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Minimality of toric arrangements

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Abstract. We prove that the complement of a toric arrangement has the homotopy type of a minimal CW-complex. As a corollary we deduce that the integer cohomology of these spaces is torsion-free.

We apply discrete Morse theory to the toric Salvetti complex, providing a sequence of cellular collapses that leads to a minimal complex.

Keywords. Toric arrangements, discrete Morse theory, minimal CW-complexes, torsion in cohomology

Introduction

A *toric arrangement* is a finite family

$$\mathscr{A} = \{K_1, \dots, K_n\}$$

of special subtori of the complex torus $(\mathbb{C}^*)^d$ (more precisely the K_i are level sets of characters, see §2.1). Given a complexified toric arrangement $\mathscr{A}$ (see Definition 2.6) we consider the space

$$M(\mathscr{A}) := (\mathbb{C}^*)^d \setminus \bigcup \mathscr{A}$$

and prove that

(a) the space $M(\mathscr{A})$ is *minimal* in the sense of [14], i.e., it has the homotopy type of a CW-complex with exactly $\beta_k = \operatorname{rk} H^k(M(\mathscr{A}); \mathbb{Z})$ cells in dimension k, for every $k \in \mathbb{N}$, hence

(b) the space $M(\mathscr{A})$ is *torsion-free*, that is, the homology and cohomology modules $H_k(X; \mathbb{Z})$, $H^k(X; \mathbb{Z})$ are torsion-free for every $k \in \mathbb{N}$.

The study of toric arrangements experienced a fresh impulse from recent work of De Concini, Procesi and Vergne [10, 9], in which toric arrangements emerge as a link between partition functions and box splines.

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In their book [9], De Concini and Procesi emphasize some similarities between toric arrangements and the well-established theory of arrangements of affine hyperplanes. The present work provides substantial new evidence in this sense.

Background

Combinatorics. The combinatorial framework for the theory of arrangements of hyperplanes is widely considered to be given by matroid theory, a well-established branch of combinatorics that has proved very useful in this context ever since the seminal work of Zaslavsky [33].

The combinatorial study of toric arrangements has quite recent roots, and is still in search of a full-fledged pertaining theory. From an enumerative point of view, the arithmetic Tutte polynomial introduced by Moci [23] summarizes previous results by Ehrenborg, Readdy and Slone [15] and of De Concini and Procesi [9]. This initiated the quest for a variation on the concept of matroid that would suit the 'toric' setting and lead D'Adderio and Moci [5] to suggest a theory of arithmetic matroids as a "combinatorialization" of the essential algebraic data of toric arrangements. Arithmetic matroids in fact encode—but, as yet, do not appear to characterize—some of the crucial combinatorial data of toric arrangements, for example the poset of layers (Definition 2.10). In this context, our work can be seen as an exploration of the properties that would be required from a (still lacking) notion of a 'toric oriented matroid'.

Topology. An important result in the theory of arrangements of hyperplanes was established by Brieskorn [3], who proved that the integer cohomology of the complement of an arrangement of complex hyperplanes is torsion-free. This allowed Orlik and Solomon to compute the integer cohomology algebra via the de Rham complex [25]. Minimality of complements of complex hyperplane arrangements was proven much later by Randell [27] and independently by Dimca and Papadima [14], with Morse-theoretic arguments. The explicit construction of such a minimal complex was studied by Yoshinaga [32], Salvetti and Settepanella [31] and the second author [12].

The present paper contributes to a similar circle of ideas for toric arrangements.

To our knowledge, the first result about the topology of toric arrangements was obtained by Looijenga [21] who deduced the Betti numbers of $M(\mathscr{A})$ from a spectral sequence computation. De Concini and Procesi [8] explicitly expressed the generators of the cohomology modules over $\mathbb{C}$ in terms of local no broken circuit sets and, for the special case of totally unimodular arrangements, were able to compute the cohomological algebra structure. A presentation of the fundamental group $\pi_1(M(\mathscr{A}))$ of complexified toric arrangements was computed by the authors in [6], based on a combinatorially defined polyhedral complex carrying the homotopy type of the complement $M(\mathscr{A})$, called the *toric Salvetti complex*. This polyhedral complex is given as the nerve of an acyclic category[1] and was introduced by the authors in [6], generalizing to arbitrary complexified toric arrangements the complex defined by Moci and Settepanella [24]. Recently, Davis

[1] For our use of the term 'acyclic category' see Remark 3.4.

and Settepanella [7] published vanishing results for cohomology of toric arrangements with coefficients in some particular local systems.

Outline

Here we prove minimality by exhibiting, for a given complexified toric arrangement $\mathscr{A}$, a minimal CW-complex that is homotopy equivalent to $M(\mathscr{A})$. This complex is obtained from the toric Salvetti complex after a sequence of cellular collapses indexed by a discrete Morse function. The construction of the discrete Morse function relies on a stratification of the toric Salvetti complex where strata are counted by 'local no-broken-circuit sets' (Definition 2.19), which are known to control the Poincaré polynomial of $M(\mathscr{A})$ by [8].

The (topological) boundary maps of the minimal complex can be recovered in principle from the discrete Morse data. The explicit computation of such boundary maps is in general difficult even in the case of hyperplane arrangements, where explicit computations are known only in dimension 2 either by following the discrete Morse gradient [17, 16] or by exploiting braid monodromy [18, 29, 30]. We leave the explicit computation of the boundary maps for our toric complex as a future direction of research.

As an application of our methods, in the last section we describe a construction of the minimal complex for complexified affine arrangements of hyperplanes that uses only the intrinsic combinatorics of the arrangement (i.e. its oriented matroid), as an alternative to the method of [31].

We close our introduction with a detailed outline of the paper.

- We begin with Section 1, where we review some known facts about the combinatorics and the topology of hyperplane arrangements and we prove some preparatory results about linear extensions of posets of regions of real arrangements.
- In Section 2 we give a short introduction to toric arrangements and we collect some results from the literature on which our work is built.
- Section 3 breaks the flow of material directly related to toric arrangements in order to develop discrete Morse theory for acyclic categories, generalizing the existing theory for posets.
- We approach the core of our work with Section 4, where we introduce a stratification and a related decomposition of the toric Salvetti complex (Definition 4.22).
- In order to understand the structure of the pieces of the decomposition of the toric Salvetti complex we need to patch together 'local' combinatorial data, which come from the theory of arrangements of hyperplanes. We do this in Section 5 using diagrams of acyclic categories.
- Our work culminates with Section 6. The keystone is Proposition 6.8, where we prove the existence of perfect acyclic matchings for the face categories of subdivisions of the compact torus given by toric arrangements. With this, we can apply the Patchwork Lemma of discrete Morse theory (in its version for acyclic categories) to our decomposition of the toric Salvetti complex in order to get an acyclic matching of the whole complex. This matching can be shown to be perfect and thus prescribes a series of cellular collapses leading to a minimal model for the complement of the toric arrangement.

- As a further application of our methods, in Section 7 we show that our methods can be used to construct a minimal complex for the complement of (finite) complexified arrangements of hyperplanes.

1. Arrangements of hyperplanes

The theory of hyperplane arrangements is an important ingredient in our treatment of toric arrangements. In order to set the stage for the subsequent considerations, we therefore introduce the language and recall some relevant results about hyperplane arrangements. A standard reference for a comprehensive introduction to the subject is [26].

1.1. Generalities

Throughout this section let V be a finite-dimensional vector space over a field $\mathbb{K}$.

An *affine hyperplane* H in V is the level set of a linear functional on V, that is, there are $\alpha \in V^*$ and $a \in \mathbb{K}$ such that $H = \{v \in V \mid \alpha(v) = a\}$. A set of hyperplanes is called *dependent* or *independent* according to whether the corresponding set of elements of V^* is dependent or not.

Definition 1.1. An *arrangement of hyperplanes* in V is a collection $\mathscr{A}$ of affine hyperplanes in V.

A hyperplane arrangement $\mathscr{A}$ is called *central* if every hyperplane $H \in \mathscr{A}$ is a linear subspace of V; *finite* if $\mathscr{A}$ is finite; *locally finite* if for every $p \in V$ the set $\{H \in \mathscr{A} \mid p \in H\}$ is finite; *real* (or *complex*, or *rational*) if V is a real (or complex, or rational) vector space.

When we will need to define a total order on the elements of a finite arrangement $\mathscr{A}$, we will do this by simply indexing the elements of $\mathscr{A}$, as $\mathscr{A} = \{H_1, \dots, H_n\}$.

Much of the theory of hyperplane arrangements is devoted to the study of the *complement* of an arrangement $\mathscr{A}$, that is, the space

$$M(\mathscr{A}) := V \setminus \bigcup \mathscr{A}.$$

Definition 1.2. Let $\mathscr{A}$ be a hyperplane arrangement. The *intersection poset* of $\mathscr{A}$ is the set

$$\mathcal{L}(\mathscr{A}) := \left\{ \bigcap \mathscr{K} \;\middle|\; \mathscr{K} \subseteq \mathscr{A} \right\} \setminus \{\emptyset\}$$

of all nonempty intersections of elements of $\mathscr{A}$, ordered by *reverse* inclusion, i.e., for $X, Y \in \mathcal{L}(\mathscr{A})$, $X \geq Y$ if $X \subseteq Y$.

Notice that the whole space V is an element of $\mathcal{L}(\mathscr{A})$ (corresponding to the empty intersection), whereas the empty set is not. The intersection poset is a meet-semilattice, and for central hyperplane arrangements it is a lattice. In this case we speak of the *intersection lattice* of $\mathscr{A}$.

1.1.1. Deletion and restriction. Consider a hyperplane arrangement $\mathscr{A}$ in the vector space V and an intersection $X \in \mathcal{L}(\mathscr{A})$. We associate to X two new arrangements:

$$\mathscr{A}_X := \{H \in \mathscr{A} \mid X \subseteq H\}, \qquad \mathscr{A}^X := \{H \cap X \mid H \in \mathscr{A} \setminus \mathscr{A}_X\}.$$

Notice that $\mathscr{A}_X$ is an arrangement in V, while $\mathscr{A}^X$ is an arrangement on X.

Remark 1.3. If a total ordering $\mathscr{A} = \{H_1, \ldots, H_n\}$ is defined, then it is clearly inherited by $\mathscr{A}_X$ for every $X \in \mathcal{L}(\mathscr{A})$. On the elements of $\mathscr{A}^X$ a total ordering is induced as follows. For $L \in \mathscr{A}^X$ define

$$_XL := \min\{H \in \mathscr{A} \mid L \subseteq H\}. \tag{1.1}$$

Then order $\mathscr{A}^X = \{L_1, \ldots, L_m\}$ so that $_XL_i < {_XL_j}$ in $\mathscr{A}$ for all $1 \leq i < j \leq m$.

1.1.2. No broken circuit sets. In this section let $\mathscr{A}$ be a central hyperplane arrangement and fix an arbitrary total ordering of $\mathscr{A}$.

Definition 1.4. A *circuit* is a minimal dependent subset $C \subseteq \mathscr{A}$. A *broken circuit* is a subset of the form

$$C \setminus \{\min C\} \subseteq \mathscr{A}$$

obtained from a circuit by removing its least element. A *no broken circuit set* (or, for short, an *nbc* set) is a subset $N \subseteq \mathscr{A}$ which does not contain any broken circuit.

Remark 1.5. An equivalent definition of nbc set is the following. A subset $N = \{H_{i_1}, \ldots, H_{i_k}\} \subseteq \mathscr{A}$ with $i_1 \leq \cdots \leq i_k$ is a *no broken circuit set* if it is independent and there is no $j < i_1$ such that $N \cup \{H_j\}$ is dependent.

Definition 1.6. We will write $\mathrm{nbc}(\mathscr{A})$ for the set of no broken circuit sets of $\mathscr{A}$ and $\mathrm{nbc}_k(\mathscr{A}) := \{N \in \mathrm{nbc}(\mathscr{A}) \mid |N| = k\}$ for the set of all no broken circuit sets of cardinality k.

1.2. Real arrangements

In this section we consider the case where $\mathscr{A}$ is an arrangement of hyperplanes in $\mathbb{R}^d$ in order to set up some notation and use the real structure to gain some deeper understanding of the combinatorics of no broken circuit sets.

It is not too difficult to verify that the complement $M(\mathscr{A})$ consists of several contractible connected components. These are called *chambers* of $\mathscr{A}$. We write $\mathcal{T}(\mathscr{A})$ for the set of all chambers of $\mathscr{A}$.

Definition 1.7. Let $\mathscr{A}$ be a real arrangement. The set of *faces* of $\mathscr{A}$ is

$$\mathcal{F}(\mathscr{A}) := \{\mathrm{relint}(\overline{C} \cap X) \mid C \in \mathcal{T}(\mathscr{A}),\ X \in \mathcal{L}(\mathscr{A})\}.$$

We partially order this set by setting $F \leq G$ if $F \subseteq \overline{G}$, and then call $\mathcal{F}(\mathscr{A})$ the *face poset* of $\mathscr{A}$.

Remark 1.8. A face $F \in \mathcal{F}(\mathscr{A})$ is an open subset of $\bigcap\{H \in \mathscr{A} \mid F \subseteq H\}$. We denote by $\overline{F}$ the topological closure of F in $\mathbb{R}^d$.

Remark 1.9. Given $F \in \mathcal{F}(\mathscr{A})$ define the subarrangement $\mathscr{A}_F := \{H \in \mathscr{A} \mid F \subseteq H\}$. We have a natural poset isomorphism $\mathcal{F}(\mathscr{A}_F) \cong \mathcal{F}(\mathscr{A})_{\geq F}$. Therefore, in the following we will identify these two posets.

One of the main enumerative questions about arrangements of hyperplanes in real space asks for the number of chambers of a given hyperplane arrangement. The answer is very elegant and somehow surprising.

Theorem 1.10 (Zaslavsky [33]). *Given a real hyperplane arrangement* $\mathscr{A}$*,*

$$|\mathcal{T}(\mathscr{A})| = |\mathrm{nbc}(\mathscr{A})|.$$

1.2.1. Taking sides. If $\mathscr{A}$ is an arrangement in a real space V, then every hyperplane H is the locus where a linear form $\alpha_H \in V^*$ takes the value a_H. This way we can associate to each $H \in \mathscr{A}$ its *positive* and *negative halfspace*:

$$H^+ := \{x \in V \mid \alpha_H(x) > a_H\}, \qquad H^- := \{x \in V \mid \alpha_H(x) < a_H\}.$$

Definition 1.11. Consider a complexified locally finite arrangement $\mathscr{A}$ with any choice of 'sides' H^+ and H^- for every $H \in \mathscr{A}$. The *sign vector* of a face $F \in \mathcal{F}(\mathscr{A})$ is the function $\gamma_F : \mathscr{A} \to \{-, 0, +\}$ defined as

$$\gamma_F(H) := \begin{cases} + & \text{if } F \subseteq H^+, \\ 0 & \text{if } F \subseteq H, \\ - & \text{if } F \subseteq H^-. \end{cases}$$

When we need to specify the arrangement $\mathscr{A}$ to which the sign vector refers, we will write $\gamma[\mathscr{A}]_F(H)$ for $\gamma_F(H)$.

Remark 1.12. The poset $\mathcal{F}(\mathscr{A})$ is isomorphic to the set $\{\gamma_F \mid F \in \mathcal{F}(\mathscr{A})\}$ with partial order given by $\gamma_F \leq \gamma_G$ if $\gamma_F(H) = \gamma_G(H)$ whenever $\gamma_G(H) \neq 0$ (see e.g. [2]).

Definition 1.13. Let $C_1, C_2 \in \mathcal{T}(\mathscr{A})$ be chambers of a real arrangement, and let $B \in \mathcal{T}(\mathscr{A})$ be a distinguished chamber. We will write

$$S(C_1, C_2) := \{H \in \mathscr{A} \mid \gamma_{C_1}(H) \neq \gamma_{C_2}(H)\}$$

for the set of hyperplanes of $\mathscr{A}$ which separate C_1 and C_2. For all $C_1, C_2 \in \mathcal{T}(\mathscr{A})$ write

$$C_1 \leq C_2 \;\Leftrightarrow\; S(C_1, B) \subseteq S(C_2, B).$$

This turns $\mathcal{T}(\mathscr{A})$ into a poset $\mathcal{T}(\mathscr{A})_B$, the *poset of regions* of the arrangement $\mathscr{A}$ with base chamber B.

Remark 1.14. Let $\mathscr{A}_0$ be a real arrangement and $B \in \mathcal{T}(\mathscr{A}_0)$. Given a subarrangement $\mathscr{A}_1 \subseteq \mathscr{A}_0$, for every chamber $C \in \mathcal{T}(\mathscr{A}_0)$ there is a unique chamber $\widehat{C} \in \mathcal{T}(\mathscr{A}_1)$ with $C \subseteq \widehat{C}$. The correspondence $C \mapsto \widehat{C}$ defines a surjective map

$$\sigma_{\mathscr{A}_1} : \mathcal{T}(\mathscr{A}_0)_B \to \mathcal{T}(\mathscr{A}_1)_{\widehat{B}}$$

such that $C \leq C'$ implies $\sigma_{\mathscr{A}_1}(C) \leq \sigma_{\mathscr{A}_1}(C')$ for all $C, C' \in \mathcal{T}(\mathscr{A}_0)$.

