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De Rham cohomology and homotopy Frobenius manifolds

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Abstract. We endow the de Rham cohomology of any Poisson or Jacobi manifold with a natural homotopy Frobenius manifold structure. This result relies on a minimal model theorem for multicomplexes and a new kind of a Hodge degeneration condition.

Keywords. De Rham cohomology, homotopy Frobenius manifold, Poisson/Jacobi/contact manifold, multicomplex, Batalin–Vilkovisky algebra

Introduction

The de Rham forms of a Poisson algebra (A-side of the Mirror Symmetry conjecture) and the Dolbeault cochain complex of a Calabi–Yau algebra (B-side) carry a square-zero order 2 differential operator endowing them with a Batalin–Vilkovisky algebra structure (see [Kos85, Man99]).

A Frobenius manifold [Man99] is an algebraic structure that amounts to the operadic action of the homology of the Deligne–Mumford–Knudsen compactification of the moduli space of genus 0 curves $H_\bullet(\overline{\mathcal{M}}_{0,n+1})$. Motivated by ideas from string theory [BCOV94], Barannikov and Kontsevich showed in [BK98] that the Dolbeault cohomology of a Calabi–Yau manifold carries a natural Frobenius manifold structure (see also [Cos05, LS07]).

Using the methods of Barannikov and Kontsevich together with a result of Mathieu [Mat95], Merkulov [Mer98] endowed the de Rham cohomology of a symplectic manifold, satisfying the hard Lefschetz condition, with a natural Frobenius manifold structure.

Getzler [Get95] proved that the Koszul dual of the operad $H_\bullet(\overline{\mathcal{M}}_{0,n+1})$ is the cohomology of the moduli space of genus 0 curves $H^{\bullet+1}(\mathcal{M}_{0,n+1})$. Hence a coherent action

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of the latter spaces defines the notion of homotopy Frobenius manifold, with the required homotopy properties [DV12].

The purpose of this paper is to prove the following theorem.

Theorem (3.7, 4.5). *The de Rham cohomology of a Poisson manifold (respectively a Jacobi manifold) carries a natural homotopy Frobenius manifold structure, which extends the product induced by the wedge product and allows one to reconstruct the algebraic homotopy type of the de Rham complex.*

This theorem extends the previous results in three directions. First, it holds for any Poisson manifolds. Then, it provides us with higher geometrical invariants which faithfully encode the initial algebraic structure. Finally, it extends to Jacobi manifolds, including the example of contact manifolds.

To prove our main result, we develop further the homotopy theory of multicomplexes [Lap01, Mey78]. The notion of a multicomplex is a certain lift of the notion of a spectral sequence. We prove a minimal model theorem for multicomplexes, which amounts to a decomposition into a product of a minimal one and an acyclic trivial one.

Furthermore, we introduce a new condition, called gauge Hodge condition, which ensures the uniform vanishing of the induced BV-operator (and its higher homotopies) on the underlying homotopy groups. This gauge Hodge condition, suggested by the Givental action formalism [Cos05, DSV13, KMS13], gives a necessary and sufficient condition for the spectral sequence of a bicomplex to degenerate at the first page.

Retrospectively, one can interpret several homotopical results as gauge-type arguments. In particular, the operator $\Delta = J d_{\mathrm{DR}} J$ in complex geometry [DGMS75] and generalised complex geometry [Cav05, Gua11], once written as $-J d_{\mathrm{DR}} J^{-1}$, can be viewed as gauge equivalent to $-d_{\mathrm{DR}}$. Formulas ensuring the degeneration of appropriate spectral sequences for cyclic homology of Poisson manifolds [Pap00] and quantum de Rham cohomology of Poisson manifolds [Shu04] have a gauge symmetry flavour to them as well. Finally, the notion of gauge equivalence for Frobenius manifolds is studied in [CZ03], where it is used to prove that the construction of Barannikov and Kontsevich applied to two quasi-isomorphic dg BV-algebras yields two Frobenius manifold structures that can be identified with one another.

Convention

Throughout the text, we work over a field $\mathbb{K}$ of characteristic 0.

