JEMS

Henning Haahr Andersen · Volodymyr Mazorchuk

Category $\mathcal{O}$ for quantum groups

Abstract. We study the BGG-categories $\mathcal{O}_q$ associated to quantum groups. We prove that many properties of the ordinary BGG-category $\mathcal{O}$ for a semisimple complex Lie algebra carry over to the quantum case.

Of particular interest is the case when q is a complex root of unity. Here we prove a tensor decomposition for simple modules, projective modules, and indecomposable tilting modules. Using the known Kazhdan–Lusztig conjectures for $\mathcal{O}$ and for finite-dimensional U_q-modules we are able to determine all irreducible characters as well as the characters of all indecomposable tilting modules in $\mathcal{O}_q$.

As a consequence, we also recover the known result that the generic quantum case behaves like the classical category $\mathcal{O}$.

Keywords. Quantized highest weights modules, specialization at roots of unity, tensor decompositions, tilting modules

1. Introduction

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{Q}$ associated to a semisimple complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$. The corresponding BGG-category $\mathcal{O}$, defined in [BGG], has been studied intensively over the last decades; see the recent monograph [Hu] for details.

In this paper we study similar categories for quantum groups. We let v denote an indeterminate and set U_v equal to the quantum group (or rather the quantized enveloping algebra) for $\mathfrak{g}$ over $\mathbb{Q}(v)$. The subcategory $\mathcal{O}_v$ of the module category for U_v is then defined in complete analogy with $\mathcal{O}_{\mathrm{int}}$, the subcategory of $\mathcal{O}$ consisting of modules with integral weights. Using Lusztig's quantum divided power version of U_v (see below) it is possible to consider also specializations $\mathcal{O}_q$ of $\mathcal{O}_v$ for any $q \in \mathbb{C} \setminus \{0\}$. Our principal interest is in the case where q is a root of unity.

In a little more detail, set $A = \mathbb{Z}[v, v^{-1}] \subset \mathbb{Q}(v)$ and let U_A be the Lusztig A-form of U_v (cf. [Lu90b]). For any nonzero $q \in \mathbb{C}$ we set $U_q = U_A \otimes_A \mathbb{C}$, where $\mathbb{C}$ is made into an A-algebra by the specialization $v \mapsto q$. Then $\mathcal{O}_q$ is the BGG-category for U_q.

H. H. Andersen: Center for Quantum Geometry of Moduli Spaces, Aarhus University, Building 530, Ny Munkegade, 8000 Aarhus C, Denmark; e-mail: mathha@qgm.au.dk

V. Mazorchuk: Department of Mathematics, Uppsala University, Box. 480, SE-75106, Uppsala, Sweden; e-mail: mazor@math.uu.se

Denote by $\mathcal{F}_q$ the subcategory of $\mathcal{O}_q$ consisting of all finite-dimensional U_q-modules of type **1**. When q is not a root of unity, this category is semisimple and it has exactly the same "combinatorics" as the category of finite-dimensional modules for $\mathfrak{g}_{\mathbb{C}}$, i.e. the characters of the simple modules are given by Weyl's character formula (see e.g. [APW91, Ja].)

Suppose now q is a root of unity. Then $\mathcal{F}_q$ has a much more complicated structure. In the present paper we assume that q is of odd order l and if $\mathfrak{g}$ contains a summand of type G_2 we assume in addition that l is prime to 3. Then the "combinatorics" of $\mathcal{F}_q$ has been worked out: Lusztig [Lu89] stated the conjecture that irreducible characters in $\mathcal{F}_q$ should be given by the values at 1 of the Kazhdan–Lusztig polynomials associated to the affine Weyl group for $\mathfrak{g}$. More specifically, the Kazhdan–Lusztig polynomials involved are the parabolic ones corresponding to affine Weyl group for the Langlands dual of $\mathfrak{g}$ relative to its finite Weyl subgroup (cf. also [So97a]). Kazhdan and Lusztig [KL94] proved that the category of finite-dimensional modules (of type **1**) for U_q is equivalent to a category of modules for the corresponding affine Kac–Moody algebra. This requires a weak restriction on l and in the non-simply laced case it has to be supplemented by Lusztig's later work [Lu94]. Then Kashiwara and Tanisaki [KT95], [KT96] proved the corresponding conjecture for affine Kac–Moody algebras. This established the above mentioned conjecture on the irreducible characters in $\mathcal{F}_q$. Soergel [So97b], [So99] subsequently determined the characters of indecomposable tilting modules in $\mathcal{F}_q$.

We prove that several classes of fundamental modules such as simple modules, indecomposable projective modules, indecomposable injective modules, and indecomposable tilting modules in $\mathcal{O}_q$ have a tensor product decomposition with a part which "comes from" $\mathcal{F}_q$ and a part which is a q-Frobenius twist of a corresponding module in $\mathcal{O}_{\text{int}}$; see Sections 3 and 4 for the precise statements. This allows us to deduce our main results: we determine all irreducible characters as well as the characters of all indecomposable tilting modules in $\mathcal{O}_q$ (see Corollaries 5.3 and 5.4).

In the process of establishing these results we prove that many of the properties of $\mathcal{O}$, e.g. finite length of all modules, the existence of enough projectives and injectives, existence of tilting modules, and Ringel self-duality, all carry over to $\mathcal{O}_q$.

One of the main features of $\mathcal{O}_q$ is that it contains a copy of $\mathcal{O}_{\text{int}}$, namely we may identify $\mathcal{O}_{\text{int}}$ with the direct sum of all "special blocks" in $\mathcal{O}_q$ (see Theorem 3.11). Once we have established this and the above mentioned properties of $\mathcal{O}_q$, we return to the generic category $\mathcal{O}_v$. Using specialization of v at 1 on the one hand and at large order roots of unity q on the other hand, we are able to identify the combinatorics (i.e. the composition factor multiplicities of simple modules in Verma modules, and the multiplicities of Verma modules in Verma flags of indecomposable tilting modules) of $\mathcal{O}_v$ with that of $\mathcal{O}_{\text{int}}$. As was pointed out to us by D. Kazhdan, in the simply laced case there is a stronger result, proved by M. Finkelberg in his thesis [Fi] from 1993, which establishes a category equivalence between $\mathcal{O}_q$ and $\mathcal{O}_{\text{int}}$ in the case when q is a nonzero non-root of unity. G. Lusztig made us aware of the paper [EK] where a generic equivalence is proved more generally for symmetrizable Kac–Moody algebras (see [EK, Theorem 4.2]). Our proof via the root of unity case is completely different.

The paper is organized as follows. In Section 2 we recall some basic facts about quantum groups at roots of unity. Then in Sections 3–4 we establish the results about $\mathcal{O}_q$ mentioned above. In particular, the tensor decompositions of simple modules, indecomposable projective or injective modules, and tilting modules are found in Theorem 3.1, Theorem 3.15, Theorem 3.18, and Corollary 4.8, respectively. Then we deduce in Section 5 the combinatorics of $\mathcal{O}_q$ before we conclude the paper in Section 6 by proving that in the generic case the combinatorics of $\mathcal{O}_v$ is the same as that of $\mathcal{O}_{\mathrm{int}}$. We complete the paper with a section which provides a parallel with the category $\mathcal{O}$ for Lie superalgebras.

2. Preliminaries on quantum groups

2.1. Quantum groups at roots of 1. For an indeterminate v denote by U_v the quantum group over $\mathbb{Q}(v)$ corresponding to a complex simple Lie algebra $\mathfrak{g}$. This is the $\mathbb{Q}(v)$-algebra with generators E_i, F_i, $K_i^{\pm 1}$, $i = 1, \ldots, n = \mathrm{rank}(\mathfrak{g})$, and relations as given in [Ja, Chapter 5].

Set $A = \mathbb{Z}[v, v^{-1}]$. Then A contains the quantum numbers $[r]_d = \frac{v^{dr} - v^{-rd}}{v^d - v^{-d}}$ for any $r, d \in \mathbb{Z}$, $d \neq 0$ as well as the corresponding q binomials $\begin{bmatrix} m \\ t \end{bmatrix}_d$, $m \in \mathbb{Z}$, $t \in \mathbb{N}$. When $r \geq 0$ we set $[r]_d! = [r]_d[r-1]_d \cdots [1]_d$. In the following we will often need these elements for $d = 1$, in which case we will omit it from the notation.

Let C be the Cartan matrix associated with $\mathfrak{g}$. We denote by D a diagonal matrix whose entries are relatively prime natural numbers d_i with the property that DC is symmetric. Then we set $E_i^{(r)} = E_i^r / [r]_{d_i}!$. With a similar expression for $F_i^{(r)}$ we now define the *A-form* U_A of U_v to be the A-subalgebra of U_v generated by the elements $E_i^{(r)}$, $F_i^{(r)}$, $K_i^{\pm 1}$, $i = 1, \ldots, n$, $r \geq 0$. This is the Lusztig divided power quantum group.

In this paper we fix throughout a primitive root of unity $q \in \mathbb{C}$ of odd order l. We assume that l is prime to 3 if $\mathfrak{g}$ has type G_2. The corresponding quantum group is then the specialization $U_q = U_A \otimes_A \mathbb{C}$ where $\mathbb{C}$ is considered as an A-module via $v \mapsto q$ (cf. [Lu90a], [Lu90b]). We abuse notation and write $E_i^{(r)}$ also for the element $E_i^{(r)} \otimes 1 \in U_q$ and similarly for $F_i^{(r)}$.

We have a triangular decomposition $U_q = U_q^- U_q^0 U_q^+$ with U_q^- and U_q^+ being the subalgebras generated by $F_i^{(r)}$ or $E_i^{(r)}$, $i = 1, \ldots, n$, $r \geq 0$, respectively. The "Cartan part" U_q^0 is the subalgebra generated by $K_i^{\pm 1}$ and $\begin{bmatrix} K_i \\ t \end{bmatrix}$, $i = 1, \ldots, n$, $t \geq 0$, where

$$\begin{bmatrix} K_i \\ t \end{bmatrix} = \prod_{j=1}^{t} \frac{K_i v^{d_i(1-j)} - K_i v^{-d_i(1-j)}}{v^{d_i j} - v^{-d_i j}}.$$

We denote the "Borel subalgebra" $U_q^0 U_q^+$ by B_q.

Recall that U_v is a Hopf algebra with comultiplication Δ, counit ϵ and antipode S (see [Ja, 4.11]). It is easy to see that their restrictions give U_A the structure of a Hopf algebra over A. Then U_q also gets an induced Hopf algebra structure.

2.2. The small quantum group. We also have the small quantum group $u_q \subset U_q$, defined as the subalgebra of U_q generated by E_i, F_i, $K_i^{\pm 1}$, $i = 1, \ldots, n$. It is also a Hopf

subalgebra. Note that U_q is generated by u_q and $E_i^{(l)}$, $F_i^{(l)}$, $i = 1, \ldots, n$, as follows from [Lu89, Proposition 3.2(a)].

The small quantum group also has a triangular decomposition $u_q = u_q^- u_q^0 u_q^+$ with the obvious definitions of the three parts. We write $b_q = u_q^0 u_q^+$. Note that u_q^- and u_q^+ are finite-dimensional. In fact, the PBW basis for U_q^- (resp. U_q^+) leads to a basis for u_q^- (resp. u_q^+): we just have to take PBW-monomials where each "root vector" has degree at most l [Lu90b, Theorem 8.3]. Also u_q^0 is finite-dimensional. In fact, $K_i^{2l} = 1$ for all i (see [Lu90a, 5.7]).

2.3. The quantum Frobenius homomorphism. Let $U_{\mathbb{C}}$ denote the enveloping algebra of $\mathfrak{g}$. It has generators e_i, f_i and h_i, $i = 1, \ldots, n$. Lusztig [Lu90b, Section 8] (see also [Lu, Part V]) has then defined a quantum Frobenius homomorphism $\mathrm{Fr}_q : U_q \to U_{\mathbb{C}}$ by

$$E_i^{(r)} \mapsto \begin{cases} e_i^{(r/l)} & \text{if } l \text{ divides } r; \\ 0 & \text{if not,} \end{cases} \qquad K_i \mapsto 1,$$

$$F_i^{(r)} \mapsto \begin{cases} f_i^{(r/l)} & \text{if } l \text{ divides } r; \\ 0 & \text{if not.} \end{cases} \qquad \begin{bmatrix} K_i \\ t \end{bmatrix} \mapsto \begin{pmatrix} h_i \\ t \end{pmatrix},$$

Here $\binom{h_i}{t} = \prod_{s=1}^t \frac{(h_i - s + 1)}{s}$.

2.4. Representations of U_q. Set $X = \mathbb{Z}^n$. Then for $\lambda \in \mathbb{Z}^n$ we define $\chi_\lambda : U_q^0 \to \mathbb{C}$ by $\chi_\lambda(K_i^{\pm 1}) = q^{\pm d_i \lambda_i}$ and $\chi_\lambda\left(\begin{bmatrix} K_i \\ t \end{bmatrix}\right) = \begin{bmatrix} \lambda_i \\ t \end{bmatrix}_{d_i}$. This is a well-defined character of U_q^0 (see e.g. [APW91, Lemma 1.1]) and it extends to B_q by mapping $E_i^{(r)}$ to 0 for all $r > 0$, $i = 1, \ldots, n$.

