure supported on Λ with $L^c(\mu) \in (-\epsilon, 0)$. By domination between E and F, if ϵ is sufficiently small, any Lyapunov exponent of μ along F is positive. On the other hand, since E^s is uniformly contracted, any Lyapunov exponent of μ along E^s is negative. Therefore, μ is a hyperbolic measure whose Oseledets splitting $E^s_\mu \oplus E^u_\mu$ coincides a.e. with $(E^s \oplus E^c) \oplus F$. Proposition 2.19 yields a sequence of hyperbolic periodic orbits O_n that are homoclinically related in U_0 and such that the invariant probability measures supported on the O_n converge to μ in the weak-$*$ topology. In particular the central exponents of all O_n belong to $(-\varepsilon, 0)$. By Corollary 2.5, $H(O_n, U_0)$ contains Λ. So assuming moreover $\varepsilon < \theta$, we get the second option of Theorem 1.2. $\square$

Continuing with the proof we have to handle the following situation that we shall assume from now on:

$(*)$ *There exists ϵ_0 such that any ergodic measure μ in Λ satisfies*

$$L^c(\mu) \notin (-\epsilon_0, 0).$$

Moreover there exists an ergodic measure μ_0 such that $L^c(\mu_0) \leq -\epsilon_0$.

5.3. 0*-CLE sets K*

A *zero central Lyapunov exponent set* (or briefly a 0*-CLE set*) is a compact invariant chain-transitive set $K \subset \Lambda$ such that $L^c(\mu) = 0$ for any ergodic measure supported on K.

Theorem 2.9 can be restated as:

Lemma 5.2. *Assume that $(*)$ holds. Then one of the following holds:*

1. *The first or second option of Theorem 1.2 is true.*
2. *There exists a* 0*-CLE set $K \subset \Lambda$.*

Proof. Let us apply Theorem 2.9 to the dominated splitting $E \oplus F = (E^s \oplus E^c) \oplus F$ on Λ. The first two cases of Theorem 2.9 give the two options of Theorem 1.2. The third case gives a 0-CLE set $K \subset \Lambda$. $\square$

Therefore, in order to prove our theorem, by the above lemma we will restrict ourselves to the following case:

$(**)$ *There exists a* 0*-CLE set $K \subset \Lambda$.*

5.4. The set X of hyperbolic points

As explained in Section 2.3, one can change the Riemannian norm and assume that there exists $\lambda \in (0, 1)$ such that for any $x \in \Lambda$ and any unitary vectors $v^{\mathrm{cs}} \in E^{\mathrm{s}}_x \oplus E^{\mathrm{c}}_x$ and $v^F \in F_x$ we have

$$\|Df.v^{\mathrm{cs}}\| \leq \lambda \|Df.v^F\|.$$

Fix $\epsilon \in (0, \epsilon_0)$ where ϵ_0 is defined by $(*)$ and assume that $e^{-\epsilon} > \lambda^{1/2}$. Let X be the set of points that are $(1, e^{-\epsilon}, E^{\mathrm{c}})$-hyperbolic for f, that is,

$$X = \{x \in \Lambda : \|Df^n_{|E^{\mathrm{c}}_x}\| \leq e^{-n\epsilon}\ \forall n \geq 0\}.$$

Notice that X is compact. The next lemma and $(*)$ show that it is non-empty.

Lemma 5.3. *For any invariant ergodic measure μ such that $L^{\mathrm{c}}(\mu) < 0$ the sets X and* $\mathrm{supp}(\mu)$ *intersect.*

Proof. The continuous function $\varphi : \mathrm{supp}(\mu) \to \mathbb{R}$ defined by $\varphi(x) = \log \|Df_{|E^{\mathrm{c}}}(x)\|$ satisfies $L^{\mathrm{c}}(\mu) = \int \varphi\, d\mu$, and $L^{\mathrm{c}}(\mu) < -\varepsilon$ by $(*)$. By Birkhoff's theorem and Lemma 2.14, the orbit of a.e. point x meets X. □

In particular any compact invariant chain-transitive set $K \subset \Lambda$ that is disjoint from X is a 0-CLE set.

5.5. The chain-unstable case

We will now consider two cases depending on whether or not there exist a 0-CLE set K and a point $y \in \Lambda$ with $\alpha(y) \subset K$ such that the chain-unstable set $\hat{W}^{\text{ch-u}}(\hat{K} \cup \mathrm{Orb}^-(\hat{y}))$ contains a non-trivial segment $\{\hat{y}\} \times [0, a]$, $a > 0$, in some central model where $\hat{K} := \alpha(\hat{y})$. When such K and y exist we apply the following proposition that will be proved in Section 6.

Proposition 5.4. *Assume that there is a* 0*-CLE set $K \subset \Lambda$ and $y \in \Lambda$ such that $\alpha(y) \subset K$ and for some central model $\hat{W}^{\text{ch-u}}(\hat{K} \cup \mathrm{Orb}^-(\hat{y}))$ contains a non-trivial segment $\{\hat{y}\} \times [0, a]$, $a > 0$, where $\hat{y}$ is a lift of y in the central model. Then for any $\theta > 0$ and any neighborhood U_Λ of Λ, there is $\eta > 0$ with the following property.*

Considering the dynamics in a central model associated to the plaques W^{c}_η, there is $x \in \Lambda$ having a chain-recurrent central segment I. Moreover, for any $z \in \mathrm{Int}(I)$ and for any neighborhood V_z of z, there is a periodic point $p \in V_z$ such that

- $\mathrm{Orb}(p) \subset U_\Lambda$,
- $L^{\mathrm{c}}(p) \in (-\theta, \theta)$.

One can thus apply Corollary 4.4. This shows that the second case of Theorem 1.2 holds.

5.6. The thin trapped case

We now consider the other case, i.e. the assumption of Proposition 5.4 does not hold. The following proposition, to be proved in Section 7, applies.

Proposition 5.5. *Assume that for any* 0*-CLE set* K_0 *and any point* $y \in \Lambda$ *such that* $\alpha(y) \subset K_0$ *we have, for any central model and any lift* $\hat{y}$ *of* y*,*

$$\hat{W}^{\text{ch-u}}(\hat{K}_0 \cup \text{Orb}^-(\hat{y})) \cap \big(\{\hat{y}\} \times [0, \infty)\big) = \{(\hat{y}, 0)\}.$$

Then, for any neighborhood U_Λ *of* Λ*, there exists a* 0*-CLE set* $K \subset \Lambda$ *and* $\eta > 0$ *such that the plaques* $W^c_\eta(x)$ *for* $x \in K$ *are contained in the chain-stable set* $W^{\text{ch-s}}_{U_\Lambda}(\Lambda)$*.*

Let K be the set provided by Proposition 5.5. If one considers a central model $(\hat{K} \times [0, \infty), \hat{f})$ associated to (K, f), the chain-unstable set of $\hat{K} \times \{0\}$ contains no non-trivial interval $\{\hat{y}\} \times [0, a]$, $a > 0$, by our assumptions. The classification Theorem 4.1(1) implies that $(\hat{K} \times [0, \infty), \hat{f})$ is thin trapped. This proves that the central dynamics of K is thin trapped.

The assumptions of Proposition 4.5 are now satisfied by the set K: there exists a periodic orbit contained in an arbitrarily small neighborhood of K and in both the local chain-stable and chain-unstable sets of Λ inside a small neighborhood U'_Λ of Λ. Since K is a 0-CLE set, the central exponents of p are contained in $(-\theta, \theta)$. By Corollary 2.5, the local homoclinic class $H(p, U_0)$ contains Λ. This shows that the second case of Theorem 1.2 holds for U_0 and θ.

This completes the proof of Theorem 1.2, assuming Propositions 5.4 and 5.5, to be proved in the next sections.

6. The chain-unstable case: proof of Proposition 5.4

Fix $\theta > 0$ and a small neighborhood U_Λ of Λ. By assumption, there exist a central model $(\hat{f}, \hat{\Lambda})$, a point $\hat{y} \in \hat{\Lambda}$ and a compact invariant set $\hat{K} \subset \hat{\Lambda}$ such that:

- $\hat{K}$ contains $\alpha(\hat{y})$ and projects by $\pi : \hat{\Lambda} \times [0, \infty) \to M$ onto a 0-CLE set $K \subset \Lambda$ of f.
- The chain-unstable set of $\hat{y}$ is non-trivial: $\hat{W}^{\text{ch-u}}(\hat{K} \cup \text{Orb}^-(\hat{y})) \setminus \{\hat{y}\} \neq \emptyset$.

We have to show the existence of $\eta > 0$ and of $x \in \Lambda$ having a chain-recurrent central segment I such that for any $z \in \text{Int}(I)$ and any neighborhood V_z of z there exists a periodic point $p \in V_z$ satisfying:

- $\text{Orb}(p) \subset U_\Lambda$.
- $L^c(p) \in (-\theta, \theta)$.

We will first select η, x and I. We then construct for any $z \in \text{Int}(I)$ and V_z a periodic point p as required by a perturbation argument (see Lemma 6.2); since f belongs to the generic set $\mathcal{G}$, the periodic point p already exists for f.

6.1. Selection of η, x and I

Here we get the chain-recurrent central segment.

