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First steps in stable Hamiltonian topology

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Abstract. We study topological properties of stable Hamiltonian structures. In particular, we prove the following results in dimension three: The space of stable Hamiltonian structures modulo homotopy is there; stable Hamiltonian structures are generically Morse–Bott (i.e. all closed orbits are Bott nondegenerate) but not Morse; the standard contact structure on S^3 is homotopic to a stable Hamiltonian structure which cannot be embedded in $\mathbb{R}^4$. Moreover, we derive a structure theorem for stable Hamiltonian structures in dimension three, study sympectic cobordisms between stable Hamiltonian structures, and discuss implications for the foundations of symplectic field theory.

Keywords. Hamiltonian structure, contact structure, integrable system

Contents

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1. Introduction

A stable Hamiltonian structure is a generalization of a contact structure as well as a taut foliation defined by a closed 1-form. Stable Hamiltonian structures were introduced by Hofer and Zehnder [33] as a condition on hypersurfaces for which the Weinstein conjecture can be proved. Later, they attained importance as the structure on a manifold needed for the compactness result in symplectic field theory [22, 9, 16]. Further interest in stable Hamiltonian structures arises from the recent proof of the Weinstein conjecture in dimension three by Hutchings and Taubes [34] (see also Rechtman [44, 45]), and from their relation to Mañé's critical values [13] and other dynamical properties [42, 14]. Stable Hamiltonian structures also appear in work by Eliashberg, Kim and Polterovich on contact non-squeezing [23], and by Wendl and coauthors on symplectic fillings [52, 43, 38, 39].

In this paper we take first steps in studying the topology of stable Hamiltonian structures. Besides their intrinsic interest, some of these topological questions are also relevant for the foundations of symplectic field theory.

Definition 1.1. A *Hamiltonian structure* (HS) on an oriented $(2n-1)$-dimensional manifold M is a closed 2-form ω of maximal rank, i.e. such that ω^{n-1} vanishes nowhere.

Associated to ω is its 1-dimensional *kernel distribution* (*foliation*) $\ker\omega := \{v \in TM \mid i_v\omega = 0\}$. We orient $\ker\omega$ using the orientation on M together with the orientation on the local transversal to $\ker\omega$ given by ω^{n-1}. A *stabilizing* 1*-form* for ω is a 1-form λ such that

$$\lambda \wedge \omega^{n-1} > 0 \quad \text{and} \quad \ker\omega \subset \ker d\lambda. \tag{1}$$

A Hamiltonian structure ω is called *stabilizable* if it admits a stabilizing 1-form λ, and the pair (ω, λ) is called a *stable Hamiltonian structure* (SHS). A SHS (ω, λ) induces a canonical *Reeb vector field* R generating $\ker\omega$ and normalized by $\lambda(R) = 1$. Note that if (ω, λ) is a SHS, then $(\omega, -\lambda)$ is a SHS inducing the opposite orientation.

We call an oriented 1-foliation $\mathcal{L}$ a *stable Hamiltonian foliation* if it is the kernel foliation of a SHS.

Note that there is no analogue for (stable) Hamiltonian structures of the stability results of Moser and Gray for symplectic resp. contact structures (see e.g. [41]) because the dynamics of the kernel foliation can change drastically under small perturbations. By "stable Hamiltonian topology" we mean the study of stable Hamiltonian structures up to homotopy in the following sense.

Definition 1.2. A *homotopy of Hamiltonian structures* is a smooth family ω_t, for t in some interval $I \subset \mathbb{R}$, such that the cohomology class of ω_t remains constant. The homotopy is called *stabilizable* if it admits a smooth family of stabilizing 1-forms λ_t, and the pair (ω_t, λ_t) is called a *homotopy of stable Hamiltonian structures*, or simply a *stable homotopy*.

Remark 1.3. Let us emphasize that for a homotopy of HS ω_t we always require $\dot\omega_t$ to be *exact*. This is the relevant notion for various reasons that will become clear in this paper (e.g. invariance considerations in symplectic field theory, see Section 6.7). We will exclusively use this notion of homotopy, with the exception of Sections 5.1 and 5.2 where we will briefly consider deformations ω_t with varying cohomology class (to which we will explicitly refer as "nonexact deformations").

Let us now describe the main results of this paper.

Discreteness. For a de Rham cohomology class $\eta \in H^2(M;\mathbb{R})$, we denote by $\mathcal{SHS}_\eta(M)$ and $\mathcal{HS}_\eta(M)$ the spaces of SHS resp. HS with $[\omega] = \eta$, equipped with the C^k-topology for some $2 \le k \le \infty$.

Theorem 1.4 (cf. Theorem 4.4). *Every SHS (ω, λ) on a closed 3-manifold has a C^5-neighbourhood in $\mathcal{SHS}_{[\omega]}(M)$ in which all SHS are stably homotopic to (ω, λ).*

This means that, for any closed 3-manifold M and cohomology class $\eta \in H^2(M;\mathbb{R})$, the space $\mathcal{SHS}_\eta(M)/\sim$ is discrete in the topological sense, which justifies the term "stable Hamiltonian topology" in the title. In particular, this implies that there are at most countably many homotopy classes of SHS representing η (Corollary 4.5).

Question 1.5. Does an analogue of Theorem 1.4 hold in higher dimensions?

Stabilization. Much of this paper is concerned with the properties of the natural map

$$\Pi : \mathcal{SHS}_\eta \to \mathcal{HS}_\eta, \qquad (\omega, \lambda) \mapsto \omega.$$

It has the following properties:

(a) Each nonempty fibre $\Pi^{-1}(\omega)$ is convex and hence contractible.
(b) Π is in general not surjective. In fact, its image $\operatorname{im}\Pi \subset \mathcal{HS}_\eta$ is in general neither closed (this is easy to see) nor open (this is much harder and proved in [14]).
(c) There exist smooth paths in $\operatorname{im}\Pi$ which have no smooth lift to $\mathcal{SHS}_\eta$. More precisely, we show

Theorem 1.6 (cf. Theorem 5.9). *On every closed oriented* 3*-manifold there exists a stable Hamiltonian foliation $\mathcal{L}$ with the following property. For any HS ω_0 defining $\mathcal{L}$ there exists a Baire set $\tilde{B}$ in the space $\tilde{E}$ of* (*exact*) *deformations $\{\omega_t\}_{t\in[0,1]}$ of ω_0 that cannot be stabilized, no matter what stabilizing* 1*-form λ_0 we take for ω_0.*

A similar result holds for lifts of convergent sequences (see Theorem 3.8). For nonexact deformations of HS, easy obstructions to stabilizability arise from foliated cohomology (see Section 5.2).

Contact structures. Every positive contact form λ induces a SHS $(d\lambda, \lambda)$ and homotopies of contact forms induce stable homotopies, so we have a natural map

$$\mathcal{CF}/\sim \to \mathcal{SHS}_0/\sim$$

from homotopy classes of positive contact forms to homotopy classes of exact SHS. Using Eliashberg's classification of contact structures on S^3 we prove

Theorem 1.7 (cf. Theorem 3.48). *On S^3 the map $\mathcal{CF}/\sim \to \mathcal{SHS}_0/\sim$ is not bijective.*

Considerations from symplectic field theory (see Section 6.7) lead us to the following

Conjecture 1.8. *The map in Theorem 1.7 is injective but not surjective.*

Structure theorem in dimension 3. In dimension 3 we have the following structure theorem.

Theorem 1.9 (cf. Corollary 4.2). *Every stable Hamiltonian structure on a closed* 3*-manifold M is stably homotopic to a SHS (ω, λ) for which there exists a* (*possibly disconnected and possibly with boundary*) *compact* 3*-dimensional submanifold $N = N^+ \cup N^- \cup N^0$ of M, invariant under the Reeb flow, and a* (*possibly empty*) *disjoint union $U = U_1 \cup \cdots \cup U_k$ of compact regions with the following properties:*

- $\operatorname{int} U \cup \operatorname{int} N = M$;
- $d\lambda = \pm\omega$ *on* $N^\pm$ *and* $d\lambda = 0$ *on* N^0;
- *on each* $U_i \cong [0, 1] \times T^2$ *the SHS* (ω, λ) *is exact and* T^2*-invariant.*

Moreover, we can always arrange that N^+ is nonempty, and if ω is exact we can arrange that N^0 is empty.

Roughly speaking, this says that M is a union of regions on which (ω, λ) is positive contact ($d\lambda = \omega$), negative contact ($d\lambda = -\omega$), or flat ($d\lambda = 0$), glued along 2-tori invariant under the Reeb flow.

Morse–Bott approximations. In order to define invariants such as symplectic field theory or Rabinowitz Floer homology, one needs to consider SHS that are *Morse* (i.e. all closed Reeb orbits are nondegenerate) or at least *Morse–Bott* (see Section 4.3 for the definition). It is well known that any contact form can be C^∞-approximated by one which is Morse. Surprisingly, this fails for more general SHS:

Theorem 1.10 (cf. Theorems 3.7, 3.36 and 3.37). *In every stable homotopy class in dimension* 3 *there exists a stable Hamiltonian structure which cannot be C^2-approximated by stable Hamiltonian structures that are Morse.*

Remark 1.11. Using the techniques in the proof of Theorem 4.6, the stable Hamiltonian structure in Theorem 1.10 can actually be chosen to be Morse–Bott.

This is bad news for the foundations of symplectic field theory: Besides the transversality problems for holomorphic curves, one faces the additional difficulty of making a SHS Morse, or at least Morse–Bott. However, this difficulty can be overcome in dimension 3:

Theorem 1.12 (cf. Theorem 4.6). *Any SHS on a closed oriented* 3*-manifold can be connected to a SHS which is Morse–Bott by a C^1-small stable homotopy.*

We discuss in Section 6.7 how Theorem 1.12 *might* be used to construct homotopy invariants of SHS in dimension 3. Let us emphasize, however, that we did not succeed in actually constructing such an invariant. In particular, we explain in Section 6.7 why, in defiance of our initial expectations, symplectic field theory does not seem to provide such an invariant. Nevertheless, we make the following

Conjecture 1.13 (cf. Conjecture 6.35). *The SHS $(d\alpha_{st}, \alpha_{st})$ and $(d\alpha_{ot}, \alpha_{ot})$ on S^3 are not stably homotopic, where α_{st} is the standard contact form and α_{ot} is an overtwisted contact form defining the same orientation.*

This conjecture would imply Conjecture 1.8 (see the proof of Theorem 3.48).

h-principle. Hamiltonian structures satisfy an h-principle ([40], see Section 2.6 for the precise statement). For stable Hamiltonian structures in dimension 3, the 0-parametric h-principle holds (Proposition 2.18). On the other hand, Conjecture 1.13 would imply that the 1-parametric h-principle fails for stable Hamiltonian structures in dimension 3.

Cobordisms. Symplectic field theory satisfies TQFT type axioms with respect to symplectic cobordisms. In order to turn it into a homotopy invariant of SHS, we need to construct suitable symplectic cobordisms from stable homotopies. The naive definition of a (*topologically trivial*) *symplectic cobordism* between HS ω_a and ω_b on M is a symplectic manifold $([a, b] \times M, \Omega)$ such that $\Omega|_{\{i\}\times M} = \omega_i$ for $i = a, b$. This definition turns out to be too restrictive (a SHS is not even cobordant to itself in this sense!). We introduce two more general notions: *strong cobordism* (replacing ω_i by $\omega_i + t_i d\lambda_i$ for sufficiently small t_i), and *weak cobordism* (replacing ω_i by more general closed 2-forms with the same kernel)—see Section 6.1. Moreover, we define a *length* of a stable homotopy (Section 6.2) and show that every sufficiently short stable homotopy gives rise to a strong cobordism (Proposition 6.6). We discuss in Section 6.7 how this might be used to turn suitable TQFTs into homotopy invariants.

In Section 6.3 we construct pairs of SHS which are homotopic as HS but not weakly cobordant, with obstructions to weak cobordisms arising from helicity and fillability. Our failure to construct such examples which are *stably* homotopic motivates the following

Question 1.14. Given a stable homotopy $(\omega_t, \lambda_t)_{t\in[0,1]}$, are ω_0 and ω_1 weakly bicobordant?

Embeddability and ambient homotopies. Given an abstract stable Hamiltonian structure $(M^{2n-1}, \omega, \lambda)$, we can ask whether it admits an embedding $\iota : M \hookrightarrow W$ into a given symplectic manifold (W^{2n}, Ω) such that $\iota^*\Omega = \omega$.

Theorem 1.15 (cf. Corollary 6.22). *There exists a SHS (ω, λ) on S^3 which is stably homotopic to $(d\alpha_{st}, \alpha_{st})$ but cannot be embedded in $(\mathbb{R}^4, \Omega_{st})$.*

This shows that abstract stable homotopies cannot in general be realized by homotopies of hypersurfaces in a fixed symplectic manifold. On the other hand, under an additional "tameness" assumption, many examples of smoothly but not tame stably homotopic hypersurfaces are constructed in [13]. In Section 3.10 we exhibit a large class of SHS that are not tame (Corollary 3.40).

Integrability in dimension 3. Although we set up the theory in arbitrary dimensions, most of our actual results concern dimension 3. The reason is that for a SHS (ω, λ) on a 3-manifold M we have $d\lambda = f\omega$ for a function $f : M \to \mathbb{R}$ which is invariant under the Reeb flow. By a version of the Arnold–Liouville theorem first observed in [1], regions where $df \neq 0$ are foliated by invariant 2-tori on which the Reeb flow is linear (Theorem 3.3). The study of SHS on these *integrable regions* reduces to the study of T^2-invariant SHS on $[0, 1] \times T^2$ (Section 3.4). On the other hand, on regions where f is constant the SHS is either positive contact ($f > 0$), negative contact ($f < 0$), or flat ($f = 0$), so it can be studied using methods from contact topology and foliations. The remaining difficulty is thus to understand the boundaries between these regions. This is overcome by our main technical result, Proposition 3.23, which allows us to replace λ by a new stabilizing 1-form $\tilde{\lambda}$ such that the boundaries of the regions where the new function $\tilde{f} = d\tilde{\lambda}/\omega$ is constant are contained in integrable regions for f.

2. Background on stable Hamiltonian structures

In this section we discuss the basic properties of stable Hamiltonian structures and collect some examples.

2.1. Basic examples

The three basic examples of stable Hamiltonian structures are the following.

Contact manifolds: (M, λ) is a contact manifold, R is the Reeb vector field, and $\omega = \pm d\lambda$.

Mapping tori: $M := W_\phi := \mathbb{R} \times W/(t, x) \sim (t + 1, \phi(x))$ is the mapping torus of a symplectomorphism ϕ of a symplectic manifold $(W, \bar{\omega})$, $R = \partial/\partial t$, $\lambda = dt$, and ω is the form on M induced by $\bar{\omega}$. Note that $d\lambda = 0$, so $\ker \lambda$ defines a *foliation*. Note that $W_\phi \cong [0, 1] \times W/(0, x) \sim (1, \phi(x))$.

Circle bundles: $\pi : M \to W$ is a principal circle bundle over a symplectic manifold $(W, \bar{\omega})$, R is the vector field generating the circle action, λ is a connection form, and $\omega = \pi^*\bar{\omega}$. Note that if $\omega = d\lambda$ is the curvature of the connection λ then we are actually in the contact case (Boothby–Wang construction).

2.2. Stable hypersurfaces

Historically, the stability condition for Hamiltonian structures first appeared as a dynamical stability condition for hypersurfaces in symplectic manifolds [33]. To see this relation, let (X, Ω) be a symplectic manifold and let $M \subset X$ be a closed hypersurface. Note that the restriction $\omega := \Omega|_M$ is a HS.

Lemma 2.1 ([16, Lemma 2.3]). *For a closed hypersurface M in a symplectic manifold (X, Ω) the following are equivalent:*

(a) *The hypersurface M is stable in the sense of* [33]*, i.e. there exists a tubular neighbourhood $(-\varepsilon, \varepsilon) \times M$ of $\{0\} \times M$ such that the* 1*-dimensional kernel foliations of $\Omega|_{\{t\}\times M}$ on $\{t\} \times M$ are all conjugate via a family of diffeomorphisms depending smoothly on t.*
(b) *There exists a vector field Y transverse to M such that* $\ker(L_Y\Omega|_M) \subset \ker(\Omega|_M)$.
(c) *The Hamiltonian structure $(M, \Omega|_M)$ is stabilizable.*

Given a SHS (ω, λ) on an abstract manifold M, we can always realize M as a stable hypersurface in some symplectic manifold (X, Ω), i.e. there exists an embedding $\iota : M \hookrightarrow X$ in a symplectic manifold (X, Ω) such that $\iota^\star\Omega = \omega$ and $\iota(M)$ is a stable hypersurface in X. For this take, for instance, the *symplectization*

$$\big(X := (-\varepsilon, \varepsilon) \times M,\ \Omega := \omega + t d\lambda + dt \wedge \lambda\big) \tag{2}$$

for $\varepsilon > 0$ small enough. In this example a transverse vector field Y in Lemma 2.1(b) can be taken to be just ∂_t and a family $\{\phi_t\}_{t\in(-\varepsilon,\varepsilon)}$ of diffeomorphisms

$$\phi_t : \{0\} \times M \to \{t\} \times M$$

with

$$\phi_{t*} \ker \Omega|_{\{0\}\times M} = \ker \Omega|_{\{t\}\times M}$$

to be the flow of ∂_t,

$$\phi_t(x,\tau) := (x, \tau + t).$$

The much more interesting question whether a given SHS can be embedded into more specific symplectic manifolds, e.g. closed ones or the standard $\mathbb{C}^n$, will be discussed in Section 3.11.

2.3. Stability and geodesibility

The property of a Hamiltonian structure ω to be stabilizable depends only on its kernel distribution $\ker\omega$. More generally, we say that an oriented 1-dimensional foliation $\mathcal{L}$ is *stabilizable* if there exists a vector field X generating $\mathcal{L}$ such that

$$\lambda(X) = 1 \quad \text{and} \quad i_X d\lambda = 0. \tag{3}$$

Definition 2.2. An orientable 1-dimensional foliation $\mathcal{L}$ is called *geodesible* if there exists a vector field X generating $\mathcal{L}$ and a Riemannian metric g such that the (naturally parametrized) flow lines of X are geodesics for g.

The following theorem gives a characterization of stabilizability which is sometimes more convenient than equation (1).

Theorem 2.3 (Wadsley [50]). *An orientable* 1*-dimensional foliation* $\mathcal{L}$ *is stabilizable if and only if it is geodesible. Given a vector field* X *generating* $\mathcal{L}$ *whose flow lines are geodesics for a metric* g *and such that* $g(X,X)\equiv 1$*, a stabilizing* 1*-form* (*called* Wadsley form) *is obtained by*

$$\lambda := i_X g.$$

Proof. Let us give the simple proof since we will need parts of it later. Given a metric $g = \langle\ ,\ \rangle$ and a vector field X, we set $\lambda := i_X g$ and compute, for a second vector field Y,

$$\begin{aligned} d\lambda(X,Y) &= X\cdot\lambda(Y) - Y\cdot\lambda(X) - \lambda([X,Y]) = X\cdot\langle X,Y\rangle - Y\cdot|X|^2 - \langle X,[X,Y]\rangle \\ &= \langle\nabla_X X, Y\rangle + \langle X, \nabla_X Y - [X,Y]\rangle - Y\cdot|X|^2 \\ &= \langle\nabla_X X, Y\rangle + \langle X, \nabla_Y X\rangle - Y\cdot|X|^2 = \langle\nabla_X X, Y\rangle - \tfrac{1}{2}Y\cdot|X|^2. \end{aligned}$$

So we have shown the formula

$$i_X d(i_X g) = i_{\nabla_X X} g - d(|X|^2/2). \tag{4}$$

Now let X be a vector field generating $\mathcal{L}$. If its flow lines are geodesics for a metric g and $g(X,X)\equiv 1$, then the right-hand side in (4) vanishes, so the 1-form $\lambda = i_X g$ satisfies (3). Conversely, if λ is a 1-form satisfying (3), pick any metric g such that $i_X g = \lambda$, i.e. X has length 1 and is perpendicular to $\ker\lambda$ with respect to g. Then $g(X,X) = 1$, so in (4) the left-hand side and the second term on the right-hand side vanish and we obtain $\nabla_X X = 0$, i.e. the flow lines of X are geodesics. □

Corollary 2.4. *On an oriented* 3*-manifold, a vector field X is the Reeb vector field of a stable Hamiltonian structure if and only if there exist a metric g and a volume form μ such that*

$$\nabla_X X = 0, \qquad g(X, X) \equiv 1, \qquad L_X \mu = 0. \tag{5}$$

Given a metric g and volume form μ satisfying (5)*, the stable Hamiltonian structure is given by*

$$\lambda := i_X g, \qquad \omega := i_X \mu.$$

Moreover, given a stable Hamiltonian structure (ω, λ) with Reeb vector field X, the metric and volume form satisfying (5) *can be chosen such that $\mu = \lambda \wedge \omega$ is the volume form induced by g.*

Proof. Given a stable Hamiltonian structure (ω, λ) with Reeb vector field X, pick a metric g such that $i_X g = \lambda$, i.e. X has length 1 and is perpendicular to $\ker \lambda$. Then $g(X, X) = 1$ and $\nabla_X X = 0$ follows from $i_X d\lambda = 0$ and (4). The volume form $\mu := \lambda \wedge \omega$ satisfies $L_X \mu = d\omega = 0$. If we pick g on $\ker \lambda$ to be $\omega(\cdot, J\,\cdot)$ for a complex structure J on $\ker \lambda$ compatible with ω, then μ is the volume form induced by g.

The converse direction is an immediate consequence of (4). □

2.4. *Relation to hydrodynamics*

In dimension 3, Reeb vector fields of stable Hamiltonian structures naturally arise as special solutions in hydrodynamics. This observation is due to Etnyre and Ghrist [26] (in the contact case, but the stable case is similar). We refer to [3] for background on hydrodynamics.

The velocity field X of an ideal incompressible fluid on a closed oriented Riemannian 3-manifold (M, g) with volume form μ (not necessarily the one induced by the metric) satisfies the *Euler equation*

$$\frac{\partial X}{\partial t} + \nabla_X X = -\nabla p, \qquad L_X \mu = 0.$$

Here the *pressure function* p is uniquely determined up to a constant by the equation $L_X \mu = 0$. Stationary (i.e. time-independent) solutions thus satisfy

$$\nabla_X X = -\nabla p, \qquad L_X \mu = 0. \tag{6}$$

In view of (4), this equation can be rewritten as

$$i_X d(i_X g) = -d\alpha, \qquad L_X \mu = 0, \tag{7}$$

where $\alpha := p + |X|^2/2$ is the Bernoulli function. It follows from the first equation that α is a first integral for X and it is shown in [1] that in regions where α is nonconstant the flow of X is completely integrable (cf. [3, 26] and the discussion in Section 3.1 below). The other extreme (and little understood) case are the *Beltrami fields*: solutions of (7) with constant α, i.e. such that

$$i_X d(i_X g) = 0, \qquad L_X \mu = 0 \tag{8}$$

for some metric g and volume form μ. The proof of the following corollary is analogous to that of Corollary 2.4.

Corollary 2.5. *On an oriented* 3*-manifold, a vector field* X *is a Beltrami field if and only if it generates the kernel foliation of a stable Hamiltonian structure. Given a metric* g *and a volume form* μ *satisfying* (8)*, the stable Hamiltonian structure is given by*

$$\lambda := i_X g, \qquad \omega := i_X \mu.$$

Moreover, given a stable Hamiltonian structure (ω, λ) *whose kernel foliation is generated by* X*, the metric and volume form satisfying* (8) *can be so chosen that* $\mu = (1/\lambda(X))\lambda \wedge \omega$ *is the volume form induced by* g*.*

Remark 2.6. (a) Note that any rescaling fX of a Beltrami field by a positive function $f : M \to \mathbb{R}$ is again a Beltrami field (just rescale g and μ accordingly), but the same is not true for the Reeb vector field of a stable Hamiltonian structure.

(b) Beltrami vector fields arise e.g. as solutions of the stationary Euler equation (6) with constant pressure, i.e. $\nabla_X X = 0$ and $L_X \mu = 0$. Indeed, the rescaled vector field $\bar{X} := X/|X|$ satisfies the same equations and in addition $g(\bar{X}, \bar{X}) \equiv 1$, so it satisfies (7) with vanishing Bernoulli function, hence $\bar{X}$ as well as X are Beltrami.

(c) The *curl* of a Beltrami field X (with respect to g, μ) is defined by the equation

$$i_{\operatorname{curl} X} \mu = d(i_X g).$$

The first equation in (8) implies $i_X i_{\operatorname{curl} X} \mu = 0$, so $\operatorname{curl} X = fX$ for a function $f : M \to \mathbb{R}$. This function f is a first integral of X and will play a crucial role in Section 3.

2.5. *Obstructions to stability*

Not every orientable 1-dimensional foliation is geodesible, and moreover not every kernel foliation of a HS is. In this paper, we will only use the second one of the following two easy obstructions.

Obstruction 1. An oriented 1-foliation $\mathcal{L}$ is not geodesible if it contains a *Reeb component*, i.e. an oriented embedded annulus A consisting of leaves of $\mathcal{L}$ such that the boundary orientation of ∂A coincides with that given by $\mathcal{L}$. Indeed, if the ambient manifold contains a Reeb component, then for any vector field X spanning $\mathcal{L}$ and any 1-form λ satisfying $i_X d\lambda = 0$ we have $\int_A d\lambda = 0$; hence by Stokes' theorem $\int_{\partial A} \lambda = 0$ and the equation $\lambda(X) = 1$ in (3) cannot be satisfied.

Obstruction 2. A Hamiltonian structure ω on a closed manifold is not stabilizable if ω has a primitive α for which

$$\alpha \wedge \omega^{n-1} \equiv 0. \tag{9}$$

Indeed, assume that there exists λ stabilizing ω. Consider

$$d(\alpha \wedge \lambda \wedge \omega^{n-2}) = \lambda \wedge \omega^{n-1} - \alpha \wedge d\lambda \wedge \omega^{n-2}.$$

The first term on the right-hand side is pointwise positive by the first condition in (3), the second term is identically zero by the second condition in (3) and (9) (all factors vanish on the Reeb vector field), and the left-hand side is exact. Integration over the manifold thus yields a contradiction.

Obstructions 1 and 2 are special cases of the following general characterization of geodesibility due to Sullivan.

Theorem 2.7 (Sullivan [48]). *An oriented foliation $\mathcal{L}$ is nongeodesible if and only if there exists a foliation cycle which can be arbitrarily well approximated by boundaries of singular* 2*-chains tangent to the foliation.*

2.6. *The h-principle for Hamiltonian structures*

Hamiltonian structures satisfy the h-principle. For this, let M be an odd-dimensional manifold. Denote by $\Omega^2_{\text{nondeg}}(M)$ the space of (not necessarily closed) 2-forms on M of maximal rank, and by $\mathcal{HS}_a(M)$ the space of *closed* 2-forms of maximal rank (i.e. Hamiltonian structures) representing the cohomology class $a \in H^2(M;\mathbb{R})$. Both spaces are equipped with the C^∞_{loc}-topology. The following h-principle was first proved by McDuff [40] (see also [24]). In fact, it is a consequence (via symplectization) of the h-principle for symplectic forms on open manifolds.

Theorem 2.8 (McDuff [40]). *The inclusion $\mathcal{HS}_a(M) \hookrightarrow \Omega^2_{\text{nondeg}}(M)$ is a homotopy equivalence. In particular, if M admits a* 2*-form of maximal rank then every $a \in H^2(M;\mathbb{R})$ is represented by a Hamiltonian structure, and two cohomologous Hamiltonian structures are homotopic iff they are homotopic through* (*not necessarily closed*) 2*-forms of maximal rank.*

Corollary 2.9. *Let M be an oriented* 3*-manifold M. Then every $a \in H^2(M;\mathbb{R})$ is represented by a Hamiltonian structure, and two cohomologous Hamiltonian structures are homotopic iff their kernel distributions are homotopic as oriented line fields. In particular, the Hamiltonian structures $d\alpha_0, d\alpha_1$ induced by contact forms α_0, α_1 on M are homotopic iff the contact distributions are homotopic as oriented plane fields.*

We will show in Proposition 2.18 that the existence part of this corollary also holds for stable Hamiltonian structures. In Theorem 3.48 we will show that, assuming transversality for holomorphic curves can be achieved, the homotopy part fails for stable Hamiltonian structures.

2.7. *Foliated cohomology*

Let $\mathcal{L}$ be an oriented 1-dimensional foliation on a closed manifold M. Fix any vector field R generating $\mathcal{L}$ and consider the subcomplex

$$\Omega^k_{\mathcal{L}}(M) := \{\alpha \in \Omega^k(M) \mid i_R\alpha = 0,\ i_R d\alpha = 0\}$$

of the de Rham complex (note that this does not depend on the choice of R). Its cohomology $H^k_{\mathcal{L}}(M)$ is the *foliated cohomology* of the foliation $\mathcal{L}$. The inclusion $\Omega^k_{\mathcal{L}}(M) \subset \Omega^k(M)$ induces a canonical map

$$\kappa : H^k_{\mathcal{L}}(M) \to H^k(M;\mathbb{R}).$$

Note that any HS ω with $\ker\omega = \mathcal{L}$ carries a cohomology class $[\omega] \in H^2_{\mathcal{L}}(M)$, and any stabilizing 1-form λ for $\mathcal{L}$ defines a cohomology class $[d\lambda] \in \ker(\kappa : H^2_{\mathcal{L}}(M) \to H^2(M;\mathbb{R}))$.

Foliated cohomology will appear throughout this paper. A first illustration of its importance is the following "foliated" version of Moser's stability theorem.

Proposition 2.10. *Let $(\omega_t)_{t\in[0,1]}$ be a smooth family of Hamiltonian structures with constant kernel foliation $\ker\omega_t = \mathcal{L}$. Suppose that the foliated cohomology $H^2_{\mathcal{L}}(M)$ is finite-dimensional and $[\omega_t] = [\omega_0] \in H^2_{\mathcal{L}}(M)$ for all t. Then there exists a smooth family $(\phi_t)_{t\in[0,1]}$ of diffeomorphisms with $\phi_0 = \mathbb{1}$, $\phi_t^*\mathcal{L} = \mathcal{L}$ and $\phi_t^*\omega_t = \omega_0$ for all t.*

Proof. Pick a vector field R generating $\mathcal{L}$ and a 1-form λ with $\lambda(R) = 1$. Since $[\omega_t] = [\omega_0] \in H^2_{\mathcal{L}}(M)$, there exist 1-forms μ_t with $\omega_0 + d\mu_t = \omega_t$ and $i_R\mu_t = 0$. According to Lemma 2.11 below, the μ_t can be chosen to depend smoothly on t. Now the proof can be concluded by a standard Moser argument: Since $i_R\dot\mu_t = 0$ and $\ker\omega_t = \mathcal{L}$, there exists a unique t-dependent vector field X_t satisfying $i_{X_t}\omega_t = -\dot\mu_t$ and $\lambda(X_t) = 0$. Its flow ϕ_t satisfies

$$\frac{d}{dt}\phi_t^*\omega_t = \phi_t^*(L_{X_t}\omega_t + \dot\omega_t) = \phi_t^* d(i_{X_t}\omega_t + \dot\mu_t) = 0$$

and hence $\phi_t^*\omega_t = \omega_0$, which in turn implies $\phi_t^*\mathcal{L} = \mathcal{L}$. □

The following lemma was used in the preceding proof.

Lemma 2.11. *Let $\mathcal{L}$ be an oriented 1-dimensional foliation on a closed manifold M such that $H^k_{\mathcal{L}}(M)$ is finite-dimensional. Then for each smooth family $\alpha_t \in \Omega^k_{\mathcal{L}}(M)$, $t \in \mathbb{R}$, with $[\alpha_t] = 0 \in H^k_{\mathcal{L}}(M)$ there exists a smooth family $\beta_t \in \Omega^{k-1}_{\mathcal{L}}(M)$ such that $d\beta_t = \alpha_t$.*

Proof. The proof uses some facts about Fréchet spaces and nuclear spaces. Here a *Fréchet space* is a metrizable locally convex topological vector space. In [46] the open mapping theorem is proved for Fréchet spaces: Any surjective continuous linear map $X \to Y$ between Fréchet spaces is open. It follows that if $T : X \to Y$ is a continuous linear map between Fréchet spaces whose image $\operatorname{im} T$ has finite codimension, then $\operatorname{im} T \subset Y$ is closed. (To see this, pick vectors $e_1, \dots, e_k \in Y$ spanning $Y/\operatorname{im} T$; then the continuous linear map $\widehat{T} : X \oplus \mathbb{R}^k \to Y$, $(x, x_1, \dots, x_k) \mapsto Tx + x_1e_1 + \dots + x_ke_k$, is surjective, hence open, so $\operatorname{im} T = \widehat{T}(X \oplus 0)$ is closed).

Next consider the Fréchet space Ω^k of smooth k-forms on M and its closed (hence Fréchet) subspaces $\mathcal{F} \subset \Omega^k_{\mathcal{L}} \subset \Omega^k$, where $\mathcal{F}$ denotes the closed forms in $\Omega^k_{\mathcal{L}}$ and we have omitted M from the notation. By hypothesis the image $\mathcal{E}$ of the continuous linear map $d : \Omega^{k-1} \to \mathcal{F}$ has finite codimension, so by the preceding discussion $\mathcal{E}$ is closed and hence Fréchet.

Now the proof is concluded by an argument of Banyaga [6]: The space $C^\infty(\mathbb{R})$ of smooth functions $\mathbb{R} \to \mathbb{R}$ is a nuclear space in the sense of Grothendieck ([31], see also [49]), so its tensor product with each Fréchet space X has a natural completion $C^\infty(\mathbb{R}) \mathbin{\widehat{\otimes}} X \cong C^\infty(\mathbb{R}, X)$. Moreover, any surjective continuous linear map $T : X \to Y$ between Fréchet spaces induces a surjective continuous linear map $\mathbb{1}\widehat{\otimes}T : C^\infty(\mathbb{R})\widehat{\otimes}X \to C^\infty(\mathbb{R})\widehat{\otimes}Y$ (see [49]). Applying this to the map $d : \Omega^{k-1}(M) \to \mathcal{E}$ above, we thus obtain a surjective continuous linear map $\mathbb{1} \mathbin{\widehat{\otimes}} d : C^\infty(\mathbb{R}, \Omega^{k-1}) \to C^\infty(\mathbb{R}, \mathcal{E})$, which gives the desired conclusion. □

Example 2.12. Suppose $\mathcal{L}$ are the orbits of a locally free circle action, so the orbit space B is an orbifold and we have a projection $\pi : M \to B$. Then $\Omega^k_{\mathcal{L}}(M) = \pi^*\Omega^k(B)$ and $H^k_{\mathcal{L}}(M) \cong H^k(B;\mathbb{R})$. The map $\kappa : H^k_{\mathcal{L}}(M) \to H^k(M;\mathbb{R})$ corresponds to the pull-back $\pi^* : H^k(B;\mathbb{R}) \to H^k(M;\mathbb{R})$, which is in general neither surjective nor injective. This example also shows that Proposition 2.10 fails under the weaker hypothesis $[\omega_t] = [\omega_0] \in H^2(M;\mathbb{R})$ (e.g. take the Hopf fibration $S^3 \to S^2$ and ω_t the pullback of an area form on S^2 of area $1+t$).

Remark 2.13. In the preceding example $H^k_{\mathcal{L}}(M)$ was finite-dimensional. We will see in Section 3.12 that for stable Hamiltonian structures in dimension three $H^2_{\mathcal{L}}(M)$ is often infinite-dimensional.

2.8. *Helicity*

Let M be a closed oriented 3-manifold. The *helicity pairing* of two exact 2-forms $\mu = d\alpha$ and $\nu = d\beta$ is

$$\mathrm{Hel}(\mu,\nu) := \int_M \alpha \wedge \nu = \int_M \beta \wedge \mu.$$

This is independent of the primitives α, β and defines a symmetric bilinear form on the space of exact 2-forms. The associated quadratic form

$$\mathrm{Hel}(\mu) = \int_M \alpha \wedge \mu$$

is called the *helicity* of μ.