If M is a U_q^0-module, then the λ-weight space of M is defined as follows:

$$M_\lambda = \{m \in M \mid um = \chi_\lambda(u)m \text{ for all } u \in U_q^0\}. \tag{2.1}$$

The module M is called a *weight module of type* **1** provided that M decomposes into a direct sum of weight spaces of the form (2.1). In this paper we consider only weight modules of type **1** and will simply call them *weight modules*.

If N is a $U_{\mathbb{C}}$-module then we may consider N also as a U_q-module via Fr_q. To distinguish it from N we denote this U_q-module by $N^{[l]}$ and call it the (q-Frobenius) *twist* of N. Note that u_q acts trivially on $N^{[l]}$. Conversely, if M is a weight U_q-module on which u_q acts trivially, then there exists a $U_{\mathbb{C}}$-module N such that $M = N^{[l]}$ [Lu90b, 8.16]. In this case we also write $N = M^{[-l]}$. Note that $N = M$ as $\mathbb{C}$-spaces and the action of e_i (resp. f_i) on a vector $v \in N$ is given by $e_i v = E_i^{(l)} v$ (resp. $f_i v = F_i^{(l)} v$).

Note that Fr_q restricts to homomorphisms $U_q^0 \to U_{\mathbb{C}}^0$ and $B_q \to B_{\mathbb{C}}$, where $U_{\mathbb{C}}^0$ is the enveloping algebra of the Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ generated by the h_i's, and $B_{\mathbb{C}}$ is the enveloping algebra of the Borel subalgebra of $\mathfrak{g}$ generated by the h_i's and e_i's. We also denote these homomorphisms by Fr_q. Using them we can twist $U_{\mathbb{C}}^0$- as well as $B_{\mathbb{C}}$-

modules. For instance, the 1-dimensional U_q^0- (or B_q-) module $\mathbb{C}_{l\lambda}$ is the twist of the 1-dimensional $U_\mathbb{C}^0$- (or $B_\mathbb{C}$-) module $\mathbb{C}_\lambda$ determined by $\lambda \in X$ (we identify X with the set of integral weights in $\mathfrak{h}^*$ in the usual way).

3. The category $\mathcal{O}_q$

3.1. Definition. Similarly to [BGG] we define the category $\mathcal{O}_q$ as the full subcategory of U_q-mod consisting of those U_q-modules M which satisfy the following conditions:

(I) M is finitely generated as a U_q-module,
(II) M is a weight module,
(III) $\dim U_q^+ m < \infty$ for all $m \in M$.

Remark. Let $\mathcal{O}_{\text{int}}$ denote the integral subcategory (i.e. the direct sum of all integral blocks) of the usual BGG category $\mathcal{O}$ for $\mathfrak{g}$ (see [BGG]). If $M \in \mathcal{O}_{\text{int}}$ then $M^{[l]} \in \mathcal{O}_q$.

For $\lambda \in X$ the *Verma U_q-module* with highest weight λ is given by the usual recipe:

$$\Delta_q(\lambda) = U_q \otimes_{B_q} \mathbb{C}_\lambda.$$

The standard arguments (see e.g. [Di, Chapter 7]) show that $\Delta_q(\lambda)$ has the following universal property:

$$\text{Hom}_{\mathcal{O}_q}(\Delta_q(\lambda), M) = \{m \in M_\lambda \mid E_i^{(r)}m = 0 \text{ for all } r > 0, i = 1, \dots, n\}.$$

Moreover, it is easily seen that $\Delta_q(\lambda)$ has a unique maximal proper submodule. The corresponding simple quotient is denoted $L_q(\lambda)$. Then the set $\{L_q(\lambda) : \lambda \in X\}$ is a complete and irredundant set of representatives of isomorphism classes of simple modules in $\mathcal{O}_q$.

3.2. Infinitesimal modules. Replacing U_q by the small quantum group u_q we get *baby Verma modules* defined by

$$\bar{\Delta}_q(\lambda) = u_q \otimes_{b_q} \mathbb{C}_\lambda, \quad \lambda \in X.$$

If we replace here u_q by the subalgebra $u_q U_q^0$ of U_q and b_q by $U_q^0 b_q = U_q^0 u_q^+$, then we have similarly

$$\hat{\Delta}_q(\lambda) = u_q U_q^0 \otimes_{U_q^0 b_q} \mathbb{C}_\lambda.$$

The module $\hat{\Delta}_q(\lambda)$ restricted to u_q coincides with $\bar{\Delta}_q(\lambda)$ and is finite-dimensional. It has dimension l^N where N is the number of positive roots (because as a vector space we may identify it with u_q^-). The module $\hat{\Delta}_q(\lambda)$ has a universal property similar to the one enjoyed by $\Delta_q(\lambda)$ and it has a unique simple quotient which we denote $\hat{L}_q(\lambda)$.

Set now $X_l = \{\lambda \in X \mid 0 \le \lambda_i < l, i = 1, \dots, n\}$. Then each $\lambda \in X$ has an 'l-adic expansion" $\lambda = \lambda^0 + l\lambda^1$ with $\lambda^0 \in X_l$ and $\lambda^1 \in X$. In the following, upper indices 0 and 1 on a weight will always refer to the components of the weight in this expansion.

We set $X^+ = \{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0$ for all positive roots $\alpha\}$. The elements of X^+ are called *dominant weights*. An *antidominant* weight is a $\lambda \in X$ for which $\lambda + \rho \in -X^+$.

We have the following remarkable fact about these infinitesimal simple modules (see [AW, Theorem 1.9]):

$$\hat{L}_q(\lambda) \simeq L_q(\lambda^0) \otimes \mathbb{C}_{\lambda^1}^{[l]}. \tag{3.1}$$

The most "special" infinitesimal simple module is the one with highest weight $(l-1)\rho$. Here, as usual, ρ is half of the sum of all positive roots. We call this module the *quantum Steinberg module* and denote it by St_l. Note that by (3.1) it is in fact a simple U_q-module; moreover,

$$\mathrm{St}_l = \hat{L}_q((l-1)\rho) = \hat{\Delta}_q((l-1)\rho) = L_q((l-1)\rho).$$

Remark. Above we could also replace u_q by $u_q B_q$. Then we get baby Verma modules for $u_q B_q$ defined by $\tilde{\Delta}_q(\lambda) = u_q B_q \otimes_{B_q} \mathbb{C}_\lambda$ with simple quotient $\tilde{L}_q(\lambda)$. When restricted to $u_q U_q^0$, these modules coincide with $\hat{\Delta}_q(\lambda)$ and $\hat{L}_q(\lambda)$, respectively. Note, in particular, that the Steinberg module St_l is also a simple $u_q B_q$-module, as it extends, in fact, to U_q.

The composition factor multiplicities of $\tilde{\Delta}_q(\lambda)$ as well as the multiplicities with which $\tilde{\Delta}_q(\lambda)$ occurs in a baby Verma filtration of an indecomposable projective $u_q B_q$-module coincide with the corresponding numbers for the Weyl module in $\mathcal{F}_q$ with highest weight λ when λ is sufficiently dominant. This follows from (3.1) (cf. [APW92, Theorem 4.6]). This fact allows us to apply the combinatorics of $\mathcal{F}_q$ mentioned in the introduction to the category of $u_q B_q$-modules.

3.3. Tensor product formula for simple modules in $\mathcal{O}_q$. Recall that U_q is a Hopf algebra. In particular, its comultiplication allows us to make the tensor product (over $\mathbb{C}$) of two modules for U_q into a U_q-module. We write $\otimes$ for $\otimes_\mathbb{C}$. Note that $\mathcal{O}_q$ is stable under tensoring with finite dimensional modules (of type **1**).

Let $M, N \in \mathcal{O}_q$. Then we consider $\mathrm{Hom}_\mathbb{C}(M, N)$ as a U_q-module in the usual way (see e.g. [APW91, Section 2.9]). The u_q-fixed points $\mathrm{Hom}_{u_q}(M, N)$ then form a U_q-submodule on which u_q acts trivially (cf. [APW92, Section 3.2]). Hence by Section 2.4 there exists a $U_\mathbb{C}$-module $P = \mathrm{Hom}_{u_q}(M, N)^{[-l]}$ with $P^{[l]} = \mathrm{Hom}_{u_q}(M, N)$.

Let us also record the following adjointness valid whenever in addition to $M, N \in \mathcal{O}_q$ we have a module $Q \in \mathcal{O}$:

$$\mathrm{Hom}_{\mathcal{O}_q}(Q^{[l]} \otimes M, N) \simeq \mathrm{Hom}_{\mathcal{O}}(Q, \mathrm{Hom}_{u_q}(M, N)^{[-l]}). \tag{3.2}$$

We now have the following theorem first proved by G. Lusztig [Lu89, Theorem 7.4]:

Theorem 3.1. *Let $\lambda \in X$. Then $L_q(\lambda) \simeq L_\mathbb{C}(\lambda^1)^{[l]} \otimes L_q(\lambda^0)$.*

Proof. Let L be any simple u_q-module (of type **1**). Recall from the previous subsection that L is the restriction of a simple U_q-module (which we also denote by L). Then for any $M \in \mathcal{O}_q$ the natural map $\mathrm{Hom}_{u_q}(L, M) \otimes L \to M$ which takes $f \otimes m$ to $f(m)$ is a U_q-homomorphism. It is in fact an injection which identifies $\mathrm{Hom}_{u_q}(L, M) \otimes L$ with

the L-isotypic component of the u_q-socle of M. By the above, this U_q-module is equal to $N^{[l]}$ for some $N \in \mathcal{O}_{\text{int}}$.

Applying these observations to $M = L_q(\lambda)$ we get $L_q(\lambda) \simeq \text{Hom}_{u_q}(L, L_q(\lambda)) \otimes L$ for some such L, i.e. $L_q(\lambda) \simeq L_1^{[l]} \otimes L$ with $L_1 \in \mathcal{O}$. Clearly L_1 must be irreducible, i.e. $L_1 = L_{\mathbb{C}}(\mu)$ for some $\mu \in X$. By Section 3.2 we have $L \simeq L_q(\nu)$ for some $\nu \in X_l$. By weight considerations and the uniqueness of the l-adic expansion of λ we get $\mu = \lambda^1$ and $\nu = \lambda^0$. □

3.4. Verma modules in $\mathcal{O}_q$. We now want to study the composition factors of Verma modules. If $M \in \mathcal{O}_q$ and $\mu \in X$, we denote by $[M : L_q(\mu)]$ the multiplicity of $L_q(\mu)$ as a composition factor of M. We use similar notation for modules in $\mathcal{O}$ and for $u_q B_q$-modules.

Lemma 3.2. *Let M be a $B_{\mathbb{C}}$-module. Then the map*

$$u \otimes m \mapsto \text{Fr}_q(u) \otimes m, \quad u \in U_q, \, m \in M,$$

is an isomorphism of U_q-modules

$$U_q \otimes_{u_q B_q} M^{[l]} \simeq (U_{\mathbb{C}} \otimes_{B_{\mathbb{C}}} M)^{[l]}.$$

Proof. The map is clearly both well-defined and a U_q-homomorphism. Note that $u_q B_q = u_q^- B_q$ and the restriction of Fr_q to U_q^- is a surjection onto $U_{\mathbb{C}}^-$ with kernel generated by the augmentation ideal of u_q^-. It follows that the two modules in question are both isomorphic as $\mathbb{C}$-spaces to $U_{\mathbb{C}}^- \otimes M$ with the claimed map identifying the two. □

Proposition 3.3. *For $\lambda \in X$ the Verma module $\Delta_q(\lambda)$ has a filtration in $\mathcal{O}_q$ with quotients of the form $\Delta_{\mathbb{C}}(\mu^1)^{[l]} \otimes \tilde{L}_q(\mu^0)$, $\mu \in X$. Each quotient $\Delta_{\mathbb{C}}(\mu^1)^{[l]} \otimes \tilde{L}_q(\mu^0)$ occurs $[\tilde{\Delta}_q(\lambda) : \tilde{L}_q(\mu)]$ times.*

Proof. Consider a composition series of $\tilde{\Delta}_q(\lambda)$

$$0 = F^r \subset F^{r-1} \subset \cdots \subset F^0 = \tilde{\Delta}_q(\lambda)$$

with quotients $F^{i-1}/F^i \simeq \tilde{L}_q(\mu_i)$. When we apply the exact functor $U_q \otimes_{u_q B_q} -$, we obtain the filtration

$$0 = U_q \otimes_{u_q B_q} F^r \subset U_q \otimes_{u_q B_q} F^{r-1} \subset \cdots \subset U_q \otimes_{u_q B_q} F^0$$

of $U_q \otimes_{u_q B_q} \tilde{\Delta}_q(\lambda) \simeq \Delta_q(\lambda)$ with quotients $U_q \otimes_{u_q B_q} \tilde{L}_q(\mu_i)$. Now we recall from Section 3.2 that for any $\mu \in X$ we have $\tilde{L}_q(\mu) \simeq \mathbb{C}_{l\mu^1} \otimes L_q(\mu^0)$. By the tensor identity we have

$$U_q \otimes_{u_q B_q} \tilde{L}_q(\mu) \simeq (U_q \otimes_{u_q B_q} \mathbb{C}_{l\mu}) \otimes L_q(\mu^0).$$