1. Homotopy theory of multicomplexes

Definition 1.1 (Mixed complex and multicomplex). A *mixed complex* (A, d, Δ) is a graded vector space A equipped with two linear operators d and Δ of respective degrees -1 and 1, satisfying

$$d^2 = \Delta^2 = d\Delta + \Delta d = 0.$$

A *multicomplex* $(A, d = \Delta_0, \Delta_1, \Delta_2, \ldots)$ is a graded vector space A endowed with a family of linear operators of respective degrees $|\Delta_n| = 2n - 1$ satisfying

$$\sum_{i=0}^{n} \Delta_i \Delta_{n-i} = 0 \quad \text{for } n \geq 0.$$

Since $d = \Delta_0$ squares to zero, (A, d) is a chain complex. We call the underlying homology groups $H(A, d)$ the *homotopy groups* of the multicomplex A. A *mixed complex* is a multicomplex where all the higher operators Δ_n vanish for $n \geq 2$.

Definition 1.2 (∞-morphism). An ∞*-morphism* $f : A \rightsquigarrow A'$ of multicomplexes is a family $\{f_n : A \to A'\}_{n \geq 0}$ of linear maps of respective degrees $|f_n| = 2n$ satisfying

$$\sum_{k+l=n} f_k \Delta_l = \sum_{k+l=n} \Delta'_k f_l \quad \text{for } n \geq 0.$$

The composite of two ∞-morphisms $f : A \rightsquigarrow A'$ and $g : A' \rightsquigarrow A''$ is given by

$$(gf)_n := \sum_{k+l=n} g_k f_l \quad \text{for } n \geq 0.$$

The associated category is denoted by ∞-multicomp.

Notice that $f_0 : (A, d) \to (A', d')$ is a chain map. When the first map f_0 is a quasi-isomorphism (respectively an isomorphism), the ∞-morphism f is called an ∞*-quasi-isomorphism* (respectively an ∞*-isomorphism*), and denoted $A \overset{\sim}{\rightsquigarrow} A'$ (respectively $A \overset{\cong}{\rightsquigarrow} A'$). The invertible morphisms of the category ∞-multicomp are the ∞-isomorphisms. An ∞-isomorphism whose first component is the identity map is called an ∞*-isotopy* and denoted $A \overset{=}{\rightsquigarrow} A'$.

Remark 1.3. Mixed complexes are differential graded modules over the free commutative algebra $D := S(\Delta)$ generated by a degree 1 element. Viewed as an associative algebra, it admits the quadratic presentation $D = T(\Delta)/(\Delta^2)$. This algebra is Koszul with Koszul dual coalgebra $D^{\text{¡}} = T^c(\delta)$, the cofree coalgebra on a degree 2 generator $\delta := s\Delta$.

Hence the notion of multicomplex is the notion of mixed complex *up to homotopy* according to Koszul duality theory: a multicomplex is a differential graded module over the cobar construction $\Omega D^{\text{¡}}$ of the Koszul dual coalgebra [LV12, Section 10.3.17].

In the same way, the above definition of ∞-morphisms coincides with the homotopy morphisms of the general theory [LV12, Section 10.2].

A *homotopy retract* consists of the following data:

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

where p is a chain map, where i is a quasi-isomorphism, and where h has degree 1, satisfying

$$ip - \mathrm{id}_A = d_A h + h d_A.$$

If moreover $pi = \mathrm{id}_H$, then it is called a *deformation retract*.

Proposition 1.4 (Homotopy Transfer Theorem [Lap01]). *Given a homotopy retract data between two chain complexes A and H, and a multicomplex structure $\{\Delta_n\}_{n\geq 1}$ on A, the following formulae define a multicomplex structure on H:*

$$\Delta'_n := \sum_{j_1+\cdots+j_k=n} p\Delta_{j_1}h\Delta_{j_2}h\ldots h\Delta_{j_k}i \quad \textit{for } n \geq 1, \tag{1}$$

an ∞-quasi-isomorphism $i_\infty = \{i_n\}_{n\geq 0}\colon H \overset{\sim}{\rightsquigarrow} A$ which extends the map i, where

$$i_n := \sum_{j_1+\cdots+j_k=n} h\Delta_{j_1}h\Delta_{j_2}h\ldots h\Delta_{j_k}i \quad \textit{for } n \geq 1,$$

and an ∞-quasi-isomorphism $p_\infty = \{p_n\}_{n\geq 0}\colon A \overset{\sim}{\rightsquigarrow} H$ which extends the map p, where

$$p_n := \sum_{j_1+\cdots+j_k=n} p\Delta_{j_1}h\Delta_{j_2}h\ldots h\Delta_{j_k}h \quad \textit{for } n \geq 1.$$

