J. Eur. Math. Soc. 17, 645–687

DOI 10.4171/JEMS/513

Joachim Cuntz · Siegfried Echterhoff · Xin Li

On the K-theory of the C*-algebra generated by the left regular representation of an Ore semigroup

Received February 29, 2012 and in revised form February 7, 2013

Abstract. We compute the K-theory of C*-algebras generated by the left regular representation of left Ore semigroups satisfying certain regularity conditions. Our result describes the K-theory of these semigroup C*-algebras in terms of the K-theory for the reduced group C*-algebras of certain groups which are typically easier to handle. Then we apply our result to specific semigroups from algebraic number theory.

Keywords. K-theory, semigroup C*-algebra, $ax + b$-semigroup, purely infinite

1. Introduction

Let P be a (discrete) semigroup. If P admits left cancellation, then left translation defines an action of P by isometries V_p, $p \in P$, on the Hilbert space $\ell^2(P)$. When P is a group, the V_p are unitaries and the reduced C*-algebra $C^*_r(P)$ generated by the operators V_p is one of the most classical objects of study in the theory of operator algebras. The analogous C*-algebra for a genuine semigroup has recently attracted attention, partly triggered by natural examples, and has been studied in various connections. We call them (reduced or regular) semigroup C*-algebras. The interested reader may consult [Li2] for a brief account of the historical background of these C*-algebras attached to semigroups.

The possibility of describing $C^*_r(P)$, for a left cancellative semigroup P, as a universal C*-algebra with generators and relations has been analyzed in [Li2] in connection with amenability properties of P. Also, such a description was discussed in detail in [C-D-L] for the important example of the "$ax + b$-semigroup" $R \rtimes R^\times$ for the ring of integers R in a number field. In this latter paper also the KMS-structure for a natural one-parameter group on $C^*_r(R \rtimes R^\times)$ was studied and it was shown that it is partly governed by the ideal class group for R.

In the present paper we set out to determine the K-theoretic invariants of $C^*_r(P)$ for a class of semigroups containing the semigroups arising from number theory that we are interested in. Here is our main result:

J. Cuntz, S. Echterhoff: Department of Mathematics, Westfälische Wilhelms-Universität Münster, Einsteinstraße 62, 48149 Münster, Germany;
e-mail: cuntz@uni-muenster.de, echters@uni-muenster.de

X. Li: School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, United Kingdom; e-mail: xin.li@qmul.ac.uk

Mathematics Subject Classification (2010): Primary 46L05, 46L80; Secondary 20Mxx, 11R04

Theorem. *Let P be a countable left Ore semigroup. Assume that the family of constructible right ideals $\mathcal{J}$ of P is independent* (§2.2)*, and that the enveloping group G of P satisfies the Baum–Connes conjecture with coefficients. Let $\mathcal{I}$ denote the G-saturation of $\mathcal{J} \setminus \{\emptyset\}$ in the power set $\mathcal{P}(G)$ of G. Then the K-theory of the semigroup C*-algebra $C_r^*(P)$ can be described as follows:*

$$K_*(C_r^*(P)) \cong \bigoplus_{[X]\in G\backslash\mathcal{I}} K_*(C_r^*(G_X)),$$

where $G_X = \{g \in G : g \cdot X = X\}$ denotes the stabilizer of $X \in \mathcal{I}$ under the action of G on $\mathcal{I}$.

In fact, we only need the Baum–Connes conjecture with coefficients in two specific G-C*-algebras. Moreover, in good situations, it turns out that $C_r^*(P)$ and $\bigoplus_{[X]\in G\backslash\mathcal{I}} C_r^*(G_X)$ are actually KK-equivalent. We refer the reader to §7 for more explanations and more precise formulations of our result. Let us now explain the basic ideas behind the proof:

As a first step, we need an embedding of $C_r^*(P)$ as a full corner of a (reduced) crossed product $D \rtimes_r G$ of a commutative C*-algebra D by an enveloping group G for P (see Section 4). The existence of such a crossed product follows from the left Ore condition on P (see [La]). As a consequence, the K-theory of $C_r^*(P)$ is isomorphic to the K-theory of $D \rtimes_r G$.

We then prove a rather general K-theoretic theorem which, in many situations, allows one to reduce the computation of $K_*(D \rtimes_r G)$ to the, often much simpler, computation of the K-theory of C*-algebras associated with certain subgroups of G. Our key technical result concerns the following situation. Assume that D is a commutative C*-algebra generated by a multiplicative family $\{e_i : i \in I\}$ of projections, satisfying a certain independence condition, and that a group G acts on D leaving the generating family invariant. We then show under the assumption that G satisfies the Baum–Connes conjecture for the coefficient algebras D and $c_0(I)$ that the computation of the K-theory of the crossed product C*-algebra $D \rtimes_r G$ is equivalent to the computation of the K-theory of the much simpler crossed product $c_0(I) \rtimes_r G$ (see Section 6). The proof uses techniques that have been developed in connection with the Baum–Connes conjecture in [C-E-O], [E-L-P-W] and [Mey-Ne]. A combined statement of the relevant results is given in [E-N-O]. Note that by [H-K] all amenable groups (among many others) satisfy the Baum–Connes conjecture, so the results apply in particular to our motivating examples $R \rtimes R^\times$.

Now, on the other hand, if a group G acts on $c_0(I)$ where I is a discrete set, then simple imprimitivity considerations show that the crossed product $c_0(I) \rtimes_r G$ is Morita equivalent to a direct sum of the (reduced) group C*-algebras of the stabilizer groups.

In the case of the crossed product $D \rtimes_r G$ connected to the left Ore semigroup P, the algebra D is generated by the set of projections $\{E_X : X \in \mathcal{I}\}$ with E_X the orthogonal projection from $\ell^2(G)$ to $\ell^2(X) \subseteq \ell^2(G)$. The independence condition for this set of projections follows from a similar independence condition on the set of constructible right ideals $\mathcal{J}$ in P. This gives the result of our theorem. Moreover, if G satisfies a certain strong version of the Baum–Connes conjecture (which again holds, among others, for all amenable groups) we can deduce the stronger result that $C_r^*(P)$ is KK-equivalent to the direct sum $\bigoplus_{[X]\in G\backslash\mathcal{I}} C_r^*(G_X)$. Note that the G-orbits in $\mathcal{I}$ and the stabilizers G_X are easily determined in specific examples.

Under the same assumptions on our semigroup P as above, there exists a natural diagonal map $C_r^*(P) \to C_r^*(P) \otimes_{\min} C_r^*(P)$. This means that, just as for a group C*-algebra, the K-homology of $C_r^*(P)$ becomes a ring via this diagonal map. The KK-equivalence between $C_r^*(P)$ and the direct sum of the C*-algebras of the stabilizer groups, which we construct assuming the strong Baum–Connes conjecture, induces in fact an isomorphism of K-homology rings.

As mentioned above, our motivating examples are the semigroups attached to a Dedekind domain R, such as the ring of integers in an algebraic number field, or function field, K. For such a ring we consider the multiplicative semigroup $R^\times$, the multiplicative semigroup of principal ideals and the $ax+b$-semigroup $R \rtimes R^\times$ (see §8). These semigroups have obvious enveloping groups $K^\times$, the group of principal fractional ideals and $K \rtimes K^\times$. The set $\mathcal{I}$ which appears when we apply our theorem can be identified with the set of fractional ideals (for both $R^\times$ and the semigroup of principal ideals), or with the translates of fractional ideals, in K, respectively. The stabilizer groups are essentially the group of invertible elements in $R^\times$, trivial or the group of invertible elements in $R \rtimes R^\times$. The orbits in $\mathcal{I}$ for the action of the enveloping group are labeled by the ideal class group Cl_K in each case. We note that in the case of multiplicative semigroups, there are natural actions of the class group on the K-theory of the corresponding semigroup C*-algebras.

Finally we turn to a study of specific structural properties of the C*-algebra $C_r^*(R \rtimes R^\times)$ for the ring of integers R in a number field (see §8.2). This algebra is of special interest for many reasons. As mentioned above, it has an intriguing KMS-structure, but it also has a unique maximal ideal and the quotient by this ideal gives the ring C*-algebra $\mathfrak{A}[R]$ studied in [Cu-Li]. This ring C*-algebra is purely infinite and simple and can be represented as a crossed product by actions on adele spaces in different ways. In [Cu-Li] we also determined its K-theory for a first class of number fields. The complete K-theoretic computation is obtained in [Li-Lü].

Using a criterion from [Pas-Rør] we can now show that $C_r^*(R \rtimes R^\times)$ is purely infinite (though of course not simple) and has the ideal property. These properties are of structural interest for a C*-algebra. Using our K-theory computation and another criterion from [Pas-Rør], we can show that $C_r^*(R \rtimes R^\times)$ on the other hand does not have real rank zero. The first named author is indebted to C. Pasnicu and G. Gong for drawing his attention to these properties.

2. Preliminaries

2.1. Semigroups. A *semigroup* is a set P together with an associative binary operation (or multiplication) $P \times P \to P$, $(p,q) \mapsto pq$. We will not consider (non-trivial) topologies on our semigroups, which means that topologically, all our semigroups will be viewed as discrete sets. A *unit element* in a semigroup P is an element e in P with the property that $ep = pe = p$ for all p in P. All the semigroups in this paper are assumed to have unit elements. In addition, since we would like to use KK-theory in §6, all our semigroups in §6 are supposed to be countable so that the semigroup C*-algebras will be separable.

Moreover, a semigroup P is called *left cancellative* if for all p, x and y in P, $px = py$ implies $x = y$. Similarly, a semigroup P is called *right cancellative* if for all p, x and y in P, $xp = yp$ implies $x = y$. A semigroup is called *cancellative* if it is both left and right cancellative.

2.2. Ideal structure. A *left ideal* of a semigroup P is a subset X of P which is invariant under left multiplication, i.e. for every x in X and p in P, px lies in X again. Similarly, a *right ideal* of a semigroup P is a subset X of P which is invariant under right multiplications, i.e. for every x in X and p in P, xp lies in X again.

In the analysis of semigroup C*-algebras, a certain family of right ideals plays an important role. It is defined as follows:

Definition 2.2.1. For a semigroup P, let $\mathcal{J}$ be the smallest family of right ideals of P satisfying

- $\emptyset, P \in \mathcal{J}$,
- $\mathcal{J}$ is closed under left multiplication and taking preimages under left multiplication $(X \in \mathcal{J},\ p \in P \Rightarrow pX,\ p^{-1}X \in \mathcal{J})$,
- $\mathcal{J}$ is closed under finite intersections $(X, Y \in \mathcal{J} \Rightarrow X \cap Y \in \mathcal{J})$.

Here for every subset X of P and for all $p \in P$ we define

$$pX := \{px : x \in X\} \quad \text{and} \quad p^{-1}X := \{q \in P : pq \in X\}.$$

It follows directly from this definition that $\mathcal{J}$ consists of $\emptyset$ and arbitrary finite intersections of right ideals of the form $q_1^{-1}p_1 \cdots q_n^{-1}p_nP$ for $q_1, \dots, q_n, p_1, \dots, p_n \in P$. Elements in $\mathcal{J}$ are called *constructible right ideals* of P.

We need the following

Definition 2.2.2. The family $\mathcal{J}$ is said to be *independent* (we also say that the constructible right ideals of P are independent) if for all right ideals $X, X_1, \dots, X_n$ in $\mathcal{J}$ with $X = \bigcup_{j=1}^n X_j$, we have $X = X_j$ for some $1 \leq j \leq n$.

In other words, $\mathcal{J}$ is independent if for every right ideal X in $\mathcal{J}$, the following holds: Given $X_1, \dots, X_n$ in $\mathcal{J}$ which are proper subsets of X ($X_j \subsetneq X$ for all $1 \leq j \leq n$), the union $\bigcup_{j=1}^n X_j$ is again a proper subset of X ($\bigcup_{j=1}^n X_j \subsetneq X$).

This independence condition plays an important role when one tries to describe amenability of semigroups in terms of semigroup C*-algebras (see [Li2]). But as we will see, it will also play a crucial role in our K-theoretic computations.

2.3. Ore semigroups. Our K-theoretic computations only work for so-called left Ore semigroups.

Definition 2.3.1. A semigroup is called *right reversible* if every pair of non-empty left ideals has a non-empty intersection.

Definition 2.3.2. A semigroup is said to satisfy the *left Ore condition* if it is cancellative and right reversible. A semigroup with these properties is called a *left Ore semigroup*.

The following result is the reason why the left Ore condition is so useful:

Theorem 2.3.3 (Ore, Dubreil). *A semigroup P can be embedded into a group G such that $G = P^{-1}P = \{q^{-1}p : p, q \in P\}$ if and only if P satisfies the left Ore condition. In this case, the group G is determined up to canonical isomorphism by the universal property that every semigroup homomorphism $P \to G'$ from P to a group G' extends uniquely to a group homomorphism $G \to G'$.*

When we write $G = P^{-1}P$ in this theorem, we are identifying P with its image in G under the embedding of P into G.

The reader may consult [Cl-Pr, Theorem 1.23] or [La, §1.1] for more explanations about this theorem. For a left Ore semigoup P, let us call the (unique up to canonical isomorphism) group G which appears in the theorem the *enveloping group* of P. It is also called the group of left quotients (which explains the terminology "left Ore semigroup").

Instead of giving a full proof of this theorem, we now describe an explicit model for the enveloping group in order to illustrate an important idea. Let P be a semigroup. We define a partial order on P by setting $p \leq q :\Leftrightarrow q \in Pp$. Here Pp is the left principal ideal of P generated by p, i.e. $Pp = \{xp : x \in P\}$. It is straightforward to see that P is right reversible if and only if P is upwards directed with respect to this partial order, which means that for all $p_1, p_2 \in P$, there exists $q \in P$ such that $p_1 \leq q$ and $p_2 \leq q$. If we further assume that P is right cancellative, then $p \leq q$ implies that there exists a unique element $r \in P$ with $q = rp$. We denote this element r by qp^{-1}. The observations made so far tell us that given a right reversible, right cancellative semigroup P, we can form an inductive system of sets indexed by the elements in P ordered by "$\leq$" in the following way:

- for every $p \in P$, the p-th set is given by P itself,
- for all $p, q \in P$ with $p \leq q$, the structure map from the p-th set to the q-th set is given by left multiplication with qp^{-1}: $P \to P, x \mapsto (qp^{-1})x$.

We can then form the set-theoretical inductive limit of this system and endow it with a binary operation so that we again obtain a semigroup. Here are the details: As a first step, we take the (set-theoretical) disjoint union $\bigsqcup_{p \in P} P$. Let us denote the embedding of P into the p-th copy of P in the disjoint union by $P \ni x \mapsto p^{-1} \cdot x \in \bigsqcup_{p \in P} P$. Then we define an equivalence relation $\sim$ by identifying $p_1^{-1} \cdot x_1$ and $p_2^{-1} \cdot x_2$ in $\bigsqcup_{p \in P} P$ if there exists p in P with $p_1 \leq p$, $p_2 \leq p$ and $(pp_1^{-1})x_1 = (pp_2^{-1})x_2$. The set of equivalence classes $(\bigsqcup_{p \in P} P)/\sim$ carries the following canonical structure of a semigroup: Given $p_1^{-1} \cdot x_1$ and $p_2^{-1} \cdot x_2$ in $\bigsqcup_{p \in P} P$, take $y \in P$ with $x_1 \leq y$ and $p_2 \leq y$ and set

$$[p_1^{-1} \cdot x_1][p_2^{-1} \cdot x_2] = [((yx_1^{-1})p_1)^{-1} \cdot ((yp_2^{-1})x_2)]. \tag{1}$$

Here $[\cdot]$ stands for equivalence class. One can check that the set $(\bigsqcup_{p \in P} P)/\sim$ together with the binary operation defined in (1) is indeed a semigroup. Let us denote it by G. The unit element in G is given by $[e^{-1} \cdot e]$ where e is the unit element of P. Moreover, by definition of the binary operation, we have

$$[p^{-1} \cdot x][x^{-1} \cdot p] = [p^{-1} \cdot p] = [e^{-1} \cdot e]$$

(take $y = x$ in (1)). So we see that we have actually defined a group. Finally, the map $P \ni p \mapsto [e^{-1} \cdot p] \in G$ defines a semigroup homomorphism which is injective if P is also left cancellative. By construction, the group G, together with this embedding of P, has all the properties of Theorem 2.3.3: Every element of G is of the form $[p^{-1} \cdot x] = [p^{-1} \cdot e][e^{-1} \cdot x] = ([e^{-1} \cdot p])^{-1}[e^{-1} \cdot x] \in P^{-1}P$. Here we are identifying P with its image in G under the embedding $P \ni p \mapsto [e^{-1} \cdot p] \in G$. Moreover, given a group G' and a semigroup homomorphism $\varphi : P \to G'$, it is straightforward to check that the map $G \to G'$, $[p^{-1} \cdot x] \mapsto \varphi(p)^{-1}\varphi(x)$, defines a group homomorphism which extends φ. Uniqueness of the extension follows from the equation $G = P^{-1}P$.