Finally, Lemma 3.2 shows that $U_q \otimes_{u_q B_q} \mathbb{C}_{l\mu} \simeq \Delta_{\mathbb{C}}(\mu^1)^{[l]}$, and the proposition follows. □

Recall that modules in $\mathcal{O}$ have finite composition series (see [Di, Chapter 7]). Moreover, by Proposition 3.1 the composition factors of $\Delta_{\mathbb{C}}(\mu^1)^{[l]} \otimes L_q(\mu^0)$ are $L_{\mathbb{C}}(\nu^1)^{[l]} \otimes L_q(\mu^0) \simeq L_q(l\nu^1 + \mu^0)$ (occurring $[\Delta_{\mathbb{C}}(\mu^1) : L_{\mathbb{C}}(\nu^1)]$ times). We thus have

Corollary 3.4. *For every* $\lambda \in X$ *the Verma module* $\Delta_q(\lambda)$ *has finite length. Moreover, for* $\mu \in X$ *we have*

$$[\Delta_q(\lambda) : L_q(\mu)] = \sum_{\nu \geq \mu^1} [\tilde{\Delta}_q(\lambda) : \tilde{L}_q(l\nu + \mu^0)][\Delta_{\mathbb{C}}(\nu) : L_{\mathbb{C}}(\mu^1)].$$

Corollary 3.5. *All modules in* $\mathcal{O}_q$ *have finite length.*

Proof. By condition (I) of $\mathcal{O}_q$ it is enough to establish this for cyclic modules M, i.e. we assume $M = U_q m$ for some $m \in M$. By conditions (II) and (III), m is contained in a finite-dimensional B_q-submodule $E \subset M$. This means that M is a quotient of $U_q \otimes_{B_q} E$ which has a finite Verma filtration (take a B_q-filtration of E with 1-dimensional quotients and apply the exact functor $U_q \otimes_{B_q} -$). It is therefore enough to check that Verma modules in $\mathcal{O}_q$ have finite length. We did this in Corollary 3.4. □

For later use we record the following consequence of Corollary 3.4.

Corollary 3.6. *Let* $\lambda, \mu \in X$. *Then for* $l \gg 0$ *we have*

$$[\Delta_q(\lambda) : L_q(\mu)] = [\tilde{\Delta}_q(\lambda) : \tilde{L}_q(\mu)].$$

Proof. Choose l so large that $\lambda - \mu \not\geq l\nu$ for any $\nu > 0$. Then the sum on the right hand side of the formula in Corollary 3.4 contains only one term, namely the term with $\nu = \mu^1$. □

3.5. Special modules in $\mathcal{O}_q$

Proposition 3.7. *Let* $\lambda \in X$. *Then* $\Delta_q(l\lambda + (l-1)\rho) \simeq \Delta_{\mathbb{C}}(\lambda)^{[l]} \otimes \mathrm{St}_l$.

Proof. We have $\tilde{\Delta}_q((l-1)\rho) \simeq \mathrm{St}_l$ (see [APW92, Lemma 2.6]). Just as in the proof of Proposition 3.3 we then get

$$\Delta_q(l\lambda + (l-1)\rho) \simeq U_q \otimes_{u_q B_q} \tilde{\Delta}_q(l\lambda + (l-1)\rho)$$
$$\simeq (U_q \otimes_{u_q B_q} \otimes \mathbb{C}_{l\lambda}) \otimes \mathrm{St}_l \simeq \Delta_{\mathbb{C}}(\lambda)^{[l]} \otimes \mathrm{St}_l. \qquad \square$$

Corollary 3.8. *If* λ *is antidominant then* $\Delta_q(l\lambda + (l-1)\rho)$ *is simple. In particular,* $\Delta_q(-\rho)$ *is simple.*

Proof. It is well-known (see e.g. [Di, Chapter 7] or [Hu, Theorem 4.4]) that $\Delta_{\mathbb{C}}(\lambda)$ is simple in $\mathcal{O}$ when λ is antidominant. □

3.6. The special block in $\mathcal{O}_q$. The considerations at the beginning of Section 3.3 allow us to define a functor $\mathrm{F} : \mathcal{O}_q \to \mathcal{O}_{\mathrm{int}}$ by

$$\mathrm{F}N = \mathrm{Hom}_{u_q}(\mathrm{St}_l, N)^{[-l]}.$$

Since St_l is projective as a u_q-module, F is exact.

Note that the map $f \otimes s \mapsto f(s)$ is a homomorphism and in fact an inclusion $(\mathrm{F}N)^{[l]} \otimes \mathrm{St}_l \to N$. The considerations in Section 3.3 prove the following:

Proposition 3.9. *Let* $\lambda \in X$. *Then*

$$\mathrm{F}\, L_q(\lambda) \simeq \begin{cases} L_{\mathbb{C}}(\lambda^1) & \text{if } \lambda = l\lambda^1 + (l-1)\rho, \\ 0 & \text{if } l \text{ does not divide } \lambda + \rho. \end{cases}$$

This is a key ingredient in the following:

Proposition 3.10. *Let* $\lambda, \mu \in X$. *Suppose* $\lambda^0 = (l-1)\rho \neq \mu^0$. *Then* $\mathrm{Ext}^i_{\mathcal{O}_q}(L_q(\lambda), L_q(\mu))$ $= 0$ *for all* i.

Proof. As $L_q(\lambda) = L_{\mathbb{C}}(\lambda^1)^{[l]} \otimes \mathrm{St}_l$ and St_l is projective as a u_q-module, for any $M \in \mathcal{O}_q$ we get, via (3.2),

$$\mathrm{Ext}^i_{\mathcal{O}_q}(L_q(\lambda), M) \simeq \mathrm{Ext}^i_{\mathcal{O}}(L_{\mathbb{C}}(\lambda^1), \mathrm{F}\, M).$$

When $M = L_q(\mu)$ we have $\mathrm{F}\, M = 0$ by Proposition 3.9, and the desired vanishing follows. $\qquad\square$

Proposition 3.10 allows us to define $\mathcal{O}_q^{\mathrm{spec}}$ to be the block in $\mathcal{O}_q$ consisting of those $M \in \mathcal{O}_q$ whose composition factors all belong to $lX + (l-1)\rho$. We call this the *special block* in $\mathcal{O}_q$ and its objects *special modules* in $\mathcal{O}_q$.

Define a functor $\mathrm{G} : \mathcal{O} \to \mathcal{O}_q^{\mathrm{spec}}$ by

$$\mathrm{G}\, N = N^{[l]} \otimes \mathrm{St}_l.$$

Note that for $N \in \mathcal{O}$, $\mathrm{G}\, N$ is indeed a special module in $\mathcal{O}_q$.

Clearly, G is exact and is in fact (left and right) adjoint to F. It is also immediate that $\mathrm{F} \circ \mathrm{G}$ is isomorphic to the identity functor on $\mathcal{O}$. Moreover, by Theorem 3.1, $\mathrm{G} \circ \mathrm{F}$ is naturally isomorphic to the identity on simple modules and hence on $\mathcal{O}_q^{\mathrm{spec}}$. We have thus proved

Theorem 3.11. *There is an equivalence of categories* $\mathcal{O}_{\mathrm{int}} \cong \mathcal{O}_q^{\mathrm{spec}}$ *given by the mutually inverse functors* F *and* G.

3.7. Projective modules in $\mathcal{O}_q$. Recall that in $\mathcal{O}_{\mathrm{int}}$ the Verma module $\Delta_{\mathbb{C}}(\lambda)$ is projective whenever $\lambda + \rho$ is dominant (cf. [Hu, Proposition 3.8]). Hence Theorem 3.11 gives

Corollary 3.12. *If* $\lambda + \rho$ *is dominant, then* $\Delta_q(l\lambda + (l-1)\rho)$ *is projective in* $\mathcal{O}_q$. *In particular,* $\Delta_q(-\rho)$ *is projective.*

More generally, let $\mu \in X$ and denote by $P_{\mathbb{C}}(\mu) \in \mathcal{O}$ a projective cover of $L_{\mathbb{C}}(\mu)$. Then Theorem 3.11 gives the following:

Proposition 3.13. *For each* $\lambda \in X$ *the module* $P_{\mathbb{C}}(\lambda)^{[l]} \otimes \mathrm{St}_l$ *is a projective cover of* $L_q(l\lambda + (l-1)\rho)$ *in* $\mathcal{O}_q$.

Having these projectives allows us to deduce the following:

Theorem 3.14. *The category* $\mathcal{O}_q$ *has enough projectives.*

Proof. This is a standard argument (cf. [Hu, 3.8]): By induction with respect to length we reduce the problem to proving that each simple module can be covered by a projective. Given $\lambda \in X$, we set $\nu = w_0 \lambda^0 + (l-1)\rho$ where w_0 denotes the longest element in the Weyl group W for $\mathfrak{g}$. Then $w_0 \nu = \lambda^0 - (l-1)\rho$ is the lowest weight of the finite-dimensional simple module $L_q(\nu)$. Therefore $\Delta_q(l\lambda^1 + (l-1)\rho) \otimes L_q(\nu)$ surjects onto $\Delta_q(l\lambda^1 + (l-1)\rho + w_0\nu) = \Delta_q(\lambda)$. Now it is an easy consequence of Proposition 3.13 that $P_{\mathbb{C}}(\lambda^1)^{[l]} \otimes \mathrm{St}_l$ surjects onto $\Delta_q((l\lambda^1 + (l-1)\rho)$. So we see that the projective module $P_{\mathbb{C}}(\lambda^1)^{[l]} \otimes \mathrm{St}_l \otimes L_q(\nu)$ surjects onto $L_q(\lambda)$. $\qquad\square$

Define $P_q(\lambda) \in \mathcal{O}_q$ as the projective cover of $L_q(\lambda)$. Then Corollary 3.12 says that $P_q(\lambda) = \Delta_q(\lambda)$ for all λ such that $\lambda + \rho \in X^+ \cap lX$. Moreover, Proposition 3.13 says that $P_q(l\lambda + (l-1)\rho) \simeq P_{\mathbb{C}}(\lambda)^{[l]} \otimes \mathrm{St}_l$ for all $\lambda \in X$. We shall now generalize this by showing that all indecomposable projectives in $\mathcal{O}_q$ have a tensor factorization.

Recall that the subcategory $\mathcal{F}_q$ consisting of all finite-dimensional modules in $\mathcal{O}_q$ also has enough projectives (see [APW92, Section 4]). Let us denote by $Q_q(\mu) \in \mathcal{F}_q$ the projective cover of $L_q(\mu)$ for $\mu \in X^+$.

Theorem 3.15. *For any $\lambda \in X$ we have $P_q(\lambda) \simeq P_{\mathbb{C}}(\lambda^1)^{[l]} \otimes Q_q(\lambda^0)$.*

Proof. By [APW92, Theorem 4.6] the restriction to u_q of $Q_q(\lambda^0)$ is the projective cover of $L_q(\lambda^0)$, i.e. for $\mu \in X_l$ we have

$$\mathrm{Hom}_{u_q}(Q(\lambda^0), L_q(\mu)) = \begin{cases} \mathbb{C} & \text{if } \mu = \lambda^0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, using (3.2) and Theorem 3.1, for any $\mu \in X$ we get

$$\mathrm{Ext}^i_{\mathcal{O}_q}(P(\lambda^1)^{[l]} \otimes Q(\lambda^0), L_q(\mu))$$
$$= \mathrm{Ext}^i_{\mathcal{O}_{\mathbb{C}}}(P_{\mathbb{C}}(\lambda^1), L_q(\mu^1) \otimes \mathrm{Hom}_{u_q}(Q(\lambda^0), L_q(\mu^0))^{[-l]})$$
$$= \begin{cases} \mathbb{C} & \text{if } \mu = \lambda \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases} \qquad\square$$

Let us also record the following important consequence of the constructions in the proof of Theorem 3.14.

Corollary 3.16. *Projective modules in $\mathcal{O}_q$ all possess Verma filtrations.*

Proof. Let $\lambda \in X$. We shall prove that the conclusion holds for $P_q(\lambda)$. When $\lambda \in lX + (l-1)\rho$ this follows from the fact that the corresponding statement is true in $\mathcal{O}$ combined with Theorem 3.11. But then the result follows in general, because the construction in the proof of Theorem 3.14 reveals that $P_q(\lambda)$ may be obtained as a summand of a projective in $\mathcal{O}_q^{\mathrm{spec}}$ tensored by a finite-dimensional module. $\qquad\square$

3.8. Injective modules in $\mathcal{O}_q$. Let M be an arbitrary U_q-module. Since the antipode S on U_q is an antihomomorphism, the dual space $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ has the natural structure of a U_q-module given by $uf(m) = f(S(u)m)$ for $u \in U_q$, $f \in M^*$, $m \in M$.