Proof. The proof is a straightforward computation. One can also prove it using the interpretation in terms of the Koszul duality theory of Remark 1.3. It is then a particular example of the general Homotopy Transfer Theorem of [LV12, Section 10.3]. □

Definition 1.5 (Hodge-to-de-Rham degeneration). Let $(A, d, \Delta_1, \Delta_2, \ldots)$ be a multicomplex. A *Hodge-to-de-Rham degeneration data* consists of a homotopy retract

$$h \circlearrowright (A, d) \underset{i}{\overset{p}{\rightleftarrows}} (H(A), 0),$$

satisfying

$$\sum_{j_1+\cdots+j_k=n} p\Delta_{j_1}h\Delta_{j_2}h\ldots h\Delta_{j_k}i = 0 \quad \text{for } n \geq 1.$$

This data amounts to the vanishing of all the transferred operators Δ'_n on the underlying homotopy groups of a multicomplex.

To any multicomplex $(A, \Delta_0, \Delta_1, \Delta_2, \ldots)$, one associates the following chain complex. Let $C_{p,q} := A_{p-q}$ and $\partial_r := \Delta_r : C_{p,q} \to C_{p-1+r,q-r}$. We consider the total complex $\widehat{\mathrm{Tot}}(C)_n := \prod_{p+q=n} C_{p,q}$, equipped with the differential $\partial := \sum_{r\geq 0} \partial_r$. (The degrees of the respective Δ_n ensure that ∂ has degree -1.) The row filtration F_n defined by considering the $C_{\bullet,k}$ for $k \leq -n$ provides us with a decreasing filtration of the total complex and thus with a spectral sequence $E^r(A)$.

Proposition 1.6 (Degeneration at page 1). *The spectral sequence $E^r(A)$ associated to a multicomplex $(A, d = \Delta_0, \Delta_1, \Delta_2, \ldots)$ degenerates at the first page if and only if there exists a Hodge-to-de-Rham degeneration data.*

Proof. If the differentials d^r vanish for $r \geq 1$, then $E^1 = E^2 = \cdots = H(A, d)$. In this case, the formulae [BT82, Chapter III] for the d^r coincide with the formulae defining the transferred Δ'_r. The other way round, one sees by induction from $r = 1$ that $E^r = H(A, d)$ and $d^r = \Delta'_r$. □

In the case of a mixed complex, $C_{\bullet,\bullet}$ is a bicomplex. So the Hodge-to-de-Rham condition is equivalent to degeneration of the usual bicomplex spectral sequence at the first page. This is the case for the classical Hodge-to-de-Rham spectral sequence of compact Kähler manifolds.

A multicomplex $(A, d = \Delta_0, \Delta_1, \Delta_2, \ldots)$ is called *minimal* when $d = \Delta_0 = 0$. It is called *acyclic* when the underlying chain complex (A, d) is acyclic, and it is called *trivial* when $\Delta_n = 0$ for $n \geq 1$.

Theorem 1.7 (Minimal model). *In the category* ∞-multicomp*, any multicomplex A is ∞-isomorphic to the product of a minimal multicomplex $H = H(A)$, given by the transferred structure, with an acyclic trivial multicomplex K.*

Proof. This theorem is a direct consequence of [LV12, Theorem 10.4.5] applied to the Koszul algebra D. More precisely, we consider a choice of representatives for the homology classes $H(A) \cong H \subset A$ and a complement $K \subset A$ of it. This decomposes the chain complex $A = H \oplus K$, where the differential on $H = H(A)$ is trivial and where the chain complex K is acyclic. Let us denote the respective projections by $p\colon A \twoheadrightarrow H$ and by $q\colon A \twoheadrightarrow K$. This induces the following homotopy retract:

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, 0).$$

Using formula (1) of Proposition 1.4, we endow H with the transferred multicomplex structure. So $(H, 0, \{\Delta'_n\}_{n\geq 1})$ is a minimal multicomplex and $(K, d_K, 0)$ is an acyclic trivial multicomplex. Their product in the category ∞-multicomp is given by $(H \oplus K, d_K, \{\Delta'_n\}_{n\geq 1})$. The projection q extends to an ∞-morphism q_∞ by $q_n := qh\Delta_n$ for $n \geq 1$. By the categorical property of the product, the maps p_∞ and q_∞ induce an ∞-isomorphism $r\colon A \rightsquigarrow H \oplus K$, explicitly given by $r_0 := p + q$ and

$$r_n := p_n + q_n = \sum_{j_1+\cdots+j_k=n} p\Delta_{j_1} h \Delta_{j_2} h \ldots h \Delta_{j_k} h + qh\Delta_n \quad \text{for } n \geq 1. \tag{2}$$