Choice of η. Let us choose $\eta > 0$ small. One can always reduce the central model to a neighborhood of $\hat{\Lambda} \times \{0\}$ such that it becomes a central model associated to the plaques W^{c}_{η}. By a conjugacy of the form $(\hat{x}, t) \mapsto (\hat{x}, c.t)$, one can "normalize" the central model so that $\hat{f}$ is defined from $\hat{\Lambda} \times [0, 1]$ to $\hat{\Lambda} \times [0, \infty)$.

Let $\hat{X}$ be the set of points of $\hat{\Lambda}$ whose projection by π belongs to X. There exists κ such that for each $x \in X$ we have

$$\begin{aligned} &f^k(W^{\mathrm{c}}_{\kappa}(x)) \subset W^{\mathrm{c}}_{\eta}(f^k(x)) \text{ for each } k \geq 0,\\ &W^{\mathrm{c}}_{\kappa}(x) \text{ is contained in the stable set of } x. \end{aligned} \tag{6.1}$$

This implies the existence of $a > 0$ such that for any $\hat{x} \in \hat{X}$ one has:

- $\hat{f}^k(\{\hat{x}\} \times [0, a]) \subset \{\hat{f}^k(\hat{x})\} \times [0, 1]$ for each $k \geq 0$.
- The length of $\hat{f}^k(\{\hat{x}\} \times [0, a])$ goes to 0 as $k \to \infty$.

Since the chain-unstable set of $\hat{y}$ is non-trivial, one can reduce $a > 0$ so that

$$\hat{W}^{\text{ch-u}}(\hat{K} \cup \mathrm{Orb}^-(\hat{y})) \supset \{\hat{y}\} \times [0, a]. \tag{6.2}$$

Choosing η small, one can also require that the η-neighborhood of Λ is contained in U_Λ and for any $x \in \Lambda$ and $z \in W^{\mathrm{c}}_{\eta}(x)$ we have

$$\left|\log \|Df_{|E^{\mathrm{c}}}(x)\| - \log \|Df_{|T_z W^{\mathrm{c}}_{\eta}(x)}(z)\|\right| < \theta/4. \tag{6.3}$$

The point x and the segment I. They are obtained by combining the following properties: the existence of a non-trivial chain-unstable set at $\hat{y}$, the uniform contraction along E^{c} at points of X and the chain-transitivity of Λ. This is done in the following lemma.

Lemma 6.1. *There exists a sequence $(\hat{x}_n)$ in $\hat{\Lambda}$ such that for each n:*

- *$\alpha(\hat{x}_n)$ projects by π onto a 0-CLE set.*
- *$\{\hat{x}_n\} \times [0, a]$ is contained in the chain-unstable set of $\hat{\Lambda} \times \{0\}$.*
- *$\{\hat{x}_n\} \times [0, a]$ is contained in the $1/n$-chain-stable set of $\hat{\Lambda} \times \{0\}$.*

In particular for each $k \geq 0$, we have

$$\hat{f}^{-k}(\{\hat{x}_n\} \times [0, a]) \subset \{\hat{f}^{-k}(\hat{x}_n)\} \times [0, 1].$$

Proof. If $\hat{y} \in \hat{X}$, it is enough to take $\hat{x}_n = \hat{y}$ for any $n \geq 0$ by properties (6.1) and (6.2). So we can assume $\hat{y} \notin \hat{X}$.

For $n \geq 1$, let $\hat{U}_n$ be the $1/n$-neighborhood of $\hat{X}$. We can assume that $\hat{y} \notin \hat{U}_n$. Fix $n \geq 1$. For any $\ell \geq 1$, consider a $1/\ell$-pseudo-orbit $O_\ell = \{p_0, \dots, p_{n_\ell}\}$ in $\hat{\Lambda}$ from $\hat{y} = p_0$ to a point $z_\ell := p_{n_\ell}$ in the closure of $\hat{U}_n$ such that $O_\ell \setminus \{z_\ell\}$ does not intersect the interior of $\hat{U}_n$. One can define some continuous maps $s \mapsto t_k(s)$ for each $0 \leq k \leq n_\ell$ such that

$t_0(s) = s$ and for each $s > 0$ small, the sequence $(p_k, t_k(s))$ is a $1/\ell$-pseudo-orbit in $\hat{\Lambda} \times [0, 1]$ between (p_0, s) and $(p_{n_\ell}, t_{n_\ell}(s))$.

When s increases from 0 to a, for some k the parameter $t_k(s)$ becomes equal to a (defined in the previous paragraph). We denote $\hat{x}_{n,\ell} := p_k$. By construction $\{\hat{x}_{n,\ell}\} \times [0, a]$ is in the $1/\ell$-chain-unstable set of $\{\hat{y}\} \times [0, s]$, hence of $\hat{\Lambda} \times \{0\}$ by (6.2). It is also in the $1/\ell$-chain-stable set of $\{z_\ell\} \times [0, t_{n_\ell}(s)] \subset \{z_\ell\} \times [0, a]$. Since z_ℓ is $1/n$-close to $\hat{X}$, the set $\{\hat{x}_{n,\ell}\} \times [0, a]$ is in the $1/n$-chain-stable set of $\hat{X} \times [0, a]$, hence of $\hat{\Lambda} \times \{0\}$ by (6.1).

Let $\hat{x}_n$ be a limit of $\hat{x}_{n,\ell}$ as $\ell \to \infty$. After passing to the limit, $\{\hat{x}_n\} \times [0, a]$ is in the chain-unstable set of $\hat{\Lambda} \times \{0\}$ and in the $1/n$-chain-stable set of $\hat{\Lambda} \times \{0\}$. By construction the set $\alpha(\hat{x}_n)$ is disjoint from the interior of $\hat{U}_n$. Thus its projection K by π is a chain-transitive compact invariant set disjoint from X. So it is a 0-CLE set. □

We define $\hat{x} \in \hat{\Lambda}$ as a limit of the points $\hat{x}_n$. We set $x = \pi(\hat{x})$, which is the limit of $x_n := \pi(\hat{x}_n)$. The segment I is the projection by π of $\{\hat{x}\} \times [0, a]$. By construction, it is a chain-recurrent central segment of x for the central model we considered.

6.2. A perturbation result

We now fix a point $z \in \mathrm{Int}(I)$ and a neighborhood V_z. Note that it is enough to prove Proposition 5.4 for a dense subset of points $z \in I$. Since f is C^1-generic, its periodic orbits are hyperbolic (see Section 5.1) and f has at most countably many periodic points. Hence one can assume that z is not periodic. The next lemma constructs the periodic point p by perturbation of f.

Lemma 6.2. *For any C^1-neighborhood $\mathcal{U}$ of f, there exists $g \in \mathcal{U}$ having a periodic orbit $O \subset U_\Lambda$ which meets V_z and whose $(\dim(E^{\mathrm{s}}) + 1)^{th}$ Lyapunov exponent belongs to $(-\theta, \theta)$.*

Recall that by the construction of x and I and the choice of z, one can find for each n a point $z_n \in W^{\mathrm{c}}_\eta(x_n)$ (where x_n has been defined in the previous step) such that:

- The sequence (z_n) converges to z.
- For each $k \geq 0$, the point $f^{-k}(z_n)$ belongs to $W^{\mathrm{c}}_\eta(f^{-k}(x_n))$.
- z_n belongs to the chain-unstable set of Λ in the η-neighborhood of Λ.

Lemma 6.2 is now a consequence of the following two results.

Lemma 6.3. *For any C^1-neighborhood $\mathcal{U}$ of f, there exists $n_0 \geq 0$ such that for any $\rho \in (0, 1)$, $\delta > 0$ and any neighborhood U_A of $A := \alpha(z_{n_0})$, there exists $g \in \mathcal{U}$ having a periodic orbit $O \subset U_\Lambda$ which meets V_z and satisfies:*

- *The C^0-distance between the restrictions of f and g to U_A is less than δ.*
- *O contains a point δ-close to A.*
- *O contains an orbit segment $\{w, g(w), \ldots, g^s(w)\}$ which is contained in U_A and whose proportion in O is larger than ρ.*

Lemma 6.4. *There exists a C^1-neighborhood $\mathcal{U}_0$ of f with the following property. For any $n_0 \geq 0$, there exist $\rho \in (0, 1)$, $\delta > 0$ and a neighborhood U_A of $A := \alpha(z_{n_0})$ such that for any diffeomorphism $g \in \mathcal{U}_0$ and any periodic orbit $O \subset U_\Lambda$ of g satisfying the conclusion of Lemma 6.3, the $(\dim(E^s) + 1)^{th}$ Lyapunov exponent of O belongs to $(-\theta, \theta)$.*

The conclusion of Lemma 6.2 is obtained by considering a small C^1-neighborhood $\mathcal{U}$ of f contained in the neighborhood $\mathcal{U}_0$ given by Lemma 6.4. We thus get from Lemma 6.3 an integer n_0 and by Lemma 6.4 we obtain constants $\rho \in (0, 1)$, $\delta > 0$ and a neighborhood U_A of $A := \alpha(z_{n_0})$. Lemma 6.3 now provides us with the diffeomorphism g and the periodic orbit O whose central Lyapunov exponent is controlled by Lemma 6.4.

6.3. The perturbation argument: proof of Lemma 6.3

First we can assume that there is $r_0 > 0$ such that $\alpha(z_n) \cap B(z, r_0) = \emptyset$ for any n and in particular $z_n \notin \alpha(z_n)$ for n large. For otherwise the fact that f is C^1-generic implies (see Section 5.1 and Theorem 2.6) that $\alpha(z_n)$ can be accumulated by periodic orbits in the Hausdorff topology. Thus, the lemma is satisfied in this case.