This is one way of constructing a model for the enveloping group. The main idea is to formally invert semigroup elements using an inductive limit procedure. Similar ideas frequently appear in the literature (compare for instance [La]), and as we will see, this idea will also play a role later on in this paper.

2.4. Reduced semigroup C*-algebras. The main goal of this paper is to compute K-theory for reduced semigroup C*-algebras of left Ore semigroups whose constructible right ideals are independent (under a certain K-theoretic assumption on the enveloping group). In this subsection, let us briefly recall the construction of reduced semigroup C*-algebras. The reader may consult [Li2] for details.

Let P be a left cancellative semigroup. Let $\ell^2(P)$ be the Hilbert space of square summable functions from P to $\mathbb{C}$ and let $\{\varepsilon_x : x \in P\}$ be the canonical orthonormal basis of $\ell^2(P)$ given by $\varepsilon_x(y) = \delta_{x,y}$ ($\delta_{x,y} = 1$ if $x = y$ and $\delta_{x,y} = 0$ if $x \neq y$). The semigroup P acts on $\ell^2(P)$ as follows: For every $p \in P$, the map $\varepsilon_x \mapsto \varepsilon_{px}$ extends to an isometry V_p on $\ell^2(P)$ because our assumption that P is left cancellative implies that $P \ni x \mapsto px \in P$ is injective. Now we simply set

Definition 2.4.1. $C_r^*(P) := C^*(\{V_p : p \in P\}) \subseteq \mathcal{L}(\ell^2(P))$. This is the *reduced semigroup C*-algebra* of P. In other words, the reduced semigroup C*-algebra is the C*-algebra generated by the left regular representation of the semigroup.

Now consider the family $\mathcal{J}$ of right ideals of P from Definition 2.2.1. For every right ideal $X \in \mathcal{J}$, we let E_X be the orthogonal projection on $\ell^2(P)$ onto the subspace $\ell^2(X) \subseteq \ell^2(P)$. As observed in [Li2, §2], the projections E_X lie in $C_r^*(P)$ for all $X \in \mathcal{J}$. Thus, the formula in

Definition 2.4.2. $D_r(P) := C^*(\{E_X : X \in \mathcal{J}\}) \subseteq \mathcal{L}(\ell^2(P))$

defines a sub-C*-algebra of $C_r^*(P)$. It is clear that $D_r(P)$ is a commutative C*-algebra, and that multiplication on the generators is given by $E_X E_Y = E_{X \cap Y}$. Moreover, $D_r(P)$ is $\mathrm{Ad}(V_p)$-invariant for every $p \in P$. Therefore, the map $\tau : P \to \mathrm{End}(D_r(P))$, $p \mapsto \tau_p := \mathrm{Ad}(V_p)|_{D_r(P)}$, defines a semigroup action of P on $D_r(P)$.

2.5. On reduced crossed products. Let us collect a few observations about reduced crossed products. These results are included for the sake of completeness and also for ease of reference. They are certainly well known and we do not claim any originality here. In what follows we always assume that G is a discrete group although most of what we say below has obvious analogues for general locally compact groups.

We denote by $\lambda : G \to \mathcal{U}(\ell^2(G))$ the left regular representation of G and by $M : c_0(G) \to \mathcal{L}(\ell^2(G))$ the representation of $c_0(G)$ by multiplication operators on $\ell^2(G)$.

Recall that the *reduced crossed product* $A \rtimes_{\alpha,r} G$ of the C*-dynamical system (A, G, α) can be defined as the sub-C*-algebra of $M(A \otimes \mathcal{K}_G)$, with $\mathcal{K}_G := \mathcal{K}(\ell^2(G))$, generated by the set

$$\{\iota_A(a)\iota_G(g) : a \in A,\ g \in G\},$$

where $\iota_G(g) = 1 \otimes \lambda_g$ and where $\iota_A : A \to M(A \otimes \mathcal{K}_G)$ is defined by the composition

$$A \xrightarrow{\tilde{\alpha}} \ell^\infty(G, A) \subseteq M(A \otimes c_0(G)) \xrightarrow{\mathrm{id}_A \otimes M} M(A \otimes \mathcal{K}_G).$$

Here $\tilde{\alpha}$ sends $a \in A$ to the function $[g \mapsto \alpha_{g^{-1}}(a)] \in \ell^\infty(G, A)$. Every representation $\rho : A \to \mathcal{L}(H)$ induces a homomorphism

$$\mathrm{Ind}\,\rho : A \rtimes_r G \to \mathcal{L}(H \otimes \ell^2(G))$$

by applying the representation $\rho \otimes \mathrm{id}_{\mathcal{K}_G} : A \otimes \mathcal{K}_G \to \mathcal{L}(H \otimes \ell^2(G))$ to $A \rtimes_{\alpha,r} G \subseteq M(A \otimes \mathcal{K}_G)$. It follows that $\mathrm{Ind}\,\rho$ is faithful if ρ is faithful. One easily checks that

$$(\mathrm{Ind}\,\rho)(\iota_A(a))(\xi \otimes \varepsilon_g) = \rho(\alpha_{g^{-1}}(a))\xi \otimes \varepsilon_g \quad \text{and} \quad (\mathrm{Ind}\,\rho)(\iota_G(g)) = 1 \otimes \lambda_g \tag{2}$$

for all $a \in A$, $\xi \in H$ and $g \in G$, where $\{\varepsilon_x : x \in G\}$ denotes the standard orthonormal basis of $\ell^2(G)$. Thus, if $\rho : A \to \mathcal{L}(H)$ is faithful, we recover the classical spatial definition of the reduced crossed product as a subalgebra of $\mathcal{L}(H \otimes \ell^2(G))$.

Our first lemma is concerned with crossed products $D \rtimes_{\tau,r} G$ where D is a closed, left-translation invariant sub-C*-algebra of $\ell^\infty(G)$ and $\tau : G \to \mathrm{Aut}(D)$ denotes the left-translation action. Let $M : D \to \mathcal{L}(\ell^2(G))$ be the representation by multiplication operators. One easily checks that (M, λ) is a covariant representation of (D, G, τ) on $\ell^2(G)$. It therefore induces a representation $M \rtimes \lambda : D \rtimes_\tau G \to \mathcal{L}(\ell^2(G))$.

Lemma 2.5.1. *Let (M, λ) be as above. Then $(M \otimes 1, \lambda \otimes 1)$ is unitarily equivalent to the regular representation $((\mathrm{Ind}\,M) \circ \iota_D, (\mathrm{Ind}\,M) \circ \iota_G)$ on $\ell^2(G \times G)$. In particular, $M \rtimes \lambda : D \rtimes_\tau G \to \mathcal{L}(\ell^2(G))$ factors through a faithful representation of $D \rtimes_{\tau,r} G$.*

Proof. Consider the unitary operator $W : \ell^2(G \times G) \to \ell^2(G \times G)$, $W(\varepsilon_x \otimes \varepsilon_y) = \varepsilon_{yx} \otimes \varepsilon_{x^{-1}}$; its adjoint is given by the formula $W^*(\varepsilon_x \otimes \varepsilon_y) = \varepsilon_{y^{-1}} \otimes \varepsilon_{xy}$. We then compute for $f \in \ell^\infty(G)$:

$$\begin{aligned} W((\mathrm{Ind}\,M) \circ \iota_D)(f)W^*(\varepsilon_x \otimes \varepsilon_y) &= W((\mathrm{Ind}\,M) \circ \iota_D)(f)(\varepsilon_{y^{-1}} \otimes \varepsilon_{xy}) \\ &= W(f(x)(\varepsilon_{y^{-1}} \otimes \varepsilon_{xy})) \\ &= f(x)(\varepsilon_x \otimes \varepsilon_y) = (M(f) \otimes 1)(\varepsilon_x \otimes \varepsilon_y) \end{aligned}$$

and

$$W(1 \otimes \lambda_g)W^*(\varepsilon_x \otimes \varepsilon_y) = W(1 \otimes \lambda_g)(\varepsilon_{y^{-1}} \otimes \varepsilon_{xy}) = W(\varepsilon_{y^{-1}} \otimes \varepsilon_{gxy}) = \varepsilon_{gx} \otimes \varepsilon_y$$
$$= (\lambda_g \otimes 1)(\varepsilon_x \otimes \varepsilon_y). \qquad \square$$

Our second lemma is about functorial properties of reduced crossed products.

Lemma 2.5.2. *Suppose that (A, G, α) and (B, H, β) are C*-dynamical systems, where G and H are discrete groups. Assume that $\varphi : A \to B$ is a homomorphism and that $j : G \to H$ is an injective homomorphism such that $\beta_{j(g)}(\varphi(a)) = \varphi(\alpha_g(a))$ for all $a \in A$ and $g \in G$. Then there exists a unique homomorphism $\varphi \rtimes_r j : A \rtimes_{\alpha,r} G \to B \rtimes_{\beta,r} H$ such that $(\varphi \rtimes_r j)(\iota_A(a)\iota_G(g)) = \iota_B(\varphi(a))\iota_H(j(g))$. If φ is faithful, then so is $\varphi \rtimes_r j$.*

Proof. We may assume without loss of generality that G is a subgroup of H and that $j : G \to H$ is the inclusion map. Restricting β to G, we first observe that we have a homomorphism $\varphi \otimes \mathrm{id}_{\mathcal{K}_G} : A \otimes \mathcal{K}_G \to B \otimes \mathcal{K}_G$ which we may extend to a homomorphism (again denoted by $\varphi \otimes \mathrm{id}_{\mathcal{K}_G}$) $A \rtimes_{\alpha,r} G \to M(B \otimes \mathcal{K}_G)$ such that

$$(\varphi \otimes \mathrm{id}_{\mathcal{K}_G})(\iota_A(a)) = \iota_B(\varphi(a)) \quad \text{and} \quad (\varphi \otimes \mathrm{id}_{\mathcal{K}_G})(\iota_G(g)) = \iota_G(g).$$

Thus $\varphi \otimes \mathrm{id}_{\mathcal{K}_G}$ maps $A \rtimes_{\alpha,r} G$ into $B \rtimes_{\beta,r} G$ and $\varphi \otimes \mathrm{id}_{\mathcal{K}_G}$ is faithful if φ is faithful.

To see that $B \rtimes_{\beta,r} G$ imbeds into $B \rtimes_{\beta,r} H$, we first observe that $\ell^2(H)$ can be identified with $\bigoplus_{[h]\in G\backslash H} \ell^2(G)$. An explicit isomorphism is given by choosing a cross section $c : G\backslash H \to H$ which induces a bijection $G \times G\backslash H \to H$, $(g, [h]) \mapsto gc([h])$, and hence an isomorphism $\ell^2(H) \cong \bigoplus_{[h]\in G\backslash H} \ell^2(G)$ by sending $\varepsilon_{gc[h]}$ to ε_g in the summand at $[h]$ for all $g \in G$ and $[h] \in G\backslash H$. Under this isomorphism, for $b \in B$ we get

$$\iota_B^H(b) = \bigoplus_{[h]\in G\backslash H} (\beta_{c[h]^{-1}} \otimes \mathrm{id}_{\mathcal{K}_G})(\iota_B^G(b)) \in \bigoplus_{[h]\in G\backslash H} M(B \otimes \mathcal{K}_G) \subseteq M(B \otimes \mathcal{K}_H)$$

and $\iota_H(g) = \bigoplus_{[h]\in G\backslash H} \iota_G(g)$ for all $g \in G$ (where the superscript H indicates that $\iota_B^H(b)$ belongs to the crossed product $B \rtimes_{\beta,r} H$). Thus we see that the subalgebra of $B \rtimes_{\beta,r} H$ generated by $\{\iota_B^H(b)\iota_H(g) : b \in B,\ g \in G\}$ equals $\bigoplus_{[h]\in G\backslash H}(\beta_{c[h]^{-1}} \otimes \mathrm{id}_{\mathcal{K}_G})(B \rtimes_{\beta,r} G)$, which is isomorphic to $B \rtimes_{\beta,r} G$ via an isomorphism sending $\iota_B^G(b)\iota_G(g)$ to $\iota_B^H(b)\iota_H(g)$ for all $b \in B$ and $g \in G$. Combining this with the first part gives the lemma. $\square$

For the proof of the following lemma we refer to [Br-Oz, Chapter 4, Proposition 1.9].

Lemma 2.5.3. *Let (A, G, α) be a C*-dynamical system with G discrete. Then there exists a unique faithful conditional expectation $E : A \rtimes_{\alpha,r} G \to A$ such that $E(\iota_A(a)\iota_G(g)) = \delta_{g,e}a$, where $\delta_{g,e} = 1$ if g is equal to the unit e of G and $\delta_{g,e} = 0$ if $g \neq e$.*

3. The strategy

Let P be a left Ore semigroup whose constructible right ideals are independent. Let G be the enveloping group of P (see Theorem 2.3.3). Using Theorem 2.3.3, we will always view P as a subsemigroup of G.

Our goal is to compute K-theory for the reduced semigroup C*-algebra of P under a K-theoretic assumption on G which we will make precise later on. Let us now present our strategy:

First, we make use of the assumption that P is a left Ore semigroup to reduce our K-theoretic problem to the problem of computing K-theory for a reduced crossed product by the enveloping group G of P. The main idea has already appeared in the previous section, namely to use inductive limit procedures to pass from P to G.

The main step is to compare the reduced crossed product we are interested in with another, but much simpler reduced crossed product. The simpler one is given by an action of G on a discrete space (simply a set). This step makes use of our K-theoretic assumption on G. It allows us to apply the machinery of Baum–Connes which will reduce the K-theoretic comparison of the reduced crossed products to the case of finite subgroups. Here our assumption that the constructible right ideals of P are independent enters the game, as we will see.

The last step is to compute K-theory for reduced crossed products associated with an action of our group G on a discrete space. This amounts to applying imprimitivity theorems.

4. Dilations of reduced semigroup C*-algebras

For what we are going to do in this section, it is enough to assume that our semigroup P satisfies the left Ore condition. We would like to describe the reduced semigroup C*-algebra $C_r^*(P)$ as a reduced crossed product by the enveloping group G, at least up to Morita equivalence. Following ideas of [La], we first of all construct a G-C*-algebra which gives rise to the reduced crossed product.

Similarly to §2.3, we consider the following inductive system of C*-algebras indexed by elements of P ordered by "$\leq$":

- the p-th C*-algebra is $D_r(P)$ for every $p \in P$,
- given $p, q \in P$ with $p \leq q$, the structure map from the p-th to the q-th C*-algebra is $\tau_{qp^{-1}} = \mathrm{Ad}(V_{qp^{-1}}) : D_r(P) \to D_r(P)$.

Let $D_r^{(\infty)}(P)$ be the inductive limit of this system, and denote by $\iota_p : D_r(P) \to D_r^{(\infty)}(P)$ the inclusion of the p-th C*-algebra into the inductive limit. As explained in [La], there is a G-action $\tau^{(\infty)}$ on $D_r^{(\infty)}(P)$ which dilates the P-action τ on $D_r(P)$. To describe $\tau^{(\infty)}$, it suffices to define $\tau_p^{(\infty)}$ for every $p \in P \subseteq G$ because the semigroup homomorphism $P \ni p \mapsto \tau_p^{(\infty)} \in \mathrm{Aut}(D_r^{(\infty)}(P))$ extends uniquely to G by Theorem 2.3.3. Now $\tau_p^{(\infty)}$ is given as follows: For $q \in P$ and $d \in D_r(P)$, let r be an element in P such that $p \leq r$ and $q \leq r$. Then we set

$$\tau_p^{(\infty)}(\iota_q(d)) := \iota_{rp^{-1}}(\tau_{rq^{-1}}(d)).$$

One can check that this formula gives rise to the desired automorphism $\tau_p^{(\infty)}$ of $D_r^{(\infty)}(P)$ and that these automorphisms give rise to the semigroup homomorphism

$P \to \mathrm{Aut}(D_r^{(\infty)}(P))$, $p \mapsto \tau_p^{(\infty)}$. Moreover, one can also verify that the automorphisms we have constructed coincide with the ones in [La, §2].