Now U_v has an automorphism ω which interchanges E_i and F_i and inverts K_i (see [Ja, 4.6]). Clearly, ω gives rise to an automorphism of U_q. Twisting M^* by ω we get the U_q-module ${}^\omega M^*$ and when $M \in \mathcal{O}_q$ we set

$$M^\star = \bigoplus_{\lambda \in X} ({}^\omega M^*)_\lambda.$$

Then $(-)^\star$ is an endofunctor on $\mathcal{O}_q$, called *duality*, with the property that for each $\lambda \in X$ we have $\dim (M^\star)_\lambda = \dim M_\lambda$. Hence $L_q^\star(\lambda) \simeq L_q(\lambda)$ (i.e. $\star$ is *simple-preserving*). The existence of $\star$ gives immediately:

Theorem 3.17. $\mathcal{O}_q$ *has enough injectives.*

We set $I_q(\lambda) = P_q^\star(\lambda)$. This is the injective envelope of $L_q(\lambda)$ in $\mathcal{O}_q$ and if we denote by $I_{\mathbb{C}}(\mu)$ the injective envelope of $L_{\mathbb{C}}(\mu)$ in $\mathcal{O}_{\text{int}}$ then Theorem 3.15 implies:

Theorem 3.18. *For any $\lambda \in X$ we have $I_q(\lambda) \simeq I_{\mathbb{C}}(\lambda^1)^{[l]} \otimes Q_q(\lambda^0)$.*

3.9. Projective-injective modules in $\mathcal{O}_q$. By a *projective-injective module* we understand a module which is both projective and injective. We have

Theorem 3.19. *Let $\lambda \in X$. Then the following assertions are equivalent:*

(a) $P_q(\lambda) \simeq I_q(\lambda)$.
(b) $L_q(\lambda)$ *occurs in the socle of a projective-injective module in $\mathcal{O}_q$.*
(c) $L_q(\lambda)$ *occurs in the top of a projective-injective module in $\mathcal{O}_q$.*
(d) $L_q(\lambda)$ *occurs in the socle of some $\Delta_q(\mu)$, $\mu \in X$.*
(e) λ *is antidominant.*

Proof. The corresponding statement for $\mathcal{O}_{\text{int}}$ (and also for its parabolic subcategories) is well-known (see e.g. [Ir, Addendum and Proposition 4.3]). Hence Theorem 3.11 implies the claim for $\lambda \in lX + (l-1)\rho$.

Note that $Q_q(\lambda^0)$ is self-dual. Hence by Theorem 3.15 and 3.18 we see that (a) holds if and only if $P_{\mathbb{C}}(\lambda^1) \simeq I_{\mathbb{C}}(\lambda^1)$.

Now it is clear that (a) implies (b) and (b) implies (c). In view of Corollary 3.16 we see that (d) is a consequence of (c). Suppose $L_q(\lambda)$ is a submodule of $\Delta_q(\mu)$ for some $\mu \in X$. Then Propositions 3.3 and 3.1 show that $L_{\mathbb{C}}(\lambda^1)$ is a submodule of $\Delta_{\mathbb{C}}(\nu)$ for some $\nu \in X$. By the $\mathcal{O}$-result this implies that λ^1 is antidominant. But this is equivalent to (e). Finally, (c) implies (a) by the observations at the beginning of the proof. $\quad\square$

The properties described in Theorem 3.19 appear frequently in various categories associated with Lie (super)algebras (see e.g. [MS, Theorem 48] and [BS, Theorems 6.1,6.2]). For Lie superalgebras this will be further clarified in Section 7.

4. BGG reciprocity, Struktursatz and Ringel self-duality

4.1. BGG reciprocity in $\mathcal{O}_q$. The dual Verma module $\Delta_q^\star(\lambda)$ is denoted $\nabla_q(\lambda)$. Then we have the following easy but very useful vanishing theorem (cf. [Hu, 6.12]).

Theorem 4.1. *Let* $\lambda, \mu \in X$. *Then*

$$\operatorname{Ext}^i_{\mathcal{O}_q}(\Delta_q(\lambda), \nabla_q(\mu)) \simeq \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ and } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Note that $\operatorname{Ext}^i_{\mathcal{O}_q}(\Delta_q(\lambda), \nabla_q(\mu)) \simeq \operatorname{Ext}^i_{\mathcal{O}_q}(\Delta_q(\mu), \nabla_q(\lambda))$ by duality. This allows us to assume that $\lambda \not< \mu$. Easy weight arguments show that the conclusion holds for $i = 0$. Now by Corollary 3.16 all projectives in $\mathcal{O}_q$ have Verma filtrations. Moreover, we have a short exact sequence $0 \to K \to P_q(\lambda) \to \Delta_q(\lambda) \to 0$ with K having a Verma filtration where all subfactors $\Delta_q(\lambda')$ have $\lambda' > \lambda$. The $i > 0$ part of the theorem then follows from this sequence by a dimension shift argument. $\square$

As a consequence we see that if $M \in \mathcal{O}_q$ has a Verma (resp. dual Verma) filtration then the number of occurrences $(M : \Delta_q(\lambda))$ (resp. $(M : \nabla_q(\lambda))$ of $\Delta_q(\lambda)$ (resp. $\nabla_q(\lambda)$) in this filtration equals the dimension of $\operatorname{Hom}_{\mathcal{O}_q}(M, \nabla_q(\lambda))$ (resp. $\operatorname{Hom}_{\mathcal{O}_q}(\Delta_q(\lambda), M)$). This immediately leads to the following BGG-reciprocity laws:

Corollary 4.2. *Let* $\lambda, \mu \in X$. *Then*

$$(P_q(\lambda) : \Delta_q(\mu)) = [\Delta_q(\mu) : L_q(\lambda)] = (I_q(\lambda) : \nabla_q(\mu)).$$

In other words, the above means that $\mathcal{O}_q$ is a highest weight category in the sense of [CPS88] (with infinitely many isomorphism classes of simple modules).

4.2. The category $\mathscr{C}$. Let $\mathscr{C}$ denote the full subcategory of $\mathcal{O}_q$ with objects $P_q(\lambda)$, $\lambda \in X$. For simplicity we will identify objects of $\mathscr{C}$ with elements in X. Then Corollary 3.5 implies that $\mathscr{C}$ is a locally finite-dimensional $\mathbb{C}$-linear category (we refer to [MOS] for generalities on representations of $\mathbb{C}$-linear categories). Moreover, from Corollary 3.5 and Theorems 3.14 and 3.17 it follows that for any $\lambda \in X$ there exist only finitely many $\mu \in X$ such that $\mathscr{C}(\lambda, \mu) \neq 0$ and that for any $\lambda \in X$ there exist only finitely many $\mu \in X$ such that $\mathscr{C}(\mu, \lambda) \neq 0$.

Let $\mathscr{C}$-mod (resp. mod-$\mathscr{C}$) denote the category of finite-dimensional left (resp. right) $\mathscr{C}$-modules, that is, covariant (resp. contravariant) functors $\mathrm{M} : \mathscr{C} \to \mathbb{C}$-mod (the latter being the category of finite-dimensional complex vector spaces) satisfying $\sum_{\lambda \in X} \dim \mathrm{M}(\lambda) < \infty$. Then abstract nonsense (see e.g. [Ga]) implies that $\mathcal{O}_q$ is equivalent to mod-$\mathscr{C}$, and the latter is equivalent to $\mathscr{C}$-mod by duality.

4.3. Dominance dimension and Soergel's Struktursatz

Proposition 4.3. *The category $\mathcal{O}_q$ has dominance dimension at least two with respect to projective-injective modules, that is, for any projective module $P \in \mathcal{O}_q$ there exists an exact sequence*

$$0 \to P \to X_1 \to X_2, \tag{4.1}$$

where both X_1 and X_2 are projective-injective.

Proof. This claim is well-known for $\mathcal{O}_{\text{int}}$ (see e.g. [KSX, 3.1]). Hence Theorem 3.11 implies the claim for $P \in \mathcal{O}_q^{\text{spec}}$. By Theorem 3.15, every indecomposable projective can be obtained by tensoring an indecomposable projective from $\mathcal{O}_q^{\text{spec}}$ with a finite-dimensional module and taking a direct summand. As this tensoring is both left and right adjoint to an exact functor, it preserves projective-injective modules. Hence such tensoring maps a sequence of the form (4.1) to a sequence of the same form, and the claim follows. $\quad\square$

Denote by $\mathscr{C}^{\text{PI}}$ the full subcategory of $\mathscr{C}$ whose objects are all antidominant $\lambda \in X$, that is, those $\lambda \in X$ for which the projective module $P_q(\lambda)$ is also injective (see Theorem 3.19). For $\lambda \in X$ define

$$\mathrm{M}_\lambda := \mathrm{Hom}_{\mathcal{O}_q}(-, P_q(\lambda)) \in \text{mod-}\mathscr{C}^{\text{PI}}.$$

Let $\overline{\mathscr{C}}$ denote the full subcategory of mod-$\mathscr{C}^{\text{PI}}$ with objects M_λ, $\lambda \in X$.

Define a functor $\Phi : \mathscr{C} \to \overline{\mathscr{C}}$ as follows: on objects we set $\Phi(\lambda) := \mathrm{M}_\lambda$, $\lambda \in X$; if $\lambda, \mu \in X$ and $\varphi \in \mathrm{Hom}_{\mathcal{O}_q}(P_q(\lambda), P_q(\mu))$, then set

$$\Phi(\varphi) := \varphi \circ - : \mathrm{Hom}_{\mathcal{O}_q}(-, P_q(\lambda)) \to \mathrm{Hom}_{\mathcal{O}_q}(-, P_q(\mu)).$$

The following result generalizes [So90, Struktursatz].

Theorem 4.4. *The functor Φ is an isomorphism of categories.*

Proof. By definition, Φ induces a bijection on objects. So we need only check that it induces a bijection on morphisms, that is, for any $\lambda, \mu \in X$ the map $\Phi_{\lambda,\mu} : \mathrm{Hom}_{\mathcal{O}_q}(P_q(\lambda), P_q(\mu)) \to \overline{\mathscr{C}}(\mathrm{M}_\lambda, \mathrm{M}_\mu)$ is an isomorphism. This is clear if both $P_q(\lambda)$ and $P_q(\mu)$ are injective.

By Proposition 4.3, the injective envelope of $P_q(\mu)$ is projective. Observe that if $\varphi \in \mathrm{Hom}_{\mathcal{O}_q}(P_q(\lambda), P_q(\mu))$ is nonzero, then the image of φ contains a simple submodule L in the socle of $P_q(\mu)$. By Theorem 3.19, L is a homomorphic image of some projective-injective module P. By the projectivity of P, the surjection $f : P \to L$ lifts to a map $f' : P \to P_q(\lambda)$ such that $f = \varphi \circ f'$. This implies that $\Phi_{\lambda,\mu}(\varphi)$ is nonzero and hence $\Phi_{\lambda,\mu}$ is injective.

To prove surjectivity let $\lambda, \mu \in X$ and $f \in \overline{\mathscr{C}}(\mathrm{M}_\lambda, \mathrm{M}_\mu)$. By Proposition 4.3, there are exact sequences

$$0 \to P_q(\lambda) \to X_1 \to X_2, \quad 0 \to P_q(\mu) \to Y_1 \to Y_2$$

in $\mathcal{O}_q$ such that X_1, X_2, Y_1 and Y_2 are projective-injective. Applying the covariant functor $\mathrm{Hom}_{\mathcal{O}_q}(-,-)$ to these exact sequences yields injective resolutions for both M_λ and M_μ in mod-$\mathscr{C}^{\mathrm{PI}}$. The map f admits lifts giving the following commutative diagram:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{M}_\lambda & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_q}(-,X_1) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_q}(-,X_2) \\
& & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f''} \\
0 & \longrightarrow & \mathrm{M}_\mu & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_q}(-,Y_1) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_q}(-,Y_2)
\end{array}
$$

As X_1, X_2, Y_1 and Y_2 are projective-injective, the right hand square of the latter diagram is the image of the right hand square of some commutative diagram of the form

$$
\begin{array}{ccccccc}
0 & \longrightarrow & P_q(\lambda) & \longrightarrow & X_1 & \longrightarrow & X_2 \\
& & \vdots{\scriptstyle \varphi} & & \downarrow & & \downarrow \\
0 & \longrightarrow & P_q(\mu) & \longrightarrow & Y_1 & \longrightarrow & Y_2
\end{array}
$$

As both rows are exact, the commutative right hand square of the latter diagram induces a unique $\varphi : P_q(\lambda) \to P_q(\mu)$ making the diagram commutative and we have $\Phi_{\lambda,\mu}(\varphi) = f$. This proves surjectivity and completes the proof. $\qquad\square$

Statements similar to Proposition 4.3 and Theorem 4.4 appear frequently and play an important role in Lie-theoretic context (sometimes in disguise); see e.g. [St03, Theorem 10.1], [St06, Theorem 3.9] and [Ma, Corollary 2]. Our proof above follows the approach of [KSX]. Making a parallel with the results of [MM], we propose the following conjecture:

Conjecture 4.5. The category $\mathscr{C}^{\mathrm{PI}}$ is symmetric, i.e. the bimodules $\mathscr{C}^{\mathrm{PI}}(-,-)^*$ and $\mathscr{C}^{\mathrm{PI}}(-,-)$ are isomorphic.