We then explain how to choose n_0. From the Connecting Lemma (Theorem 2.2) we get an $L \in \mathbb{N}$ associated to $\mathcal{U}$ and two perturbation neighborhoods $B_z \subset \hat{B}_z$ of z small enough such that:

- $\hat{B}_z \subset V_z$.
- $U_{L,z} := \bigcup_{i=0}^{L-1} f^i(\hat{B}_z)$ is disjoint from $\alpha(z_n)$ for any $n \in \mathbb{N}$ (using that $\alpha(z_n) \cap B(z, r_0) = \emptyset$).
- $U_{L,z} \subset U_\Lambda$.

Now, we let n_0 be such that $z_{n_0} \in B_z$ and set $A := \alpha(z_{n_0})$. We know that A is disjoint from $U_{L,z}$. Let U_A be any neighborhood of A such that $U_A \cap U_{L,z} = \emptyset$ and $\overline{U_A} \subset U_\Lambda$.

The strategy. In order to perform the required perturbation we would like to be in the following setting (see Figure 1):

- a is a point in A and b in $U_A \setminus A$ outside the backward orbit of z_{n_0}.
- $B_b \subset \hat{B}_b$ are perturbation neighborhoods of b such that $U_{L,b} := \bigcup_{i=0}^{L-1} f^i(\hat{B}_b)$ is contained in U_A and disjoint from the backward orbit of z_{n_0} and from A.
- $\{y_{n_1}, \dots, f^{l_{n_1}}(y_{n_1})\}$ is an orbit segment U_Λ connecting B_z to B_b.
- $B_a \subset \hat{B}_a$ are arbitrarily small perturbation neighborhoods of a such that $U_{L,a} := \bigcup_{i=0}^{L-1} f^i(\hat{B}_a)$ is contained in U_A and is disjoint from $U_{L,b}$ and from the orbit segment above.
- $\{f^{l_{n_2}}(y_{n_2}), \dots, f^{k_{n_2}}(y_{n_2})\}$ is an orbit segment in U_A connecting B_b to B_a.

Once we have this setting we perform a connecting argument using the Connecting Lemma (Theorem 2.2) three times.

Choice of connecting segments. Let us obtain the above setting.

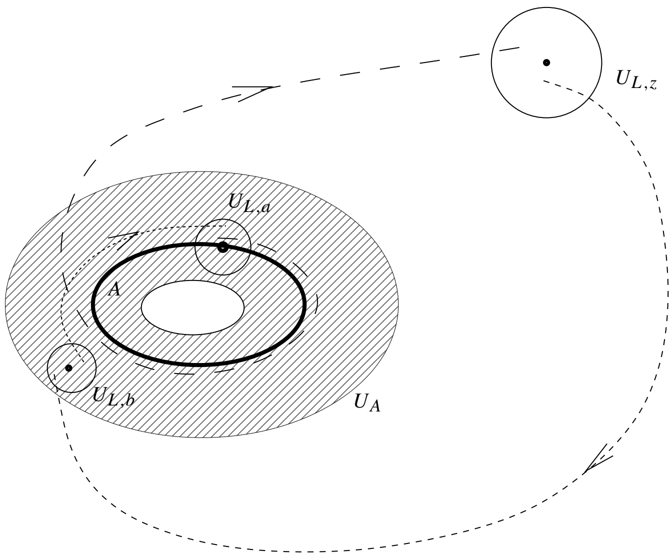

Fig. 1. The connecting orbits.

The points a and b. Since z and Λ are in the same chain-transitive set contained in U_Λ and since $A = \alpha(z_{n_0})$ is in the local chain-unstable set of Λ in the η-neighborhood of Λ, together with the fact that f is C^1-generic (see Section 5.1 and Theorem 2.4), there is a sequence of orbit segments

$$\{y_n, f(y_n), \dots, f^{k_n}(y_n)\}_{n\in\mathbb{N}} \subset U_\Lambda$$

such that:

- (y_n) converges to z.
- $(f^{k_n}(y_n))$ converges to some $a \in A$.

Let us consider a smaller neighborhood U'_A of A whose closure is contained in the interior of U_A. For each n, define $l_n \in [0, k_n] \cap \mathbb{N}$ such that:

- $f^{l_n-1}(y_n) \notin U'_A$.
- $f^m(y_n) \in U'_A$ for any $l_n \le m \le k_n$.

By taking a subsequence if necessary, one can assume that:

- $(f^{l_n}(y_n))$ converges to some point $b \in \overline{U'_A}$.
- the forward iterations of b are all contained in U_A (and different from z_{n_0}).

The perturbation domain at b and the connecting orbit between z and b. By Theorem 2.2, there are two neighborhoods $B_b \subset \hat{B}_b$ of b small enough such that:

- The connected components of $U_{L,b}$ each have diameter smaller than δ.
- $U_{L,b} \subset U_A$.
- $U_{L,b}$ is disjoint from A and from the backward orbit of z_{n_0}.

We then choose n_1 such that $y_{n_1} \in B_z$ and $f^{l_{n_1}}(y_{n_1}) \in B_b$. We also choose $m_1 \geq 1$ such that the backward orbit of $f^{-m_1}(z_{n_0})$ is contained in U_A, and disjoint from $U_{L,b}$.

The perturbation domain at a and the connecting orbit between b and a. By Theorem 2.2, there are two neighborhoods $B_a \subset \hat{B}_a$ of a small enough such that:

- The connected components of $U_{L,a}$ each have diameter smaller than δ.
- $U_{L,a} \subset U_A$.
- $U_{L,a}$ is disjoint from $U_{L,b}$.

We then choose n_2 such that $f^{l_{n_2}}(y_{n_2}) \in B_b$ and $f^{k_{n_2}}(y_{n_2}) \in B_a$. The orbit segment $\{f^{l_{n_2}}(y_{n_2}), \dots, f^{k_{n_2}}(y_{n_2})\}$ is contained in U_A.

It is important to note that the two neighborhoods $B_a \subset \hat{B}_a$ can be chosen arbitrarily small after choosing the integers l_{n_1} and m_1.

The perturbation. We now realize successively three perturbations of f in $\mathcal{U}$ supported in the disjoint domains $U_{L,z}, U_{L,b}, U_{L,a}$ given by Theorem 2.2. The composition of the three perturbations provides a diffeomorphism g which also belongs to $\mathcal{U}$ (see [BC, Remark 4.3]).

The perturbation at z. We apply the Connecting Lemma (Theorem 2.2) to the perturbation domain B_z, $\hat{B}_z$ and the points $f^{-m_1}(z_{n_0})$ and $f^{l_{n_1}}(y_{n_1})$.

We obtain a diffeomorphism $f_1 \in \mathcal{U}$ (coinciding with f on the complement of $U_{L,z}$, hence on U_A), a point $p \in \hat{B}_z \subset V_z$ and $m, l \geq 0$ such that:

- $f_1^{-m}(p) = f^{-m_1}(z_{n_0})$ and $f_1^l(p) = f^{l_{n_1}}(y_{n_1})$.
- Any iterate $f_1^i(p)$ with $-m \leq i \leq l$ belongs to U_Λ.
- The backward f_1-orbit of p is disjoint from $U_{L,b}$.
- The backward f_1-orbit of $f_1^{-m}(p)$ is contained in U_A, its α-limit is A.
- $m \leq m_1$ and $l \leq l_{n_1}$.

The perturbation at b. We apply Theorem 2.2 to the perturbation domains B_b, $\hat{B}_b$ and the points p and $f^{k_{n_2}}(y_{n_2})$ for the map f_1, which coincides with f on $U_{L,b}$.

We obtain a diffeomorphism $f_2 \in \mathcal{U}$ (coinciding with f on the complement of $U_{L,z} \cup U_{L,b}$, hence on $U_{L,a}$) and $m, l', k \geq 0$ such that:

- $f_2^{-m}(p) = f^{-m_1}(z_{n_0})$ and $f_2^k(p) = f^{k_{n_2}}(y_{n_2})$.
- Any iterate $f_2^i(p)$ with $-m \leq i \leq k$ belongs to U_Λ.
- The backward f_2-orbit of $f_2^{-m}(p)$ is contained in U_A, its α-limit is A.
- The forward f_2-orbit of $f_2^{l'}(p)$ is contained in U_A.
- $m \leq m_1$ and $l' \leq l_{n_1}$.

The perturbation at a. We apply Theorem 2.2 to the perturbation domains B_a, $\hat{B}_a$ and to the forward and backward orbits of p for the map f_2, which coincides with f on $U_{L,a}$.

We obtain a diffeomorphism $f_3 \in \mathcal{U}$ such that p is periodic. Its orbit O is contained in U_Λ and intersects $\hat{U}_z$. Moreover $O \setminus \{f^{-m_1}(p), \dots, f^{l_{n_1}}(p)\} \subset U_A$.

End of proof of Lemma 6.3. It remains to check that the three properties stated in Lemma 6.3 are satisfied.

Since the orbit O intersects the set $\hat{U}_a$ which meets A and has diameter smaller than δ, it contains a point that is δ-close to A.

Since all the perturbation domains $U_{L,b}, U_{L,a} \subset U_A$ have connected components with diameter smaller than δ and since $U_{L,z}$ is disjoint from U_A, the C^0-distance between f and g in U_A is smaller than δ.