In the following, we construct a covariant representation for the C*-dynamical system $(D_r^{(\infty)}(P), G, \tau^{(\infty)})$. First, we obtain a canonical faithful representation of $D_r^{(\infty)}(P)$ on $\ell^2(G)$ as follows: Using the inductive limit structure of $D_r^{(\infty)}(P)$, it suffices to construct a family $\{\pi_p\}_{p \in P}$ of faithful representations of $D_r(P)$ on $\ell^2(G)$ which are compatible with the structure maps. As $D_r(P)$ acts on $\ell^2(P)$ by construction, we can conjugate the identity representation of $D_r(P)$ by the canonical isometric embedding $\ell^2(P) \hookrightarrow \ell^2(G)$ to obtain a faithful representation π of $D_r(P)$ on $\ell^2(G)$. Then define for every $p \in P$ the representation $\pi_p := \mathrm{Ad}(\lambda_p^*) \circ \pi$. Here for every $g \in G$, we denote by λ_g the unitary on $\ell^2(G)$ given by $\lambda_g(\varepsilon_x) = \varepsilon_{gx}$ for the canonical orthonormal basis $\{\varepsilon_x : x \in G\}$ of $\ell^2(G)$. In other words, λ_g is the image of $g \in G$ under the left regular representation λ of G. These representations π_p are faithful by construction. For a subset Y of G, let $E_Y \in \mathcal{L}(\ell^2(G))$ be the orthogonal projection onto the subspace $\ell^2(Y)$ of $\ell^2(G)$. It is then immediate that for every $X \in \mathcal{J}$, we have

$$\pi_p(E_X) = E_{p^{-1} \cdot X}. \tag{3}$$

Note that $p^{-1} \cdot X$ is the subset $\{p^{-1}x : x \in X\}$ of G; it should not be confused with $p^{-1}X = \{q \in P : pq \in X\}$. From (3), it follows that the representations π_p are compatible with the structure maps, in the sense that for all $p, q \in P$ with $p \leq q$, we have

$$\pi_q \circ \mathrm{Ad}(V_{qp^{-1}}) = \pi_p.$$

Therefore, the faithful representations $\{\pi_p\}_{p \in P}$ give rise to a faithful representation $\pi^{(\infty)}$ of $D_r^{(\infty)}(P)$ on $\ell^2(G)$. This representation is determined by $\pi^{(\infty)}(\iota_q(E_X)) = E_{q^{-1} \cdot X}$.

We claim that this representation $\pi^{(\infty)}$, together with the left regular representation λ of G, is a covariant representation of $(D_r^{(\infty)}(P), G, \tau^{(\infty)})$. To show this, take $p, q, r \in P$ with $p, q \leq r$, $X \in \mathcal{J}$ and compute

$$\begin{aligned}
\lambda_p(\pi^{(\infty)}(\iota_q(E_X)))\lambda_p^* &= \lambda_p E_{q^{-1} \cdot X} \lambda_p^* = E_{p \cdot q^{-1} \cdot X} \\
&= E_{p \cdot r^{-1} \cdot r \cdot q^{-1} \cdot X} = E_{(rp^{-1})^{-1} \cdot (rq^{-1}) \cdot X} \\
&= \pi^{(\infty)}(\iota_{rp^{-1}}(\tau_{rq^{-1}}(E_X))) = \pi^{(\infty)}(\tau_p^{(\infty)}(\iota_q(E_X))).
\end{aligned}$$

So far, we have constructed a covariant representation $(\pi^{(\infty)}, \lambda)$ of the C*-dynamical system $(D_r^{(\infty)}(P), G, \tau^{(\infty)})$. Next we claim:

Lemma 4.1. *The covariant representation $(\pi^{(\infty)}, \lambda)$ gives rise to a faithful representation $\pi^{(\infty)} \rtimes_r \lambda$ of the reduced crossed product $D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)}, r} G$ on $\ell^2(G)$. This representation is determined by $(\pi^{(\infty)} \rtimes_r \lambda)(dU_g) = \pi^{(\infty)}(d)\lambda_g$ for all $d \in D_r^{(\infty)}(P)$ and $g \in G$. Here U_g are the canonical unitaries in the multiplier algebra of $D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)}, r} G$ implementing $\tau^{(\infty)}$.*

Proof. Apply Lemma 2.5.1 to $D = \pi^{(\infty)}(D_r^{(\infty)}(P))$. □

Using this representation $\pi^{(\infty)} \rtimes_r \lambda$, we will always think of $D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G$ as a concrete C*-algebra acting on $\ell^2(G)$.

Now consider the orthogonal projection $E_P \in \mathcal{L}(\ell^2(G))$ onto the subspace $\ell^2(P) \subseteq \ell^2(G)$. This projection lies in $D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G$.

Lemma 4.2. *The projection E_P is a full projection in $D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G$, and the corner $E_P(D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G)E_P$ can be identified with $C_r^*(P)$ via*

$$C_r^*(P) \ni V_p \mapsto E_P U_p E_P \in E_P(D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G)E_P.$$

Proof. The C*-algebra $D_r^{(\infty)}(P)$ (or rather $\pi^{(\infty)}(D_r^{(\infty)}(P))$) is generated by the projections $\{E_{q^{-1}\cdot X} : q \in P, X \in \mathcal{J}\}$. Thus the net $(E_{q^{-1}\cdot P})_{q\in P}$ is an approximate unit of $D_r^{(\infty)}(P)$, hence of $D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G$. As

$$E_{q^{-1}\cdot P} = U_q^* E_P U_q \in (D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G)E_P(D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G),$$

our first claim follows.

Now let us prove that the assignment $V_p \mapsto E_P U_p E_P$ extends to an isomorphism

$$C_r^*(P) \to E_P(D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G)E_P.$$

The assignment $V_p \mapsto E_P U_p E_P$ first of all extends to a homomorphism $C_r^*(P) \to E_P(D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G)E_P$ because the operator $E_P U_p E_P$, viewed as an operator on $\ell^2(P) \subseteq \ell^2(G)$, is really nothing other than the isometry V_p itself (note that U_p is just λ_p since we view $D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G$ as a concrete C*-algebra acting on $\ell^2(G)$ via $\pi^{(\infty)} \rtimes_r \lambda$). This observation implies that the resulting homomorphism $C_r^*(P) \to E_P(D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G)E_P$ must be injective. To show surjectivity, it is enough to prove that for all $p, q_1, q_2 \in P$ and $X \in \mathcal{J}$, the element $E_P E_{q_1^{-1}\cdot X} U_{q_2^{-1}p} E_P \in E_P(D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G)E_P$ lies in the image. But

$$\begin{aligned} E_P E_{q_1^{-1}\cdot X} U_{q_2^{-1}p} E_P &= (E_P E_{q_1^{-1}\cdot X} E_P)(E_P U_{q_2}^* E_P)(E_P U_p E_P) \\ &= (E_P E_{P\cap(q_1^{-1}\cdot X)} E_P)(E_P U_{q_2} E_P)^*(E_P U_p E_P) \\ &= (E_P E_{q_1^{-1}\cdot X} E_P)(E_P U_{q_2} E_P)^*(E_P U_p E_P) \end{aligned}$$

is the image of $E_{q_1^{-1}X} V_{q_2}^* V_p$. □

Corollary 4.3. *The embedding $\iota : C_r^*(P) \to D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G$ determined by $\iota(V_p) = E_P U_p E_P$ induces a KK-equivalence in $KK(C_r^*(P), D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G)$.*

5. From concrete to abstract

Corollary 4.3 tells us that if we are interested in the K-theory of $C_r^*(P)$, we can equally well study the reduced crossed product $D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G$. The situation is as follows:

(i) $D_r^{(\infty)}(P)$ is a commutative C*-algebra generated by the projections

$$\{E_{q^{-1}\cdot X} : q \in P, \emptyset \neq X \in \mathcal{J}\}.$$

As P is countable, this family of projections is countable as well ($\mathcal{J}$ is countable as P is). Moreover,

$$\{E_{q^{-1}\cdot X} : q \in P,\ X \in \mathcal{J}\} = \{E_{q^{-1}\cdot X} : q \in P,\ \emptyset \neq X \in \mathcal{J}\} \cup \{0\}$$

is multiplicatively closed because given $q_1, q_2 \in P$ and $X_1, X_2 \in \mathcal{J}$, we can choose $q \in P$ with $q_1 \leq q$ and $q_2 \leq q$, and then

$$(q_1^{-1} \cdot X_1) \cap (q_2^{-1} \cdot X_2) = q^{-1} \cdot \underbrace{((qq_1^{-1} \cdot X_1) \cap (qq_2^{-1} \cdot X_2))}_{\in \mathcal{J}},$$

so that

$$E_{q_1{}^{-1}\cdot X_1} E_{q_2{}^{-1}\cdot X_2} = E_{q^{-1}\cdot((qq_1^{-1}\cdot X_1)\cap(qq_2^{-1}\cdot X_2))}$$

lies in $\{E_{q^{-1}\cdot X} : q \in P,\ X \in \mathcal{J}\}$.

(ii) Assume that the constructible right ideals of P are independent. Then we can prove the following:
For all projections $E, E_1, \dots, E_n$ in $\{E_{q^{-1}\cdot X} : q \in P, X \in \mathcal{J}\}$, the strict inequalities $E_1, \dots, E_n \lneqq E$ imply $\bigvee_{j=1}^n E_j \lneqq E$. Here $\bigvee_{j=1}^n E_j$ is the smallest projection in $D_r^{(\infty)}(P)$ which is greater than or equal to $E_1, \dots, E_n$.
Here is the proof: Let $E_j = E_{q_j{}^{-1}\cdot X_j}$, $j = 1, \dots, n$, and let $E = E_{q^{-1}\cdot X}$ with $q, q_1, \dots, q_n \in P$ and $X, X_1, \dots, X_n \in \mathcal{J}$. It follows that $\bigvee_{j=1}^n E_{q_j{}^{-1}\cdot X_j} = E_{\bigcup_{j=1}^n q_j^{-1}\cdot X_j}$. We claim that $E_{q_j{}^{-1}\cdot X_j} \lneqq E_{q^{-1}\cdot X}$ for all $1 \leq j \leq n$ implies $\bigvee_{j=1}^n E_{q_j{}^{-1}\cdot X_j} \lneqq E_{q^{-1}\cdot X}$. Since for $Y_1, Y_2 \subseteq G$, the inequality $E_{Y_1} \leq E_{Y_2}$ is equivalent to $Y_1 \subseteq Y_2$, we have to show that $q_j^{-1} \cdot X_j \subsetneq q^{-1} \cdot X$ for all $1 \leq j \leq n$ implies $\bigcup_{j=1}^n q_j^{-1} \cdot X_j \subsetneq q^{-1} \cdot X$. But this follows from our assumption that the constructible right ideals of P are independent: Choose $r \in P$ such that $q \leq r$ and $q_j \leq r$ for all $1 \leq j \leq n$. Then $q_j^{-1} \cdot X_j \subsetneq q^{-1} \cdot X$ for all $1 \leq j \leq n$ implies that $(rq_j^{-1}) \cdot X_j \subsetneq (rq^{-1}) \cdot X$ for all $1 \leq j \leq n$. But $(rq^{-1}) \cdot X = (rq^{-1})X$ and $(rq_j^{-1}) \cdot X_j = (rq_j^{-1})X_j$ lie in $\mathcal{J}$ for all $1 \leq j \leq n$. Thus the independence condition tells us that

$$r\Big(\bigcup_{j=1}^n q_j^{-1} \cdot X_j\Big) = \bigcup_{j=1}^n (rq_j^{-1})X_j \subsetneq (rq^{-1})X = r(q^{-1} \cdot X).$$

Since left multiplication by r is injective, we deduce $\bigcup_{j=1}^n q_j^{-1} \cdot X_j \subsetneq q^{-1} \cdot X$, as claimed.

(iii) The G-action $\tau^{(\infty)}$ on $D_r^{(\infty)}(P)$ leaves the set of projections

$$\{E_{q^{-1}\cdot X} : q \in P,\ \emptyset \neq X \in \mathcal{J}\}$$

invariant.

This is the situation we are interested in. In the following section, we look at it from an abstract point of view.

6. The general K-theoretic result

We first formulate our assumptions:

(I) D is a commutative C*-algebra generated by a countable family $\{e_i\}_{i\in I}$ of pairwise distinct (commuting) non-zero projections. Moreover, $\{e_i\}_{i\in I} \cup \{0\}$ is *multiplicatively closed* (i.e. for all e_i, e_j in $\{e_i\}_{i\in I}$, either $e_i e_j = 0$ or there exists $k \in I$ such that $e_i e_j = e_k$).

(II) The family $\{e_i\}_{i\in I}$ is *independent*, i.e. given $e \in \{e_i\}_{i\in I}$ and finitely many $e_1, \dots, e_n \in \{e_i\}_{i\in I}$ with $e_1, \dots, e_n \lneq e$, we always have $\bigvee_{i=1}^n e_i \lneq e$, i.e. $e - \bigvee_{i=1}^n e_i$ is a non-zero projection. Here $\bigvee_{i=1}^n e_i$ is the smallest projection in D which is greater than (or equal to) all the e_i, $1 \leq i \leq n$. Note that since D is commutative, $\bigvee_{i=1}^n e_i = \sum_{\emptyset \neq J \subseteq \{1,\dots,n\}} (-1)^{|J|-1} \prod_{j\in J} e_j$.

(III) G is a discrete countable group and τ is an action of G on D which leaves $\{e_i\}_{i\in I}$ invariant. This means that there is an action of G on the index set I such that $\tau_g(e_i) = e_{g\cdot i}$.

Assume that we have a C*-dynamical system (D, G, τ) satisfying (I)–(III). In this situation, the homomorphisms $\phi_i : \mathbb{C} \to D, 1 \mapsto e_i$ (for $i \in I$), give rise to a KK-element in $KK(\bigoplus_{i\in I} \mathbb{C}, D) \cong \prod_{i\in I} KK(\mathbb{C}, D)$ which can be viewed as an element in equivariant KK-theory. This means that with respect to the G-action σ on $\bigoplus_{i\in I} \mathbb{C}$ given by shifting the index set I and the G-action τ on D, the ϕ_i yield in a canonical way an element $\mathbf{x} \in KK^G(\bigoplus_{i\in I} \mathbb{C}, D)$. This KK-element will be described in detail in §6.1. Here is our main result:

Theorem 6.1. *Assume that we are in the situation described above. Then for every finite subgroup H of G, the element $j^H(\mathrm{res}_H^G(\mathbf{x})) \in KK((\bigoplus_{i\in I} \mathbb{C}) \rtimes_\sigma H, D \rtimes_\tau H)$ is a KK-equivalence. Here res_H^G is the canonical restriction map $KK^G \to KK^H$ and j^H is the descent $KK^H(\bigoplus_{i\in I} \mathbb{C}, D) \to KK((\bigoplus_{i\in I} \mathbb{C}) \rtimes_\sigma H, D \rtimes_\tau H)$.*

The proof of this theorem is the content of §6.1 to §6.4.

Just a remark on notation: From now on, we write $c_0(I)$ for $\bigoplus_{i\in I} \mathbb{C}$ and $c_0(I, D)$ for $\bigoplus_{i\in I} D$. Moreover, given a Hilbert module Z, we write $\ell^2(I, Z)$ for $\bigoplus_{i\in I} Z$ (where the direct sum is taken in the sense of Hilbert modules).

6.1. Description of the KK-element. Our goal is to describe the element $\mathbf{x} \in KK^G(c_0(I), D)$. First of all, the element in $KK(c_0(I), D)$ given by the homomorphisms $\phi_i : \mathbb{C} \to D, 1 \mapsto e_i$ $(i \in I)$, can be represented by the Kasparov module $(\ell^2(I, D), \phi, 0)$. The left action of $c_0(I)$ on the Hilbert D-module $\ell^2(I, D)$ is given by

$$\phi := \bigoplus \phi_i : c_0(I) \to c_0(I, D) \subseteq \mathcal{K}(\ell^2(I, D)) \subseteq \mathcal{L}(\ell^2(I, D)).$$

Here $c_0(I, D)$ acts on $\ell^2(I, D)$ by diagonal operators. Let us write $\mathbb{1}_i \in c_0(I)$ for the element whose i-th component is 1 and whose other components are 0, and $\mathbb{1}_j \otimes d \in \ell^2(I, D)$ for the element whose j-th component is $d \in D$ and whose remaining components vanish. Then

$$\phi(\mathbb{1}_i)(\mathbb{1}_j \otimes d) = (\mathbb{1}_i \otimes \phi_i(1))(\mathbb{1}_j \otimes d) = (\mathbb{1}_i \otimes e_i)(\mathbb{1}_j \otimes d) = \delta_{i,j}(\mathbb{1}_i \otimes e_i d).$$

Since $\mathrm{Im}(\phi)$ is contained in the set of compact operators on $\ell^2(I, D)$, the operator in our Kasparov module may be chosen to be 0.