Definition 1.15. Let $\mathscr{A}_0$ be a real arrangement and let $\succ_0$ denote any total ordering of $\mathcal{T}(\mathscr{A}_0)$. Consider a subarrangement $\mathscr{A}_1 \subseteq \mathscr{A}_0$. The section

$$\mu[\mathscr{A}_1, \mathscr{A}_0] : \mathcal{T}(\mathscr{A}_1) \to \mathcal{T}(\mathscr{A}_0), \quad C \mapsto \min_{\succ_0}\{K \in \mathcal{T}(\mathscr{A}_0) \mid K \subseteq C\},$$

of $\sigma_{\mathscr{A}_1}$ defines a total ordering $\succ_{0,1}$ on $\mathcal{T}(\mathscr{A}_1)$ by

$$C \succ_{0,1} D \;\Leftrightarrow\; \mu[\mathscr{A}_1, \mathscr{A}_0](C) \succ_0 \mu[\mathscr{A}_1, \mathscr{A}_0](D)$$

that we call *induced by* $\succ_0$.

Lemma 1.16. *Consider real arrangements $\mathscr{A}_2 \subseteq \mathscr{A}_1 \subseteq \mathscr{A}_0$, a given total ordering $\succ_0$ of $\mathcal{T}(\mathscr{A}_0)$ and the induced total ordering $\succ_{0,1}$ of $\mathcal{T}(\mathscr{A}_1)$. Then*

$$\mu[\mathscr{A}_1, \mathscr{A}_0] \circ \mu[\mathscr{A}_2, \mathscr{A}_1] = \mu[\mathscr{A}_2, \mathscr{A}_0].$$

Proof. Take any $C \in \mathcal{T}(\mathscr{A}_2)$ and define

$$\begin{aligned} &C_0 := \mu[\mathscr{A}_2, \mathscr{A}_0](C), \quad C_1 := \sigma_{\mathscr{A}_1}(C_0), \text{ so } \mu[\mathscr{A}_1, \mathscr{A}_0](C_1) = C_0,\\ &C_2 := \mu[\mathscr{A}_2, \mathscr{A}_1](C), \quad C_3 := \mu[\mathscr{A}_1, \mathscr{A}_0](C_2). \end{aligned}$$

We have to show that $C_0 = C_3$. First, notice that $C_0 \preceq_0 C_3$ because $C_3 \subseteq C_2 \subseteq C$. For the reverse inequality notice that $C_1, C_2 \subseteq C$, which implies $C_2 \preceq_{0,1} C_1$ and so, by definition of the induced ordering, $C_3 = \mu[\mathscr{A}_1, \mathscr{A}_0](C_2) \preceq_0 \mu[\mathscr{A}_1, \mathscr{A}_0](C_1) = C_0$. □

Proposition 1.17. *Let a base chamber B of $\mathscr{A}_0$ be chosen. If $\succ_0$ is a linear extension of $\mathcal{T}(\mathscr{A}_0)_B$, then $\succ_{0,1}$ is a linear extension of $\mathcal{T}(\mathscr{A}_1)_{\widehat{B}}$.*

Proof. We have to prove that for all $C, D \in \mathcal{T}(\mathscr{A}_1)$, $C \leq D$ in $\mathcal{T}(\mathscr{A}_1)_{\widehat{B}}$ implies $C \preceq_{0,1} D$, i.e., $\mu[\mathscr{A}_0, \mathscr{A}_1](C) \preceq_0 \mu[\mathscr{A}_0, \mathscr{A}_1](D)$.

We argue by induction on $k := |\mathscr{A}_0 \setminus \mathscr{A}_1|$, the claim being evident when $k = 0$. Suppose then that $k > 0$, choose $H \in \mathscr{A}_0 \setminus \mathscr{A}_1$ and set $\mathscr{A}_0' := \mathscr{A}_0 \setminus \{H\}$. By induction hypothesis we have

$$\mu[\mathscr{A}_0', \mathscr{A}_1](C) \preceq_0' \mu[\mathscr{A}_0', \mathscr{A}_1](D),$$

which by definition means

$$\mu[\mathscr{A}_0, \mathscr{A}_0'](\mu[\mathscr{A}_0', \mathscr{A}_1](C)) \preceq_0 \mu[\mathscr{A}_0, \mathscr{A}_0'](\mu[\mathscr{A}_0', \mathscr{A}_1](D))$$

and thus, via Lemma 1.16, $\mu[\mathscr{A}_0, \mathscr{A}_1](C) \preceq_0 \mu[\mathscr{A}_0, \mathscr{A}_1](D)$. □

1.3. Complex(ified) arrangements

We turn to the case of complex hyperplane arrangements, where the space $M(\mathscr{A})$ has subtler topology. For the sake of concision here we deliberately disregard the chronological order in which the relevant theorems were proved, and start with the minimality result.

Definition 1.18. Let X be a topological space. For $j \geq 0$, the *j-th Betti number* is $\beta_j(X) := \operatorname{rk} H^j(M(\mathscr{A}); \mathbb{Z})$. The space X is called *minimal* if it is homotopy equivalent to a CW-complex with $\beta_j(X)$ cells of dimension j, for all $j \geq 0$. Such a CW-complex is also called minimal.

Theorem 1.19 (Randell [27], Dimca and Papadima [14]). *The space $M(\mathscr{A})$ is minimal.*

Corollary 1.20. *The cohomology groups $H^k(M(\mathscr{A}); \mathbb{Z})$ are torsion-free.*

Proof. Theorem 1.19 asserts the existence of a minimal complex for $M(\mathscr{A})$. The (algebraic) boundary maps of the chain complex constructed from this minimal complex are all zero, thus torsion cannot arise in homology. □

Corollary 1.20 can be traced back to the seminal work of Brieskorn [3], where also the following other important fact about the cohomology of affine arrangements of hyperplanes was proved.

Theorem 1.21 (Brieskorn [3]). *Let $\mathscr{A}$ be a finite affine hyperplane arrangement. Then, for every $p \in \mathbb{N}$,*

$$H^p(M(\mathscr{A}); \mathbb{Z}) \cong \bigoplus_{X \in \mathcal{L}(\mathscr{A})_p} H^p(M(\mathscr{A}_X); \mathbb{Z}),$$

where $\mathcal{L}(\mathscr{A})_p := \{X \in \mathcal{L}(\mathscr{A}) \mid \operatorname{codim} X = p\}$.

Intimitely related to this torsion-freeness is the fact that it is enough to compute de Rham cohomology in order to know the cohomology with integer coefficients, the so-called *Orlik–Solomon algebra* introduced in [25]. Here, too, no broken circuit sets enter the picture as most handy combinatorial invariants.

Theorem 1.22. *Let $\mathscr{A}$ be a complex central hyperplane arrangement. Then the Poincaré polynomial of $M(\mathscr{A})$ satisfies*

$$P_{\mathscr{A}}(t) := \sum_{j \geq 0} \operatorname{rk} H^j(M(\mathscr{A}); \mathbb{Z})\, t^j = \sum_{j \geq 0} |\mathrm{nbc}_j(\mathscr{A})|\, t^j.$$

Remark 1.23. In particular, the numbers $|\mathrm{nbc}_k(\mathscr{A})|$ do not depend on the chosen ordering of $\mathscr{A}$.

Remark 1.24 ([19]). Combining Theorem 1.21 with Theorem 1.22 we get the following formula for the Poincaré polynomial of the complement of an arbitrary finite affine complex arrangement:

$$P_{\mathscr{A}}(t) = \sum_{X \in \mathcal{L}(\mathscr{A})} |\mathrm{nbc}_{\operatorname{codim} X}(\mathscr{A}_X)|\, t^{\operatorname{codim} X}.$$

We now turn to a special class of arrangements in complex space.

Definition 1.25. An arrangement $\mathscr{A}$ in $\mathbb{C}^d$ is called *complexified* if every hyperplane $H \in \mathscr{A}$ is the complexification of a real hyperplane, i.e. there are $\alpha_H \in (\mathbb{R}^d)^*$ and $a_H \in \mathbb{R}$ with

$$H = \{x \in \mathbb{C}^d \mid \alpha_H(\Re(x)) + i\alpha_H(\Im(x)) = a_H\}.$$

Let $\mathscr{A}$ be a complexified arrangement and consider its real part

$$\mathscr{A}_{\mathbb{R}} := \{H \cap \mathbb{R}^d \mid H \in \mathscr{A}\},$$

an arrangement of hyperplanes in $\mathbb{R}^d$. Notice that $\mathcal{L}(\mathscr{A}) \cong \mathcal{L}(\mathscr{A}_{\mathbb{R}})$ and therefore $\mathrm{nbc}(\mathscr{A}) = \mathrm{nbc}(\mathscr{A}_{\mathbb{R}})$.

If $\mathscr{A}$ is a complexified arrangement, one can use the combinatorial structure of $\mathscr{A}_{\mathbb{R}}$ to study the topology of $M(\mathscr{A})$. Therefore we will write $\mathcal{F}(\mathscr{A}) = \mathcal{F}(\mathscr{A}_{\mathbb{R}})$ and $\mathcal{T}(\mathscr{A}) = \mathcal{T}(\mathscr{A}_{\mathbb{R}})$.

1.3.1. The homotopy type of complexified arrangements. Using combinatorial data about $\mathscr{A}_{\mathbb{R}}$, Salvetti [28] defined a cell complex which embeds in the complement $M(\mathscr{A})$ as a deformation retract. We explain Salvetti's construction.

Definition 1.26. Given a face $F \in \mathcal{F}(\mathscr{A})$ and a chamber $C \in \mathcal{T}(\mathscr{A})$, define $C_F \in \mathcal{T}(\mathscr{A})$ as the unique chamber such that, for $H \in \mathscr{A}$,

$$\gamma_{C_F}(H) = \begin{cases} \gamma_F(H) & \text{if } \gamma_F(H) \neq 0, \\ \gamma_C(H) & \text{if } \gamma_F(H) = 0. \end{cases}$$

The reader may think of C_F as the one among the chambers adjacent to F that "faces" C.

Definition 1.27. Consider an affine complexified locally finite arrangement $\mathscr{A}$ and define the *Salvetti poset* as follows:

$$\mathrm{Sal}(\mathscr{A}) := \{[F, C] \mid F \in \mathcal{F}(\mathscr{A}),\ C \in \mathcal{T}(\mathscr{A}),\ F \leq C\},$$

with the order relation

$$[F_1, C_1] \leq [F_2, C_2] \ \Leftrightarrow\ F_2 \leq F_1 \text{ and } (C_2)_{F_1} = C_1.$$

Definition 1.28. Let $\mathscr{A}$ be an affine complexified locally finite hyperplane arrangement. Its *Salvetti complex* is $\mathcal{S}(\mathscr{A}) = \Delta(\mathrm{Sal}(\mathscr{A}))$.

Theorem 1.29 (Salvetti [28]). *The complex $\mathcal{S}(\mathscr{A})$ is homotopically equivalent to $M(\mathscr{A})$. More precisely $\mathcal{S}(\mathscr{A})$ embeds in $M(\mathscr{A})$ as a deformation retract.*

Remark 1.30. In fact, $\mathrm{Sal}(\mathscr{A})$ is the face poset of a regular cell complex (of which $\mathcal{S}(\mathscr{A})$ is the barycentric subdivision) whose maximal cells correspond to the pairs

$$\{[P, C] \mid P \in \min \mathcal{F}(\mathscr{A}),\ C \in \mathcal{T}(\mathscr{A})\}.$$

It is this complex that Salvetti describes in [28]. When we need to distinguish between the two complexes we will speak of the *cellular* and *simplicial Salvetti complex*.

1.3.2. Minimality. In the case of complexified arrangements, explicit constructions of a minimal CW-complex for $M(\mathscr{A})$ were given in [31] and [12]. We review the material of [12, §4] that will be useful for our later purposes.

Lemma 1.31 ([12, Theorem 4.13]). *Let $\mathscr{A}$ be a central arrangement of real hyperplanes, let $B \in \mathcal{T}(\mathscr{A})$ and let $\preceq$ be any linear extension of the poset $\mathcal{T}(\mathscr{A})_B$. The subset of $\mathcal{L}(\mathscr{A})$ given by all intersections X such that*

$$S(C, C') \cap \mathscr{A}_X \neq \emptyset \quad \text{for all } C' \prec C$$

is an order ideal of $\mathcal{L}(\mathscr{A})$. In particular, it has a well defined and unique minimal element which we will call X_C.

Remark 1.32. Note that X_C depends on the choice of B and of the linear extension of $\mathcal{T}(\mathscr{A})_B$.

Corollary 1.33. *For all $C \in \mathcal{T}(\mathscr{A})$ we have*

$$C = \min_{\preceq}\{K \in \mathcal{T}(\mathscr{A}) \mid K_{X_C} = C_{X_C}\},$$

where, for $Y \in \mathcal{L}(\mathscr{A})$ and $K \in \mathcal{T}(\mathscr{A})$, we define $K_Y := \sigma_{\mathscr{A}_Y}(K)$.

Now recall the (cellular) Salvetti complex of Definition 1.28 and Remark 1.30. In particular, its maximal cells correspond to the pairs $[P, C]$ where P is a point and C is a chamber. When $\mathscr{A}$ is a central arrangement, the maximal cells correspond to the chambers in $\mathcal{T}(\mathscr{A})$. In this case we can stratify the Salvetti complex assigning to each chamber $C \in \mathcal{T}(\mathscr{A})$ the corresponding maximal cell of $\mathcal{S}(\mathscr{A})$, together with its faces. Let us make this precise.

Definition 1.34. Let $\mathscr{A}$ be a central complexified hyperplane arrangement and write $\min \mathcal{F}(\mathscr{A}) = \{P\}$. Define a stratification of the cellular Salvetti complex $\mathcal{S}(\mathscr{A}) = \bigcup_{C \in \mathcal{T}(\mathscr{A})} \mathcal{S}_C$ through

$$\mathcal{S}_C := \bigcup \{[F, K] \in \mathrm{Sal}(\mathscr{A}) \mid [F, K] \leq [P, C]\}.$$

Given an arbitrary linear extension $(\mathcal{T}(\mathscr{A}), \preceq)$ of $\mathcal{T}(\mathscr{A})_B$, for all $C \in \mathcal{T}(\mathscr{A})$ define

$$\mathcal{N}_C := \mathcal{S}_C \setminus \bigcup_{D \prec C} \mathcal{S}_D.$$

In particular the poset $\mathrm{Sal}(\mathscr{A})$ can be partitioned as

$$\mathrm{Sal}(\mathscr{A}) = \bigsqcup_{C \in \mathcal{T}(\mathscr{A})} \mathcal{N}_C(\mathscr{A}).$$

Theorem 1.35 ([12, Lemma 4.18]). *There is an isomorphism of posets*

$$\mathcal{N}_C \cong \mathcal{F}(\mathscr{A}^{X_C})^{\mathrm{op}},$$

where X_C is the intersection defined via Lemma 1.31 by the same choice of base chamber and of linear extension of $\mathcal{T}(\mathscr{A})_B$ used to define the subposets $\mathcal{N}_C$.

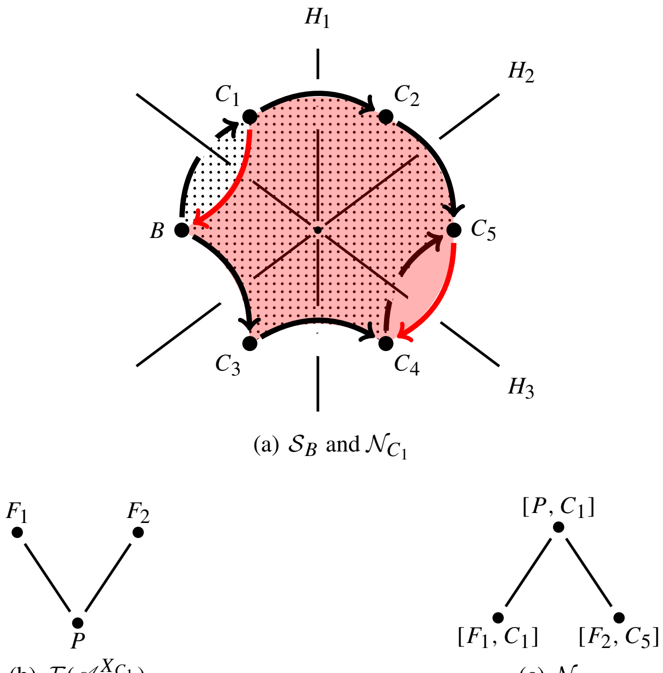
Remark 1.36. The alternative proof given in [12] of minimality of $M(\mathscr{A})$ for $\mathscr{A}$ a complexified central arrangement follows from Theorem 1.35 by an application of discrete Morse theory (see Section 3). Indeed, from a shelling order of $\mathcal{F}(\mathscr{A}^{X_C})$ one can construct a sequence of cellular collapses of the induced subcomplex of $\mathcal{S}_C$ that leaves only one 'surviving' cell. Via the Patchwork Lemma (Lemma 3.7 below) these sequences of collapses can be concatenated to give a sequence of collapses on the cell complex $\mathcal{S}(\mathscr{A})$. The resulting complex after the collapses has one cell for every $\mathcal{N}_C$, namely $|\mathrm{nbc}(\mathcal{A})| = P_{\mathscr{A}}(1)$ cells, and is thus minimal.