Since $U_{L,a}$ can be chosen arbitrarily small after one has built the orbit segments $\{f^{-m_1}(z_{n_0}), \dots, z_{n_0}\}$ and $\{y_{n_1}, \dots, f^{l_{n_1}}(y_{n_1})\}$, the period of O can be chosen arbitrarily large whereas m_1, l_{n_1} are fixed. This shows that the orbit segment $O \setminus \{f^{-m_1}(p), \dots, f^{l_{n_1}}(p)\} \subset U_A$ has proportion within O close to 1, larger than ρ.

6.4. Control of the central Lyapunov exponent: proof of Lemma 6.4

One first chooses the neighborhood $\mathcal{U}_0$ and finds some $\delta_0 > 0$ so that for any diffeomorphism $g \in \mathcal{U}_0$, for any point x whose g-orbit is contained in U_Λ and any point $z \in \Lambda$ such that x and z are δ_0-close, the spaces E^c_x for g and E^c_z for f are close and $\|Dg_{|E^c}(x)\|$ and $\|Df_{|E^c}(z)\|$ are $\theta/3$-close.

Let us now fix some $n_0 \geq 0$. We have the following control on the central Lyapunov exponents of the set $\alpha(z_{n_0})$.

Sublemma 6.5. *Any ergodic measure supported on $A := \alpha(z_{n_0})$ has central Lyapunov exponent in $(-\theta/3, \theta/3)$ and A does not contain any periodic orbit.*

Proof. Consider any ergodic measure μ supported on A. Since the bundle E^c is one-dimensional, its central Lyapunov exponent is equal to

$$L^c(\mu) = \int \log(\|Df_{|E^c}\|)\, d\mu.$$

Recall that each backward iterate $f^{-i}(z_{n_0})$ of z_{n_0} belongs to the central plaque $W^c_\eta(f^{-i}(x_{n_0}))$ and that the tangent space of $W^c_\eta(f^{-i}(x_{n_0}))$ at $f^{-i}(z_{n_0})$ converges as $i \to \infty$ to the central bundle of $A = \alpha(z_{n_0})$. Hence there exists an orbit segment $\{f^{-\ell}(z_{n_0}), \dots, f^{-k}(z_{n_0})\}$, with k large and $\ell > k$, such that

$$\frac{1}{\ell - k} \log \|Df^{\ell-k}_{|TW^c_\eta(f^{-\ell}(x_{n_0}))}(f^{-\ell}(z_{n_0}))\|$$

is close to $L^c(\mu)$. By our choice of η (see (6.3)), this last quantity is $\theta/4$-close to

$$\frac{1}{\ell - k} \log \|Df^{\ell-k}_{|E^c}(f^{-\ell}(x_{n_0}))\|,$$

which is arbitrarily close to zero if k and $\ell - k$ are large since $\alpha(x_{n_0})$ is a 0-CLE set.

Assume for contradiction that A contains a periodic point p. Observe that there exists $q \in \alpha(x_{n_0})$ such that $f^{-i}(p) \in W^c_\eta(f^{-i}(q))$ for any $i \geq 0$ and the α-limit set of q is a periodic orbit contained in the 0-CLE set $\alpha(x_{n_0})$. Such a periodic orbit has a zero Lyapunov exponent, which contradicts our genericity assumptions made on f in Section 5.1. $\square$

As a consequence there exists $N_1 \geq 1$ such that

$$\frac{1}{N_1}\sum_{i=0}^{N_1-1} \log \|Df|_{E^c(f^i(y))}\| \in (-\theta/3, \theta/3).$$

There exist $\delta > 0$ and a neighborhood U_A of A such that any diffeomorphism $g \in \mathcal{U}_0$ whose C^0-distance to f on U_A is smaller than δ has the following property: any orbit segment $\{w, g(w), \dots, g^{N_1}(w)\} \subset U_A$ of a point w whose orbit is contained in U_Λ is δ_0-close to an orbit segment $\{y, f(y), \dots, f^{N_1}(y)\}$ of f in A. In particular by our choice of $\mathcal{U}_0$ we have

$$\frac{1}{N_1}\sum_{i=0}^{N_1-1} \log \|Dg|_{E^c(g^i(w))}\| - \frac{1}{N_1}\sum_{i=0}^{N_1-1} \log \|Df|_{E^c(f^i(y))}\| \in (-\theta/3, \theta/3).$$

From these estimates, for such an orbit segment of g we get

$$-\frac{2}{3}\theta < \frac{1}{N_1}\sum_{i=0}^{N_1-1} \log \|Dg|_{E^c(g^i(w))}\| < \frac{2}{3}\theta. \tag{6.4}$$

There exist $T \geq 1$ large and a proportion $\rho \in (0, 1)$ close to 1 such that any periodic orbit O satisfying the conclusion of Lemma 6.3 and of period P larger than T, contains an orbit segment in U_A whose length is a multiple of N_1 and is comparable to the period P. From (6.4) one deduces that for $p \in O$,

$$\frac{1}{P}\sum_{i=0}^{P-1} \log \|Dg|_{E^c(g^i(x))}\| \in (-\theta, \theta). \tag{6.5}$$

Hence the central Lyapunov exponent of O has modulus smaller than θ as required.

Note that $\delta > 0$ can be reduced so that the previous properties still hold with the same constants T, ρ, N_1 and the same neighborhood U_A. Since by Sublemma 6.5 the set A does not contain any periodic orbit for f, if δ is small enough, any periodic orbit of g containing a point δ-close to A has period larger than T. Hence (6.5) holds for any periodic orbit as in the statement of Lemma 6.4. $\square$

6.5. Conclusion of the proof of Proposition 5.4

Note that the neighborhoods U_Λ and V_z can be chosen in the countable basis $\mathcal{B}$ introduced in Section 5.1. For any C^1-neighborhood $\mathcal{U}$ of f, Lemma 6.2 provides us with a diffeomorphism $g \in \mathcal{U}$ having a hyperbolic periodic orbit $O \subset U_\Lambda$ which meets V_z and whose $(\dim(E^s) + 1)^{\text{th}}$ Lyapunov exponent belongs to $(-\theta, \theta)$. Since f belongs to the dense G_δ set $\mathcal{G}$ defined at (5.1), the same property holds for f, ending the proof of Proposition 5.4. $\square$

7. The thin trapped case: proof of Proposition 5.5

We start with a lemma which allows us to select some 0-CLE sets which are related to the set X. This comes from the chain-transitivity of Λ, the existence of X and the existence of 0-CLE sets. At this point we do not use the fact that we are in the thin trapped case.

Lemma 7.1. *There exist constants $C > 0$, $\sigma \in (0, 1)$, a sequence (K_n) of 0-CLE sets, a sequence (z_n) of points in Λ and a point $x \in X$ such that:*

(i) *For each n, the α-limit set of z_n is K_n.*
(ii) *The sequence (z_n) converges to x.*
(iii) *For each n, the point z_n is (C, σ, F)-hyperbolic for f^{-1}.*

Proof. We first construct two sequences (y_n) and (K_n) and a point $y \in X$ satisfying (i) and (ii). For that, we will divide the proof into two cases: the non-isolated case and the isolated case.

- Non-isolated case: there is a sequence (K_n) of 0-CLE sets such that there are $y_n \in K_n$ so that $y = \lim_{n\to\infty} y_n$ exists and $y \in X$.
- Isolated case: there is a neighborhood U of X such that $K \cap U = \emptyset$ for any 0-CLE set K.

In the non-isolated case, (i) and (ii) are satisfied by (K_n), (y_n) and y.

Now we discuss the isolated case. For any neighborhood U of X, let $K_{\Lambda\setminus U} = \bigcap_{n\in\mathbb{Z}} f^n(\Lambda \setminus U)$ be the maximal invariant set in $\Lambda \setminus U$. By Lemma 5.3, the central Lyapunov exponent of any ergodic measure supported on $K_{\Lambda\setminus U}$ is 0. Take a sequence of neighborhoods U_n of X such that $\bigcap_{n\in\mathbb{N}} \overline{U_n} = X$ and fix n. For any $m \in \mathbb{N}$, since Λ is chain-transitive and Λ contains a 0-CLE set K, there is a $1/m$-pseudo-orbit from K to X. As a consequence, there is a $1/m$-pseudo-orbit $\{z_a\}_{a=0}^{b(m)}$ such that:

- $z_{b(m)} \in U_n$ and $z_a \notin U_n$ for any $0 \le a \le b(m) - 1$.
- z_0 is $1/m$-close to K. Thus, $b(m) \to \infty$ as $m \to \infty$.

By taking a subsequence if necessary, one can assume that $y_n = \lim_{m\to\infty} z_{b(m)}$ exists and:

- $y_n \in \overline{U_n}$.
- $\mathrm{Orb}^-(y_n) \subset \Lambda \setminus U_n$. Hence, $\alpha(y_n) \subset \Lambda \setminus U_n$. By the isolated assumption, $\alpha(y_n) \subset K_{\Lambda\setminus U_n}$ is a 0-CLE set.

By taking a subsequence if necessary, one can assume that $y := \lim_{n\to\infty} y_n$ exists and belongs to X. This gives (i) and (ii) in the isolated case.

Now, for both cases (non-isolated and isolated), one gets a sequence (K_n) of 0-CLE sets, a sequence (y_n) of points in Λ and $y \in X$ such that:

- For each n, the α-limit set of y_n is K_n.
- (y_n) converges to y.