Now we want to interpret this Kasparov module as an element in $KK^G(c_0(I), D)$. So we introduce a G-action on the Hilbert module $\ell^2(I, D)$ which is compatible with the action τ of G on D so that ϕ becomes G-equivariant.

We let σ be the G-action on $c_0(I)$ determined by $\sigma_g(\mathbb{1}_i) = \mathbb{1}_{g\cdot i}$. The G-action $\sigma \otimes \tau$ on the Hilbert module $\ell^2(I, D)$ is given by

$$(\sigma \otimes \tau)_g(\mathbb{1}_i \otimes d) = \mathbb{1}_{g\cdot i} \otimes \tau_g(d).$$

It can be checked immediately that this G-action $\sigma \otimes \tau$ is compatible with the Hilbert D-module structure on $\ell^2(I, D)$, in the sense that $\langle(\sigma\otimes\tau)_g(\xi), (\sigma\otimes\tau)_g(\eta)\rangle_D = \tau_g(\langle\xi, \eta\rangle_D)$ and $(\sigma \otimes \tau)_g(\xi \cdot d) = (\sigma \otimes \tau)_g(\xi) \cdot \tau_g(d)$ for all $g \in G$, $\xi, \eta \in \ell^2(I, D)$ and $d \in D$. Conjugation yields a G-action $\mathrm{Ad}(\sigma \otimes \tau)$ of G on $\mathcal{L}(\ell^2(I, D))$ given by $G \ni g \mapsto \mathrm{Ad}((\sigma \otimes \tau)_g) \in \mathrm{Aut}(\mathcal{L}(\ell^2(I, D)))$. To check that ϕ is G-equivariant with respect to the G-action σ on $c_0(I)$ and $\mathrm{Ad}(\sigma \otimes \tau)$, it suffices to consider elements $\mathbb{1}_i \in c_0(I)$ and $\mathbb{1}_j \otimes d \in \ell^2(I, D)$. We compute

$$\begin{aligned}
(\phi(\sigma_g(\mathbb{1}_i)))(\mathbb{1}_j \otimes d) &= (\phi(\mathbb{1}_{g\cdot i}))(\mathbb{1}_j \otimes d) = (\mathbb{1}_{g\cdot i} \otimes e_{g\cdot i})(\mathbb{1}_j \otimes d)\\
&= \delta_{g\cdot i, j}\mathbb{1}_{g\cdot i} \otimes (e_{g\cdot i}d) = \delta_{i, g^{-1}\cdot j}\mathbb{1}_{g\cdot i} \otimes (e_{g\cdot i}d) = (\sigma \otimes \tau)_g\big(\delta_{i, g^{-1}\cdot j}\mathbb{1}_i \otimes (e_i\tau_{g^{-1}}(d))\big)\\
&= \mathrm{Ad}(\sigma \otimes \tau)_g(\phi(\mathbb{1}_i))(\mathbb{1}_j \otimes d).
\end{aligned}$$

This shows that the Kasparov module $(\ell^2(I, D), \phi, 0)$ together with the G-action $\sigma \otimes \tau$ really gives rise to an element $\mathbf{x} \in KK^G(c_0(I), D)$.

Let us summarize our construction in the following

Definition 6.1.1. Let $\mathbf{x} \in KK^G(c_0(I), D)$ (where G acts on $c_0(I)$ and D via σ and τ) be represented by the Kasparov G-module for $(c_0(I), D)$ consisting of

- the Hilbert D-module $\ell^2(I, D)$ with G-action $\sigma \otimes \tau$ given by

$$(\sigma \otimes \tau)_g(\mathbb{1}_i \otimes d) = \mathbb{1}_{g\cdot i} \otimes \tau_g(d),$$

- the equivariant homomorphism

$$\phi : c_0(I) \to \mathcal{K}(\ell^2(I, D)) \subseteq \mathcal{L}(\ell^2(I, D))$$

  determined by $(\phi(\mathbb{1}_i))(\mathbb{1}_j \otimes d) = \delta_{i,j}\mathbb{1}_i \otimes e_i d$,
- the operator $0 \in \mathcal{L}(\ell^2(I, D))$.

6.2. Descent of the restriction. Let $H \subseteq G$ be a subgroup. Our goal is to describe the element $j_r^H(\mathrm{res}_H^G(\mathbf{x})) \in KK(c_0(I) \rtimes_{\sigma,r} H, D \rtimes_{\tau,r} H)$ given by the descent of the restriction of $\mathbf{x}$ to H.

Proposition 6.2.1. *For every subgroup H of G, the KK-element $j_r^H(\mathrm{res}_H^G(\mathbf{x}))$ in $KK(c_0(I) \rtimes_{\sigma,r} H, D \rtimes_{\tau,r} H)$ is represented by the Kasparov $(c_0(I) \rtimes_{\sigma,r} H, D \rtimes_{\tau,r} H)$-module consisting of*

- *the Hilbert* $D \rtimes_{\tau,r} H$*-module* $\ell^2(I, D \rtimes_{\tau,r} H)$*,*
- *the homomorphism* $\phi \rtimes_r H : c_0(I) \rtimes_{\sigma,r} H \to \mathcal{L}(\ell^2(I, D \rtimes_{\tau,r} H))$ *given by* $(\phi \rtimes_r H)(\mathbb{1}_i U_h) = (\mathbb{1}_i \otimes e_i) \circ (\sigma_h \otimes U_h)$*,*
- *the operator* $0 \in \mathcal{L}(\ell^2(I, D \rtimes_{\tau,r} H))$*.*

Here U_h *are the canonical unitaries in the multiplier algebra of* $c_0(I) \rtimes_{\sigma,r} H$ *which implement* σ*.*

Proof. First of all, to obtain a Kasparov H-module for $(c_0(I), D)$ with respect to the restricted actions $\sigma|_H$ and $\tau|_H$ (we will denote these actions again by σ and τ) which represents $\mathrm{res}_H^G(\mathbf{x})$, we can just take the Kasparov G-module from Definition 6.1.1 and restrict the G-action $\sigma \otimes \tau$ to H.

We now describe the element $j_r^H(\mathrm{res}_H^G(\mathbf{x})) \in KK(c_0(I) \rtimes_{\sigma,r} H, D \rtimes_{\tau,r} H)$ following [Kas, §3.7]. The construction for full crossed products is also described in [Bla, Chapter VIII, §20.6], and it is very similar to the one for reduced crossed products. Of course, in the case of finite subgroups (which in view of Theorem 6.1 is the most interesting), it does not matter at all whether we take full or reduced crossed products.

By definition, $j_r^H(\mathrm{res}_H^G(\mathbf{x}))$ is represented by the Kasparov module

$$(\ell^2(I, D) \rtimes_{\tau,r} H, \psi, 0).$$

Let us start with the Hilbert $D \rtimes_{\tau,r} H$-module $\ell^2(I, D) \rtimes_{\tau,r} H$. It is the completion of the pre-Hilbert $C_c(H, D)$-module whose underlying vector space $C_c(H, \ell^2(I, D))$ consists of all functions from H to $\ell^2(I, D)$ with finite support (we are in the discrete case). Given such a function $\xi \in C_c(H, \ell^2(I, D))$ and an element $b \in C_c(H, D)$, the right action of $C_c(H, D)$ on $C_c(H, \ell^2(I, D))$ is given by

$$(\xi \bullet b)(h) = \sum_{\tilde{h} \in H} \xi(\tilde{h}) \tau_{\tilde{h}}(b(\tilde{h}^{-1} h)). \tag{4}$$

Given two functions $\xi, \eta \in C_c(H, \ell^2(I, D))$, the $C_c(H, D)$-valued inner product on $C_c(H, \ell^2(I, D))$ is given by

$$\langle \xi, \eta \rangle_{C_c(H,D)}(h) = \sum_{\tilde{h} \in H} \tau_{\tilde{h}^{-1}}(\langle \xi(\tilde{h}), \eta(\tilde{h} h) \rangle_D). \tag{5}$$

Consider $D \rtimes_{\tau,r} H$ as a Hilbert module over itself and form the direct sum $\ell^2(I, D \rtimes_{\tau,r} H)$. We claim that the map

$$\Theta : C_c(H, \ell^2(I, D)) \to \ell^2(I, D \rtimes_{\tau,r} H), \quad \xi \mapsto ([h \mapsto (\xi(h))_i])_i,$$

extends to an isomorphism $\Theta : \ell^2(I, D) \rtimes_{\tau,r} H \xrightarrow{\cong} \ell^2(I, D \rtimes_{\tau,r} H)$ of Hilbert $D \rtimes_{\tau,r} H$-modules. Here we view functions from H to D with finite support as elements of $D \rtimes_{\tau,r} H$, and a function $f : H \to D$ is often denoted by $[h \mapsto f(h)]$.

As Θ obviously has dense image in $\ell^2(I, D \rtimes_{\tau,r} H)$, it suffices to check that Θ preserves the right $D \rtimes_{\tau,r} H$-actions as well as the inner products. It certainly suffices to check this for elements in $C_c(H, \ell^2(I, D))$ of the form $(\mathbb{1}_i \otimes d) U_h = [\tilde{h} \mapsto \delta_{\tilde{h},h} \mathbb{1}_i \otimes d]$.

Such an element corresponds to $\mathbb{1}_i \otimes (dU_h) \in \ell^2(I, D \rtimes_{\tau,r} H)$ under Θ. Hereafter, U_h is the characteristic function of $h \in H$. For the right $D \rtimes_{\tau,r} H$-actions, it certainly suffices to look at elements in $C_c(H, D) \subseteq D \rtimes_{\tau,r} H$ of the form $b = d_b U_{h_b}$. For $\xi = (\mathbb{1}_i \otimes d_\xi) U_{h_\xi} \in C_c(H, \ell^2(I, D))$ and $b = d_b U_{h_b}$, by (4) we have

$$
\begin{aligned}
(\xi \bullet b)(h) &= \sum_{\tilde{h} \in H} \delta_{\tilde{h}, h_\xi} (\mathbb{1}_i \otimes d_\xi) \tau_{\tilde{h}} (\delta_{\tilde{h}^{-1} h, h_b} d_b) = \delta_{h_\xi^{-1} h, h_b} (\mathbb{1}_i \otimes d_\xi)(\tau_{h_\xi}(d_b)) \\
&= \delta_{h_\xi^{-1} h, h_b} \mathbb{1}_i \otimes (d_\xi \tau_{h_\xi}(d_b)) = (\mathbb{1}_i \otimes d_\xi \tau_{h_\xi}(d_b)) U_{h_\xi h_b}(h),
\end{aligned}
$$

so that

$$
\begin{aligned}
\Theta(\xi \bullet b) &= \mathbb{1}_i \otimes (d_\xi \tau_{h_\xi}(d_b) U_{h_\xi h_b}) = \mathbb{1}_i \otimes (d_\xi U_{h_\xi})(d_b U_{h_b}) = (\mathbb{1}_i \otimes (d_\xi U_{h_\xi}))(d_b U_{h_b}) \\
&= (\Theta(\xi)) \cdot b,
\end{aligned}
$$

where in the last line, we let b act on $\Theta(\xi) \in \ell^2(I, D \rtimes_\tau H)$ using the right $D \rtimes_\tau H$-module structure of $\ell^2(I, D \rtimes_\tau H)$.

Moreover, for $\xi = (\mathbb{1}_i \otimes d_\xi) U_{h_\xi}$ and $\eta = (\mathbb{1}_j \otimes d_\eta) U_{h_\eta}$ in $C_c(H, \ell^2(I, D))$, by (5) we have

$$
\begin{aligned}
\langle \xi, \eta \rangle_{C_c(H,D)}(h) &= \sum_{\tilde{h} \in H} \tau_{\tilde{h}^{-1}} (\langle \delta_{\tilde{h}, h_\xi} \mathbb{1}_i \otimes d_\xi, \delta_{\tilde{h} h, h_\eta} \mathbb{1}_j \otimes d_\eta \rangle_D) \\
&= \delta_{h_\xi h, h_\eta} \tau_{h_\xi^{-1}} (\delta_{i,j} d_\xi^* d_\eta) = \delta_{i,j} \delta_{h, h_\xi^{-1} h_\eta} \tau_{h_\xi^{-1}} (d_\xi^* d_\eta) \\
&= \delta_{i,j} (d_\xi U_{h_\xi})^* (d_\eta U_{h_\eta})(h) \\
&= \langle \mathbb{1}_i \otimes (d_\xi U_{h_\xi}), \mathbb{1}_j \otimes (d_\eta U_{h_\eta}) \rangle_{D \rtimes_{\tau,r} H}(h) \\
&= \langle \Theta(\xi), \Theta(\eta) \rangle_{D \rtimes_{\tau,r} H}(h).
\end{aligned}
$$

This proves our claim that Θ extends to an isomorphism of Hilbert $D \rtimes_{\tau,r} H$-modules.

Finally, it remains to describe ψ, i.e. to describe the left $c_0(I) \rtimes_{\sigma,r} H$-action on the Hilbert module. Let $a \in C_c(H, c_0(I)) \subseteq c_0(I) \rtimes_{\sigma,r} H$. Then for $\xi \in C_c(H, \ell^2(I, D))$, $\psi(a)\xi$ is given by

$$
(\psi(a)\xi)(h) = \sum_{\tilde{h} \in H} \phi(a(\tilde{h})) \big((\sigma \otimes \tau)_{\tilde{h}} \xi(\tilde{h}^{-1} h) \big)
$$

(see [Kas, §3.7] or [Bla, Chapter VIII, §20.6]). To explicitly compute the action, we again take $\xi = (\mathbb{1}_j \otimes d_\xi) U_{h_\xi}$ and $a = \mathbb{1}_i U_{h_a}$. Then

$$
\begin{aligned}
(\psi(a)\xi)(h) &= \sum_{\tilde{h} \in H} \delta_{\tilde{h}, h_a} \phi(\mathbb{1}_i) \big((\sigma \otimes \tau)_{\tilde{h}} \delta_{\tilde{h}^{-1} h, h_\xi} \mathbb{1}_j \otimes d_\xi \big) \\
&= \delta_{h_a^{-1} h, h_\xi} \phi_i(\mathbb{1}_i)(\mathbb{1}_{h_a \cdot j} \otimes \tau_{h_a}(d_\xi)) \\
&= \delta_{h_a^{-1} h, h_\xi} \delta_{i, h_a \cdot j} (\mathbb{1}_i \otimes e_i \tau_{h_a}(d_\xi)) \\
&= \delta_{i, h_a \cdot j} \mathbb{1}_i \otimes ((e_i U_{h_a})(d_\xi U_{h_\xi})(h)) \\
&= \Theta^{-1}((\mathbb{1}_i \sigma_{h_a} \otimes e_i U_{h_a})(\mathbb{1}_j \otimes (d_\xi U_{h_\xi})))(h).
\end{aligned}
$$

Thus $\Theta \circ \psi(a) \circ \Theta^{-1} = (\mathbb{1}_i \otimes e_i) \circ (\sigma_{h_a} \otimes U_{h_a})$.