Example 1.37. Consider the arrangement of Figure 1. We have

$$\mathcal{L}(\mathscr{A}) = \{\mathbb{R}^2, H_1, H_2, H_3, P\}$$

where $P = H_1 \cap H_2 \cap H_3$. The chambers are ordered according to their indices, B being the base chamber. Then $X_B = \mathbb{R}^2$, $X_{C_1} = H_3$, $X_{C_2} = H_1$, $X_{C_3} = H_2$, $X_{C_4} = X_{C_5} = P$.

Recall the construction of the *cellular* Salvetti complex (e.g. from [6, Definition 2.4]). Figure 1(a) shows the stratum $\mathcal{S}_B = \mathcal{N}_B$ (dotted shading) and the stratum $\mathcal{N}_{C_1}$ (solid shading). The stratum $\mathcal{N}_{C_1}$ consists of two 1-dimensional faces and one 2-dimensional face. Its poset structure is showed in Figure 1(c) and it is isomorphic, as a poset, to the order dual of $\mathcal{F}(\mathscr{A}^{X_{C_1}})$, depicted in Figure 1(b).



(a) $\mathcal{S}_B$ and $\mathcal{N}_{C_1}$

(b) $\mathcal{F}(\mathscr{A}^{X_{C_1}})$

(c) $\mathcal{N}_{C_1}$

Fig. 1. Example of stratification.

2. Toric arrangements

2.1. Introduction

We have presented arrangements of hyperplanes in affine space as families of level sets of linear forms. Now, we want to explain in which sense this idea generalizes to a toric setting.

Our ambient spaces will be the *complex torus* $(\mathbb{C}^*)^d$ and the *compact* (or *real*) *torus* $(S^1)^d$, where we consider S^1 as the unit circle in $\mathbb{C}$. We consider *characters* of the torus, i.e., maps $\chi : (\mathbb{C}^*)^d \to \mathbb{C}^*$ given by

$$\chi(x_1, \ldots, x_d) = x_1^{\alpha_1} \cdots x_d^{\alpha_d} \qquad \text{for an } \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d.$$

The characters form a lattice, which we denote by Λ, under pointwise multiplication. Notice that the assignment $\alpha \mapsto x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ provides an isomorphism $\mathbb{Z}^d \to \Lambda$.

We consider subtori defined as level sets of characters, that is, hypersurfaces in $(\mathbb{C}^*)^d$ of the form

$$K = \{x \in (\mathbb{C}^*)^d \mid \chi(x) = a\} \quad \text{with } \chi \in \Lambda, a \in \mathbb{C}^*. \tag{2.1}$$

Notice that, if $a \in S^1$, the intersection $K \cap (S^1)^d$ is also a level set of a character (described by the same equation).

Definition 2.1. A (*complex*) *toric arrangement* $\mathscr{A}$ in $(\mathbb{C}^*)^d$ is a finite set

$$\mathscr{A} = \{K_1, \ldots, K_n\}$$

of hypersufaces of the form (2.1) in $(\mathbb{C}^*)^d$.

Definition 2.2. Let $\mathscr{A}$ be a toric arrangement in $(\mathbb{C}^*)^d$. Its *complement* is

$$M(\mathscr{A}) := (\mathbb{C}^*)^d \setminus \bigcup \mathscr{A}.$$

Definition 2.3. A *real toric arrangement* $\mathscr{A}$ in $(S^1)^d$ is a finite set

$$\mathscr{A}^c = \{K_1^c, \ldots, K_n^c\}$$

of hypersufaces K_i^c in $(S^1)^d$ of the form (2.1) with $a \in S^1$. If a complex toric arrangement restricts to a real toric arrangement on $(S^1)^d$ we will call $\mathscr{A}$ *complexified.*

We will often use this interplay between the complex and the 'real' hypersurfaces in the same vein that one exploits properties of the real part of complexified arrangements to gain insight into the complexification.

2.2. An abstract approach

We now introduce an equivalent but more abstract approach to toric arrangements. Being able to switch the point of view according to the situation will make our considerations below considerably more transparent.

Definition 2.4. Let $\Lambda \cong \mathbb{Z}^d$ be a finite rank lattice. The corresponding *complex torus* is

$$T_\Lambda := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}^*).$$

The *compact* (or *real*) *torus* corresponding to Λ is

$$T_\Lambda^c := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, S^1),$$

where, again, $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$.

The choice of a basis $\{u_1, \dots, u_d\}$ of Λ gives isomorphisms

$$\begin{aligned} \Phi : T_\Lambda &\to (\mathbb{C}^*)^d, & \Phi^c : T_\Lambda^c &\to (S^1)^d, \\ \varphi &\mapsto (\varphi(u_1), \dots, \varphi(u_d)), & \varphi &\mapsto (\varphi(u_1), \dots, \varphi(u_d)). \end{aligned} \tag{2.2}$$

Remark 2.5. Consider a finite rank lattice Λ and the corresponding torus T_Λ. The *characters* of T_Λ are the functions

$$\chi_\lambda : T_\Lambda \to \mathbb{C}^*, \qquad \chi_\lambda(\varphi) = \varphi(\lambda) \qquad \text{with } \lambda \in \Lambda.$$

The characters form a lattice under pointwise multiplication, and this lattice is naturally isomorphic to Λ. Therefore in the following we will identify the character lattice of T_Λ with Λ.

Now, the 'abstract' definition of toric arrangements is the following.

Definition 2.6. Consider a finite rank lattice Λ. A *toric arrangement* in T_Λ is a finite set of pairs

$$\mathscr{A} = \{(\chi_1, a_1), \dots, (\chi_n, a_n)\} \subset \Lambda \times \mathbb{C}^*.$$

A toric arrangement $\mathscr{A}$ is called *complexified* if $\mathscr{A} \subset \Lambda \times S^1$.

Remark 2.7. The abstract definition is clearly equivalent to Definition 2.1 via the isomorphisms in (2.2) and by

$$K_i := \{x \in T_\Lambda \mid \chi_i(x) = a_i\}. \tag{2.3}$$

Accordingly, we have $M(\mathscr{A}) := T_\Lambda \setminus \bigcup\{K_1, \dots, K_n\}$.

Definition 2.8. Let Λ be a finite rank lattice. A *real* toric arrangement in T_Λ^c is a finite set of pairs

$$\mathscr{A}^c = \{(\chi_1, a_1), \dots, (\chi_n, a_n)\} \subset \Lambda \times S^1.$$

Remark 2.9. A complexified toric arrangement $\mathscr{A}$ in T_Λ induces a real toric arrangement $\mathscr{A}^c$ in the sense of Definition 2.3 via (2.2) and

$$K_i^c := \{x \in T_\Lambda^c \mid \chi_i(x) = a_i\}.$$

Furthermore, embedding $T_\Lambda^c \hookrightarrow T_\Lambda$ in the obvious way, we have $K_i^c = K_i \cap T_\Lambda^c$ and thus a complexified toric arrangement as in Definition 2.3.

We now illustrate what has been proposed [8, 22] as the 'toric analogue' of the intersection poset (see Definition 1.2).

Definition 2.10. Let $\mathscr{A} = \{(\chi_1, a_1), \dots, (\chi_n, a_n)\}$ be a toric arrangement on T_Λ. A *layer* of $\mathscr{A}$ is a connected component of a nonempty intersection of some of the subtori K_i (defined in Remark 2.7). The set of all layers of $\mathscr{A}$ ordered by reverse inclusion is the *poset of layers* of the toric arrangement, denoted by $\mathcal{C}(\mathscr{A})$.

Notice that, as in the case of hyperplane arrangements, the torus T_Λ itself is a layer, while the empty set is not.

Definition 2.11. Let Λ be a rank d lattice and let $\mathscr{A}$ be a toric arrangement on T_Λ. The *rank* of $\mathscr{A}$ is $\operatorname{rk}(\mathscr{A}) := \operatorname{rk}\langle \chi \mid (\chi, a) \in \mathscr{A}\rangle$.

(a) A character $\chi \in \Lambda$ is called *primitive* if, for all $\psi \in \Lambda$, $\chi = \psi^k$ only if $k \in \{-1, 1\}$.
(b) The toric arrangement $\mathscr{A}$ is called *primitive* if for each $(\chi, a) \in \mathscr{A}$, χ is primitive.
(c) The toric arrangement $\mathscr{A}$ is called *essential* if $\operatorname{rk}(\mathscr{A}) = d$.

Remark 2.12. For every nonprimitive arrangement there is a primitive arrangement which has the same complement. Furthermore, if $\mathscr{A}$ is a nonessential arrangement, then there is an essential arrangement $\mathscr{A}'$ such that

$$M(\mathscr{A}) \cong (\mathbb{C}^*)^{d-l} \times M(\mathscr{A}') \quad \text{where} \quad l = \operatorname{rk}(\mathscr{A}').$$

Therefore the topology of $M(\mathscr{A})$ can be derived from the topology of $M(\mathscr{A}')$.

In view of Remark 2.12, our study of the topology of complements of toric arrangements will not loose in generality by stipulating the next assumption.

Assumption 2.13. *From now on we assume that every toric arrangement is* primitive *and* essential.

2.2.1. Deletion and restriction. Let Λ be a finite rank lattice and $\mathscr{A}$ be a toric arrangement in T_Λ.

Definition 2.14. For every sublattice $\Gamma \subseteq \Lambda$ we define the arrangement

$$\mathscr{A}_\Gamma := \{(\chi, a) \mid \chi \in \Gamma\},$$

and for every layer $X \in \mathcal{C}(\mathscr{A})$ a sublattice

$$\Gamma_X := \{\chi \in \Lambda \mid \chi \text{ is constant on } X\} \subseteq \Lambda.$$

Definition 2.15. Let X be a layer of $\mathscr{A}$. We define toric arrangements

$$\mathscr{A}_X := \mathscr{A}_{\Gamma_X} \quad \text{on } T_{\Gamma_X},$$

and

$$\mathscr{A}^X := \{K_i \cap X \mid X \not\subseteq K_i\} \quad \text{on the torus } X.$$

Remark 2.16. Notice that for a layer $X \in \mathcal{C}(\mathscr{A})$ and a hypersurface K of $\mathscr{A}$, the intersection $K \cap X$ need not be connected.

In general $K \cap X$ consists of several connected components, each of which is a level set of a character in the torus X. In particular $\mathscr{A}^X$ is a toric arrangement in the sense of Definition 2.6.

2.2.2. Covering space. We now recall a construction of [6] which we need in the following. For more details we refer to [6, §3.2]. Consider the covering map

$$p : \mathbb{C}^d \cong \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}) \to \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}^*) = T_\Lambda, \qquad \varphi \mapsto \exp \circ \varphi. \tag{2.4}$$

Notice that by identifying $\operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}) \cong \mathbb{C}^d$, p becomes the universal covering map

$$(t_1, \ldots, t_d) \mapsto (e^{2\pi i t_1}, \ldots, e^{2\pi i t_d})$$

of the torus T_Λ. Also, this map restricts to a universal covering map

$$\mathbb{R}^d \cong \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{R}) \to \operatorname{Hom}_{\mathbb{Z}}(\Lambda, S^1) \cong (S^1)^d.$$

Consider now a toric arrangement $\mathscr{A}$ on T_Λ. Its preimage under p is a locally finite affine hyperplane arrangement on $\operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C})$,

$$\mathscr{A}^{\upharpoonright} := \{(\chi, a') \in \Lambda \times \mathbb{C} \mid (\chi, e^{2\pi i a'}) \in \mathscr{A}\}.$$

If we write it in coordinates, $\mathscr{A}^{\upharpoonright}$ becomes the arrangement on $\mathbb{C}^d$ defined as

$$\mathscr{A}^{\upharpoonright} = \{H_{\chi,a'} \mid (\chi, e^{2\pi i a'}) \in \mathscr{A}\} \quad \text{with} \quad H_{\chi,a'} = \left\{x \in \mathbb{C}^n \;\middle|\; \sum \alpha_i x_i = a'\right\},$$

where we expanded $\chi(x) = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$.

Remark 2.17. If the toric arrangement $\mathscr{A}$ is complexified, so is the hyperplane arrangement $\mathscr{A}^{\upharpoonright}$.

2.3. *Combinatorics*

As in the case of hyperplanes, one would like to describe the topology of the complement in terms of the combinatorics of the arrangement.

Lemma 2.18. *Let $\mathscr{A}$ be a toric arrangement, and $X \in \mathcal{C}(\mathscr{A})$ a layer. Then the subposet $\mathcal{C}(\mathscr{A})_{\leq X}$ is the intersection poset of a central hyperplane arrangement $\mathscr{A}[X]$. If $\mathscr{A}$ is complexified, then so is $\mathscr{A}[X]$.*

Proof. This is implicit in much of [8, 22]; the proof follows by lifting the layer X to $\mathscr{A}^{\upharpoonright}$. A formally precise definition of $\mathscr{A}[Y]$ can also be found in Section 4.1 below. □

In other words, lower intervals of posets of layers are intersection lattices of (central) hyperplane arrangements. The following definition is then natural.

Definition 2.19 ([8, 22]). Let $\mathscr{A}$ be a toric arrangement of rank d and fix a total ordering on $\mathscr{A}$. A *local no broken circuit set* of $\mathscr{A}$ is a pair

$$(X, N) \text{ with } X \in \mathcal{C}(\mathscr{A}),\ N \in \mathrm{nbc}_k(\mathscr{A}(X)) \text{ where } k = d - \dim X.$$

We will write $\mathscr{N}$ for the set of local no broken circuits, and partition it into subsets

$$\mathscr{N}_j := \{(X, N) \in \mathscr{N} \mid \dim X = d - j\}.$$

Remark 2.20. Let $X \in \mathcal{C}(\mathscr{A})$ and $N \subseteq \mathscr{A}(X)$. If we consider the 'list' $\mathscr{X}$ of all pairs (χ_i, a_i) with $\chi_{i|X} \equiv a_i$, then the elements of N index a 'sublist' $\mathscr{X}_N$. Then (X, N) is a local no broken circuit set if and only if $\mathscr{X}_N$ is a basis of $\mathscr{X}$ with no *local external activity* in the sense of d'Adderio and Moci [5, Section 5.3].

2.4. Cohomology

The cohomology (with complex coefficients) of the complements of toric arrangements was studied by Looijenga [21] and De Concini and Procesi [8].

Theorem 2.21 ([8, Theorem 4.2]). *Consider a toric arrangement $\mathscr{A}$. The Poincaré polynomial of $M(\mathscr{A})$ can be expressed as follows:*

$$P_{\mathscr{A}}(t) := \sum_{j=0}^{\infty} \dim H^j(M(\mathscr{A}); \mathbb{C})\, t^j = \sum_{j=0}^{\infty} |\mathscr{N}_j|(t+1)^{k-j} t^j.$$

This result was obtained in [8] by computing de Rham cohomology, and in [21] via spectral sequence computations. In the special case of (totally) unimodular arrangements, De Concini and Procesi also determine the algebra structure of $H^*(M(\mathscr{A}); \mathbb{C})$ by formality of $M(\mathscr{A})$ [8, Section 5].

2.5. The homotopy type of complexified toric arrangements

From now on, we will think of $\mathscr{A}$ as being a complexified (primitive, essential) toric arrangement.

The complement of a complexified toric arrangement $\mathscr{A}$ has the homotopy type of a finite cell complex, defined from the stratification of the real torus T_Λ into *chambers* and *faces* induced by the associated 'real' arrangement $\mathscr{A}^c$.

Definition 2.22. Let $\mathscr{A} = \{(\chi_1, a_1), \dots, (\chi_n, a_n)\}$ be a complexified toric arrangement. Its *chambers* are the connected components of $M(\mathscr{A}^c)$. We denote the set of chambers of $\mathscr{A}$ by $\mathcal{T}(\mathscr{A})$.

The *faces* of $\mathscr{A}$ are the connected components of the intersections

$$\mathrm{relint}(\overline{C} \cap X) \quad \text{with } C \in \mathcal{T}(\mathscr{A}),\ X \in \mathcal{C}(\mathscr{A}).$$

The faces of $\mathscr{A}$ are the cells of a polyhedral complex, which we denote by $\mathcal{D}(\mathscr{A})$.