Now consider a 1-dimensional oriented foliation $\mathcal{L}$. An exact 2-form ω with $\mathcal{L} \subset \ker\omega$ carries a helicity $\mathrm{Hel}(\omega)$. If $\omega = d\alpha$ with $\alpha|_{\mathcal{L}} = 0$ then $\alpha \wedge \omega$ contracts to zero with $\mathcal{L}$ and thus vanishes identically, so we have $\mathrm{Hel}(\omega) = 0$. Thus helicity descends to a quadratic form on

$$D_{\mathcal{L}} := \ker(\kappa : H^2_{\mathcal{L}}(M) \to H^2(M;\mathbb{R})).$$

Note that for each stabilizing 1-form λ for $\mathcal{L}$ we have $d\lambda \in D_{\mathcal{L}}$ and thus a helicity $\mathrm{Hel}(d\lambda)$.

Example 2.14. If $\omega = d\alpha$ for a positive contact form α, its helicity satisfies

$$\mathrm{Hel}(\omega) = \int_M \alpha \wedge d\alpha > 0.$$

Example 2.15. For $\mathcal{L}$ corresponding to the fibres of a circle bundle $\pi : M \to B$ over a closed oriented surface B as in Example 2.12 we distinguish two cases according to its Euler number $e \in \mathbb{Z}$. If $e = 0$ the pullback $\pi^* : H^2(B;\mathbb{R}) \to H^2(M;\mathbb{R})$ is injective, thus $D_{\mathcal{L}} = 0$ and the helicity vanishes identically. If $e \neq 0$ the pullback $\pi^* : H^2(B;\mathbb{R}) \to H^*(M;\mathbb{R})$ is the zero map, thus $D_{\mathcal{L}} \cong H^2(B;\mathbb{R})$ and the helicity pairing is given by

$$\mathrm{Hel} : H^2(B;\mathbb{R}) \times H^2(B;\mathbb{R}) \to \mathbb{R}, \qquad (x, y) \mapsto \frac{\int_B x \int_B y}{e}.$$

2.9. Left-invariant Hamiltonian structures on $PSL(2,\mathbb{R})$

This example is a particular case of the contact one. We consider left-invariant 1-forms on the Lie group $PSL(2,\mathbb{R})$. To be more concrete, we choose a basis

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

of the Lie algebra $sl(2,\mathbb{R})$. The structural equations are

$$[H, E_+] = 2E_+, \quad [H, E_-] = -2E_-, \quad [E_+, E_-] = H.$$

Let (h, e^+, e^-) be the basis of $sl(2,\mathbb{R})^\star$ dual to (H, E_+, E_-). We extend h, e^+ and e^- to left-invariant 1-forms on $PSL(2,\mathbb{R})$ by left multiplication and denote the 1-forms by the same letters. The structural equations above can be rewritten in terms of the 1-forms h, e^+ and e^- as follows:

$$dh = e^- \wedge e^+, \quad de^+ = 2e^+ \wedge h, \quad de^- = 2h \wedge e^-.$$

This shows that if $0 \neq \alpha \in sl(2,\mathbb{R})^*$, then $d\alpha$ is nowhere zero and so for dimensional reasons defines a Hamiltonian structure. For

$$\alpha = ah + a_+e^+ + a_-e^- \neq 0$$

we compute

$$\begin{aligned} \alpha \wedge d\alpha &= (ah + a_+e^+ + a_-e^-) \wedge (adh + a_+de^+ + a_-de^-) \\ &= a^2 h \wedge e^- \wedge e^+ + a_+a_- 2e^+ \wedge h \wedge e^- + a_-a_+ 2e^- \wedge e^+ \wedge h \\ &= (a^2 + 4a_+a_-) h \wedge e^- \wedge e^+. \end{aligned}$$

We fix an orientation of $PSL(2,\mathbb{R})$ (and its compact quotients) by specifying the volume form $h \wedge e^- \wedge e^+$. It is clear from the computation above that α is a positive contact form when $a^2 + 4a_+a_- > 0$, negative contact when $a^2 + 4a_+a_- < 0$, and defines a foliation when $a^2 + 4a_+a_- = 0$. The cone

$$C := \{\alpha \in sl(2,\mathbb{R})^* \mid a^2 + 4a_+a_- = 0\}$$

separates the space $sl(2,\mathbb{R})^*$ into three connected components: one consisting of positive contact forms, and the other two consisting of negative contact forms. For a contact form

$\alpha \in sl(2,\mathbb{R})^*$ its Reeb vector field (viewed as an element of $sl(2,\mathbb{R})$) is given by $R = X/(a^2 + 4a_+a_-)$ with

$$X = aH + 2a_-E_+ + 2a_+E_- = \begin{pmatrix} a & 2a_- \\ 2a_+ & -a \end{pmatrix}.$$

Note that the spectrum of X consists of two distinct real eigenvalues if α is positive contact and of two conjugate imaginary eigenvalues if α is negative contact. Let Γ be a lattice in $PSL(2,\mathbb{R})$ such that the quotient $\Gamma\backslash PSL(2,\mathbb{R})$ is smooth and compact. Then the space of left-invariant forms on $PSL(2,\mathbb{R})$ descends to a certain space V of exterior forms on the quotient $\Gamma\backslash PSL(2,\mathbb{R})$ and the natural identification is an isomorphism of graded commutative algebras. Thus all our equalities involving left-invariant forms or left-invariant vector fields on $PSL(2,\mathbb{R})$ induce the corresponding equalities of their descendants on $\Gamma\backslash PSL(2,\mathbb{R})$. In particular, the cone C descends to a cone in V (still denoted by C) which separates V into three connected components as before. Moreover, Obstruction 2 in Section 2.3 yields: *For any nonzero $\alpha \in C \subset V$ (i.e., α defines a foliation) the HS $d\alpha$ is not stabilizable.*

The dynamics on the quotient is well understood in view of

Theorem 2.16 ([4, Theorem 5.3]).

(a) *For a negative contact form $\alpha_- \in V$ the Reeb flow is periodic.*
(b) *For a positive contact form $\alpha_+ \in V$ the Reeb flow is ergodic.*

We elaborate a bit on the contact cases.

Case 1: $\alpha_- \in V$ is a negative contact form. By Theorem 2.16(a), its Reeb vector field R defines a (locally free) circle action on $M := \Gamma\backslash PSL(2,\mathbb{R})$, so its orbit space Σ is a closed 2-dimensional oriented orbifold. Denote by $\pi : M \to \Sigma$ the projection and by $\mathcal{L}$ the foliation defined by R. According to Example 2.12, the foliated cohomology is given by $H^2_{\mathcal{L}}(M) \cong H^2(\Sigma;\mathbb{R}) \cong \mathbb{R}$. Since $d\alpha_-$ is nowhere vanishing on planes transverse to $\mathcal{L}$ it represents a nonzero class $[d\alpha_-] \in H^2_{\mathcal{L}}(M)$ and therefore generates $H^2_{\mathcal{L}}(M)$. If θ is a closed 2-form on M with

$$i_R\theta = 0, \tag{10}$$

then $[\theta] = c[d\alpha_-] \in H^2_{\mathcal{L}}(M)$ for a constant $c \in \mathbb{R}$, hence there exists a 1-form ρ on M such that

$$\theta = c\,d\alpha_- + d\rho, \qquad i_R\rho = 0. \tag{11}$$

Case 2: $\alpha_+ \in V$ is a positive contact form. Here the corresponding discussion is even easier. Namely, let R be the Reeb vector field of α_+ and let θ be a closed 2-form on M satisfying (10). Then θ is proportional to $d\alpha$, i.e. $\theta = f\,d\alpha$ for a function $f : M \to \mathbb{R}$. Now the vector field R preserves both $d\alpha$ and θ, so it must preserve the function f. By ergodicity of R (Theorem 2.16(b)), the function f is then constant, i.e.

$$\theta = c\,d\alpha_+ \tag{12}$$

for some $c \in \mathbb{R}$. In particular, we see that in both cases the foliated cohomology $H^2_{\mathcal{L}}(M)$ is 1-dimensional and generated by $[d\alpha_\pm]$, so $\kappa : H^2_{\mathcal{L}}(M) \to H^2(M;\mathbb{R})$ is the zero map. We will come back to this example in Section 5 and in Section 6.3.

2.10. *Magnetic flows*

A large and interesting class of SHS arises in magnetic flows. For this, consider the cotangent bundle $\tau : T^*Q \to Q$ of a closed n-manifold Q. Given a closed 2-form σ on Q (the "magnetic field"),

$$\Omega = dp \wedge dq + t\tau^*\sigma$$

defines a symplectic form on T^*Q. Consider on T^*Q a classical Hamiltonian of the form

$$H(q, p) = \tfrac{1}{2}|p|^2$$

for a metric $g = |\ |$ on Q. Solutions of the Hamiltonian system defined by Ω and H describe the motion of a charged particle in the magnetic field σ. For each $t > 0$ the level set $M_t := H^{-1}(t)$ is canonically diffeomorphic (by rescaling in p) to the unit cotangent bundle S^*Q, so $\omega_t := \Omega|_{M_t}$ defines a Hamiltonian structure on S^*Q. We assume that $\sigma|_{\pi_2(Q)} = 0$. Then the classical theory of Hamiltonian systems associates to this situation a *Mañé critical value* $c = c(g, \sigma) \in (0, \infty]$ (see e.g. [13]). The following facts are proved or illustrated by examples in [13].

(1) In all known examples except one ($Q = T^2$), M_t is stable for $t < c$ and nonstable for $t = c$. For $t > c$, M_t is sometimes stable and sometimes not.
(2) In most known examples, M_t is stable except for finitely many t. However, there exist examples in which M_t is nonstable for all t in an open interval (a, ∞).
(3) There exist examples of level sets M_s, M_t with $s < c < t$ that are tame stable but not tame stably homotopic (see Section 6.6).

Note that if $\tau^*\sigma$ is nonexact then M_t cannot be of contact type, so magnetic flows provide a large class of stable Hamiltonian structures that are not contact. We conclude this subsection with an explicit example (see [13]).

Example 2.17. Let (Q, g) be a closed hyperbolic surface and σ the area form defined by the hyperbolic metric g. Then the Mañé critical value is $c = 1/2$. For $t > 1/2$, M_t is of contact type with positive contact form $p\,dq$, and the Reeb flow is ergodic with periodic orbits representing all nontrivial free homotopy classes in T^*Q. For $t < 1/2$, M_t is of negative contact type, i.e. $\omega_t = d\alpha_t$ for a contact form defining the opposite orientation, and all Reeb orbits are periodic and contractible in T^*Q. $M_{1/2}$ is nonstable and has no periodic orbits at all (the flow on $M_{1/2}$ is the famous "horocycle flow" and nonstability follows from Obstruction 2 in Section 2.5). Note that this example can also be recovered from Theorem 2.16 since $S^*Q = \Gamma\backslash PSL(2, \mathbb{R})$ for a lattive Γ.

2.11. *An existence result*

The following existence result was independently proved in [43] in dimension 3; we thank C. Wendl for suggesting its generalization to higher dimensions.

Proposition 2.18. *For any $(2n-1)$-dimensional closed contact manifold (M, ξ) and any rational cohomology class $\eta \in H^2(M; \mathbb{Q})$ there exists a closed 2-form ω with $[\omega] = \eta$ and a contact form λ defining ξ such that (ω, λ) is a stable Hamiltonian structure. In the case $n = 2$ the same holds for any real cohomology class $\eta \in H^2(M; \mathbb{R})$.*

Proof. We proceed by induction on n. Let $n \geq 2$ and assume the statement has been established for all contact manifolds of dimension $2(n-1)-1$ (for $n=2$ this hypothesis is vacuous). Let (M, ξ) be a contact manifold of dimension $2n-1$ and $\eta \in H^2(M; \mathbb{Q})$. After rescaling we may assume $\eta \in H^2(M; \mathbb{Z})$. By [35] we can represent the Poincaré dual of the cohomology class η by an embedded closed contact submanifold $N \subset M$. We apply the induction hypothesis to $(N, \xi|_N, \eta|_N)$. This gives us a contact form λ_N on N and a closed 2-form ω_N on N representing the class $\eta|_N$ such that (ω_N, λ_N) is a SHS (for $n=2$ we take any λ_N and $\omega_N = 0$).

Pick some Riemannian metric on N. Let $\pi : D_N \to N$ denote the disk bundle associated to the normal bundle of N in M and let $S_N := \partial D_N$ be the corresponding circle bundle. Note that $\eta|_N$ is the Euler class $e(S_N) \in H^2(N; \mathbb{Z})$ of the circle bundle S_N. We proceed in two steps. First, we write down a normal form for a contact form λ on D_N using a connection form on S_N. Then we use the same connection form to write down a Thom form for D_N and add it to $d\lambda$ to obtain the desired 2-form ω.

Constructing a contact form on D_N. We can always realize a closed 2-form representing the Euler class of a principal S^1-bundle in terms of a connection form. Thus, there exists a connection 1-form δ on S_N such that $d\delta = -\pi^*\omega_N$ (here the connection form δ is normalized to have integral 1 over the fibre). Let r be the radial coordinate in a fibre of D_N normalized such that $\partial D_N = \{r = 1\}$. Let $\dot{D}_N := D_N \setminus N$ be the punctured disk bundle. We extend the 1-form δ from S_N to $\dot{D}_N$ in an r-invariant way. Then the 1-form $r^2\delta$ on $\dot{D}_N$ extends as zero over N to a smooth 1-form on D_N.

Lemma 2.19. *For $\varepsilon > 0$ sufficiently small the 1-form*

$$\lambda_\varepsilon := \pi^*\lambda_N + \varepsilon r^2 \delta$$

defines a contact form on D_N. Moreover, for each compactly supported function $F : [0,1) \to \mathbb{R}$ constant near 0 there exists a constant $K_F > 0$ such that for all $K \geq K_F$ we have

$$\lambda_\varepsilon \wedge \big(d\lambda_\varepsilon - d[K^{-1}F(r)\delta]\big)^{n-1} > 0.$$

Proof. Let us write out

$$d\lambda_\varepsilon = d\pi^*\lambda_N + \varepsilon r^2 d\delta + 2\varepsilon r dr \wedge \delta = [\pi^* d\lambda_N - \varepsilon r^2 \pi^*\omega_N] + 2\varepsilon r dr \wedge \delta$$

and

$$d\lambda_\varepsilon - d[K^{-1}F(r)\delta] = [\pi^* d\lambda_N + (K^{-1}F(r) - \varepsilon r^2)\pi^*\omega_N] + [2\varepsilon r - K^{-1}F'(r) dr] \wedge \delta.$$

It follows that

$$\lambda_\varepsilon \wedge \big(d\lambda_\varepsilon - d[K^{-1}F(r)\delta]\big)^{n-1} = (n-1)\alpha_1 \wedge \alpha_2,$$

where

$$\alpha_1 := \pi^*\lambda_N \wedge \big(\pi^* d\lambda_N + [K^{-1}F(r) - \varepsilon r^2]\pi^*\omega_N\big)^{n-2},$$
$$\alpha_2 := [2\varepsilon r - K^{-1}F'(r)] dr \wedge \delta.$$

Since (ω_N, λ_N) is a SHS on N and λ_N is contact, for small ε and large K we get

$$\ker\bigl(\pi^* d\lambda_N + [K^{-1}F(r) - \varepsilon r^2]\pi^*\omega_N\bigr) = \pi^* \ker d\lambda_N.$$

Hence the kernel of α_1 is exactly the tangent space to the fibre. Since α_2 restricts to the fibre as a positive area form for K large, the lemma follows. □

Note that in the preceding proof we have shown (for $F \equiv 0$)

$$\ker(\pi^* d\lambda_N - \varepsilon r^2 \pi^*\omega_N) = \pi^* \ker d\lambda_N.$$

The expression for $d\lambda_\varepsilon$ now shows that

$$\pi^* \ker d\lambda_N \cap \ker dr \cap \ker\delta \subset \ker d\lambda_\varepsilon,$$

and since λ_ε is contact we have in fact equality:

$$\ker d\lambda_\varepsilon = \pi^* \ker d\lambda_N \cap \ker dr \cap \ker\delta.$$

Constructing a Thom form for the bundle D_N. Let now F be any nonnegative compactly supported nonincreasing function on $[0, 1)$ with $F \equiv 1$ near $r = 0$. Then

$$T_F := -d(F(r)\delta)$$

is a Thom form on D_N. Set

$$\omega_F := K d\lambda_\varepsilon + T_F$$

on D_N, for some constant $K \geq K(F) > 0$ as in Lemma 2.19. Then Lemma 2.19 yields $\lambda_\varepsilon \wedge \omega_F^{n-1} > 0$. On the other hand, we have

$$\ker d\lambda_\varepsilon = \pi^* \ker d\lambda_N \cap \ker dr \cap \ker\delta \subset \ker\bigl(-F'(r)dr \wedge \delta + F(r)\pi^*\omega_N\bigr) = \ker T_F,$$

hence $\ker d\lambda_\varepsilon \subset \ker\omega_F$. Since λ_ε is contact, this shows that $(\omega_F, \lambda_\varepsilon)$ is a SHS on D_N.

Having defined the forms λ_ε, T_F and ω_F on D_N, we now finish the argument as follows. By the contact neighbourhood theorem (see e.g. [28, Theorem 2.5.15]) there exists a positive $r_1 < 1$ and an embedding

$$i : \{r \leq r_1\} \to M$$

such that $i_* \ker\lambda_\varepsilon = \xi$. Thus there exists a positive $r_0 < r_1$ and a contact form λ on M defining ξ that coincides with $i_*\lambda_\varepsilon$ on $i(\{r \leq r_0\}) \subset M$. We choose the function F in such a way that $\mathrm{supp}(F) \subset \{r < r_0\}$ and thus

$$\mathrm{supp}(T_F) \subset \{r < r_0\}.$$

The pushforward i_*T_F extends as zero to the complement of $i_*(\{r < r_0\})$. Finally we set

$$\omega := K d\lambda + i_*T_F$$

on M (with K as above). The pair (ω, λ) then constitutes the desired SHS and Proposition 2.18 is proved for rational cohomology classes.

The case $n = 2$**.** In the case $n = 2$ the argument can be adjusted to work for any $\eta \in H^2(M;\mathbb{R})$ as follows. Pick an integer basis $C_1,\dots,C_k$ of the free part of $H^2(M;\mathbb{Z})$ and write $\eta = \sum_{i=1}^k a_i C_i$ with coefficients $a_i \in \mathbb{R}$. Represent the Poincaré duals of $C_1,\dots,C_k$ in $H_1(M;\mathbb{Z})$ by disjoint embedded loops $N_1,\dots,N_k$ that are positively transverse to ξ, i.e. $\lambda(\dot N_i) > 0$. Pick a contact form λ defining ξ which is standard near the N_i and Thom forms T_i near N_i as above. Then

$$\omega := K d\lambda + \sum_{i=1}^k a_i T_i$$

with a suficiently large constant K yields the desired SHS (ω,λ). □

3. Stable Hamiltonian structures in dimension three

3.1. Integrability

When the underlying manifold M is 3-dimensional, there are a number of simplifications. First, ω being maximally nondegenerate means simply that it is nowhere zero. Second, the second condition in (1) simplifies to

$$d\lambda = f\omega, \tag{13}$$

where f is a smooth function on M. Since the Reeb vector field R of the SHS (ω,λ) preserves both ω and λ, it must also preserve the proportionality coefficient f between $d\lambda$ and ω. In other words, the function f is an integral of motion for the vector field R.

Remark 3.1. The function f already appeared in Remark 2.6(c) as the proportionality coefficient between R and its curl. To see that this is really the same function, write $\operatorname{curl} R = fR$ and compute, using the volume form $\mu = \lambda\wedge\omega$,

$$d\lambda = i_{\operatorname{curl} R}\mu = f\, i_R\mu = f\omega.$$

If f is constant we have either $f \equiv 0$ or $f \equiv c \neq 0$. In the first case the closed 1-form λ defines a taut foliation, in the second case λ defines a contact structure and $\omega = c^{-1} d\lambda$ is exact. Regions where f is nonconstant are foliated by the level sets of f. The level sets of f are 2-tori, so any connected region where $df \neq 0$ is simply an interval times T^2. This motivates the following

Definition 3.2. Let $\mathcal{L}$ be a stable Hamiltonian foliation on M. Let $I \subset \mathbb{R}$ be an interval (open, closed or half-open) and $U \subset M$ be a region diffeomorphic to $I \times T^2$ such that $\mathcal{L}$ is a subfoliation of the foliation by T^2's. Then U is called an *integrable region* for $\mathcal{L}$ (and for any SHS defining $\mathcal{L}$).

A Hamiltonian structure ω on $I\times T^2$ is called T^2*-invariant* if ω is invariant under the obvious action of T^2. Similarly a SHS (ω,λ) is called T^2*-invariant* if both forms ω and λ are invariant under the action of T^2. These are discussed in detail in Section 3.4. For now we state the following consequence of the Arnold–Liouville theorem [2].

Theorem 3.3. *Let (ω, λ) be a SHS with Reeb vector field R on an integrable region $I \times T^2$. Then there exists an orientation preserving diffeomorphism Φ of $I \times T^2$ preserving the tori $\{r\} \times T^2$ such that $\Phi^*\omega$, Φ^*R and the restrictions $\Phi^*\lambda|_{\{r\}\times T^2}$ are T^2-invariant. If in addition $f = d\lambda/\omega$ is constant on the tori $\{r\} \times T^2$, then $\Phi^*\lambda$ is T^2-invariant. Moreover, for any $k \geq 1$ the assignment $(\omega, \lambda) \mapsto \Phi$ is continuous with respect to the C^k-topology on the space of stable Hamiltonian structures and the C^{k-1}-topology on the group of diffeomorphisms.*

Proof. We denote by $\mathcal{E}$ the space of SHS having $I \times T^2$ as an integrable region, equipped with the C^k-topology. Consider some $(\omega, \lambda) \in \mathcal{E}$. We denote $U := I \times T^2$ and view the flow of the Reeb vector field R as a Hamiltonian system as follows. Consider the product $(-\varepsilon, \varepsilon) \times U$ with the symplectic form

$$\Omega = \omega + t d\lambda + dt \wedge \lambda. \tag{14}$$

Then R is the Hamiltonian vector field with respect to Ω of the Hamiltonian function given by the projection onto the first factor

$$H : (-\varepsilon, \varepsilon) \times U \to \mathbb{R}, \qquad H(t, x) := t.$$

Let p denote the projection to the first factor of $U := I \times T^2$. This gives us an integral of motion for this Hamiltonian vector field

$$F : (-\varepsilon, \varepsilon) \times U \to \mathbb{R}, \qquad F(t, x) := p(x).$$

Vanishing of the Poisson bracket $\{H, F\} = 0$ is automatic as for any integral of motion with the Hamiltonian, so the pair (H, F) gives us a complete system of integrals on $(-\varepsilon, \varepsilon) \times U$. Now we argue as in the proof of the Arnold–Liouville Theorem in [2]. Let X_H and X_F be the Hamiltonian vector fields on $(-\varepsilon, \varepsilon) \times U$ of H and F. Both vector fields X_H and X_F preserve H and F, thus the vector fileds are tangent to the tori $\{t, r\} \times T^2$. Vanishing of $\{H, F\}$ implies vanishing of the commutator $[X_H, X_F]$. Linear independence of dH and dF at every point implies that the vector fields X_H and X_F are pointwise linearly independent. Altogether, the flows of X_H and X_F define an $\mathbb{R}^2$ action on $(-\varepsilon, \varepsilon) \times I \times T^2$ whose orbits are the tori $\{(t, r)\} \times T^2$. For each torus $\{(t, r)\} \times T^2$ the corresponding stabilizer group $\Gamma^{(\omega,\lambda)}_{(t,r)}$ that leaves one (and then any) point of $\{(t, r)\} \times T^2$ fixed is a lattice in $\mathbb{R}^2$. Moreover, by the implicit function theorem, these lattices vary continuously with $((t, r), (\omega, \lambda))$, and smoothly with (t, r) for fixed (ω, λ). Let

$$\Phi(X_H, X_F, \cdot) : \mathbb{R}^2 \to \mathrm{Diff}_+((-\varepsilon, \varepsilon) \times U)$$

denote the 2-flow of the pair (X_H, X_F) of vector fields. Let $i : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\Gamma^{(\omega,\lambda)}_{(t,r)}$ be a linear orientation preserving identification varying continuously with $((t, r), (\omega, \lambda))$. Given $\tau \in T^2 = \mathbb{R}^2/\mathbb{Z}^2$ let $\tau^{(\omega,\lambda)}_{(t,r)}$ denote its image in $\mathbb{R}^2/\Gamma^{(\omega,\lambda)}_{(t,r)}$ under the identification i. This gives us a free T^2-action parametrized by $(\omega, \lambda) \in \mathcal{E}$,

$$\rho : T^2 \times \mathcal{E} \to \mathrm{Diff}_+((-\varepsilon, \varepsilon) \times U), \qquad (\tau, (\omega, \lambda)) \mapsto \Phi(X_H, X_F, \tau^{(\omega,\lambda)}_{(t,r)}),$$

transitive on every torus. The parametrization by $(\omega, \lambda) \in \mathcal{E}$ is continuous because the triple $(X_H, X_F, \tau_{(t,r)}^{(\omega,\lambda)})$ depends continuously on $((t,r),(\omega,\lambda))$ and Φ is a continuous function of its arguments. Here the space of vector fields and the space of diffeomorphisms are given the C^{k-1}-topologies.

Fix a smooth section s (independent of (ω, λ)) of the trivial fibration $(-\varepsilon, \varepsilon) \times I \times T^2 \to T^2$. We use the action ρ and the section s to define an orientation preserving self-diffeomorphism of $(-\varepsilon, \varepsilon) \times U$ by the formula

$$\Psi(t,r,\theta,\phi) := \rho(\theta,\phi)(t,r,s(t,r)).$$

Note that the first two coordinates of $\Psi(t,r,\theta,\phi)$ are (t,r). Pulling back the action ρ we get the action $\Psi^{-1}\rho\Psi$ which is just the standard action of T^2 on $(-\varepsilon, \varepsilon) \times U$ by shifts in θ and ϕ. So given a differential-geometric object on $(-\varepsilon, \varepsilon) \times U$ invariant under the action of ρ, its pullback under Ψ is invariant under the standard T^2-action (below "T^2-invariant" always means invariance under the standard T^2-action).

By construction of ρ the vector fields X_H and X_F are ρ-invariant, hence $\Psi^* X_H$ and $\Psi^* X_F$ are T^2-invariant. In other words, $\Psi^* X_H$ and $\Psi^* X_F$ are linear on the tori $\{t, r\} \times T^2$. The symplectic form Ω is invariant under the flows of X_H and X_F and thus under ρ, hence the pullback $\Psi^*\Omega$ is T^2-invariant. We view $\Phi := \Psi|_{\{0\}\times U}$ as a diffeomorphism of U. Formula (14) shows that $\Psi^*\Omega|_{\{0\}\times U} = \Phi^*\omega$ and $X_H|_{\{t=0\}} = R$. This shows T^2-invariance of $\Phi^*\omega$ and of $\Phi^* R$. Set $X := X_F|_{\{t=0\}}$ and note that $\Phi^* X$ is T^2-invariant. We contract formula (14) with X_F at $t = 0$ to get

$$\iota_X \Omega = \iota_X \omega - \lambda(X) dt = -dr.$$

This shows that $\lambda(X) = 0$ and $\iota_X \omega = -dr$. We view the last two equalities as living on U. Pulling back with Φ yields

$$(\Phi^*\lambda)(\Phi^* X) = 0, \qquad \iota_{\Phi^* X} \Phi^*\omega = dr.$$

On the other hand, we have $\Phi^*\lambda(\Phi^* R) = \lambda(R) = 1$, so the restriction of $\Phi^*\lambda$ to a torus $\{r\} \times T^2$ is the unique 1-form which takes value 0 on $\Phi^* X$ and 1 on $\Phi^* R$. Since the vector fields $\Phi^* X$ and $\Phi^* R$ are T^2-invariant, this shows T^2-invariance of $\Phi^*\lambda|_{\{r\}\times T^2}$.

To analyze invariance of $\Phi^*\lambda$, we work in standard coordinates (r, θ, ϕ) on $(-\varepsilon, \varepsilon) \times T^2$ and rename $\Phi^*\omega$, $\Phi^* R$, $\Phi^* X$ and $\Phi^*\lambda$ back to ω, R, X and λ respectively. The invariance properties allow us to write ω and λ uniquely as

$$\omega = dr \wedge (k_1(r)d\theta + k_2(r)d\phi), \qquad \lambda = g_1(r)d\theta + g_2(r)d\phi + g_3(r,\theta,\phi)dr$$

(cf. the proof of Lemma 3.9 below). Here k_1, k_2, g_1, g_2 are functions of r only, but g_3 will in general depend on (r, θ, ϕ)—see Remark 3.4 below. However, if we in addition assume that $f = d\lambda/\omega$ is constant on the tori $r \times T^2$, then we can deduce more. Indeed, the relation $d\lambda = f\omega$ writes out as

$$g_1' - \frac{\partial g_3}{\partial \theta} = f k_1, \qquad g_2' - \frac{\partial g_3}{\partial \phi} = f k_2.$$

Since the functions g_1, g_2, k_1, k_2, f are T^2-invariant, so are $\partial g_3/\partial\theta$ and $\partial g_3/\partial\phi$, which therefore vanish by periodicity of g_3. So in this case g_3, and thus λ, is T^2-invariant. □

Remark 3.4. In general, we cannot achieve in Theorem 3.3 that $\Phi^*\lambda$ is T^2-invariant. To see this, consider

$$\lambda = r^2 d\theta + (1 - r^2)d\phi$$

on $(0, 1) \times T^2$ and

$$\omega = d\lambda = 2rdr \wedge (d\theta - d\phi).$$

Let $\eta := 2\sin(\theta - \phi)rdr$. Note that

$$d\eta = 2\cos(\theta - \phi)(d\theta - d\phi)r \wedge dr = \cos(\theta - \phi)\omega.$$

Thus for small $\varepsilon > 0$ the form λ_ε stabilizes ω and

$$f_\varepsilon = d\lambda_\varepsilon/\omega = 1 + \varepsilon\cos(\theta - \phi).$$

This function is not constant on the tori $\{r\} \times T^2$. So for any diffeomorphism Φ of $(0, 1) \times T^2$ preserving the tori $\{r\} \times T^2$ we see that $\Phi^* d\lambda_\varepsilon/\Phi^*\omega = \Phi^* f_\varepsilon$ is not constant on the tori $\{r\} \times T^2$. Thus the form $\Phi^*\lambda_\varepsilon$ cannot be T^2-invariant (assuming that $\Phi^*\omega$ is T^2-invariant).

3.2. *Slope functions*

In standard coordinates (r, θ, ϕ) on an integrable region $I \times T^2$, the pullback $\Phi^* R$ of the Reeb vector field in Theorem 3.3 has the form

$$\Phi^* R = w_1(r)\partial_\theta + w_2(r)\partial_\phi.$$

In particular, it is T^2-invariant, linear on each invariant torus $\{r\} \times T^2$, and it preserves the standard area form $d\theta \wedge d\phi$. In this subsection we associate a "slope function" to such a vector field.

Consider a nowhere zero vector field X on a 2-torus T preserving an area form σ. By Cartan's formula,

$$\beta := i_X\sigma$$

is a closed 1-form on T.

Lemma 3.5. (a) *There exists a closed 1-form α on T with $\alpha(X) > 0$.*
(b) *There exist coordinates (θ, ϕ) on $T \cong T^2$ in which X defines a linear foliation. In particular, the flow lines of X are either all dense or all closed.*
(c) *Upon replacing σ by a different X-invariant area form of the same sign, the cohomology class $[i_X\sigma] \in H^1(T; \mathbb{R})$ gets multiplied by a positive constant.*
(d) *For any vector field $\tilde{X}$ defining the same oriented 1-foliation as X and any $\tilde{X}$-invariant area form $\tilde{\sigma}$ of the same sign as σ, the cohomology class $[i_{\tilde{X}}\tilde{\sigma}] \in H^1(T; \mathbb{R})$ is a positive multiple of $[i_X\sigma] \in H^1(T; \mathbb{R})$.*

Proof. (a) First note that β is *transitive*, i.e. any two points in T can be connected by a path γ with $\beta(\dot{\gamma}) > 0$. (This was remarked by Calabi [11], and it can be proved by combining two results in [27]: By [27, Theorem 9.6], either all leaves of the foliation $\ker\beta$ are noncompact, or all leaves are compact. In the first case transitivity follows from [27, Theorem 9.13]. In the second case the leaves are the fibres of a fibration $T \to S^1$, and sections of this fibration provide paths with $\beta(\dot{\gamma}) > 0$ between any two points.) By a theorem of Calabi [11], transitivity implies the existence of a metric on T for which β is harmonic. Then $\alpha := -*\beta$, where $*$ is the Hodge star operator of the metric, has the desired properties.

(b) Let α be a 1-form as in (a). The vector field $\bar{X} := \alpha(X)^{-1}X$ satisfies $\beta(\bar{X}) = 0$ and $\alpha(\bar{X}) = 1$. Define a vector field Y on T by $\alpha(Y) = 0$ and $\beta(Y) = 1$. Then $\bar{X}$ and Y are commuting vector fields which are linearly independent at every point, so they are linear in suitable angular coordinates (θ, ϕ) on $T \cong T^2$.

(c) If $f\sigma$ is a different X-invariant area form of the same sign, f is a positive function on T invariant under X. If all flow lines of X are dense this implies that f is constant and (c) follows. Otherwise we may choose coordinates (θ, ϕ) such that X points in the direction ∂_θ; then f is a function of ϕ only and β is a θ-independent multiple of $d\phi$, so $f\beta$ is cohomologous to a positive multiple of β.

(d) Let f be any positive smooth function on T^2 and consider the vector field $\tilde{X} := fX$. Assume that $\tilde{X}$ preserves an area form $\tilde{\sigma}$ of the same sign as σ. Then $i_{\tilde{X}}\tilde{\sigma} = i_X f\tilde{\sigma}$ is closed. In particular, $f\tilde{\sigma}$ is an X-invariant area form of the same sign as σ and the result follows from part (c). □

Let X and σ be as above. Note that $i_X\sigma$ is nowhere vanishing, so in particular it is not exact (if $i_X\sigma = d\chi$ for a function $\chi \in C^\infty(T)$, the maximum of χ would give a zero of $i_X\sigma$). We say that a and b in $H^1(T;\mathbb{R})$ are *equivalent* and write $a \sim b$ if one is the positive multiple of the other. The quotient

$$PH^1(T;\mathbb{R}) := H^1(T;\mathbb{R}) \setminus \{0\}/\sim$$

is called the *projectivization*. The descendant of $a \neq 0 \in H^1(T;\mathbb{R})$ in the projectivization will be denoted by Pa. Since the cohomology class $[i_X\sigma]$ is not zero it descends to the projectivization to give a quantity

$$k_X := -P[i_X\sigma] \in PH^1(T;\mathbb{R})$$

called the *slope* of X. Note that by Lemma 3.5(d), the slope k_X depends only on the oriented foliation defined by X. The integral cohomology $H^1(T,\mathbb{Z})$ sits inside $H^1(T,\mathbb{R})$ as an integer lattice; we call images of nonzero integral cohomology classes in the projectivization *rational* points. By Lemma 3.5(b) we have the following dichotomy: Either k_X is rational and all the orbits of X are closed, or k_X is irrational and all the orbits of X are dense.