□

2. Gauge Hodge condition

We consider the algebra $\mathrm{End}(A)[[z]] := \mathrm{Hom}(A, A) \otimes \mathbb{K}[[z]]$ of formal power series with coefficients in the endomorphism algebra of A. One can view the ∞-endomorphisms of a multicomplex A as elements of $\mathrm{End}(A)[[z]]$. Under this interpretation, their composite corresponds to the product of the associated series.

Theorem 2.1. *A multicomplex* $(A, d, \Delta_1, \Delta_2, \ldots)$ *admits a Hodge-to-de-Rham degeneration data if and only if there exists an element* $R(z) := \sum_{n\geq 1} R_n z^n$ *in* $\mathrm{End}(A)[[z]]$ *satisfying*

$$e^{R(z)} d e^{-R(z)} = d + \Delta_1 z + \Delta_2 z^2 + \cdots . \tag{3}$$

Proof. The proof is built from the following three equivalences.

Step 1. We first prove that there exists a series $R(z) := \sum_{n\geq 1} R_n z^n \in \mathrm{End}(A)[[z]]$ satisfying condition (3) if and only if there exists an ∞-isotopy

$$(A, d, 0, \ldots) \overset{=}{\rightsquigarrow} (A, d, \Delta_1, \Delta_2, \ldots)$$

between A with trivial structure and A with its multicomplex structure.

Condition (3) is equivalent to $e^{R(z)} d = (d + \Delta_1 z + \Delta_2 z^2 + \cdots) e^{R(z)}$, which means that $e^{R(z)}$ is the required ∞-isotopy.

Step 2. Given a deformation retract for A onto its underlying homotopy groups $H(A)$, there exists an ∞-isotopy $(A, d, 0, \ldots) \overset{=}{\rightsquigarrow} (A, d, \Delta_1, \Delta_2, \ldots)$ if and only if there exists an ∞-isotopy

$$(H(A), 0, 0, \ldots) \overset{=}{\rightsquigarrow} (H(A), 0, \Delta'_1, \Delta'_2, \ldots).$$

The homotopy transfer theorem of Proposition 1.4 provides us with the following diagram in ∞-multicomp:

$$\begin{array}{ccc}
(A, d, 0, \ldots) & \overset{=}{\underset{\varphi}{\rightsquigarrow}} & (A, d, \Delta_1, \Delta_2, \ldots) \\
i \Big\uparrow \wr & & \wr \Big\downarrow p_\infty \\
(H(A), 0, 0, \ldots) & \overset{=}{\underset{\psi}{\rightsquigarrow}} & (H(A), 0, \Delta'_1, \Delta'_2, \ldots)
\end{array}$$

So given an ∞-isotopy φ, the composite $\psi := p_\infty \varphi i$ is an ∞-isotopy. Conversely, suppose we are given an ∞-isotopy ψ. The map $p + q : A \to H(A) \oplus K$ is a map of chain complexes, hence it is an ∞-isomorphism between these two trivial multicomplexes. Then the map $\psi + \mathrm{id}_K$ defined by $\mathrm{id}_H + \mathrm{id}_K$ for $n = 0$ and by ψ_n for $n \geq 1$ defines an ∞-isomorphism between $H(A) \oplus K$ with trivial multicomplex structure to $H(A) \oplus K$ with the transferred structure. Finally, we consider the inverse ∞-isomorphism $r^{-1} : H(A) \oplus K \overset{\cong}{\rightsquigarrow} A$ of the ∞-isomorphism r given at (2) in the proof of Theorem 1.7. The composite $r^{-1} (\psi + \mathrm{id}_K)(p + q)$ of these three maps provides us with the required ∞-isotopy.

Step 3. Let us now prove that an ∞-isotopy

$$(H(A), 0, 0, \ldots) \overset{=}{\rightsquigarrow} (H(A), 0, \Delta'_1, \Delta'_2, \ldots)$$

exists if and only if the (transferred) operators Δ'_n vanish for $n \geq 1$.