In [MM] it is shown that some similar categories associated to certain Lie superalgebras are symmetric, using a description of the Serre functor for the corresponding category $\mathcal{O}$ via Harish-Chandra bimodules. A similar approach for $\mathscr{C}^{\mathrm{PI}}$ would require developing the theory of Harish-Chandra bimodules in the quantum case.

4.4. Tilting modules in $\mathcal{O}_q$. A module $M \in \mathcal{O}_q$ is called *tilting* if M has both a Verma filtration and a dual Verma filtration. In $\mathcal{O}_{\mathrm{int}}$ there exists, for each $\lambda \in X$, a unique indecomposable tilting module $T_{\mathbb{C}}(\lambda)$ which has λ as its unique highest weight. The same is true in $\mathcal{O}_q$:

Theorem 4.6. *For each $\lambda \in X$ there exists an indecomposable tilting module $T_q(\lambda)$ with λ as its unique highest weight. Every indecomposable tilting module in $\mathcal{O}_q$ is isomorphic to $T_q(\lambda)$ for some $\lambda \in X$.*

There are various ways to prove this (compare e.g. with [So99]); we choose the one which we think is the shortest.

Proof. The functor $G : \mathcal{O}_{\mathrm{int}} \to \mathcal{O}_q^{\mathrm{spec}}$ clearly takes tilting modules in $\mathcal{O}_{\mathrm{int}}$ to tilting modules in $\mathcal{O}_q$ (see Proposition 3.7). Hence for $\mu \in X$ we set $T_q(l\mu + (l-1)\rho) = T_{\mathbb{C}}(\mu)^{[l]} \otimes \mathrm{St}_l$.

For general $\lambda \in X$ we set $\mu = \lambda^1 - \rho$ and consider $T = T_{\mathbb{C}}(\mu)^{[l]} \otimes \mathrm{St}_l \otimes L_q(\lambda^0 + \rho)$. Then T is a tilting module and its highest weight is λ occurring with multiplicity 1. So we set $T_q(\lambda)$ equal to the unique indecomposable summand of T which has a nonzero λ-weight space.

This gives the existence of $T_q(\lambda)$. The second statement is then seen by standard arguments (see [Hu, Theorem 11.2]). $\qquad\square$

For $N \in \mathcal{O}_q$ we denote by $\mathrm{Tr}_{\mathrm{PI}}(N)$ the *trace* in N of all projective-injective modules, that is, the sum of the images of all homomorphisms from M to N, where M is projective-injective. Note that for every finite-dimensional $V \in \mathcal{O}_q$ the functor $V \otimes -$ preserves the category of projective-injective modules. This implies that for any $N \in \mathcal{O}_q$ we have $\mathrm{Tr}_{\mathrm{PI}}(V \otimes N) \cong V \otimes \mathrm{Tr}_{\mathrm{PI}}(N)$. Titling modules in $\mathcal{O}_q$ can be alternatively described as follows:

Theorem 4.7. (i) *For every $\lambda \in X$ the module $\mathrm{Tr}_{\mathrm{PI}}(P_q(\lambda))$ is an indecomposable tilting module.*
(ii) *Every indecomposable tilting module is isomorphic to $\mathrm{Tr}_{\mathrm{PI}}(P_q(\lambda))$ for some $\lambda \in X$.*
(iii) (Ringel self-duality) *For all $\lambda, \mu \in X$ we have*

$$\mathrm{Hom}_{\mathcal{O}_q}(P_q(\lambda), P_q(\mu)) \cong \mathrm{Hom}_{\mathcal{O}_q}(\mathrm{Tr}_{\mathrm{PI}}(P_q(\lambda)), \mathrm{Tr}_{\mathrm{PI}}(P_q(\mu))).$$

In the classical case, Ringel self-duality is due to Soergel [So97b].

Proof. This is well-known for $\mathcal{O}_{\mathbb{C}}$ (see e.g. [So97b, FKM]). Hence Theorem 3.11 implies the claim for $\mathcal{O}_q^{\mathrm{spec}}$. Using translation and Theorem 3.15 we deduce that $\mathrm{Tr}_{\mathrm{PI}}(P_q(\lambda))$ is a tilting module for every $\lambda \in X$.

From Theorem 4.4 it follows that for all $\lambda, \mu \in X$ the restriction map

$$\mathrm{Hom}_{\mathcal{O}_q}(P_q(\lambda), P_q(\mu)) \to \mathrm{Hom}_{\mathcal{O}_q}(\mathrm{Tr}_{\mathrm{PI}}(P_q(\lambda)), \mathrm{Tr}_{\mathrm{PI}}(P_q(\mu)))$$

is bijective. This proves (iii) and implies that every $\mathrm{Tr}_{\mathrm{PI}}(P_q(\lambda))$ is indecomposable, proving (i). Claim (ii) follows from the fact that every tilting module occurs as a direct summand of a simple tilting module from $\mathcal{O}_q^{\mathrm{spec}}$ tensored with a finite-dimensional module. $\qquad\square$

Theorem 4.7(i) combined with Theorem 3.15 implies a tensor product formula for indecomposable tilting modules, similar to Theorems 3.15 and 3.17. Namely, let $\lambda^0 \in X_l$ and write $\tilde{\lambda}^0 = l\rho + w_0 \cdot \lambda^0$.

Corollary 4.8. *For each $\lambda \in X$ we have $T_q(\lambda) \simeq T_{\mathbb{C}}(\lambda^1 - \rho)^{[l]} \otimes Q_q(\tilde{\lambda}^0)$.*

5. Characters and Kazhdan–Lusztig data

5.1. Character formulas. Consider the group ring $\mathbb{Z}[X]$ in which we denote the basis element corresponding to $\lambda \in X$ by e^λ. Multiplication is then determined by $e^\lambda e^\mu = e^{\lambda+\mu}$.

We extend this ring by defining its "completion" $\widehat{\mathbb{Z}[X]}$ to consist of all expressions $\sum_\lambda c_\lambda e^\lambda$ where $c_\lambda \in \mathbb{Z}$ for all λ and there exist $\lambda_1, \ldots, \lambda_r \in X$ such that $c_\lambda = 0$ unless $\lambda \leq \lambda_i$ for some i (here $\leq$ is the usual order on X). Alternatively, this is the set of $\mathbb{Z}$-valued functions on X whose support is contained in a finite union of subsets of the form $X_{\leq\mu} = \{\lambda \in X \mid \lambda \leq \mu\}$. Clearly, multiplication on $\mathbb{Z}[X]$ extends to $\widehat{\mathbb{Z}[X]}$.

If $f = \sum a_\lambda e^\lambda \in \widehat{\mathbb{Z}[X]}$, we set $f^{[l]} = \sum a_\lambda e^{l\lambda}$. If $M \in \mathcal{O}_{\mathrm{int}}$ or $M \in \mathcal{O}_q$, we set $\mathrm{ch}\, M = \sum_\mu (\dim M_\mu) e^\mu \in \widehat{\mathbb{Z}[X]}$ and call this the *character* of M. Then for $M \in \mathcal{O}_{\mathrm{int}}$ we get $\mathrm{ch}(M^{[l]}) = (\mathrm{ch}\, M)^{[l]}$.

Using the notation from Section 3.4, for $M \in \mathcal{O}_{\mathrm{int}}$ we have

$$\mathrm{ch}\, M = \sum_\mu [M : L_\mathbb{C}(\mu)]\,\mathrm{ch}\, L_\mathbb{C}(\mu), \tag{5.1}$$

and similarly for $M \in \mathcal{O}_q$ we have

$$\mathrm{ch}\, M = \sum_\mu [M : L_q(\mu)]\,\mathrm{ch}\, L_q(\mu). \tag{5.2}$$

These sums are finite (cf. Corollary 3.5). If we take $M = \Delta_\mathbb{C}(\lambda)$, then the sum in (5.1) has a unique highest term, namely $1 \cdot \mathrm{ch}\, L_\mathbb{C}(\lambda)$. We can therefore "invert" these equations and obtain

$$\mathrm{ch}\, L_\mathbb{C}(\lambda) = \sum_\mu p^\mathbb{C}_{\mu,\lambda}\,\mathrm{ch}\, \Delta_\mathbb{C}(\mu) \tag{5.3}$$

for some unique $p^\mathbb{C}_{\mu,\lambda} \in \mathbb{Z}$. Similarly, we get

$$\mathrm{ch}\, L_q(\lambda) = \sum_\mu p^q_{\mu,\lambda}\,\mathrm{ch}\, \Delta_q(\mu) \tag{5.4}$$

for some unique $p^q_{\mu,\lambda} \in \mathbb{Z}$.

Note that whereas the sum in (5.3) is finite for all $\lambda \in X$ (we have $p^\mathbb{C}_{\mu,\lambda} = 0$ unless $\mu \in W \cdot \lambda$), this is not so in (5.4) (as blocks of $\mathcal{O}_q$ could have infinitely many simples). For instance in the $\mathfrak{sl}_2$-case we have

$$\mathrm{ch}\, L_q(-2) = \Delta_q(-2) + \sum_{m \leq -1} (\mathrm{ch}\, \Delta_q(2ml) - \mathrm{ch}\, \Delta_q(2ml-2)). \tag{5.5}$$

Similarly, we may consider the characters of (finite-dimensional) $u_q B_q$-modules. Here we obtain the analogous formulas

$$\mathrm{ch}\, M = \sum_\mu [M : \tilde{L}_q(\mu)]\,\mathrm{ch}\, \tilde{L}_q(\mu), \tag{5.6}$$

and

$$\mathrm{ch}\, \tilde{L}_q(\lambda) = \sum_\mu \tilde{p}^q_{\mu,\lambda}\,\mathrm{ch}\, \tilde{\Delta}_q(\mu) \tag{5.7}$$

for some unique $\tilde{p}_{\mu,\lambda}^q \in \mathbb{Z}$. Again, (5.6) clearly involves only finite sums for any finite-dimensional M (and is in fact a formula in $\mathbb{Z}[X]$), whereas the sum in (5.7) may well be infinite.

Finally, we observe the following obvious identities, valid for all $\lambda, \mu \in X$:

$$(\operatorname{ch} \Delta_C(\lambda))e^\mu = \operatorname{ch} \Delta_{\mathbb{C}}(\lambda + \mu), \tag{5.8}$$

$$(\operatorname{ch} \Delta_q(\lambda))e^\mu = \operatorname{ch} \Delta_q(\lambda + \mu), \tag{5.9}$$

$$(\operatorname{ch} \tilde{\Delta}_q(\lambda))e^\mu = \operatorname{ch} \tilde{\Delta}_q(\lambda + \mu). \tag{5.10}$$

5.2. Characters of simple modules in $\mathcal{O}_q$. Using the notation of Section 5.1 we have:

Theorem 5.1. *For all $\lambda \in X$ we have the following:*

(i) $\operatorname{ch} L_q(\lambda) = \sum_{\nu,\eta} p_{\nu,\lambda^1}^{\mathbb{C}} \tilde{p}_{\eta,\lambda^0}^q \operatorname{ch} \Delta_q(l\nu + \eta);$

(ii) $p_{\mu,\lambda}^q = \sum_{l\nu+\eta=\mu} p_{\nu,\lambda^1}^{\mathbb{C}} \tilde{p}_{\eta,\lambda^0}^q = \sum_{w\in W^{\lambda^1}} p_{w\cdot\lambda^1,\lambda^1}^{\mathbb{C}} \tilde{p}_{\mu-lw\cdot\lambda^1,\lambda^0}^q$, *where W^{λ^1} denotes the set of shortest coset representatives in $W/\operatorname{Stab}_{W\cdot}(\lambda^1)$.*

Proof. By Proposition 3.1 combined with (5.3) and (5.7) we find

$$\operatorname{ch} L_q(\lambda) = (\operatorname{ch} L_{\mathbb{C}}(\lambda^1)^{[l]} \operatorname{ch} L_q(\lambda^0) = \sum_\nu p_{\nu,\lambda^1}^{\mathbb{C}} (\operatorname{ch} \Delta_{\mathbb{C}}(\nu))^{[l]} \sum_\eta \tilde{p}_{\eta,\lambda^0}^q \operatorname{ch} \tilde{\Delta}_q(\eta)$$

$$= \sum_\mu \Big(\sum_{l\nu+\eta=\mu} p_{\nu,\lambda^1}^{\mathbb{C}} \tilde{p}_{\eta,\lambda^0}^q \Big)(\operatorname{ch} \Delta_{\mathbb{C}}(\nu))^{[l]} \operatorname{ch} \tilde{\Delta}_q(\eta).$$

So, to establish (i) we should only check that

$$(\operatorname{ch} \Delta_{\mathbb{C}}(\nu))^{[l]} \operatorname{ch} \tilde{\Delta}_q(\eta) = \operatorname{ch} \Delta_q(l\nu + \eta). \tag{5.11}$$

However, by (5.10) we have $\operatorname{ch} \tilde{\Delta}_q(\eta) = (\operatorname{ch} \operatorname{St}_l)e^{\eta-(l-1)\rho}$ (because $\operatorname{St}_l = \tilde{\Delta}_q((l-1)\rho)$). Hence using Proposition 3.7 we find

$$(\operatorname{ch} \Delta_{\mathbb{C}}(\nu))^{[l]} \operatorname{ch} \tilde{\Delta}_q(\eta) = (\operatorname{ch} \Delta_{\mathbb{C}}(\nu))^{[l]}(\operatorname{ch} \operatorname{St}_l)e^{\eta-(l-1)\rho}$$

$$= (\operatorname{ch} \Delta_q(l\nu + (l-1)\rho))e^{\eta-(l-1)\rho} = \operatorname{ch} \Delta_q(l\nu + \eta).$$

Here we have used (5.9) for the last equality.