There is a more concrete coordinate way to describe the first cohomology of T and its projectivization, which will enable us to write a formula for k_X in terms of X. Namely, let $T \cong \mathbb{R}^2/\mathbb{Z}^2$ be coordinatized by (θ, ϕ) and oriented by $d\theta \wedge d\phi$. The form $i_X\sigma$ can be written as

$$i_X\sigma = a_1(\theta, \phi)d\theta + a_2(\theta, \phi)d\phi, \qquad \frac{\partial a_1}{\partial \phi} = \frac{\partial a_2}{\partial \theta}.$$

In the identification $H^1(T; \mathbb{R}) \cong \mathbb{R}^2$ via the basis $[d\theta]$, $[d\phi]$ its cohomology class is given by

$$[i_X\sigma] = \left(\int_{T^2} a_1\, d\theta\, d\phi, \int_{T^2} a_2\, d\theta\, d\phi\right)$$

and the slope by

$$k_X = -P[i_X\sigma] = -|i_X\sigma|^{-1}\left(\int_{T^2} a_1\, d\theta\, d\phi, \int_{T^2} a_2\, d\theta\, d\phi\right), \tag{15}$$

where

$$|i_X\sigma| := \left(\left(\int_{T^2} a_1\, d\theta\, d\phi\right)^2 + \left(\int_{T^2} a_2\, d\theta\, d\phi\right)^2\right)^{1/2}.$$

Consider now $I \times T$ for an interval $I \subset \mathbb{R}$, open or closed or half-open. Let X be a vector field on $I \times T$ tangent to the tori $\{r\} \times T$ and preserving a volume form V on $I \times T$. The vector field X can also be viewed as a family $\{X_r\}_{r\in I}$ of vector fields on T parametrized by I. Similarly, we can write the volume form as

$$V = dr \wedge \sigma_r,$$

where r is the coordinate on I and σ_r is a smooth family of area forms on T preserved by X_r. This gives rise to a smooth family of 1-forms $\{i_{X_r}\sigma_r\}_{r\in I}$ as above, and formula (15) applied to X_r in place of X shows that k_{X_r} depends smoothly on r (since a_1 and a_2 depend smoothly on r).

Definition 3.6. The smooth function

$$k : I \to S^1, \qquad r \mapsto k_{X_r},$$

is called the *slope function* of the oriented foliation $\mathcal{L}$ defined by X.

We note one useful simplification of (15) for future use. Indeed, by Lemma 3.5(b) we may assume that the vector field X is linear in coordinates (θ, ϕ), say $X = b_1\partial_\theta + b_2\partial_\phi$ for constants $b_1, b_2 \in \mathbb{R}$. By Lemma 3.5(d) we may choose $\sigma := d\theta \wedge d\phi$, then $a_1 = -b_2$ and $a_2 = b_1$. So (15) simplifies to

$$k_X = (b_2, -b_1)/\sqrt{b_1^2 + b_2^2}. \tag{16}$$

The situation considered in this subsection arises in integrable 4-dimensional Hamiltonian systems as follows. Consider a symplectic 4-manifold (W, ω) and two Poisson commuting functions H, F on W whose differentials are linearly independent at every point. Then each compact connected component of a level set of (H, F) is a 2-torus T. Define a 2-form σ on T by $\sigma(X_H, X_F) := 1$. A short computation using $[X_H, X_F] = 0$

shows that σ is preserved by the flows of X_H and X_F. So we can consider the slope function of the foliation defined by X_H.

For a stable Hamiltonian structure (ω, λ) on a 3-manifold M this situation arises in regions where the proportionality factor $f = d\lambda/\omega$ is nonconstant. The unique vector field $X_f \in \ker\lambda$ defined by $i_{X_f}\omega + df = 0$ commutes with the Reeb vector field R and is linearly independent of R where $df \neq 0$. So $\sigma(R, X_f) := 1$ defines an invariant area form on level sets of f and we can consider the slope function of the foliation defined by the Reeb vector field R. Of course, this description is related to the 4-dimensional picture by symplectization, as described in Section 3.1.

3.3. *Persistence of invariant tori*

The following theorem states that integrable regions persist under C^2-small perturbations of a SHS. Moreover, if the slope function is nonconstant then the perturbed SHS has rational as well as irrational invariant tori, so in particular it is not Morse. In Section 3.9 we will show that any SHS is homotopic to one satisfying the hypotheses of this theorem.

Theorem 3.7. *Let (ω_0, λ_0) be a SHS on M and assume that it has an integrable region $K_0 \cong [a, b] \times T^2 \subset M$ on which (ω_0, λ_0) is T^2-invariant and the proportionality coefficient $f_0 := d\lambda_0/\omega_0 : K_0 \to [a, b]$ is the projection onto the first factor. Denote by μ the Lebesgue measure on $[a, b] \times T^2$. Then:*

(a) *There exist constants $C, \delta_0 > 0$ such that for any $\delta < \delta_0$, any SHS (ω, λ) which is δ-close to (ω_0, λ_0) in the C^2-metric has an integrable region K with $\mu(K) \geq \mu(K_0) - C\delta$.*
(b) *If (ω, λ) as in* (a) *is actually C^k-close to (ω_0, λ_0) for some $k \geq 4$, then there exists a diffeomorphism Ψ of M, C^{k-3}-close (and thus isotopic) to the identity, such that $\Psi([a, b]\times T^2) = K$ and the pullback SHS $(\Psi^*\omega, \Psi^*\lambda)$ is T^2-invariant on $[a, b]\times T^2$.*
(c) *If in addition the slope function k_0 of $\mathcal{L}_0$ on K_0 is not constant, then $\delta_0 > 0$ can be chosen such that for any SHS (ω, λ) as in* (a) *the kernel foliation $\mathcal{L}$ contains rational as well as irrational invariant tori.*

Proof. Consider (ω, λ) as in (a). The hypothesis implies that the function $f := d\lambda/\omega : K_0 \to \mathbb{R}$ is C^1-close to f_0, more precisely, $\|f - f_0\|_{C^1} < C\delta$ for some constant C independent of δ. To simplify notation, we will replace $C\delta$ by δ and drop C in the following. Set $I := [a, b]$ and

$$I_\eta := [a + \eta, b - \eta], \qquad \eta > 0.$$

For the first step we will only need that $\|f - f_0\|_{C^0} < \delta$. This implies

$$f^{-1}(I_\delta) \subset \mathrm{int}(I \times T^2)$$

and

$$I_{3\delta} \subset f(I_{2\delta} \times T^2) \subset I_\delta.$$

Combining these inclusions, we obtain

$$f^{-1}(f(I_{2\delta} \times T^2)) \subset I \times T^2,$$

i.e. if a level set of f meets $I_{2\delta} \times T^2$, then it is contained in the interior of $I \times T^2$. If δ is small enough, then C^1-closeness of f and f_0 implies that all points in $I \times T^2$ are regular for f. Combining this with the above inclusion gives us an integrable region

$$K := f^{-1}(f(I_{2\delta} \times T^2)) \supset I_{2\delta} \times T^2$$

for $\mathcal{L}$ containing $I_{2\delta} \times T^2$. This proves part (a).

Set $J := f(I_{2\delta} \times T^2)$ and define

$$\Phi : K \to J \times T^2, \qquad (r, x) \mapsto (f(r, x), x).$$

This is a diffeomorphism C^1-close to the identity because the function f is C^1-close to f_0, the projection onto the first factor.

For part (b), assume that (ω, λ) is in fact C^k-close to (ω_0, λ_0) for some $k \geq 4$. Then Φ is C^{k-1}-close to the identity. We extend $f|_K$ to a submersion $\tilde{f} : [a, b] \times T^2 \to [a, b]$ which is C^{k-1}-close to f_0 and coincides with f_0 near $\partial[a, b]$. Using the last displayed formula with $\tilde{f}$ in place of f we extend Φ to a self-diffeomorphism $\tilde{\Phi}$ of $[a, b] \times T^2$ which is C^{k-1}-close to the identity and equals the identity near the boundary. We extend $\tilde{\Phi}$ to a self-diffeomorphism (still called $\tilde{\Phi}$) of M in the obvious way. The pushforward SHS $\Phi_*(\omega, \lambda)$ is then C^{k-2}-close to (ω_0, λ_0), and $J \times T^2$ is an integrable region for $\Phi_*(\omega, \lambda)$. Thus by Theorem 3.3 there exists a diffeomorphism Θ of $J \times T^2$, C^{k-3}-close to the indentity, such that $\Theta_*\Phi_*(\omega, \lambda)$ is T^2-invariant. The diffeomorphism Θ extends to a diffeomorphism of $[a, b] \times T^2$ which is C^{k-3}-close to the identity and equals the identity near the boundary, and thus to a diffeomorphism of M (called $\tilde{\Theta}$) in the obvious way. Let $\Gamma : J \times T^2 \to [a, b] \times T^2$ be the mapping which is the identity on the second factor and a linear isomorphism on the first factor. Since the integrable region $[a, b] \times T^2$ can be slightly extended, the diffeomorphism Γ can be extended to a self-diffeomorphism $\tilde{\Gamma}$ of M, C^{k-3}-close to identity. Now the diffeomorphism $\Psi := (\tilde{\Gamma} \circ \tilde{\Theta} \circ \tilde{\Phi})^{-1}$ has the required properties and part (b) follows.

For part (c), assume that the slope function k_0 of $\mathcal{L}_0$ is not constant on I. By taking δ_0 small enough, we can ensure that k_0 is not constant on $I_{3\delta}$. Note that $I_{3\delta} \subset J = f(I_{2\delta} \times T^2)$ by the proof of part (a). In particular, k_0 is not constant on J. The pushforward foliation $\Phi_*\mathcal{L}$ on $J \times T^2$ is the kernel foliation of $\Phi_*(\omega, \lambda)$ and thus tangent to the tori $\{r\} \times T^2$. Let $k \colon J \to S^1$ be the slope function of $\Phi_*\mathcal{L}$ on $J \times T^2$. Since the diffeomorphism Φ is C^1-close to the identity and the foliation $\mathcal{L}$ on K is C^0-close to $\mathcal{L}_0$, we see that the foliation $\Phi_*\mathcal{L}$ is C^0-close to $\mathcal{L}_0$, so the slope function k is C^0-close to the restriction of the slope function k_0 to J, which is nonconstant. Thus k is not constant and so attains rational and irrational values, therefore the foliation $\Phi_\star\mathcal{L}$ and thus $\mathcal{L}$ has rational as well as irrational invariant tori. □

As a first application of Theorem 3.7, we illustrate the difference between a convergent sequence of stabilizable Hamiltonian structures and a convergent sequence of stable Hamiltonian structures.

Theorem 3.8. *There exists a stable Hamiltonian structure (ω_0, λ_0) on $\mathbb{R}P^3$ and a sequence of stabilizable Hamiltonian structures ω_n which C^2-converges to ω_0, but such that there is no sequence of 1-forms λ_n stabilizing ω_n which C^2-converges to λ_0.*

Proof. We view $\mathbb{R}P^3$ as the unit cotangent bundle for the standard round metric g_0 on S^2. The Liouville form α_0 defines a contact form on $\mathbb{R}P^3$ whose Reeb flow is the geodesic flow for g_0. This is a periodic flow, so it gives $\mathbb{R}P^3$ the structure of a principal S^1-bundle over a closed surface $\Sigma \cong S^2$ (the space of oriented great circles on S^2). Let $\pi : \mathbb{R}P^3 \to \Sigma$ denote the corresponding bundle projection. Note that α_0 defines a connection 1-form on this circle bundle and $d\alpha_0 = \pi^*\sigma$ for an area form σ on Σ. We choose a 1-form ρ on Σ such that $d\rho$ is somewhere zero and somewhere nonzero. Then $\lambda_0 := \alpha_0 + \pi^*\rho$ defines another connection form on the circle bundle, so λ_0 stabilizes the HS $\omega_0 := d\alpha_0$. Since $d\lambda_0 = \pi^*(\sigma + d\rho)$, the proportionality coefficient $f_0 := d\lambda_0/\omega_0$ is given by $(\sigma + d\rho)/\sigma$, which is nonconstant because $d\rho$ is somewhere zero and somewhere nonzero. This produces an integrable region K_0 for the kernel foliation $\mathcal{L}_0$ of ω_0 meeting the conditions of Theorem 3.7(a). It follows that for any SHS (ω, λ) sufficiently C^2-close to (ω_0, λ_0) there persist invariant tori, and moreover, they fill a region of measure approximately $\mu(K_0)$.

Now we invoke the following theorem of Katok [36]: For each $k \geq 2$ and $\varepsilon, \delta > 0$ there exists a Finsler metric g on S^2, δ-close to g_0 in the C^k-norm, such that the geodesic flow of g is ergodic on an open invariant region U of the unit cotangent bundle $S_g^*S^2$ whose complement $S_g^*S^2 \setminus U$ has measure $< \varepsilon$. This implies that the invariant tori for the geodesic flow of g constitute a set of measure at most ε.

Now pick a sequence g_n of such Finsler metrics which C^3-converges to g_0 and such that the set of invariant tori for the geodesic flow of g_n has measure at most $\mu(K_0)/2$ for all n. Let α_n be the contact form on $\mathbb{R}P^3$ obtained by restricting the Liouville form to the unit cotangent bundle $S_{g_n}^*S^2$. Then $\omega_n := d\alpha_n$ is a sequence of stabilizable (by α_n) Hamiltonian structures which C^2-converges to $\omega_0 = d\alpha_0$. This sequence cannot be stabilized by a family of 1-forms λ_n which C^2-converges to λ_0 because, by the discussion above, this would imply that for large n the set of invariant tori for ω_n would have measure greater than $\mu(K_0)/2$, contradicting the choice of the g_n. □

3.4. T^2-invariant Hamiltonian structures on $I \times T^2$

Let I be an interval in $\mathbb{R}$ (open or closed or half-open). Consider $I \times T^2$ with coordinates (r, θ, ϕ) and the T^2-action by shift in (θ, ϕ). We orient $I \times T^2$ by the volume form $dr \wedge d\theta \wedge d\phi$.

Hamiltonian structures. For a path $h = (h_1, h_2) : I \to \mathbb{R}^2$ consider the T^2-invariant 1-form

$$\alpha_h := h_1(r)d\theta + h_2(r)d\phi$$

and the 2-form

$$\omega_h := d\alpha_h = h_1'(r)dr \wedge d\theta + h_2'(r)dr \wedge d\phi.$$

This defines a HS iff $h'(r) \neq 0$ for all $r \in I$, and in that case its oriented kernel foliation is

$$\mathcal{L}_h = \mathrm{Span}_{\mathbb{R}}\{-h_2'(r)\partial_\theta + h_1'(r)\partial_\phi\}. \tag{17}$$

Note that α_h is a positive contact form if and only if h always turns clockwise. To see this, we view $\mathbb{R}^2$ as $\mathbb{C}$. Then the contact condition $\alpha_h \wedge d\alpha_h > 0$ reads $\langle h, ih'\rangle > 0$, where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product on $\mathbb{R}^2$. Writing $h = \rho(r)e^{i\sigma(r)}$ we find $\langle h, ih'\rangle = -|h|^2\sigma'(r)$, so the contact condition is equivalent to $\sigma' < 0$.

Slope functions. We define the slope function of the HS ω_h on $I \times T^2$ as the slope function of its oriented kernel foliation $\ker\omega_h = \mathcal{L}_h$ given by (17). In terms of the function h, formula (16) (with $b_1 = -h_2'$ and $b_2 = h_1'$) translates to

$$k(r) = h'(r)/|h'(r)|.$$

Note that the slope function and all its properties are intrinsic to the foliation $\mathcal{L}_h$ and do not depend on the defining HS.

Winding numbers. We define the *winding number* of h by $w(h) := \sigma(b) - \sigma(a) \in \mathbb{R}$, where $h'(r)/|h'(r)| = e^{i\sigma(r)}$ for a function $\sigma : I = [a,b] \to \mathbb{R}$. Note that for immersions h_0, h_1 which agree near ∂I we have $w(h_0) - w(h_1) \in \mathbb{Z}$, and h_0, h_1 are regularly homotopic rel ∂I iff $w(h_0) = w(h_1)$. For a T^2-invariant HS ω defining the foliation $\mathcal{L}_h$ we define its winding number by $w(\omega) := w(h)$.

Stabilizing 1-forms. For another T^2-invariant 1-form

$$\lambda_g := g_1(r)d\theta + g_2(r)d\phi$$

we have

$$d\lambda_g = g_1'(r)dr \wedge d\theta + g_2'(r)dr \wedge d\phi, \qquad \lambda_g \wedge \omega_h = (h_1'g_2 - h_2'g_1)dr \wedge d\theta \wedge d\phi.$$

So λ_g stabilizes ω_h iff

$$g_1'h_2' - g_2'h_1' = 0, \qquad h_1'g_2 - h_2'g_1 > 0.$$

Viewing $\mathbb{R}^2$ as $\mathbb{C}$, this can also be written as

$$\langle g', ih'\rangle = 0, \qquad \langle g, ih'\rangle > 0. \tag{18}$$

Lemma 3.9. *Fix an interval I and a relatively compact subinterval $J \subset\subset I$.*

(a) *Any T^2-invariant exact 2-form ω on $I \times T^2$ can be written as $\omega = \omega_h$ for a function $h : I \to \mathbb{C}$ which is unique up to adding a constant.*
(b) *Any 1-form α with $d\alpha = \omega$ T^2-invariant can be modified rel boundary to $\tilde\alpha$ with $d\tilde\alpha = \omega$ satisfying $\tilde\alpha|_J = \alpha_h$ for a function $h : J \to \mathbb{R}$. Moreover, if α is C^k-small we can choose $\tilde\alpha$ C^k-small as well.*
(c) *Any T^2-invariant 1-form λ stabilizing ω_h is homotopic rel boundary through T^2-invariant stabilizing 1-forms to $\tilde\lambda$ satisfying $\tilde\lambda|_J = \lambda_g$ for a function $g : J \to \mathbb{R}$.*

Proof. (a) Any T^2-invariant 2-form ω can be written as

$$\omega = k_1(r)dr \wedge d\theta + k_2(r)dr \wedge d\phi + k_3(r)d\phi \wedge d\theta$$

for smooth functions $k_1, k_2, k_3 : I \to \mathbb{R}$. If ω is exact, then $k_3 = 0$ and integrating k_1 and k_2 we see that there exists a function $h : I \to \mathbb{R}^2$, unique up to adding a constant, such that $\omega = d\alpha_h$.

(b) Consider a 1-form α with $d\alpha = \omega$ T^2-invariant. Denote by $\bar{\alpha}$ the T^2-invariant 1-form on $I \times T^2$ obtained by averaging α. Then $d\bar{\alpha} = \omega = d\alpha$, so $\alpha - \bar{\alpha}$ is closed. Since ω vanishes on each torus $\{r\} \times T^2$, the restriction $\alpha|_{\{r\}\times T^2}$ is closed. Its de Rham cohomology class is preserved under averaging, so we have $[\bar{\alpha}|_{\{r\}\times T^2}] = [\alpha|_{\{r\}\times T^2}] \in H^1(\{r\} \times T^2; \mathbb{R})$. This shows that $[\bar{\alpha} - \alpha] = 0 \in H^1(I \times T^2; \mathbb{R})$, so we can write $\bar{\alpha} - \alpha = df$ for a function $f : I \times T^2 \to \mathbb{R}$. Pick a cutoff function $\rho : I \to [0, 1]$ which equals 0 near ∂I and 1 on a neighbourhood K of $J \subset\subset I$. Then $\hat{\alpha} := \alpha + d(\rho(r) f)$ satisfies $d\hat{\alpha} = \omega$, agrees with α near $\partial I \times T^2$ and is T^2-invariant on $K \times T^2$. Thus $\hat{\alpha}|_{K\times T^2}$ can be written in the form

$$h_1(r)d\theta + h_2(r)d\phi + h_3(r)dr = \alpha_h + h_3(r)dr$$

for smooth functions $h_1, h_2, h_3 : K \to \mathbb{R}$. Pick a function $\sigma : I \to [0, 1]$ which equals 0 outside K and 1 on J. Then $\tilde{\alpha} := \hat{\alpha} - \sigma(r)h_3(r)dr$ is the desired 1-form. The construction shows that $\tilde{\alpha}$ will be C^k-small if α is.

(c) Any T^2-invariant 1-form λ on $I \times T^2$ can be written in the form

$$\lambda = g_1(r)d\theta + g_2(r)d\phi + g_3(r)dr = \lambda_g + g_3(r)dr$$

for smooth functions $g_1, g_2, g_3 : I \to \mathbb{R}$. It stabilizes ω_h if and only if $g = (g_1, g_2)$ satisfies (18). Pick a function $\rho : I \to [0, 1]$ which equals 1 near ∂I and 0 on J. Then $\tilde{\lambda} := \lambda_g + \rho(r)g_3(r)dr$ is the desired 1-form and $(1-t)\lambda + t\tilde{\lambda}$ the desired homotopy. □

In the remainder of this section we investigate the question of stabilizability: Given an immersion $h : I \to \mathbb{C}$, does there exist a function $g : I \to \mathbb{C}$ satisfying (18) with prescribed values near ∂I?

It turns out that the answer depends on the slope function $k = h'/|h'| : I \to S^1$.

Stabilization for constant slope. Let us first consider the case of *constant slope*, i.e. with constant slope function $k = h'/|h'| \in S^1$. In this case (18) is equivalent to

$$h'/|h'| \equiv k, \qquad \langle g, ik\rangle \equiv c > 0 \tag{19}$$

for a constant $c > 0$. Thus h moves along a straight line in direction k, and the function g can move freely on the straight line $\langle g, ik\rangle \equiv c$. Note that convex combinations of pairs satisfying (19) with the same slope k (but possibly different constants c) again satisfy (19). So we obtain the following answer to the stabilizability question:

Let $h : [0, 1] \to \mathbb{C}$ be an immersion with constant slope k and g_0, g_1 be functions near 0, 1 satisfying (19) with constants c_0, c_1. Then there exist a function $g : I \to \mathbb{C}$ satisfying (18) which agrees with g_0 near 0 and g_1 near 1 if and only if $c_0 = c_1$.

More generally, we can show a stabilization result for constant slopes near the boundary. Fix $0 < \delta < \varepsilon$ and $\bar h, \bar g : [0, \varepsilon] \cup [1 - \varepsilon, 1] \to \mathbb{C}$ satisfying

$$\begin{aligned} \bar h'/|\bar h'| &\equiv k_- \text{ on } [0, \varepsilon], \qquad & \bar h'/|\bar h'| &\equiv k_+ \text{ on } [1 - \varepsilon, 1], \\ \langle \bar g, ik_-\rangle &\equiv c_- \text{ on } [0, \varepsilon], \qquad & \langle \bar g, ik_+\rangle &\equiv c_+ \text{ on } [1 - \varepsilon, 1], \end{aligned}$$

for unit vectors $k_\pm \in \mathbb{C}$ and constants $c_\pm > 0$. The preceding discussion shows that we cannot hope for a general stabilization result unless $c_- = c_+$, which turns out to be also sufficient:

Lemma 3.10. *Suppose $c_- = c_+$. Then there exists a continuous map that assigns to every immersion $h : [0, 1] \to \mathbb{C}$ with $h = \bar h$ on $[0, \varepsilon] \cup [1-\varepsilon, 1]$ a function $g : [0, 1] \to \mathbb{C}$ satisfying (18) and $g = \bar g$ on $[0, \delta] \cup [1 - \delta, 1]$.*

Proof. Note that a special solution to conditions (18) is given by the formula

$$g(r) := ich'(r)/|h'(r)| \tag{20}$$

for some constant $c > 0$. In other words, each immersion h can be stabilized by $g : I \to \mathbb{C}$ via (20). To apply this, pick a function $\rho : [0, 1] \to [0, 1]$ which equals 0 on $[0, \delta] \cup [1 - \delta, 1]$ and 1 on $[\varepsilon, 1 - \varepsilon]$. With $c := c_- = c_+$, we define g by $(1 - \rho)\bar g + \rho ci \bar h'/|\bar h'|$ on $[0, \varepsilon] \cup [1 - \varepsilon, 1]$ and by (20) on $[\varepsilon, 1 - \varepsilon]$. □

Remark 3.11. The 1-form λ_g corresponding to g defined by (20) is the Wadsley form associated to the (suitably scaled) flat metric on $I \times T^2$.

The following corollary spells out the special case $\bar g = \bar h$.

Corollary 3.12. *Fix $0 < \delta < \varepsilon$, $c > 0$ and unit vectors $k_\pm \in \mathbb{C}$. Suppose that $\bar h : [0, \varepsilon] \cup [1-\varepsilon, 1] \to \mathbb{C}$ satisfies $\bar h'/|\bar h'| \equiv k_-$ and $\langle \bar h, ik_-\rangle \equiv c$ on $[0, \varepsilon]$, and $\bar h'/|\bar h'| \equiv k_+$ and $\langle \bar h, ik_+\rangle \equiv c$ on $[1 - \varepsilon, 1]$. Then there exists a smooth map that assigns to every immersion $h : [0, 1] \to \mathbb{C}$ with $h = \bar h$ on $[0, \varepsilon] \cup [1 - \varepsilon, 1]$ a function $g : [0, 1] \to \mathbb{C}$ satisfying (18) and $g = \bar h$ on $[0, \delta] \cup [1 - \delta, 1]$.*

One useful example to which the corollary applies is $\bar h(r) = \pm(r^2, 1 - r^2)$.

Remark 3.13. Moving the function g on the line we can achieve that $g \equiv$ const on some subinterval $J \subset I$. Then we can perturb h slightly on J, so that (h, g) still satisfies (18) and the slope $h'/|h'|$ is nonconstant on J.

Stabilization for nonconstant slope. We have seen that for constant slope there is an obstruction to stabilizability. On the other hand, we will show that stabilization is always possible if the slope function is nonconstant:

Proposition 3.14. *Let $h : [0, 1] \to \mathbb{C}$ be an immersion such that $h'/|h'|$ is not constant on $[\varepsilon, 1 - \varepsilon]$. Let $\bar g : [0, \varepsilon] \cup [1 - \varepsilon] \to \mathbb{C}$ be given such that $(h, \bar g)$ satisfies (18) on $[0, \varepsilon] \cup [1 - \varepsilon, 1]$. Then there exists a function $g : [0, 1] \to \mathbb{C}$ which agrees with $\bar g$ on $[0, \varepsilon] \cup [1 - \varepsilon, 1]$ such that (h, g) satisfies (18).*

The proof is based on two lemmata. The first lemma gives a criterion when a T^2-invariant Hamiltonian perturbation of a T^2-invariant SHS is again stabilizable. It will play a crucial role in Section 4.

Lemma 3.15. (a) *Let* $\bar{h}, \bar{g} : [0,1] \to \mathbb{C}$ *satisfy* (18) *and suppose that* $\bar{h}'/|\bar{h}'|$ *is not constant on* $[\varepsilon, 1-\varepsilon]$. *Then for every* $h : [0,1] \to \mathbb{C}$ *sufficiently* C^1*-close to* $\bar{h}$ *and* $g_0 : [0,\varepsilon] \cup [1-\varepsilon, 1] \to \mathbb{C}$ *sufficiently* C^1*-close to* $\bar{g}|_{[0,\varepsilon]\cup[1-\varepsilon,1]}$ *such that* (h, g_0) *satisfies* (18) *on* $[0,\varepsilon] \cup [1-\varepsilon, 1]$ *there exists* $g : [0,1] \to \mathbb{C}$ *which is* C^1*-close to* $\bar{g}$ *and agrees with* g_0 *on* $[0,\varepsilon] \cup [1-\varepsilon, 1]$ *such that* (h, g) *satisfies* (18). *Moreover, for fixed* $\bar{g}$ *the assignment* $(h, g_0) \mapsto g$ *works smoothly in families.*

(b) *Let* C^1*-small* $\xi : [0,1] \to \mathbb{C}$ *be given. Set* $g_0 = \bar{g}$ *and consider the family* $h_t = \bar{h} + t\xi$, $t \in [0,1]$. *Then for the corresponding functions* g_t *from* (a) *we have an estimate* $|\dot{g}_t|_{C^1} \le C|\xi|_{C^1}$ *for some constant* $C > 0$ *and all* $t \in [0,1]$.

Proof. (a) By the first condition in (18) we have $\bar{g}'(r) = \bar{\rho}(r)\bar{h}'(r)$ for a function $\bar{\rho} : [0,1] \to \mathbb{R}$, and $g_0'(r) = \rho_0(r)h'(r)$ for a function $\rho_0 : [0,\varepsilon] \cup [1-\varepsilon, 1] \to \mathbb{R}$ which is C^0-close to $\bar{\rho}|_{[0,\varepsilon]\cup[1-\varepsilon,1]}$. Let us extend ρ_0 to a function $\rho_0 : [0,1] \to \mathbb{R}$ which is C^0-close to $\bar{\rho}$. We look for g satisfying $g'(r) = \rho(r)h'(r)$ for a function $\rho : [0,1] \to \mathbb{R}$ which agrees with ρ_0 on $[0,\varepsilon] \cup [1-\varepsilon, 1]$, so that the first condition in (18) holds. Given such a ρ we set

$$g(r) := g_0(0) + \int_0^r \rho(s)h'(s)\,ds.$$

This agrees with g_0 on $[0,\varepsilon]$, and it agrees with g_0 on $[1-\varepsilon, \varepsilon]$ iff

$$\int_0^1 \rho(s)h'(s)\,ds = g_0(1) - g_0(0).$$

Denote by V the space of smooth functions $\sigma : [0,1] \to \mathbb{R}$ with support in $[\varepsilon, 1-\varepsilon]$, equipped with the C^0 norm. Writing $\rho = \rho_0 + \sigma$, the preceding condition is equivalent to

$$\delta := g_0(1) - g_0(0) - \int_0^1 h'(s)\rho_0(s)\,ds$$

being in the image of the linear map

$$L_{h'} : V \to \mathbb{C}, \qquad \sigma \mapsto \int_0^1 \sigma(s)h'(s)\,ds.$$

Now a unit vector $v \in \mathbb{C}$ is orthogonal to the image of $L_{h'}$ iff $\langle k'(s), v\rangle = 0$ for all $s \in [\varepsilon, 1-\varepsilon]$, which can only happen if the slope $h'/|h'|$ is constant on $[\varepsilon, 1-\varepsilon]$. But this is not the case since by assumption the slope $\bar{h}'/|\bar{h}'|$ is not constant on $[\varepsilon, 1-\varepsilon]$ and h' is C^0-close to $\bar{h}'$. Thus $L_{h'}$ is surjective. Closer inspection shows that we can find a right inverse for $L_{h'}$ whose norm is uniformly bounded for h' in a C^0-neighbourhood of $\bar{h}'$. Indeed, let $K \subset V$ be the kernel of the operator $L_{\bar{h}'}$ and T be a 2-dimensional algebraic complement, so $L_{\bar{h}'}|_T : T \to \mathbb{C}$ is an isomorphism. Since the map $h' \mapsto L_{h'}$ is continuous with respect to the operator norm, $L_{h'}|_T$ is invertible with uniformly bounded inverses for h' in a C^0-neighbourhood of $\bar{h}'$. Thus $(L_{h'}|_T)^{-1} : \mathbb{C} \to T \hookrightarrow V$ are the required right inverses of $L_{h'}$.

Note that, since the pair (g_0, ρ_0) is C^0-close to the pair $(\bar{g}, \bar{\rho})$, the complex number δ is small and hence we can find g which is C^1-close to $\bar{g}$ such that the first condition in (18) holds. The second condition in (18) now follows from C^1-closeness.

For (b) note that g_t is defined by

$$g_t(r) := \bar{g}(0) + \int_0^r \rho_t(s)h_t'(s)\,ds, \qquad h_t(s) = \bar{h}(s) + t\xi(s), \qquad \rho_t(s) = \rho_0(s) + \sigma_t(s).$$

Thus

$$\dot{g}_t(r) := \int_0^r [\dot{\sigma}_t(s)h_t'(s) + \rho_t(s)\xi'(s)]\,ds.$$

Since h_t' is C^0-bounded and ξ' is C^0-small, for C^1-smallness of $\dot{g}_t$ it remains to show C^0-smallness of $\dot{\sigma}_t$.

Note that σ_t is defined by

$$L_{h_t'}(\sigma_t) = L_{\bar{h}'}(\sigma_t) + tL_{\xi'}(\sigma_t) = \delta_t$$

with

$$\delta_t = \bar{g}(1) - \bar{g}(0) - \int_0^1 (\bar{h} + t\xi)'(s)\rho_0(s)\,ds, \qquad L_{\xi'}(\sigma_t) = \int_0^1 \sigma_t(s)\xi'(s)\,ds.$$

It follows that

$$L_{h_t'}(\dot{\sigma}_t) + L_{\xi'}(\sigma_t) = \dot{\delta}_t = -\int_0^1 \xi'(s)\rho_0(s)\,ds.$$

Since ξ' is C^0-small, we see that $\dot{\delta}_t$ is small (and independent of t). Since the operators $L_{h_t'}$ have uniformly bounded right inverses, it follows that

$$\dot{\sigma}_t = (L_{h_t'})^{-1}(\dot{\delta}_t - L_{\xi'}(\sigma_t))$$

is uniformly C^0-small for all $t \in [0, 1]$. This finishes the proof of Lemma 3.15. □

Lemma 3.16. *Let $h, g_0, g_1 : [0, 1] \to \mathbb{C}$ be given such that the pairs (h, g_0) and (h, g_1) both satisfy (18). Suppose that $h'/|h'|$ is not constant. Then there exists a function $g : [0, 1] \to \mathbb{C}$ which agrees with g_0 near 0 and with g_1 near 1 such that (h, g) satisfies (18).*

Proof. By assumption, there exists a point $p \in (0, 1)$ at which the slope function $k = h'/|h'|$ has nonzero derivative. We pick a small interval $I \subset (0, 1)$ on which $k' \neq 0$, so the unit vectors $k_0 = k(0)$ and $k_1 = k(1)$ are linearly independent and close to each other, and rescale I back to $[0, 1]$. We set $c_j = \langle g_j(j), ik_j\rangle$ for $j = 0, 1$. Linear independence of k_0 and k_1 implies that there exists a unique pair $(s_0, s_1) \in \mathbb{R}^2$ such that

$$g_0(0) - g_1(1) = s_1k_1 - s_0k_0. \tag{21}$$

We pick a small $\varepsilon > 0$ and define the following piecewise constant function σ on $[0, 1]$: on $[0, \varepsilon)$ we set $\sigma = \varepsilon^{-1}s_0$, on $[\varepsilon, 1-\varepsilon]$ we set $\sigma = 0$, and finally on $(1-\varepsilon, 1]$ we set $\sigma = -\varepsilon^{-1}s_1$. We set

$$g(r) := g_0(0) + \int_0^r \sigma(s)k(s)\,ds.$$

We claim the following:

(i) $g(1) - g_1(1)$ is small and
(ii) the pair (h, g) satisfies (18).

For (i) we introduce the averages $A_0 := \varepsilon^{-1}\int_0^\varepsilon k(s)\,ds$, $A_1 := \varepsilon^{-1}\int_{1-\varepsilon}^1 k(s)\,ds$ and note that $\lim_{\varepsilon\to 0} A_0 = k_0$ and $\lim_{\varepsilon\to 0} A_1 = k_1$. Using the explicit formula for g we write

$$g(1) - g_1(1) = g_0(0) - g_1(1) + s_0A_0 - s_1A_1.$$

This together with the above convergence properties of A_0 and A_1 and equation (21) implies (i) for small enough ε.

For (ii) we first consider $r \in [0, \varepsilon]$ and write out the estimate

$$\begin{aligned}|\langle g(r) - g_0(0), ik(r)\rangle| &\le |s_0|\varepsilon^{-1}\int_0^\varepsilon |\langle k(s), ik(r)\rangle|\,ds \\ &\le |s_0| \max_{r_1,r_2\in[0,\varepsilon]} |\langle k(r_1), ik(r_2)\rangle|.\end{aligned}$$

Note that the latter maximum tends to zero as $\varepsilon \to 0$. This shows that if ε is small enough, then $\langle g(r), ik(r)\rangle$ is close to $\langle g_0(0), ik(r)\rangle$, and the latter is close to $\langle g_0(0), ik_0\rangle = c_0 > 0$. Altogether, $\langle g(r), ik(r)\rangle$ is close to c_0 and thus in particular positive. Similarly for $r \in [1-\varepsilon, 1]$, we write $g(r) - g(1) = \varepsilon^{-1}s_1\int_r^1 k(s)\,ds$, use that $g(1)$ is close to $g_1(1)$ and that $\max_{r_1,r_2\in[1-\varepsilon,1]} |\langle k(r_1), ik(r_2)\rangle|$ tends to zero as $\varepsilon \to 0$ to conclude that $\langle g(r), ik(r)\rangle$ is close to c_1 and thus in particular positive. For the intermediate region $r \in [\varepsilon, 1-\varepsilon]$ recall that $k' \ne 0$ and thus any $k(r)$ "sits between" $k(\varepsilon)$ and $k(1-\varepsilon)$. Since g is constant on $[\varepsilon, 1-\varepsilon]$, the number $\langle g(r), ik(r)\rangle$ belongs to the interval bounded by $\langle g(\varepsilon), ik(\varepsilon)\rangle$ and $\langle g(1-\varepsilon), ik(1-\varepsilon)\rangle$, which implies the required positivity.