Let us denote by $f\colon (H(A),0,0,\ldots) \overset{=}{\rightsquigarrow} (H(A),0,\Delta'_1,\Delta'_2,\ldots)$ the given ∞-isotopy. The defining condition

$$\sum_{k+l=n} \Delta'_k f_l = \sum_{k+l=n} f_k \Delta_l = 0 \quad \text{for } n \geq 1$$

implies $\Delta'_n = 0$ for $n \geq 1$, by a direct induction. Conversely, the identity $\mathrm{id}_{H(A)}$ provides us the required ∞-isotopy. ◻

We define the *gauge Hodge condition* to be the existence of a series $R(z) \in \mathrm{End}(A)[[z]]$ satisfying the conjugation condition (3).

Remarks 2.2. ◇ Notice that this proof actually shows that, under the gauge Hodge condition, *every* deformation retract is a Hodge-to-de-Rham degeneration data. In this case, the transferred multicomplex structure vanishes uniformly, i.e. independently of the choices of representatives of the homotopy groups. This theorem solves a question raised at the end of [DSV13].

◇ When (A,d,Δ) is a mixed complex equipped with a Hodge-to-de-Rham degeneration data, the series $R(z)$ defined by the formula

$$R(z) := -\log\Big(1 - h\Delta z + \sum_{n\geq 1} ip(\Delta h)^n z^n\Big)$$

satisfies relation (3) of Theorem 2.1. Explicitly, $R(z) = \sum_{n\geq 1} r_n z^n$, where

$$r_n = \frac{(h\Delta)^n}{n} - n\sum_{l=1}^{n} \frac{(h\Delta)^{l-1} ip(\Delta h)^{n-l+1}}{l}.$$

◇ A BV-algebra equipped with a series $R(z) := \sum_{n\geq 1} R_n z^n$ satisfying (3) is called a *BV/Δ-algebra* in [KMS13], where this notion is studied in detail.

3. De Rham cohomology of Poisson manifolds

Definition 3.1 (Frobenius manifold, [Man99]). A (*formal*) *Frobenius manifold* is an algebra over the operad $H_\bullet(\overline{\mathcal{M}}_{0,n+1})$ made up of the homology of the Deligne–Mumford–Knudsen moduli spaces of stable genus 0 curves.

This algebraic structure amounts to giving a collection of symmetric multilinear maps $\mu_n\colon A^{\otimes n} \to A$, for $n \geq 2$, of degrees $|\mu_n| := 2(n-2)$ satisfying some quadratic relations (see [Man99] for instance). It is also called a *hypercommutative algebra* in the literature. (Notice that we do not require here any non-degenerate pairing, nor any unit).

The operad $H_\bullet(\overline{\mathcal{M}}_{0,n+1})$ is Koszul, with Koszul dual cooperad $H^{\bullet+1}(\mathcal{M}_{0,n+1})$, the cohomology groups of the moduli spaces of genus 0 curves. Algebras over the linear dual operad $H_\bullet(\mathcal{M}_{0,n+1})$ are called *gravity algebras* in the literature. The operadic cobar construction $\Omega H^\bullet(\mathcal{M}_{0,n+1}) \xrightarrow{\sim} H_\bullet(\overline{\mathcal{M}}_{0,n+1})$ provides a resolution of the former operad (see [Get95]).

Definition 3.2 (Homotopy Frobenius manifold). A *homotopy Frobenius manifold* is an algebra over the operad $\Omega H^\bullet(\mathcal{M}_{0,n+1})$.

The operations defining such a structure are parametrised by $H^\bullet(\mathcal{M}_{0,n+1})$. Hence, a homotopy Frobenius manifold structure on a chain complex with trivial differential is made up of an infinite sequence of strata of multilinear operations, whose first stratum forms a Frobenius manifold.

Definition 3.3 (dg BV-algebra). A *dg* BV-*algebra* $(A, d, \wedge, \Delta)$ is a differential graded commutative algebra equipped with a square-zero degree 1 operator Δ of order less than 2.

The data of a dg BV-algebra amounts to a mixed complex data (A, d, Δ) together with a compatible commutative product. We refer the reader to [LV12, Section 13.7] for more details on this notion.

To any homotopy Frobenius manifold H, we can associate a *rectified* dg BV-algebra $\mathrm{Rec}(H)$ (see [DV12, Section 6.3]).