In particular, by definition of $\lambda \in (0, 1)$ (see Section 2.3) we have

$$\limsup_{k\to\infty} \sum_{i=0}^{k-1} \frac{1}{k} \log \|Df^{-1}|_{E^c(f^{-i}(y_n))}\| < \log(\lambda^{-1/2}).$$

The same estimate also holds along the bundle $E = E^s \oplus E^c$, hence the first condition of Lemma 2.15 is satisfied with $\rho = \lambda^{1/2}$ and $\sigma \in (\lambda^{1/2}, 1)$.

We now explain how to get the points (z_n) satisfying (iii) from the sequence (y_n), by applying Lemma 2.15. If the first conclusion of Lemma 2.15 is satisfied, the conclusion of Lemma 7.1 holds directly for $z_n = y_n$ and $x = y$.

If the second conclusion of Lemma 2.15 is satisfied, it provides us with a sequence (z_n) of points which

- are $(1, \sigma, F)$-hyperbolic for f^{-1},
- are in the backward orbit of the points y_n,
- (on taking a subsequence) converge to a point x that is $(1, \rho, E^c)$-hyperbolic for f.

Since $\rho = \lambda^{1/2} < e^{-\epsilon}$, the point x belongs to X as required. Thus (z_n) satisfies all the requirements of Lemma 7.1. □

Now we will discuss the thin trapped case. Let us recall the statement of the proposition.

Proposition 5.5. *Assume that for any* 0*-CLE set* K_0 *and any point* $y \in \Lambda$ *such that* $\alpha(y) \subset K_0$*, for any central model and any lift* $\hat{y}$ *of* y *we have*

$$\hat{W}^{\text{ch-u}}(\hat{K}_0 \cup \text{Orb}^-(\hat{y})) \cap (\{\hat{y}\} \times [0, \infty)) = \{(\hat{y}, 0)\}.$$

Then, for any neighborhood U_Λ *of* Λ*, there exists a* 0*-CLE set* $K \subset \Lambda$ *and* $\eta > 0$ *such that the plaques* $W^c_\eta(x)$ *for* $x \in K$ *are contained in the chain-stable set* $W^{\text{ch-s}}_{U_\Lambda}(\Lambda)$.

The idea of the proof is the following. We choose $y = z_n$ and $K = K_n$ given by the previous lemma for n large and we consider the points y' in a central plaque of y. Two cases appear.

If the points y' "escape in the past": the distance $d(f^{-k}(y), f^{-k}(y'))$ does not remain small as $k \to \infty$, then K is already thin trapped.

If $d(f^{-k}(y), f^{-k}(y'))$ remains bounded as $k \to \infty$, the unstable manifold of y' (tangent to F) is large and intersects the stable manifold of some $x \in X$ (a uniform disk tangent to $E^s \oplus E^c$). This proves that the α-limit set of y' is contained in the chain-stable set Λ. Since we are in the thin trapped case, $d(f^{-k}(y), f^{-k}(y'))$ remains bounded away from zero, and the union of the $\alpha(y')$ when y' varies in the central plaque of y covers the central plaques of $\alpha(y) = K$. See Figure 2.

We now give a precise argument.

Proof of Proposition 5.5. First we make some preliminary choices. Lemma 7.1 gives some constants $C > 0$ and $\sigma \in (0, 1)$, some sequences (z_n), (K_n) and a point $x \in X$. There exists a curve I in the central plaque of x and containing x in its interior, which is exponentially contracted by forward iterations. In particular I is contained in $\Lambda^+ := \bigcap f^{-k}(\overline{U_\Lambda})$ (see Section 2.3) and E^s is uniformly contracted at points of I by forward iterations. We choose any $\sigma' \in (\sigma, 1)$ and a constant $r > 0$ given by Lemma 2.18 adapted to the constants of hyperbolicity C, σ' and to x, I. We choose $\chi > 0$ and work with central plaques W^c_χ. We require several conditions on χ:

- $\chi < r/2$.
- $W^c_\chi(w) \subset U_\Lambda$ for any $w \in \Lambda$.
- For any (C, σ, F)-hyperbolic point $y \in \Lambda$ for f^{-1}, any point y' such that $f^{-k}(y') \in W^c_\chi(f^{-k}(y))$ for each $k \geq 0$ is (C, σ', F)-hyperbolic for f^{-1}.

By taking n large, the point $y := z_n$ and the 0-CLE set $K := K_n$ satisfy:

- $\alpha(y) = K$.
- $d(x, y) < r/2$.
- y is (C, σ, F)-hyperbolic for f^{-1}.

Let us consider any central model $(\hat{f}, \hat{\Lambda} \times [0, \infty))$ of Λ associated to the plaques W^c_χ and let $\hat{y}$ be any lift of y in the central model. Then $\hat{K} = \alpha(\hat{y})$ is a lift of K.

The following claim is the key point of the proof.

Claim. *There is $\delta_{\hat{K}} > 0$ such that $\pi(\hat{K} \times [0, \delta_{\hat{K}}]) \in W^{\text{ch-s}}_{U_\Lambda}(\Lambda)$.*

Proof of the claim. One can assume that $\hat{f}$ is defined from $\hat{\Lambda} \times [0, 1]$ to $\hat{\Lambda} \times [0, \infty)$. There are two possibilities:

(1) There is a sequence $\{\delta_m\}$ and a sequence $\{k_m\} \subset \mathbb{N}$ such that:

- $\delta_m \to 0$ as $m \to \infty$.
- $f^{-k}(\{\hat{y}\} \times [0, \delta_m]) \subset \{\hat{f}^{-k}(\hat{y})\} \times [0, 1]$ for any $0 \leq k \leq k_m$.
- $\hat{f}^{-k_m}(\{\hat{y}\} \times [0, \delta_m]) = \{\hat{f}^{-k_m}(\hat{y})\} \times [0, 1]$.

(2) There is $\delta_y > 0$ such that $\hat{f}^{-k}(\{\hat{y}\} \times [0, \delta_y]) \subset \{\hat{f}^{-k}(\hat{y})\} \times [0, 1]$ for any $k \in \mathbb{N}$.

In case (1), the chain-stable set of $\hat{K} \times \{0\}$ contains an interval $\{\hat{z}\} \times [0, 1]$, which is the limit of $\hat{f}^{-k_m}(\{\hat{y}\} \times [0, \delta_m])$. By our assumptions, the chain-unstable set contains no non-trivial interval $\{\hat{w}\} \times [0, 1]$ with $\hat{w} \in \hat{K}$, hence the central dynamics of $\hat{K} \times \{0\}$ is thin trapped by the classification Theorem 4.1(1). Arguing as at the beginning of the proof of Proposition 4.5, one deduces that there exists $a > 0$ such that $\hat{K} \times [0, a]$ is in the chain-stable set of $\hat{K} \times \{0\}$ in the central model. Since the plaques W^c_χ are contained in U_Λ, the projection $\pi(\hat{K} \times [0, 1])$ is contained in $W^{\text{ch-s}}_{U_\Lambda}(\Lambda)$.

In case (2), the length of $\hat{f}^{-k}(\{\hat{y}\} \times [0, \delta_y])$, $k \geq 0$, is bounded away from zero, since otherwise $\{\hat{y}\} \times [0, \delta_y]$ would be contained in the chain-unstable set of $\hat{K} \cup \text{Orb}^-(\hat{y})$ in the central model, which would contradict our assumption that $\hat{W}^{\text{ch-u}}(\hat{y})$ reduces to $\{\hat{y}\}$. We let $\delta_{\hat{K}}$ be a lower bound for the length of $\hat{f}^{-k}(\{\hat{y}\} \times [0, \delta_y])$, $k \geq 0$.

Let $(\hat{w}, a) \in \hat{K} \times [0, \delta_{\hat{K}}]$ and fix $\alpha > 0$. By construction, there exists $k_m \geq 0$ arbitrarily large, $t \in [0, \delta_y]$ and $y' := \pi(\hat{y}, t)$ such that $f^{-k_m}(y')$ is α-close to $\pi(\hat{w}, a)$. Since $\hat{f}^{-k}(\hat{y}, t)$ are defined for any $k \geq 0$, the backward iterates of y' belong to the plaques $f^{-k}(W^c_\chi(y))$. In particular $y' \in \Lambda^- := \bigcap_{k \geq 0} f^k(\overline{U_\Lambda})$ and by our choice of χ

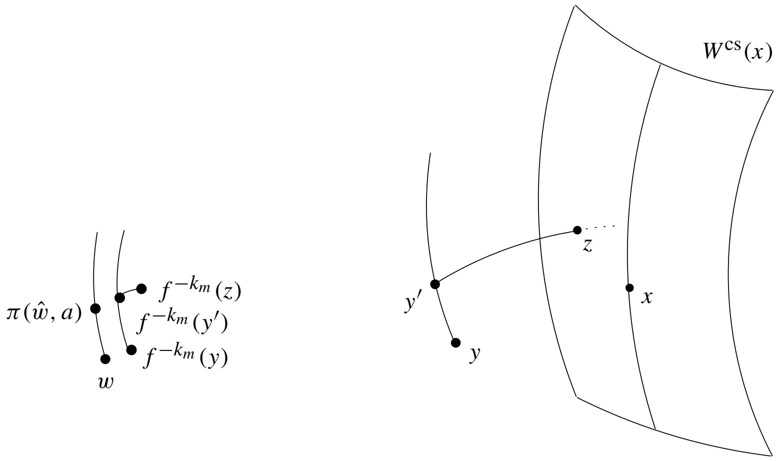

Fig. 2. Second case of the proof of Proposition 5.5.

and y we have $d(y', x) < r$ and y' is (C, σ', F)-hyperbolic for f^{-1}. So we can apply Lemma 2.18: the local unstable manifold of y' and the local stable manifold of x intersect at some point z whose orbit is contained in U_Λ. See Figure 2.