The topology of a (nonregular) polyhedral complex is encoded in an acyclic category, called the *face category* of the complex (see [6, §2.2.2] for some details on face categories, our Section 3 below for some basics about acyclic categories, and [20] for a more comprehensive treatment).

Definition 2.23. The face category of a complexified toric arrangement is $\mathcal{F}(\mathscr{A}) = \mathcal{F}(\mathcal{D}(\mathscr{A}))$, i.e. the face category of the polyhedral complex $\mathcal{D}(\mathscr{A})$.

The lattice Λ acts on $\mathbb{C}^n$ and on $\mathbb{R}^n$ as the group of automorphisms of the covering map p of (2.4) above. Consider now the map $q : \mathcal{F}(\mathscr{A}^{\upharpoonright}) \to \mathcal{F}(\mathscr{A})$ induced by p.

Proposition 2.24 ([6, Lemma 4.8]). *Let $\mathscr{A}$ be a complexified toric arrangement. The map $q : \mathcal{F}(\mathscr{A}^{\upharpoonright}) \to \mathcal{F}(\mathscr{A})$ induces an isomorphism of acylic categories*

$$\mathcal{F}(\mathscr{A}) \cong \mathcal{F}(\mathscr{A}^{\upharpoonright})/\Lambda.$$

2.5.1. The Salvetti category. Recall that the Salvetti complex for affine hyperplane arrangements makes use of the operation of Definition 1.26. We need a suitable analogue for toric arrangements.

Proposition 2.25 ([6, Proposition 3.12]). *Let Λ be a finite rank lattice, and Γ a sublattice of Λ. Let $\mathscr{A}$ a complexified toric arrangement on T_Λ and recall the arrangement $\mathscr{A}_\Gamma$ from Definition 2.14. The projection $\pi_\Gamma : T_\Lambda \to T_\Gamma$ induces a morphism of acyclic categories*

$$\pi_\Gamma : \mathcal{F}(\mathscr{A}) \to \mathcal{F}(\mathscr{A}_\Gamma).$$

Consider now a face $F \in \mathcal{F}(\mathscr{A})$. We associate to it the sublattice

$$\Gamma_F = \{\chi \in \Lambda \mid \chi \text{ is constant on } F\} \subseteq \Lambda.$$

Definition 2.26. Consider a toric arrangement $\mathscr{A}$ on T_Λ and a face $F \in \mathcal{F}(\mathscr{A})$. The *restriction* of $\mathscr{A}$ to F is the arrangement $\mathscr{A}_F = \mathscr{A}_{\Gamma_F}$ on T_{Γ_F}.

We will write $\pi_F = \pi_{\Gamma_F} : \mathcal{F}(\mathscr{A}) \to \mathcal{F}(\mathscr{A}_F)$.

Definition 2.27 ([6, Definition 4.1]). Let $\mathscr{A}$ be a toric arrangement on a complex torus T_Λ. The *Salvetti category* of $\mathscr{A}$ is the category $\mathrm{Sal}(\mathscr{A})$ defined as follows.

(a) The objects are the morphisms in $\mathcal{F}(\mathscr{A})$ between faces and chambers:

$$\mathrm{Ob}(\mathrm{Sal}(\mathscr{A})) = \{m : F \to C \mid m \in \mathrm{Mor}(\mathcal{F}(\mathscr{A})),\ C \in \mathcal{T}(\mathscr{A})\}.$$

(b) The morphisms are the triples $(n, m_1, m_2) : m_1 \to m_2$, where $m_1 : F_1 \to C_1, m_2 : F_2 \to C_2 \in \mathrm{Ob}(\mathrm{Sal}(\mathscr{A}))$, $n : F_2 \to F_1 \in \mathrm{Mor}(\mathcal{F}(\mathscr{A}))$ and m_1, m_2 satisfy the condition

$$\pi_{F_1}(m_1) = \pi_{F_1}(m_2).$$

(c) Composition of morphisms is defined as

$$(n', m_2, m_3) \circ (n, m_1, m_2) = (n \circ n', m_1, m_3)$$

whenever n and n' are composable.

Remark 2.28. The Salvetti category is an acyclic category in the sense of Definition 3.1.

Definition 2.29. Let $\mathscr{A}$ be a complexified toric arrangement; its *Salvetti complex* is the nerve $\mathcal{S}(\mathscr{A}) := \Delta(\mathrm{Sal}(\mathscr{A}))$.

Theorem 2.30 ([6, Theorem 4.3]). *The Salvetti complex $\mathcal{S}(\mathscr{A})$ embeds in the complement $M(\mathscr{A})$ as a deformation retract.*

Remark 2.31. As for the case of affine arrangements, the Salvetti category is the face category of a polyhedral complex, of which the toric Salvetti complex is a subdivision. If we need to distinguish between the two, we will call the first the *cellular Salvetti complex* and the second the *simplicial Salvetti complex*.

3. Discrete Morse theory

Our proof of minimality will consist in describing a sequence of cellular collapses on the toric Salvetti complex, which is not necessarily a regular cell complex. We thus need to extend discrete Morse theory from posets to acyclic categories.

The setup used in the textbook of Kozlov [20] happens to lend itself very nicely to such a generalization—in fact, once the right definitions are made, even the proofs given in [20] just need some minor additional observation.

Definition 3.1. An *acyclic category* is a small category where the only endomorphisms are the identities, and these are the only invertible morphisms.

An *indecomposable morphism* in an acyclic category is a morphism that cannot be written as the composition of two nontrivial morphisms. The *length* of a morphism m in an acyclic category is the maximum number of members in a decomposition of m into nontrivial morphisms. The *height* of an acyclic category is the maximum of the lengths of its morphisms: here we will restrict ourselves to acyclic categories of finite height.

A *rank function* on an acyclic category $\mathcal{C}$ is a function $\mathrm{rk} : \mathrm{Ob}(\mathcal{C}) \to \mathbb{N}$ such that $\mathrm{rk}^{-1}(0) \neq \emptyset$ and for every indecomposable morphism $x \to y$, $\mathrm{rk}(x) = \mathrm{rk}(y) - 1$. An acyclic category is called *ranked* if it admits a rank function.

A *linear extension* $\prec$ of an acyclic category is a total order on its set of objects such that

$$\mathrm{Mor}(x, y) \neq \emptyset \quad \text{implies} \quad x \prec y.$$

Remark 3.2 (Acyclic categories and posets). Every partially ordered set can be viewed as an acyclic category whose objects are the elements of the poset and $|\mathrm{Mor}(x, y)| = 1$ if $x \leq y$, $|\mathrm{Mor}(x, y)| = 0$ else (see [20, Exercise 4.9]).

Conversely, to every acyclic category $\mathcal{C}$ there is naturally associated a partial order on the set $\mathrm{Ob}(\mathcal{C})$ defined by $x \leq y$ if and only if $\mathrm{Mor}(x, y) \neq \emptyset$. We denote by $\underline{\mathcal{C}}$ this poset and by $\underline{\cdot} : \mathcal{C} \to \underline{\mathcal{C}}$ the natural functor, with $\underline{\mathcal{C}}$ viewed as a category as above. We say $\mathcal{C}$ *is* a poset if this functor is an isomorphism.

In the following sections we will freely switch between the categorical and set-theoretical view of posets.

Remark 3.3 (Face categories). The acyclic categories we will be concerned with will arise mostly as face categories of polyhedral complexes. Intuitively, a polyhedral complex is a CW-complex X whose cells are polyhedra, and the attaching maps of a cell x restrict to homeomorphisms on every boundary face of x. The face category then has an object for every cell of X and an arrow $x \to y$ for every boundary cell of y that is attached to x. See [6, Definitions 2.6 and 2.8] for the precise definition.

Notice that the face category of a polyhedral complex is naturally ranked by the dimension of the cells.

Remark 3.4 (Terminology). We take the term *acyclic category* from [20]. The same name, in other contexts, is given to categories with acyclic nerve. The reader should be warned: acyclic categories as defined here need not have acyclic nerve.

On the other hand, the reader should be aware that what we call "acyclic category" appears in the literature also as *loopless category* or as *scwol* (for "small category without loops").

The data about the cellular collapses that we will perform are stored in so-called *acyclic matchings*.

Definition 3.5. A *matching* of an acyclic category $\mathcal{C}$ is a set $\mathfrak{M}$ of indecomposable morphisms such that, for all $m, m' \in \mathfrak{M}$, the sources and the targets of m and m' are four distinct objects of $\mathcal{C}$. A *cycle* of a matching $\mathfrak{M}$ is an ordered sequence of morphisms

$$a_1 b_1 a_2 b_2 \cdots a_n b_n$$

where

(1) for all i, $a_i \notin \mathfrak{M}$ and $b_i \in \mathfrak{M}$,
(2) for all i, the targets of a_i and b_i coincide and the sources of a_{i+1} and b_i coincide, as also do the sources of a_1 and b_n.

A matching $\mathfrak{M}$ is called *acyclic* if it has no cycles. A *critical element* of $\mathfrak{M}$ is any object of $\mathcal{C}$ that is neither the source nor the target of any $m \in \mathfrak{M}$.

Lemma 3.6. *A matching $\mathfrak{M}$ of an acyclic category $\mathcal{C}$ is acyclic if and only if*

(a) *for all $x, y \in \mathrm{Ob}(\mathcal{C})$, $m \in \mathfrak{M} \cap \mathrm{Mor}(x, y)$ implies $\mathrm{Mor}(x, y) = \{m\}$,*
(b) *there is a linear extension of $\mathcal{C}$ where the source and target of every $m \in \mathfrak{M}$ are consecutive.*

Proof. Recall from Remark 3.2 the poset $\underline{\mathcal{C}}$, and notice that for every matching $\mathfrak{M}$ of $\mathcal{C}$, the set $\underline{\mathfrak{M}}$ is a matching of $\underline{\mathcal{C}}$. Moreover, by Theorem 11.1 of [20], condition (b) above is equivalent to $\underline{\mathfrak{M}}$ being acyclic.

To prove the statement, let first $\mathfrak{M}$ be a matching of $\mathcal{C}$ satisfying (a) and (b). Because of (a), every cycle of $\mathfrak{M}$ maps to a cycle of $\underline{\mathfrak{M}}$. Since $\underline{\mathfrak{M}}$ is acyclic because of (b), $\mathfrak{M}$ must be acyclic too.

Let now $\mathfrak{M}$ be an acyclic matching of $\mathcal{C}$. Then $\underline{\mathfrak{M}}$ must be acyclic, thus (b) holds. If (a) fails, say because of some $x, y \in \mathrm{Ob}(\mathcal{C})$ with $\mathrm{Mor}(x, y) \supseteq \{m, m'\}$ and $m \in \mathfrak{M}$, then

$m' \notin \mathfrak{M}$ (because $\mathfrak{M}$ is a matching) and the sequence $m'm$ is a cycle of $\mathfrak{M}$, contradicting the assumption. □

A handy tool for dealing with acyclic matchings is the following result, which generalizes [20, Theorem 11.10].

Lemma 3.7 (Patchwork Lemma). *Consider a functor $\varphi : \mathcal{C} \to \mathcal{C}'$ of acyclic categories and suppose that for each object c of $\mathcal{C}'$ an acyclic matching $\mathfrak{M}_c$ of $\varphi^{-1}(c)$ is given. Then the matching $\mathfrak{M} := \bigcup_{c\in \mathrm{Ob}(\mathcal{C}')} \mathfrak{M}_c$ of $\mathcal{C}$ is acyclic.*

Proof. We apply Lemma 3.6. Since $\mathrm{Mor}_{\varphi^{-1}(c)}(x, y) = \mathrm{Mor}_{\mathcal{C}}(x, y)$ for all $c \in \mathrm{Ob}(\mathcal{C}')$ and all $x, y \in \mathrm{Ob}(\varphi^{-1}(c))$, condition (a) holds for $\mathfrak{M}$ because it holds for $\mathfrak{M}_c$.

Property (b) for $\mathfrak{M}$ is proved via the linear extension of $\mathcal{C}$ obtained by concatenation of the linear extensions given by the $\mathfrak{M}_c$ on the categories $\varphi(c)$. □

The topological gist of discrete Morse theory is the so-called "Fundamental Theorem" (see e.g. [20, §11.2.2]). Here we state the part of it that will be needed below.

Theorem 3.8. *Let $\mathcal{F}$ be the face category of a finite polyhedral complex X, and let $\mathfrak{M}$ be an acyclic matching of $\mathcal{F}$. Then X is homotopy equivalent to a CW-complex X' with, for all k, one cell of dimension k for every critical element of $\mathfrak{M}$ of rank k.*

Proof. A proof can be obtained applying [20, Theorem 11.15] to the filtration of X with i-th term $F_i(X) = \bigcup_{j\leq i} x_j$, where $x_0, x_1, \ldots$ is an enumeration of the cells of X corresponding to a linear extension of $\mathcal{F}(X)$ in which the source and target of every $m \in \mathfrak{M}$ are consecutive (such a linear extension exists by Lemma 3.6(b)). □

Remark 3.9. Let $\mathfrak{M}$ be an acyclic matching of a polyhedral complex X.

(i) The boundary maps of the complex X' in Theorem 3.8 can be explicitly computed by tracking the individual collapses, as in [20, Theorem 11.13(c)].
(ii) We will call $\mathfrak{M}$ *perfect* if the number of its critical elements of rank k is $\beta_k(X)$, the k-th Betti number of X. Note that if the face category of a complex X admits a perfect acyclic matching, then X is minimal in the sense of [14].

4. Stratification of the toric Salvetti complex

We now work our way towards the proof of minimality of complements of toric arrangements. We start by defining a stratification of the toric Salvetti complex, in which each stratum corresponds to a local nonbroken circuit. Then, in the next section, we will exploit the structure of this stratification to define a perfect acyclic matching on the Salvetti category.

4.1. Local geometry of complexified toric arrangements

We start by introducing the key combinatorial tool in order to have a 'global' control of the local contributions.

Consider a rank d complexified toric arrangement $\mathscr{A} = \{(\chi_1, a_1), \ldots, (\chi_n, a_n)\}$. As usual write $\chi_i(x) = x^{\alpha_i}$ for $\alpha_i \in \mathbb{Z}^d$ and $K_i = \{x \in T_\Lambda \mid \chi_i(x) = a_i\}$.

Define

$$\mathscr{A}_0 := \{H_i = \ker \langle \alpha_i, \cdot \rangle \mid i = 1, \ldots, n\},$$

a central hyperplane arrangement in $\mathbb{R}^d$.

From now on, fix a chamber $B \in \mathcal{T}(\mathscr{A}_0)$ and a linear extension $\prec_0$ of $\mathcal{T}(\mathscr{A}_0)_B$.

Next, we introduce some central arrangements associated with the 'local' data.

Definition 4.1. For every face $F \in \mathcal{F}(\mathscr{A})$ define the arrangement

$$\mathscr{A}[F] := \{H_i \in \mathscr{A}_0 \mid \chi_i(F) = a_i\}.$$

If $Y \in \mathcal{C}(\mathscr{A})$ define

$$\mathscr{A}[Y] := \{H_i \in \mathscr{A}_0 \mid Y \subseteq K_i\}.$$

Remark 4.2. The linear extension $\prec_0$ of $\mathcal{T}(\mathscr{A}_0)_B$ induces as in Proposition 1.17 linear extensions $\prec_F$ of $\mathcal{T}(\mathscr{A}[F])_{B_F}$ and $\prec_Y$ of $\mathcal{T}(\mathscr{A}[Y])_{B_Y}$, for every $F \in \mathcal{F}(\mathscr{A})$ and every $Y \in \mathcal{C}(\mathscr{A})$.

Moreover, for $F \in \mathcal{F}(\mathscr{A})$ and $C, C' \in \mathcal{T}(\mathscr{A}[F])$ we denote by $S_F(C, C')$ the set of separating hyperplanes of the arrangement $\mathscr{A}[F]$, as introduced in Definition 1.13.

Definition 4.3. Given $Y \in \mathcal{C}(\mathscr{A})$ let $\widetilde{Y} \in \mathcal{L}(\mathscr{A}_0)$ be defined as

$$\widetilde{Y} := \bigcap_{Y \subseteq K_i} H_i.$$

Moreover, for $C \in \mathcal{T}(\mathscr{A}[Y])$ let $X(Y, C) \supseteq Y$ be the layer determined by the intersection defined by Lemma 1.31 from $\prec_Y$. Analogously, for $C \in \mathcal{T}(\mathscr{A}[F])$ let $X(F, C)$ be defined with respect to $\prec_F$.

We write $\widetilde{X}(Y, C)$ and $\widetilde{X}(F, C)$ for the corresponding elements of $\mathcal{L}(\mathscr{A}[Y])$ and $\mathcal{L}(\mathscr{A}[F])$.