So all in all, we have computed that $j_r^H(\mathrm{res}_H^G(\mathbf{x})) \in KK(c_0(I) \rtimes_{\sigma,r} H, D \rtimes_{\tau,r} H)$ is represented by the Kasparov module

$$(\ell^2(I, D \rtimes_{\tau,r} H), \phi \rtimes_r H, 0)$$

where $\phi \rtimes_r H : c_0(I) \rtimes_{\sigma,r} H \to \mathcal{L}(\ell^2(I, D \rtimes_{\tau,r} H))$ is given by

$$c_0(I) \rtimes_{\sigma,r} H \ni \mathbb{1}_i U_h \mapsto (\mathbb{1}_i \otimes e_i) \circ (\sigma_h \otimes U_h) \in \mathcal{L}(\ell^2(I, D \rtimes_{\tau,r} H)). \qquad \square$$

6.3. Direct sum decomposition. Let H be a subgroup of G. It is possible to decompose $c_0(I) \rtimes_{\sigma,r} H$ into direct summands corresponding to the H-orbits on I, i.e.

$$c_0(I) \rtimes_{\sigma,r} H = \bigoplus_{[i] \in H \backslash I} (c_0(H \cdot i) \rtimes_{\sigma,r} H).$$

Let us denote the summand $c_0(H \cdot i) \rtimes_{\sigma,r} H$ corresponding to $[i] \in H \backslash I$ by $C_{[i]}$ and let $\iota_{[i]}$ be the embedding $C_{[i]} \to c_0(I) \rtimes_{\sigma,r} H$.

As explained in [Bla, Theorem 19.7.1],

$$\prod_{[i] \in H \backslash I} (KK(\iota_{[i]}) \otimes \sqcup) : KK(c_0(I) \rtimes_{\sigma,r} H, D \rtimes_{\tau,r} H) \to \prod_{[i] \in H \backslash I} KK(C_{[i]}, D \rtimes_{\tau,r} H)$$

is an isomorphism. Here "$\otimes$" stands for the Kasparov product.

It is immediate that under this isomorphism, the element $j_r^H(\mathrm{res}_H^G(\mathbf{x}))$ corresponds to $(\mathbf{x}_{[i]})_{[i] \in H \backslash I}$ where $\mathbf{x}_{[i]} \in KK(C_{[i]}, D \rtimes_{\tau,r} H)$ is represented by the Kasparov module

$$(\ell^2(H \cdot i, D \rtimes_{\tau,r} H), (\phi \rtimes_r H)_{[i]}, 0) \tag{6}$$

with $(\phi \rtimes_r H)_{[i]}$ given by $C_{[i]} \ni \mathbb{1}_j U_h \mapsto (\mathbb{1}_j \otimes e_j) \circ (\sigma_h \otimes U_h) \in \mathcal{L}(\ell^2(H \cdot i, D \rtimes_{\tau,r} H))$.

In other words, we have

$$\mathbf{x}_{[i]} = KK(\iota_{[i]}) \otimes j_r^H(\mathrm{res}_H^G(\mathbf{x})). \tag{7}$$

We describe $\mathbf{x}_{[i]}$ alternatively as follows: Let $\varphi_{[i]}$ be the homomorphism

$$\varphi_{[i]} : C_{[i]} \to \mathcal{K}(\ell^2(H \cdot i)) \otimes (D \rtimes_{\tau,r} H), \quad \mathbb{1}_j U_h \mapsto e_{j,h^{-1} \cdot j} \otimes e_j U_h, \tag{8}$$

where $e_{j,h^{-1} \cdot j}$ is the rank 1 operator $\langle \sqcup, \varepsilon_{h^{-1} \cdot j} \rangle \varepsilon_j \in \mathcal{L}(\ell^2(H \cdot i))$ ($\{\varepsilon_j : j \in H \cdot i\}$ is the canonical orthonormal basis of $\ell^2(H \cdot i)$).

Existence of $\varphi_{[i]}$ can be seen as follows: Using a faithful representation of D on a Hilbert space $\mathcal{H}$, we can view D as a sub-C*-algebra of $\mathcal{L}(\mathcal{H})$. Hence, according to the definition of the reduced crossed product, the C*-algebra $\mathcal{K}(\ell^2(H \cdot i)) \otimes_{\min} (D \rtimes_{\tau,r} H)$ acts on the Hilbert space $\ell^2(H \cdot i) \otimes \mathcal{H} \otimes \ell^2(H)$. At the same time, using the definition of the reduced crossed product $C_{[i]} = c_0(H \cdot i) \rtimes_{\sigma,r} H$, we obtain a faithful representation π of $C_{[i]}$ sending $\mathbb{1}_j U_h \in C_{[i]}$ to the operator $\pi(\mathbb{1}_j)(1 \otimes 1 \otimes \lambda_h)$ on $\ell^2(H \cdot i) \otimes \mathcal{H} \otimes \ell^2(H)$ where $\pi(\mathbb{1}_j)$ is given by $\pi(\mathbb{1}_j)(\varepsilon_k \otimes \xi \otimes \varepsilon_x) = (e_{x^{-1} \cdot j, x^{-1} \cdot j} \otimes e_{x^{-1} \cdot j} \otimes 1)(\varepsilon_k \otimes \xi \otimes \varepsilon_x)$ for $j, k \in H \cdot i$, $\xi \in \mathcal{H}$ and $x \in H$. Here $e_{x^{-1} \cdot j, x^{-1} \cdot j}$ is the rank 1 projection corresponding to the basis vector $\varepsilon_{x^{-1} \cdot j} \in \ell^2(H \cdot i)$. Now, applying Fell's absorption principle or rather adapting its proof, we consider the unitary W on $\ell^2(H \cdot i) \otimes \mathcal{H} \otimes \ell^2(H)$ given by $W(\varepsilon_k \otimes \xi \otimes \varepsilon_x) = \varepsilon_{x \cdot k} \otimes \xi \otimes \varepsilon_x$. Then a direct computation shows

$$\mathrm{Ad}(W) \circ (\pi(\mathbb{1}_j)(1 \otimes 1 \otimes \lambda_h)) = e_{j,h^{-1} \cdot j} \otimes e_j U_h.$$

Therefore, $\mathrm{Ad}(W) \circ \pi$ is the desired homomorphism $\varphi_{[i]}$.

The homomorphism $e_{i,i} \otimes \mathrm{id}_{D \rtimes_{\tau,r} H} : D \rtimes_{\tau,r} H \to \mathcal{K}(\ell^2(H \cdot i)) \otimes (D \rtimes_{\tau,r} H)$, $b \mapsto e_{i,i} \otimes b$, gives a KK-equivalence between $D \rtimes_{\tau,r} H$ and $\mathcal{K}(\ell^2(H \cdot i)) \otimes (D \rtimes_{\tau,r} H)$.

Lemma 6.3.1. $\mathbf{x}_{[i]} = KK(\varphi_{[i]}) \otimes KK(e_{i,i} \otimes \mathrm{id}_{D \rtimes_{\tau,r} H})^{-1}$ *where* $\otimes$ *is the Kasparov product.*

Proof. Viewing $D \rtimes_{\tau,r} H$ as a full corner in $\mathcal{K}(\ell^2(H \cdot i)) \otimes (D \rtimes_{\tau,r} H)$ via $e_{i,i} \otimes \mathrm{id}_{D \rtimes_{\tau,r} H}$, it is clear that $KK(e_{i,i} \otimes \mathrm{id}_{D \rtimes_{\tau,r} H})^{-1}$ is represented by the Kasparov module given by the $(\mathcal{K}(\ell^2(H \cdot i)) \otimes (D \rtimes_{\tau,r} H))$-$D \rtimes_{\tau,r} H$-imprimitivity bimodule $\ell^2(H \cdot i, D \rtimes_{\tau,r} H)$. This Kasparov module is explicitly given by the Hilbert $D \rtimes_{\tau,r} H$-module $\ell^2(H \cdot i, D \rtimes_{\tau,r} H)$ and the left action

$$\mathcal{K}(\ell^2(H \cdot i)) \otimes (D \rtimes_{\tau,r} H) \to \mathcal{K}(\ell^2(H \cdot i, D \rtimes_{\tau,r} H)), \quad e_{j,h^{-1} \cdot j} \otimes b \mapsto \mathbb{1}_j \sigma_h \otimes b.$$

Using the descriptions of $\mathbf{x}_{[i]}$ and $\varphi_{[i]}$ from (6) and (8), it is clear that

$$\mathbf{x}_{[i]} = KK(\varphi_{[i]}) \otimes KK(e_{i,i} \otimes \mathrm{id}_{D \rtimes_{\tau,r} H})^{-1}. \qquad \square$$

Corollary 6.3.2. *Let B be a sub-C*-algebra of $D \rtimes_{\tau,r} H$ such that for all $j \in H \cdot i$ and $h \in H$, $e_j U_h$ lies in B. Let ι be the inclusion $B \hookrightarrow D \rtimes_{\tau,r} H$, let $\varphi_{[i]}|^B$ be the homomorphism $C_{[i]} \to \mathcal{K}(\ell^2(H \cdot i)) \otimes B$, $a \mapsto \varphi_{[i]}(a)$ (we just restrict the image of $\varphi_{[i]}$), and denote by $e_{i,i} \otimes \mathrm{id}_B$ the homomorphism $B \to \mathcal{K}(\ell^2(H \cdot i)) \otimes B$, $b \mapsto e_{i,i} \otimes b$. Then*

$$KK(\varphi_{[i]}|^B) \otimes KK(e_{i,i} \otimes \mathrm{id}_B)^{-1} \otimes KK(\iota) = \mathbf{x}_{[i]}.$$

Proof. We have

$$\begin{aligned}
&KK(\varphi_{[i]}|^B) \otimes KK(e_{i,i} \otimes \mathrm{id}_B)^{-1} \otimes KK(\iota) \otimes KK(e_{i,i} \otimes \mathrm{id}_{D \rtimes_{\tau,r} H}) \\
&\quad = KK(\varphi_{[i]}|^B) \otimes KK(e_{i,i} \otimes \mathrm{id}_B)^{-1} \otimes KK(e_{i,i} \otimes \mathrm{id}_B) \otimes KK(\mathrm{id}_{\mathcal{K}(\ell^2(H \cdot i))} \otimes \iota) \\
&\quad = KK(\varphi_{[i]}|^B) \otimes KK(\mathrm{id}_{\mathcal{K}(\ell^2(H \cdot i))} \otimes \iota) \\
&\quad = KK(\varphi_{[i]}).
\end{aligned}$$

Now multiply on the right with $KK(e_{i,i} \otimes \mathrm{id}_{D \rtimes_{\tau,r} H})^{-1}$ and use Lemma 6.3.1. $\square$

6.4. KK-equivalences for all finite subgroups. Now we consider finite subgroups. Since in this case, we do not have to distinguish between full and reduced crossed products, we can omit the index r everywhere. Our goal is to prove

Theorem 6.4.1. *For every finite subgroup H of G, the element $j^H(\mathrm{res}^G_H(\mathbf{x}))$ in $KK(c_0(I) \rtimes_\sigma H, D \rtimes_\tau H)$ is a KK-equivalence.*

As both $c_0(I) \rtimes_\sigma H$ and $D \rtimes_\tau H$ satisfy the UCT being crossed products of commutative C*-algebras by amenable groups, it suffices to prove that $j^H(\mathrm{res}^G_H(\mathbf{x}))$ induces an isomorphism on K-theory. To show this, the strategy is to reduce everything to finite-dimensional sub-C*-algebras. Therefore, we write both $c_0(I) \rtimes_\sigma H$ and $D \rtimes_\tau H$ as inductive limits of finite-dimensional C*-algebras and consider the corresponding inductive limit descriptions of their K-groups.

In what follows, we write K_* for the direct sum of K_0 and K_1 viewed as a $\mathbb{Z}/2\mathbb{Z}$-graded abelian group.

We start with $c_0(I) \rtimes_\sigma H$. We have already seen in §6.3 the decomposition $c_0(I) \rtimes_\sigma H = \bigoplus_{[i] \in H \backslash I} C_{[i]}$. Thus, it is clear that we have $c_0(I) \rtimes_\sigma H \cong \varinjlim_F \bigoplus_{[i] \in [F]} C_{[i]}$, where the limit is taken over the finite subsets F of I and we denote the image of F under the projection $I \to H \backslash I$ by $[F]$. Therefore we obtain $\varinjlim_F \bigoplus_{[i] \in [F]} K_*(C_{[i]}) \cong K_*(c_0(I) \rtimes_\sigma H)$, and this identification is induced by the homomorphisms

$$\sum_{[i] \in [F]} \iota_{[i]} : \bigoplus_{[i] \in [F]} K_*(C_{[i]}) \to K_*(c_0(I) \rtimes_\sigma H).$$

Now we consider $D \rtimes_\tau H$. For a finite subset F of I, let $(D \rtimes_\tau H)_F$ be the sub-C*-algebra of $D \rtimes_\tau H$ which is generated by $\{e_i U_h : i \in H \cdot F,\ h \in H\}$. As before, we certainly have $D \rtimes_\tau H \cong \varinjlim_F (D \rtimes_\tau H)_F$ and thus $\varinjlim_F K_*((D \rtimes_\tau H)_F) \cong K_*(D \rtimes_\tau H)$. This identification is realized by the homomorphisms induced by the canonical inclusions $(D \rtimes_\tau H)_F \hookrightarrow D \rtimes_\tau H$ on K-theory.

We now compare these direct limit decompositions. Given a finite subset F of I, we set

$$\mathbf{x}^F_{[i]} := KK(\varphi_{[i]}|^{(D \rtimes_\tau H)_F}) \otimes KK(e_{i,i} \otimes \mathrm{id}_{(D \rtimes_\tau H)_F})^{-1} \tag{9}$$

using the notation from Corollary 6.3.2. Let $K_*(\mathbf{x}^F_{[i]})$ be the homomorphism induced on K-theory by $\mathbf{x}^F_{[i]}$. By (7) and Corollary 6.3.2, the diagram

$$\begin{array}{ccc} K_*(C_{[i]}) & \xrightarrow{K_*(\mathbf{x}^F_{[i]})} & K_*((D \rtimes_\tau H)_F) \\ \downarrow {\scriptstyle K_*(\iota_{[i]})} & & \downarrow \\ K_*(c_0(I) \rtimes_\sigma H) & \xrightarrow{K_*(j^H(\mathrm{res}^G_H(\mathbf{x})))} & K_*(D \rtimes_\tau H) \end{array} \tag{10}$$

commutes, where the right vertical arrow is induced by the canonical inclusion $(D \rtimes_\tau H)_F \hookrightarrow D \rtimes_\tau H$. Therefore, for every finite subset F of I, we have a homomorphism $\sum_{[i] \in [F]} K_*(\mathbf{x}^F_{[i]}) : K_*(\bigoplus_{[i] \in [F]} C_{[i]}) \to K_*((D \rtimes_\tau H)_F)$, and these homomorphisms induce a homomorphism

$$\varinjlim_F \sum_{[i] \in [F]} K_*(\mathbf{x}^F_{[i]}) : \varinjlim_F \bigoplus_{[i] \in [F]} K_*(C_{[i]}) \to \varinjlim_F K_*((D \rtimes_\tau H)_F)$$

by a similar computation to the one in Corollary 6.3.2. By commutativity of (10), the diagram

$$\begin{array}{ccc} \varinjlim_F \bigoplus_{[i]\in[F]} K_*(C_{[i]}) & \xrightarrow{\varinjlim_F \sum_{[i]\in[F]} K_*(\mathbf{x}^F_{[i]})} & \varinjlim_F K_*((D \rtimes_\tau H)_F) \\ \downarrow\cong & & \downarrow\cong \\ K_*(c_0(I) \rtimes_\sigma H) & \xrightarrow{K_*(j^H(\mathrm{res}^G_H(\mathbf{x})))} & K_*(D \rtimes_\tau H) \end{array} \tag{11}$$

commutes as well.

In these inductive limits, it clearly suffices to only take those finite subsets F which satisfy the condition that $\{e_i : i \in H \cdot F\} \cup \{0\}$ is multiplicatively closed. Now the point is that we will prove in the next proposition that for these finite subsets F, $\sum_{[i]\in[F]} K_*(\mathbf{x}^F_{[i]}) : \bigoplus_{[i]\in[F]} K_*(C_{[i]}) \to K_*((D \rtimes_\tau H)_F)$ is an isomorphism. This will then imply that the homomorphism $\varinjlim_F \sum_{[i]\in[F]} K_*(\mathbf{x}^F_{[i]}) : \varinjlim_F \bigoplus_{[i]\in[F]} K_*(C_{[i]}) \to \varinjlim_F K_*((D \rtimes_\tau H)_F)$ is an isomorphism, where we take the inductive limit over those F satisfying the condition that $\{e_i : i \in H \cdot F\} \cup \{0\}$ is multiplicatively closed. Because diagram (11) commutes, this will then imply our main observation that $K_*(j^H(\mathrm{res}^G_H(\mathbf{x})))$ is an isomorphism.