On the one hand, $d(f^k(z), \Lambda) \to 0$ as k goes to ∞. On the other hand, $d(f^{-k_m}(z), f^{-k_m}(y'))$ is exponentially small in k_m since y' is (C, σ', F)-hyperbolic for f^{-1}, by Lemma 2.16 applied to f^{-1}. If k_m has been chosen large enough, $f^{-k_m}(z)$ is at distance smaller than 2α from $\pi(\hat{w}, a)$. Since α is arbitrarily small, the point $\pi(\hat{w}, a)$ belongs to $W^{\text{ch-s}}_{U_\Lambda}(\Lambda)$, proving the claim. □

By changing the lift of $\hat{y}$ (in case the central dynamics of Λ is non-orientable) or by changing the central model, we get another lift $\hat{K}$ and another constant $\delta'_{\hat{K}'}$ such that the union of the projections $\pi(\hat{K} \times [0, \delta_{\hat{K}}]) \cup \pi(\hat{K}' \times [0, \delta'_{\hat{K}'}])$ covers the plaques $W^c_\eta(x)$ with $x \in K$ for some $\eta > 0$ small. This implies the proposition. □

8. Consequences

In this section we prove the corollaries stated in the introduction.

8.1. Central bundles of homoclinic classes. Proof of Corollary 1.9

Let us consider a homoclinic class $H(p)$ for a C^1-generic diffeomorphism $f \in \text{Diff}^1(M) \setminus \overline{\text{HT}}$, and a one-dimensional central bundle E^c. Note that if $H(p)$ contains periodic orbits such that E^c is stable and others such that E^c is unstable, then the class contains a chain-transitive central segment associated to E^c. Otherwise, the next result shows that the central dynamics along the bundle E^c is thin trapped for f or f^{-1} (implying that E^c has (H)-attracting or (H)-repelling type according to the classification of [C_3, Section 2.2] and that it is "chain-hyperbolic", according to the definition in [CP, Definition 7]).

Proposition 8.1. *Let f be a diffeomorphism in a dense G_δ subset of* $\mathrm{Diff}^1(M)$ *and let $H(p)$ be a homoclinic class endowed with a dominated splitting $T_{H(p)}M = E \oplus E^c \oplus F$ such that:*

- *The minimal index of $H(p)$ equals* $\dim(E \oplus E^c)$.
- *There exist periodic orbits related to p whose Lyapunov exponent along E^c is arbitrarily close to* 0.

Then the central dynamics along E^c is thin trapped. In particular E is uniformly contracted.

Proof. Note that since $H(p)$ contains periodic orbits such that E^c is stable, there is no central model associated to E^c such that the dynamics is trapped for $\hat{f}^{-1}$. According to Theorem 4.1(2) we have to rule out the existence of a chain-recurrent central segment I for $H(p)$. Since $I \subset H(p)$, for any z in the interior of I there exists a sequence of periodic points p_n converging to z and whose central Lyapunov exponent is close to zero. Since f is C^1-generic, by Lemma 2.1 one can choose the points p_n with index $\dim(E)$. By Corollary 4.4 the points p_n belong to $H(p)$ and we obtain a contradiction. □

We now prove the result about the ergodic closing lemma.

Proof of Corollary 1.9. By Theorem 1.1, there exists a dominated splitting $T_{H(p)}M = E^s \oplus E^c \oplus F$ with $\dim(E^c) = 1$ and $\dim(E^s \oplus E^c) = i$. Moreover, there exist periodic orbits in $H(p)$ whose Lyapunov exponent along E^c is close to zero. By the previous lemma the central dynamics along E^c is thin trapped.

By Mañé's ergodic closing lemma [M1, ABC], there exists a sequence of periodic orbits $\bar{O}_n$ whose associated measures converge to μ. Note that they are contained in an arbitrarily small neighborhood of $H(p)$. Consider a plaque family $\mathcal{D}^{cs}$ tangent to $E^s \oplus E^c$ over the maximal invariant set in a small neighborhood of $H(p)$, which is trapped by f and whose plaques are small. For some $\sigma \in (0, 1)$, there exists a $(1, \sigma, F)$-hyperbolic point x in the support of μ. Its unstable manifold meets the plaque $\mathcal{D}^{cs}_{\bar{q}_n}$ of some point $\bar{q}_n \in \bar{O}_n$ for large n. Since the plaques are trapped, it meets the stable manifold of some hyperbolic periodic point $q_n \in \mathcal{D}^{cs}_{\bar{q}_n}$. This implies that q_n belongs to the chain-unstable set of $H(p)$. Considering smaller plaque families, this construction shows that there exists a sequence of hyperbolic periodic orbits O_n whose associated measures converge to μ and that are contained in the chain-unstable set of $H(p)$.

On the other hand, since the central exponent of O_n is close to zero, some point $q'_n \in O_n$ has a uniform unstable manifold tangent to F. One can assume that (q'_n) converges to some point $x \in H(p)$. Since the central dynamics along E^c is thin trapped, Lemma 4.3 implies that the orbits O_n also lie in the chain-stable set of $H(p)$. As a consequence, $H(p)$ contains periodic orbits whose measures converge to μ. □

8.2. More arguments about aperiodic classes

We now provide an alternative argument for item (1) of Theorem 1.1 about aperiodic classes, which is shorter since it does not use Theorem 1.2 (nor Lemma 3.1) and Sections 5 to 7.

Proposition 8.2. *For any diffeomorphism f in a dense G_δ subset of* $\mathrm{Diff}^1(M) \setminus \overline{\mathrm{HT}}$, *any aperiodic class $\mathcal{C}$ is partially hyperbolic with a one-dimensional central bundle.*

Proof. By Theorem 2.6, the class $\mathcal{C}$ is the limit of a sequence of hyperbolic periodic orbit O_n for the Hausdorff distance.

Assume for contradiction that these periodic orbits do not have a weak Lyapunov exponent. By Theorem 2.7, this induces a dominated splitting $T_{\mathcal{C}}M = E \oplus F$ where $\dim(E)$ equals the index of the periodic orbits and there exist $C > 0$, $\sigma \in (0, 1)$ and $N \geq 1$ such that inequalities (2.1) in Theorem 2.7 hold for any point of the orbits O_n. We will set $\lambda = 1/2$ and we can assume that $\sigma^2 > \lambda = 1/2$.

On passing to the limit, this implies that $\mathcal{C}$ has a (C, σ, E)-hyperbolic point y for f^N. Consider a sequence of periodic points y_k in the union of the orbits O_n which converges to y. By Lemma 2.15, one can arrange y_k to be (C, σ, F)-hyperbolic for f^{-N} and y to be (C, σ, E)-hyperbolic for f^N. Lemma 2.17 implies that for n large enough, the periodic orbits O_n lie in the chain-stable set of $\mathcal{C}$.

The symmetric argument for f^{-1} shows that for n large enough, the periodic orbits O_n are in the chain-unstable set of $\mathcal{C}$. This proves that the O_n are contained in $\mathcal{C}$, contradicting the assumption that $\mathcal{C}$ is aperiodic.

We have shown that the periodic orbits O_n have a weak Lyapunov exponent, hence induce by Theorem 2.7 a splitting $E \oplus E^c \oplus F$ on $\mathcal{C}$ with $\dim(E^c) = 1$. If the conclusion of Proposition 8.2 does not hold, either F is not uniformly expanded or E is not uniformly contracted. One can assume for instance that F is not uniform. By domination, this implies that $\mathcal{C}$ contains an E^c-hyperbolic point for f. In particular, for any central model, the dynamics cannot be thin trapped for f^{-1}. By Corollary 4.4, the set $\mathcal{C}$ has no chain-recurrent central segment. Thus by Theorem 4.1(2), the central dynamics of $\mathcal{C}$ is thin trapped for f. By domination, we conclude that E is uniformly contracted. The existence of an E^c-hyperbolic point for f in $\mathcal{C}$ also implies that there exists a central plaque of $\mathcal{C}$ contained in the chain-stable set of $\mathcal{C}$.

Passing to the limit with the periodic orbits O_n, we also deduce that there exists a (C, σ, F)-hyperbolic point for f^{-N} in $\mathcal{C}$. Thus the result stated in Remark 4.3 applies, showing that $\mathcal{C}$ contains a periodic point, which is a contradiction. The proof is now complete. □

8.3. *Weak periodic points: proof of Corollary 1.4*

By Theorem 1.1, for any C^1-generic f which is away from ones exhibiting a homoclinic tangency, every homoclinic class $H(p)$ admits a dominated splitting $E \oplus F$ such that $\dim(E)$ equals the index of p. If $H(p)$ is not hyperbolic, either E is not uniformly contracted or F is not uniformly expanded. We will assume the former. Theorem 1.1 gives a dominated splitting $E = E' \oplus E^c$ with $\dim(E^c) = 1$.