Definition 4.4. Let

$$\mathscr{Y} := \{(Y, C) \mid Y \in \mathcal{C}(\mathscr{A}),\ C \in \mathcal{T}(\mathscr{A}[Y]),\ X(Y, C) = Y\}.$$

For $i = 0, \ldots, d$ let $\mathscr{Y}_i := \{(Y, C) \in \mathscr{Y} \mid \dim Y = i\}$.

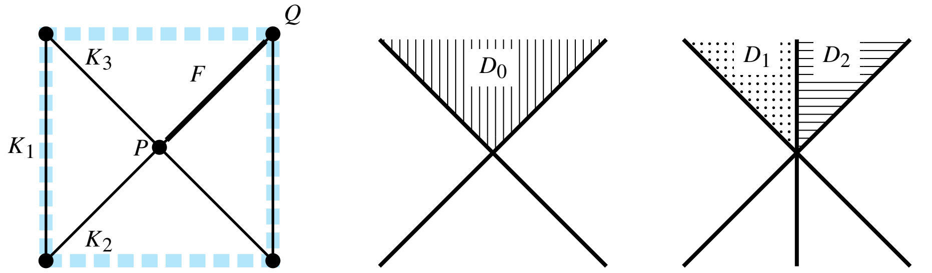
Example 4.5. Consider the toric arrangement $\mathscr{A} = \{(x, 1), (xy^{-1}, 1), (xy, 1)\}$ of Figure 2(a). In this and in the following pictures we consider the compact torus $(S^1)^2$ as a quotient of the square. Therefore we draw toric arrangements in a square (pictured with a dashed line), where the opposite sides are identified.

The layer poset consists of the following elements:

$$\mathcal{C}(\mathscr{A}) = \{P, Q, K_1, K_2, K_3, (\mathbb{C}^*)^2\}.$$

Figures 2(b) and 2(c) show respectively the arrangements $\mathscr{A}[P]$ and $\mathscr{A}[Q] = \mathscr{A}_0$.

Let $\mathscr{Y}$ be as in Definition 4.4. There is one element $(P, D_0) \in \mathscr{Y}$ and two elements $(Q, D_1), (Q, D_2) \in \mathscr{Y}$. Furthermore we have an element for each 1-dimensional layer $(K_i, D_i) \in \mathscr{Y}$.



(a) A toric arrangement (b) The arrangement $\mathscr{A}[P]$ (c) The arrangement $\mathscr{A}[Q]$

Fig. 2. A toric arrangement and some of its associated hyperplane arrangements.

Lemma 4.6. *Let $\mathscr{A}$ be a rank d toric arrangement. For all $i = 0, \ldots, d$, we have $|\mathscr{Y}_i| = |\mathscr{N}_i|$.*

Proof. This follows because for every $i = 0, \ldots, d$,

$$|\mathscr{N}_i| = \sum_{\substack{Y \in \mathcal{C}(\mathscr{A}) \\ \dim Y = i}} |\mathrm{nbc}_i(\mathscr{A}[Y])|.$$

Every summand on the right hand side counts the number of generators in top degree cohomology or—equivalently—the number of top-dimensional cells of a minimal CW-model of the complement of the complexification of $\mathscr{A}[Y]$. By [12, Lemma 4.18 and Proposition 2] these top-dimensional cells correspond bijectively to chambers $C \in \mathcal{T}(\mathscr{A}[Y])$ with $X(Y, C) = Y$. Therefore

$$|\mathscr{N}_i| = \sum_{\substack{Y \in \mathcal{C}(\mathscr{A}) \\ \dim Y = i}} |\{C \in \mathcal{T}(\mathscr{A}[Y]) \mid X(Y, C) = Y\}| = |\mathscr{Y}_i|. \qquad \square$$

Definition 4.7. Recall Definition 1.15 and define a function

$$\xi_0 : \mathscr{Y} \to \mathcal{T}(\mathscr{A}_0)_B, \qquad (Y, C) \mapsto \mu[\mathscr{A}[Y], \mathscr{A}_0](C).$$

Fix a total order $\dashv$ on $\mathscr{Y}$ that makes this function order preserving (i.e., for $y_1, y_2 \in \mathscr{Y}$, by definition $\xi_0(y_1) \prec_0 \xi_0(y_2)$ implies $y_1 \dashv y_2$).

We now examine the local properties of the ordering $\dashv$.

Definition 4.8. For $F \in \mathcal{F}(\mathscr{A})$ let $\mathscr{Y}_F := \{(Y, C) \in \mathscr{Y} \mid F \subseteq Y\}$. Since $F \subseteq Y$ implies $\mathscr{A}[Y] \subseteq \mathscr{A}[F]$, we can define a function

$$\xi_F : \mathscr{Y}_F \to \mathcal{T}(\mathscr{A}[F]), \qquad (Y, C) \mapsto \mu[\mathscr{A}[Y], \mathscr{A}[F]](C).$$

Remark 4.9. By Lemma 1.16, $\mu[\mathscr{A}[F], \mathscr{A}_0] \circ \xi_F = \xi_0$ on $\mathscr{Y}_F$. Therefore, for $y_1, y_2 \in \mathscr{Y}_F$, $\xi_F(y_1) \prec_F \xi_F(y_2)$ implies $\xi_0(y_1) \prec_0 \xi_0(y_2)$, and thus $y_1 \dashv y_2$.

Proposition 4.10. *For all $F \in \mathcal{F}(\mathscr{A})$ and every $y = (Y, C) \in \mathscr{Y}_F$,*

$$X(F, \xi_F(y)) = Y.$$

Proof. We will use the lattice isomorphisms $\mathcal{L}(\mathscr{A}[F])_{\leq \widetilde{Y}} \cong \mathcal{L}(\mathscr{A}[Y]) \cong \mathcal{C}(\mathscr{A})_{\leq Y}$. By definition we have

$$\xi_F(y) = \mu[\mathscr{A}[Y], \mathscr{A}[F]](C) = \min_{\prec_F}\{K \in \mathcal{T}(\mathscr{A}[F]) \mid K \subseteq C\}$$

and therefore $\mathscr{A}[F]_{\widetilde{Y}} \cap S_F(\xi_F(y), C_1) \neq \emptyset$ for all $C_1 \prec_F \xi_F(y)$, which shows that $\widetilde{Y} \geq \widetilde{X}(F, \xi_F(y))$ in $\mathcal{L}(\mathscr{A}[F])$ and thus $Y \geq X(F, \xi_F(y))$ in $\mathcal{C}(\mathscr{A})$. Now, for every layer Z with $Z < Y$ we have $\mathscr{A}[Z] \subseteq \mathscr{A}[Y]$. Because by definition $Y = X(Y, C)$, we have $\widetilde{Z} < \widetilde{Y} = \widetilde{X}(Y, C)$ in $\mathcal{L}(\mathscr{A}[Y])$ and so there is $C_2 \prec_Y C$ with $S_Y(C_2, C) \cap A[Y]_{\widetilde{Z}} = \emptyset$.

Let $C_3 := \mu[\mathscr{A}[Y], \mathscr{A}[F]](C_2)$. We have $C_3 \subseteq C_2$ and $\xi_F(y) \subseteq C$, therefore $S_F(C_3, \xi_F(y)) \cap \operatorname{supp}(\widetilde{Z}) = \emptyset$, and $C_3 \prec_F \xi_F(y)$ by $C_2 \prec_Y C$. This means $Z \not\geq X(F, \xi_F(y))$, and the claim follows. □

Lemma 4.11. *For $F \in \mathcal{F}(\mathscr{A})$ and $C \in \mathcal{T}(\mathscr{A}[F])$ we have*

$$\xi_F(X_C, \sigma_{\mathscr{A}[X_C]}(C)) = C.$$

In particular $\xi_F : \mathscr{Y}_F \to \mathcal{T}(\mathscr{A}[F])$ is a bijection.

Proof. Using the definition of ξ_F and Corollary 1.33 we have

$$\begin{aligned}\xi_F(X_C, \sigma_{\mathscr{A}[X_C]}(C)) &= \mu[\mathscr{A}[X_C], \mathscr{A}[F]](\sigma_{\mathscr{A}[X_C]}(C)) \\ &= \min\{K \in \mathcal{T}(\mathscr{A}[F]) \mid K_{X_C} = C_{X_C}\} = C.\end{aligned}$$

Letting $\beta_F : \mathcal{T}(\mathscr{A}[F]) \to \mathscr{Y}_F$ be defined by $C \mapsto (X_C, \sigma_{\mathscr{A}[X_C]}(C))$, the above means $\xi_F \circ \beta_F = \mathrm{id}$, therefore the map ξ_F is surjective. Injectivity of ξ_F amounts now to proving $\beta_F \circ \xi_F = \mathrm{id}$, which is an easy check on the definitions. □

Corollary 4.12. *For $y_1, y_2 \in \mathscr{Y}_F$, $y_1 \dashv y_2$ if and only if $\xi_F(y_1) \preceq_F \xi_F(y_2)$.*

4.2. *Lifting faces and morphisms*

We now relate our constructions to the covering $\mathscr{A}^{\upharpoonright}$ of $\mathscr{A}$ of §2.2.2. Recall that Λ acts freely on $\mathcal{F}(\mathscr{A}^{\upharpoonright})$ and that $q : \mathcal{F}(\mathscr{A}^{\upharpoonright}) \to \mathcal{F}(\mathscr{A}) = \mathcal{F}(\mathscr{A}^{\upharpoonright})/\Lambda$ is the projection to the quotient (compare Proposition 2.24).

Remark 4.13. Fix a face $F \in \mathrm{Ob}(\mathcal{F}(\mathscr{A}))$, and choose a lifting $F^{\upharpoonright}$ in $\mathcal{F}(\mathscr{A}^{\upharpoonright})$. Then the arrangements $\mathscr{A}^{\upharpoonright}_{F^{\upharpoonright}}$ and $\mathscr{A}[F]$ differ only by a translation. Thus we have natural isomorphisms of posets

$$\mathcal{F}(\mathscr{A}[F]) \cong \mathcal{F}(\mathscr{A}^{\upharpoonright}_{F^{\upharpoonright}}) \cong \mathcal{F}(\mathscr{A}^{\upharpoonright})_{\geq F^{\upharpoonright}}.$$

In the following we will identify these posets and, in particular, define a functor of acyclic categories $q : \mathcal{F}(\mathscr{A}[F]) \to \mathcal{F}(\mathscr{A})$ according to the restriction of $q : \mathcal{F}(\mathscr{A}^{\upharpoonright}) \to \mathcal{F}(\mathscr{A})$ to $\mathcal{F}(\mathscr{A}^{\upharpoonright})_{\geq F^{\upharpoonright}}$.

Given a face G of $\mathcal{F}(\mathscr{A}[F])$ we will write $q(G)$ for the image under the covering q (see Proposition 2.24) of the corresponding face of $\mathcal{F}(\mathscr{A}^{\upharpoonright})_{\geq F^{\upharpoonright}}$.

Remark 4.14 (Notation). Recall that we identify posets (such as $\mathcal{F}(\mathscr{A}^{\upharpoonright})$ or $\mathcal{F}(\mathscr{A}[F])$) with the associated acyclic categories, as explained in Remark 3.2. In particular, if x, y are elements in a poset with $x \leq y$, we will take the notation $x \leq y$ also to stand for the unique morphism $x \to y$ in the associated category.

Now, given a morphism $m : F \to G$ of $\mathcal{F}(\mathscr{A})$, for every choice of $F^{\upharpoonright} \in \mathcal{F}(\mathscr{A}^{\upharpoonright})$ lifting F, there is a unique morphism $F^{\upharpoonright} \leq G^{\upharpoonright}$ lifting m. We have $\mathcal{F}(\mathscr{A}^{\upharpoonright}_{G^{\upharpoonright}}) \subseteq \mathcal{F}(\mathscr{A}^{\upharpoonright}_{F^{\upharpoonright}})$ (see Remark 1.9)

Definition 4.15. Consider a toric arrangement $\mathscr{A}$ on $T_\Lambda \cong (\mathbb{C}^*)^k$ and a morphism $m : F \to G$ of $\mathcal{F}(\mathscr{A})$. Because of the freeness of the action of Λ, for every choice of $F^{\upharpoonright}$ in $\mathcal{F}(\mathscr{A}^{\upharpoonright})$ lifting F, there is a unique morphism $F^{\upharpoonright} \leq G^{\upharpoonright}$ lifting m.

To m we associate

(a) the order preserving function

$$i_m : \mathcal{F}(\mathscr{A}[G]) \to \mathcal{F}(\mathscr{A}[F])$$

corresponding to the inclusion $\mathcal{F}(\mathscr{A}^{\upharpoonright}_{G^{\upharpoonright}}) \subseteq \mathcal{F}(\mathscr{A}^{\upharpoonright}_{F^{\upharpoonright}})$ (see Remark 1.9) under the identification of Remark 4.13;

(b) the face $F_m \in \mathcal{F}(\mathscr{A}[F])$ defined by

$$F_m := i_m(\widehat{G})$$

where $\widehat{G}$ denotes the unique minimal element of $\mathcal{F}(\mathscr{A}[G])$.

Clearly then $\widehat{G} = F_{\mathrm{id}_G}$. In the following we will abuse notation for the sake of transparency and, given a face G of $\mathcal{F}(\mathscr{A})$, we will write G_{id} for F_{id_G}.

Example 4.16. Consider the arrangement $\mathscr{A}$ of Figure 2. Figure 3 illustrates the maps i_m and i_n for the morphisms $m : P \to F$ and $n : Q \to F$.

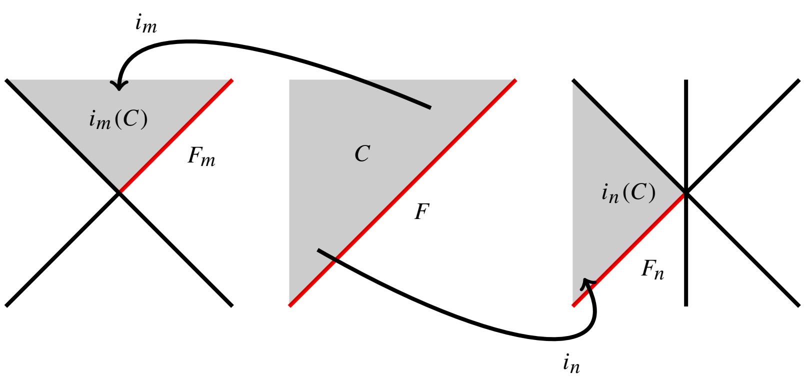


Fig. 3. F_m and the map i_m.

Remark 4.17. Every choice of positive sides for the elements of $\mathscr{A}_0$ determines a corresponding choice for all the elements of $\mathscr{A}^{\upharpoonright}$. Then given $m : F \to G$ and any lift $G^{\upharpoonright}$ of G, in terms of sign vectors and identifying each $H \in \mathscr{A}[F]$ with its unique translate which contains $G^{\upharpoonright}$, we have

$$\gamma_{F_m}[\mathscr{A}[F]] = \gamma_{G^{\upharpoonright}}[\mathscr{A}^{\upharpoonright}]_{|\mathscr{A}[F]}.$$

In particular, if G is a chamber, then so is F_m.

Lemma 4.18. *Recall the setup of Definition 4.15.*

(a) *If* $F \xrightarrow{m} G \xrightarrow{n} K$ *are morphisms of* $\mathcal{F}(\mathscr{A})$*, then*

$$i_{n\circ m} = i_m \circ i_n, \quad \textit{thus} \quad i_m(F_n) = F_{n\circ m}.$$

(b) *Let* $m : F \to G$ *be a morphism of* $\mathcal{F}(\mathscr{A})$*. Then, for every morphism* n *of* $\mathscr{A}[G]$*, we have* $q(i_m(n)) = q(n)$*, and in particular* $q(i_m(K)) = q(K)$ *for every face* K *of* $\mathscr{A}[G]$*.*

(c) *Let* $m : G \leq K$ *be a morphism of* $\mathcal{F}(\mathscr{A}[F])$*. Then there are morphisms* $n : F \to q(G)$ *and* $m^{\downharpoonleft}$ *of* $\mathcal{F}(\mathscr{A})$ *with*

$$i_n(q(G)_{\mathrm{id}} \leq F_{m^{\downharpoonleft}}) = m.$$

Proof. Parts (a) and (b) are immediate rephrasing of the definitions. For part (c) let $n := q(F_{\mathrm{id}} \leq G)$ and $m^{\downharpoonleft} := q(m)$. □

4.3. Definition of the strata

Each stratum will be associated to an element of $\mathscr{Y}$, and we will think of the Salvetti category as being 'built up' from strata according to the ordering of $\mathscr{Y}$.

Definition 4.19. Define the map $\theta : \mathrm{Sal}(\mathscr{A}) \to \mathscr{Y}$ as follows:

$$\theta : (m : F \to C) \mapsto (X(F, F_m), \sigma_{\mathscr{A}[X(F,F_m)]}(F_m)).$$

Remark 4.20. For every object $m : F \to C$ of $\mathrm{Sal}(\mathscr{A})$ we have $\xi_F(\theta(m)) = F_m$.