Proposition 6.4.2. *Let F be a finite subset of I such that $\{e_j : j \in H \cdot F\} \cup \{0\}$ is multiplicatively closed. Then the KK-elements $\mathbf{x}^F_{[i]}$, $[i] \in [F]$, induce a K-theoretic isomorphism*

$$\sum_{[i]\in[F]} K_*(\mathbf{x}^F_{[i]}) : \bigoplus_{[i]\in[F]} K_*(C_{[i]}) \to K_*((D \rtimes_\tau H)_F).$$

Proof. We decompose $(D \rtimes_\tau H)_F$ into direct summands as follows: For every $j \in F$, set $e(j) := e_j - \bigvee_{k\in H\cdot F,\, e_k \lneq e_j} e_k$, and for every $i \in F$, define $e([i]) = \sum_{j\in H\cdot i} e(j)$. By construction, the following facts hold:

- For every j in F, $e(j) \neq 0$ as $\{e_i\}_{i\in I}$ is independent (see (II)) and because of our assumption that $e_j \neq 0$ for all $j \in I$.
- For $i, j \in F$ with $[i] \neq [j]$, $e([i]) \perp e([j])$.
- $\bigvee_{j\in H\cdot F} e_j = \sum_{[i]\in[F]} e([i])$.
- For every i in F, $e([i])$ is H-invariant with respect to the action τ.

The last fact implies that these projections $e([i])$ are central in $(D \rtimes_\tau H)_F$. Thus, using this, the second and third fact and also our condition that $\{e_i : i \in H \cdot F\} \cup \{0\}$ is multiplicatively closed, we deduce

$$(D \rtimes_\tau H)_F = \bigoplus_{[i]\in[F]} e([i])((D \rtimes_\tau H)_F)e([i]).$$

Using the first two facts, it is straightforward to check that $e([i])((D \rtimes_\tau H)_F)e([i])$ is generated as a C*-algebra by the elements $e(j)U_h$ for $j \in H \cdot i$ and $h \in H$, and that we can identify $e([i])((D \rtimes_\tau H)_F)e([i])$ with $c_0(H \cdot i) \rtimes_\sigma H = C_{[i]}$ via

$$e(j)U_h \mapsto \mathbb{1}_j U_h. \tag{12}$$

Thus we obtain an isomorphism

$$(D \rtimes_\tau H)_F \cong \bigoplus_{[i]\in[F]} C_{[i]}.$$

Let $\pi_F^{[i]} : (D \rtimes_\tau H)_F \cong \bigoplus_{[i]\in[F]} C_{[i]} \to C_{[i]}$ be the composition of this isomorphism with the canonical projection $\bigoplus_{[i]\in[F]} C_{[i]} \to C_{[i]}$. It follows that in K-theory, $\bigoplus_{[i]\in[F]} K_*(\pi_F^{[i]}) : K_*((D \rtimes_\tau H)_F) \to \bigoplus_{[i]\in[F]} K_*(C_{[i]})$ is an isomorphism. This means that to show that $\sum_{[i]\in[F]} K_*(\mathbf{x}_{[i]}^F)$ is an isomorphism, we can equally well prove that the composition

$$\bigoplus_{[i]\in[F]} K_*(C_{[i]}) \xrightarrow{\sum_{[i]\in[F]} K_*(\mathbf{x}_{[i]}^F)} K_*((D \rtimes_\tau H)_F) \xrightarrow{\bigoplus_{[i]\in[F]} K_*(\pi_F^{[i]})} \bigoplus_{[i]\in[F]} K_*(C_{[i]})$$

is an isomorphism.

This composition can be described by the $[F] \times [F]$-matrix whose $([i],[j])$-th entry is $K_*(\pi_F^{[i]}) \circ K_*(\mathbf{x}_{[j]}^F)$ (here $\circ$ is composition of homomorphisms). Going through our constructions, it is clear that

$$K_*(\pi_F^{[i]}) \circ K_*(\mathbf{x}_{[j]}^F) \neq 0 \text{ only if } \bigvee_{l\in H\cdot j} e_l \geq e([i]) \Leftrightarrow \bigvee_{l\in H\cdot j} e_l \geq \bigvee_{k\in H\cdot i} e_k. \tag{13}$$

It is immediate that $[j] \geq [i] :\Leftrightarrow \bigvee_{l\in H\cdot j} e_l \geq \bigvee_{k\in H\cdot i} e_k$ defines a partial order relation on $[F]$. If we arrange the elements of $[F]$ in increasing order with respect to this partial order (increasing means that the elements $[j]$ which come after an element $[i]$ do not satisfy $[j] \leq [i]$), then (13) tells us that $(K_*(\pi_F^{[i]}) \circ K_*(\mathbf{x}_{[j]}^F))_{[i],[j]}$ becomes an upper triangular matrix. Hence the $[F]\times[F]$-matrix describing $(\bigoplus_{[i]\in[F]} K_*(\pi_F^{[i]}))\circ(\sum_{[i]\in[F]} K_*(\mathbf{x}_{[i]}^F))$ is the sum of a nilpotent matrix and a diagonal matrix whose $([i],[i])$-th entry is $K_*(\pi_F^{[i]}) \circ K_*(\mathbf{x}_{[i]}^F)$. To prove that the matrix $(K_*(\pi_F^{[i]}) \circ K_*(\mathbf{x}_{[j]}^F))_{[i],[j]}$ is invertible, it remains to prove that the diagonal entries of this matrix are invertible, i.e. $K_*(\pi_F^{[i]}) \circ K_*(\mathbf{x}_{[i]}^F) : K_*(C_{[i]}) \to K_*(C_{[i]})$ is an isomorphism for all $[i] \in [F]$.

Recall that

$$\mathbf{x}_{[i]}^F = KK(\varphi_{[i]}|^{(D\rtimes_\tau H)_F}) \otimes KK(e_{i,i} \otimes \mathrm{id}_{(D\rtimes_\tau H)_F})^{-1},$$

so that

$$K_*(\mathbf{x}_{[i]}^F) = K_*(e_{i,i} \otimes \mathrm{id}_{(D\rtimes_\tau H)_F})^{-1} \circ K_*(\varphi_{[i]}|^{(D\rtimes_\tau H)_F}).$$

As

$$(e_{i,i} \otimes \mathrm{id}_{C_{[i]}}) \circ \pi_F^{[i]} = (\mathrm{id}_{\mathcal{L}(\ell^2(H\cdot i))} \otimes \pi_F^{[i]}) \circ (e_{i,i} \otimes \mathrm{id}_{(D\rtimes_\tau H)_F}),$$

we obtain

$$\begin{aligned} K_*(\pi_F^{[i]}) \circ K_*(\mathbf{x}_{[i]}^F) &= K_*(\pi_F^{[i]}) \circ K_*(e_{i,i} \otimes \mathrm{id}_{(D\rtimes_\tau H)_F})^{-1} \circ K_*(\varphi_{[i]}|^{(D\rtimes_\tau H)_F}) \\ &= K_*(e_{i,i} \otimes \mathrm{id}_{C_{[i]}})^{-1} \circ K_*(\mathrm{id}_{\mathcal{L}(\ell^2(H\cdot i))} \otimes \pi_F^{[i]}) \circ K_*(\varphi_{[i]}|^{(D\rtimes_\tau H)_F}) \\ &= K_*(e_{i,i} \otimes \mathrm{id}_{C_{[i]}})^{-1} \circ K_*((\mathrm{id}_{\mathcal{L}(\ell^2(H\cdot i))} \otimes \pi_F^{[i]}) \circ \varphi_{[i]}|^{(D\rtimes_\tau H)_F}). \end{aligned}$$

Note that $\mathcal{L}(\ell^2(H\cdot i)) = \mathcal{K}(\ell^2(H\cdot i))$ as H is finite.

To prove that $K_*(\pi_F^{[i]}) \circ K_*(\mathbf{x}_{[i]}^F)$ is an isomorphism, it therefore suffices to check that $K_*((\mathrm{id}_{\mathcal{L}(\ell^2(H\cdot i))} \otimes \pi_F^{[i]}) \circ \varphi_{[i]}|^{(D\rtimes_\tau H)_F})$ is an isomorphism. By (8) and (12), the element $(\mathrm{id}_{\mathcal{L}(\ell^2(H\cdot i))} \otimes \pi_F^{[i]}) \circ \varphi_{[i]}|^{(D\rtimes_\tau H)_F}$ is given by the homomorphism

$$C_{[i]} \to \mathcal{L}(\ell^2(H\cdot i)) \otimes C_{[i]}, \qquad \mathbb{1}_j U_h \mapsto e_{j,h^{-1}\cdot j} \otimes \mathbb{1}_j U_h.$$

Let $s : H\cdot i \to H$ be a map satisfying $s(h\cdot i)\cdot i = h\cdot i$. Define

$$W := \sum_{j\in H\cdot i} \sigma_{s(j)} \otimes \mathbb{1}_j \in \mathcal{L}(\ell^2(H\cdot i)) \otimes C_{[i]}.$$

We finally claim that W is a unitary such that

$$\mathrm{Ad}(W^*) \circ (\mathrm{id}_{\mathcal{L}(\ell^2(H\cdot i))} \otimes \pi_F^{[i]}) \circ \varphi_{[i]}|^{(D\rtimes_\tau H)_F} = e_{i,i} \otimes \mathrm{id}_{C_{[i]}}.$$

This follows from the following computations:

$$W^*W = \sum_{j\in H\cdot i} \sigma_{s(j)}^* \sigma_{s(j)} \otimes \mathbb{1}_j = 1 \otimes 1,$$
$$WW^* = \sum_{j\in H\cdot i} \sigma_{s(j)} \sigma_{s(j)}^* \otimes \mathbb{1}_j = 1 \otimes 1$$

and

$$W^*(e_{j,h^{-1}\cdot j} \otimes \mathbb{1}_j U_h)W = \sum_{k,l\in H\cdot i} \sigma_{s(k)}^* e_{j,h^{-1}\cdot j} \sigma_{s(l)} \otimes \underbrace{\mathbb{1}_k \mathbb{1}_j U_h \mathbb{1}_l}_{=\delta_{k,j}\delta_{j,hl}\mathbb{1}_j U_h}$$
$$= \sigma_{s(j)}^* e_{j,h^{-1}\cdot j} \sigma_{s(h^{-1}\cdot j)} \otimes \mathbb{1}_j U_h = e_{i,i} \otimes \mathbb{1}_j U_h = (e_{i,i} \otimes \mathrm{id}_{C_{[i]}})(\mathbb{1}_j U_h).$$

Thus $(\mathrm{id}_{\mathcal{L}(\ell^2(H\cdot i))} \otimes \pi_F^{[i]}) \circ \varphi_{[i]}|^{(D\rtimes_\tau H)_F}$ is unitarily equivalent to $e_{i,i} \otimes \mathrm{id}_{C_{[i]}}$. As $e_{i,i} \otimes \mathrm{id}_{C_{[i]}}$ induces an isomorphism on K-theory, we are done. □

Proof of Theorem 6.1. Theorem 6.1 now follows from Proposition 6.4.2 and commutativity of diagram (11). □

6.5. Baum–Connes. Under certain K-theoretic assumptions on our group G, we may now apply the Baum–Connes machinery to our situation.

Corollary 6.5.1. *Assume that the conditions of Theorem 6.1 are satisfied, i.e. conditions* (I)–(III) *from* §6 *hold. Moreover, assume that the group G satisfies the Baum–Connes conjecture with coefficients in $c_0(I)$ and D with respect to the G-actions σ and τ. Then the descent $j_r^G(\mathbf{x}) \in KK(c_0(I) \rtimes_{\sigma,r} G, D \rtimes_{\tau,r} G)$ induces an isomorphism on K-theory.*

Proof. We have proven in Theorem 6.1 that for all finite subgroups H of G, the descent $j^H(\mathrm{res}_H^G(\mathbf{x}))$ is a KK-equivalence. Now our corollary follows from [E-N-O, Proposition 2.1(i)]. □

Under additional assumptions, we even obtain

Corollary 6.5.2. *In addition to the requirements of the previous corollary, assume that both reduced crossed products $c_0(I) \rtimes_{\sigma,r} G$ and $D \rtimes_{\tau,r} G$ satisfy the UCT. Then $j_r^G(\mathbf{x})$ is a KK-equivalence.*

Proof. This follows immediately from the previous corollary. □

To conclude that $j_r^G(\mathbf{x})$ is a KK-equivalence, we can also proceed as follows:

Corollary 6.5.3. *In addition to the requirements of Corollary 6.5.1, assume that G satisfies the strong Baum–Connes conjecture with coefficients in $c_0(I)$ and D with respect to the G-actions σ and τ. Then $j_r^G(\mathbf{x})$ is a KK-equivalence.*

Proof. This follows from Theorem 6.1 and [E-N-O, Proposition 2.1(iii)]. □

The conditions of this corollary are for instance satisfied if G is amenable.

6.6. Imprimitivity theorems. Consider the direct sum decomposition

$$c_0(I) \rtimes_{\sigma,r} G = \bigoplus_{[i] \in G \backslash I} (c_0(G \cdot i) \rtimes_{\sigma,r} G).$$

As before, we denote the summand $c_0(G \cdot i) \rtimes_{\sigma,r} G$ corresponding to $[i] \in G\backslash I$ by $C_{[i]}$, and let $\iota_{[i]}$ be the embedding $C_{[i]} \to c_0(I) \rtimes_{\sigma,r} G$.

Under the isomorphism

$$\prod_{[i] \in G\backslash I} (KK(\iota_{[i]}) \otimes \sqcup) : KK(c_0(I) \rtimes_{\sigma,r} G, D \rtimes_{\tau,r} G) \to \prod_{[i] \in G\backslash I} KK(C_{[i]}, D \rtimes_{\tau,r} G)$$

from [Bla, Theorem 19.7.1], the element $j_r^G(\mathbf{x}) \in KK(c_0(I) \rtimes_{\sigma,r} G, D \rtimes_{\tau,r} G)$ corresponds to $(\mathbf{x}_{[i]})_{[i] \in G\backslash I}$ with $\mathbf{x}_{[i]} := KK(\iota_{[i]}) \otimes j_r^G(\mathbf{x})$. By Lemma 6.3.1, we have

$$\mathbf{x}_{[i]} = KK(\varphi_{[i]}) \otimes KK(e_{i,i} \otimes \mathrm{id}_{D \rtimes_{\tau,r} G})^{-1} \tag{14}$$

where the homomorphisms $\varphi_{[i]}$ and $e_{i,i} \otimes \mathrm{id}_{D \rtimes_{\tau,r} G}$ are given by

$$\varphi_{[i]} : C_{[i]} \to \mathcal{K}(\ell^2(G \cdot i)) \otimes_{\min} (D \rtimes_{\tau,r} G), \quad \mathbb{1}_j U_g \mapsto e_{j, g^{-1} \cdot j} \otimes e_j U_g,$$

and

$$e_{i,i} \otimes \mathrm{id}_{D \rtimes_{\tau,r} G} : D \rtimes_{\tau,r} G \to \mathcal{K}(\ell^2(G \cdot i)) \otimes_{\min} (D \rtimes_{\tau,r} G), \quad T \mapsto e_{i,i} \otimes T.$$

To further examine $\mathbf{x}_{[i]}$, let us now describe $C_{[i]} = c_0(G \cdot i) \rtimes_{\sigma,r} G$ up to Morita equivalence with the help of concrete homomorphisms. For $i \in I$, let G_i be the stabilizer of i, i.e. $G_i := \{g \in G : g \cdot i = i\}$. Then we have a bijection $G/G_i \cong G \cdot i$, $gG_i \mapsto g \cdot i$, which is G-equivariant. Thus we can identify $C_{[i]} = c_0(G \cdot i) \rtimes_{\sigma,r} G$ with $c_0(G/G_i) \rtimes_r G$ where we take the translation action of G on $c_0(G/G_i)$ for the second reduced crossed product. Moreover, the homomorphism

$$C_r^*(G_i) \to c_0(G/G_i) \rtimes_r G, \quad \lambda_g \mapsto \mathbb{1}_{eG_i} U_g,$$

exists by Lemma 2.5.2 and induces a KK-equivalence in $KK(C_r^*(G_i), c_0(G/G_i) \rtimes_r G)$. The last assertion follows from the observation that the projection $\mathbb{1}_{eG_i} \in c_0(G/G_i) \rtimes_r G$ is a full projection and that the above homomorphism yields an isomorphism $C_r^*(G_i) \cong \mathbb{1}_{eG_i}(c_0(G/G_i) \rtimes_r G)\mathbb{1}_{eG_i}$, $\lambda_g \mapsto \mathbb{1}_{eG_i} U_g$ (injectivity follows from Lemma 2.5.2, and surjectivity can be seen immediately).