If there exist periodic points in $H(p)$ of index $\dim(E')$, then by Theorem 2.11 there exist periodic orbits in $H(p)$ with the same index as p whose Lyapunov exponent along E^c is arbitrarily weak. By Corollary 2.3, these periodic points are homoclinically related to p as required.

Otherwise the index of p coincides with the minimal index of the class. By Theorem 1.1, there exist periodic orbits in $H(p)$ with the same index as p whose Lyapunov exponent along E^{c} is arbitrarily weak. We conclude in the same way.

8.4. About the Palis conjecture: proof of Corollary 1.5

For a C^1-generic f which is away from ones exhibiting a homoclinic tangency or a heterodimensional cycle, Theorem 1.1 applies and any homoclinic class $H(p)$ has a partially hyperbolic splitting $T_{H(p)}M = E^{\mathrm{s}} \oplus E_1^{\mathrm{c}} \oplus \cdots \oplus E_k^{\mathrm{c}} \oplus E^{\mathrm{u}}$. If $k \leq 2$ we are done, otherwise the class contains periodic points of different indices and Theorem 2.12 gives a contradiction.

8.5. Lyapunov stable classes: proof of Corollary 1.6

Consider a C^1-generic diffeomorphism f in $\mathrm{Diff}^1(M) \setminus \overline{\mathrm{HT}}$ and a Lyapunov stable chain-recurrence class $\mathcal{C}$.

Claim ([Y_1]). *The class $\mathcal{C}$ is a homoclinic class.*

Indeed, if it is an aperiodic class, by Theorem 1.1 it has a partially hyperbolic decomposition $T_{\mathcal{C}}M = E^{\mathrm{s}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{u}}$, and this contradicts Theorem 5 of [BGW].

Claim ([Y_1]). *If $T_{\mathcal{C}}M = E^{\mathrm{s}} \oplus E_1^{\mathrm{c}} \oplus \cdots \oplus E_k^{\mathrm{c}} \oplus E^{\mathrm{u}}$ is the partially hyperbolic structure on $\mathcal{C}$, then the class $\mathcal{C}$ contains periodic points of stable index* $\dim(E^{\mathrm{s}}) + k$. *In particular if it is not a sink, the bundle E^{u} is non-degenerate.*

Proof. The proof is by contradiction. Since it is similar to [CP, Corollary 2.3], we only give the idea. Since f is C^1-generic and $\mathcal{C}$ contains periodic points of index $\dim(E^{\mathrm{s}}) + k - 1$ with Lyapunov exponent along E_k^{c} arbitrarily close to zero, Lemma 2.1 implies that there exist periodic points q of index $\dim(E^{\mathrm{s}})+k$ arbitrarily close to $\mathcal{C}$. By Proposition 8.1 the central dynamics along E_k^{c} is thin trapped by f^{-1}. Hence there exists a (small) plaque family $\mathcal{D}^{\mathrm{cu}}$ tangent to $E^{\mathrm{cu}} = E_k^{\mathrm{c}} \oplus E^{\mathrm{u}}$ that is trapped by f^{-1}. Also fix a (small) plaque family $\mathcal{D}^{\mathrm{cs}}$ tangent to $E^{\mathrm{cs}} = E^{\mathrm{s}} \oplus E_1^{\mathrm{c}} \oplus \cdots \oplus E_k^{\mathrm{c}}$. All the Lyapunov exponents of q along E^{cs} are uniformly bounded away from zero. Lemmas 2.14 and 2.16 imply that up to replacing q by one of its iterates, the stable manifold of q contains $\mathcal{D}_q^{\mathrm{cs}}$. Similarly, $\mathcal{C}$ contains a dense subset of periodic points x whose Lyapunov exponents along E^{cu} are uniformly bounded away from zero, implying that $\mathcal{D}^{\mathrm{cu}}_{f^k(x)} \subset W^{\mathrm{u}}(f^k(x))$ for some iterate $f^k(x)$ of x. The trapping property on $\mathcal{D}^{\mathrm{cu}}$ implies that also $\mathcal{D}_x^{\mathrm{cu}} \subset W^{\mathrm{u}}(x)$. Since q is close to $\mathcal{C}$, there exists such a point $x \in \mathcal{C}$ with $\mathcal{D}_q^{\mathrm{cs}}$ intersecting $\mathcal{D}_x^{\mathrm{cu}}$, hence q belongs to the closure of the unstable set of x. Using that $\mathcal{C}$ is Lyapunov stable one deduces that $\mathcal{C}$ contains the periodic point q. This contradicts the fact that $\mathcal{C}$ does not contain any periodic point of index $\dim(E^{\mathrm{s}}) + k$. $\square$

Proof of Corollary 1.6. Assume that a Lyapunov stable class $H(p)$ has no robust heterodimensional cycle. Arguing as in the previous section and using the previous claim, we find that $H(p)$ has a dominated splitting $T_{H(p)}M = E^{\mathrm{s}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{s}}$, $\dim(E^{\mathrm{c}}) \le 1$, all its periodic points have index $\dim(E^{\mathrm{s}} \oplus E^{\mathrm{c}})$, and the central dynamics along the bundle E^{c} is thin trapped for f. One can thus apply [CP, Theorem 13] and deduce that E^{c} is trivial. Hence $H(p)$ is hyperbolic. □

8.6. *Bound on the number of classes: proof of Corollary 1.7*

We could argue as in [C_3, Section 6, p. 724] but instead we give a different argument. Assume for contradiction that there exists a countable collection of homoclinic classes $H(p_n)$ having the same dominated splitting $T_{H(p_n)}M = E \oplus E^{\mathrm{c}}_1 \oplus E^{\mathrm{c}}_2 \oplus F$, such that the E^{c}_i are one-dimensional, and there exists for each n a periodic orbits O^i_n homoclinically related to p_n and whose Lyapunov exponent along E^{c}_i is arbitrarily weak. One can assume that $H(p_n)$ converges to a chain-transitive set Λ. We claim that for n large, the set Λ is contained in the chain-unstable set of $H(p_n)$. Arguing similarly, Λ is contained in the chain-stable set of $H(p_n)$, hence all the chain-recurrence classes $H(p_n)$ contain Λ and should coincide, giving a contradiction.

Fix $\sigma \in (0, 1)$ close to 1. Domination and the fact that O^2_n is weak along E^{c}_2 imply that there exists $x_n \in O^2_n$ that is a $(1, \sigma, E \oplus E^{\mathrm{c}}_1)$-hyperbolic point for f. One can assume that the sequence (x_n) converges to $x \in \Lambda$. One can also consider $y_n \in H(p_n) \cap W^{\mathrm{u}}(O^1_n)$ arbitrarily close to x. Since $O^1_n = \alpha(y_n)$ is weak along E^{c}_1, by Lemma 2.15, we can assume that there exists $C > 0$ such that the points y_n are $(C, \sigma, E^{\mathrm{c}}_2 \oplus F)$-hyperbolic point for f^{-1} and x is $(1, \sigma, E \oplus E^{\mathrm{c}}_1)$-hyperbolic point for f. As a consequence, the unstable manifold of y_n (and of $O^1_n \subset H(p_n)$) meets the stable manifold of $x \in \Lambda$. This proves the claim and ends the proof.

8.7. *Index completeness: proof of Corollary 1.8*

Let Λ be as in the statement of Corollary 1.8. By Theorem 1.1, it has a splitting $T_\Lambda M = E^{\mathrm{s}} \oplus E^{\mathrm{c}}_1 \oplus \cdots \oplus E^{\mathrm{c}}_k \oplus E^{\mathrm{u}}$. We fix $\delta > 0$ and want to prove the existence of a periodic orbit that is δ-close to Λ for the Hausdorff distance and of any index in $\{\dim(E^{\mathrm{s}}), \dots, \dim(E^{\mathrm{s}}) + k\}$. We fix a neighborhood U_0 of Λ contained in the δ-neighborhood U of Λ.

If $k = 0$, the set Λ is hyperbolic and the shadowing lemma shows that $\mathrm{ind}(\Lambda) = \dim(E^{\mathrm{s}})$.

If $k = 1$ and all the invariant measures have central Lyapunov exponent zero, then by Theorem 2.6 the set Λ is the Hausdorff limit of a sequence of periodic orbits. Their central Lyapunov exponent is arbitrarily close to zero, so by Lemma 2.1 both indices $\dim(E^{\mathrm{s}})$ and $\dim(E^{\mathrm{s}}) + 1$ appear, as required.

In the remaining cases, $k \ge 1$ and there exists an invariant measure whose exponent along E^{c}_1 is non-zero. Then Theorem 1.2 and Lemma 2.1 imply that $\dim(E^{\mathrm{s}}) \in \mathrm{ind}(\Lambda)$. Moreover, there exists a periodic point p whose orbit is contained in U_0 and is δ-close to Λ for the Hausdorff distance, whose local homoclinic class $H(p, U_0)$ contains Λ and such that the index of p is $\dim(E^{\mathrm{s}})$ or $\dim(E^{\mathrm{s}}) + 1$.

Similarly, $\dim(E^s) + k \in \operatorname{ind}(\Lambda)$, and there exists a periodic point q whose orbit is contained in U_0, whose local homoclinic class $H(q, U_0)$ contains Λ and such that the index of q is equal to $\dim(E^s) + k$ or $\dim(E^s) + k - 1$.