Now approximate σ by an L^1-close smooth function (still denoted by σ) and define g as above. The new function g is smooth and still satisfies conditions (i) and (ii). Recall that k' is nonconstant. Now condition (i) and the equality $g(0) = g_0(0)$ allow us to use Lemma 3.15 to adjust σ so that $g = g_0$ near 0 and $g = g_1$ near 1. □

Proof of Proposition 3.14. We extend $\bar{g}$ from $[0, \varepsilon]$ to a larger interval $[0, r_0]$ such that (18) still holds and the derivative of the slope function $k = h'/|h'|$ is nonzero at r_0. (If k is not constant near ε this can be done for r_0 slightly larger than ε, and if k is constant near ε we extend $\bar{g}$ satisfying equation (19) until k becomes nonconstant). Similarly, we extend $\bar{g}$ from $[1-\varepsilon, 1]$ to a larger interval $[r_1, 1]$ so that (18) still holds and $k'(r_1) \ne 0$. Now we use Lemma 3.16 to find $g : [0, r_0] \cup [r_1, 1] \to \mathbb{C}$ satisfying (18) which agrees with $\bar{g}$ on $[0, \varepsilon] \cup [1-\varepsilon, 1]$ and with ik near r_0 and r_1, so we can extend it as ik over $[r_0, r_1]$ to the desired function $g : [0, 1] \to \mathbb{C}$. □

Proposition 3.14 answers the stabilization question for a single function $h : [0, 1] \to \mathbb{C}$. The following proposition answers the question for homotopies.

Proposition 3.17. *Let $h_t : [0,1] \to \mathbb{C}$, $t \in [a,b]$, be a homotopy of immersions such that for each t the slope $h'_t/|h'_t|$ restricted to $[\varepsilon, 1-\varepsilon]$ is nonconstant. Let $\bar{g}_t : [0,\varepsilon] \cup [1-\varepsilon, 1]$, $t \in [a,b]$, be a homotopy such that $(h_t, \bar{g}_t)$ satisfies (18) on $[0,\varepsilon] \cup [1-\varepsilon,1]$ for all $t \in [a,b]$. Then there exists a homotopy g_t which agrees with $\bar{g}_t$ on $[0,\varepsilon] \cup [1-\varepsilon,1]$ such that (h_t, g_t) satisfies (18) on $[0,1]$ for all $t \in [a,b]$.*

Proof. We apply Proposition 3.14 for each time $s \in [a,b]$ to get a (possibly discontinuous) family $\{\tilde{g}_s\}_{s\in[a,b]}$ stabilizing h_s on $[0,1]$ (i.e. $(h_s, \tilde{g}_s)$ satisfies (18)) and restricting to $[0,\varepsilon] \cup [1-\varepsilon,1]$ as $\bar{g}_s$. Lemma 3.15 applied with $\bar{g} := \tilde{g}_s$ shows that each $s \in [a,b]$ has an open neighbourhood $U \subset [a,b]$ and a smooth family g_t^U, $t \in U$, such that (h_t, g_t^U) satisfies (18) and $g_t^U|_{[0,\varepsilon]\cup[1-\varepsilon,1]} = \bar{g}_t$ for all $t \in U$. Finitely many such neighbourhoods U_i cover $[a,b]$. Let $\{\rho_i\}$ be a finite partition of unity subordinate to this covering and set $g_t := \sum_i \rho_i g_t^{U_i}$. This family is smooth as a function of $t \in [a,b]$. Moreover, for each t it is a finite sum of functions stabilizing h_t, thus the pair (h_t, g_t) satisfies (18). On $[0,\varepsilon] \cup [1-\varepsilon,1]$ we have $g_t = \sum_i \rho_i \bar{g}_t = \bar{g}_t$. This completes the proof. □

Corollary 3.18. *Let $h_0, h_1 : [0,1] \to \mathbb{C}$ be two immersions with $h_0 = h_1$ on $[0,\varepsilon] \cup [1-\varepsilon,1]$ and the same winding number. Let $g_0 : [0,\varepsilon] \cup [1-\varepsilon,1] \to \mathbb{C}$ be such that (h_0, g_0) satisfies (18). Then there exists a homotopy (h_t, g_t), $t \in [0,1]$, satisfying (18) and fixed on $[0,\varepsilon] \cup [1-\varepsilon,1]$, such that h_t connects h_0 and h_1.*

Proof. Since the immersions h_0, h_1 have the same winding number, they are homotopic through immersions h_t, $t \in [0,1]$, fixed on $[0,\varepsilon] \cap [1-\varepsilon,1]$. If h_0 has constant slope on $[\varepsilon, 1-\varepsilon]$ we use Remark 3.13 to make it nonconstant, and similarly for h_1. Since the condition to have constant slope on $[\varepsilon, 1-\varepsilon]$ is of infinite codimension, we can choose the homotopy of immersions h_t to have nonconstant slope on $[\varepsilon, 1-\varepsilon]$ for all $t \in [0,1]$. Now the stabilizing family g_t is provided by Proposition 3.17. □

Remark 3.19. Clearly, Proposition 3.17 remains true with the interval $[a,b]$ replaced by any compact manifold with boundary. Hence the proof of Corollary 3.18 shows that the space of pairs $h, g : [0,1] \to \mathbb{C}$ satisfying (18), fixed on $[0,\varepsilon] \cup [1-\varepsilon,1]$ and with fixed winding number, is weakly contractible.

3.5. T^2-invariant Hamiltonian structures on T^3 and S^3

To illustrate the techniques developed in Section 3.4, we now classify exact T^2-invariant Hamiltonian structures on T^3 and S^3 up to T^2-invariant stable homotopy.

We begin with the 3-torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ with coordinates (r, θ, ϕ). We let T^2 act on T^3 via shift along θ and ϕ, and consider exact T^2-invariant Hamiltonian structures on T^3. Any exact T^2-invariant Hamiltonian structure ω on T^3 can be written as $\omega = \omega_h$, where $h : \mathbb{R} \to \mathbb{C}$ is an immersion with periodic h'. Now exact T^2-invariant HS on T^3 are classified up to T^2-invariant homotopy by the winding number $w(h) \in \mathbb{Z}$, i.e. the degree of the map $h'/|h'| : S^1 \to S^1$. Since every homotopy ω_{h_t} of exact T^2-invariant HS on T^3 can be stabilized by λ_{g_t} with g_t obtained from h_t by formula (20), we have shown

Corollary 3.20. *Two exact T^2-invariant SHS on T^3 are connected by a T^2-invariant stable homotopy if and only if they have the same winding number.*

Next we consider the 3-sphere

$$S^3 = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \mid x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\}.$$

We introduce the radial coordinate $r \in [0, 1]$ and two angular coordinates ϕ and θ on S^3 as follows:

$$r := (x_1^2 + y_1^2)^{1/2}, \quad \tan\theta = y_1/x_1, \quad \tan\phi = y_2/x_2.$$

Let T^2 act on S^3 by rotations in (x_1, y_1) and (x_2, y_2) planes, i.e. by shifts along θ and ϕ. The goal of this subsection is to classify T^2-invariant SHS up to T^2-invariant stable homotopy.

Consider first a T^2-invariant HS ω on S^3. Its Reeb vector field R is tangent to the level sets $r = \text{const}$, in particular $\{r = 0\}$ and $\{r = 1\}$ are closed Reeb orbits. Define two signs s_0, s_1 by $s_0 = +$ iff the orientations on $\{r = 0\}$ induced by R and ∂_ϕ coincide, and $s_1 = +$ iff the orientations on $\{r = 1\}$ induced by R and ∂_θ coincide. Clearly, these signs remain constant during a homotopy of T^2-invariant HS.

Recall from Lemma 3.9 that ω can be written on $\{0 < r < 1\}$ as $\omega = \omega_h = h_1'(r)dr \wedge d\theta + h_2'(r)dr \wedge d\phi$ for an immersion $h : (0, 1) \to \mathbb{C}$ which is unique up to adding a constant. Smoothness and nondegeneracy at $r = 0$ imply that $h_1'(r) = ar + O(r^2)$ near $r = 0$ for some $a \neq 0$. Note that the sign of a is s_0. Thus we can T^2-invariantly homotope $\omega = \omega_h$ until $h'(r) = (s_0 r, 0)$ near $r = 0$, i.e. ω agrees with $s_0 r\, dr \wedge d\theta$ near $r = 0$. Note that the orientation preserving diffeomorphism of S^3 mapping $x_1 \mapsto x_2$, $x_2 \mapsto x_1$, $y_1 \mapsto y_2$ and $y_2 \mapsto y_1$ exchanges θ with ϕ and r^2 with $1 - r^2$. In particular, rdr pulls back to $-rdr$. Thus, arguing symmetrically we can further T^2-invariantly homotope $\omega = \omega_h$ until it agrees with $-s_1 rdr \wedge d\phi$ near $r = 1$. After this standardization, we define the winding number as in Section 3.4. Then two T^2-invariant HS are T^2-invariantly homotopic iff they have the same signs and winding number.

Next consider a T^2-invariant SHS (ω, λ) on S^3. After a T^2-invariant stable homotopy as in Remark 3.13, we may assume that the slope function $h'/|h'|$ is nonconstant. By definition of the signs s_0, s_1, the HS ω is stabilized by the 1-form $s_0 d\phi$ near $r = 0$ and by $s_1 d\theta$ near $r = 1$. By Proposition 3.14, there exists a T^2-invariant stabilizing 1-form $\tilde{\lambda}$ for ω which agrees with $s_0 d\phi$ near $r = 0$ and with $s_1 d\theta$ near $r = 1$. Fixing the stabilizing 1-form $\tilde{\lambda}$, we can now homotope ω near $r = 0, 1$ to $\tilde{\omega}$ which agrees with $s_0 rdr \wedge d\theta$ near $r = 0$ and with $-s_1 rdr \wedge d\phi$ near $r = 1$.

Finally, consider two T^2-invariant SHS (ω_i, λ_i), $i = 0, 1$, on S^3 with the same signs s_0, s_1 and the same winding number. After applying the stable homotopies in the previous paragraph, we may assume that both (ω_i, λ_i) agree with $(s_0 rdr \wedge d\theta, s_0 d\phi)$ near $r = 0$ and with $(-s_1 rdr \wedge d\phi, s_1 d\theta)$ near $r = 1$. By assumption, ω_0 and ω_1 have the same winding number. Hence, by Corollary 3.18, (ω_0, λ_0) and (ω_1, λ_1) are connected by a T^2-invariant stable homotopy fixed near $r = 0, 1$. So we have shown

Corollary 3.21. *Two T^2-invariant SHS on S^3 are connected by a T^2-invariant stable homotopy if and only if they have the same signs and winding number.*

Remark 3.22. Similar arguments (cf. Remark 3.19) show that the space of exact T^2-invariant SHS on T^3 (resp. S^3) with fixed winding number (resp. signs and winding number) is weakly contractible.

3.6. Thickening a level set

In this subsection we prove the following technical result which will play a crucial role in the remainder of this paper.

Proposition 3.23. *Let (ω, λ) be a SHS on a closed 3-manifold M and set $f := d\lambda/\omega$. Let $Z \subset \mathbb{R}$ be any set of Lebesgue measure zero containing a value $a \in Z \cap \operatorname{im} f$. Then there exists a stabilizing form $\tilde{\lambda}$ for ω such that $\tilde{f} := d\tilde{\lambda}/\omega$ can be written as $\tilde{f} = \sigma \circ f$ for a function $\sigma : \mathbb{R} \to \mathbb{R}$ which is locally constant on a open neighbourhood of Z (and thus $\tilde{f}$ is locally constant on an open neighbourhood of $f^{-1}(Z)$) and $\tilde{f} \equiv a$ on an open neighbourhood of $f^{-1}(a)$. Moreover, for every $s \in [0, 1)$ we can achieve that $\tilde{\lambda}$ is C^{1+s}-close to λ.*

We will use two special cases of this result. The first one is $Z = \{a\}$ for some value a of f:

Corollary 3.24. *Let (ω, λ) be a SHS on a closed 3-manifold M and set $f := d\lambda/\omega$. Let $a \in \operatorname{im} f$ be any (singular or regular) value of f. Then there exists a stabilizing form $\tilde{\lambda}$ for ω such that $\tilde{f} := d\tilde{\lambda}/\omega$ satisfies $\tilde{f} \equiv a$ on an open neighbourhood of $f^{-1}(a)$.*

The second special case arises for Z the set of critical values:

Corollary 3.25. *Let (ω, λ) be a SHS on a closed 3-manifold M and set $f := d\lambda/\omega$. Then there exists a (possibly disconnected and possibly with boundary) compact 3-dimensional submanifold N of M, invariant under the Reeb flow, a finite family $\{U_i\}_{i=1,\dots,k}$ of disjoint open integrable regions and a stabilizing 1-form $\tilde{\lambda}$ for ω with the following properties:*

- $\bigcup_i U_i \cup N = M$;
- *$\tilde{\lambda}$ is C^1-close to λ;*
- *the proportionality coefficient $\tilde{f} := d\tilde{\lambda}/\omega$ is constant on each connected component of N;*
- *on each $U_i \cong (a_i, b_i) \times T^2$ the function f is given by the projection onto the first factor and for r sufficiently close to a_i or b_i we have $\{r\} \times T^2 \subset N$.*

Moreover, $\tilde{f} = \sigma \circ f$ for a function $\sigma : \mathbb{R} \to \mathbb{R}$ which is C^0-close to the identity.

Proof. Let the 1-form $\tilde{\lambda}$ and the proportionality coefficient $\tilde{f} = d\tilde{\lambda}/\omega = \sigma \circ f$ be obtained from Proposition 3.23 with Z the set of critical values of f (and any value $a \in Z$). Since Z is compact, it is covered by finitely many open intervals (c_j, d_j), $j = 1, \dots, n$, on which σ is constant. By shrinking these intervals slightly if necessary we may assume that all c_j, d_j are regular values of f. Set $J := \bigcup_{j=1}^n [c_j, d_j]$ and $N := f^{-1}(J)$. Let d be the minimal distance between any two of the intervals $[c_j, d_j]$. Let $I = \bigcup_{i \in \mathbb{N}} (a_i, b_i)$ be the set of regular values of f, written as a countable union of disjoint intervals. As the a_i, b_i are singular values, those intervals (a_i, b_i) with length $b_i - a_i$ strictly smaller than d must be contained in one of the intervals $[c_j, d_j]$. After renumbering we may asume that (a_i, b_i) for $i = 1, \dots, m$ are all the intervals from I with finite length $\geq d$. Set $I_d := \bigcup_{i=1}^m (a_i, b_i)$. It is clear that the image of f is contained in $I_d \cup J$. Thus setting $U := f^{-1}(I_d)$ gives $N \cup U = M$. The connected components U_i of U are diffeomorphic to $(a_i, b_i) \times T^2$ with f being the obvious projection. Finally, note that each a_i and b_i is itself a *singular* value of f and thus must be contained in (c_j, d_j) for some j. □

The proof of Proposition 3.23 consists in analysis of suitable linear function spaces. We begin with a lemma about functions on the real line.

Lemma 3.26. *Let σ be a smooth real valued function on an interval $[c, d]$. Let $Z \subset [c, d]$ be a set of Lebesgue measure zero, $a \in Z$, and $s \in [0, 1)$. Then there exists a sequence of smooth functions σ_n on $[c, d]$ converging to σ in the C^s-norm such that each σ_n is locally constant on an open neighbourhood of Z and $\sigma_n(a) = \sigma(a)$ for all n.*

Proof. Since Z has Lebesgue measure 0, it has an open neighbourhood U of arbitrarily small Lebesgue measure. We can represent U as a countable union $\bigcup_{i\in\mathbb{N}} U_i$ of intervals. We can assume that all U_i are disjoint by replacing any two of them which intersect by their union. Moreover, we can assume that the distance between any two of them is positive by replacing any pair (x_1, x_2), (x_2, x_3) of intervals by the interval (x_1, x_3). Then we can choose a set of intervals V_i with the following properties. For each i we have $\bar{U}_i \subset V_i$, all the V_i have positive distance from each other, and the measure of $V := \bigcup_{i\in\mathbb{N}} V_i$ does not exceed twice the measure of U. For each i we choose a compactly suported cutoff function $\chi_i : V_i \to [0, 1]$ which equals 1 on U_i. These cutoff functions patch together to a cutoff function χ which is compactly supported in V and equals 1 on U.

We set $h_U := (1 - \chi)\sigma'$. By construction h_U is a smooth function vanishing on U. Now $\sigma(x) = \sigma(a) + \int_a^x \sigma'(y)\, dy$. Set $\sigma_U(x) := \sigma(a) + \int_a^x h_U(y)\, dy$. This defines a smooth function on $[c, d]$ with $\sigma_U(a) = \sigma(a)$. Note that σ' and $\sigma_U' = (1 - \chi)\sigma'$ are both bounded and differ only on the support of χ, which is a set of small measure. Thus σ' and σ_U' are L^p-close to each other for any $1 \le p < \infty$. Next set $C := \max_{y\in[c,d]} |\sigma'(y)|$ and compute

$$|\sigma(x) - \sigma_U(x)| = \left| \int_a^x (\sigma'(y) - h_U(y))\, dy \right| \le \int_{[c,d]} |\chi(y)|\, |\sigma'(y)|\, dy \le C|V|.$$

Thus σ and σ_U are C^0-close. Combined with L^p-closeness of their derivatives this also gives $W^{1,p}$-closeness for any $1 \le p < \infty$. The Sobolev embedding theorem yields a continuous embedding $W^{1,p}([c, d]) \hookrightarrow C^s([c, d])$ for $s \le 1 - 1/p$, so by choosing p large we get C^s-closeness for any given $s \in [0, 1)$. □

Remark 3.27. If Z is compact it is covered by finitely many of the intervals U_α in the proof, so in this case we can achieve that for each n the neighbourhood of Z on which σ_n is constant is a *finite* union of intervals.

Proof of Proposition 3.23. Now we fix some interval $[c, d]$ containing im f and let Z, a, s be as in the proposition. Set

$$\mathcal{C} := \{\sigma \in C^\infty([c, d]) \mid \sigma \text{ is constant on some open set containing } Z\},$$

Consider the space $C^\infty(M)$ of smooth functions on M. Let

$$f^*: C^\infty([c,d]) \to C^\infty(M)$$

denote the linear operator given by composing with the function f on the right, and introduce the linear subspaces

$$\mathcal{D} := f^*(\mathcal{C}) \subset \mathcal{E} := f^*(C^\infty([c,d])) \subset C^\infty(M).$$

We introduce the following evaluation linear functional:

$$a^*: \mathcal{E} \to \mathbb{R}, \qquad h = \sigma \circ f \mapsto \sigma(a).$$

In other words, $a^*(h)$ is the value the function h takes on the level set $f^{-1}(a)$. For any $b \in \mathbb{R}$ and any subset $S \subset \mathcal{E}$ we denote $S^b := S \cap (a^*)^{-1}(b)$.

We equip $\mathcal{E}$ with the C^s-topology (note that $\mathcal{E}$ is not complete with this topology) and denote by $\bar{S} \subset \mathcal{E}$ the closure of a subset $S \subset \mathcal{E}$ in $\mathcal{E}$ with respect to this topology. We claim that for any $b \in \mathbb{R}$,

$$\bar{\mathcal{D}}^b = \mathcal{E}^b. \tag{22}$$

Indeed, the estimate $\|\sigma \circ f\|_{C^s(M)} \leq \|\sigma\|_{C^s([c,d])}\|f\|^s_{C^1(M)}$ shows continuity of $f^*: C^\infty([c,d]) \to \mathcal{E}$ with respect to the C^s-norms, so the claim follows from Lemma 3.26. In particular, we have

$$\bar{\mathcal{D}} = \mathcal{E}. \tag{23}$$

The key observation is that for any $h = \sigma \circ f \in \mathcal{E}$ the 2-form $h\omega$ is closed:

$$d(h\omega) = \sigma' df \wedge \omega = \sigma' d(f\omega) = \sigma' d(d\lambda) = 0.$$

This allows us to define the operator

$$H: \mathcal{E} \to F \subset H^2(M;\mathbb{R}),$$

where $H(h) := [h\omega]$ is the de Rham cohomology class and $F := H(\mathcal{E})$. At this point we fix some reference Riemannian metric. In terms of the Hodge decomposition, the operator of taking cohomology is just the L^2-projection from the space of closed forms to the space of harmonic forms and thus is continuous with respect to the L^2 topology on the space of closed forms. Since the topology on $\mathcal{E}$ is stronger than L^2, we deduce that the operator H is continuous on $\mathcal{E}$ and from (23) we get

$$H(\mathcal{D}) = H(\mathcal{E}) = F.$$

The real vector space F is finite-dimensional as a subspace of the finite-dimensional space $H^2(M;\mathbb{R})$ and we set $k := \dim F$. Set

$$K := \ker H, \qquad K_{\mathcal{D}} := \ker(H|_{\mathcal{D}}) = K \cap \mathcal{D}.$$

We claim that the codimension of $K_{\mathcal{D}}$ in $\mathcal{D}$ is k. Indeed, assume there were $k+1$ linearly independent vectors in $\mathcal{D}$ spanning a subspace intersecting $K_{\mathcal{D}}$ trivially; then the restriction of H to this subspace would give us an injective map from this space to the space F

of dimension one less. On the other hand, the codimension of $K_{\mathcal{D}}$ in $\mathcal{D}$ cannot be smaller than k, for otherwise the image of H would have dimension smaller than k, contradicting $H(\mathcal{D}) = F$. Thus there exists a (nonunique) k-dimensional subspace T of $\mathcal{D}$ such that we have the following (algebraic, not topological) direct sum decomposition:

$$\mathcal{D} = K_{\mathcal{D}} \oplus T, \tag{24}$$

with H restricting as an isomorphism to T. Continuity of H implies that

- the kernel K of H is closed in $\mathcal{E}$, and
- the projection from $\mathcal{E}$ onto T along K (understood as the composition of H and the finite-dimensional inverse of $H|_T$) is continuous.

Now we use the freedom in the choice of T to see that we can without loss of generality assume that either $K_{\mathcal{D}} \subset \mathcal{D}^0$ or $T \subset \mathcal{D}^0$. Indeed, if for all $h \in K_{\mathcal{D}}$ we have $a^*(h) = 0$, then the first case is realized. Otherwise, there exists $h \in K_{\mathcal{D}}$ with $a^*(h) = c \neq 0$. Let $t_1, \dots, t_k \in T$ be a basis of T. For any $j = 1, \dots, k$ set $c_j := a^*(t_j)$ and $\hat{t}_j := t_j - (c_j/c)h$. Now $a^*(\hat{t}_j) = c_j - (c_j/c)c = 0$, so $\hat{t}_j \in \mathcal{D}^0$. Moreover, $\{\hat{t}_j\}_{j=1,\dots,k}$ is the basis of a linear subspace space $\hat{T}$ which complements $K_{\mathcal{D}}$ in $\mathcal{D}$ because $H(\hat{t}_j) = H(t_j)$ and $\{H(t_j)\}_{j=1,\dots,k}$ is a basis of F.

Now we come to the crucial assertion. We claim that for each $b \in \mathbb{R}$,

$$\bar{K}^b_{\mathcal{D}} = K^b. \tag{25}$$

To see this let $h \in K^b$ be arbitrary. According to (22) we find a sequence $\{h_n\}_{n\in\mathbb{N}} \subset \mathcal{D}^b$ converging to h. According to (24) each h_n can be uniquely decomposed as $h_n = h^K_n + h^T_n$, with $h^K_n \in K_{\mathcal{D}}$ and $h^T_n \in T$. Continuity of the projection from $\mathcal{E}$ to T along K and $h \in K$ implies that the sequence $\{h^T_n\}_{n\in\mathbb{N}} \subset T$ converges to $0 \in T$. Since K is closed, we conclude

$$h^K_n = h_n - h^T_n \;\to\; h \in K.$$

Next, recall that one of the spaces in the direct sum $K_{\mathcal{D}} \oplus T$ is a subspace of $\mathcal{D}^0$. Now if $K_{\mathcal{D}} \subset \mathcal{D}^0$, then $a^*(h^K_n) = 0$ for all n and thus $a^*(h) = 0$. If $T \subset \mathcal{D}^0$, then $a^*(h^K_n) = a^*(h_n - h^T_n) = a^*(h_n) = b = a^*(h)$. In any case,

$$a^*(h^K_n) = a^*(h)$$

for all n. This shows (25).

Note that $f \in K^a$. Therefore, by (25) the function f can be arbitrarily well C^s-approximated by some $\tilde{f} \in K^a_{\mathcal{D}}$. Then the difference $\beta := \tilde{f}\omega - f\omega$ is C^s-small and exact. By Lemma 3.28 below, β has a primitive 1-form α that is C^{1+s}-small. The desired 1-form is now $\tilde{\lambda} := \lambda + \alpha$. It is C^{1+s}-close to λ, so in particular it evaluates positively on the Reeb vector field of (ω, λ). As $d\tilde{\lambda} = \tilde{f}\omega$ by construction, we see that $\tilde{\lambda}$ stabilizes ω. By definition of $\mathcal{D}^a$, the function $\tilde{f} = \sigma \circ f$ is locally constant on a neighbourhood of $f^{-1}(Z)$ and takes value a on $f^{-1}(a)$. This proves Proposition 3.23. □

Lemma 3.28. *Let β be a C^s-small exact 2-form on a closed manifold M, for some $s \in (0, 1)$. Then β has a primitive 1-form α that is C^{1+s}-small.*

Proof. A primitive 1-form α is obtained from β by first applying the Green operator G for the Laplace–Beltrami operator Δ and then taking the co-differential d^*. Since Δ is elliptic (see e.g. [51]), it satisfies for each $0 < s < 1$ and all smooth 2-forms β an estimate of Hölder norms

$$\|\beta\|_{2+s} \leq C(\|\Delta\beta\|_s + \|\beta\|_s).$$

Indeed, such an estimate is proved in [29] for domains in $\mathbb{R}^n$. From this the estimate on a compact manifold M follows using a finite partition of unity ϕ_i via

$$\begin{aligned}\|\beta\|_{2+s} &\leq \sum_i \|\phi_i\beta\|_{2+s} \leq C\sum_i(\|\Delta(\phi_i\beta)\|_s + \|\phi_i\beta\|_s) \\ &\leq C'\Big(\sum_i \|\phi_i\Delta\beta\|_s + \|\beta\|_{s+1}\Big) \leq C''(\|\Delta\beta\|_s + \|\beta\|_{s+1})\end{aligned}$$

and applying the same estimate to $\|\beta\|_{s+1}$. Now standard Hodge theory arguments (see [51]) imply that the Green operator satisfies an estimate

$$\|G\beta\|_{C^{2+s}(M)} \leq C\|\beta\|_{C^s(M)}$$

for all exact 2-forms β, from which the claim follows. □

3.7. Taut foliations

In this subsection we investigate the special case of a SHS (ω, λ) with $d\lambda = 0$, so $\ker\lambda$ defines a taut foliation. After a small deformation of the closed 1-form we may assume that λ represents a rational cohomology class, and after rescaling we may assume that $[\lambda] \in H^1(M;\mathbb{Z})$. Then integration of λ over paths from a fixed base point yields a fibration $\pi : M \to S^1 = \mathbb{R}/\mathbb{Z}$ over the circle. It follows that the restriction $\bar\omega$ of ω to a fibre W is symplectic. Thus $M = W_\psi$ is the mapping torus of a symplectomorphism ψ of $(W, \bar\omega)$ with ω induced by $\bar\omega$ and $\lambda = \pi^*d\phi$, where ϕ is the coordinate on S^1. Now every isotopy of symplectomorphisms ψ_t induces a stable homotopy on M. (For this, we always identify W_{ψ_0} with W_{ψ_t} by the diffeomorphism $(\phi, x) \mapsto (\phi, \psi_{t\rho(\phi)} \circ \psi_0^{-1}(x))$ for some fixed function $\rho : [0,1] \to [0,1]$ which equals 0 near 0 and 1 near 1.) Using Moser's theorem, this can be used to classify SHS with $d\lambda = 0$ in dimension 3. Here we content ourselves with the following observation that will be needed later.

Lemma 3.29. *For any symplectomorphism ψ of a closed symplectic manifold (W, ω) there exists a symplectic isotopy ψ_t such that $\psi_0 = \phi$ and $\psi_1 = \mathbb{1}$ on some open subset $U \subset W$.*

Proof. After a Hamiltonian isotopy we may assume that ψ has a fixed point p. Now consider the graph $\mathrm{gr}(\psi)$ as a Lagrangian submanifold of $(W \times W, \omega \oplus -\omega)$. A neighbourhood of the diagonal $\Delta \subset W \times W$ is symplectomorphic to a neighbourhood of the zero section in the cotangent bundle $T^*\Delta$. Since $\mathrm{gr}(\psi)$ intersects the zero section at (p, p) it can be written nearby as the graph of an exact 1-form dH on Δ with $H(p, p) = 0$ and $dH(p, p) = 0$. Replacing H by a function which vanishes identically near (p, p) thus yields a symplectic isotopy from ψ to a symplectomorphism ψ_1 which equals the identity near p. □

Corollary 3.30. *Any SHS* (ω,λ) *with* $d\lambda=0$ *on a closed* 3*-manifold* M *is stably homotopic to* (ω_1,λ_1) *such that there exists an embedded solid torus* $S^1\times D$ *in* M *on which* $(\omega_1,\lambda_1)=(d\alpha_{\rm st},\alpha_{\rm st})$ *with*

$$\alpha_{\rm st}=r^2d\theta+(1-r^2)d\phi.$$

Proof. By the discussion preceding Lemma 3.29, after a stable homotopy we may assume that $M=W_\psi$ is the mapping torus of a symplectomorphism ψ of $(W,\bar\omega)$ with ω induced by $\bar\omega$ and $\lambda=\pi^*d\phi$. By Lemma 3.29, we may further assume that $\psi=\mathbb{1}$ on some open region $U\subset W$. Pick a disk $D=\{r\le r_0\}\subset U$ on which $\bar\omega=d(r^2d\theta)$ in polar coordinates, so on $D_\psi\cong S^1\times D$ we have $\omega=d(r^2d\theta)$ and $\lambda=d\phi$. Define a new trivialization of $D_\psi\cong S^1\times D$ by composing the previous one with the diffeomorphism $(r,\theta,\phi)\mapsto(r,\theta-\phi,\phi)$ of $S^1\times D$. In this trivialization we then have

$$\omega=d(r^2d\theta+(1-r^2)d\phi)=d\alpha_{\rm st},\qquad \lambda=d\phi.$$

Since ω has constant slope $(1-i)/\sqrt{2}$ on $S^1\times D$, we can homotope the stabilizing form λ to make it restrict as $r^2d\theta+(1-r^2)d\phi$ on a smaller solid torus $S^1\times D'$. □

3.8. *Contact regions*

Proposition 3.31. *Any SHS* (ω,λ) *on a closed* 3*-manifold* M *is stably homotopic to* (ω_1,λ_1) *such that there exists an embedded solid torus* $S^1\times D$ *in* M *on which* $(\omega_1,\lambda_1)=(d\alpha_{\rm st},\alpha_{\rm st})$ *with*

$$\alpha_{\rm st}=r^2d\theta+(1-r^2)d\phi.$$

Proof. Corollary 3.30 allows us to assume that the function $f=d\lambda/\omega$ is not identically zero. After applying Corollary 3.24 to a value $a\neq 0$ of f we may assume that $d\lambda=a\omega$ on some open region $U\subset M$. Suppose first that $a>0$, so after rescaling we may assume $d\lambda=\omega$ on U. Pick any contractible transverse knot γ in U. Pick a neighbourhood $S^1\times D$ of γ on which the contact structure is given by $\ker\lambda=\ker\alpha_{\rm st}$. Then there exists a contact homotopy λ_t rel ∂U (with fixed kernel on $S^1\times D$) from λ to a contact form λ_1 which equals $\alpha_{\rm st}$ on a neighbourhood $S^1\times D$ of γ, so we are done in this case.

If $a<0$ we rescale so that $d\lambda=-\omega$ on U. Let $\bar U$ be the neighbourhood $U\subset M$ with the orientation reversed (opposite to that of M). Then λ is a *positive* contact form on $\bar U$. Thus, there exists a contact homotopy λ_t rel ∂U from λ to a contact form λ_1 which equals $r^2d\theta+(1-r^2)d\phi$ on a neighbourhood $S^1\times D^2$ of a contractible transverse knot γ with $d\phi\wedge dr\wedge d\theta$ defining the opposite orientation on M. Now consider another embedding of $S^1\times D^2$ in M by composing the old one with the flip map $\theta\mapsto-\theta$ on the right. In the new coordinates (ϕ,r,θ) the form λ_1 equals $-r^2d\theta+(1-r^2)d\phi$ near γ and the volume form $d\phi\wedge dr\wedge d\theta$ defines the positive orientation on M. Define $\omega_t:=-d\lambda_t$. Note that $\omega_1=-d\lambda_1=2rdr\wedge(d\theta+d\phi)$. Now we homotope rel $\partial(S^1\times D^2)$ the stabilizing form λ_1 to λ_2 (still stabilizing ω_1) that restricts as $d\phi$ to a neighbourhood of γ. (This uses constancy of the slope of ω_1.) Then we homotope the HS ω_1 to ω_2 (supported in the neighbourhood $\{\lambda_2=d\phi\}$ of γ) restricting as $2rdr\wedge(d\theta-d\phi)$ to an even smaller neighbourhood of γ (by changing the $d\phi$ summand only). The last step is to homotope (supported in $\{\omega_2=2rdr\wedge(d\theta-d\phi)\}$) the stabilizing form λ_2 to λ_3 that restricts as $r^2d\theta+(1-r^2)d\phi$ to a neighbourhood of γ. □

3.9. Genuine stable Hamiltonian structures

Let $h : I \to \mathbb{C}^2$ be an immersion. In the notation introduced at the beginning of Section 3.4 set $\omega_h := d\alpha_h$. Set $\mathcal{L}_h := \ker \omega_h$.

Definition 3.32. We say that the slope function k of ω_h

- *twists* if $k(r) \in S^1$ makes one full turn clockwise and one full turn counterclockwise as r runs through I;
- *is nowhere constant* if for any interval $(a, b) \subset I$ the restriction $k|_{(a,b)}$ is not constant.

Lemma 3.33. *If k twists, then the form α_h is not contact.*

Proof. Assume for contradiction that α_h is contact. Recall that this means that $\langle h, ih' \rangle \neq 0$ or equivalently that h always turns clockwise (counterclockwise). Monotonicity of the angle of h and twisting of $h'/|h'|$ imply that for some $r \in I$ the vectors $h(r)$ and $h'(r)$ are real multiples of each other, i.e. $\langle h(r), ih'(r) \rangle \neq 0$, which is a contradiction. □

Lemma 3.34. *Let (ω, λ) be any SHS defining the foliation $\mathcal{L}_h$ above. If k twists, then the proportionality coefficient $f := d\lambda/\omega$ is not globally constant.*

Proof. Assume for contradiction that $f = d\lambda/\omega$ is constant. Then by Theorem 3.3 we can choose coordinates in which both the Hamiltonian structure ω and the stabilizing form λ are T^2-invariant. So we can write

$$\lambda = g_1 d\theta + g_2 d\phi + g_3 dr = \lambda_g + g_3 dr$$

for some $g = (g_1, g_2) : I \to \mathbb{C}$ and $g_3 : I \to \mathbb{R}$. If $f \neq 0$ then λ is contact. Now $\lambda \wedge d\lambda = (\lambda_g + g_3 dr) \wedge d\lambda_g = \lambda_g \wedge d\lambda_g$, so λ_g is contact and $\ker d\lambda_g = \mathcal{L}_h$. The contradiction follows from Lemma 3.33 with g in place of h. If $f = 0$, then the functions g_1 and g_2 above are constant. The Reeb vector field R spans $\mathcal{L}_h$ and thus is proportional to ik. So $\lambda(R)$ is a multiple of $\langle g, ik \rangle$. Constancy of g and twisting of k force the last expression to vanish for some $r \in I$, contradicting the condition $\lambda(R) = 1$. □

Definition 3.35. Let $\mathcal{L}$ be a stable Hamiltonian foliation on a 3-manifold M. We say that $\mathcal{L}$ is *genuine* if for *every* SHS (ω, λ) defining $\mathcal{L}$ the proportionality coefficient $f := d\lambda/\omega$ is not globally constant on M. A SHS defining a genuine 1-foliation is called *genuine*.