The first equality in (ii) is immediate from (i), and the second comes from the fact that $p_{\nu,\lambda^1}^{\mathbb{C}} = 0$ unless $\nu \in W \cdot \lambda^1$. $\qquad\square$

5.3. Characters of indecomposable tilting modules in $\mathcal{O}_q$. By Corollary 4.8 we get, for any $\lambda \in X$,

$$\operatorname{ch} T_q(\lambda) = \sum_{\nu,\eta} (T_{\mathbb{C}}(\lambda^1 - \rho) : \Delta_{\mathbb{C}}(\nu))(Q_q(\tilde{\lambda}^0) : \tilde{\Delta}_q(\eta)) \operatorname{ch}(\Delta_{\mathbb{C}}(\nu)^{[l]}) \operatorname{ch} \tilde{\Delta}_q(\eta).$$

Applying (5.11) in this formula we get

Theorem 5.2. *For all $\lambda, \mu \in X$ we have*

$$(T_q(\lambda) : \Delta_q(\mu)) = \sum_{\nu,\eta,l\nu+\eta=\mu} (T_{\mathbb{C}}(\lambda^1 - \rho) : \Delta_C(\nu))(Q_q(\tilde{\lambda}^0) : \tilde{\Delta}_q(\eta)).$$

5.4. Kazhdan–Lusztig theory for $\mathcal{O}_q$. Fix an antidominant weight $\lambda \in X$. For each $\mu \in W \cdot \lambda$ we pick $w \in W$ minimal such that $w \cdot \lambda = \mu$. Then the Kazhdan–Lusztig conjecture [KL] proved independently by Beilinson and Bernstein [BB] and by Brylinski and Kashiwara [BK] says that (for each such minimal $y, w \in W$)

$$p^{\mathbb{C}}_{y \cdot \lambda, w \cdot \lambda} = (-1)^{l(yw)} P_{y,w}(1). \tag{5.12}$$

Here $P_{y,w}$ is the Kazhdan–Lusztig polynomial associated to y, w (see [KL]).

As discussed in the introduction, the analogous Lusztig conjecture for finite-dimensional simple modules in $\mathcal{F}_q$ has also been settled.

Let W_l be the affine Weyl group (of dual Langlands type). Set

$$A_l^- = \{\lambda \in X \mid -l < \langle \lambda + \rho, \alpha^\vee \rangle < 0 \text{ for all positive roots } \alpha\}.$$

This is the top antidominant alcove. Fix $\lambda \in \bar{A}_l^-$ and choose for each $\mu \in W_l \cdot \lambda$ a minimal $x \in W_l$ such that $\mu = x \cdot \lambda$. Then in analogy with (5.12) for all such minimal $z, x \in W_l$ for which $z \cdot \lambda, x \cdot \lambda \in X^+$ we have

$$p^q_{z \cdot \lambda, x \cdot \lambda} = (-1)^{l(zx)} P_{z,x}(1). \tag{5.13}$$

Here $P_{z,x}$ is again the Kazhdan–Lusztig polynomial associated to the pair (z, x) in the affine Weyl group W_l. Note that this gives us only some of the coefficients; the remaining coefficients (for $x \cdot \lambda \in X^+$ fixed) can be obtained using the Weyl character formula for quantum Weyl modules.

Combining the above two formulas allows us to formulate Theorem 5.1 as follows:

Corollary 5.3. *Let $\lambda \in X$ and suppose $w \in W$, resp. $x \in W_l$, is minimal such that $w^{-1} \cdot \lambda^1$ is antidominant, resp. $x^{-1} \cdot \lambda^0 \in \bar{A}_l^-$. Then the character of $L_q(\lambda)$ equals*

$$\sum_{r \in W} \sum_{y \in W^{w^{-1} \cdot \lambda^1}} \sum_{\substack{z \in W_l^{x^{-1} \cdot \lambda^0} \\ zx^{-1} \cdot \lambda^0 \in X^+}} (-1)^{l(yw)+l(zx)+l(r)} P_{y,w}(1) P_{z,x}(1)$$
$$\cdot \operatorname{ch} \Delta_{\mathbb{C}}(yw^{-1} \cdot \lambda^1)^{[l]} \operatorname{ch} \Delta_q(rzx^{-1} \cdot \lambda^0)$$

Similarly, the result in Theorem 5.2 leads to the following expression for the characters of indecomposable tilting modules in $\mathcal{O}_q$. The formula involves the "inverse" Kazhdan–Lusztig polynomials $Q_{x,y}$, i.e. the polynomials determined by the equations

$$\sum_z (-1)^{l(z)-l(y)} P_{y,z} Q_{z,w} = \delta_{y,w}.$$

Corollary 5.4. *Let $\lambda \in X$. Assume that $\lambda^1 - \rho$ is regular, i.e. belongs to the interior of a chamber so that there is a unique $w \in W$ with $w^{-1} \cdot (\lambda^1 - \rho)$ antidominant. Likewise assume that λ^0 is l-regular so that there is a unique $x \in W_l$ with $x^{-1} \cdot \lambda^0 \in A_l^-$. Then*

$$(T_q(\lambda) : \Delta_q(\mu)) = \sum_{y,z} P_{y,w}(1) Q_{z,x}(1)$$

where the sum runs over those $y \in W$, $z \in W_l$ for which $\mu = lyw^{-1} \cdot \lambda^1 + zx^{-1} \cdot \lambda^0$.

Proof. According to [So97a, Conjecture 7.1] (proved in [So97b]) we have $(T_{\mathbb{C}}(\lambda^1 - \rho) : \Delta_{\mathbb{C}}(yw^{-1} \cdot (\lambda^1 - \rho)) = P_{y,w}(1)$ and $(Q_q(\tilde{\lambda}^0) : \tilde{\Delta}_q(zx^{-1} \cdot \lambda^0 + l\rho) = Q_{z,x}(1)$. $\square$

Remark. Using [So97a, Remark 7.2.2] it is possible to generalize this corollary to include weights λ without the stated regularity assumptions.

Example 5.5. Consider the simplest possible case where the Lie algebra $\mathfrak{g}$ is $\mathfrak{sl}_2$. Then $X = \mathbb{Z}$ with $X^+ = \mathbb{Z}_{\geq 0}$. Suppose q is a complex root of unity of odd order $l > 1$. Then we have the following description of the Verma modules and the indecomposable tilting modules in $\mathcal{O}_q$:

Case 1. Suppose $\lambda \in \mathbb{Z}$ is l-singular, i.e. $\lambda \equiv -1 \pmod{l}$. Then we are in the special block of $\mathcal{O}_q$ and just as in the classical case we have (cf. Theorem 3.11) $\Delta_q(\lambda) = T_q(\lambda) = L_q(\lambda)$ when $\lambda < 0$. On the other hand, for $\lambda \geq 0$ we have two short exact sequences

$$0 \to L_q(-\lambda - 2) \to \Delta_q(\lambda) \to L_q(\lambda) \to 0,$$
$$0 \to \Delta_q(\lambda) \to T_q(\lambda) \to \Delta_q(-\lambda - 2) \to 0.$$

Case 2. Suppose λ is l-regular and write $\lambda = \lambda^0 + l\lambda^1$ with $0 \leq \lambda^0 < l - 1$. If $\lambda < 0$ we have two exact sequences

$$0 \to L_q(-\lambda^0 - 2 + l\lambda^1) \to \Delta_q(\lambda) \to L_q(\lambda) \to 0,$$
$$0 \to \Delta_q(\lambda) \to T_q(\lambda) \to \Delta_q(-\lambda^0 - 2 + l\lambda^1) \to 0.$$

On the other hand, if $\lambda \geq 0$ then $\Delta_q(\lambda)$ has composition factors $L_q(\lambda)$, $L_q(-\lambda^0 - 2 + l\lambda^1)$, $L_q(-\lambda - 2)$, and $L_q(\lambda^0 + l(-\lambda^1 - 2))$. Note that if $\lambda^1 = 0$ the second and the third factors coincide and so $\Delta_q(\lambda)$ has in this case only three composition factors. Similarly the tilting module $T_q(\lambda)$ has Verma filtration factors $\Delta_q(\lambda)$, $\Delta_q(-\lambda^0 - 2 + l\lambda^1)$, $\Delta_q(-\lambda - 2)$, and $\Delta_q(\lambda^0 - l\lambda^1)$. If $\lambda^1 = 0$ then the first and the last factors are identical and so are the second and the third, so that in this case there are only two factors.

Although this example is maybe too simple to catch the full flavor of the behavior of Verma modules and tilting modules, it does illustrate some important features: In the most singular case (the special block) the modules are as in the classical case (with the Kazhdan–Lusztig polynomials for the finite ordinary Weyl group governing the combinatorics). At the other extreme, when the weight λ in question is l-regular and far from the walls of the Weyl chamber C containing it, the Verma module $\Delta_q(\lambda)$, respectively the tilting module $T_q(\lambda)$, has composition factors, respectively Verma filtration factors, belonging to "clusters" in all the Weyl chambers below C (in the Bruhat order). These clusters look like the alcove patterns in [Lu80]. When λ is close to some walls there are (rather complicated) cancellations among these clusters. Moreover, there are several degrees of l-singularity (depending on the facette for the affine Weyl group to which λ belongs) which also influence the patterns. This is all encoded in Corollaries 5.3 and 5.4.

6. The generic case

6.1. The category $\mathcal{O}_v$. We define the category $\mathcal{O}_v$ to be the full subcategory of the category of U_v-modules consisting of those modules which satisfy the analogues of (I)–(III) in Section 3.1.

Among the objects in $\mathcal{O}_v$ we have the generic Verma modules $\Delta_v(\lambda)$, $\lambda \in X$, defined in the usual way. They have unique simple quotients $L_v(\lambda)$, and these are up to isomorphism a complete set of simple modules in $\mathcal{O}_v$.

The category $\mathcal{O}_v$ has properties completely analogous to $\mathcal{O}_{\mathrm{int}}$ (see e.g. [C-P, Chapters 9–10]). In particular all modules in $\mathcal{O}_v$ have finite length, the Verma module $\Delta_v(\lambda)$ has composition factors $L_v(\mu)$ with $\mu \in W \cdot \lambda$, and in fact $\mathcal{O}_v$ splits into blocks,

$$\mathcal{O}_v = \bigoplus_\lambda \mathcal{O}_v^\lambda,$$

where the block $\mathcal{O}_v^\lambda$ consists of those modules from $\mathcal{O}_v$ whose composition factors have highest weights in $W \cdot \lambda$, and where the sum runs over the set of all λ for which $\lambda + \rho$ are dominant.

In analogy to $\mathcal{O}_q$ we have a duality $(-)^\star$ on $\mathcal{O}_v$ which fixes simple modules. The dual Verma module $\Delta_v^\star(\lambda)$ is denoted $\nabla_v(\lambda)$.

6.2. A-lattices. Clearly the Verma module $\Delta_v(\lambda)$ has an A-lattice, namely the Verma module for U_A defined by

$$\Delta_A(\lambda) = U_A \otimes_{B_A} A_\lambda.$$

Here B_A is the Borel subalgebra of U_A defined in analogy to B_q, and A_λ denotes the free rank one A-module with B_A-action given by the analogue over A of the character χ_λ from Section 2.4.

Similarly, $\nabla_v(\lambda)$ has an A-lattice $\nabla_A(\lambda)$ defined as the A-dual of $\Delta_A(\lambda)$ (with the appropriate U_A-structure).