Composing the homomorphism $C_r^*(G_i) \to c_0(G/G_i) \rtimes_r G$ and the above canonical identification $c_0(G/G_i) \rtimes_r G \cong c_0(G \cdot i) \rtimes_{\sigma,r} G = C_{[i]}$, we obtain the homomorphism

$$\varphi_i : C_r^*(G_i) \to C_{[i]}, \qquad \lambda_g \mapsto \mathbb{1}_i U_g. \tag{15}$$

By our observations, $KK(\varphi_i)$ is a KK-equivalence in $KK(C_r^*(G_i), C_{[i]})$. For two different choices of the representative i of the class $[i] \in G \backslash I$, the stabilizers will be different in general, but they will always be conjugate. So the choices of the particular representatives do not really matter.

Finally, let us compute the Kasparov product $KK(\varphi_i) \otimes \mathbf{x}_{[i]}$. As a preparation, note that $(\varphi_{[i]} \circ \varphi_i)(\lambda_g) = e_{i,i} \otimes e_i U_g \in \mathcal{K}(\ell^2(G \cdot i)) \otimes_{\min} (D \rtimes_{\tau,r} G)$ for $g \in G_i$. Thus composing $\varphi_{[i]} \circ \varphi_i$ with the canonical identification $e_{i,i} \otimes D \rtimes_{\tau,r} G \cong D \rtimes_{\tau,r} G$, we obtain a homomorphism $\Phi_i : C_r^*(G_i) \to D \rtimes_{\tau,r} G$, $\lambda_g \mapsto e_i U_g$. By construction,

$$\varphi_{[i]} \circ \varphi_i = (e_{i,i} \otimes \mathrm{id}_{D \rtimes_{\tau,r} G}) \circ \Phi_i. \tag{16}$$

Thus

$$\begin{aligned} KK(\varphi_i) \otimes \mathbf{x}_{[i]} &= KK(\varphi_i) \otimes KK(\iota_{[i]}) \otimes j_r^G(\mathbf{x}) \\ &\overset{(14)}{=} KK(\varphi_i) \otimes KK(\varphi_{[i]}) \otimes KK(e_{i,i} \otimes \mathrm{id}_{D \rtimes_{\tau,r} G})^{-1} \\ &\overset{(16)}{=} KK(\Phi_i) \otimes KK(e_{i,i} \otimes \mathrm{id}_{D \rtimes_{\tau,r} G}) \otimes KK(e_{i,i} \otimes \mathrm{id}_{D \rtimes_{\tau,r} G})^{-1} \\ &= KK(\Phi_i). \end{aligned} \tag{17}$$

Let us summarize our observations.

Proposition 6.6.1. *Let $\mathcal{R}$ be a complete system of representatives for $G \backslash I$. The homomorphism*

$$\bigoplus_{i \in \mathcal{R}} \varphi_i : \bigoplus_{i \in \mathcal{R}} C_r^*(G_i) \to \bigoplus_{i \in \mathcal{R}} C_{[i]} = c_0(I) \rtimes_{\sigma,r} G$$

(the φ_i are given by (15)) induces a KK-equivalence in $KK(\bigoplus_{i \in \mathcal{R}} C_r^*(G_i), c_0(I) \rtimes_{\sigma,r} G)$. Moreover,

$$KK(\varphi_i) \otimes KK(\iota_{[i]}) \otimes j_r^G(\mathbf{x}) = KK(\Phi_i) \tag{18}$$

where Φ_i is the homomorphism

$$\Phi_i : C_r^*(G_i) \to D \rtimes_{\tau,r} G, \qquad \lambda_g \mapsto e_i U_g.$$

Proof. Since each of the φ_i identifies $C_r^*(G_i)$ with a full corner of $C_{[i]}$, the homomorphism $\bigoplus_{i \in \mathcal{R}} \varphi_i$ identifies the direct sum $\bigoplus_{i \in \mathcal{R}} C_r^*(G_i)$ with a full corner of $\bigoplus_{i \in \mathcal{R}} C_{[i]} = c_0(I) \rtimes_{\sigma,r} G$. This proves our first assertion. The second one is just (17). □

7. From abstract to concrete

Let us now go back to the situation of semigroup C*-algebras and summarize what we have obtained so far. We just have to apply our general results from the previous section to the case of reduced semigroup C*-algebras. We use the same notation as in §2. For a left Ore semigroup P whose constructible right ideals are independent, set $P^{-1} \cdot (\mathcal{J} \setminus \{\emptyset\}) = \{q^{-1} \cdot X : q \in P,\ \emptyset \neq X \in \mathcal{J}\}$ and let G be the enveloping group of P. The G-action on $P^{-1} \cdot (\mathcal{J} \setminus \{\emptyset\})$ via left multiplication (i.e. $g \cdot (q^{-1} \cdot X) = g \cdot q^{-1} \cdot X$) induces in a canonical way a G-action on $c_0(P^{-1} \cdot (\mathcal{J} \setminus \{\emptyset\}))$ by shifting indices. In the following, we will consider the conditions

(A$_1$) P is a left Ore semigroup whose constructible right ideals are independent, and the enveloping group G of P satisfies the Baum–Connes conjecture with coefficients in the G-C*-algebras $c_0(P^{-1} \cdot (\mathcal{J} \setminus \{\emptyset\}))$ and $D_r^{(\infty)}(P)$.

(A$_2$) Condition (A$_1$) holds, and $c_0(P^{-1} \cdot (\mathcal{J} \setminus \{\emptyset\})) \rtimes_{\sigma,r} G$ and $D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G$ satisfy the UCT or G satisfies the strong Baum–Connes conjecture with coefficients in the G-C*-algebras $c_0(P^{-1} \cdot (\mathcal{J} \setminus \{\emptyset\}))$ and $D_r^{(\infty)}(P)$.

Theorem 7.1. *If condition* (A$_1$) *is satisfied, then the descent*

$$j_r^G(\mathbf{x}) \in KK\big(c_0(P^{-1} \cdot (\mathcal{J} \setminus \{\emptyset\})) \rtimes_{\sigma,r} G,\ D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G\big)$$

of the element $\mathbf{x}$ *from Definition 6.1.1 induces an isomorphism on K-theory. If the stronger assumption* (A$_2$) *is valid, then* $j_r^G(\mathbf{x})$ *is a KK-equivalence.*

Proof. We have checked at the beginning of §5 that under the present assumptions, all the conditions in Theorem 6.1 are satisfied. Hence the first part of the present theorem follows from Corollary 6.5.1, and the second part follows from Corollaries 6.5.2 and 6.5.3. □

Recall that the embedding $\iota : C_r^*(P) \to D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G$, $V_p \mapsto E_P U_p E_P$, induces a KK-equivalence in $KK(C_r^*(P), D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G)$ by Corollary 4.3. Also recall that for $X \in \mathcal{J} \setminus \{\emptyset\}$, we have introduced the homomorphism $\varphi_X : C_r^*(G_X) \to c_0(G \cdot X) \rtimes_{\sigma,r} G$, $\lambda_g \mapsto 1\!\!1_X U_g$, in §6.6. Here $G_X = \{g \in G : g \cdot X = X\}$. Let $\iota_{[X]}$ be the embedding $c_0(G \cdot X) \rtimes_{\sigma,r} G \hookrightarrow c_0(P^{-1} \cdot \mathcal{J} \setminus \{\emptyset\}) \rtimes_{\sigma,r} G$.

Lemma 7.2. *For every* X *in* $\mathcal{J} \setminus \{\emptyset\}$, *there exists a homomorphism*

$$\Psi_X : C_r^*(G_X) \to C_r^*(P), \qquad \lambda_{q^{-1}p} \mapsto E_X V_q^* V_p E_X,$$

which satisfies

$$KK(\Psi_X) = KK(\varphi_X) \otimes KK(\iota_{[X]}) \otimes j_r^G(\mathbf{x}) \otimes KK(\iota)^{-1}. \tag{19}$$

Proof. Let X be an element of $\mathcal{J} \setminus \{\emptyset\}$. Recall that Φ_X is the homomorphism $C_r^*(G_X) \to D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G$, $\lambda_g \mapsto E_X U_g = E_X U_g E_X$. It is clear that $\mathrm{Im}(\Phi_X) \subseteq \mathrm{Im}(\iota)$, so that we can define $\Psi_X := \iota^{-1} \circ (\Phi_X|^{\mathrm{Im}(\iota)})$. This homomorphism has the desired properties. Equation (19) follows from $\iota \circ \Psi_X = \Phi_X$ (by construction) and (18). □

Now let $\mathcal{X}$ be a complete system of representatives for $G\backslash(P^{-1}\cdot(\mathcal{J}\setminus\{\emptyset\}))$ such that $\mathcal{X}\subseteq\mathcal{J}\setminus\{\emptyset\}$. The homomorphisms $\{\Psi_X\}_{X\in\mathcal{X}}$ from the previous lemma give rise to the Kasparov $(\bigoplus_{X\in\mathcal{X}} C_r^*(G_X), C_r^*(P))$-module $(\ell^2(\mathcal{X}, C_r^*(P)), \bigoplus_{X\in\mathcal{X}}\Psi_X, 0)$ with the homomorphism

$$\bigoplus_{X\in\mathcal{X}}\Psi_X : \bigoplus_{X\in\mathcal{X}} C_r^*(G_X)\to c_0(\mathcal{X}, C_r^*(P))\subseteq\mathcal{K}(\ell^2(\mathcal{X}, C_r^*(P)))\subseteq\mathcal{L}(\ell^2(\mathcal{X}, C_r^*(P))).$$

Here $c_0(\mathcal{X}, C_r^*(P))$ acts as diagonal multiplication operators on the Hilbert $C_r^*(P)$-module $\ell^2(\mathcal{X}, C_r^*(P))$. Let Ψ be the KK-element in $KK(\bigoplus_{X\in\mathcal{X}} C_r^*(G_X), C_r^*(P))$ represented by the Kasparov module $(\ell^2(\mathcal{X}, C_r^*(P)), \bigoplus_{X\in\mathcal{X}}\Psi_X, 0)$. Let ι_{G_X} be the inclusion $C_r^*(G_X)\hookrightarrow\bigoplus_{X\in\mathcal{X}} C_r^*(G_X)$. By construction,

$$KK(\iota_{G_X})\otimes\Psi = KK(\Psi_X). \tag{20}$$

Theorem 7.3. *If condition* (A_1) *is valid, then the above KK-element* Ψ *induces an isomorphism on K-theory. If the stronger assumption* (A_2) *holds, then* Ψ *is a KK-equivalence.*

Proof. By Corollary 4.3, $KK(\iota)$ is a KK-equivalence. By the first part of Proposition 6.6.1, $KK(\bigoplus_{X\in\mathcal{X}}\varphi_X)$ is a KK-equivalence. And going through the identification

$$KK\Big(\bigoplus_{X\in\mathcal{X}} C_r^*(G_X), C_r^*(P)\Big)\cong\prod_{X\in\mathcal{X}} KK(C_r^*(G_X), C_r^*(P))$$

from [Bla, Theorem 19.7.1], it follows from equation (19) of the previous lemma and (20) that

$$\Psi = KK\Big(\bigoplus_{X\in\mathcal{X}}\varphi_X\Big)\otimes j_r^G(\mathbf{x})\otimes KK(\iota)^{-1}.$$

Therefore, the first part of the present theorem follows from the first part of Theorem 7.1, and the second part follows from the second part of the same theorem. □

Corollary 7.4. *If condition* (A_1) *is satisfied, then the homomorphism*

$$\sum_{X\in\mathcal{X}} K_*(\Psi_X) : \bigoplus_{X\in\mathcal{X}} K_*(C_r^*(G_X))\to K_*(C_r^*(P))$$

is an isomorphism. And under the stronger assumption (A_2)*, the homomorphism*

$$\prod_{X\in\mathcal{X}} K^*(\Psi_X) : K^*(C_r^*(P))\to\prod_{X\in\mathcal{X}} K^*(C_r^*(G_X))$$

is an isomorphism. Here K_* *is* $K_0\oplus K_1$ *and* K^* *is* $K^0\oplus K^1$*, viewed as* $\mathbb{Z}/2\mathbb{Z}$*-graded abelian groups.*

Proof. By (20), these homomorphisms are just the compositions of taking the Kasparov product with Ψ and the canonical isomorphisms

$$\bigoplus_{X\in\mathcal{X}} K_*(C_r^*(G_X))\cong K_*\Big(\bigoplus_{X\in\mathcal{X}} C_r^*(G_X)\Big)\text{ and } K^*\Big(\bigoplus_{X\in\mathcal{X}} C_r^*(G_X)\Big)\cong\prod_{X\in\mathcal{X}} K^*(C_r^*(G_X)).$$

□

Our last goal in this section is to show that whenever P is a left Ore semigroup whose constructible right ideals are independent, there exists a canonical ring structure on the K-homology of $C_r^*(P)$, and the isomorphism $\prod_{X \in \mathcal{X}} K^*(\Psi_X)$ from the last corollary is a ring isomorphism.

Lemma 7.5. *Let P be a left Ore semigroup whose constructible right ideals are independent. Then there exists a homomorphism*

$$\Delta_P : C_r^*(P) \to C_r^*(P) \otimes_{\min} C_r^*(P) \quad \textit{determined by} \quad V_p \mapsto V_p \otimes V_p.$$

Note that we always have such a homomorphism in the case where the left regular representation $C^*(P) \to C_r^*(P)$ is an isomorphism because an analogous homomorphism always exists on the full semigroup C*-algebra (see [Li2, proof of Proposition 2.24]).

Proof. Since the constructible right ideals of P are independent, there exists a homomorphism $D_r(P) \to D_r(P) \otimes_{\min} D_r(P)$ sending E_X to $E_X \otimes E_X$ for all $X \in \mathcal{J}$. This can be seen as follows: By [Li2, Corollary 2.26], the restriction of the left regular representation to the commutative sub-C*-algebra $D(P)$ of the full semigroup C*-algebra $C^*(P)$ yields an isomorphism $D(P) \cong D_r(P)$ if (and only if) the constructible right ideals of P are independent. But we can always construct a homomorphism $D(P) \to D(P) \otimes_{\min} D(P)$, $e_X \mapsto e_X \otimes e_X$, by restricting the homomorphism $C^*(P) \to C^*(P) \otimes_{\min} C^*(P)$, $v_p \mapsto v_p \otimes v_p$, to $D(P)$ (as observed above, such a homomorphism always exists; see also [Li2, proof of Proposition 2.24]).