Theorem 3.36. *If a stable Hamiltonian foliation $\mathcal{L}$ on M has an integrable region K in which the slope function k twists, then for any SHS (ω, λ) defining $\mathcal{L}$ the proportionality coefficient $f := d\lambda/\omega$ is not constant on K. In particular, $\mathcal{L}$ is genuine. If in addition k is nowhere constant, then any SHS (ω, λ) defining $\mathcal{L}$ satisfies the assumptions of Theorem 3.7*(c).

Proof. The first assertion follows from Lemma 3.34. Assume in addition that S is nowhere constant and write $K \cong I \times T^2$. Then for any SHS (ω, λ) the proportionality coefficient $f = d\lambda/\omega$ must be constant on irrational tori and thus on all the tori $\{r\} \times T^2$, so by shrinking I, if necessary, we can achieve that f is a submersion. This matches the assumptions of Theorem 3.7(c). □

The following is the main result of this subsection.

Theorem 3.37. *Any SHS on a closed* 3*-manifold can be homotoped to a SHS* (ω, λ) *with the following properties. The* 1*-dimensional foliation* $\mathcal{L}$ *associated with* (ω, λ) *satisfies the assumptions of Theorem* 3.36 (*has an integrable region* K *in which the slope function is nowhere constant and twists*).

Moreover, we can achieve that $K = \gamma \times (D^2 \setminus \{0\})$ *for an embedded solid torus* $\gamma \times D^2$ *around a contractible periodic orbit* γ *of* (ω, λ).

Proof. In view in Proposition 3.31, we can achieve that (ω, λ) restricts to some embedded solid torus $S^1 \times D^2 \cong W \subset M$ (where $D^2 = \{r \leq r_0\} \subset \mathbb{R}^2$ for some $r_0 < 1$) as $(d\alpha_{\text{st}}, \alpha_{\text{st}})$, with α_{st} as in Proposition 3.31. Moreover, we can arrange that the periodic orbit $\gamma := S^1 \times (0, 0)$ is contractible in M. We write $\alpha_{\text{st}} = \alpha_{h^0}$ for $h^0 = (h_1^0, h_2^0) = (r^2, 1 - r^2)$. Note that $W \setminus \gamma$ is diffeomorphic to $(0, r_0] \times T^2$ with coordinates (r, θ, ϕ) and we fall into the setup of Section 3.4. We consider a homotopy of immersions $h^s = (h_1^s, h_2^s) : (0, r_0] \to \mathbb{R}^2$ as shown in Figure 1. (The curve h^0 is dashed, the curve h^1 is

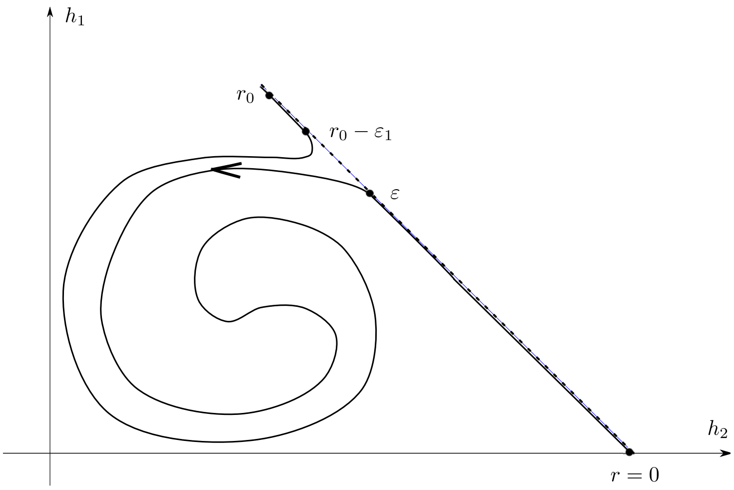

Fig. 1. The slope function twists.

bold, and the points $r = 0, \varepsilon, r_0 - \varepsilon_1, r_0$ follow the direction of increase of the parameter r on the curve h^1.) This homotopy is fixed and equal to $(r^2, 1 - r^2)$ near $r = 0$ and $r = r_0$, so we can stabilize it by Corollary 3.12. Since the resulting homotopy of SHS is supported in the interior of $W \setminus \gamma$, it extends to the whole of M. By construction the kernel foliation $\mathcal{L}$ of ω satisfies the assumptions of Theorem 3.36 on the integrable region $W \setminus \gamma$. □

3.10. Nontame stable Hamiltonian structures

As a byproduct of the techniques developed in Section 3.4, we obtain examples of SHS which are not weakly tame in the sense of [13].

Consider a closed $(2n-1)$-manifold M with a Hamiltonian structure ω. Assume that $[\omega]$ vanishes on $\pi_2(M)$. Denote by $\mathcal{R}$ the space of contractible closed Reeb orbits. For $\gamma \in \mathcal{R}$ we define the *ω-energy* by

$$E_\omega(\gamma) := \int_D \bar{\gamma}^*\omega,$$

where $\bar{\gamma}\colon D \to M$ is a smooth map from the unit disk with $\bar{\gamma}|_{\partial D} = \gamma$. Note that this definition is unambiguous because of the assumption on $[\omega]$. For a stabilizing 1-form λ we define the *λ-energy* of γ by

$$E_\lambda(\gamma) := \int_\gamma \lambda.$$

Definition 3.38. Following [13], we say that a HS ω as above is

- *weakly tame* if for any compact interval $[a,b] \subset \mathbb{R}$ the set of $\gamma \in \mathcal{R}$ with $E_\omega(\gamma) \in [a,b]$ is compact (with respect to the C^∞-topology);
- *tame* if for some (and hence every) stabilizing 1-form λ there exists a constant $c_\lambda > 0$ such that $E_\lambda(\gamma) \le c_\lambda |E_\omega(\gamma)|$ for all $\gamma \in \mathcal{R}$.

Remark 3.39. Phrased differently, weak tameness means that the function $E_\omega : \mathcal{R} \to \mathbb{R}$ is proper. It follows from Ascoli–Arzelà theorem that for any SHS the function $(E_\omega, E_\lambda) : \mathcal{R} \to \mathbb{R}^2$ is proper. Thus a SHS is weakly tame whenever E_λ is bounded in terms of E_ω on $\mathcal{R}$, e.g. if (ω,λ) is tame. Note that a SHS induced by a contact structure is tame because $E_\lambda = E_\omega$ in that case.

Corollary 3.40. *Let (ω,λ) be a stable Hamiltonian structure as in Theorem 3.37, constructed from a contractible knot γ. The (ω,λ) is never tame, and it can be arranged not to be weakly tame.*

Proof. We use the notation from the proof of Theorem 3.37. By construction, $\omega = d\beta$ for a 1-form β on M which is of the form $h_1(r)d\phi + h_2(r)d\theta$ on the region $(0, r_0] \times T^2$ around the contractible knot γ. Moreover, we can arrange that γ is contractible and $E_\omega(\gamma) = \int_\gamma \beta$.

By construction, there exists an interval $(a,b) \subset (0, r_0]$ such that on $(a,b) \times T^2$ the slope function of β is nowhere constant and twists. By Lemma 3.33, twisting implies that β is not contact on $(a,b)\times T^2$, so there exists $r^* \in (a,b)$ such that $\beta \wedge d\beta = 0$ along $\{r^*\} \times T^2$. In other words, $\beta(R)|_{\{r^*\}\times T^2} \equiv 0$ for the Reeb vector field R of (ω,λ). Since the slope function S is not constant near r^*, there exists a sequence $\{r_p\}_{p\in\mathbb{N}} \subset S^{-1}(\mathbb{Q})$ with $r_p \to r^*$ as $p \to \infty$. Then the foliation $\mathcal{L}$ is rational on the tori $\{r_p\} \times T^2$ and $\varepsilon_p := \beta(R)|_{\{r_p\}\times T^2} \to 0$ as $p \to \infty$. Let γ_p be a periodic orbit of R on $\{r_p\} \times T^2$ of period T_p. Then its λ-energy and ω-energy satisfy

$$E_\lambda(\gamma_p) = \int_0^{T_p} \lambda(R)\,dt = T_p, \qquad E_\omega(\gamma_p) = \int_0^{T_p} \beta(R)\,dt = \varepsilon_p T_p,$$

which shows that (ω,λ) is never tame.

For the last statement, we choose the function $h(r)$ in the proof of Theorem 3.37 such that for some r^* the following holds: $h'(r^*)$ is rational and $h(r^*)$ is proportional to $h'(r^*)$ (e.g. $h(r^*) = 0$). Then each leaf γ of $\mathcal{L}$ on the torus $\{r^*\} \times T^2$ is closed and $\int_\gamma \beta = 0$, so ω is not weakly tame. □

3.11. Embeddability

Lemma 3.41. *Let $h = (h_1, h_2)\colon (0,1) \to \mathbb{R}_+ \times \mathbb{R}_+$ be an* embedding *such that $(h_1, h_2)|_{(0,\varepsilon)\cup(1,1-\varepsilon)} = (r^2, 1-r^2)$ for some $\varepsilon > 0$. Then the corresponding T^2-invariant Hamiltonian structure $d(h_1(r)d\theta + h_2(r)d\phi)$ on $(0,1) \times T^2$ can be realized as a hypersurface in the symplectization $(\mathbb{R}_+ \times (0,1) \times T^2, d(s\alpha_{\mathrm{st}}))$. Here s is the coordinate on $\mathbb{R}_+$ and $\alpha_{\mathrm{st}} = r^2 d\theta + (1-r^2)d\phi$.*

Proof. Consider the map $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \times (0,1)$,

$$f(x_1, x_2) := \left(x_1 + x_2, \sqrt{\frac{x_1}{x_1 + x_2}}\right).$$

This map is a diffeomorphism with inverse given by

$$f^{-1}(s, r) = (sr^2, s(1-r^2)).$$

Thus it induces a diffeomorphism

$$F : \mathbb{R}_+ \times \mathbb{R}_+ \times T^2 \to \mathbb{R}_+ \times (0,1) \times T^2, \qquad (x_1, x_2, \theta, \phi) \mapsto (f(x_1, x_2), \theta, \phi),$$

which satisfies

$$F^*(s\alpha_{\mathrm{st}}) = F^*(sr^2 d\theta + s(1-r^2)d\phi) = x_1 d\theta + x_2 d\phi.$$

The embedding $h = (h_1, h_2) : (0,1) \to \mathbb{R}_+ \times \mathbb{R}_+$ induces an embedding

$$H : (0,1) \times T^2 \to \mathbb{R}_+ \times \mathbb{R}_+ \times T^2, \qquad (r, \theta, \phi) \mapsto (h_1(r), h_2(r), \theta, \phi).$$

Hence the composition

$$F \circ H : (0,1) \times T^2 \to \mathbb{R}_+ \times (0,1) \times T^2$$

is an embedding satisfying

$$(F \circ H)^*(s\alpha_{\mathrm{st}}) = H^*(x_1 d\theta + x_2 d\phi) = h_1(r)d\theta + h_2(r)d\phi.$$

Taking the exterior derivative on both sides gives the lemma. □

As an application of this lemma, let us apply the construction of Theorem 3.37 in Section 3.9 to the SHS $(\omega = d\alpha, \lambda = \alpha)$ corresponding to a Stein fillable contact manifold (M, α). Here we choose the function h^1 to take values only in the first quadrant, as shown in Figure 1. Since the symplectization of (M, α) then embeds into the completion of the Stein filling, Lemma 3.41 implies

Corollary 3.42. *Let* $(\omega = d\alpha, \lambda = \alpha)$ *be the SHS corresponding to a Stein fillable contact* 3*-manifold* (M, α)*. Then the resulting stable Hamiltonian structure* (ω_1, λ_1) *in Theorem* 3.37 *embeds as a hypersurface in the completion of the same Stein manifold. For example, applying this to the standard contact structure on* S^3*, we obtain a hypersurface bounding a ball in standard* $\mathbb{R}^4$ *whose characteristic foliation has the properties in Theorem* 3.36*.*

Remark 3.43. Lemma 3.41 fails without the assumption that (h_1, h_2) takes values only in the positive quadrant—see Remark 6.23.

3.12. Foliated cohomology of integrable regions

Proposition 3.44. *Let* ω *be a Hamiltonian structure on a closed* 3*-manifold* M *with* $\ker\omega = \mathcal{L}$*. If* $\mathcal{L}$ *possesses an integrable region on which the slope function is nonconstant, then the foliated cohomology* $H^2_{\mathcal{L}}(M)$ *is infinite-dimensional.*

Remark 3.45. It follows that under the hypotheses of Proposition 3.44, $\ker(\kappa : H^2_{\mathcal{L}}(M) \to H^2(M;\mathbb{R}))$ is infinite-dimensional.

Proof of Proposition 3.44. Let $I \times T^2$ be an open integrable region on which $\omega = d\alpha$ with

$$\alpha = h_1(r)d\theta + h_2(r)d\phi$$

and nonconstant slope function $S(r) = h'(r)/|h'(r)|$. After shrinking I we may assume that $S(r)$ is nowhere constant in I. Note that the kernel foliation $\mathcal{L}$ is positively generated by the vector field

$$R = h_1'(r)\partial_\phi - h_2'(r)\partial_\theta.$$

Consider first a 2-form $\beta \in \Omega^2_{\mathcal{L}}(I \times T^2)$. Then $\beta = f d\alpha$ for an R-invariant function f. As f must be constant on each irrational torus, and irrational tori are dense (because the slope of R is nowhere constant), $f = f(r)$ depends only on r. This shows that

$$\Omega^2_{\mathcal{L}}(I \times T^2) = \ker(d : \Omega^2_{\mathcal{L}} \to \Omega^3_{\mathcal{L}}) = \{f(r)d\alpha \mid f : I \to \mathbb{R}\}.$$

Next we compute the image of $d : \Omega^1_{\mathcal{L}} \to \Omega^2_{\mathcal{L}}$. Thus consider $\mu \in \Omega^1_{\mathcal{L}}$. The condition $i_R\mu = 0$ implies

$$\mu = a dr + b(h_1'(r)d\theta + h_2'(r)d\phi)$$

for functions $a, b : I \times T^2 \to \mathbb{R}$. By the discussion above, $i_R d\mu = 0$ implies that $d\mu = f(r)d\alpha$ for a function $f : I \to \mathbb{R}$, which is equivalent to the conditions

$$\begin{aligned}
\frac{\partial}{\partial r}(bh_1') - \frac{\partial a}{\partial\theta} &= f(r)h_1'(r),\\
\frac{\partial}{\partial r}(bh_2') - \frac{\partial a}{\partial\phi} &= f(r)h_2'(r),\\
h_1'(r)\frac{\partial b}{\partial\phi} - h_2'(r)\frac{\partial b}{\partial\theta} &= 0.
\end{aligned}$$

The last equation shows that b is invariant under R and thus a function $b = b(r)$ of r only. The first two equations then show that $\partial a/\partial\theta$ and $\partial a/\partial\phi$ depend only on r, so by periodicity $a = a(r)$ depends only on r. The first two equations now combine to

$$b(r)h''(r) + b'(r)h'(r) = \frac{d}{dr}(b(r)h'(r)) = f(r)h'(r).$$

Since $h''(r)$ is linearly independent of $h'(r)$ for almost all r, this implies $b \equiv 0$, and therefore $\mu = a(r)dr$ and $d\mu = 0$. So we have shown

$$\mathrm{im}(d : \Omega^1_{\mathcal{L}} \to \Omega^2_{\mathcal{L}}) = 0 \quad \text{and} \quad H^2_{\mathcal{L}}(I \times T^2) = \{f(r)d\alpha \mid f : I \to \mathbb{R}\}.$$

Now if $f \in C_0^\infty(I)$ has compact support the 2-form $f(r)d\alpha$ extends by zero to a closed form in $\Omega^2_{\mathcal{L}}(M)$ which by the preceding argument is exact iff $f \equiv 0$, so $H^2_{\mathcal{L}}(M)$ contains the infinite-dimensional subspace $\{f(r)d\alpha \mid f \in C_0^\infty(I)\}$. □

3.13. Overtwisted disks

Legendrian and transverse knots play a central role in contact topology. The basic reason for this is that, as a consequence of Gray's stability theorem, the homotopy types of the spaces of Legendrian and transverse knots do not change during contact homotopies (see below for the precise formulation). In this subsection we show that any SHS in dimension 3 is homotopic to one containing an overtwisted disk, and as a consequence, the homotopy types of the spaces of Legendrian and transverse knots may change during stable homotopies.

Recall that an *overtwisted disk* in a contact 3-manifold $(M, \xi = \ker\lambda)$ is an embedded disk $D_{\mathrm{ot}} \subset M$ such that $TD_{\mathrm{ot}} = \xi$ along ∂D_{ot}.

Proposition 3.46. *Any SHS (ω_0, λ_0) on a closed oriented 3-manifold M is stably homotopic to a SHS (ω_1, λ_1) such that $\omega_1 = d\lambda_1$ on an open region $U \subset M$ and there exists an overtwisted disk $D_{\mathrm{ot}} \subset U$.*

Proof. By Proposition 3.31, after a stable homotopy we may assume that there exists an embedded solid torus $S^1 \times D$ in M on which $(\omega_0, \lambda_0) = (d\alpha_{\mathrm{st}}, \alpha_{\mathrm{st}})$ with

$$\alpha_{\mathrm{st}} = r^2 d\theta + (1 - r^2)d\phi.$$

Here (ϕ, r, θ) are polar coordinates on $S^1 \times D$ and $D = \{r \le r_2\}$ for some $r_2 > 0$.

Let $h_t = (h_{1t}, h_{2t}) : [0, r_2] \to \mathbb{R}^2$ be a family of immersions and $0 < r_* < r_0 < r_1 < r_2$ be points with the following properties (see Figure 2; the curve h_0 is dashed, the curve h_1 is bold, and the points $r = 0, r_*, r_0, r_1, r_2$ follow the direction of increase of the parameter r on the curve h_1; note that the h_1-axis is vertical and the h_2-axis horizontal):

- $h_0(r) = (r^2, 1 - r^2)$;
- $h_t(r) = (r^2, 1 - r^2)$ near $r = 0$ and $r = r_2$ for all t;
- $h_1(r) = (r^2, 1 - r^2)$ for $r \in [r_0, r_1]$;

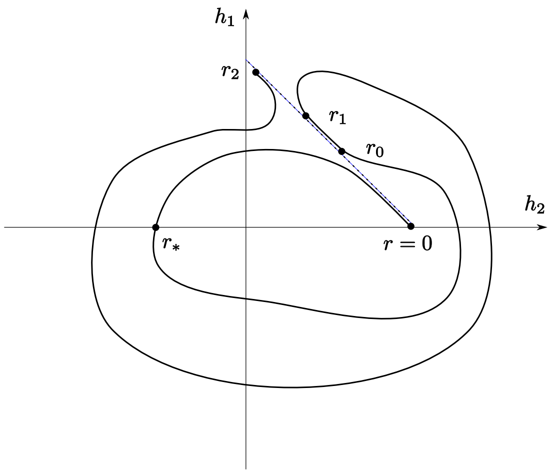

Fig. 2. Overtwisted disk.

- $h'_{11}h_{21} - h'_{21}h_{11} > 0$ on the interval $(0, r_1]$;
- $h_{11}(r_*) = 0$ and $h_{21}(r_*) < 0$.

This induces a homotopy of T^2-invariant HS $\omega_t := d\alpha_t$ with

$$\alpha_t := h_{1t}(r)d\theta + h_{2t}(r)d\phi.$$

Note that $\alpha_t = \alpha_{\text{st}}$ near $r = 0$ and $r = r_2$ for all t, and $\alpha_1 = \alpha_{\text{st}}$ for $r \in [r_0, r_1]$. Moreover, α_1 is a positive contact form on $\{r \leq r_1\}$. We apply Corollary 3.12 twice. First, we apply it to the homotopy $\{\omega_t\}_{t\in[0,1]}$ to get a family of T^2-invariant 1-forms λ_t stabilizing ω_t. Second, we apply Corollary 3.12 to the constant homotopy $\omega_1|_{\{r\in[r_0,r_2]\}}$. This gives us a stabilizing form $\hat{\lambda}$ for $\omega_1|_{\{r\in[r_0,r_2]\}}$ which coincides with α_1 on $\{r \in [r_0, r_0 + \varepsilon]\}$ (both forms equal α_{st} there). Since α_1 is a contact form on $\{r \leq r_1\}$, the form $\hat{\lambda}$ can be extended from $\{r \in [r_0, r_2]\}$ to $\{r \leq r_1\} = S^1 \times D$ as α_1. Thus $\hat{\lambda}$ stabilizes ω_1 and $\omega_1 = d\hat{\lambda}$ on $U := \{r < r_0 + \varepsilon\}$. We join λ_1 and $\hat{\lambda}$ by a linear homotopy. This shows that (ω_0, λ_0) is stably homotopic to the SHS $(\omega_1, \hat{\lambda})$. We claim that for each angle ϕ_*, the set $D_{\text{ot}} := \{r \leq r_*, \phi = \phi_*\} \subset U$ is an overtwisted disk for the contact form $\hat{\lambda}|_U = \alpha_1|_U$. Indeed, the conditions $h_{11}(r_*) = 0$ and $h_{21}(r_*) < 0$ imply that along ∂D_{ot} the contact planes $\xi_1 = \ker \alpha_1$ and the tangent spaces TD_{ot} are both spanned by the vectors ∂_r and ∂_θ. Renaming $\lambda_1 := \hat{\lambda}$ thus concludes the proof. □

Now consider a nowhere vanishing 1-form λ on an oriented 3-manifold M with kernel distribution $\xi = \ker \lambda$. Following the terminology from contact topology (see e.g. [25]), we call an oriented knot $\gamma : S^1 \to M$ *Legendrian* if $\lambda(\dot{\gamma}) \equiv 0$, and (*positively*) *transverse* if $\lambda(\dot{\gamma}) > 0$. Denote by Ω^1_{nv} the space of nowhere vanishing 1-forms on M and by Λ the space of embeddings $S^1 \hookrightarrow M$, both equipped with the C^∞-topology, and consider the projections

$$\pi^{\text{Leg}} : \{(\lambda,\gamma) \in \Omega^1_{\text{nv}} \times \Lambda \mid \lambda(\dot\gamma) \equiv 0\} \to \Omega^1_{\text{nv}},$$
$$\pi^{\text{trans}} : \{(\lambda,\gamma) \in \Omega^1_{\text{nv}} \times \Lambda \mid \lambda(\dot\gamma) > 0\} \to \Omega^1_{\text{nv}}.$$

Thus the fibres $(\pi^{\text{Leg}})^{-1}(\lambda)$ resp. $(\pi^{\text{trans}})^{-1}(\lambda)$ are the spaces of Legendrian resp. transverse knots for λ. As a consequence of Gray's stability theorem, the restrictions of π^{Leg} and π^{trans} to the preimages of the space of contact forms are locally trivial fibrations. By contrast, denote by $\mathcal{SHS}$ the space of SHS and consider the projections

$$\Pi^{\text{Leg}} : \{(\omega,\lambda,\gamma) \in \mathcal{SHS} \times \Lambda \mid \lambda(\dot\gamma) \equiv 0\} \to \mathcal{SHS},$$
$$\Pi^{\text{trans}} : \{(\omega,\lambda,\gamma) \in \mathcal{SHS} \times \Lambda \mid \lambda(\dot\gamma) > 0\} \to \mathcal{SHS}.$$

Corollary 3.47. *The projections Π^{Leg} and Π^{trans} are not Serre fibrations.*

Proof. By Proposition 3.46 there exists a homotopy (ω_t,λ_t) of SHS on M with $(\omega_0,\lambda_0) = (d\lambda_{\text{st}},\lambda_{\text{st}})$ such that $\omega_1 = d\lambda_1$ on an open region $U \subset M$ and there exists an overtwisted disk $D_{\text{ot}} \subset U$. We claim that this path in $\mathcal{SHS}$ cannot be lifted to a continuous path γ_t in the total space of Π^{Leg} with γ_1 parametrizing ∂D_{ot}.

Indeed, suppose such a lift γ_t exists. For each t let $\text{tb}(\gamma_t)$ be the *Thurston–Bennequin invariant*, defined as the linking number of γ_t with its pushoff in the Reeb direction. Then $t \mapsto \text{tb}(\gamma_t)$ is continuous and integer valued, hence constant, and therefore $\text{tb}(\gamma_0) = \text{tb}(\gamma_1) = 0$. But this contradicts Bennequin's inequality $\text{tb}(\gamma_0) \le -1$ for every Legendrian knot γ_0 in $(S^3,\lambda_{\text{st}})$.

This proves that Π^{Leg} is not a Serre fibration. An analogous argument shows that the path (ω_t,λ_t) cannot be lifted to a continuous path γ_t in the total space of Π^{trans} with γ_1 parametrizing a positive transverse pushoff of the negatively oriented boundary ∂D_{ot}.

Indeed, suppose such a lift γ_t exists. For each t let $\text{sl}(\gamma_t)$ be the *self-linking number*, defined as the linking number of γ_t with its pushoff in the direction of a nowhere vanishing section of $\ker\lambda$ over a spanning disk for γ_t. Then $t \mapsto \text{sl}(\gamma_t)$ is continuous and integer valued, hence constant, and therefore $\text{sl}(\gamma_0) = \text{sl}(\gamma_1) = 1$ (see [5]). But this contradicts Bennequin's inequality $\text{sl}(\gamma_0) \le -1$ for every transverse knot γ_0 in $(S^3,\lambda_{\text{st}})$. □

3.14. Contact structures as stable Hamiltonian structures

Every positive contact form λ induces a SHS $(d\lambda,\lambda)$ and homotopies of contact forms induce stable homotopies, so we have a natural map

$$\mathcal{CF}/\!\sim\; \to \mathcal{SHS}_0/\!\sim$$

from homotopy classes of positive contact forms to homotopy classes of exact SHS.

Theorem 3.48. *On S^3 the map $\mathcal{CF}/\!\sim\; \to \mathcal{SHS}_0/\!\sim$ is not bijective.*

For the proof, let us fix some conventions. All contact structures will be oriented, so that the underlying formal structures are oriented plane fields. For a contact form α the orientation on the contact structure $\ker\alpha$ is given by $d\alpha$. More generally, for any plane field ξ a 2-form γ restricting as an area form on ξ induces an orientation on ξ. The notation (ξ, γ) will stand for the corresponding oriented plane field. The corresponding homotopy class of oriented plane fields will be denoted by $[(\xi, \gamma)]$.

Consider

$$S^3 := \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \mid x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\}$$

with its standard orientation. Consider the orientation reversing involution

$$\Psi : S^3 \to S^3, \quad (x_1, y_1, x_2, y_2) \mapsto (-x_1, y_1, x_2, y_2).$$

For a 1-form α on S^3 we set $\bar{\alpha} := \Psi^*\alpha$. So the standard contact form and its pullback are given by

$$\begin{aligned} \lambda_{\mathrm{st}} &= x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2, \\ \bar{\lambda}_{\mathrm{st}} &= -x_1 dy_1 + y_1 dx_1 + x_2 dy_2 - y_2 dx_2. \end{aligned}$$

Lemma 3.49. *The standard contact form and its pullback under Ψ represent different homotopy classes of oriented plane fields:*

$$\mathcal{C} := [(\ker\lambda_{\mathrm{st}}, d\lambda_{\mathrm{st}})] \neq \bar{\mathcal{C}} := [(\ker\bar{\lambda}_{\mathrm{st}}, d\bar{\lambda}_{\mathrm{st}})].$$

Proof. The vector fields

$$\begin{aligned} R_+ &:= -y_1\partial_{x_1} + x_1\partial_{y_1} - y_2\partial_{x_2} + x_2\partial_{y_2}, \\ R_- &:= -y_1\partial_{x_1} + x_1\partial_{y_1} + y_2\partial_{x_2} - x_2\partial_{y_2} \end{aligned}$$

are transverse to $\ker\lambda_{\mathrm{st}}$ and $\ker\bar{\lambda}_{\mathrm{st}}$, respectively. Moreover, the transverse orientations defined by these vector fields together with the corresponding orientations of the plane fields define the standard orientation on S^3. We show that (in some trivialization) the Hopf invariants of these vector fields are different. For this we choose a framing of TS^3 as follows:

$$\begin{aligned} e_1 &:= R_+, \\ e_2 &:= (-x_2\partial_{x_1} + y_2\partial_{y_1} + x_1\partial_{x_2} - y_1\partial_{y_2}), \\ e_3 &:= (-y_2\partial_{x_1} - x_2\partial_{y_1} + y_1\partial_{x_2} + x_1\partial_{y_2}). \end{aligned}$$

In this framing R_+ is constant and thus has Hopf invariant zero. It remains to compute R_-. For this we use the Riemannian metric $\langle\ ,\ \rangle$ on S^3 induced by the standard Euclidean metric on $\mathbb{R}^4$. This gives us

$$\begin{aligned} \langle R_-, e_1\rangle &= (x_1^2 + y_1^2) - (x_2^2 + y_2^2), \\ \langle R_-, e_2\rangle &= 2(y_1x_2 + x_1y_2), \\ \langle R_-, e_3\rangle &= 2(y_1y_2 - x_1x_2). \end{aligned}$$

So in complex coordinates $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ the map $S^3 \to S^2$ induced by R_- is given by

$$(z_1, z_2) \mapsto (|z_1|^2 - |z_2|^2, -2iz_1z_2)$$

if we identify S^2 with the unit sphere in $\mathbb{R} \times \mathbb{C}$. Up to a factor of $-i$ this is just the Hopf fibration and thus has Hopf invariant 1. □

Proof of Theorem 3.48. Let λ_{ot} be an overtwisted contact form on S^3 with the same underlying class of oriented plane fields as λ_{st}. We distinguish two cases.

Case 1: $(d\lambda_{\text{st}}, \lambda_{\text{st}})$ and $(d\lambda_{\text{ot}}, \lambda_{\text{ot}})$ are stably homotopic. Then the map $\mathcal{CF}/\!\sim\, \to \mathcal{SHS}_0/\!\sim$ is not injective.

Case 2: $(d\lambda_{\text{st}}, \lambda_{\text{st}})$ and $(d\lambda_{\text{ot}}, \lambda_{\text{ot}})$ are not stably homotopic. In this case, observe that $[(\ker \bar{\lambda}_{\text{ot}}, d\bar{\lambda}_{\text{ot}})] = [(\ker \bar{\lambda}_{\text{st}}, d\bar{\lambda}_{\text{st}})] = \bar{\mathcal{C}}$. Consider the positive SHS $(d\bar{\lambda}_{\text{st}}, -\bar{\lambda}_{\text{st}})$ and $(d\bar{\lambda}_{\text{ot}}, -\bar{\lambda}_{\text{ot}})$ on S^3 and note that their underlying classes of oriented plane fields are both equal to $\bar{\mathcal{C}}$. Assume for contradiction that the map $\mathcal{CF}/\!\sim\, \to \mathcal{SHS}_0/\!\sim$ is surjective. Then there exist positive contact forms α_1 and α_2 on S^3 such that the SHS $(d\bar{\lambda}_{\text{st}}, -\bar{\lambda}_{\text{st}})$ is homotopic to $(d\alpha_1, \alpha_1)$ and the SHS $(d\bar{\lambda}_{\text{ot}}, -\bar{\lambda}_{\text{ot}})$ is homotopic to $(d\alpha_2, \alpha_2)$. Note that the underlying oriented plane field class for both α_1 and α_2 is $\bar{\mathcal{C}}$ and the class of λ_{st} is $\mathcal{C} \neq \bar{\mathcal{C}}$ by Lemma 3.49. Thus, by uniqueness of the positive tight contact structure on S^3 (see [19]), both α_1 and α_2 must be overtwisted. The classification of overtwisted contact structures [20] implies that α_1 and α_2 are homotopic through contact forms. By transitivity, this implies that the SHS $(d\bar{\lambda}_{\text{st}}, -\bar{\lambda}_{\text{st}})$ and $(d\bar{\lambda}_{\text{ot}}, -\bar{\lambda}_{\text{ot}})$ on S^3 are homotopic, so the SHS $(d\lambda_{\text{st}}, \lambda_{\text{st}})$ and $(d\lambda_{\text{ot}}, \lambda_{\text{ot}})$ are homotopic, contradicting the assumption of Case 2.

□

Considerations from symplectic field theory (see Section 6.7) suggest that the SHS $(d\lambda_{\text{ot}}, \lambda_{\text{ot}})$ and $(d\lambda_{\text{st}}, \lambda_{\text{st}})$ on S^3 are not stably homotopic. Then Case 1 in the preceding proof would not occur and the discussion of Case 2 shows that the map $\mathcal{CF}/\!\sim\, \to \mathcal{SHS}_0/\!\sim$ would not be surjective. Moreover, this map would be injective in view of the classification of contact structures on S^3 in [19, 20]. This leads us to

Conjecture 3.50. *The map in Theorem 1.7 is injective but not surjective.*

4. Structure results in dimension three

4.1. A structure theorem

Theorem 4.1. *Let (ω, λ) be a SHS on a closed 3-manifold M and set $f := d\lambda/\omega$. Then there exists a (possibly disconnected and possibly with boundary) compact 3-dimensional submanifold N of M, invariant under the Reeb flow, a (possibly empty) disjoint union $U = U_1 \cup \cdots \cup U_k$ of compact integrable regions and a stabilizing 1-form $\tilde{\lambda}$ for ω with the following properties:*

- $\operatorname{int} U \cup \operatorname{int} N = M$;
- *the proportionality coefficient $\tilde{f} := d\tilde{\lambda}/\omega$ is constant on each connected component of N;*

- *on each* $U_i \cong [0,1] \times T^2$ *the SHS* (ω, λ) *is exact and* T^2*-invariant and* $f(r,z) = \alpha_i r + \beta_i$ *for constants* $\alpha_i > 0$, $\beta_i \in \mathbb{R}$;
- $\tilde{\lambda}$ *is* C^1*-close to* λ.

Moreover, if ω *is exact we can arrange that* $\tilde{f}$ *attains only nonzero values on* N.

Proof. Let $\tilde{f}$ and the decomposition $M = \bigcup_{i=1}^k U_i' \cup N = M$ be obtained from Corollary 3.25. Recall that the $U_i' \cong (a_i', b_i') \times T^2$ are open and for r sufficiently close to a_i' or b_i' we have $\{r\} \times T^2 \subset N$. Thus we may replace each interval by a slightly smaller closed interval $[a_i, b_i] \subset (a_i', b_i')$ such that the closed integrable regions $U_i := [a_i, b_i] \times T^2$ satisfy $\bigcup_i \operatorname{int} U_i \cup \operatorname{int} N = M$.

Next recall from Corollary 3.25 that on each $U_i \cong [a_i, b_i] \times T^2$ the function f is given by the projection onto the first factor and $\tilde{f} = \sigma \circ f$ for a function $\sigma : \mathbb{R} \to \mathbb{R}$. So the function $\tilde{f}$ is constant on the tori $\{r\} \times T^2$, $r \in [a_i, b_i]$. Therefore, for each integrable region U_i the full version of Theorem 3.3 applies to give an identification $U_i \cong [0,1] \times T^2$ (linear on the first factor) in which both ω and $\tilde{\lambda}$ are T^2-invariant and ω is exact.

For the last statement, denote by N^+, N^0, N^- the union of components of N on which $\tilde{f}$ is positive (resp. zero, negative). We want to get rid of N^0 if $[\omega] = 0$. Pick a primitive β of ω and consider the 1-form $\lambda_t := \tilde{\lambda} + t\beta$ for small positive t. Then $d\lambda_t/\omega = \tilde{f} + t$, so λ_t stabilizes ω and has proportionality factor shifted by t. We choose t such that $0 < t < -\min_{N^-} \tilde{f}$. Then $\tilde{f} + t$ is positive on $N^+ \cup N^0$ and negative on N^-, so the stabilizing 1-form λ_t is nonzero on N. Clearly, we can choose λ_t C^1-close to $\tilde{\lambda}$ and hence to λ. Renaming back λ_t to $\tilde{\lambda}$ concludes the proof of Theorem 4.1. □

Theorem 4.1 is the formulation of the structure theorem that will be most useful for applications. The following corollary gives an alternative formulation in which the structure is more restrictive but we lose C^1-closeness.