Theorem 3.4 ([DV12]). *Let $(A, d, \wedge, \Delta)$ be a dg* BV*-algebra equipped with a Hodge-to-de-Rham degeneration data. The underlying homotopy groups $H(A, d)$ carry a homotopy Frobenius manifold structure whose rectified dg* BV*-algebra is homotopy equivalent to A.*

This result shows that the transferred homotopy Frobenius manifold faithfully encodes the homotopy type of the dg BV-algebra A. It provides a refinement of a result of Barannikov and Kontsevich [BK98], where only the underlying Frobenius manifold structure is considered. This first stratum of operations can be described in terms of sums of labelled graphs (see [LS07]).

Proposition 3.5 ([Kos85]). *Let (M, ω) be a Poisson manifold. Its de Rham complex $(\Omega^\bullet(M), d_{\mathrm{DR}}, \wedge, \Delta)$ is a dg* BV*-algebra, with the operator Δ defined by*

$$\Delta := i(\omega)d_{\mathrm{DR}} - d_{\mathrm{DR}}i(\omega) = [i(\omega), d_{\mathrm{DR}}],$$

where $i(-)\colon \Omega^\bullet(M) \to \Omega^{\bullet-2}(M)$ denotes the contraction operator.

In particular, $\Omega^\bullet(M)$ becomes a mixed complex, the *canonical double complex* of Brylinski [Bry88].

Koszul's proof of this result relies on the following relation between the contraction operators, the Schouten–Nijenhuis bracket, and the de Rham differential, which we shall use throughout the paper.

Proposition 3.6 ([Kos85]). *For every smooth manifold M, and any polyvector fields ω_1, ω_2,*

$$i([\omega_1, \omega_2]) = -[[i(\omega_2), d], i(\omega_1)].$$

Theorem 3.7. *The de Rham cohomology of a Poisson manifold* (M, ω) *carries a natural homotopy Frobenius manifold structure whose rectified dg* BV*-algebra is homotopy equivalent to* $(\Omega^\bullet(M), d_{\mathrm{DR}}, \wedge, \Delta)$.

Proof. The operators of Proposition 3.5 satisfy

$$[i(\omega), [i(\omega), d_{\mathrm{DR}}]] = -[[i(\omega), d_{\mathrm{DR}}], i(\omega)] = i([\omega, \omega]) = 0,$$

where $i(-)$ denotes the contraction of differential forms by vector fields. This, together with the fact that

$$e^{R(z)} d e^{-R(z)} = e^{\mathrm{ad}_{R(z)}}(d) \quad \text{for any } R(z) \in \mathrm{End}(A)[[z]],$$

immediately implies that

$$e^{i(\omega)z} d_{\mathrm{DR}} e^{-i(\omega)z} = d_{\mathrm{DR}} + \Delta z.$$

So by Theorem 2.1, $\Omega^\bullet(M)$ admits a Hodge-to-de-Rham degeneration data, and Theorem 3.4 applies. □

This result refines the Lie formality theorem of [ST08], since the transferred L_∞-algebra structure is trivial. Note that gauge-theoretic methods were already used (independently) in [FM12] to obtain a conceptual proof of the former theorem.

A direct corollary of Theorem 3.7 and Proposition 1.6 is that the spectral sequence for the double complex $(\Omega^\bullet(M), d_{\mathrm{DR}}, \Delta)$ degenerates on the first page for every Poisson manifold M [FIdL96].

4. De Rham cohomology of Jacobi manifolds

We extend the above arguments even further to treat the case of Jacobi manifolds, which is a generalisation of the notion of Poisson manifolds including the case of contact manifolds.

Definition 4.1 (Jacobi manifold [Lic78]). A *Jacobi manifold* is a smooth manifold M equipped with a pair

$$(\omega, E) \in \Gamma(\Lambda^2(TM)) \times \Gamma(TM),$$

for which

$$[\omega, \omega] = 2E \wedge \omega, \quad [E, \omega] = 0.$$

We consider again the space $\Omega(M)$ of differential forms equipped with the order 2 operator $\Delta := [i(\omega), d_{\mathrm{DR}}]$. It is easy to check that it anticommutes with the de Rham differential: $d_{\mathrm{DR}}\Delta + \Delta d_{\mathrm{DR}} = 0$. Unlike the previous case of Poisson manifolds, the operator Δ does not square to 0 on every form of a Jacobi manifold. The vector field E induces a homotopy for this relation. Hence the differential forms actually carry a *homotopy* BV-algebra structure.