Lemma 4.21. *For* $m : G \to C$ *and* $m' : G \to C' \in \zeta$*, if* $\theta(m) \dashv \theta(m')$ *then* $F_m \prec_G F_{m'}$*.*

Proof. If $\theta(m) \dashv \theta(m')$, then by Remark 4.20 and Corollary 4.12, $F_m = \xi_G(\theta(m)) \prec_G \xi_G(\theta(m')) = F_{m'}$. □

Definition 4.22. Given a complexified toric arrangement $\mathscr{A}$ on $(\mathbb{C}^*)^d$, we consider the stratification $\mathrm{Sal}(\mathscr{A}) = \bigcup_{(Y,C)\in\mathscr{Y}} \mathcal{S}_{(Y,C)}$ indexed by $\mathscr{Y}$, where

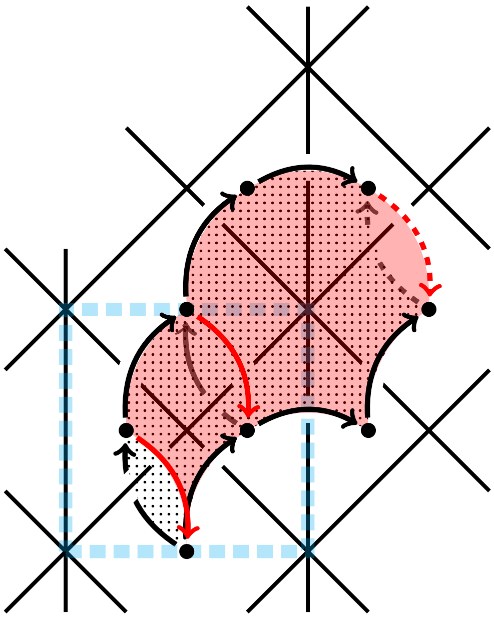
$$\mathcal{S}_{(Y,C)} := \{m \in \mathrm{Sal}(\mathscr{A}) \mid \exists (m \to n) \in \mathrm{Mor}(\mathrm{Sal}(\mathscr{A})),\ n \in \theta^{-1}(Y, C)\}.$$

Moreover, recall from Definition 4.7 the total ordering $\vdash$ on $\mathscr{Y}$ and define

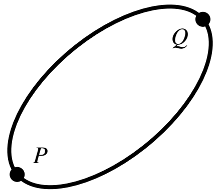
$$\mathcal{N}_y := \mathcal{S}_y \setminus \bigcup_{y' \dashv y} \mathcal{S}_{y'}.$$

Example 4.23. Consider the toric arrangement $\mathscr{A}$ of Figure 2. Figure 4(a) shows two strata of the stratification of $\mathrm{Sal}(\mathscr{A})$ of Definition 4.22.

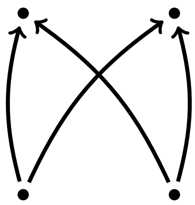
The stratum $\mathcal{S}_{((\mathbb{C}^*)^2,D)}$ appears with a dotted shading, while the stratum $\mathcal{N}_{(K_2,D_2)}$ has a solid shading. Thus $\mathcal{N}_{(K_2,D_2)}$ consists of two 1-dimensional layers and two 2-dimensional layers. Figure 4(b) depicts the rank 1 arrangement $\mathscr{A}^{K_2}$. The category $\mathcal{N}_{(K_2,D_2)}$ is showed in Figure 4(c) and it is isomorphic to $\mathcal{F}(\mathscr{A}^{K_2})^{\mathrm{op}}$ (in this case $\mathcal{F}(\mathscr{A}^{K_2})^{\mathrm{op}} \cong \mathcal{F}(\mathscr{A}^{K_2})$).



(a) Stratification of the toric Salvetti complex



(b) The arrangement $\mathscr{A}^{K_2}$ on S^1



(c) $\mathcal{N}_{(K_2,D_2)} \cong \mathcal{F}(\mathscr{A}^{K_2})^{\mathrm{op}}$

Fig. 4. Stratification of the toric Salvetti complex (cf. Figure 2).

5. The topology of the strata

We now want to show that, for $y \in \mathscr{Y}$, the category $\mathcal{N}_y$ is isomorphic to the face category of a complexified toric arrangement. The main result of this section is the following.

Theorem 5.1. *Consider a complexified toric arrangement $\mathscr{A}$ and for $y = (Y, C) \in \mathscr{Y}$ let $\mathcal{N}_y$ be as in Definition 4.22. Then there is an isomorphism of acyclic categories*

$$\mathcal{N}_{(Y,C)} \cong \mathcal{F}(\mathscr{A}^Y)^{\mathrm{op}}.$$

The main idea for proving this theorem is to use the 'local' combinatorics of the (hyperplane) arrangements $\mathscr{A}[F]$ to understand the 'global' structure of the strata in $\mathrm{Sal}(\mathscr{A})$. We carry out this 'local-to-global' approach by using the language of diagrams.

*5.1. The category **AC***

Let ***Cat*** denote the category of small categories. We define ***AC*** to be the full subcategory of ***Cat*** consisting of acyclic categories (see Definition 3.1, cf. [20]).

Colimits in $\boldsymbol{AC}$ do not coincide with colimits taken in $\boldsymbol{Cat}$. In the following, we will need an explicit description of colimits in $\boldsymbol{AC}$, at least for the special class of diagrams with which we will be concerned.

Definition 5.2. Let $\mathcal{I}$ be an acyclic category. A diagram $\mathscr{D} : \mathcal{I} \to \boldsymbol{AC}$ of acyclic categories is called *geometric* if

(i) • for every $X \in \mathrm{Ob}(\mathcal{I})$, $\mathscr{D}(X)$ is ranked and
• for every $f \in \mathrm{Mor}(\mathcal{I})$, $\mathscr{D}(f)$ is rank-preserving;

(ii) for every $X \in \mathrm{Ob}(\mathcal{I})$ and every $x \in \mathrm{Mor}(\mathscr{D}(X))$ there exist

- $\widehat{X} \in \mathrm{Ob}(\mathcal{I})$,
- $f \in \mathrm{Mor}_{\mathcal{I}}(\widehat{X}, X)$ and
- $\widehat{x} \in \mathrm{Mor}(\mathscr{D}(\widehat{X}))$ with $\mathscr{D}(f)(\widehat{x}) = x$

such that for every morphism $g \in \mathrm{Mor}_{\mathcal{I}}(Y, X)$ and every $y \in \mathscr{D}(g)^{-1}(x)$ there exists a morphism $\widehat{g} \in \mathrm{Mor}_{\mathcal{I}}(\widehat{X}, Y)$ such that $\mathscr{D}(\widehat{g})(\widehat{x}) = y$.

Remark 5.3. From the definition it follows that the morphism $\widehat{x}$ in (ii) is unique.

Definition 5.4. Define a relation $\sim$ on $\coprod_{X \in \mathrm{Ob}(\mathcal{I})} \mathrm{Mor}(\mathscr{D}(X))$ as follows: for $x \in \mathrm{Mor}(\mathscr{D}(X))$ and $y \in \mathrm{Mor}(\mathscr{D}(Y))$ let $x \sim y$ if there are

- an object $Z \in \mathrm{Ob}(\mathcal{I})$, a morphism $z \in \mathrm{Mor}(\mathscr{D}(Z))$ and
- morphisms $f_X : Z \to X$, $f_Y : Z \to Y$ of $\mathcal{I}$

such that $\mathscr{D}(f_X)(z) = x$ and $\mathscr{D}(f_Y)(z) = y$.

Moreover, define a relation $\approx$ on $\coprod_{X \in \mathrm{Ob}(\mathcal{I})} \mathrm{Ob}(\mathscr{D}(X))$ by setting $a \approx b$ if $\mathrm{id}_a \sim \mathrm{id}_b$.

Remark 5.5. If $\mathscr{D}$ is a geometric diagram of acyclic categories, the observation that $x \sim y$ if and only if $\widehat{x} = \widehat{y}$, together with Remark 5.3, shows that $\sim$ and $\approx$ are in fact equivalence relations.

Proposition 5.6. *Let $\mathscr{D} : \mathcal{I} \to \boldsymbol{AC}$ be a geometric diagram of acyclic categories. Then the colimit of $\mathscr{D}$ exists and is given by the co-cone $(\mathcal{C}, (\gamma_X)_{X \in \mathrm{Ob}(\mathcal{I})})$ with*

$$\mathrm{Ob}(\mathcal{C}) = \coprod_{X \in \mathrm{Ob}(\mathcal{I})} \mathrm{Ob}(\mathscr{D}(X))\big/\!\approx, \qquad \mathrm{Mor}(\mathcal{C}) = \coprod_{X \in \mathrm{Ob}(\mathcal{I})} \mathrm{Mor}(\mathscr{D}(X))\big/\!\sim$$

(where $[m]_\sim : [x]_\approx \to [y]_\approx$ whenever $m : x \to y$), and for every $X \in \mathrm{Ob}(\mathcal{I})$, $x \in \mathrm{Ob}(\mathscr{D}(X))$ and $m \in \mathrm{Mor}(\mathscr{D}(X)$,

$$\gamma_X(x) = [x]_\approx, \qquad \gamma_X(m) = [m]_\sim.$$

Proof. One easily checks that $\mathcal{C}$ is a well-defined small category. We have to prove two claims.

Claim 1. *$\mathcal{C}$ is acyclic.*

Proof. Because the definition of a geometric diagram requires $\mathscr{D}(f)$ to be rank-preserving for all $f \in \mathrm{Mor}(\mathcal{I})$, we can define for all $[x]_\approx \in \mathrm{Ob}(\mathcal{C})$ a value $\nu([x]_\approx) := \mathrm{rk}(x)$, where

x is any representative and the rank is taken in the appropriate category. Now, for every $X \in \mathrm{Ob}(\mathcal{I})$, every nonidentity morphism $m \in \mathrm{Mor}_{\mathscr{D}(X)}(x, y)$ has $\mathrm{rk}(x) < \mathrm{rk}(y)$ and thus $\nu([x]_\approx) < \nu([y]_\approx)$—in particular, $[m]_\sim$ is not an identity. This implies directly that the only endomorphisms of $\mathcal{C}$ are the identities. Moreover, if the morphism $[m]_\sim$ above is an invertible nonidentity, then its inverse whould be a morphism $[y]_\approx \to [x]_\approx$—but since $\nu([x]_\approx) < \nu([y]_\approx)$, no such morphism exists.

Claim 2. *The co-cone $(\mathcal{C}, (\gamma_X)_{X \in \mathrm{Ob}\,\mathcal{I}})$ has the universal property.*

Proof. Let $(\mathcal{C}', (\gamma'_X)_{X\in\mathrm{Ob}\,\mathcal{I}})$ be a co-cone over $\mathscr{D}$. We have to show that there is a unique morphism of co-cones $\Psi : (\mathcal{C}, (\gamma_X)_{X\in\mathrm{Ob}\,\mathcal{I}}) \to (\mathcal{C}', (\gamma'_X)_{X\in\mathrm{Ob}\,\mathcal{I}})$.

In order to do so, notice that if $y \in [x]_\sim \in \mathrm{Mor}(\mathcal{C})$, there are $X, Y, Z \in \mathrm{Ob}(\mathcal{I})$, $f_X, f_Z \in \mathrm{Mor}(\mathcal{I})$ and $z \in \mathrm{Mor}(\mathscr{D}(Z))$ as in Definition 5.4 such that

$$\gamma'_X(x) = \gamma'_Z f_X(z) = \gamma'_Y f_Y(z) = \gamma'_Y(y).$$

This proves that the assignments

$$\Psi[x]_\approx := \gamma'_X(x), \qquad \Psi[m]_\sim := \gamma'_X(m),$$

where X is such that x is in $\mathscr{D}(X)$, do not depend on the choice of the representative x and thus define a function $\Psi : \mathcal{C} \to \mathcal{C}'$. A routine check shows functoriality and uniqueness of Ψ. □

5.2. Proof of Theorem 5.1

Throughout this section let $\mathscr{A}$ be a complexified toric arrangement and recall the notational conventions of Section 4.2, in particular Remarks 4.13 and 4.14.

Definition 5.7 (A diagram for the face category of the compact torus).

$$\begin{aligned} \mathscr{F}(\mathscr{A}) = \mathscr{F} : \mathcal{F}(\mathscr{A})^{\mathrm{op}} &\to \boldsymbol{AC}, \\ F &\mapsto \mathcal{F}(\mathscr{A}[F]), \\ (m : F \to G) &\mapsto (i_m : \mathcal{F}(\mathscr{A}[G]) \to \mathcal{F}(\mathscr{A}[F])). \end{aligned}$$

After these preparations, we turn to diagrams.

Lemma 5.8. *For the diagram $\mathscr{F}$ of Definition 5.7 we have*

$$\mathrm{colim}\, \mathscr{F}(\mathscr{A}) = \mathcal{F}(\mathscr{A}).$$

Proof. We begin by noticing that $\mathscr{F}$ is a geometric diagram. Indeed, for a morphism $m : G \le K$ of $\mathscr{F}(F)$ let n and $m^\downarrow$ be obtained as in Lemma 4.18(c). Then

$$\widehat{F} := q(G), \qquad f := n^{\mathrm{op}}, \qquad \widehat{m} := (q(G)_{\mathrm{id}} \le F_{m^\downarrow}) \tag{5.1}$$

satisfy the requirements of Definition 5.2.

Accordingly, the objects and morphisms of colim $\mathscr{F}$ are given as in Proposition 5.6, with the relation $\sim$ generated by $n \sim \mathscr{F}(m)(n)$ for every morphism $m : F \to G$ of $\mathcal{F}(\mathscr{A})$

and every morphism $n : G' \to G''$ of $\mathcal{F}(\mathscr{A}[G])$ and, accordingly, the relation $\approx$ generated by $G' \approx \mathscr{F}(m)(G')$ for all morphisms $(m : F \to G) \in \mathrm{Mor}(\mathcal{F}(\mathscr{A}))$ and all $G' \in \mathrm{Ob}(\mathcal{F}(\mathscr{A}[G]))$. For the sake of notational transparency we will omit explicit reference to $\sim$ and $\approx$ and denote equivalence classes with respect to these equivalence relations simply by $[\![\cdot]\!]$, to avoid confusion with the square brackets used to identify elements of the Salvetti complex.

We prove the lemma by constructing an isomorphism $\Phi : \mathcal{F}(\mathscr{A}) \to \mathrm{colim}\,\mathscr{F}$ as follows. For every object $F \in \mathcal{F}(\mathscr{A})$ define $\Phi(F) := [\![F_{\mathrm{id}}]\!]$ (recall from Definition 4.15 that F_{id} is a face in $\mathcal{F}(\mathscr{A}[F])$), and for every morphism $m : F \to G$ in $\mathcal{F}(\mathscr{A})$ define

$$\Phi(m) := [\![F_{\mathrm{id}} \le F_m]\!].$$

The bijectivity of Φ is easily seen, so we only need to show the functoriality of Φ. To this end consider two composable morphisms $F \xrightarrow{m} G \xrightarrow{n} H$. Using Lemma 4.18(a) we get

$$\begin{aligned}\Phi(n) \circ \Phi(m) &= [\![G_{\mathrm{id}} \le G_n]\!] \circ [\![F_{\mathrm{id}} \le F_m]\!] = [\![\mathscr{F}(m)(G_{\mathrm{id}} \le G_n)]\!] \circ [\![F_{\mathrm{id}} \le F_m]\!] \\ &= [\![i_m(G_{\mathrm{id}}) \le i_m(G_n)]\!] \circ [\![F_{\mathrm{id}} \le F_m]\!] \\ &= [\![F_m \le F_{n\circ m}]\!] \circ [\![F \le F_m]\!] = [\![F \le F_{n\circ m}]\!] = \Phi(n \circ m). \qquad \square\end{aligned}$$

Definition 5.9 (A diagram for the Salvetti category).

$$\begin{aligned}\mathscr{D}(\mathscr{A}) = \mathscr{D} : \mathcal{F}(\mathscr{A})^{\mathrm{op}} &\to \boldsymbol{AC}, \\ F &\mapsto \mathrm{Sal}(\mathcal{A}[F]), \\ (m : F \to G) &\mapsto j_m : \mathrm{Sal}(\mathscr{A}[G]) \hookrightarrow \mathrm{Sal}(\mathscr{A}[F]),\end{aligned}$$

where $j_m([G, C]) = [i_m(G), i_m(C)]$.