Corollary 4.2. *Every stable Hamiltonian structure on a closed* 3*-manifold* M *is stably homotopic to a SHS* (ω, λ) *for which there exists a (possibly disconnected and possibly with boundary) compact* 3*-dimensional submanifold* $N = N^+ \cup N^- \cup N^0$ *of* M*, invariant under the Reeb flow, and a (possibly empty) disjoint union* $U = U_1 \cup \cdots \cup U_k$ *of compact integrable regions with the following properties:*

- $\operatorname{int} U \cup \operatorname{int} N = M$;
- $d\lambda = \pm\omega$ *on* $N^\pm$ *and* $d\lambda = 0$ *on* N^0;
- *on each* $U_i \cong [0,1] \times T^2$ *we have* $(\omega, \lambda) = (\omega_h, \lambda_g)$ *for functions* $h, g : [0,1] \to \mathbb{C}$ *satisfying* (18).

Moreover, we can always arrange that N^+ *is nonempty, and if* ω *is exact we can arrange that* N^0 *is empty.*

Proof. Let a SHS (ω, λ) be given. After applying the homotopy in Proposition 3.31, we may assume that $d\lambda = \omega$ on some open region $V \subset M$, so 1 is a singular value of the function $f = d\lambda/\omega$.

After applying Theorem 4.1 and renaming $\tilde{\lambda}, \tilde{f}$ back to λ, f we may assume that there exist invariant compact 3-dimensional submanifolds N with the following properties:

- $\operatorname{int} U \cup \operatorname{int} N = M$;
- on each connected component N_i of N we have $f \equiv c_i$ for some constant $c_i \in \mathbb{R}$;
- on each $U_i \cong [0,1] \times T^2$ the SHS (ω, λ) is exact and T^2-invariant.

Moreover, if ω is exact we can arrange that $c_i \neq 0$ for all i. Since 1 was a singular value of the original function f, there will be some positive c_i.

By Lemma 3.9, after shrinking the U_i may assume that on each $U_i \cong [0,1] \times T^2$ we have $(\omega, \lambda) = (\omega_h, \lambda_g)$ for functions $h, g : [0,1] \to \mathbb{C}$. By Remark 3.13, after a stable homotopy supported in $\operatorname{int} U$ we can assume that the slope function $h'/|h'|$ is nonconstant on each $U_i \setminus N$.

Now we define a homotopy of stabilizing 1-forms λ_t, $t \in [0,1]$, for ω as follows. On each component N_i with $c_i = 0$ we set $\lambda_t := \lambda$. On each component N_i with $c_i \neq 0$ we set

$$\lambda_t := (1-t)\lambda + t|c_i|^{-1}\lambda.$$

We use Proposition 3.17 to extend this homotopy to a homotopy of stabilizing 1-forms over all integrable regions U_i. The homotopy starts at $\lambda_0 = \lambda$, and the proportionality coefficient $f_1 = d\lambda_1/\omega$ of λ_1 takes only values $0, 1, -1$ on N. Renaming λ_1 back to λ concludes the proof of Corollary 4.2. □

The following version of the structure theorem will be used in [17].

Corollary 4.3. *Every stable Hamiltonian structure on a closed 3-manifold M is stably homotopic to a SHS (ω, λ) for which there exists a (possibly disconnected and possibly with boundary) compact 3-dimensional submanifold $N = N^+ \cup N^- \cup N^0$ of M, invariant under the Reeb flow, and a (possibly empty) disjoint union $U = U_1 \cup \cdots \cup U_k$ of compact integrable regions with the following properties:*

- *$\operatorname{int} U \cup \operatorname{int} N = M$;*
- *the proportionality coefficient $f := d\lambda/\omega$ is constant positive resp. negative on each connected component of N^+ resp. N^-;*
- *on each $U_i \cong [0,1] \times T^2$ the SHS (ω, λ) is exact and T^2-invariant and f is nowhere zero;*
- *on N^0 there exists a closed 1-form $\bar\lambda$ representing a primitive integer cohomology class $\bar\lambda \in H^1(N^0; \mathbb{Z})$ such that $\bar\lambda \wedge \omega > 0$ and $\bar\lambda$ is T^2-invariant near ∂N^0.*

Proof. Consider a SHS (ω, λ) and apply the Structure Theorem 4.1 to find a new stabilizing 1-form $\tilde\lambda$. Denote by N^+, N^0, N^- the union of components of N on which $\tilde f = d\tilde\lambda/\omega$ is positive (resp. zero, negative). If the original proportionality coefficient $f = d\lambda/\omega$ is nowhere zero, then we may assume that the new proportionality coefficient $\tilde f$ is nowhere zero and $(\omega, \tilde\lambda)$ has all the desired properties (with $N^0 = \emptyset$). So suppose that f has nonempty zero set $f^{-1}(0)$. The proof of Theorem 4.1 (using Proposition 3.23 with Z the set of critical values together with the value $a = 0$) allows us to arrange that $f^{-1}(0)$ is contained in the interior of the flat part N^0.

The new stabilizing form $\tilde\lambda$ restricts as a closed form to N^0. We C^1-perturb $\tilde\lambda|_{N^0}$ to get a 1-form $\hat\lambda$ on N^0 representing a rational cohomology class. Let $V \cong [0,1] \times T^2 \subset N^0$ be a part of an integrable region sitting in N^0 as a collar neighbourhood of one of its boundary components $\{1\} \times T^2$. Since the restriction $\tilde{\lambda}|_V$ is T^2-invariant and $\hat{\lambda}$ is C^1-close to $\tilde{\lambda}$, the T^2 average λ_{inv} of $\hat{\lambda}$ on V is C^1-close to $\hat{\lambda}$. As λ_{inv} and $\hat{\lambda}$ represent the same cohomology class on V, we can write $\hat{\lambda} = \lambda_{\text{inv}} + d\chi$ for a smooth function χ on V. Moreover, C^1-closeness of λ_{inv} and $\hat{\lambda}$ allows us to choose χ also C^1-small. Let ρ be a cutoff function on $[0, 1]$ which equals 1 near 0 and 0 near 1. Set $\bar{\lambda} := \lambda_{\text{inv}} + d(\rho\chi)$ on V and extend this form as $\hat{\lambda}$ inside N^0. The closed form $\bar{\lambda}$ is T^2-invariant near the boundary of N^0. Note also that

$$\bar{\lambda} - \hat{\lambda} = \lambda_{\text{inv}} + d(\rho\chi) - (\lambda_{\text{inv}} + d\chi) = d(\rho\chi) - d\chi = d(\chi(\rho - 1))$$

on V. Therefore, the difference $\bar{\lambda} - \hat{\lambda}$ is exact on N^0 and thus $\bar{\lambda}$ represents a rational cohomology class. Assume the procedure above has been performed near all boundary components of N^0. Now C^1-smallness of χ implies C^1-smallness of $\rho\chi$. This together with the computation above and the C^1-smallness of the difference $\hat{\lambda} - \tilde{\lambda}$ ensures that $\bar{\lambda} \wedge \omega > 0$ on N^0. After multiplying $\bar{\lambda}$ with a rational number we may assume that it represents a primitive integer cohomology class in $H^1(N^0; \mathbb{Z})$.

The set N^0 and the 1-form $\bar{\lambda}$ on it have the desired properties. However, the new proportionality coefficient $\tilde{f}$ may still vanish on some integrable region U_i. To remedy this, we choose $\delta > 0$ so small that $\{f < \delta\} \subset N^0$. Now we apply Theorem 4.1 *again* to the original SHS (ω, λ) to obtain new $\tilde{\lambda}$, $\tilde{f}$ and new regions $\tilde{N}^\pm$, $\tilde{N}^0$, $\tilde{U}_i$. Moreover, we can make $\|\tilde{\lambda} - \lambda\|_{C^1}$ so small that $\tilde{N}^0 \subset \{\tilde{f} = 0\} \subset \{f < \delta\} \subset N^0$. In particular, $\tilde{f}$ does not vanish on a neighbourhod W of $M \setminus \text{int}\, N^0$. Now the SHS $(\omega, \tilde{\lambda})$, the *old* set N^0 and 1-form $\bar{\lambda}$, and the intersections of the *new* sets $\tilde{U}_i$ and $\tilde{N}^\pm$ with W satisfy all conditions in Corollary 4.3. □

4.2. Discreteness of homotopy classes of stable Hamiltonian structures

Theorem 4.4. *Let $(\bar{\omega}, \bar{\lambda})$ be a SHS on a closed 3-manifold M. Then for every SHS (ω, λ) which is sufficiently C^5-close to $(\bar{\omega}, \bar{\lambda})$ and satisfies $[\omega] = [\bar{\omega}] \in H^2(M; \mathbb{R})$ there exists a stable homotopy (ω_t, λ_t), $t \in [0, 1]$, such that $\omega_0 = \bar{\omega}$ and $\omega_1 = \omega$. Moreover, the homotopy ω_t can be chosen C^1-small.*

Corollary 4.5. *For any closed oriented 3-manifold M and cohomology class $\eta \in H^2(M; \mathbb{R})$ there are at most countably many homotopy classes of SHS representing η.*

Proof. Consider a family of pairwise nonhomotopic SHS $(\omega_i, \lambda_i)_{i \in I}$ representing the class η, for some index set I. Consider the space $\mathcal{SHS}_\eta$ of SHS representing η, equipped with the C^5-topology. By Theorem 4.4, each ω_i has an open neighbourhood $\mathcal{U}_i$ in $\mathcal{SHS}_\eta$ such that all elements in $\mathcal{U}_i$ are stably homotopic to (ω_i, λ_i). Hence $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$ for $i \neq j$, so the $(\mathcal{U}_i)_{i \in I}$ are disjoint open sets in $\mathcal{SHS}_\eta$. Since $\mathcal{SHS}_\eta$ is second countable (as a subset of the second countable linear space of pairs of 2- and 1-forms), this is only possible if I is countable. □

Proof of Theorem 4.4. **Step 1.** Fix a SHS $(\bar{\omega}, \bar{\lambda})$ on a closed 3-manifold M. Pick an adapted decomposition $M = \bigcup_i U_i \cup N$ as in Theorem 4.1.

Consider a SHS (ω, λ) sufficiently C^5-close to $(\bar\omega, \bar\lambda)$ with $[\omega] = [\bar\omega]$. By Theorem 3.7(b), after pulling back (ω, λ) by a diffeomorphism C^2-close to identity, we may assume that (ω, λ) is exact and T^2-invariant on U_i and C^1-close to $(\bar\omega, \bar\lambda)$. By Remark 3.13 we can adjust $(\bar\omega, \bar\lambda)$ so that it has nonconstant slope on each U_i.

By Lemma 3.28 we can write $\omega = \bar\omega + d\alpha$ for a C^1-small 1-form α. After applying Lemma 3.9(b) and shrinking the U_i, we may assume that α is T^2-invariant on each U_i (and the slope of $\bar\omega$ is still nonconstant on the new U_i).

Now we replace $\bar\lambda$ by the new stabilizing 1-form provided by Theorem 4.1 (C^1-close to the old one and still denoted by $\bar\lambda$) such that $\bar f = d\bar\lambda/\bar\omega$ is constant on each connected component of N.

Step 2. Consider first the region N. Fix a smooth function $\tau : [0,1] \to [0,1]$ which equals 0 near 0 and 1 near 1 and define $\omega_t := \bar\omega + \tau(t) d\alpha$ on N. Denote by N^c the union of components of N on which $\bar f \equiv c \in \mathbb{R}$ and define 1-forms λ_t on N by $\lambda_t := \bar\lambda + \tau(t) c\alpha$ on N^c. These 1-forms are C^1-close to $\bar\lambda$ and satisfy $d\lambda_t = c(\bar\omega + \tau(t) d\alpha) = c\omega_t$ on N^c, so λ_t stabilizes ω_t on N. It remains to extend the homotopy (ω_t, λ_t) over the integrable regions U_i.

Step 3. Consider an integrable region $U_i \cong [0,1] \times T^2$. After applying Lemma 3.9(c) and shrinking the U_i, we can write $\bar\omega = d\alpha_{\bar h}$, $\bar\lambda = \lambda_{\bar g}$, $\omega = d\alpha_h$, $\lambda = \lambda_g$, and $\alpha = \alpha_\xi$ for functions $\bar h, \bar g, \xi, h = \bar h + \xi, g : [0,1] \to \mathbb{C}$, where the pairs $(\bar h, \bar g)$ and (h, g) satisfy (18) and ξ is C^1-small. For some small ε the homotopy ω_t is already defined on $[0,\varepsilon] \cup [1-\varepsilon, 1] \times T^2$ and we can write it as $\omega_t = d\alpha_{h_t}$ for $h_t = \bar h + \tau(t)\xi$. Using Remark 3.13 we adjust ξ on $[\varepsilon, 1-\varepsilon]$ so that the slope $(\bar h + \xi)'/|(\bar h + \xi)'|$ is nonconstant. We extend the homotopy h_t from $[0,\varepsilon] \cup [1-\varepsilon, 1]$ to $[0,1]$ so that the slope $h_t'/|h_t'|$ is nonconstant on $[\varepsilon, 1-\varepsilon]$ throughout the homotopy and h_t stays in a small C^1-neighbourhood of $\bar h$. Since on $[0,\varepsilon] \cup [1-\varepsilon, 1] \times T^2$ the homotopy ω_t is stabilized by α_t, Proposition 3.17 allows us to extend this homotopy throughout $[0,1] \times T^2$. This concludes the proof of Theorem 4.4. □

4.3. Approximation of stable Hamiltonian structures by Morse–Bott ones

Theorem 4.6. *For any SHS (ω, λ) on a closed oriented 3-manifold M and any $\varepsilon > 0$ there exists a stable homotopy $\gamma = \{(\omega_t = \omega + d\mu_t, \lambda_t)\}_{t \in [0,1]}$ such that (ω_1, λ_1) is Morse–Bott and*

$$\|\gamma\|_{C^1} := \max_{t \in [0,1]} (\|\dot\mu_t\|_{C^1} + \|\dot\lambda_t\|_{C^1}) < \varepsilon.$$

Let us first recall the definition of Morse–Bott from [9]. A SHS (ω, λ) with Reeb vector field R is called *Morse–Bott* if the following holds: For all $T > 0$ the set $\mathcal{N}_T \subset M$ formed by the T-periodic Reeb orbits is a closed submanifold, the rank of $\omega|_{\mathcal{N}_T}$ is locally constant, and $T_p\mathcal{N}_T = \ker(D_p\Phi_T - \mathbb{1})$ for all $p \in \mathcal{N}_T$, where Φ_t is the Reeb flow. Note that the case $\dim \mathcal{N}_T = 1$ corresponds to nondegeneracy of closed Reeb orbits. In this case we call the SHS *Morse*.

Next we prove a version of Theorem 4.6 for T^2-invariant SHS. Consider an integrable region $I \times T^2$ with SHS (ω_h, λ_g) as in Section 3.4, $h = (h_1, h_2)$ and $g = (g_1, g_2)$. Its

Reeb vector field is given by

$$R = \frac{-h_2'\partial_\theta + h_1'\partial_\phi}{h_1'g_2 - h_2'g_1}.$$

So T^2-families of periodic Reeb orbits occur whenever the slope $h'/|h'|$ is rational.

Lemma 4.7. (a) *The SHS (ω_h, λ_g) on $I \times T^2$ is Morse–Bott iff $h_1''h_2' - h_1'h_2'' \neq 0$ whenever $h'/|h'|$ is rational.*

(b) *Assume that (ω_h, λ_g) is already Morse–Bott in a neighbourhood of $\partial I \times T^2$. Then for each $\varepsilon > 0$ there exists a stable homotopy $\gamma = \{(\omega_{h_t}, \lambda_{g_t})\}_{t\in[0,1]}$ such that $(\omega_{h_1}, \lambda_{g_1})$ is Morse–Bott and $\|\gamma\|_{C^1} < \varepsilon$.*

Proof. (a) The Reeb flow for time T is given by

$$\Phi_T \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} = \begin{pmatrix} r \\ \theta - k_2(r)T \\ \phi + k_1(r)T \end{pmatrix}, \qquad k_i := \frac{h_i'}{h_1'g_2 - h_2'g_1},$$

and its linearization equals

$$D\Phi_T = \begin{pmatrix} 1 & 0 & 0 \\ -k_2'T & 1 & 0 \\ k_1'T & 0 & 1 \end{pmatrix}.$$

Thus $\ker(D\Phi_T - \mathbb{1}) = \text{span}\{\partial_\theta, \partial_\phi\}$ iff $(k_1', k_2') \neq (0, 0)$. A short computation shows that the latter condition is equivalent to $h_1''h_2' - h_1'h_2'' \neq 0$.

(b) Suppose that (ω_h, λ_g) is Morse–Bott near ∂I, but not on all of I (otherwise there is nothing to show). Then $h'/|h'|$ is not constant on I. By replacing h with $\tilde{h} := h + \xi$ for a C^∞-small perturbation ξ supported away from the boundary we can arrange that $\tilde{h}_1''\tilde{h}_2' - \tilde{h}_1'\tilde{h}_2'' \neq 0$ whenever $\tilde{h}'/|\tilde{h}'|$ is rational, and $\tilde{h}'/|\tilde{h}'|$ is still nonconstant. Then according to Lemma 3.15 we can C^1-perturb g accordingly rel boundary so that (18) continues to hold.

It remains to check C^1-smallness of the corresponding stable homotopy γ. It can be written as $(\omega_t, \lambda_t) = (d\alpha_h + td\alpha_\xi, \lambda_{g_t})$, $t \in [0, 1]$, where g_t is obtained from $h + t\xi$ by Lemma 3.15. We can take as primitives $\mu_t = t\alpha_\xi$, so that $\dot{\mu}_t = \alpha_\xi$, which is C^1-small if ξ is. The derivative $\dot{\lambda}_{g_t}$ is C^1-small because $\dot{g}_t$ is, due to the estimate $\|\dot{g}_t\|_{C^1} \leq \|\xi\|_{C^1}$ from Lemma 3.15(b). □

Proof of Theorem 4.6. We will construct the homotopy γ in four steps, each step providing a C^1-small homotopy $\{\gamma_i(t)\}_{t\in[0,1]}$. The desired homotopy $\{\gamma(t)\}_{t\in[0,1]}$ will then be given by $\gamma(t) = \gamma_i(4t)$ for $t \in [(i-1)/4, i/4]$. Since $\|\gamma\|_{C^1} \leq 4 \max_i \|\gamma_i\|_{C^1}$, the homotopy γ will be C^1-small as well.

Step 1. Given (ω, λ), let $\tilde{\lambda}$ be the stabilizing 1-form for ω given by Theorem 4.1. We linearly homotope λ to $\tilde{\lambda}$ by $\lambda_t := (1-t)\lambda + t\tilde{\lambda}$. Since $\tilde{\lambda}$ is C^1-close to λ, the corresponding stable homotopy $\gamma = \{\omega, \lambda_t\}_{t\in[0,1]}$ is C^1-small. We rename $\tilde{\lambda}$ back to λ and $\tilde{f}$ back to f.

Step 2. Let $U_i \cong [0,1] \times T^2$ be one of the integrable regions from Theorem 4.1. Recall that in the situation of Theorem 4.1, for some $\varepsilon > 0$ and some constants $c_-, c_+ \in \mathbb{R}$ we have $d\lambda = c_-\omega$ for $r \in [0, 2\varepsilon]$ and $d\lambda = c_+\omega$ for $r \in [1-2\varepsilon, 1]$. Since (ω, λ) is exact and T^2-invariant on $[0,1] \times T^2$, by Lemma 3.9 we can write $\omega = d\alpha_h$ and $\lambda = \lambda_g + g_3(r)dr$ for functions $h, g : [0,1] \to \mathbb{C}$ satisfying (18) and $g_3 : [0,1] \to \mathbb{R}$. (The term $g_3(r)dr$ does not affect the stabilization property and will remain unchanged throughout the following discussion.) Moreover, we have $g = c_\pm h + d_\pm$ on $[0, 2\varepsilon]$ resp. $[1-2\varepsilon, 1]$, for constants $d_\pm \in \mathbb{C}$. Consider a C^∞-small perturbation ξ supported in $(0, 2\varepsilon) \cup (1-2\varepsilon, 1)$ of h such that the slope $(h+\xi)'/|(h+\xi)'|$ is constant irrational near ε and $1-\varepsilon$. The corresponding stable homotopy $\gamma = \{\omega_{h_t}, \lambda_{g_t} + g_3(r)dr\}_{t\in[0,1]}$ with $h_t = h + t\xi$ and $g_t = c_\omega h_t + d_\pm$ on $[0, 2\varepsilon]$ resp. $[1-2\varepsilon, 1]$ is clearly C^1-small. We rename (h_1, g_1) back to (h, g).

Step 3. After Step 2 we may assume that the slope $h'/|h'|$ is constant irrational near ε and $1-\varepsilon$. According to Lemma 4.7, we can modify (h, g) by a C^1-small homotopy supported in $(\varepsilon, 1-\varepsilon)$ to make the SHS Morse–Bott on a neighbourhood of $[\varepsilon, 1-\varepsilon] \times T^2$. After having performed this homotopy, denote by $Y \subset M$ the union of the $[\varepsilon, 1-\varepsilon] \times T^2$ for all integrable regions. Then $M \setminus \operatorname{int} Y = \bigcup_i N_i$ is a disjoint union of compact regions N_i on which $d\lambda = c_i\omega$ for constants $c_i \in \mathbb{R}$. Moreover, each boundary component of N_i is a 2-torus lying in an integrable region in which the Reeb vector field has irrational slope.

Step 4. Consider a region $N = N_i$ as in Step 3 on which $d\lambda = c\omega$ for some constant $c \in \mathbb{R}$. Note that the Reeb vector field has no periodic orbits near ∂N. According to [12, Theorem B.1], we can find a C^∞-small 1-form ν compactly supported in $\operatorname{int} N$ such that all periodic orbits of $\omega + d\nu$ in N are nondegenerate. (This is proved in [12] in the case without boundary; it carries over to the case with boundary provided there are no periodic orbits near the boundary.) We replace λ by the stabilizing form $\lambda + c\nu$ on N. The corresponding stable homotopy $\{\omega + td\nu, \lambda + tc\nu\}_{t\in[0,1]}$ is C^1-small. Performing such a perturbation on every region N_i, we obtain the desired Morse–Bott SHS, and Theorem 4.6 is proved. □

5. Deformations of stable Hamiltonian structures

Throughout this section, (ω, λ) is a fixed stable Hamiltonian structure with Reeb vector field R.

Definition 5.1. A (*possibly nonexact*) *deformation* of ω is (the germ of) a smooth family $\{\omega_t\}_{t\in[0,\varepsilon)}$ of closed 2-forms with $\omega_0 = \omega$. It is called *exact* if $[\dot\omega_t] = 0$. A (*possibly nonexact*) *stable deformation* (ω, λ) is (the germ of) a smooth family $\{(\omega_t, \lambda_t)\}_{t\in[0,\varepsilon)}$ of SHS with $(\omega_0, \lambda_0) = (\omega, \lambda)$. We call a deformation $\{\omega_t\}_{t\in[0,\varepsilon)}$ on ω *stabilizable* if for some $\delta \le \varepsilon$ there exists a smooth family $\{\lambda_t\}_{t\in[0,\delta)}$ of stabilizing 1-forms for ω_t with $\lambda_0 = \lambda$.

Note that, by contrast to the rest of the paper, in this section we consider exact as well as nonexact deformations.

Our goal in this section is to describe the set of those deformations that are stabilizable. In full generality this question seems to be quite hard and we have been able to answer it only in some special cases. In particular, we provide examples of (exact as well as nonexact) deformations that are not stabilizable.

5.1. Linear stabilizability

Linearizing at $t = 0$ gives a necessary linear condition for stabilizability. To derive it, consider a (possibly nonexact) stable deformation $\{(\omega_t, \lambda_t)\}_{t\in[0,\varepsilon)}$ of $(\omega_0, \lambda_0) = (\omega, \lambda)$ with Reeb vector fields R_t. Differentiating the equations $i_{R_t}\omega_t = 0 = i_{R_t}d\lambda_t$ at $t = 0$ yields

$$i_{\dot{R}_0}\omega + i_R\dot{\omega}_0 = 0 = i_{\dot{R}_0}d\lambda + i_R d\dot{\lambda}_0.$$

For given $\dot{\omega}_0$, we consider the first equation

$$i_{\hat{R}}\omega = -i_R\dot{\omega}_0 \tag{26}$$

as an equation for $\hat{R} = \dot{R}_0$. Note, first, that equation (26) is always solvable because its right-hand side evaluates to zero on the kernel of ω and, second, that the difference between any two solutions of this equation is a multiple of R. Fixing a solution $\hat{R}$ of (26), we consider the second equation

$$i_R d\hat{\lambda} = -i_{\hat{R}}d\lambda \tag{27}$$

as an equation for the 1-form $\hat{\lambda}$. Note that the right-hand side of (27) does not depend on the choice of $\hat{R}$ because the difference between any two such choices is a multiple of R contracting to zero with $d\lambda$.

We call a deformation $\{\omega_t\}_{t\in[0,\varepsilon)}$ of ω *linearly stabilizable* if equation (27) has a solution, where $\hat{R}$ is determined by (26). Note that this condition depends only on $\dot{\omega}_0$. We can reformulate it as follows:

Lemma 5.2. *A (possibly nonexact) deformation $\{\omega_t\}_{t\in[0,\varepsilon)}$ is linearly stabilizable if and only if there exists a smooth family $\{(\lambda_t, R_t)\}_{t\in[0,\varepsilon)}$ of 1-forms and vector fields satisfying $(\lambda_0, R_0) = (\lambda, R)$, $i_{R_t}\omega_t = 0$, and $i_{R_t}d\lambda_t = O(t^2)$.*

Proof. The "if" follows from the preceding discussion. For the "only if", assume that there exists $\hat{\lambda}$ solving (27). Consider the linear family of 1-forms $\lambda_t := \lambda_0 + t\hat{\lambda}$ for $t \in [0, \varepsilon)$. Pick a smooth family of vector fields R_t spanning $\ker \omega_t$ with $R_0 = R$ (we may normalize it by $\lambda_t(R_t) = 1$ for small t). As above, differentiating $i_{R_t}\omega_t = 0$ at $t = 0$ shows that $\hat{R} = \dot{R}_0$ solves (26). In order to check that $i_{R_t}d\lambda_t = O(t^2)$ we differentiate the expression $i_{R_t}d\lambda_t$ at $t = 0$. This gives us

$$\frac{d}{dt}\bigg|_{t=0} i_{R_t}d\lambda_t = \frac{d}{dt}\bigg|_{t=0} i_{R_t}(d\lambda_0 + td\hat{\lambda}) = i_{\dot{R}_0}d\lambda + i_R d\hat{\lambda}$$

and the last expression is zero because $\hat{\lambda}$ solves (27). □

5.2. Deformations of contact structures

In this subsection we assume that the stable Hamiltonian structure is positive or negative contact, i.e. $\omega = \pm d\lambda$. To warm up, suppose that the deformation $\{\omega_t\}_{[0,\varepsilon)}$ is exact, i.e. $[\dot\omega_t] = 0$. Then we can choose a smooth path of primitives σ_t of $\omega_t - \omega$ and $\lambda_t := \lambda \pm \sigma_t$ is a path of contact forms stabilizing $\pm d\lambda_t = \omega_t$. So every exact deformation of a contact structure is stabilizable.

For nonexact deformations, the linearized problem simplifies because of the extra information that $\omega = \pm d\lambda$. Indeed, (27) can be rewritten using (26) as

$$i_R d\hat\lambda = -i_{\hat R} d\lambda = \mp i_{\hat R}\omega = \pm i_R \dot\omega_0,$$

and hence

$$i_R(d\hat\lambda \mp \dot\omega_0) = 0. \tag{28}$$

Solvability of this equation for $\hat\lambda$ is equivalent to existence of a closed 2-form θ $(= \dot\omega_0 \mp d\hat\lambda)$ with

$$[\theta] = [\dot\omega_0] \in H^2(M;\mathbb{R}), \qquad i_R\theta = 0. \tag{29}$$

This condition has a natural interpretation in terms of foliated cohomology (see Section 2.7) of the foliation $\mathcal{L}$ defined by R: For a deformation $\{\omega_t\}$ of a positive or negative contact SHS, condition (29) for linear stabilizability is equivalent to

$$[\dot\omega_0] \in \operatorname{im}[\kappa : H^2_{\mathcal{L}}(M) \to H^2(M;\mathbb{R})]. \tag{30}$$

Thus any deformation with $[\dot\omega_0] \notin \operatorname{im}\kappa$ will not be stabilizable. Below we give some examples in which this happens.

Example 5.3. As in Section 2.9, let M be a compact quotient of $PSL(2,\mathbb{R})$. Let $\alpha_\pm$ be a positive resp. negative contact form coming from a left-invariant form on $PSL(2,\mathbb{R})$, and let $\mathcal{L}$ be the foliation generated by its Reeb vector field R. By the discussion at the end of Section 2.9, $\kappa : H^2_{\mathcal{L}}(M) \to H^2(M;\mathbb{R})$ is the zero map. Hence according to (30), a deformation ω_t of the SHS $(\omega,\lambda) = (d\alpha_\pm, \pm\alpha_\pm)$ is linearly stabilizable if and only if $[\dot\omega_0] = 0$.

As another example, let T^n be the n-torus with the standard flat Euclidean metric and $\pi : S^*T^n \cong T^n \times S^{n-1} \to T^n$ its unit cotangent bundle. Let

$$i : T^n \to T^n \times S^{n-1} = S^*T^n$$

be the inclusion of $T^n \times \{p\}$ for a fixed point $p \in S^{n-1}$. Let R be the Reeb vector field of the (contact) Liouville form λ on S^*T^n and $\mathcal{L}$ the foliation generated by R. The following lemma was pointed out to us by J. Bowden.

Lemma 5.4. *In the notation above,*

$$\operatorname{im}[\kappa : H^2_{\mathcal{L}}(S^*T^n) \to H^2(S^*T^n;\mathbb{R})] = \ker[i^* : H^2(S^*T^n;\mathbb{R}) \to H^2(T^n;\mathbb{R})].$$

Therefore, a deformation $\{\omega_t\}$ of $(d\lambda,\lambda)$ with $i^*[\dot\omega_0] \neq 0$ is not linearly stabilizable.

Proof. Assume first that θ is a closed 2-form on S^*T^n which satisfies $\iota_R\theta = 0$. We claim that $i^*[\theta] = 0$, i.e. $\langle[\theta], i_*b\rangle = 0$ for each $b \in H_2(T^n)$. This immediately follows from the fact that $i_*\big(H_2(T^n)\big)$ is generated by homology classes which are represented by smooth tori invariant under the Reeb flow. To see the latter fact, write T^n as $\mathbb{R}^n/\mathbb{Z}^n$ and let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$. It is well known that the homology of T^n in degree two is generated by the 2-dimensional subtori

$$T_{jk} = \langle e_j, e_k\rangle/\mathbb{Z}^2 \subset T^n, \quad 1 \le j < k \le n.$$

Hence we can choose as generators of $i_*(H_2(S^*T^n))$ the 2-tori

$$\widetilde{T}_{jk} = T_{jk} \times \{e_j\} \subset T^n \times S^{n-1}, \quad 1 \le j < k \le n.$$

The Reeb flow ϕ_R^τ for $\tau \in \mathbb{R}$ on S^*T^n is given by the geodesic flow. Since geodesics on the torus are just straight lines, we obtain the simple formula

$$\phi_R^\tau(x, y) = (x + \tau y, y), \quad (x, y) \in T^n \times S^{n-1}, \quad \tau \in \mathbb{R}.$$

Hence the 2-tori $\widetilde{T}_{jk}$ are invariant under the geodesic flow. This proves the claim and thus $\operatorname{im}\kappa \subset \ker i^*$.

For $n \ge 3$ we are done since in that case $i^* : H^2(S^*T^n; \mathbb{R}) \to H^2(T^n; \mathbb{R})$ is an isomorphism and therefore $\operatorname{im}\kappa = \{0\}$. It remains to show $\ker i^* \subset \operatorname{im}\kappa$ in the case $n = 2$. For this, pick coordinates (x, y, ϕ) on S^*T^2, where x and y are coordinates on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $\phi \in S^1$ is the angular coordinate on the fibre. Then the Reeb vector field is given by

$$R = \cos\phi\, \partial_x + \sin\phi\, \partial_y.$$

A short computation shows that a 2-form θ is closed and satisfies $i_R\theta = 0$ precisely if it is of the form

$$\theta = f(\phi)(\cos\phi\, dy - \sin\phi\, dx) \wedge d\phi$$

for a 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$. Then the values of $[\theta]$ on the subtori $C_x = \{x = 0\}$ and $C_y = \{y = 0\}$ are

$$[\theta](C_x) = \int_0^{2\pi} f(\phi)\cos\phi\, d\phi, \quad [\theta](C_y) = \int_0^{2\pi} f(\phi)\sin\phi\, d\phi.$$

Since these values can be arbitrarily prescribed by the choice of f, this shows that every cohomology class in $\ker i^*$ can be represented by such a form θ, and hence $\ker i^* \subset \operatorname{im}\kappa$. □

The case $n = 2$ is particularly interesting:

Proposition 5.5. *In the notation of Lemma 5.4, let β be the standard area form on $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ and consider the (nonexact) family of Hamiltonian structures $\omega_t = d\lambda + t\beta$ for $t \in \mathbb{R}$. Then*

(i) *each ω_t is stabilizable, but*
(ii) *there is no smooth family $\{\lambda_t\}_{t\in\mathbb{R}}$ of 1-forms λ_t stabilizing ω_t. (More precisely, smoothness always fails at $t = 0$.)*

Put simply: stabilizing forms exists, but they never form a smooth family. We emphasize that in (ii) it is *not* assumed that $\lambda_0 = \lambda$.

Proof. (i) To see stabilizability, note that $\omega_0 = d\lambda$ is stabilizable because λ is a contact form. For $t \neq 0$ consider the connection form λ_{st} for the flat metric on the torus. In coordinates (x, y, ϕ) where x and y are coordinates on T^2 and ϕ is the angular coordinate on the fibre, the Liouville form, the connection form and the standard area form on T^2 write out as

$$\lambda = \cos\phi\, dx \wedge + \sin\phi\, dy, \qquad \lambda_{\text{st}} = d\phi, \qquad \beta = dx \wedge dy.$$

It follows that $d\lambda_{\text{st}} = 0$ and $\lambda_{\text{st}} \wedge \omega_t = t\, dx \wedge dy \wedge d\phi$, so the 1-form $\operatorname{sign}(t)\lambda_{\text{st}}$ defines a taut foliation and stabilizes ω_t for $t \neq 0$.

(ii) To see the second assertion, assume for contradiction there exists a smooth family $\{\lambda_t\}_{t\in\mathbb{R}}$ stabilizing ω_t. We distinguish two cases according to the proportionality coefficient $f_0 = d\lambda_0/\omega_0$ for $t = 0$.

Case 1: f_0 is constant. Note that in this case $f_0 \neq 0$: otherwise, the closed stabilizing form λ_0 would define a foliation with the exact 2-form $d\lambda$ restricting as an area form to the leaves, which is impossible. Thus we are in the contact situation for $t = 0$, and therefore the existence of a smooth family $\{\lambda_t\}_{t\in\mathbb{R}}$ stabilizing $\{\omega_t\}_{t\in\mathbb{R}}$ contradicts Lemma 5.4.