Definition 4.2 (Commutative BV_∞-algebra [Kra00]). A *commutative* BV_∞*-algebra*

$$(A, \wedge, d = \Delta_0, \Delta_1, \Delta_2, \dots)$$

is a dg commutative algebra A equipped with operators Δ_n of degree $2n-1$ and order at most $n+1$, satisfying

$$\sum_{i=0}^{n} \Delta_i \Delta_{n-i} = 0 \quad \text{for } n \geq 0.$$

In particular, the operators Δ_n of a commutative BV_∞-algebra A make it a multicomplex. A commutative BV_∞-algebra is an example of a homotopy BV-algebra [GCTV12, DV12], where all the structural operations vanish except for the commutative product and the operators Δ_n.

Theorem 4.3. *Let (M, ω, E) be a Jacobi manifold. Its de Rham complex*

$$\left(\Omega^\bullet(M), \wedge, d_{\mathrm{DR}}, \Delta_1 := [i(\omega), d_{\mathrm{DR}}], \Delta_2 := i(E)i(\omega)\right)$$

is a commutative BV_∞*-algebra.*

Proof. Let us denote $\Delta_0 = d_{\mathrm{DR}}$, $\Delta_1 = \Delta$, and $\Delta_2 = i(E)i(\omega)$. Clearly, $(\Delta_0)^2 = 0$, and since

$$d_{\mathrm{DR}}\Delta + \Delta d_{\mathrm{DR}} = d_{\mathrm{DR}}(i(\omega)d_{\mathrm{DR}} - d_{\mathrm{DR}}i(\omega)) + (i(\omega)d_{\mathrm{DR}} - d_{\mathrm{DR}}i(\omega))d_{\mathrm{DR}} = 0,$$

we have $\Delta_0\Delta_1 + \Delta_1\Delta_0 = 0$. A similar computation shows that the de Rham differential anti-commutes with $i(E)$:

$$\begin{aligned} i(E)\Delta + \Delta i(E) &= i(E)(i(\omega)d_{\mathrm{DR}} - d_{\mathrm{DR}}i(\omega)) + (i(\omega)d_{\mathrm{DR}} - d_{\mathrm{DR}}i(\omega))i(E) \\ &= i(\omega)i(E)d_{\mathrm{DR}} + (d_{\mathrm{DR}}i(E) - L_E)i(\omega) + i(\omega)(-i(E)d_{\mathrm{DR}} + L_E) - d_{\mathrm{DR}}i(E)i(\omega) \\ &= -L_E i(\omega) + i(\omega)L_E = -i([E, \omega]) = 0. \end{aligned}$$

We note that

$$[i(\omega), \Delta] = -[\Delta, i(\omega)] = -[[i(\omega), d_{\mathrm{DR}}], i(\omega)] = i([\omega, \omega]) = 2i(E \wedge \omega) = 2i(E)i(\omega). \tag{4}$$

Furthermore, we have

$$\Delta^2 = d_{\mathrm{DR}}i(\omega)d_{\mathrm{DR}}i(\omega) + i(\omega)d_{\mathrm{DR}}i(\omega)d_{\mathrm{DR}} - d_{\mathrm{DR}}i(\omega)^2 d_{\mathrm{DR}}. \tag{5}$$

To simplify the latter expression, we compute

$$\begin{aligned} [i(\omega), \Delta] &= i(\omega)(i(\omega)d_{\mathrm{DR}} - d_{\mathrm{DR}}i(\omega)) - (i(\omega)d_{\mathrm{DR}} - d_{\mathrm{DR}}i(\omega))i(\omega) \\ &= -2i(\omega)d_{\mathrm{DR}}i(\omega) + i(\omega)^2 d_{\mathrm{DR}} + d_{\mathrm{DR}}i(\omega)^2, \end{aligned}$$

hence

$$2i(\omega)d_{\mathrm{DR}}i(\omega) = i(\omega)^2 d_{\mathrm{DR}} + d_{\mathrm{DR}}i(\omega)^2 - 2i(E)i(\omega),$$

which allows us to simplify formula (5) into

$$\Delta^2 + i(E)i(\omega)d_{\mathrm{DR}} + d_{\mathrm{DR}}i(E)i(\omega) = 0,$$

which is $\Delta_1^2 + \Delta_2\Delta_0 + \Delta_0\Delta_2 = 0$. Also,

$$\begin{aligned}\Delta_1\Delta_2 + \Delta_2\Delta_1 &= \Delta i(E)i(\omega) + i(E)i(\omega)\Delta \\ &= -i(E)\Delta i(\omega) + i(E)i(\omega)\Delta = i(E)[i(\omega),\Delta] = 2i(E)^2 i(\omega) = 0\end{aligned}$$