Lemma 5.10.

$$\mathrm{colim}\,\mathscr{D}(\mathscr{A}) = \mathrm{Sal}(\mathscr{A}).$$

Proof. The proof follows the outline of the proof of Lemma 5.8, and starts by noticing that the diagram $\mathscr{D}$, too, is geometric. Indeed, let $x : [K_1, C_1] \le [K_2, C_2]$ be a morphism in $\mathrm{Sal}(\mathscr{A}[F])$. Correspondingly, we have morphisms $m_0 : K_2 \le K_1$, $m_1 : K_1 \le C_1$, $m_2 : K_2 \le C_2$ of $\mathcal{F}(\mathscr{A}[F])$. For $i = 0, 1, 2$ let $n_i, m_i^{\downarrow}$ be obtained from m_i as in Lemma 4.18(c). Then a straightforward check on the definitions shows that the assignment

$$\widehat{F} := q(K_2), \qquad f := n^{\mathrm{op}}, \qquad \widehat{x} : [(K_2)_{\mathrm{id}}, F_{m_2^{\downarrow}}] \le [F_{m_0^{\downarrow}}, F_{m_1^{\downarrow}\circ m_0^{\downarrow}}]$$

is well-defined and satisfies the requirement of Definition 5.2.

Thus Proposition 5.6 again applies and, accordingly, we write objects and morphisms of $\mathrm{colim}\,\mathscr{D}$ as equivalence classes of the appropriate relations, which we will again denote by $[\![\cdot]\!]$.

An isomorphism $\Psi : \mathrm{Sal}(\mathscr{A}) \to \mathrm{colim}\,\mathscr{D}$ can now be defined as follows. For an object $m : F \to C$ of $\mathrm{Sal}(\mathscr{A})$ define $\Psi(m) = [\![[F_{\mathrm{id}}, F_m]]\!]$ (notice that, considering m as a morphism of $\mathcal{F}(\mathscr{A})$, we have $\Psi(m) = \Phi(m)$). For a morphism (n, m_1, m_2) of $\mathrm{Sal}(\mathscr{A})$ with $m_i : F_i \to C_i$ and $n : F_2 \to F_1$ define

$$\begin{aligned}\Psi(n, m_1, m_2) &= [\![\mathscr{D}(n)([(F_1)_{\mathrm{id}}, F_{m_1}]) \leq [(F_2)_{\mathrm{id}}, F_{m_2}]]\!] \\ &= [\![[i_n((F_1)_{\mathrm{id}}), i_n(F_{m_1})] \leq [(F_2)_{\mathrm{id}}, F_{m_2}]]\!] = [\![[F_n, F_{m_1 \circ n}] \leq [(F_2)_{\mathrm{id}}, F_{m_2}]]\!],\end{aligned}$$

where in the last equality we have used Lemma 4.18(a). □

Remark 5.11. Using Remark 4.18(c) we find that every element $\varepsilon \in \mathrm{Ob}(\mathrm{colim}\,\mathscr{D}(\mathscr{A}))$ has a (unique) representative $[F_{\mathrm{id}}, C] \in \mathcal{S}(\mathscr{A}[F])$ such that for any other representative $[G, K]$ with $\varepsilon = [\![G', K]\!]$ there is a unique face $G \in \mathcal{F}(\mathscr{A})$ and a unique morphism $m : F \to G$ with $[G', K] = [F_m, i_m(C)]$.

Lemma 5.12. *Let $m : F \to G$ be a morphism of $\mathcal{F}(\mathscr{A})$ and consider a $(Y, C) \in \mathscr{Y}_F$. Then the inclusion $j_m : \mathrm{Sal}(\mathscr{A}[G]) \to \mathrm{Sal}(\mathscr{A}[F])$ restricts to an inclusion*

$$j_m : \mathcal{S}_{\xi_G(Y,C)} \to \mathcal{S}_{\xi_F(Y,C)}.$$

Remark 5.13. Note that, given any chamber C of $\mathscr{A}[G]$ and any chamber C' of $\mathscr{A}[F]$, there is a natural inclusion $\mathcal{S}(\mathscr{A}[G])_C \hookrightarrow \mathcal{S}(\mathscr{A}[F])_{C'} \subseteq \mathcal{S}(\mathscr{A}[F])$ if and only if $S(i_m(C), C') \cap \mathscr{A}[G] = \emptyset$.

Proof of Lemma 5.12. With Remark 5.13 we only need to show that

$$S(i_m(\xi_G(Y, C)), \xi_F(Y, C)) \cap \mathscr{A}[G] = \emptyset.$$

Let $H \in \mathscr{A}[G]$. Then

$$\gamma_{i_m(\xi_G(Y,C))}(H) = \gamma_{\xi_G(Y,C)}(H) = \gamma_{\xi_F(Y,C)}(H), \quad \text{so} \quad H \notin S(i_m(\xi_G(Y, C)), \xi_F(Y, C)),$$

where the last equality follows from the fact that $\xi_F(Y, C) \subseteq \xi_G(Y, C)$. □

Lemma 5.12 allows us to state the following definition.

Definition 5.14. Given $(Y, C) \in \mathscr{Y}$ let

$$\begin{aligned}\mathscr{E}_{(Y,C)} : \mathcal{F}(\mathscr{A}^Y)^{\mathrm{op}} &\to \boldsymbol{AC}, \\ F &\mapsto \mathcal{S}(\mathscr{A}[F])_{\xi_F(Y,C)}, \\ (m : F \to G) &\mapsto (j_m)_{|\mathscr{E}_{(Y,C)}(G)}.\end{aligned}$$

Lemma 5.15. *Let $(Y, C) \in \mathscr{Y}$, then*

$$\mathrm{colim}\,\mathscr{E}_{(Y,C)} = \mathcal{S}_{(Y,C)}.$$

Proof. We consider the isomorphism $\Psi : \mathrm{Sal}(\mathscr{A}) \to \mathrm{colim}\,\mathscr{D}$ of Lemma 5.10. We want to show that $\Psi(\mathcal{S}_{(Y,C)}) = \mathrm{colim}\,\mathscr{E}_{(Y,C)}$, and we do this in two steps.

Step 1: $\operatorname{colim} \mathscr{E}_{(Y,C)} \subseteq \Psi(\mathcal{S}_{(Y,C)})$. Let $[\![G, K]\!] \in \operatorname{colim} \mathscr{E}_{(Y,C)}$. Then (recall Remark 5.11) there is a morphism $m : F \to G$ of $\mathcal{F}(\mathscr{A})$ such that $[F_m, i_m(K)] \in \mathcal{S}_{\xi_F(Y,C)} \subseteq \operatorname{Sal}(\mathscr{A}[F])$, i.e.

$$[F_m, i_m(K)] \le [F_{\mathrm{id}}, \xi_F(Y, C)].$$

Taking the preimage under Ψ of this relation we get a morphism

$$\Psi^{-1}([\![G, K]\!]) \to \Psi^{-1}([\![F_{\mathrm{id}}, \xi_F(Y, C)]\!]) \in \operatorname{Mor}(\operatorname{Sal}(\mathscr{A})).$$

Now, using Proposition 4.10 we have

$$\begin{aligned}\theta(\Psi^{-1}([\![F_{\mathrm{id}}, \xi_F(Y, C)]\!])) &= (X(F, \xi_F(Y, C)), \sigma_{\mathscr{A}[Y]}\xi_F(Y, C)) \\ &= (Y, \sigma_{\mathscr{A}[Y]} \circ \mu[\mathscr{A}[Y], \mathscr{A}[F]](C)) = (Y, C).\end{aligned}$$

Therefore $\Psi^{-1}([\![G, K]\!]) \in \mathcal{S}_{(Y,C)}$, so $[\![G, K]\!] \in \Psi(\mathcal{S}_{(Y,C)})$, as was to be proved.

Step 2: $\Psi(\mathcal{S}_{(Y,C)}) \subseteq \operatorname{colim} \mathscr{E}_{(Y,C)}$. Consider now $(m : G \to K) \in \mathcal{S}_{(Y,C)}$. Then there is a morphism $(h, m, n) : m \to n \in \operatorname{Mor}(\operatorname{Sal}(\mathscr{A}))$ with $n : F \to K'$, $h : F \to G$ and $\theta(n) = (Y, C)$. In particular, in view of Remark 4.20, we get $F_n = \xi_F(\theta(n)) = \xi_F(Y, C)$.

Applying Ψ to the morphism (h, m, n), in $\operatorname{Sal}(\mathscr{A}[F])$ we obtain

$$j_n([G, G_m]) \le [F, F_n] = [F, \xi_F(Y, C)], \quad \text{thus} \quad j_n([G, G_m]) \in \mathcal{S}_{\xi_F(Y,C)},$$

and we conclude that

$$\Psi(m) = [\![G, G_m]\!] = [\![j_n([G, G_m])]\!] \in \operatorname{colim} \mathscr{E}_{(Y,C)},$$

as required. □

Definition 5.16. Given $(Y, C) \in \mathscr{Y}$, define

$$\begin{aligned}\mathscr{G}_{(Y,C)} : \mathcal{F}(\mathscr{A}^Y)^{\mathrm{op}} &\to \boldsymbol{AC}, \\ F &\mapsto \mathcal{N}_{\xi_F(Y,C)}, \\ (m : F \to G) &\mapsto (j_m)_{|\mathscr{G}_{(Y,C)}(G)}.\end{aligned}$$

Remark 5.17. To prove that the diagram $\mathscr{G}_{(Y,C)}$ is well defined, we have to show that for every morphism $m : F \to G$ of $\mathcal{F}(\mathscr{A}^Y)$,

$$j_m(\mathcal{N}_{\xi_G(Y,C)}) \subseteq \mathcal{N}_{\xi_F(Y,C)}. \tag{5.2}$$

This follows because by Proposition 4.10 we have $X(F, \xi_F(Y, C)) = Y$, and thus with [12, Lemma 4.18] we can rewrite

$$\mathcal{N}_{\xi_F(Y,C)} = \{[G, K] \in \operatorname{Sal}(\mathscr{A}[F]) \mid G \in \mathcal{F}(\mathscr{A}[F]^{\widetilde{Y}}),\ K_G = \xi_F(Y, C)_G\}.$$

Now let $[G', C'] \in \mathcal{N}_{\xi_G(Y,C)}$. Then since $G' \subseteq \widetilde{Y}$ we have $i_m(G') \in \mathcal{F}(\mathscr{A}[F]^{\widetilde{Y}})$, and from $\xi_F(Y, C) \subseteq \xi_G(Y, C)$ we conclude $i_m(C')_{G'} = \xi_F(Y, C)_{G'}$. Therefore $j_m([G', C']) = [i_m(G'), i_m(C')] \in \mathcal{N}_{\xi_F(Y,C)}$, and the inclusion (5.2) is proved.

Lemma 5.18.

$$\operatorname{colim} \mathscr{G}_{(Y,C)} = \mathcal{N}_{(Y,C)}.$$

Proof. Again the proof is in two steps.

Step 1: $\operatorname{colim} \mathscr{G}_{(Y,C)} \subseteq \mathcal{N}_{(Y,C)}$. Let $[\![F, K]\!] \in \operatorname{colim} \mathscr{G}_{(Y,C)}$ and suppose $[\![F, K]\!] \notin \mathcal{N}_{(Y,C)}$. Then $[\![F, K]\!] \in \operatorname{colim} \mathscr{E}_{(Y',C')}$ for some $(Y', C') < (Y, C)$. As $[\![F, K]\!] \in \operatorname{colim} \mathscr{G}_{(Y,C)}$, there exist a point $P \in \mathcal{F}(\mathscr{A})$ and a morphism $m : P \to F$ with $[P_m, i_m(K)] \in \mathcal{N}_{\xi_P(Y,C)}$. Therefore, in $\mathscr{A}[P]$ we have $[P_m, i_m(K)] \leq [P, \xi_P(Y, C)]$, which implies that $K_{P_m} = \xi_P(Y, C)_{P_m}$, and thus $K = \sigma_{\mathscr{A}[F]}(K_{P_m}) = \xi_F(Y, C)$.

Similarly, since $[\![F, K]\!] \in \operatorname{colim} \mathscr{E}_{(Y',C')}$ there is a point $Q \in \mathcal{F}(\mathscr{A})$ and a morphism $n : Q \to F$ with $[Q_n, i_n(K)] \in \mathcal{S}_{\xi_Q(Y',C')}$. Then, as above, $K = \xi_F(Y', C')$.

From the bijectivity proven in Lemma 4.11 we conclude that $(Y, C) = (Y', C')$, which contradicts $(Y', C') < (Y, C)$, proving that $[\![F, K]\!] \in \mathcal{N}_{(Y,C)}$, as desired.

Step 2: $\mathcal{N}_{(Y,C)} \subseteq \operatorname{colim} \mathscr{G}_{(Y,C)}$. Suppose that $[F, K] \in \mathcal{N}_{(Y,C)} \setminus \operatorname{colim} \mathscr{G}_{(Y,C)}$. Then $[F, K] \in \mathcal{S}_{\xi_P(Y',C')}$ for some $P \in \mathcal{F}(\mathscr{A})$ and some $(Y', C') < (Y, C)$. But then we have $[F, K] \in \operatorname{colim} \mathscr{E}_{(Y',C')}$, thus $[F, K] \notin \mathcal{N}_{(Y,C)}$. □

Lemma 5.19. *There is an equivalence of diagrams*

$$\mathscr{G}_{(Y,C)} \cong \mathscr{F}(\mathscr{A}^Y)^{\mathrm{op}}.$$

Proof. For each $F \in \mathcal{F}(\mathscr{A}^Y)$ define isomorphisms $\mathscr{G}_{(Y,C)}(F) \to \mathscr{F}(\mathscr{A}^Y)^{\mathrm{op}}(F)$ as follows:

$$\mathscr{G}_{(Y,C)}(F) = \mathcal{N}_{\xi_F(Y,C)} \cong \mathcal{F}(\mathscr{A}[F]^{\widetilde{Y}})^{\mathrm{op}} = \mathcal{F}(\mathscr{A}^Y[F])^{\mathrm{op}} = \mathscr{F}(\mathscr{A}^Y)^{\mathrm{op}}(F),$$

where the isomorphism in the middle comes from Theorem 1.35. It can be easily checked that these isomorphisms indeed induce morphisms of diagrams. □

Proof of Theorem 5.1. As a consequence of Lemma 5.19,

$$\mathcal{N}_{(Y,C)} = \operatorname{colim} \mathscr{G}_{(Y,C)} \cong \operatorname{colim} \mathscr{F}(\mathscr{A}^Y)^{\mathrm{op}} = \mathcal{F}(\mathscr{A}^Y)^{\mathrm{op}}.$$ □

6. Minimality of toric arrangements

In this section we will construct a perfect acyclic matching of the Salvetti category of a complexified toric arrangement. By Remark 3.9 this will imply minimality and, with it, torsion-freeness of the arrangement's complement.

6.1. *Perfect matchings for the compact torus*

Let $\mathscr{A}$ be a complexified toric arrangement in T_Λ and recall the notation of Section 2.1. Choose a point $P \in \max \mathcal{C}(\mathscr{A})$. Up to a biholomorphic transformation we may suppose that P is the origin of the torus.

Let then $(\chi_1, a_1), \dots, (\chi_d, a_d) \in \mathscr{A}$ be such that $\alpha_1, \dots, \alpha_d$ are ($\mathbb{Q}$-) linearly independent and $P \in K_i$ for all $i = 1, \dots, d$. For $i = 1, \dots, d$ let H_i^1 denote the hyperplane of $\mathscr{A}^{\upharpoonright}$ lifting K_i at the origin of $\mathrm{Hom}(\Lambda, \mathbb{R}) \cong \mathbb{R}^d$. For ease of notation we identify $\Lambda \cong \mathbb{Z}^d \subseteq \mathbb{R}^d$, and in particular think of α_i as the normal vector to H_i^1.

For $j \in [d] := \{1, \dots, d\}$ we consider the rank $j - 1$ lattice

$$\Lambda_j := \mathbb{Z}^d \cap \bigcap_{i \geq j} H_i^1.$$

Lemma 6.1. *There is a basis $u_1, \dots, u_d$ of Λ such that for all $i = 1, \dots, d$, the elements $u_1, \dots, u_{i-1}$ are a basis of Λ_i.*

Proof. The proof is by repeated application of the Invariant Factor Theorem (e.g. [4, Theorem 16.18]) to the free $\mathbb{Z}$-submodule Λ_j of Λ_{j-1}. □

Let $(H_i^1)^+ := \{x \in \mathbb{R}^d \mid \langle x, \alpha_i \rangle \geq 0\}$.

Remark 6.2. In particular, $u_i \notin H_i^1$, hence $u_i(H_i^1) \neq H_i^1$. Moreover, without loss of generality we may suppose $u_i \in (H_i^1)^+$.