Note that $\mathrm{Hom}_{U_A}(\Delta_A(\lambda), \nabla_A(\lambda)) \simeq A$. We let c_λ denote a generator of this module and set $K_A(\lambda)$, respectively $L_A(\lambda)$ and $C_A(\lambda)$, equal to the kernel, respectively the image and the cokernel, of c_λ. Then we get the following two short exact sequences in $\mathcal{O}_v$:

$$0 \to K_A(\lambda) \to \Delta_A(\lambda) \to L_A(\lambda) \to 0,$$
$$0 \to L_A(\lambda) \to \nabla_A(\lambda) \to C_A(\lambda) \to 0.$$

Tensoring by the fraction field $\mathbb{Q}(v)$ of A we see that $L_A(\lambda) \otimes_A \mathbb{Q}(v) \simeq L_v(\lambda)$ because $L_v(\lambda)$ is the image of $c_\lambda \otimes 1 : \Delta_v(\lambda) \to \nabla_v(\lambda)$. On the other hand, if we specialize to a root of unity $q \in \mathbb{C}$ (i.e. apply $- \otimes_A \mathbb{C}_q$ with $\mathbb{C}_q$ denoting $\mathbb{C}$ made into an A-module by mapping v to q) then we obtain the following two exact sequences in $\mathcal{O}_v$:

$$K_A(\lambda) \otimes_A \mathbb{C}_q \to \Delta_q(\lambda) \to L_A(\lambda) \otimes_A \mathbb{C}_q \to 0,$$
$$0 \to \mathrm{Tor}_1^A(C_A(\lambda), \mathbb{C}_q) \to L_A(\lambda) \otimes_A \mathbb{C}_q \to \nabla_q(\lambda) \to C_A(\lambda) \otimes_A \mathbb{C}_q \to 0.$$

As $L_A(\lambda) \otimes_A \mathbb{C}_q$ is a nonzero quotient of $\Delta_q(\lambda)$ it has $L_q(\lambda)$ as a quotient but it may be larger.

Proposition 6.1. *Let $\lambda, \mu \in X$. Then*

$$\dim_{\mathbb{C}} L_q(\lambda)_\mu \leq \dim_{\mathbb{Q}(v)} L_v(\lambda)_\mu$$

for all l. Equality holds if $l \gg 0$.

Proof. The inequality follows from the above considerations. They also show that we have equality if and only if $\text{Tor}_1^A(C_A(\lambda)_\mu, \mathbb{C}_q) = 0$. But $C_A(\lambda)_\mu$ is a finitely generated A-module, so this Tor vanishes for all but at most finitely many q. □

6.3. Generic multiplicities. From the above discussion we deduce the following result, comparing combinatorics of $\mathcal{O}_v$ with that of $\mathcal{O}$ (see [HK] for some other results in the same spirit).

Theorem 6.2. *Let $\lambda \in X$. Then for all $\mu \in X$,*

$$[\Delta_v(\lambda) : L_v(\mu)] = [\Delta_\mathbb{C}(\lambda) : L_\mathbb{C}(\mu)].$$

Proof. Recall that both sides are 0 unless $\mu \in W \cdot \lambda$. Let therefore $\mu \in W \cdot \lambda$. Choose l so large that we have equality in Proposition 6.1 for all these finitely many μ's. Then

$$[\Delta_v(\lambda) : L_v(\mu)] = [\Delta_q(\lambda) : L_q(\mu)].$$

By Corollary 3.6 this equals $[\tilde{\Delta}_q(\lambda) : \tilde{L}_q(\mu)]$ (for large l). But if we also assume that l is so large that both λ and μ belong to l-alcoves adjacent to $-\rho$ then we claim that the composition factor multiplicities in these (baby) Verma modules for $u_q B_q$ agree with their counterparts in $\mathcal{O}$. This is a consequence of Corollary 5.3 because when $x \in W$ the Kazhdan–Lusztig polynomials $P_{z,x}$ clearly vanish unless $z \in W$. □

Clearly, this theorem implies

Corollary 6.3. *The result in Theorem 6.2 holds not only for an indeterminate v but also for any specialization $v \mapsto q$ where $q \in \mathbb{C}$ is transcendental.*

Remark. The results of this section also follow from the fact that, after extending the scalars to $\mathbb{C}$, for any transcendental $q \in \mathbb{C} \setminus \{0\}$ the categories $\mathcal{O}_q$ and $\mathcal{O}_{\text{int}}$ are equivalent (each block of such a category can be realized as modules over an associative algebra with relations defined over $\mathbb{Q}$). As was pointed out to us by D. Kazhdan, such an equivalence was established (in the simply laced case) by M. Finkelberg in his thesis [Fi]. G. Lusztig made us aware of the paper [EK] where such an equivalence is established more generally for symmetrizable Kac–Moody algebras (see [EK, Theorem 4.2]). In our case this result gives that if v is an indeterminate then the category $\mathcal{O}_v$ is equivalent to the integral block in the category $\mathcal{O}$ for $\mathfrak{g} \otimes_\mathbb{Q} K$ where K is the fraction field of $\mathbb{Q}[[v]]$.

The methods used in both [Fi] and [EK] are completely different from our approach.

7. A parallel with Lie superalgebras

There are several similarities between general properties of $\mathcal{O}_q$ and those of the category $\mathcal{O}$ for finite-dimensional Lie superalgebras. These similarities played an important role in forming our intuition for the results of the present paper and in this section we try to make them more precise, following a suggestion of the referee and the editor. We refer the reader e.g. to [Mu] for more details on Lie superalgebras and their modules. For a Lie superalgebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ the corresponding enveloping algebra.

7.1. Super setup. Let $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ be a Lie superalgebra over $\mathbb{C}$. We assume that $\mathfrak{g}_{\overline{0}}$ is a finite-dimensional reductive Lie algebra and $\mathfrak{g}_{\overline{1}}$ is a semisimple $\mathfrak{g}_{\overline{0}}$-module. We denote by $\mathfrak{g}$-smod the abelian category of $\mathfrak{g}$-supermodules (where morphisms are homogeneous $\mathfrak{g}$-homomorphisms of degree 0). Fix some triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ with the induced triangular decomposition $\mathfrak{g}_{\overline{0}} = \mathfrak{n}_{\overline{0}}^- \oplus \mathfrak{h}_{\overline{0}} \oplus \mathfrak{n}_{\overline{0}}^+$. Our two basic examples are: the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ and the queer Lie superalgebra $\mathfrak{q}_n$.

The superalgebra $\mathfrak{gl}(m|n)$ consists of $(n + m) \times (n + m)$ matrices naturally divided into $n \times n$, $n \times m$, $m \times n$ and $m \times m$ blocks. The operation is the usual super-commutator of matrices. The diagonal blocks form the even part while the off-diagonal blocks form the odd part. The standard triangular decomposition corresponds to taking lower triangular, diagonal and upper triangular matrices. Note that the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices is purely even.

The superalgebra $\mathfrak{q}_n$ consists of $2n \times 2n$ matrices of the form

$$\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array} \right)$$

with respect to the usual super-commutator of matrices. The even part corresponds to $B = 0$ while the odd part corresponds to $A = 0$. The triangular decomposition is induced by the standard triangular decomposition for A and B (simultaneously). In this example the Cartan subalgebra $\mathfrak{h}$ has a nonzero odd component, in particular, it is not commutative.

7.2. Category $\mathcal{O}$. To avoid technicalities and complicated notation, we will describe the situation for the classical category $\mathcal{O}$ for $\mathfrak{g}$. All properties transfer mutatis mutandis to the parabolic versions of $\mathcal{O}$.

For a Lie superalgebra with triangular decomposition as above we can consider the corresponding category $\mathcal{O}$ defined as the full subcategory of $\mathfrak{g}$-smod containing all objects M with the following properties:

(i) M is finitely generated;
(ii) the action of $\mathfrak{h}_{\overline{0}}$ on M is diagonalizable;
(iii) the action of $U(\mathfrak{n}^+)$ on M is locally finite.

Let V be a simple $\mathfrak{h}_0$-diagonalizable $\mathfrak{h}$-supermodule. Set $\mathfrak{n}^+ \cdot V = 0$ and define the *Verma* or *proper standard* supermodule $\overline{\Delta}(V)$ as usual via

$$\overline{\Delta}(V) := U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}^+)} V.$$

Standard arguments (see [Di, Hu]) show that $\overline{\Delta}(V)$ has simple top, which we denote by $L(V)$. Then the map $V \mapsto L(V)$ sets up a bijection from the set isomorphism classes of simple $\mathfrak{h}_0$-diagonalizable $\mathfrak{h}$-supermodules to the set of isomorphism classes of simple objects in $\mathcal{O}$.

Let $\mathcal{O}_{\overline{0}}$ denote the category $\mathcal{O}$ for $\mathfrak{g}_{\overline{0}}$. As $U(\mathfrak{g})$ is finite over $U(\mathfrak{g}_{\overline{0}})$ by the PBW theorem, we have the usual induction functor

$$\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}} : \mathcal{O}_{\overline{0}} \to \mathcal{O}.$$

Our first observation is the following (compare with Corollary 3.5):

Proposition 7.1. (a) *The restriction functor* $\mathrm{Res}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}$ *maps* $\mathcal{O}$ *to* $\mathcal{O}_{\overline{0}}$.
(b) *Every object in* $\mathcal{O}$ *has finite length.*

Proof. From the PBW Theorem it follows that $\mathrm{Res}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}} \overline{\Delta}(V)$ is in $\mathcal{O}_{\overline{0}}$, which implies $\mathrm{Res}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}} L(V) \in \mathcal{O}_{\overline{0}}$. By adjunction, a projective cover $P(V)$ for $L(V)$ is a direct summand of some module of the form $\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}} X$, where $X \in \mathcal{O}_{\overline{0}}$.

By the PBW Theorem, the composition $\mathrm{Res}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}} \circ \mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}$ is naturally isomorphic to tensoring with the finite-dimensional $\mathfrak{g}_{\overline{0}}$-module $\bigwedge \mathfrak{g}_{\overline{1}}$. This means that $P(V)$ is of finite length already as a $\mathfrak{g}_{\overline{0}}$-module, in particular, it is of finite length as an object in $\mathcal{O}$. This implies claim (b), and claim (a) follows by induction on the length of a module. $\qquad\square$

Both $\mathrm{Res}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}$ and $\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}$ are exact by the PBW Theorem (and even biadjoint up to parity shifts, see e.g. [Go00]). In particular, the two functors take projectives to projectives and injectives to injectives. The PBW Theorem also implies that they map modules with Verma filtrations to modules with Verma filtrations.

Usual arguments imply that $\mathcal{O}$ has a block decomposition into a direct sum of indecomposable subcategories. The first principal difference with $\mathcal{O}_{\overline{0}}$ is that blocks of $\mathcal{O}$ might contain infinitely many isomorphism classes of simple objects (however, always at most countably many). Description of the linkage principle for simples in $\mathcal{O}$ is a very hard problem in full generality. However, it is known for several special cases: see e.g. [Br03] for $\mathfrak{gl}(m|n)$ or [Br04] for $\mathfrak{q}_n$.

7.3. Standardly stratified structure. Let V be a simple $\mathfrak{h}_0$-diagonalizable $\mathfrak{h}$-supermodule and $\hat{V}$ be its projective cover in the category of $\mathfrak{h}_0$-diagonalizable $\mathfrak{h}$-supermodules. Set $\mathfrak{n}^+ \cdot \hat{V} = 0$ and define the *standard* module $\Delta(V)$ as follows:

$$\Delta(V) := U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}^+)} \hat{V}.$$

From the PBW Theorem it follows that every projective in $\mathcal{O}$ has a *standard filtration*, that is, a filtration with standard subquotients. Clearly, each $\Delta(V)$ has a filtration by Verma (proper standard) supermodules. Note that $\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{\mathfrak{g}}$ maps modules with Verma filtrations to modules with standard filtrations.

Dually, one defines the costandard module $\nabla(V)$ and the proper costandard module $\overline{\nabla}(V)$. Using adjunction and the fact that $\mathcal{O}_{\overline{0}}$ is a highest weight category, we get the following ext-vanishing property:

$$\mathrm{Ext}_{\mathcal{O}}^i(\Delta(V), \overline{\nabla}(V')) \cong \begin{cases} \mathbb{C} & \text{if } V \cong V', \\ 0 & \text{otherwise.} \end{cases}$$

This implies that the associative algebra describing a block of $\mathcal{O}$ is standardly stratified in the sense of [CPS96]. In particular, we have the following BGG-reciprocity in $\mathcal{O}$:

$$(P(V) : \Delta(V')) = [\overline{\nabla}(V') : L(V)]$$

(compare with Corollary 4.2). Another consequence is that $\mathcal{O}$ has tilting modules (in the sense of [Fr07-1]). Tilting modules are modules which have both a standard filtration

and a proper costandard filtration. Both $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}$ and $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}$ map tilting modules to tilting modules. It follows that tilting modules are also cotilting.

We refer the reader to [Fr07-2] for a very detailed $\mathfrak{q}_n$-example where $\Delta(V)$ and $\overline{\Delta}(V)$ are explicitly computed and compared.