and

$$\Delta_2^2 = i(\omega)i(E)i(\omega)i(E) = i(\omega)^2 i(E)^2 = 0.$$

Therefore, the operators $\Delta_0, \Delta_1, \Delta_2$ and $\Delta_n = 0$ for $n > 2$ endow $\Omega^\bullet(M)$ with a structure of a multicomplex. It is clear that $\Delta_0 = d_{\mathrm{DR}}$ is a differential operator of order at most 1, and that Δ_1 and Δ_2 are differential operators of order at most 2 and at most 3 respectively. □

The following statement is a special case of the general homotopy transfer theorem [DV12, Th. 6.2] for homotopy BV-algebras.

Proposition 4.4 ([DSV13, Prop. 10]). *Let $(A, \wedge, d = \Delta_0, \Delta_1, \Delta_2, \ldots)$ be a commutative BV_∞-algebra admitting a Hodge-to-de-Rham degeneration data. The underlying homotopy groups $H(A, d)$ carry a homotopy Frobenius manifold structure extending the induced commutative product.*

We shall use this result to deduce the following theorem.

Theorem 4.5. *The de Rham cohomology of a Jacobi manifold carries a natural homotopy Frobenius manifold structure extending the product induced by the wedge product.*

Proof. By formula (4),

$$[i(\omega), [i(\omega), d_{\mathrm{DR}}]] = [i(\omega), \Delta_1] = 2\Delta_2$$

and $[i(\omega), [i(\omega), [i(\omega), d_{\mathrm{DR}}]]] = [i(\omega), 2i(E)i(\omega)] = 0$. Therefore,

$$e^{i(\omega)z}\, d_{\mathrm{DR}}\, e^{-i(\omega)z} = e^{\mathrm{ad}_{i(\omega)z}}(d_{\mathrm{DR}}) = \Delta_0 + \Delta_1 z + \Delta_2 z^2.$$

By Theorem 2.1, we conclude that $\Omega^\bullet(M)$ admits a Hodge-to-de-Rham degeneration data, so Theorem 4.3 and Proposition 4.4 apply, which completes the proof. □

Remarks 4.6. ⋄ One can also consider the subspace of basic differential forms $\Omega_B^\bullet(M)$ defined by $i(E)(\alpha) = i(E)(d_{\mathrm{DR}}\alpha) = 0$. The various operators restrict to this space and $(\Omega_B^\bullet(M), d_{\mathrm{DR}}, \wedge, \Delta)$ forms a dg BV-algebra [CMdL98]. Hence the basic de Rham cohomology of a Jacobi manifold carries a natural homotopy Frobenius manifold structure, whose rectified dg BV-algebra is homotopy equivalent to the basic de Rham algebra $(\Omega_B^\bullet(M), d_{\mathrm{DR}}, \wedge, \Delta)$.

For the so-called regular Jacobi manifolds [CMdL98], this result is literally contained in Theorem 3.7. A Jacobi manifold is *regular* if the space of leaves $\widetilde{M} = M/E$ can be defined as a smooth manifold; in this case, it automatically inherits a Poisson structure from the Jacobi structure on M, and $\Omega_B^\bullet(M) \simeq \Omega^\bullet(\widetilde{M})$.

◇ The homotopy Frobenius structure on the de Rham cohomology of a Poisson manifold, as in Theorem 3.7, and on the basic de Rham cohomology of a Jacobi manifold, can also be described in a different way. In both cases, there is a structure of a BV/Δ-algebra, a notion defined in [KMS13], on the level of differential forms. In [KMS13], an explicit formula for a quasi-isomorphism between the operads $H_\bullet(\overline{\mathcal{M}}_{0,n+1})$ and BV/Δ is given. Therefore, the de Rham algebra carries a Frobenius manifold structure, and this structure induces a homotopy Frobenius manifold structure on the de Rham cohomology.
It is an interesting question whether it is possible to match the two approaches on the level of formulas. Since one of the ways to obtain this quasi-isomorphism uses Givental theory, one natural idea would be to describe the BV_∞-structure in terms of cohomological field theory and infinitesimal Givental operators. The first step in that direction is made in [DSV13], where this kind of description is given for commutative BV_∞-algebras.

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