Choose any $i \in \{\operatorname{ind}(p), \ldots, \operatorname{ind}(q)\}$. By Corollary 2.3, since $H(p, U_0)$ and $H(q, U_0)$ intersect, for any neighborhood $U_1 \subset U$ of $\overline{U_0}$, the class $H(p, U_1)$ contains q. Then, by Theorem 2.11 and Remark 2.1, one can choose $U_2 \subset U$ containing U_1 such that $H(p, U_2)$ contains a hyperbolic periodic point z of index i. By Corollary 2.3, the class $H(z, U)$ contains p and a periodic point x arbitrarily close to p, whose orbit is contained in U and of any index i. By construction this orbit is δ-close to Λ for the Hausdorff distance. Hence Λ is index complete.

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References

[ABC] Abdenur, F., Bonatti, C., Crovisier, S.: Non-uniform hyperbolicity for C^1-generic diffeomorphisms. Israel J. Math. **183**, 1–60 (2011) Zbl 1246.37040 MR 2811152

[ABD] Abdenur, F., Bonatti, C., Díaz, L.: Non-wandering sets with non-empty interiors. Nonlinearity **17**, 175–191 (2004) Zbl 1083.37020 MR 2023438

[ABCDW] Abdenur, F., Bonatti, C., Crovisier, S., Díaz, L., Wen, L.: Periodic points and homoclinic classes. Ergodic Theory Dynam. Systems **27**, 1–22 (2007) Zbl 1144.37007 MR 2297084

[Bon] Bonatti, C.: Towards a global view of dynamical systems, for the C^1-topology. Ergodic Theory Dynam. Systems **31**, 959–993 (2011) Zbl 1256.37006 MR 2818683

[BC] Bonatti, C., Crovisier, S.: Récurrence et généricité. Invent. Math. **158**, 33–104 (2004) Zbl 1071.37015 MR 2090361

[BCDG] Bonatti, C., Crovisier, S., Díaz, L., Gourmelon, N.: Internal perturbations of homoclinic classes:non-domination, cycles, and self-replication. Ergodic Theory Dynam. Systems **33**, 739–776 (2013) Zbl 06176618 MR 3062900

[BD_1] Bonatti, C., Díaz, L.: Connexions hétéroclines et généricité d'une infinité de puits et de sources. Ann. Sci. École Norm. Sup. **32**, 135–150 (1999) Zbl 0944.37012 MR 1670524

[BDK] Bonatti, C., Díaz, L., Kiriki, S.: Stabilization of heterodimensional cycles. Nonlinearity **25**, 931–960 (2012) Zbl 1284.37020 MR 2891553

[BDPR] Bonatti, C., Díaz, L., Pujals, E., Rocha, J.: Robustly transitive sets and heterodimensional cycles. In: Geometric Methods in Dynamics. I, Astérisque **286**, 187–222 (2003) Zbl 1056.37024 MR 2052302

[BGLM] Bonatti, C., Gan, S., Li, M., Yang, D.: A generalization of the notion of topological attractor: residual attractor. In preparation

[BGW] Bonatti, C., Gan, S., Wen, L.: On the existence of non-trivial homoclinic classes. Ergodic Theory Dynam. Systems **27**, 1473–1508 (2007) Zbl 1128.37021 MR 2358974

[BGY] Bonatti, C., Gan, S., Yang, D.: On the hyperbolicity of homoclinic classes. Discrete Contin. Dynam. Systems **25**, 1143–1162 (2009) Zbl 1200.37023 MR 2552132

[Bow_1] Bowen, R.: Entropy-expansive maps. Trans. Amer. Math. Soc. **164**, 323–331 (1972) Zbl 0229.28011 MR 0285689

[Bow_2] Bowen, R.: Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. Lecture Notes in Math. 470, Springer (1977) Zbl 1172.37001 MR 2423393

[BFF] Boyle, M., Fiebig, D.,, Fiebig, U.: Residual entropy, conditional entropy, and subshift covers. Forum Math. **14**, 713–757 (2002) Zbl 1030.37012 MR 1924775

[CY] Cowieson, W., Young, L.-S.: SRB measures as zero-noise limits. Ergodic Theory Dynam. Systems **25**, 1115–1138 (2005) Zbl 1098.37020 MR 2158399

[C_1] Crovisier, S.: Periodic orbits and chain-transitive sets of C^1-diffeomorphisms. Publ. Math. Inst. Hautes Études Sci. **104**, 87–141 (2006) Zbl 1226.37007 MR 2264835

[C_2] Crovisier, S.: Birth of homoclinic intersections: a model for the central dynamics of partially hyperbolic systems. Ann. of Math. **172**, 1641–1677 (2010) Zbl 1233.37009 MR 2726096

[C_3] Crovisier, S.: Partial hyperbolicity far from homoclinic bifurcations. Adv. Math. **226**, 673–726 (2011) Zbl 1215.37018 MR 2735772

[CP] Crovisier, S., Pujals, E.: Essential hyperbolicity and homoclinic bifurcations: a dichotomy phenomenon/mechanism for diffeomorphisms. Invent. Math. (online)

[DFPV] Díaz, L., Fisher, T., Pacifico, M., Vieitez, J.: Entropy-expansiveness for partially hyperbolic diffeomorphisms. Discrete Contin. Dynam. Systems **32**, 4195-4207 (2012) Zbl 1256.37025 MR 2966742

[F] Franks, J.: Necessary conditions for stability of diffeomorphisms. Trans. Amer. Math. Soc. **158**, 301–308 (1971) Zbl 0219.58005 MR 0283812

[GW] Gan, S., Wen, L.: Heteroclinic cycles and homoclinic closures for generic diffeomorphisms. J. Dynam. Differential Equations **15** (2003), 451–471. Zbl 1034.37013 MR 2046726

[GYW] Gan, S., Wen, L., Yang, D.: Minimal non-hyperbolicity and index completeness. Discrete Contin. Dynam. Systems **25**, 1349–1366 (2009) Zbl 1214.37021 MR 2552145

[GY] Gan, S., Yang, D.: Expansive homoclinic classes. Nonlinearity **22**, 729–734 (2009) Zbl 1158.37301 MR 2486353

[G_1] Gourmelon, N.: Adapted metrics for dominated splittings. Ergodic Theory Dynam. Systems **27**, 1839–1849 (2007) Zbl 1127.37031 MR 2371598

[G_2] Gourmelon, N.: Generation of homoclinic tangencies by C^1-perturbations. Discrete Contin. Dynam. Systems **26**, 1–42 (2010) Zbl 1178.37025 MR 2552776

[H] Hayashi, S.: Connecting invariant manifolds and the solution of the C^1-stability and Ω-stability conjectures for flows. Ann. of Math. **145**, 81–137 (1997) and **150**, 353–356 (1999) Zbl 0871.58067 MR 1715329

[HPS] Hirsch, M., Pugh, C., Shub, M.: Invariant Manifolds. Lecture Notes in Math. 583, Springer (1977) Zbl 0355.58009 MR 0501173

[LVY] Liao, G., Viana, M., Yang, J.: The entropy conjecture for diffeomorphisms away from tangencies. J. Eur. Math. Soc. **15**, 2043–2060 (2013) Zbl 06243065 MR 3120734

[M_1] Mañé, R.: An ergodic closing lemma. Ann. of Math. **116**, 503–540 (1982) Zbl 0511.58029 MR 0678479

[M$_2$] Mañé, R.: A proof of the C^1 stability conjecture. Publ. Math. Inst. Hautes Études Sci. **66**, 161–210 (1988) Zbl 0678.58022 MR 0932138

[Pa] Palis, J.: Homoclinic bifurcations, sensitive-chaotic dynamics and strange attractors. In: Dynamical Systems and Related Topics (Nagoya, 1990). Adv. Ser. Dynam. Systems 9, World Sci., River Edge, NJ, 466–472 (1991)

[PT] Palis, J., Takens, F.: Hyperbolicity & Sensitive Chaotic Dynamics at Homoclinic Bifurcations. Cambridge Stud. Adv. Math. 35, Cambridge Univ. Press (1993) Zbl 0790.58014 MR 1237641

[Po] Potrie, R.: Generic bi-Lyapunov stable homoclinic classes. Nonlinearity **23**, 1631–1649 (2010) Zbl 1198.37027 MR 2652474

[PoS] Potrie, R., Sambarino, M.: Codimension one generic homoclinic classes with interior. Bull. Brazil. Math. Soc. **41**, 125–138 (2010) Zbl 1196.37058 MR 2609214

[PuS] Pujals, E., Sambarino, M.: Homoclinic tangencies and hyperbolicity for surface diffeomorphisms. Ann. of Math. **151**, 961–1023 (2000) Zbl 0959.37040 MR 1779562

[S] Shub, M.: Topological transitive diffeomorphism on $\mathbb{T}^4$. In: Lecture Notes in Math. 206, Springer, 39–40 (1971) Zbl 0311.58008

[W$_1$] Wen, L.: Homoclinic tangencies and dominated splittings. Nonlinearity **15**, 1445–1469 (2002) Zbl 1011.37011 MR 1925423

[W$_2$] Wen, L.: Generic diffeomorphisms away from homoclinic tangencies and heterodimensional cycles. Bull. Brazil. Math. Soc. **35**, 419–452 (2004) Zbl 1099.37024 MR 2106314

[Y$_1$] Yang, J.: Lyapunov stable chain recurrent classes. arXiv:0712.0514 (2007)

[Y$_2$] Yang, J.: Aperiodic classes. Preprint IMPA (2009)