7.4. Dominance dimension and Soergel's Struktursatz. For category $\mathcal{O}$ one has a direct analogue of all our results from Subsection 4.3. We claim that every projective P in $\mathcal{O}$ admits a two-step coresolution

$$0 \to P \to X_1 \to X_2,$$

where both X_1 and X_2 are projective-injective (compare with Proposition 4.3). It is certainly enough to prove this for $P(V)$. From the above we know that the latter module occurs as a direct summand of a module induced from a projective module in $\mathfrak{g}_{\bar{0}}$. Since induction is exact and maps projectives to projectives and injectives to injectives, the claim follows from the corresponding classical claim for $\mathcal{O}_{\bar{0}}$.

Pick a representative in each isomorphism class of indecomposable projective-injective modules in $\mathcal{O}$ and let $\mathscr{C}^{\mathrm{PI}}$ be the full subcategory of $\mathcal{O}$ formed by these. Taking homomorphisms in $\mathcal{O}$ into these representatives defines a contravariant functor Φ from the additive category of projective objects in $\mathcal{O}$ into mod-$\mathscr{C}^{\mathrm{PI}}$ (see Subsection 4.3 for more details).

Theorem 7.2. *The functor Φ is fully faithful.*

Proof. Mutatis mutandis the proof of Theorem 4.4. $\square$

7.5. Irving's theorems. We also have the following natural super-analogue of Theorem 3.19.

Theorem 7.3. *Let V be a simple $\mathfrak{h}_0$-diagonalizable $\mathfrak{h}$-supermodule. Then the following assertions are equivalent:*

(a) *$P(V)$ is isomorphic to $I(V)$ up to parity shift.*
(b) *$L(V)$ occurs in the socle of a projective-injective module in $\mathcal{O}$.*
(c) *$L(V)$ occurs in the top of a projective-injective module in $\mathcal{O}$.*
(d) *$L(V)$ occurs in the socle of some $\Delta(V')$.*

Proof. This follows by adjunction from the corresponding properties of $\mathcal{O}_{\bar{0}}$. $\square$

7.6. Constructive differences. The most important difference between category $\mathcal{O}_q$ in the quantum case and category $\mathcal{O}$ in the super case is that for the latter we do not know of any analogue for our decomposition statements: Theorem 3.1, Theorem 3.15, Theorem 3.17 and Corollary 4.8. It seems that it is unreasonable to expect such analogues. To some extent the situation in the super case is opposite to the quantum case. In the quantum case we have one nice subcategory (the special block) and all other subcategories are more complicated, however, they are "more complicated in the same way".

For Lie superalgebras the situation is the opposite: generically, a Lie superalgebra "behaves" like its even part (see e.g. [Go02, FM]), that is, in an "easy" way. However, there are nongeneric degenerations of various kinds (e.g. atypical modules, nondiagonalizable $\mathfrak{h}$-supermodules, projective-injectives which are self-dual only up to a parity shift etc.) and for these degenerations the behavior is more complicated, however, in different ways. Therefore we do not think that our tensor decomposition statements have any chance to generalize to the super case.

References

[APW91]   Andersen, H. H., Polo, P., Wen, K.: Representations of quantum algebras. Invent. Math. **104**, 1–59 (1991)   Zbl 0724.17012   MR 1094046

[APW92]   Andersen, H. H., Polo, P., Wen, K.: Injective modules for quantum algebras. Amer. J. Math. **114**, 571–604 (1992)   Zbl 0755.17005   MR 1165354

[AW]   Andersen, H. H., Wen, K.: Representations of quantum algebras. The mixed case. J. Reine Angew. Math. **427**, 35–50 (1992)   Zbl 0771.17010   MR 1162431

[BB]   Beilinson, A., Bernstein, J.: Localisation de $\mathfrak{g}$-modules. C. R. Acad. Sci. Paris Sér. I Math. **292**, 15–18 (1981)   Zbl 0476.14019   MR 0610137

[BGG]   Bernstein, I., Gelfand, I., Gelfand, S.: A certain category of $\mathfrak{g}$-modules. Funktsional. Anal. i Prilozhen. **10**, no. 2, 1–8 (1976) (in Russian)   Zbl 0353.18013   MR 0407097

[Br03]   Brundan, J.: Kazhdan–Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{gl}(m|n)$. J. Amer. Math. Soc. **16**, 185–231 (2003)   Zbl 1050.17004   MR 1937204

[Br04]   Brundan, J.: Kazhdan–Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{q}(n)$. Adv. Math. **182**, 28–77 (2004)   Zbl 1048.17003   MR 2028496

[BS]   Brundan, J., Stroppel, C.: Highest weight categories arising from Khovanov's diagram algebra. II. Koszulity. Transform. Groups **15**, 1–45 (2010)   Zbl 1205.17010   MR 2600694

[BK]   Brylinski, J.-L., Kashiwara, M.: Kazhdan–Lusztig conjecture and holonomic systems. Invent. Math. **64**, 387–410 (1981)   Zbl 0473.22009   MR 0632980

[C-P]   Chari, V., Pressley, A.: A Guide to Quantum Groups. Cambridge Univ. Press (1994)   Zbl 0839.17009   MR 1300632

[CPS88]   Cline, E., Parshall, B., Scott, L.: Finite-dimensional algebras and highest weight categories. J. Reine Angew. Math. **391**, 85–99 (1988)   Zbl 0657.18005   MR 0961165

[CPS96]   Cline, E., Parshall, B., Scott, L.: Stratifying endomorphism algebras. Mem. Amer. Math. Soc. **124**, no. 591 (1996)   Zbl 0888.16006   MR 1350891

[Di]   Dixmier, J.: Enveloping Algebras. Revised reprint of the 1977 translation, Grad. Stud. Math. 11, Amer. Math. Soc., Providence, RI (1996)   Zbl 0867.17001   MR 1393197

[EK] Etingof, P., Kazhdan, D.: Quantization of Lie bialgebras. VI. Quantization of generalized Kac–Moody algebras. Transform. Groups **13**, 527–539 (2008) Zbl 1191.17004 MR 2452604

[Fi] Finkelberg, M.: Fusion categories. Ph.D Thesis, Harvard Univ., 50 pp. (1993) MR 2689590

[Fr07-1] Frisk, A.: Dlab's theorem and tilting modules for stratified algebras. J. Algebra **314**, 507–537 (2007) Zbl 1127.16009 MR 2344576

[Fr07-2] Frisk, A.: Typical blocks of the category $\mathcal{O}$ for the queer Lie superalgebra. J. Algebra Appl. **6**, 731–778 (2007) Zbl 1236.17009 MR 2355618

[FM] Frisk, A., Mazorchuk, V.: Regular strongly typical blocks of $\mathcal{O}_q$. Comm. Math. Phys. **291**, 533–542 (2009) Zbl 1269.17005 MR 2530171

[FKM] Futorny, V., König, S., Mazorchuk, V.: $\mathcal{S}$-subcategories in $\mathcal{O}$. Manuscripta Math. **102**, 487–503 (2000) Zbl 1018.17005 MR 1785327

[Ga] Gabriel, P.: Indecomposable representations. II. In: Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971), Academic Press, London, 81–104 (1973) Zbl 0276.16001 MR 0340377

[Go00] Gorelik, M.: On the ghost centre of Lie superalgebras. Ann. Inst. Fourier (Grenoble) **50**, 1745–1764 (2000) Zbl 1063.17006 MR 1817382

[Go02] Gorelik, M.: Strongly typical representations of the basic classical Lie superalgebras. J. Amer. Math. Soc. **15**, 167–184 (2002) Zbl 0985.17011 MR 2866851

[HK] Heckenberger, I., Kolb, S.: On the Bernstein–Gelfand–Gelfand resolution for Kac–Moody algebras and quantized enveloping algebras. Transform. Groups **12**, 647–655 (2007) Zbl 1138.17010 MR 2365438

[Hu] Humphreys, J. E.: Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$. Grad. Stud. Math. 94, Amer. Math. Soc., Providence, RI (2008) Zbl 1177.17001 MR 2428237

[Ir] Irving, R.: Projective modules in the category $\mathcal{O}_S$: self-duality. Trans. Amer. Math. Soc. **291**, 701–732 (1985) Zbl 0594.17005 MR 0800259

[Ja] Jantzen, J. C.: Lectures on Quantum Groups. Grad. Stud. Math. 6, Amer. Math. Soc., Providence, RI (1996) Zbl 0842.17012 MR 1359532

[KT95] Kashiwara, M., Tanisaki, T.: Kazhdan–Lusztig conjecture for affine Lie algebras with negative level. Duke Math. J. **77**, 21–62 (1995) Zbl 0829.17020 MR 1317626

[KT96] Kashiwara, M., Tanisaki, T.: Kazhdan–Lusztig conjecture for affine Lie algebras with negative level. II. Nonintegral case. Duke Math. J. **84**, 771–813 (1996) Zbl 0929.17027 MR 1408544

[KL] Kazhdan, D., Lusztig, G.: Representations of Coxeter groups and Hecke algebras. Invent. Math. **53**, 165–184 (1979) Zbl 0499.20035 MR 0560412

[KL94] Kazhdan, D., Lusztig, G.: Tensor structures arising from affine Lie algebras I–IV. J. Amer. Math. Soc. **6**, 905–947, 949–1011 and **7**, 335–381, 383–453 (1994) Zbl 0802.17008 MR 1239507

[KSX] König, S., Slungård, I., Xi, C.: Double centralizer properties, dominant dimension, and tilting modules. J. Algebra **240**, 393–412 (2001) Zbl 0980.17003 MR 1830559

[Lu80] Lusztig, G.: Hecke algebras and Jantzen's generic decomposition patterns. Adv. Math. **37**, 121–164 (1980) Zbl 0448.20039 MR 0591724

[Lu89] Lusztig, G.: Modular representations and quantum groups. In: Classical Groups and Related Topics (Beijing, 1987), Contemp. Math. 82, Amer. Math. Soc., 59–77 (1989) Zbl 0665.20022 MR 0982278

[Lu90a] Lusztig, G.: Finite dimensional Hopf algebras arising from quantized universal enveloping algebras. J. Amer. Math. Soc. **3**, 257– 296 (1990) Zbl 0695.16006 MR 1013053

[Lu90b]   Lusztig, G.: Quantum groups at roots of 1. Geom. Dedicata **35**, 89–114 (1990) Zbl 0714.17013   MR 1066560

[Lu]   Lusztig, G.: Introduction to Quantum Groups. Progr. Math. 110, Birkhäuser Boston, Boston, MA (1993)   Zbl 0788.17010   MR 1227098

[Lu94]   Lusztig, G.: Monodromic systems on affine flag manifolds, Proc. Roy. Soc. London Ser. A **445**, 231–246 (1994)   Zbl 0829.17018   MR 1276910

[Ma]   Mazorchuk, V.: Some homological properties of the category $\mathcal{O}$. II. Represent. Theory **14**, 249–263 (2010)   Zbl 1200.16013   MR 2602033

[MM]   Mazorchuk, V., Miemietz, V.: Serre functors for Lie algebras and superalgebras. Ann. Inst. Fourier (Grenoble) **62**, 47–75 (2012)   Zbl 06064511   MR 2986264

[MS]   Mazorchuk, V., Stroppel, C.: Categorification of (induced) cell modules and the rough structure of generalised Verma modules. Adv. Math. **219**, 1363–1426 (2008) Zbl 1234.17007   MR 2450613

[MOS]   Mazorchuk, V., Ovsienko, S., Stroppel, C.: Quadratic duals, Koszul dual functors, and applications. Trans. Amer. Math. Soc. **361**, 1129–1172 (2009)   Zbl 1229.16018 MR 2457393

[Mu]   Musson, I.: Lie Superalgebras and Enveloping Algebras. Grad. Stud. Math. 131, Amer. Math. Soc., Providence, RI (2012)   Zbl 1255.17001   MR 2906817

[So90]   Soergel, W.: Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. J. Amer. Math. Soc. **3**, 421–445 (1990)   Zbl 0747.17008   MR 1029692

[So97a]   Soergel, W.: Kazhdan–Lusztig-Polynome und eine Kombinatorik für Kipp-Moduln. Represent. Theory **1**, 37–68 (1997)   Zbl 0886.05124   MR 1445511

[So97b]   Soergel, W.: Charakterformeln für Kipp-Moduln über Kac–Moody-Algebren. Represent. Theory **1**, 115–132 (1997)   Zbl 0964.17019   MR 1445716

[So99]   Soergel, W.: Character formulas for tilting modules over quantum groups at roots of one. In: Current Developments in Mathematics, 1997 (Cambridge, MA), Int. Press, Boston, MA, 161–172 (1999)   Zbl 0964.17020   MR 1698856

[St03]   Stroppel, C.: Category $\mathcal{O}$: quivers and endomorphism rings of projectives. Represent. Theory **7**, 322–345 (2003)   Zbl 1050.17005   MR 2017061

[St06]   Stroppel, C.: TQFT with corners and tilting functors in the Kac–Moody case. arXiv:math/0605103.