Case 2: f_0 is not constant. We will show that this case cannot occur. For this, we need some dynamical information about the kernel foliation $\ker\omega_t$ of ω_t. Since

$$\omega_t = -\sin\phi\, d\phi \wedge dx + \cos\phi\, d\phi \wedge dy + t\, dx \wedge dy,$$

the lift of $\ker\omega_t$ from $S^1 \times T^2$ to $S^1 \times \mathbb{R}^2$ is generated by the vector field

$$X_t = \cos\phi\, \partial_x + \sin\phi\, \partial_y - t\, \partial_\phi.$$

It is easy to see that the projection of an orbit of X_t to the (x, y)-plane is a curve, parametrized by arc length, with the property that its tangent vector is rotating with constant speed t. This means that the projection of any orbit to the (x, y)-plane is a circle. Thus the kernel foliation of ω_t consists of closed leaves. In particular, $\ker\omega_t$ does not possess any irrational invariant tori. But this contradicts Theorem 3.7(c) applied to $(d\lambda, \lambda_0)$. Namely, fix some $\phi_* \in S^1$. Then $(S^1 \setminus \{\phi_*\}) \times T^2 \cong (0, 2\pi) \times T^2$ and $\ker\omega_0$ descends to each torus $\{\phi\} \times T^2$ as a linear foliation with slope function $S_0(\phi) = \phi$. In particular, $(0, 2\pi) \times T^2$ is an integrable region on which the slope function of $\ker\omega_0$ is nowhere constant. The latter implies that f_0 is T^2-invariant. Since by assumption f_0 is nonconstant, we can apply Theorem 3.7(c) to deduce the existence of irrational invariant tori. This contradiction shows that Case 2 does not occur and concludes the proof of Proposition 5.5. □

We have seen several examples of nonexact deformations which are not stabilizable. Our next goal is to give an example of an *exact* nonstabilizable deformation. For this we need some preparation from functional analysis.

5.3. Preliminaries from analysis

Let V be a smooth real vector bundle over a compact manifold M (possibly with boundary). We consider the space $\Gamma(V)$ of all smooth sections of V as a Fréchet space and let F be any closed linear subspace of $\Gamma(V)$. Fix a base point $x_0 \in F$ and set

$$\tilde{F} := \{x \in C^\infty([0,1], F) \mid x(0) = x_0\}$$

to be the smooth path space of F. Our next goal is give the affine space $\tilde{F}$ the structure of an affine Fréchet space. Indeed, let $\pi : [0,1] \times M \to M$ be the natural projection. Any path $x \in \tilde{F}$ can be considered as a section s_x of the pullback bundle $\pi^* V$ as follows: for $t \in [0,1]$ and $p \in M$ we set

$$s_x(t,p) := x(t)(p) \in \pi^* V.$$

We identify the path space $\tilde{F}$ with the corresponding affine subspace in $\Gamma(\pi^* V)$. Now the space of sections $\Gamma(\pi^* V)$ is Fréchet and the subspace $\tilde{F}$ is closed because F is closed in $\Gamma(V)$. Thus $\tilde{F}$ is itself an affine Fréchet space. The following straightforward lemma will be crucial for us.

Lemma 5.6. *For each $T \in (0,1]$ the evaluation map*

$$\mathrm{ev}_T : \tilde{F} \to F, \quad \mathrm{ev}_T(x) := x(T),$$

is open and continuous.

Proof. The map ev_T is a surjective linear map between affine Fréchet spaces. It is continuous because it is obtained by composing on the right with the map i_T sending $p \in M$ to $(T,p) \in [0,1] \times M$ as follows: $\mathrm{ev}_T(s_x) = s_x \circ i_T$. Finally, ev_T is open by the open mapping theorem. □

Note that for any open and continuous map its restriction to any open subset of the domain remains open and continuous. We apply this to the map ev_T as follows: Let $E \subset F$ be any open subset with $x_0 \in E$. The set

$$\tilde{E} := \{x \in C^\infty([0,1], E) \mid x(0) = x_0\} \tag{31}$$

is an open subset of $\tilde{F}$, so the restriction of ev_T to $\tilde{E}$ is open and continuous.

Recall that a subset B of a topological space X is called a *Baire set* (or of second category) if it contains a countable intersection of open and dense sets. Note that if we have a continuous open map $f : X \to Y$ between topological spaces and $B \subset Y$ is a Baire set, then $f^{-1}(B)$ is Baire in X. Indeed, the preimage of an open set is open because the map is continuous, and the preimage of a dense set is dense because the map is open, so the preimage of a countable intersection of open and dense sets is a countable intersection of open and dense sets. We apply this to the map ev_T and summarize our discussion as follows.

Lemma 5.7. *Let M be a closed manifold, $V \to M$ a smooth real vector bundle, $F \subset \Gamma(V)$ a closed linear subspace, $E \subset F$ an open subset, $x_0 \in E$ a base point, $B \subset E$ a Baire set, and $T \in (0,1]$ a point. Then for $\tilde{E}$ defined by (31) and the evaluation map $\mathrm{ev}_T : \tilde{E} \to E$ at T the preimage $\mathrm{ev}_T^{-1}(B)$ is Baire in $\tilde{E}$.*

The next step is to see that $\tilde{E}$ is a *Baire space*, i.e. any Baire subset is dense. This holds because $\tilde{E}$ is an open subset of the affine Fréchet space $\tilde{F}$, and open subsets of complete metrizable spaces have the Baire property.

Now we fix a Baire set $B \subset E$ and define the following subset of $\tilde{E}$:

$$\tilde{B} := \{x \in \tilde{E} \mid \exists\, t_n \to 0 : x(t_n) \in B\}.$$

Clearly,

$$\tilde{B} \supset \bigcap_{n=1}^{\infty} \{x \in \tilde{E} \mid x(1/n) \in B\} = \bigcap_{n=1}^{\infty} \mathrm{ev}_{1/n}^{-1}(B).$$

Each set $\mathrm{ev}_{1/n}^{-1}(B)$ is Baire by Lemma 5.7, so their intersection is Baire, thus $\tilde{B}$ is a Baire set. In particular, $\tilde{B}$ is dense in $\tilde{E}$.

5.4. *Exact deformations*

We now apply the discussion of the previous subsection to Hamiltonian structures on a closed oriented manifold M as follows (using the same notation):

- $V = \Lambda^2(M)$ is the bundle of exterior 2-forms over M;
- $F \subset \Gamma(V) = \Omega^2(M)$ is the space of closed 2-forms of a given cohomology class η;
- $E = \mathcal{HS}_\eta \subset F$ is the open subset of Hamiltonian structures of cohomology class η;
- the base point x_0 is a fixed Hamiltonian structure ω_0.

Lemma 5.8. *The subset $B \subset \mathcal{HS}_\eta$ of Hamiltonian structures of cohomology class η that are* Morse*, i.e. all periodic orbits are nondegenerate, is a Baire set.*

Proof. Consider a HS $\omega \in \mathcal{HS}_\eta$. As in Section 2.2, we realize M as the hypersurface $\{0\} \times M$ in its symplectization

$$(X := [-\varepsilon, \varepsilon] \times M,\ \Omega := \omega + t d\lambda + dt \wedge \lambda).$$

Nearby hypersurfaces $M' \subset X$ are graphs over $\{0\} \times M$, so pulling back $\Omega|_{M'}$ under the projection $M \to M'$ along $\mathbb{R}$ yields HS ω' on M cohomologous to ω. By [12, Theorem B.1], there exist hypersurfaces M' C^∞-close to $\{0\} \times M$ for which $\Omega|_{M'}$ is Morse. This shows that the set B is dense.

Let us fix a Riemannian metric on M and parametrize all orbits of $\ker \omega$, for $\omega \in \mathcal{HS}_\eta$, with unit speed. For $N \in \mathbb{N}$ denote by $B_N \subset \mathcal{HS}_\eta$ the set of HS of class η for which all periodic orbits of period $\leq N$ are nondegenerate. A standard argument shows that B_N is open. (Consider a sequence $\omega_k \notin B_N$ converging to $\omega \in \mathcal{HS}_\eta$; let γ_k be degenerate periodic orbits of ω_k of period $\leq N$; by the Ascoli–Arzelà theorem a subsequence of γ_k converges to a degenerate periodic orbit of ω of period $\leq N$; hence $\omega \notin B_N$.) Since $B \subset B_N$ and B is dense, we see that B_N is dense. Hence $B = \bigcap_{N \in \mathbb{N}} B_N$ is a Baire set. □

We keep the notation of the previous subsection. Thus

- $\tilde{E}$ is the space of smooth paths $\{\omega_t\}_{t\in[0,1]}$ in $\mathcal{HS}_\eta$ starting at ω_0;
- $\tilde{B} \subset \tilde{E}$ is the subset of those paths for which there exists a sequence $t_n \to 0$ such that ω_{t_n} is Morse.

We will refer to $\tilde{E}$ as the *space of exact deformations of* ω_0. By the discussion in the previous subsection, $\tilde{B}$ is a Baire set. This is what we mean when we say "a Hamiltonian structure can be deformed slightly to make it Morse".

If the HS ω_0 is contact, i.e. $\omega_0 = d\lambda_0$ for a contact form λ_0, the above argument justifies the folklore lemma that we can always deform a contact form slightly to make it Morse. This is in sharp contrast to the situation with general SHS: Theorem 3.37 provides ("quite a lot of") SHS which cannot even be C^2-approximated by Morse SHS because any C^2-close SHS must have rational invariant tori by Theorem 3.7(c). In particular, for a SHS (ω_0, λ_0) as in Theorem 3.7(c) the deformations in the Baire set $\tilde{B}$ cannot be stabilized. This discussion shows

Theorem 5.9. *Consider a* 1*-dimensional foliation* $\mathcal{L}$ *as in Theorem* 3.36*. Then for any HS* ω_0 *defining* $\mathcal{L}$ *there exists a Baire set* $\tilde{B}$ *in the space* $\tilde{E}$ *of exact deformations of* ω_0 *that cannot be stabilized, no matter what stabilizing* 1*-form* λ_0 *we take for* ω_0. □

6. Homotopies and cobordisms of stable Hamiltonian structures

A smooth homotopy of contact forms $(\lambda_t)_{t\in[0,1]}$ on a closed manifold M gives rise to a symplectic cobordism $([0,1]\times M, d(e^{ct}\lambda_t))$ from $(M, d\lambda)$ to $(M, e^c d\lambda)$ for a sufficiently large constant $c > 0$. The corresponding statement for stable Hamiltonian structures fails (see e.g. Corollary 6.21 below). In this section we investigate the relation between stable homotopies and various generalized notions of symplectic cobordism. We discuss obstructions to symplectic cobordisms, as well as obstructions to ambient stable homotopies arising from Rabinowitz Floer homology and potential obstructions arising from symplectic field theory.

Recall that for all homotopies of HS ω_t we assume that the cohomology class of ω_t is constant.

6.1. Various notions of cobordism

For a SHS (ω, λ) with $\ker\omega = \mathcal{L}$ we introduce the following notation:

$$\begin{aligned}
\delta^+_{(\omega,\lambda)} &:= \sup\{\tau > 0 \mid \ker(\omega + td\lambda) = \mathcal{L} \text{ for all } t \in [0,\tau)\},\\
\delta^-_{(\omega,\lambda)} &:= \sup\{\tau > 0 \mid \ker(\omega + td\lambda) = \mathcal{L} \text{ for all } t \in (-\tau,0]\},\\
\delta_{(\omega,\lambda)} &:= \min\{\delta^+_{(\omega,\lambda)}, \delta^-_{(\omega,\lambda)}\},\\
I_{(\omega,\lambda)} &:= (-\delta^-_{(\omega,\lambda)}, \delta^+_{(\omega,\lambda)}),\\
C_{(\omega,\lambda)} &:= \{\omega + td\lambda \mid t \in I_{(\omega,\lambda)}\}.
\end{aligned}$$

Moreover, for a HS ω we define

$$D^+_\omega := \{\hat\omega \in \Omega^2(M) \mid d\hat\omega = 0,\ [\hat\omega] = [\omega] \in H^2(M;\mathbb{R}),\ \ker\hat\omega = \ker\omega,\ \hat\lambda \wedge \hat\omega^{n-1} > 0\},$$

where $\hat\lambda$ is any 1-form with $\hat\lambda|_{\ker\omega} > 0$. Note that the definition of D^+_ω only depends on the oriented kernel foliation $\ker\omega$ and the cohomology class of ω. Also note that $C_{(\omega,\lambda)} \subset D^+_\omega$ for a SHS (ω, λ).

Now we turn to symplectic cobordisms. We will only consider cobordisms that are *topologically trivial*, i.e. of the form $[a, b] \times M$.

Definition 6.1. A (*topologically trivial*) *symplectic cobordism* between HS ω_a and ω_b on M is a symplectic manifold $([a, b] \times M, \Omega)$ such that $\Omega|_{\{i\}\times M} = \omega_i$ for $i = a, b$. A symplectic cobordism $([a, b] \times M, \Omega)$ is called *trivial* if

$$\Omega = \omega + d(f(t)\lambda)$$

for some SHS (ω, λ) on M and an increasing function $f : [a, b] \to I_{(\omega,\lambda)}$. A *homotopy of symplectic cobordisms* between ω_a and ω_b is a smooth family $\{\Omega^s\}_{s\in[0,1]}$ of symplectic forms on $[a, b] \times M$ with $\Omega^s|_{\{i\}\times M} = \omega_i$ for $i = a, b$ and all s.

Note that if $([a, b] \times M, \Omega)$ is a symplectic cobordism, then $\omega_t := \Omega|_{\{t\}\times M}$ defines a homotopy of HS from ω_a to ω_b. Note that the ω_t represent the same cohomology class, so there exists a smooth path of 1-forms μ_t with $\mu_0 = 0$ and $\omega_t = \omega_a + d\mu_t$. In the following *all homotopies will come with a chosen smooth path of primitives* μ_t. Conversely, a homotopy of HS $\omega_t = \omega_a + d\mu_t$ induces a closed 2-form

$$\Omega := \omega_a + d\mu_t + dt \wedge \dot\mu_t$$

on $[a, b] \times M$, which is symplectic inducing the orientation of $[a, b] \times M$ iff

$$\dot\mu_t|_{\ker\omega_t} > 0. \tag{32}$$

The above definition of symplectic cobordism is too rigid. For example, a HS is not cobordant to itself with this definition. Therefore, we now introduce two more flexible notions.

Definition 6.2. A *strong symplectic cobordism* between SHS (ω_a, λ_a) and (ω_b, λ_b) on M is a symplectic manifold $([a, b] \times M, \Omega)$ such that $\Omega|_{\{i\}\times M} \in C_{(\omega_i,\lambda_i)}$ for $i = a, b$. A *weak symplectic cobordism* between HS ω_a and ω_b on M is a symplectic manifold $([a, b] \times M, \Omega)$ such that $\Omega|_{\{i\}\times M} \in D^+_{\omega_i}$ for $i = a, b$. Two SHS (ω_a, λ_a) and (ω_b, λ_b) are called (*strongly* resp. *weakly*) *bicobordant* if they are (strongly resp. weakly) cobordant both ways. Homotopies of strong resp. weak cobordisms are defined analogously to Definition 6.1.

Note that a strong cobordism is in particular a weak cobordism, but not the other way around. Also note that a trivial cobordism is in particular a strong cobordism from (ω, λ) to itself and such trivial cobordisms exist for all SHS. Finally we point out that, while cobordisms in the sense of Definition 6.1 can obviously be composed, this is not true for strong or weak cobordisms. We will discuss this in Section 6.3.

6.2. Short homotopies and strong cobordisms

Consider now a homotopy of *stable* HS $(\omega_t = \omega_a + d\mu_t, \lambda_t)$ with Reeb vector fields R_t. We try to build a symplectic form on $[a, b] \times M$ by

$$\Omega := \omega_a + d\nu_t + dt \wedge \dot{\nu}_t, \qquad \nu_t := \mu_t + f(t)\lambda_t$$

for some smooth function $f : [a, b] \to \mathbb{R}$. The form Ω is symplectic iff $\omega_t + f(t)d\lambda_t$ is maximally nondegenerate for all t, and ν_t satisfies (32). The first condition holds if $|f(t)| < \delta_{(\omega_t,\lambda_t)}$ (as defined in Section 6.1). Condition (32) for ν_t is equivalent to

$$(\dot{\mu}_t + \dot{f}(t)\lambda_t + f(t)\dot{\lambda}_t)(R_t) > 0.$$

We associate to the homotopy $\gamma := \{\omega_t = \omega_a + d\mu_t, \lambda_t\}$ the following quantities:

$$\delta := \delta(\gamma) := \min_t \delta_{(\omega_t,\lambda_t)}, \quad A := A(\gamma) := \max_t |\dot{\lambda}_t(R_t)|, \quad B := B(\gamma) := \max_t |\dot{\mu}_t(R_t)|.$$

Then symplecticity of Ω is implied by

$$|f(t)| < \delta, \qquad \dot{f}(t) > A|f(t)| + B.$$

These conditions are satisfied by the linear function

$$f(t) = (A\delta + B + (b-a)^{-1}\delta)(t - a/2 - b/2)$$

provided that

$$(A\delta + B)(b - a) < \delta.$$

Define the *length* of the homotopy γ as

$$L(\gamma) := (A + B/\delta)(b - a).$$

Then we have shown

Lemma 6.3. *To every homotopy $\gamma := \{(\omega_t, \lambda_t)\}_{t\in[a,b]}$ of length $L(\gamma) < 1$ we can canonically associate a symplectic cobordism*

$$C(\gamma) := ([a, b] \times M, \Omega)$$

such that for all $t \in [a, b]$ we have $\Omega|_{\{t\}\times M} \in C_{(\omega_t,\lambda_t)}$. If γ is a constant homotopy, then $C(\gamma)$ is a trivial cobordism, and if $\{\gamma^s\}_{s\in[0,1]}$ is a path of homotopies, then $C(\gamma^s)$ is a homotopy of cobordisms. □

Example 6.4. If $[a, b] = [0, 1]$ and $\omega_t = \omega$ is independent of t we have $B = 0$ and $L(\gamma) = A$. Moreover, in this case the stabilizing 1-forms λ_0 and λ_1 can be connected by a linear homotopy $\lambda_t = (1 - t)\lambda_0 + t\lambda_1$. A short computation yields

$$R_t = \frac{R_0}{1 - t + t\lambda_1(R_0)}, \qquad \dot{\lambda}_t(R_t) = \frac{\lambda_1(R_0) - 1}{1 - t + t\lambda_1(R_0)}.$$

So $L(\gamma)$ is small whenever λ_1 is C^0-close to λ_0.

Let us collect the dependence of the length on certain natural operations on homotopies.

(Reparametrization) Given a homotopy $\gamma = \{\gamma_t\}_{t\in[a,b]}$ and a smooth function $\tau : [a', b'] \to [a, b]$ consider the reparametrized homotopy $\gamma \circ \tau = \{\gamma_{\tau(t)}\}_{t\in[a',b']}$. With $K := \max_t |\dot\tau(t)|$ the constants change as follows:

$$A(\gamma \circ \tau) \le K A(\gamma), \qquad B(\gamma \circ \tau) \le K B(\gamma),$$
$$\delta(\gamma \circ \tau) = \delta(\gamma), \qquad L(\gamma \circ \tau) \le \frac{K(b' - a')}{(b - a)} L(\gamma).$$

In particular, reparametrization by a nonconstant linear function τ does not change the length. Moreover, we can reparametrize a homotopy to make it constant near a and b, or to turn a piecewise smooth homotopy into a smooth one, with arbitrarily small change in length.

(Rescaling) Replacing the stabilizing 1-forms λ_t by $c\lambda_t$ for a constant t has the following effect on the constants:

$$R_t \mapsto R_t/c, \quad A \mapsto A, \quad B \mapsto B/c, \quad \delta \mapsto \delta/c, \quad L \mapsto L.$$

Replacing the 2-forms ω_t by $c\omega_t$ for a constant t has the following effect on the constants:

$$\mu_t \mapsto c\mu_t, \quad A \mapsto A, \quad B \mapsto cB, \quad \delta \mapsto c\delta, \quad L \mapsto L.$$

In particular, the length is invariant under both rescalings.

(Concatenation) Two homotopies $\gamma = \{\gamma_t\}_{t\in[a,b]}$ and $\gamma' = \{\gamma'_t\}_{t\in[b,c]}$ with $\gamma_b = \gamma'_b$ can be concatenated in the obvious way to a homotopy $\gamma \# \gamma'$ on the interval $[a, c]$. Then we have

$$A(\gamma \# \gamma') = \max\{A(\gamma), A(\gamma')\}, \qquad B(\gamma \# \gamma') = \max\{B(\gamma), B(\gamma')\},$$
$$\delta(\gamma \# \gamma') = \min\{\delta(\gamma), \delta(\gamma')\}, \qquad L(\gamma \# \gamma') \ge L(\gamma) + L(\gamma').$$

(Restriction) Restricting a homotopy γ to a subinterval $[a', b'] \subset [a, b]$ does not increase A and B and does not decrease δ, so

$$L(\gamma|_{[a',b']}) \le \frac{b' - a'}{b - a} L(\gamma).$$

In particular, any homotopy γ can be decomposed into homotopies of arbitrarily small length.

(Reversal) For a homotopy $\gamma = \{\gamma_t\}_{t\in[a,b]}$ the reversed homotopy $\gamma^{-1} = \{\gamma_{2b-t}\}_{t\in[b,2b-a]}$ retains the same constants and in particular the same length. The homotopy $\gamma \# \gamma^{-1}$, which runs from γ_1 to γ_b and back to γ_a over the interval $[a, 2b - a]$, has length

$$L(\gamma \# \gamma^{-1}) = 2L(\gamma).$$

We see that for concatenation there is no triangle type inequality in general. Nevertheless, the following lemma provides an upper bound on $L(\gamma \# \gamma')$ in terms of $L(\gamma)$ and $L(\gamma')$ for certain concatenations. For a stable homotopy $\gamma = \{\omega_t = \omega_0 + d\mu_t, \lambda_t\}_{t\in[0,1]}$ we define the quantity

$$\|\gamma\|_{C^1} := \max_{t\in[0,1]} (\|\dot\mu_t\|_{C^1} + \|\dot\lambda_t\|_{C^1}).$$

We say that γ is C^1-*small* if $\|\gamma\|_{C^1}$ is small.

Lemma 6.5. *For every SHS (ω, λ) and every $\varepsilon > 0$ there exists $\rho > 0$ with the following property. Whenever $\gamma = \{\omega_t, \lambda_t\}_{t\in[-1,0]}$ is a stable homotopy ending at (ω, λ) and $\gamma' = \{\omega_t, \lambda_t\}_{t\in[0,1]}$ is a stable homotopy starting at (ω, λ) such that $\|\gamma'\|_{C^1} < \rho$, then $L(\gamma \# \gamma') \le L(\gamma) + \varepsilon$.*

Proof. It follows imediately from the definition that the assignment $(\omega, \lambda) \mapsto \delta_{(\omega,\lambda)}$ is lower semicontinuous with respect to the $C^0 \times C^1$-topology, i.e. for each SHS (ω, λ) and $\varepsilon > 0$ there exists a $C^0 \times C^1$-neighbourhood V of (ω, λ) such that for all $(\tilde\omega, \tilde\lambda) \in V$ we have

$$\delta_{(\tilde\omega,\tilde\lambda)} \ge \delta_{(\omega,\lambda)} - \varepsilon/2.$$

Note that for each $t \in [0, 1]$ we have $\|\omega_t - \omega\|_{C^0} + \|\lambda_t - \lambda\|_{C^1} \le \|\gamma'\|_{C^1}$, hence for $\rho > \|\gamma'\|_{C^1}$ sufficiently small we have $(\omega_t, \lambda_t) \in V$ and thus

$$\delta(\gamma') \ge \delta_{(\omega,\lambda)} - \varepsilon/2 \ge \delta(\gamma) - \varepsilon/2.$$

Moreover, by the (Concatenation) properties we have

$$A(\gamma \# \gamma') \le A(\gamma) + A(\gamma') \le A(\gamma) + \rho, \qquad B(\gamma \# \gamma') \le B(\gamma) + B(\gamma') \le B(\gamma) + \rho.$$

These estimates combine to $L(\gamma \# \gamma') \le L(\gamma) + \varepsilon$ for ρ sufficiently small. □

The main result of this subsection is

Proposition 6.6. *Let $\gamma = \{\omega_t, \lambda_t\}_{t\in[0,1]}$ be a homotopy of SHSs of length $L(\gamma) < 1/3$. Then there exists a symplectic cobordism $([0, 3] \times M, \Omega)$ with the following properties:*

- $\Omega|_{\{t\}\times M} \in C_{(\omega_{\tau(t)},\lambda_{\tau(t)})}$, *where $\tau : [0, 3] \to [0, 1]$ is the function*

$$\tau(t) = \begin{cases} t, & t \in [0, 1], \\ 2 - t, & t \in [1, 2], \\ t - 2, & t \in [2, 3]. \end{cases}$$

- $([0, 2] \times M, \Omega)$ *and* $([1, 3] \times M, \Omega)$ *are homotopic to trivial cobordisms.*

More colloquially, this means that short stable homotopies give rise to strong bicobordisms whose compositions are defined and homotopic to trivial cobordisms.

Proof. Since $|\tau'| \le 1$, the reparametrized homotopy $\gamma \circ \tau$ has length $L(\gamma \circ \tau) \le 3L(\gamma) < 1$. Hence by Lemma 6.3 there exists a symplectic cobordism $C(\gamma \circ \tau) = ([0, 3] \times M, \Omega)$ with $\Omega|_{\{t\}\times M} \in C_{(\omega_{\tau(t)},\lambda_{\tau(t)})}$. By the same lemma, the path of homotopies $\{\gamma \circ \tau^s\}_{s\in[0,2]}$ with fixed endpoints, where $\tau^s(t) = s\tau(t)$, induces a homotopy of cobordisms $C(\gamma \circ \tau^s)$ from $([0, 2] \times M, \Omega)$ to a trivial cobordism. An analogous argument applies to $([1, 3] \times M, \Omega)$. □

6.3. Large homotopies and weak cobordisms

If we drop the shortness condition on the homotopy, both Lemma 6.3 and Proposition 6.6 fail.

Proposition 6.7. *On any closed oriented 3-manifold M there exist a homotopy $(\omega_t, \lambda_t)_{t\in[0,1]}$ of exact SHS such that there exists no strong symplectic cobordism from (ω_0, λ_0) to (ω_1, λ_1).*

We will discuss in this subsection two obstructions to symplectic cobordisms. The proof of Proposition 6.7 is based on

First obstruction: helicity. Suppose that $\dim M = 3$ and recall from Section 2.8 the helicity $\mathrm{Hel}(\omega) = \int_M \alpha\wedge\omega$ of an exact 2-form $\omega = d\alpha$. Consider a symplectic cobordism $([0, 1] \times M, \Omega)$ with $\Omega|_{\{i\}\times M} = \omega_i = d\alpha_i$ for $i = 0, 1$. This Ω is exact, i.e. $\Omega = d\gamma$ for some 1-form γ on $[0, 1] \times M$, and we obtain

$$0 < \int_{[0,1]\times M} \Omega \wedge \Omega = \int_{[0,1]\times M} d(\gamma \wedge \Omega) = \int_{\{1\}\times M} \gamma \wedge d\alpha_1 - \int_{\{0\}\times M} \gamma \wedge d\alpha_0$$
$$= \int_M \alpha_1 \wedge \omega_1 - \int_M \alpha_0 \wedge \omega_0 = \mathrm{Hel}(\omega_1) - \mathrm{Hel}(\omega_0).$$

Hence helicity is monotone under symplectic cobordisms. To obtain from this an obstruction to strong cobordisms we need to control the helicity $\mathrm{Hel}(\hat\omega)$ for $\hat\omega \in C_{(\omega,\lambda)}$. In the presence of integrable regions, this can be done by choosing a sufficiently "exotic" stabilizing 1-form:

Lemma 6.8. *Let $(\omega = d\alpha, \lambda_0)$ be an exact SHS on a closed 3-manifold M. Suppose that there exists an integrable region $I \times T^2$ in M on which $(d\alpha, \lambda_0)$ has a constant slope $v \in S^1$ in the sense of Section 3.4. Then for every $\varepsilon > 0$ there exists a stabilizing 1-form λ for ω such that $|\mathrm{Hel}(\hat\omega) - \mathrm{Hel}(\omega)| < \varepsilon$ for all $\hat\omega \in C_{(\omega,\lambda)}$.*

Proof. Consider $\hat\omega \in C_{(\omega,\lambda)}$, i.e. $\hat\omega = \omega + td\lambda$ with $t \in I_{(\omega,\lambda)}$. Its helicity is

$$\mathrm{Hel}(\hat\omega) = \int_M (\alpha + t\lambda) \wedge (d\alpha + td\lambda) = \mathrm{Hel}(\omega) + 2t \int_M \lambda \wedge \omega + t^2 \int_M \lambda \wedge d\lambda.$$

Now suppose that $(\alpha, \lambda_0) = (\alpha_h, \lambda_{g_0})$ has a constant unit slope $v \in \mathbb{C}$ on $I \times T^2$, i.e. $ih'/|h'| \equiv v$ and $\langle g_0, v\rangle \equiv c > 0$. Given $\varepsilon > 0$, we deform g rel ∂I, keeping the condition $\langle g, v\rangle \equiv c$, such that $g'(r_\pm) = \pm h'(r_\pm)/\varepsilon$ for some $r_\pm \in I$. Now $\omega + td\lambda \neq 0$ iff $h'(r) + tg'(r) \neq 0$ for all $r \in I$, which implies $|t| \leq \varepsilon$. The term $\lambda_g \wedge \omega_h = \langle g, ih'\rangle dr \wedge d\theta \wedge d\phi = c\,\mathrm{vol}$ is independent of ε, and the term $\lambda_g \wedge d\lambda_g = \langle g, ig'\rangle\,\mathrm{vol}$ is of order $1/\varepsilon$. Hence the above formula for the helicity yields $\mathrm{Hel}(\hat\omega) = \mathrm{Hel}(\omega) + O(\varepsilon)$, and the lemma is proved. □

Proof of Proposition 6.7. By Proposition 3.31, there exists an exact SHS (ω_0, λ_0) on M ((ω_1, λ_1) in the notation of Proposition 3.31) restricting as $(d\alpha_{\mathrm{st}}, \alpha_{\mathrm{st}})$ to some embedded solid torus $S^1 \times D^2$ with $\alpha_{\mathrm{st}} = \alpha_{h_0}$ for $h_0 = (r^2, 1 - r^2)$. Set $\gamma := S^1 \times \{(0, 0)\}$ and write

$S^1 \times D^2 \setminus \gamma = I \times T^2$. Let $h_1 : I \to \mathbb{C}$ be an immersion which is homotopic to h_0 rel ∂I and such that the signed area ΔA enclosed between the curves h_0 and h_1 is positive. We use Corollary 3.12 to find a connecting homotopy (h_t, g_t), $t \in [0,1]$, satisfying (18). Thus the SHS (ω_t, λ_t) defined by (h_t, g_t) satisfy

$$\mathrm{Hel}(\omega_1) - \mathrm{Hel}(\omega_0) = \int_I h_1^*(ydx - xdy) - \int_I h_0^*(ydx - xdy) = -2\Delta A < 0,$$

where $x+iy$ denote coordinates on $\mathbb{C}$. Now pick $0 < \varepsilon < \Delta A$ and replace λ_j, $j = 0, 1$, by the stabilizing forms (still denoted λ_j) provided by Lemma 6.8. Then for all $\hat{\omega}_i \in C_{(\omega_i,\lambda_i)}$ we have

$$\mathrm{Hel}(\hat{\omega}_1) - \mathrm{Hel}(\hat{\omega}_0) \leq -2\Delta A + 2\varepsilon < 0,$$

so by monotonicity of helicity there exists no strong symplectic cobordism from (ω_0, λ_0) to (ω_1, λ_1). □

Proposition 6.7 shows that the notion of strong cobordism is too rigid for large homotopies. This motivates the following

Question 6.9 (Large Cobordism Question). Given a stable homotopy $(\omega_t, \lambda_t)_{t\in[0,1]}$, are ω_0 and ω_1 weakly bicobordant?

A positive answer to this question would be very useful for the following reason: Obstructions to homotopies of SHS beyond the homotopy class of almost contact structures are hard to construct (the only known ones arise in a more restricted setting from Rabinowitz Floer homology, see Section 6.6 below). On the other hand, there are simple obstructions to weak symplectic cobordisms, two of which we will now describe.

The first obstruction is again based on monotonicity of helicity on cobordisms. To apply this to weak cobordisms, suppose that we have two exact HS's ω_0 and ω_1 such that

$$\mathrm{Hel}(\hat{\omega}_0) > 0, \qquad \mathrm{Hel}(\hat{\omega}_1) \leq 0$$

for all $\hat{\omega}_i \in D^+_{\omega_i}$. Then ω_0 and ω_1 are not weakly cobordant. This situation occurs in the example discussed in Section 2.9: Let M be a compact quotient of $PSL(2,\mathbb{R})$ and let α_+ be a positive contact form coming from a left-invariant one on $PSL(2,\mathbb{R})$. Since any $\hat{\omega} \in D^+_{d\alpha_+}$ is a closed form contracting to zero with the Reeb vector field of α_+, equation (12) holds with $\hat{\omega}$ in place of θ, so $\hat{\omega} = c\, d\alpha_+$ for some constant $c \neq 0$. Thus

$$\mathrm{Hel}(\hat{\omega}) = \mathrm{Hel}(cd\alpha_+) = c^2\mathrm{Hel}(d\alpha_+) = c^2 \int_M \alpha_+ \wedge d\alpha_+ > 0.$$

Let α_- be a negative contact form coming from a left-invariant one on $PSL(2,\mathbb{R})$. Since $[d\alpha_-]$ generates $\ker\kappa = H^2_{\mathcal{L}}(M)$, any $\hat{\omega} \in D^+_{d\alpha_-}$ has foliated cohomology class $[\hat{\omega}] = c[d\alpha_-] \in \ker\kappa$ for a constant $c \in \mathbb{R}$, and therefore

$$\mathrm{Hel}(\hat{\omega}) = \mathrm{Hel}(c\, d\alpha_-) = c^2\mathrm{Hel}(d\alpha_-) \leq 0$$

because α_- is a negative contact form. On the other hand, $d\alpha_+$ and $d\alpha_-$ are homotopic as HS because any two nonzero left-invariant 2-forms on $PLS(2,\mathbb{R})$ are homotopic through nonzero left-invariant 2-forms and this homotopy descends to any compact quotient. Therefore, we have shown

Proposition 6.10. *The stable Hamiltonian structures* $(d\alpha_+, \alpha_+)$ *and* $(d\alpha_-, -\alpha_-)$ *above on* $\Gamma\backslash PSL(2, \mathbb{R})$ *are homotopic as HS, but there exists no weak symplectic cobordism from* $d\alpha_+$ *to* $d\alpha_-$. □

We conjecture that the two SHS $(d\alpha_+, \alpha_+)$ and $(d\alpha_-, -\alpha_-)$ are not stably homotopic.

The second obstruction to weak symplectic cobordisms is fillability: We will show that the standard tight contact structure on S^3 is not weakly cobordant to any overtwisted one. Before showing this, we first discuss

Composability of weak cobordisms. Let us consider two symplectic cobordisms $([a, b] \times M, \Omega_-)$ and $([b, c] \times M, \Omega_+)$ with $\Omega_\pm|_{\{b\}\times M} \in D^+_\omega$. Assume that there exists a constant $C > 0$ and an orientation preserving diffeomorphism $\Phi : M \to M$ such that

$$\Phi^*(\Omega_+)|_{\{b\}\times M} = C\Omega_-|_{\{b\}\times M}.$$

Then

$$W := ([a, b] \times M) \cup_\Phi ([b, c] \times M)$$

with the symplectic form given by Ω_+ on $[b, c] \times M$ and by $C\Omega_-$ on $[a, b] \times M$ defines a symplectic cobordism from $(\{a\} \times M, C\Omega_-)$ to $(\{c\} \times M, \Omega_+)$. In general, however, such a diffeomorphism need not exist.

Example 6.11. Consider a circle bundle $\pi : M \to W$ over a closed symplectic manifold $(W, \bar{\omega})$ with the pullback (stabilizable) Hamiltonian structure $\omega = \pi^*\bar{\omega}$. Then forms in D^+_ω descend to the quotient and D^+_ω is homeomorphic to the space of positive symplectic forms on W whose pullback to M is cohomologous to ω. In general, two such symplectic forms need not be diffeomorphic.