The lattice Λ acts on $\mathbb{R}^d$ by translations. Given $u \in \Lambda$, let the corresponding translation be

$$t_u : \mathbb{R}^d \to \mathbb{R}^d. \quad x \mapsto t_u(x) := x + u.$$

Corollary 6.3. *For all $x \in \mathbb{R}^d$ and all $i < j \in [d]$, $\langle t_{u_i}(x), \alpha_{d-j} \rangle = \langle x, \alpha_{d-j} \rangle$.*

Proof. We have $u_i \in \Lambda_j \subseteq H_{d-j}^1$, therefore $\langle u_i, \alpha_{d-j} \rangle = 0$ and thus

$$\langle t_{u_i}(x), \alpha_{d-j} \rangle = \langle x + u_i, \alpha_{d-j} \rangle = \langle x, \alpha_{d-j} \rangle + \langle u_i, \alpha_{d-j} \rangle = \langle x, \alpha_{d-j} \rangle + 0. \qquad \square$$

For $i = 1, \dots, d$ let $(H_i^2)^+ := t_{u_i}((H_i^1)^+)$, and define

$$Q := \bigcap_{i=1}^{d} [(H_i^1)^+ \setminus (H_i^2)^+].$$

Lemma 6.4. *The region Q is a fundamental region for the action of Λ on $\mathbb{R}^d$.*

Proof. For $i = 1, \dots, d$, write

$$l_i := \langle u_i, \alpha_i \rangle.$$

Then $Q = \{x \in \mathbb{R}^d \mid 0 \leq \langle x, \alpha_i \rangle < l_i \text{ for all } i = 1, \dots, d\}$. It is clear that Q can contain at most one point for each orbit of the action of Λ.

Now fix an $x \in \mathbb{R}^d$. We want to construct a $y \in Q$ such that $x \in y + \Lambda$.

To this end write $x_0 := x$ and let $\lambda_d := \lfloor \langle x_0, \alpha_d \rangle / l_d \rfloor$. Then let

$$x_1 := x_0 - \lambda_d u_d, \quad \text{thus} \quad 0 \leq \langle x_1, \alpha_d \rangle < l_d.$$

For every $i \in \{1, \ldots, d-1\}$ define now recursively $\lambda_{d-i} := \lfloor \langle x_i, \alpha_{d-i} \rangle / l_{d-i} \rfloor$ and $x_{i+1} := x_i - \lambda_{d-i} u_{d-i}$, so that

$$0 \leq \langle x_{i+1}, \alpha_{d-i} \rangle < l_{d-i}$$

and so, by Corollary 6.3, for every $j < i$,

$$\langle x_{i+1}, \alpha_{d-j} \rangle = \langle t_{u_{d-i}}^{-\lambda_{d-i}} \cdots t_{u_{d-j-1}}^{-\lambda_{d-j-1}}(x_{j+1}), \alpha_{d-j} \rangle = \langle x_{j+1}, \alpha_{d-j} \rangle \in [0, l_{d-j}[.$$

After d steps, we will have reached x_d with

$$0 \leq \langle x_d, \alpha_i \rangle < l_i \qquad \text{for all } i = 1, \ldots, d.$$

Hence $y := x_d \in Q$ is the required point because, setting $u := \sum_{i=1}^{d} \lambda_i u_i$, we have by construction $x_d = t_{-u}(x)$ and so $x = t_u(y) \in y + \Lambda$. □

Definition 6.5. Let $\mathscr{A}$ be a rank d toric arrangement, and let $\mathcal{B}_d$ be the 'Boolean poset on d elements', i.e., the acyclic category of the subsets of $[d]$ with the inclusion morphisms. Since $\mathcal{B}_d$ is a poset, the function

$$\mathrm{Ob}(\mathcal{F}(\mathscr{A})) \to \mathrm{Ob}(\mathcal{B}_d), \qquad F \mapsto \{i \in [d] \mid F \subseteq K_i\},$$

induces a well-defined functor of acyclic categories

$$\mathcal{I} : \mathcal{F}(\mathscr{A}) \to \mathcal{B}_d^{\mathrm{op}}.$$

For every $I \subseteq [d]$ define the category

$$\mathcal{F}_I := \mathcal{I}^{-1}(I).$$

Our main technical result about the category $\mathcal{F}_I$ is the following.

Lemma 6.6. *For all $I \subseteq [d]$, the subcategory $\mathcal{F}_I$ is a poset admitting an acyclic matching with only one critical element* (*in top rank*).

We postpone the proof of this lemma after some preparatory steps. Fix $I \subset [d]$, and let $k := |I|$.

We consider

$$Q_I := Q \cap \Big(\bigcap_{i \in I} H_i^1\Big) \setminus \bigcup_{j \notin I} (H_j^1 \cup H_j^2).$$

The set $\mathscr{B} := \{H \cap X \mid H \in \mathscr{A}^{\upharpoonright}, H \cap Q \neq \emptyset\}$ is a finite arrangement of affine hyperplanes in the affine hull X of Q_I. This arrangement determines a (regular) polyhedral decomposition $\mathcal{D}(\mathscr{B})$ of $\mathbb{R}^{d-k}$ that coincides with $\mathcal{D}(\mathscr{A}_X^{\upharpoonright})$ on Q.

The covering of Section 2.2.2 maps Q_I homeomorphically to its image, hence $\mathcal{F}_I$ is the face category of the set of cells of the decomposition of Q_I by $\mathcal{D}(\mathscr{B})$. Regularity of $\mathcal{D}(\mathscr{B})$ implies that $\mathcal{F}_I$ is a poset. Indeed, if $\mathcal{D}(\mathscr{B})^{\vee}$ is the (regular) CW-decomposition dual to the one induced by $\mathscr{B}$, then $\mathcal{F}_I^{\mathrm{op}}$ is the poset of cells of Y_I (subcomplex of $\mathcal{D}(\mathscr{B})$) that is entirely contained in Q_I.

Let $\mathcal{Q}$ be the subdivision induced by $\mathscr{B}$ on the closure $\overline{Q_I}$.

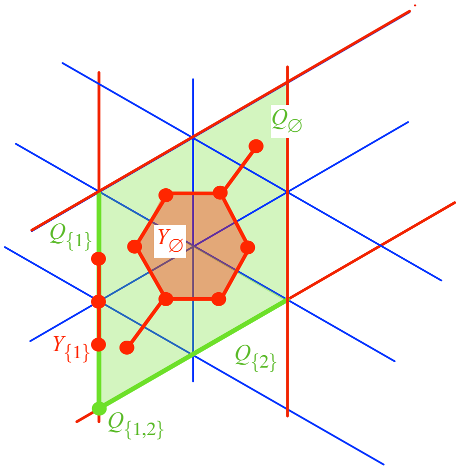


Fig. 5. The case of the toric Weyl arrangement of type A_2.

Lemma 6.7. *The complex $\mathcal{Q}$ is shellable.*

Proof. Coning the arrangement $\mathscr{B}$ (as in [26, Definition 1.15]) we obtain a central arrangement $\widehat{\mathscr{B}} = \{\widehat{H} \mid H \in \mathscr{B}\}$ which subdivides the unit sphere into a regular cell complex $\mathcal{K}$. Then $\mathcal{Q}$ is isomorphic to the subcomplex of $\mathcal{K}$ given by

$$\bigcap_{i \notin I} (\widehat{H_i^1})^+ \cap \bigcap_{i \notin I} (\widehat{H_i^2})^-,$$

which, by [2, Proposition 4.2.6(c)], is shellable. □

Proof of Lemma 6.6. The pseudomanifold $\mathcal{Q}$ is constructible because it is shellable. By [1, Theorem 4.1], it is also endo-collapsible, i.e., it admits an acyclic matching where the critical cells are precisely the cells on the boundary plus one single cell in the interior of $\mathcal{Q}$. But this restricts to an acyclic matching of the subposet $\mathcal{F}_I \subseteq \mathcal{F}(\mathcal{Q})$ with exacly one critical cell.

In turn, this gives an acyclic matching of $\mathcal{F}_I^{\mathrm{op}}$ with exactly one critical cell. Since $\mathcal{F}_I^{\mathrm{op}}$ is the face poset of the CW-complex Y_I, the critical cell must be in bottom rank—thus in top rank of $\mathcal{F}_I$, as required. □

Proposition 6.8. *For any complexified toric arrangement $\mathscr{A}$, the acyclic category $\mathcal{F}(\mathscr{A})$ admits a perfect acyclic matching.*

Proof. Let $\mathscr{A}$ be of rank d. The proof is a straightforward application of the Patchwork Lemma 3.7 in order to merge the 2^d acyclic matchings described in Lemma 6.6 along the map $\mathcal{I}$ of Definition 6.5. The resulting 'global' acyclic matching has 2^d critical elements and is thus perfect. □

6.2. Perfect matchings for the toric Salvetti complex

Let $\mathscr{A}$ be a (complexified) toric arrangement.

Proposition 6.9. *The Salvetti category* $\mathrm{Sal}(\mathscr{A})$ *admits a perfect acyclic matching.*

Proof. Let the set $\mathcal{Y}$ be totally ordered according to Definition 4.7. Let P denote the acyclic category given by the $|\mathscr{Y}|$-chain. We define a functor of acyclic categories

$$\varphi : \mathrm{Sal}(\mathscr{A}) \to P, \qquad m \mapsto (Y, C) \text{ for } m \in \mathcal{N}_{(Y,C)},$$

and by Theorem 5.1 we have an isomorphism of acyclic categories $\varphi^{-1}((Y,C)) = \mathcal{N}_{(Y,C)} \cong \mathcal{F}(\mathscr{A}^Y)^{\mathrm{op}}$. Then, by Proposition 6.8, $\varphi^{-1}((Y,C))$ has an acyclic matching with $2^{d-\mathrm{rk}\,X}$ critical cells.

An application of the Patchwork Lemma 3.7 yields an acyclic matching on $\mathrm{Sal}(\mathscr{A})$ with

$$\sum_j |\mathscr{Y}_j| 2^{d-j} = \sum_j |\mathscr{N}_j| 2^{d-j} = P_{\mathscr{A}}(1)$$

critical cells, where the first equality is given by Lemma 4.6. This matching is thus perfect. □

Corollary 6.10. *The complement* $M(\mathscr{A})$ *is a minimal space.*

Proof. The cellular collapses given by the acyclic matching of Proposition 6.9 show that the complement $M(\mathscr{A})$ is homotopy equivalent to a complex whose cells are counted by the Betti numbers. □

Corollary 6.11. *The homology and cohomology groups* $H_k(M(\mathscr{A}); \mathbb{Z})$*,* $H^k(M(\mathscr{A}); \mathbb{Z})$ *are torsion-free for all* k*.*

Proof. See Corollary 1.20. □

7. Application: minimality of affine arrangements

After the existence proofs of Dimca and Papadima [14] and Randell [27], the first step towards an explicit characterization of the minimal model for complements of hyperplane arrangements was taken by Yoshinaga [32] who, for complexified arrangements, identified the cells of the minimal complex using their incidence with a general position flag in real space and studied their boundary maps. Salvetti and Settepanella [31] obtained a complete description of the minimal complex by using a 'polar ordering' determined by a general position flag to define a perfect acyclic matching on the Salvetti complex.

In this section we explain how to use our techniques in order to extend the idea of [12] to affine complexified hyperplane arrangements. We thus obtain a minimal complex that is defined only in terms of the arrangement's (affine) oriented matroid and is less cumbersome than the one described in [13].

Consider a finite affine complexified arrangement $\mathscr{A} = \{K_1, \dots, K_n\}$. Define the central arrangements $\mathscr{A}_0$ and $\mathscr{A}[F]$ for $F \in \mathcal{F}(\mathscr{A})$ in analogy to those of Section 4.1. Choose a base chamber $B \in \mathcal{T}(\mathscr{A}_0)$, fix a total ordering $\prec_0$ on $\mathscr{A}_0$ and define $\prec_F$, $\prec_Y$ for $F \in \mathcal{F}(\mathscr{A})$, $Y \in \mathcal{L}(\mathscr{A})$ as in Section 4.1. Moreover, let $\mathscr{Y}$ be as in Definition 4.4.

Remark 7.1. Notice that, given the affine oriented matroid of $\mathscr{A}$, the oriented matroid of $\mathscr{A}_0$ can be recovered without referring to geometry. For instance, the tope poset of $\mathscr{A}_0$ can be defined in terms of the tope poset of $\mathscr{A}$ based at any unbounded chamber (see [2] for terminology and basics on oriented matroids).

Lemma 7.2. *Let $\mathscr{A}$ be a finite complexified affine hyperplane arrangement, and $\mathscr{Y}$ as above. Then*

$$|\mathscr{Y}| = \sum_{k\in\mathbb{N}} \operatorname{rk} H^k(M(\mathscr{A}); \mathbb{Z}).$$

Proof. As in Lemma 4.6, applying [12, Lemma 4.18 and Proposition 2], for all $Y \in \mathcal{L}(\mathscr{A})$ we have

$$|\{C \in \mathcal{T}(\mathscr{A}[Y]) \mid X(Y, C) = Y\}| = \operatorname{rk} H^{\operatorname{codim} Y}(M(\mathscr{A}_Y); \mathbb{Z}).$$

The claim follows from Theorem 1.21. □

We now define the analogue of the map θ of Definition 4.19.

Definition 7.3. Let $F, G \in \mathcal{F}(\mathscr{A})$ with $F \leq G$ and identify

$$\mathscr{A}[F] = \mathscr{A}_F = \{H \in \mathscr{A} \mid F \subseteq H\},$$

in particular we have an inclusion $\mathscr{A}[G] \subseteq \mathscr{A}[F]$ and, correspondingly, a function $i_{F\leq G} : \mathcal{F}(\mathscr{A}[G]) \to \mathcal{F}(\mathscr{A}[F])$ as in Definition 4.15, which induces a function $j_{F\leq G}$: $\operatorname{Sal}(\mathscr{A}[F]) \to \operatorname{Sal}(\mathscr{A}[G])$ as in Definition 5.9.

Theorem 7.4 ([11, Lemma 3.2.8 and Theorem 4.2.1]). *The assignment $\mathscr{E} : \mathcal{F}(\mathscr{A}) \to \boldsymbol{AC}^{\mathrm{op}}$, $\mathscr{E}(F) := \operatorname{Sal}(\mathscr{A}[F])$, $\mathscr{E}(F \leq G) := j_{F\leq G}$, defines a diagram of posets such that* $\operatorname{colim}\mathscr{E}$ *is poset isomorphic to* $\operatorname{Sal}(\mathscr{A})$.

The stratification of $\operatorname{Sal}(\mathscr{A})$ is also defined along the lines of the preceding sections.

Definition 7.5. Define the map $\theta : \operatorname{Sal}(\mathscr{A}) \to \mathscr{Y}$ as follows:

$$\theta([F, C]) := (X(F, i_{F\leq G}(G)), \sigma_{\mathscr{A}[X(F, i_{F\leq G}(G))]}(G)),$$

where we have identified $G = \min \mathcal{L}(\mathscr{A}[G])$.

Definition 7.6. Let $\mathscr{A}$ be a finite complexified affine hyperplane arrangement and define a total ordering $\dashv$ on $\mathscr{Y}$ as in Definition 4.7. Define

$$\mathcal{S}_{(Y,C)} := \left\{[F, C] \in \operatorname{Sal}(\mathscr{A}) \middle| \begin{array}{c}\text{there is } [G, K] \in \operatorname{Sal}(\mathscr{A}) \text{ with} \\ [F, C] \leq [G, K] \text{ and } \theta([G, K]) = (Y, C)\end{array}\right\},$$

$$\mathcal{N}_{(Y,C)} := \mathcal{S}_{(Y,C)} \setminus \bigcup_{(Y',C')\dashv(Y,C)} \mathcal{S}_{(Y',C')}.$$

The arguments of Section 5 can now be adapted to the affine case, giving the following analogue of Theorem 5.1.

Theorem 7.7. *Let $\mathscr{A}$ be a finite complexified affine hyperplane arrangement. There is an isomorphism of posets*

$$\mathcal{N}_{(Y,C)} \cong \mathcal{F}(\mathscr{A}^Y)^{\mathrm{op}} \quad \textit{for all } (Y,C) \in \mathscr{Y}.$$

The analogue of Proposition 6.8 is proved in [2, Theorem 4.5.7 and Corollary 4.5.8], from which it follows that the poset $\mathcal{N}_{(Y,C)}^{\mathrm{op}}$ is shellable, and therefore $\mathcal{N}_{(Y,C)}$ admits an acyclic matching with one critical cell in top dimension. Applying the Patchwork Lemma as in Proposition 6.9 we obtain a perfect acyclic matching $\mathfrak{M}$ of $\mathrm{Sal}(\mathscr{A})$. We summarize:

Proposition 7.8. *Let $\mathscr{A}$ be a finite complexified affine hyperplane arrangement. The (affine) oriented matroid data of $\mathscr{A}$ intrinsically define a discrete Morse function on* $\mathrm{Sal}(\mathscr{A})$ *that collapses the Salvetti complex to a minimal complex.*

Remark 7.9. The considerations of this section carry over to the general case of non-stretchable affine oriented matroids, as in [12] for the nonaffine case (compare Remark 7.1).

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