However, in dimension 3, Proposition 2.10 and the preceding discussion implies

Lemma 6.12. *Suppose that* $\dim M = 3$ *and the foliated cohomology* $H^2_{\mathcal{L}}(M)$ *for* $\mathcal{L} = \ker\omega$ *is* 1*-dimensional. Then for all* $\omega_0, \omega_1 \in D^+_\omega$ *there exists a constant* $C > 0$ *and an orientation preserving diffeomorphism* $\Phi : M \to M$ *such that*

$$\Phi^*\omega_1 = C\omega_0.$$

Thus weak cobordisms as above can be composed at (M, ω) *after rescaling. In particular, this situation occurs for a circle bundle* $\pi : M \to W$ *over a closed symplectic* 2*-manifold* $(W, \bar{\omega})$ *with the pullback (stabilizable) Hamiltonian structure* $\omega = \pi^*\bar{\omega}$. □

Second obstruction: fillability. As a first application of this observation, we now have

Corollary 6.13. *There is no weak symplectic cobordism from* $(d\alpha_{\rm st}, \alpha_{\rm st})$ *to* $(d\alpha_{\rm ot}, \alpha_{\rm ot})$*, where* $\alpha_{\rm st}$ *is the standard tight contact form on* S^3 *and* $\alpha_{\rm ot}$ *is an overtwisted contact form defining the same orientation.*

Proof. For contradiction, suppose $([0,1]\times S^3,\Omega)$ is a symplectic cobordism with $\omega_0 := \Omega|_{\{0\}\times S^3}\in D^+_{d\alpha_{\rm st}}$ and $\omega_1 := \Omega|_{\{1\}\times S^3}\in D^+_{d\alpha_{\rm ot}}$. By Lemma 6.12, we can glue the standard symplectic 4-ball $(B^4, C\Omega_{\rm st})$, rescaled by some constant $C>0$, to $([0,1]\times S^3,\Omega)$ along $\{0\}\times S^3$ to get a symplectic form Ω on B^4 with $\Omega|_{\partial B^4}\in D^+_{d\alpha_{\rm ot}}$. By definition of $D^+_{d\alpha_{\rm ot}}$, the form $\Omega|_{\partial B^4}$ restricts positively to $\ker\alpha_{\rm ot}$. Thus (B^4,Ω) is a weak symplectic filling of $(S^3,\alpha_{\rm ot})$, which contradicts the theorem by Eliashberg and Gromov [18, 30] that weakly fillable contact manifolds are tight. □

6.4. T^2-invariant cobordisms in dimension three

We return to the setup of Section 3.4. For an interval $I\subset\mathbb{R}$ we consider the 4-manifold

$$[0,1]\times I\times T^2$$

with coordinates (t,r,θ,ϕ), equipped with the T^2-action given by shift in (θ,ϕ), viewed as a topologically trivial cobordism between $\{0\}\times I\times T^2$ and $\{0\}\times I\times T^2$. Consider a T^2-invariant exact 2-form Ω on $[0,1]\times I\times T^2$. We can write it as

$$\Omega = dt\wedge\beta_t + \omega_t$$

for t-dependent T^2-invariant forms β_t,ω_t on $I\times T^2$. Exactness of Ω is equivalent to exactness of ω_t and $d\beta_t = \dot\omega_t$. By Lemma 3.9, ω_t can be written as $\omega_t = d\alpha_t$ for a smooth family of 1-forms α_t, $t\in[0,1]$, of the form

$$\alpha_t = h_1(t,r)d\theta + h_2(t,r)d\phi.$$

It follows that $d\beta_t = \dot\omega_t = d\dot\alpha_t$ and thus $\beta_t = \dot\alpha_t + \gamma_t$ for a family of T^2-invariant closed 1-forms γ_t. After writing $\gamma_t = \dot\delta_t$ for a family of closed 1-forms

$$\delta_t = a_1(t)d\theta + a_2(t)d\phi + f_t(r)dr$$

and replacing α_t by

$$\alpha_t + a_1 d\theta + a_2 d\phi,$$

we thus have

$$\Omega = dt\wedge(\dot\alpha_t + \dot f_t dr) + d\alpha_t.$$

Denoting t-derivatives by $\dot h_i$ and r-derivatives by h_i', we compute

$$\begin{aligned}\Omega\wedge\Omega &= 2dt\wedge\dot\alpha_t\wedge d\alpha_t = 2dt\wedge(\dot h_1 d\theta + \dot h_2 d\phi)\wedge dr\wedge(h_1' d\theta + h_2' d\phi)\\ &= 2(h_1'\dot h_2 - h_2'\dot h_1)dt\wedge dr\wedge d\theta\wedge d\phi.\end{aligned}$$

Hence Ω is a positive symplectic form iff

$$h_1'\dot h_2 - h_2'\dot h_1 > 0$$

for all (t, r). Geometrically, this condition means that the velocity vector $h_t' = (h_1', h_2')$ and the t-derivative $\dot{h}_t = (\dot{h}_1, \dot{h}_2)$ of the family of curves $h_t = (h_1(t, \cdot), h_2(t, \cdot)) : I \to \mathbb{C}$, $t \in [0, 1]$, satisfy

$$\langle \dot{h}_t, ih_t' \rangle > 0, \tag{33}$$

i.e. $(\dot{h}_t, h_t')$ is a positive basis of $\mathbb{C}$ for all t. Note that the functions f_t do not enter condition (33). Therefore, from now on we will assume $f_t = 0$ and only consider Ω of the form

$$\Omega = dt \wedge \dot{\alpha}_t + d\alpha_t. \tag{34}$$

Definition 6.14. A *monotone homotopy* is a smooth family of curves $h_t : I \to \mathbb{C}$ satisfying condition (33).

The preceding discussion shows

Lemma 6.15. *Two T^2-invariant Hamiltonian structures $d\alpha_0$ and $d\alpha_1$ on $I \times T^2$ are cobordant by a T^2-invariant cobordism iff there exists a monotone homotopy from $h_0 : I \to \mathbb{R}^2$ to $h_1 : I \to \mathbb{R}^2$.*

Let us now restrict our attention to curves $h : [0, 1] \to \mathbb{C}$ which are *standardized* in the sense that $h(r)$ is a positive multiple of $(r^2, 1 - r^2)$ near $\partial[0, 1]$. Note that each monotone homotopy h_t is in particular a regular homotopy, and if the h_t are standardized then by Corollary 3.12 the corresponding homotopy of Hamiltonian structures $d\alpha_t$ can be stabilized by T^2-invariant 1-forms. On the other hand, we will now construct examples of standardized immersions h_0, h_1 which are regularly homotopic (i.e. have the same rotation number), but for which there exists no standardized monotone homotopy from h_0 to a positive multiple of h_1.

The first obstruction to monotone homotopies comes from winding numbers as defined in Section 3.4.

Lemma 6.16. *Let $h_0, h_1 : [0, 1] \to \mathbb{C} \setminus \{0\}$ be standardized immersions missing the origin. If there exists a standardized monotone homotopy from h_0 to a positive multiple of h_1, then $w(h_1) \le w(h_0)$. Moreover, if in addition $w(h_1) < w(h_0)$, then the corresponding symplectic cobordism contains an exact Lagangian 2-torus.*

Proof. Condition (33) implies that during a standardized monotone homotopy h_t, the winding number around 0 decreases by 1 each time h_t passes through the origin. This proves the first statement. If in addition $w(h_1) < w(h_0)$, then there exists a (t, r) with $h_t(r) = 0$, so the primitive α of the symplectic form Ω given by (34) vanishes on the torus $\{(t, r)\} \times T^2$. □

Example 6.17. Figure 3 shows a curve h which is obtained from $h_{\text{st}}(r) = (r^2, 1 - r^2)$ by a standardized monotone homotopy. (The curve h_{st} is dashed, the curve h is bold. The direction of increase of the parameter r on the curve h_{st} is designated by the two arrows. The curve h is oriented accordingly.) On the other hand, it has $w(h) = -1$, so there is no standardized monotone homotopy from h to a positive multiple of h_{st}.

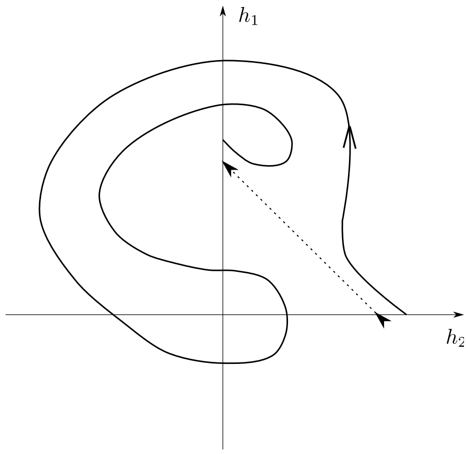

Fig. 3. Monotone homotopy.

Remark 6.18. Given two standardized immersions $h_0, h_1 : [0, 1] \to \mathbb{C}$ it is in general not easy to see whether they can be connected by a standardized monotone homotopy. Indeed, J. Kukla [37] has shown that this question is equivalent to a well-known problem in the topology of curves: Connect the curves h_0, h_1 along the coordinate axes to a 4-gon and smoothen the corners to obtain an immersion $\gamma : S^1 \to \mathbb{C}$ of the circle. Then the curves $h_{0,1}$ can be connected by a standardized monotone homotopy if and only if γ extends to an immersion of the disk. There is no simple invariant controlling the existence of such an extension, although there are several algorithms for deciding this in the literature.

6.5. An exotic symplectic ball

The following construction was communicated to the second author by Y. Chekanov, who remembers having seen it somewhere in the literature, but we have not been able to trace its origins.

We consider $\mathbb{R}^4$ with its standard T^2-action and the standard symplectic form

$$\Omega_{\rm st} := d\alpha_{\rm st}, \qquad \alpha_{\rm st} := \frac{1}{2}\sum_{j=1}^{2}(x_j dy_j - y_j dx_j).$$

Denote by $B^4(r)$ the ball of radius r around the origin and by $S^3(r) := \partial B^4(r)$ its boundary sphere.

Proposition 6.19. *There exists a symplectic form Ω on $B^4(1)$ with the following properties:*

(i) *Ω is T^2-invariant and $\Omega = \Omega_{\rm st}$ on $B^4(1/2)$.*
(ii) *$\Omega|_{S^3(r)}$, $r \in [1/2, 1]$, can be stabilized to a T^2-invariant stable homotopy from $\frac{1}{2}(d\alpha_{\rm st}, \alpha_{\rm st})$ to the induced stable Hamiltonian structure (ω, λ) on S^3.*
(iii) *(B^4, Ω) cannot be symplectically embedded into $(\mathbb{R}^4, \Omega_{\rm st})$.*

Remark 6.20. The existence of an exotic symplectic ball is of course not new: An example can be obtained by cutting a large ball out of an exotic symplectic $\mathbb{R}^4$. Historically, the first exotic symplectic structure on $\mathbb{R}^{2n}$ was found by Gromov [30]. Later an easy explicit construction of an exotic symplectic structure on $\mathbb{R}^4$ was given by Bates and Peschke [7].

Proof of Proposition 6.19. The proof combines the construction in Example 6.17 with a result of Gromov. Let h_t, $t \in [1/2, 1]$, be a standardized monotone homotopy from $h_{1/2}(r) := \frac{1}{2}h_{\mathrm{st}}(r) = \frac{1}{2}(r^2, 1-r^2)$ to the function $h_1 = h$ in Example 6.17. Due to the standardization, the associated 1-forms α_t extend to S^3, so (34) defines an exact symplectic structure Ω on $[1/2, 1] \times S^3$. Since $w(h) = -1$, $([1/2, 1] \times S^3, \Omega)$ contains an exact Lagrangian 2-torus. Since $h_{1/2} = \frac{1}{2}h_{\mathrm{st}}$, we can glue a standard ball of radius $1/2$ to this cobordism to obtain an exact symplectic manifold (B^4, Ω). Properties (i) and (ii) in the proposition follow immediately from the construction and Corollary 3.12. Property (iii) follows from the fact that (B^4, Ω) contains an exact Lagrangian 2-torus, but $(\mathbb{R}^4, \Omega_{\mathrm{st}})$ does not by a theorem of Gromov [30]. □

Corollary 6.21. *There exists no symplectic cobordism* $([1, 2] \times S^3, \Omega)$ *with* $\Omega_{\{1\}\times S^3} = \omega$ *the Hamiltonian structure in Proposition 6.19 and* $\Omega_{\{2\}\times S^3} \in D^+_{d\alpha_{\mathrm{st}}}$. □

Proof. Suppose such a cobordism exists. Then we can glue this cobordism to the ball in Proposition 6.19 to obtain a symplectic manifold $(B^4 \cup [1, 2] \times S^3, \Omega)$, which in turn can be glued using Lemma 6.12 to $(\mathbb{R}^4 \setminus S^3(R), \Omega_{\mathrm{st}})$ along a sphere $S^3(R)$ of suitable radius R. This yields an exact symplectic structure Ω on $\mathbb{R}^4$ with $\Omega = \Omega_{\mathrm{st}}$ outside $B^4(R)$. By Gromov's uniqueness theorem for symplectic structures on $\mathbb{R}^4$ standard at infinity [30], $(\mathbb{R}^4, \Omega)$ is symplectomorphic to $(\mathbb{R}^4, \Omega_{\mathrm{st}})$, contradicting property (iii) in Proposition 6.19. □

Corollary 6.22. *The SHS* (ω, λ) *on* S^3 *in Corollary 6.21 is stably homotopic to* $(d\alpha_{\mathrm{st}}, \alpha_{\mathrm{st}})$ *but cannot be embedded in* $(\mathbb{R}^4, \Omega_{\mathrm{st}})$.

Remark 6.23. Note that this corollary does not contradict Lemma 3.41 because the curve $h : [0, 1] \to \mathbb{C}$ defining ω does not remain in the positive quadrant.

Remark 6.24. $(d\alpha_{\mathrm{st}}, \alpha_{\mathrm{st}})$ and (ω, λ) in Proposition 6.19 are candidates for stable Hamiltonian structures that are stably homotopic and such that there exists no weak cobordism from ω to $d\alpha_{\mathrm{st}}$. However, the above argument breaks down if we relax the assumption $\Omega_{\{1\}\times S^3} = \omega$ in Corollary 6.21 to $\Omega_{\{1\}\times S^3} \in D^+_\omega$ because the foliated cohomology $H^2_{\mathcal{L}}(S^3)$ for $\mathcal{L} = \ker \omega$ is infinite-dimensional (Proposition 3.44), so we cannot apply Lemma 6.12 to glue the cobordisms.

6.6. Ambient homotopies and Rabinowitz Floer homology

Any embedding $\psi : M \hookrightarrow W$ of a hypersurface in a symplectic manifold (W, Ω) induces a Hamiltonian structure $\omega = \psi^*\Omega$ on M, and any smooth isotopy of embeddings $\psi_t : M \hookrightarrow W$ induces a homotopy of HS $\omega_t = \psi_t^*\Omega$. We call such a homotopy ω_t an *ambient* *homotopy*. Corollary 6.22 shows that not every homotopy of $\omega = \psi^*\Omega$ can be realized by an ambient homotopy.

Now we turn to the question of ambient stable homotopies: Given two stable hypersurfaces M_0, M_1 in a symplectic manifold (W, Ω), are they stably homotopic? Under an additional technical hypothesis, this question has a negative answer and can be addressed by Rabinowitz Floer homology [13]. We first give the relevant definitions.

For the remainder of this subsection, let (W, Ω) be a fixed symplectic manifold of dimension $2n$. We assume that $\Omega|_{\pi_2(M)} = 0$ and (W, Ω) is convex at infinity or geometrically bounded (see [13]). For a hypersurface $M \subset W$ denote by $\mathcal{R}(M)$ the space of periodic orbits on M that are contractible in W. Recall that to $\gamma \in \mathcal{R}(M)$ and a 1-form λ on M we associate the energies

$$E_\Omega(\gamma) = \int_{D^2} \bar{\gamma}^*\Omega, \qquad E_\lambda(\gamma) = \int_\gamma \lambda,$$

where $\bar{\gamma} : D^2 \to W$ with $\bar{\gamma}|_{\partial D^2} = \gamma$. All hypersurfaces $M \subset W$ are assumed to be closed and *separating*, i.e. $W \subset M$ consists of two connected components.

Definition 6.25. A hypersurface M in (W, Ω) is called *stable* if the HS $\Omega|_M$ is stabilizable. A smooth homotopy $(M_t)_{t\in[0,1]}$ of hypersurfaces in W is called *stable* if there exists a smooth homotopy of stabilizing 1-forms λ_t for $\Omega|_{M_t}$. Recall that a stable hypersurface M is called *tame* if for some (and hence every) stabilizing 1-form λ there exists a constant $c_\lambda > 0$ such that $E_\lambda(\gamma) \le c_\lambda |E_\Omega(\gamma)|$ for all $\gamma \in \mathcal{R}(M)$. A stable homotopy M_t is called *tame* if there exists a smooth homotopy of stabilizing 1-forms λ_t and a constant c such that $E_{\lambda_t}(\gamma) \le c|E_\Omega(\gamma)|$ for all $\gamma \in \mathcal{R}(M_t)$ and all $t \in [0, 1]$.

Rabinowitz Floer homology (RFH) [12] associates to every (closed, separating) stable tame hypersurface $M \subset W$ a $\mathbb{Z}_2$-vector space $RFH(M)$ which is invariant under tame stable homotopies. Moreover, it has the following properties:

(i) If M is displaceable (by a Hamiltonian isotopy) then $RFH(M) = 0$.
(ii) If $RFH(M) = 0$ then M carries a periodic orbit contractible in W.

In particular, if $RFH(M_0) \neq RFH(M_1)$ for two tame stable hypersurfaces, then they are not tame stably homotopic. Using this, many examples of smoothly but not tame stably homotopic hypersurfaces are constructed in [13]. Here we just give one example.

Example 6.26 ([13, Theorem 1.6]). Let G be the 3-dimensional Heisenberg group of matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{R}$. The 1-form $\gamma := dz - x dy$ is left-invariant and we let $\sigma := d\gamma$ be the exact magnetic field. If Γ is a co-compact lattice in G, $Q := \Gamma\backslash G$ is a closed 3-manifold and σ descends to an exact 2-form on Q. Equip the cotangent bundle $\tau : T^*Q \to Q$ with the symplectic form $\Omega = dp \wedge dq + \tau^*\sigma$ and the left-invariant Hamiltonian

$$H := \tfrac{1}{2}\big(p_x^2 + (p_y + x p_z)^2 + p_z^2\big).$$

Then each level set $M_t = H^{-1}(t)$ for $t \neq 1/2$ is stable and tame. For $t > 1/2$, M_t has no contractible periodic orbits and thus $RFH(M_t) \neq 0$. For $t < 1/2$, M_t is displaceable and thus $RFH(M_t) = 0$. Therefore, two level sets M_s, M_t with $s < 1/2 < t$ are smoothly homotopic and tame stable, but not tame stably homotopic.

Remark 6.27. In the preceding example, $t = 1/2$ is the Mañé critical value of the Hamiltonian system defined by Ω and H (see [13]). It appears to be a general feature (though not a proven theorem) that level sets below and above the Mañé critical value are not tame stably homotopic.

We expect that the hypersurfaces M_s, M_t with $s < 1/2 < t$ in the preceding example are not stably homotopic (i.e. without the tameness assumption), and more generally $\Omega|_{M_s}, \Omega|_{M_t}$ are not stably homotopic through SHS on the unit cotangent bundle S^*Q. However, this appears to lie outside the scope of established techniques such as RFH.

6.7. *Symplectic TQFTs*

In this section we introduce the notion of a "symplectic TQFT" and show how it gives rise to a homotopy invariant for stable Hamiltonian structures. This notion was motivated by symplectic field theory (SFT) introduced in [22], which indeed satisfies three of the four axioms of a symplectic TQFT. However, as we explain below, it is unclear whether there exists a version of SFT satisfying the fourth axiom. We include this section in the hope that it may be helpful in finding some homotopy invariant for SHS in the future.

Definition 6.28. A *symplectic TQFT* is a contravariant functor $\mathcal{F}$ from the category of SHS to a category $\mathcal{C}$. Thus $\mathcal{F}$ associates

- to every SHS (M, ω, λ) an object $\mathcal{F}(M, \omega, \lambda)$ in $\mathcal{C}$;
- to every symplectic cobordism (W, Ω) a morphism $\mathcal{F}(W, \Omega)$ in $\mathcal{C}$;
- to homotopic symplectic cobordisms the same morphism in $\mathcal{C}$;
- to a trivial cobordism $([0, 1] \times M, \omega + d(f(t)\lambda))$ an isomorphism in $\mathcal{C}$

so that the usual functoriality properties hold.

Remark 6.29. (a) We are a little sloppy when speaking about the "category of SHS". For example, this category has no identity morphisms, and morphisms should be concatenations of cobordisms rather that single cobordisms. See [21] for a more thorough discussion.

(b) For concreteness, we will assume that objects in $\mathcal{C}$ are vector spaces with some additional structure, and morphisms are linear maps preserving this structure, so that a morphism is invertible iff it is injective and surjective. This is the case for the algebraic formulation of SFT in [15].

For two HS ω_0, ω_1 with the same kernel $\mathcal{L}$ and a stabilizing 1-form λ for $\mathcal{L}$ we write $\omega_0 <_\lambda \omega_1$ iff $\omega_1 = \omega_0 + \tau d\lambda$ for a $\tau > 0$ such that $\ker(\omega_0 + td\lambda) = \mathcal{L}$ for all $t \in [0, \tau]$. Note that for fixed λ this defines a partial ordering on HS with kernel $\mathcal{L}$, and a complete ordering on each equivalence class $C(\omega, \lambda)$.

Proposition 6.30. *For a symplectic TQFT $\mathcal{F}$ and SHS (ω_i, λ_i) on M the following hold:*

(a) *For $\omega_0 <_\lambda \omega_1$ there exist canonical isomorphisms*

$$\psi_{(\omega_0,\omega_1;\lambda)} : \mathcal{F}(\omega_1, \lambda_1) \to \mathcal{F}(\omega_0, \lambda_0)$$

such that for $\omega_0 <_\lambda \omega_1 <_\lambda \omega_2$ we have

$$\psi_{(\omega_0,\omega_1;\lambda)}\psi_{(\omega_1,\omega_2;\lambda)} = \psi_{(\omega_0,\omega_2;\lambda)}.$$

(b) *Each stable homotopy $\gamma = \{\omega_t, \lambda_t\}_{t\in[0,1]}$ induces a canonical isomorphism*

$$\phi_\gamma : \mathcal{F}(\omega_1, \lambda_1) \to \mathcal{F}(\omega_0, \lambda_0)$$

such that for stable homotopies γ, γ' with $\gamma_1 = \gamma'_0$ we have

$$\phi_\gamma \phi_{\gamma'} = \phi_{\gamma\#\gamma'}.$$

Proof. (a) For $\omega_0 <_\lambda \omega_1$ there exists a trivial cobordism from ω_0 to ω_1 which induces an isomorphism $\psi_{(\omega_0,\omega_1;\lambda)} : \mathcal{F}(\omega_1, \lambda_1) \to \mathcal{F}(\omega_0, \lambda_0)$. This isomorphism does not depend on the trivial cobordism since any two trivial cobordisms from ω_0 to ω_1 are homotopic. The composition property follows from functoriality of $\mathcal{F}$ and the fact that the composition of trivial cobordisms for the same λ is again a trivial cobordism.

(b) Consider a stable homotopy $\gamma = \{\omega_t, \lambda_t\}_{t\in[0,1]}$. Assume first that $L(\gamma) < 1/3$ and let $([0, 3] \times M, \Omega)$ be the symplectic cobordism provided by Proposition 6.6. This is the composition $W_1W_2W_3$ of three cobordisms from ω_0^- to ω_1^- to ω_0^+ to ω_1^+, where $\omega_i^\pm \in C_{(\omega_i,\lambda_i)}$. Without loss of generality we may assume $\omega_i^- <_{\lambda_i} \omega_i <_{\lambda_i} \omega_i^+$. Denote by $\mathcal{F}(W_i)$ the induced morphisms. Since W_1W_2 and W_2W_3 are homotopic to trivial cobordisms, the compositions $\mathcal{F}(W_1)\mathcal{F}(W_2)$ and $\mathcal{F}(W_2)\mathcal{F}(W_3)$ are isomorphisms, hence

$$\mathcal{F}(W_2) : \mathcal{F}(\omega_0^+, \lambda_0) \to \mathcal{F}(\omega_1^-, \lambda_1)$$

is an isomorphism (see Remark 6.29). Define the isomorphism

$$\phi_\gamma := \psi^{-1}_{(\omega_1^-,\omega_1;\lambda_1)}\mathcal{F}(W_2)\psi^{-1}_{(\omega_0,\omega_0^+;\lambda_0)} : \mathcal{F}(\omega_0, \lambda_0) \to \mathcal{F}(\omega_1, \lambda_1).$$

By the composition property in (a), this isomorphism does not depend on the choice of ω_0^+ and ω_1^- and is thus canonically defined.

Now let us drop the assumption $L(\gamma) < 1/3$. By the (Restriction) property in Section 6.2, γ can be written as a concatenation $\gamma = \gamma_1 \# \cdots \# \gamma_N$ of homotopies of length $L(\gamma_i) < 1/3$. We define

$$\phi_\gamma := \phi_{\gamma_1} \cdots \phi_{\gamma_N} : \mathcal{F}(\omega_0, \lambda_0) \to \mathcal{F}(\omega_1, \lambda_1)$$

with the isomorphisms ϕ_{γ_i} defined above. We need to show that this is independent of the decomposition of γ into short homotopies γ_i. After taking a common refinement of two decompositions, this reduces to showing $\phi_\gamma = \phi_{\gamma_1}\phi_{\gamma_2}$ for a homotopy $\gamma = \gamma_1 \# \gamma_2$ of length $L(\gamma) < 1/3$. After reparametrization, we may assume that $\gamma = \{(\omega_t, \lambda_t)\}_{t\in[0,2]}$ and γ_1, γ_2 are the restrictions to the intervals $[0, 1]$ and $[1, 2]$, respectively. By definition,

$$\phi_\gamma = \psi^{-1}_{(\omega_0^-,\omega_0;\lambda_0)} \mathcal{F}(W) \psi^{-1}_{(\omega_2,\omega_2^+;\lambda_2)} : \mathcal{F}(\omega_2, \lambda_2) \to \mathcal{F}(\omega_0, \lambda_0)$$

for a cobordism (W, Ω) from ω_0^- to ω_2^+ provided by Proposition 6.6. Without loss of generality we may assume that $\Omega_{\{1\}\times M} = \omega_1$, so $W = W_1 W_2$ for cobordisms W_1 from ω_0^- to ω_1 and W_2 from ω_1 to ω_2^+. Then we have

$$\phi_{\gamma_1} = \psi^{-1}_{(\omega_0^-,\omega_0;\lambda_0)} \mathcal{F}(W_1), \qquad \phi_{\gamma_2} = \mathcal{F}(W_2) \psi^{-1}_{(\omega_2^+,\omega_2;\lambda_2)},$$

which together with $\mathcal{F}(W_1)\mathcal{F}(W_2) = \mathcal{F}(W)$ yields

$$\phi_{\gamma_1}\phi_{\gamma_2} = \psi^{-1}_{(\omega_0^-,\omega_0;\lambda_0)} \mathcal{F}(W_1)\mathcal{F}(W_2) \psi^{-1}_{(\omega_2^+,\omega_2;\lambda_2)} = \phi_\gamma.$$

This proves that the isomorphism ϕ_γ is independent of the decomposition of γ into short homotopies and thus canonically defined. The same argument also shows the composition property. □

Remark 6.31. More generally, one could define a "homotopical" symplectic TQFT from the 2-category of SHS to a 2-category $\mathcal{C}$, with homotopies of cobordisms giving rise to homotopies in $\mathcal{C}$ and a trivial cobordism inducing a homotopy equivalence. Then the arguments in the proof of Proposition 6.30 show that stable homotopies give rise to homotopy equivalences.

Let us now discuss to what extent SFT as introduced in [22] gives rise to a symplectic TQFT in the sense above. For this, let us first consider only SHS (M, ω, λ) that are *Morse–Bott*.

An $\mathbb{R}$-invariant almost complex structure J on $\mathbb{R} \times M$ is called *adjusted to* (ω, λ) if $J(\partial_r) = R$, $J(\xi) = \xi$, and $J|_\xi$ is *tamed* by $\omega|_\xi$ in the sense that $\omega(v, Jv) > 0$ for all $0 \neq v \in \xi$. SFT associates to such an (M, J) an algebraic object (Poisson algebra, Weyl algebra etc. depending on the version of SFT) $\mathrm{SFT}(M, J)$ by counting suitable holomorphic curves in $(\mathbb{R} \times M, J)$ asymptotic to closed Reeb orbits.

Next consider a symplectic cobordism (W, Ω) with stable boundaries $(M_\pm, \omega_\pm = \Omega|_{M_\pm}, \lambda_\pm)$. Note that we have canonical vector fields $X_\pm$ along $M_\pm$ defined by $i_{X_\pm}\Omega = \lambda_\pm$, so X_+ is outward pointing and X_- is inward pointing. An almost complex structure J on W is called *adjusted to* $(\Omega, \lambda_\pm)$ if it is tamed by Ω and $J(X_\pm) = R_\pm$, $J(\xi_\pm) = \xi_\pm$ along $M_\pm$. Denote by $(\hat{W}, \hat{J})$ the almost complex manifold obtained by gluing semi-infinite collars $R_\pm \times M_\pm$ to W along $M_\pm$ and extending J $\mathbb{R}$-invariantly by $J_\pm$ to the collars. SFT associates to such a (W, J) a morphism $\mathrm{SFT}(W, J)$ from (M_+, J_+) to (M_-, J_-) by counting suitable holomorphic curves in $(\hat{W}, \hat{J})$.

Composition of cobordisms induces composition of morphisms, and homotopies of J_t adjusted to the same $(\Omega, \lambda_\pm)$ give rise to the same morphisms. Since the space of J adjusted to $(\Omega, \lambda_\pm)$ is contractible, this implies that SFT is in fact independent of the adjusted J, so objects and morphisms can be written as $\mathrm{SFT}(M, \omega, \lambda)$ and $\mathrm{SFT}(W, \Omega)$.

By construction, SFT will satisfy the first three axioms of a symplectic TQFT.

For the last axiom, consider a trivial cobordism $(W = [0,1]\times M, \Omega = \omega + d(f(t)\lambda)$ between SHS $(\omega_0 = \omega + f(0)d\lambda, \lambda)$ and $(\omega_1 = \omega + f(1)d\lambda)$ on M. We wish to show that SFT(W, Ω) is an isomorphism. After cutting $[0,1]$ into smaller intervals and using functoriality, we may assume that the $\omega_t := \Omega|_{\{t\}\times M}$, $t \in [0,1]$, are arbitrarily C^1-close. Since taming is an open condition, we then find an almost complex structure J on M that is tamed by all the ω_t. For this t-independent J the morphism SFT(W, J) is counting only trivial cylinders over closed Reeb orbits. This seems to suggest that SFT(W, J) is the identity and the fourth axiom holds. However, a more careful inspection of the definition of SFT shows that it involves Novikov completions depending on integrals of the 2-form ω over holomorphic curves. As a result, the spaces SFT(M, ω_0, λ) and SFT(M, ω_1, λ) might differ and SFT(W, J) need not be an isomorphism.

Remark 6.32. At the time of writing, SFT has not yet been rigorously defined. The solution of the relevant transversality problems for holomorphic curves is work in progress by Hofer, Wysocki and Zehnder [32]. Apart from these well-known problems, we wish to point out another issue that needs to be addressed. A priori, the construction of SFT only works for stable Hamiltonian structures that are *Morse–Bott*. On the other hand, in the proof of homotopy invariance in Proposition 6.30 we need to cut a stable homotopy into short ones and we cannot guarantee that the SHS at which we cut are Morse–Bott. So in order to construct a homotopy invariant of SHS out of SFT, one needs to define SFT also for SHS that are not Morse–Bott. Moreover, in view of Theorem 1.10 one cannot approximate a given SHS by *Morse* ones. The counterexample in Theorem 1.10 is still Morse–Bott, which would suffice to define SFT, but it appears hopeless to try to prove that any SHS can be approximated by Morse–Bott ones. One way out may be perturbing the holomorphic curve equation used in SFT by a suitable Hamiltonian term.

Dimension 3. In dimension three, the problem in Remark 6.32 can be overcome due to Theorem 4.6 and the following consequence of it.

Corollary 6.33. *Let M be a closed oriented 3-manifold M. Then any stable homotopy γ between Morse–Bott SHS on M can be written as a concatenation $\gamma = \gamma_1 \# \cdots \# \gamma_N$ of stable homotopies γ_i between Morse–Bott SHS of length $L(\gamma_i) < 1/3$, $i = 1, \dots, N$.*

Proof. By the (Restriction) property in Section 6.2, γ can be written as a concatenation $\gamma = \bar\gamma_1 \# \cdots \# \bar\gamma_N$ of homotopies of length $L(\bar\gamma_i) < 1/3$. Denote the end points of γ_i by $(\omega_{i-1}, \lambda_{i-1})$ and (ω_i, λ_i). Thus (ω_0, λ_0) and (ω_N, λ_N) are Morse–Bott but the other ones need not be. Since length is invariant under linear rescaling, we may assume that each γ_i is parametrized over an interval of length 1.

According to Theorem 4.6, there exists for each $i = 1, \dots, N-1$ a stable homotopy δ_i (parametrized over $[0,1]$) from (ω_i, λ_i) to a Morse–Bott SHS with arbitrarily small $\|\delta_i\|_{C^1}$. We let $\delta_0 = (\omega_0, \lambda_0)$ and $\delta_N = (\omega_N, \lambda_N)$ be the constant homotopies. Applying Lemma 6.5 twice for each $i = 1, \dots, N$, we can thus achieve that each concatenation $\gamma_i := \delta_{i-1}^{-1} \# \bar\gamma_i \# \delta_i$ has length $L(\gamma_i) < 1/3$. Since the end points of each γ_i are Morse–Bott, this concludes the proof. □

Using Corollary 6.33, the proof of Proposition 6.30 can be carried out on the full subcategory of Morse–Bott SHS to obtain an invariant of Morse–Bott SHS up to stable homotopies (through SHS that need not be Morse–Bott). Since every SHS is stably homotopic to a Morse–Bott one, this invariant extends to all SHS, and we have shown:

Corollary 6.34. *In dimension* 3*, every symplectic TQFT defined on the full subcategory of Morse–Bott SHS gives rise to a homotopy invariant of SHS.* □

We conclude this section with the following

Conjecture 6.35. *The SHS* $(d\alpha_{\rm st}, \alpha_{\rm st})$ *and* $(d\alpha_{\rm ot}, \alpha_{\rm ot})$ *on* S^3 *are not stably homotopic, where* $\alpha_{\rm st}$ *is the standard contact form and* $\alpha_{\rm ot}$ *is an overtwisted contact form defining the same orientation.*

This conjecture is motivated by rational symplectic field theory (RSFT) introduced in [22]: It is well-known (see e.g. [53, 10]) that $\mathrm{RFST}(S^3, \alpha_{\rm ot}) = \{0\}$. On the other hand, for the standard contact form $\alpha_{\rm st}$ all closed Reeb orbits have even degree, so the RSFT Hamiltonian (without differential forms) for $\alpha_{\rm st}$ vanishes and $\mathrm{RSFT}(S^3, \alpha_{\rm st})$ is the free Poisson algebra generated by the closed Reeb orbits. Hence $(d\alpha_{\rm st}, \alpha_{\rm st})$ and $(d\alpha_{\rm ot}, \alpha_{\rm ot})$ have different RSFT.

Remark 6.36. SFT is certainly *not* invariant under *nonexact* stable homotopies (ω_t, λ_t), i.e. with varying cohomology class ω_t. For example, consider T^3 with coordinates (x, y, z) and the nonexact stable homotopy

$$\omega_t = dx \wedge dy + t\, dx \wedge dz, \qquad \lambda_t = dz.$$

(This corresponds to the mapping torus of a rotation of T^2 by angle t.) Note that the Reeb vector field is $R_t = \partial_z - t\partial_y$. For $t = 0$ all Reeb orbits are closed and rational SFT is nontrivial. On the other hand, for t irrational there are no closed Reeb orbits and hence rational SFT is trivial.

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