The homomorphism $D_r(P) \to D_r(P) \otimes_{\min} D_r(P)$, $E_X \mapsto E_X \otimes E_X$, is obviously equivariant with respect to the P-actions τ and $\tau \otimes \tau$. By definition of $D_r^{(\infty)}(P)$ (see the beginning of §4), we obtain a homomorphism $D_r^{(\infty)}(P) \to D_r^{(\infty)}(P) \otimes_{\min} D_r^{(\infty)}(P)$, $E_Y \mapsto E_Y \otimes E_Y$ (for $Y \in P^{-1} \cdot \mathcal{J}$). This homomorphism is again obviously G-equivariant with respect to the actions $\tau^{(\infty)}$ and $\tau^{(\infty)} \otimes \tau^{(\infty)}$. Therefore, applying Lemma 2.5.2 to this homomorphism and the diagonal embedding $G \hookrightarrow G \times G$, we obtain the homomorphism

$$\begin{aligned} D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G &\to (D_r^{(\infty)}(P) \otimes_{\min} D_r^{(\infty)}(P)) \rtimes_{\tau^{(\infty)} \otimes \tau^{(\infty)},r} (G \times G), \\ E_Y U_g &\mapsto (E_Y \otimes E_Y) U_{(g,g)}. \end{aligned}$$

Composing this map with the canonical identification

$$\begin{aligned} &(D_r^{(\infty)}(P) \otimes_{\min} D_r^{(\infty)}(P)) \rtimes_{\tau^{(\infty)} \otimes \tau^{(\infty)},r} (G \times G) \\ &\qquad \cong (D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G) \otimes_{\min} (D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G), \\ &(E_{Y_1} \otimes E_{Y_2}) U_{(g_1,g_2)} \mapsto E_{Y_1} U_{g_1} \otimes E_{Y_2} U_{g_2}, \end{aligned}$$

we obtain the homomorphism

$$\begin{aligned} D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G &\to (D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G) \otimes_{\min} (D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G), \\ E_Y U_g &\mapsto E_Y U_g \otimes E_Y U_g. \end{aligned}$$

Since this map sends $E_P U_p E_P$ to $E_P U_p E_P \otimes E_P U_p E_P$, we just have to restrict this homomorphism to $E_P(D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G)E_P$ and to use the identification

$$C_r^*(P) \cong E_P(D_r^{(\infty)}(P) \rtimes_{\tau^{(\infty)},r} G)E_P, \quad V_p \mapsto E_P U_p E_P,$$

from Lemma 4.2 to obtain our desired homomorphism Δ_P. □

Now, whenever there exists such a diagonal homomorphism

$$\Delta_P : C_r^*(P) \to C_r^*(P) \otimes_{\min} C_r^*(P), \quad V_p \mapsto V_p \otimes V_p,$$

we obtain a canonical graded ring structure on $K^*(C_r^*(P))$ in analogy to the group case. Multiplication in this ring structure is given by the following composition:

$$\begin{aligned} K^i(C_r^*(P)) \times K^j(C_r^*(P)) &\cong KK^i(C_r^*(P), \mathbb{C}) \times KK^j(C_r^*(P), \mathbb{C}) \\ &\xrightarrow{\otimes} KK^{i+j}(C_r^*(P) \otimes_{\min} C_r^*(P), \mathbb{C}) \\ &\xrightarrow{KK(\Delta_P)\otimes \sqcup} KK^{i+j}(C_r^*(P), \mathbb{C}) \cong K^{i+j}(C_r^*(P)). \end{aligned}$$

And on $\prod_{X\in\mathcal{X}} K^*(C_r^*(G_X))$, there is a canonical ring structure given by the canonical ring structure on each of the K-homology groups $K^*(C_r^*(G_X))$ (it is constructed in the same way as for $K^*(C_r^*(P))$). Our last observation in this section is that the isomorphism on K-homology from the last corollary is compatible with these ring structures.

Theorem 7.6. *If condition* (A$_2$) *is satisfied, then the homomorphism*

$$\prod_{X\in\mathcal{X}} K^*(\Psi_X) : K^*(C_r^*(P)) \to \prod_{X\in\mathcal{X}} K^*(C_r^*(G_X))$$

is a ring isomorphism.

Proof. In view of the last corollary, all we have to prove is that $\prod_{X\in\mathcal{X}} K^*(\Psi_X)$ is multiplicative. Let us check this for K^0; the remaining cases are similar. Let Δ_{G_X} be the diagonal homomorphism $C_r^*(G_X) \to C_r^*(G_X) \otimes_{\min} C_r^*(G_X)$, $\lambda_g \mapsto \lambda_g \otimes \lambda_g$. Using the natural identification $K^0(\cdot) \cong KK(\cdot, \mathbb{C})$ and the definition of the multiplicative structures, our assertion amounts to saying that for all $X \in \mathcal{X}$ and all $\mathbf{y}, \mathbf{z}$ in $KK(C_r^*(P), \mathbb{C})$, we have

$$KK(\Psi_X) \otimes (KK(\Delta_P) \otimes (\mathbf{y} \otimes \mathbf{z})) = KK(\Delta_{G_X}) \otimes \big((KK(\Psi_X) \otimes \mathbf{y}) \otimes (KK(\Psi_X) \otimes \mathbf{z})\big).$$

It is immediate that

$$\Delta_P \circ \Psi_X = (\Psi_X \otimes_{\min} \Psi_X) \circ \Delta_{G_X}. \tag{21}$$

Thus

$$
\begin{aligned}
&KK(\Delta_{G_X}) \otimes \big((KK(\Psi_X) \otimes \mathbf{y}) \otimes (KK(\Psi_X) \otimes \mathbf{z})\big) \\
&\qquad = KK(\Delta_{G_X}) \otimes (KK(\Psi_X) \otimes KK(\Psi_X)) \otimes (\mathbf{y} \otimes \mathbf{z}) \\
&\qquad = KK(\Delta_{G_X}) \otimes KK(\Psi_X \otimes_{\min} \Psi_X) \otimes (\mathbf{y} \otimes \mathbf{z}) \\
&\qquad \overset{(21)}{=} KK(\Psi_X) \otimes (KK(\Delta_P) \otimes (\mathbf{y} \otimes \mathbf{z})). \qquad \square
\end{aligned}
$$

8. Semigroups attached to Dedekind domains

In this section, we apply our general K-theoretic results from §7 to specific semigroups attached to Dedekind domains. Let R be a Dedekind domain. This means that R is a noetherian, integrally closed integral domain with the property that every non-zero prime ideal is a maximal ideal (compare [Neu, Chapter I, Definition (3.2)]). By an integral domain, we mean a commutative ring without zero divisors.

We would like to treat the multiplicative semigroup $R^\times = R \setminus \{0\}$, the semigroup of principal ideals of R and the $ax+b$-semigroup $R \rtimes R^\times$. The semidirect product $R \rtimes R^\times$ is taken with respect to the multiplicative action of the multiplicative semigroup $R^\times$ on the additive group R.

Examples of Dedekind domains are given by rings of integers in number fields or function fields. These rings and the corresponding semigroups have actually been our motivating examples.

Since it will be important later on, let us briefly recall the definition of the class group of R. Let $Q(R)$ be the quotient field of R.

Definition 8.1. A *fractional ideal* of $Q(R)$ (or R) is a non-zero, finitely generated sub-R-module of $Q(R)$. A *principal fractional ideal* of $Q(R)$ (or R) is a fractional ideal of the form $a \cdot R$ for some $a \in Q(R)^\times = Q(R) \setminus \{0\}$.

As explained in [Neu, Chapter I, §3], the set of fractional ideals of $Q(R)$ is an abelian group under multiplication. Furthermore, the subset of principal fractional ideals of $Q(R)$ is multiplicatively closed, hence it forms a subgroup.

Definition 8.2. The *ideal class group* (or simply *class group*) $Cl_{Q(R)}$ of $Q(R)$ is the quotient of the group of fractional ideals by the subgroup of principal fractional ideals of $Q(R)$.

Remark 8.3. It follows directly from the definition that we can equivalently describe $Cl_{Q(R)}$ (at least as a set) as follows: The multiplicative group $Q(R)^\times = Q(R) \setminus \{0\}$ acts on the set of fractional ideals of $Q(R)$ by multiplication, and $Cl_{Q(R)}$ is given by the set of orbits of this action.

8.1. Multiplicative semigroups. We first consider the multiplicative semigroup $R^\times$. Let R^* be the group of units in R, or in other words, R^* is the subgroup of invertible elements of $R^\times$.

Our goal is to apply our general K-theoretic results from §7 to prove:

Theorem 8.1.1. *$C_r^*(R^\times)$ and $\bigoplus_{\gamma \in Cl_{Q(R)}} C_r^*(R^*)$ are KK-equivalent. Furthermore, choose for every $\gamma \in Cl_{Q(R)}$ an ideal I_γ of R which represents γ. Then there is a homomorphism $\Psi_{I_\gamma} : C_r^*(R^*) \to C^*(R^\times)$ determined by $\Psi_{I_\gamma}(V_a) = E_{I_\gamma} V_a$. These homomorphisms give rise to isomorphisms*

$$\sum_{\gamma \in Cl_{Q(R)}} (\Psi_{I_\gamma})_* : \bigoplus_{\gamma \in Cl_{Q(R)}} K_*(C_r^*(R^*)) \to K_*(C_r^*(R^\times))$$

and

$$\prod_{\gamma \in Cl_{Q(R)}} (\Psi_{I_\gamma})^* : K^*(C_r^*(R^\times)) \to \prod_{\gamma \in Cl_{Q(R)}} K^*(C_r^*(R^*)).$$

The last isomorphism $\prod_{\gamma \in Cl_{Q(R)}} (\Psi_{I_\gamma})^*$ on K-homology is a ring isomorphism.

Proof. We just have to check the assumptions in Theorem 7.3. First of all, $R^\times$ is a left Ore semigroup because it is cancellative and abelian. Moreover, the constructible right ideals of $R^\times$ are independent. This can be proven analogously to [Li2, Lemma 2.29]. The enveloping group of $R^\times$ is $Q(R)^\times$. Since $Q(R)^\times$ is abelian, it is amenable, hence it satisfies the strong Baum–Connes conjecture for all coefficients. Therefore the conditions in the second part of Theorem 7.3 are satisfied. For the semigroup $R^\times$, $\mathcal{J} \setminus \{\emptyset\}$ is given by all non-zero ideals of R. This can be proven analogously to the case of the $ax+b$-semigroup over R which is explained in [Li2, second half of §2.4]. Therefore, $(R^\times)^{-1} \cdot (\mathcal{J} \setminus \{\emptyset\})$ is the set of fractional ideals of $Q(R)$, and the set of orbits $Q(R)^\times \backslash (P^{-1} \cdot (\mathcal{J} \setminus \{\emptyset\}))$ coincides with $Cl_{Q(R)}$ by Remark 8.3. And finally, for a non-zero ideal I of R, the stabilizer $Q(R)_I^\times = \{a \in Q(R)^\times : a \cdot I = I\}$ is given by R^*. The first part of our theorem now follows from the second part of Theorem 7.3 and from the second part of Corollary 7.4. That $\prod_{\gamma \in Cl_{Q(R)}} (\Psi_{I_\gamma})^*$ is a ring isomorphism follows from Theorem 7.6. □

Remark 8.1.2. Let L be an ideal in R. We define a (non-unital) endomorphism α_L of $C_r^*(R^\times)$ by $V_p \mapsto V_p E_L$, $E_I \mapsto E_{LI}$. Then $\alpha_J \alpha_L = \alpha_{JL}$ and α_L is inner if L is a principal ideal.

As a consequence we obtain an action of the class group $Cl_{Q(R)}$ on the K-theory and K-homology of $C_r^*(R^\times)$ (in fact this defines a multiplicative map $Cl_{Q(R)} \to KK(C_r^*(R^\times), C_r^*(R^\times))$). It is clear that this action of $Cl_{Q(R)}$ corresponds under $\sum_{\gamma \in Cl_{Q(R)}} (\Psi_{I_\gamma})_*$ to the obvious action of $Cl_{Q(R)}$ on $\bigoplus_{\gamma \in Cl_{Q(R)}} K_*(C_r^*(R^*))$, and similarly on K-homology.

We also discuss the multiplicative semigroup of principal ideals over a Dedekind domain R. It is clear that this semigroup can be identified with $R^\times / R^*$. Note that the family $\mathcal{J}_{R^\times / R^*}$ of ideals for this semigroup can be identified with the corresponding family $\mathcal{J}_{R^\times}$ for the multiplicative semigroup of the ring R via $\mathcal{J}_{R^\times / R^*} \ni X/R^* \leftrightarrow X \in \mathcal{J}_{R^\times}$, where X/R^* is the image of X in $R^\times / R^*$ under the canonical projection $R^\times \twoheadrightarrow R^\times / R^*$. With this observation, we can, in complete analogy to the case of $R^\times$, apply Theorem 7.3, Corollary 7.4 and Theorem 7.6 to deduce

Theorem 8.1.3. *$C_r^*(R^\times/R^*)$ and $\bigoplus_{\gamma \in Cl_{Q(R)}} \mathbb{C}$ are KK-equivalent. Furthermore, choose for every $\gamma \in Cl_{Q(R)}$ an ideal I_γ of R which represents γ. Then the canonical homomorphisms $\Psi_{I_\gamma} : \mathbb{C} \to C^*(R^\times/R^*)$ determined by $\Psi_{I_\gamma}(1) = E_{I_\gamma/R^*}$ give rise to isomorphisms*

$$\sum_{\gamma \in Cl_{Q(R)}} (\Psi_{I_\gamma})_* : \bigoplus_{\gamma \in Cl_{Q(R)}} \mathbb{Z} \to K_*(C_r^*(R^\times))$$

and

$$\prod_{\gamma \in Cl_{Q(R)}} (\Psi_{I_\gamma})^* : K^*(C_r^*(R^\times)) \to \prod_{\gamma \in Cl_{Q(R)}} \mathbb{Z}.$$

The last isomorphism $\prod_{\gamma \in Cl_{Q(R)}} (\Psi_{I_\gamma})^*$ is a ring isomorphism, where we take the canonical ring structure on $\mathbb{Z}$.

We also remark that we obtain an analogous action of the class group $Cl_{Q(R)}$ on the K-theory and K-homology of $C_r^*(R^\times/R^*)$ as in the previous remark.

8.2. $ax + b$**-semigroups.** Let us now treat the case of the $ax + b$-semigroup $R \rtimes R^\times$ over R. First, we apply our general results from §7 to compute K-theory, and secondly, we show that the corresponding semigroup C*-algebras are purely infinite.

Again, let R^* be the group of units in R and choose for every $\gamma \in Cl_{Q(R)}$ an ideal I_γ of R which represents γ.

Applying our general K-theoretic results from §7, we obtain

Theorem 8.2.1. *The C*-algebras $C_r^*(R \rtimes R^\times)$ and $\bigoplus_{\gamma \in Cl_{Q(R)}} C_r^*(I_\gamma \rtimes R^*)$ are KK-equivalent. Here we form the semidirect product $I_\gamma \rtimes R^*$ with respect to the multiplicative action of R^* on the additive group I_γ.*

Moreover, for every $\gamma \in Cl_{Q(R)}$ there is a homomorphism $\Psi_{I_\gamma} : C_r^*(I_\gamma \rtimes R^*) \to C^*(R \rtimes R^\times)$ determined by $\Psi_{I_\gamma}(V_{(b,a)}) = E_{I_\gamma \times I_\gamma^\times} V_{(b,a)}$. These homomorphisms give rise to isomorphisms

$$\sum_{\gamma \in Cl_{Q(R)}} (\Psi_{I_\gamma})_* : \bigoplus_{\gamma \in Cl_{Q(R)}} K_*(C_r^*(I_\gamma \rtimes R^*)) \to K_*(C_r^*(R \rtimes R^\times))$$

and

$$\prod_{\gamma \in Cl_{Q(R)}} (\Psi_{I_\gamma})^* : K^*(C_r^*(R \rtimes R^\times)) \to \prod_{\gamma \in Cl_{Q(R)}} K^*(C_r^*(I_\gamma \rtimes R^*)).$$

The last isomorphism $\prod_{\gamma \in Cl_{Q(R)}} (\Psi_{I_\gamma})^*$ on K-homology is a ring isomorphism.

Proof. Again, we just have to check the assumptions in Theorem 7.3. First of all, $R \rtimes R^\times$ is a left Ore semigroup by [Li1, §5.1]. And the constructible right ideals of $R \rtimes R^\times$ are independent by [Li2, Lemma 2.29]. The enveloping group of $R \rtimes R^\times$ is given by the $ax+b$-group $Q(R) \rtimes Q(R)^\times$ over $Q(R)$. Since $Q(R) \rtimes Q(R)^\times$ is solvable, it is amenable, hence it satisfies the strong Baum–Connes conjecture for all coefficients. Therefore the conditions in the second part of Theorem 7.3 are fulfilled. For the semigroup $R \rtimes R^\times$,

$\mathcal{J} \setminus \{\emptyset\}$ is given by $\{(r+I) \times I^{\times} : r \in R,\ (0) \neq I \triangleleft R\}$. This is explained in the second half of §2.4 of [Li2]. Therefore,

$$(R \rtimes R^{\times})^{-1} \cdot (\mathcal{J} \setminus \{\emptyset\}) = \{(a^{-1}b + a^{-1}I) \times (a^{-1}I)^{\times} : (b,a) \in R \rtimes R^{\times}, (0) \neq I \triangleleft R\},$$

and we see using Remark 8.3 that

$$Cl_{Q(R)} \to (Q(R) \rtimes Q(R)^{\times}) \backslash ((R \rtimes R^{\times})^{-1} \cdot (\mathcal{J} \setminus \{\emptyset\})), \qquad J \mapsto [J \times J^{\times}],$$

is a bijection. And finally, for a non-zero ideal I of R, the stabilizer $(Q(R) \rtimes Q(R)^{\times})_{I \times I^{\times}} = \{(b,a) \in Q(R) \rtimes Q(R)^{\times} : b + a \cdot I = I\}$ is given by $I \rtimes R^{*} \subseteq R \rtimes R^{\times}$. The first part of our theorem now follows from the second part of Theorem 7.3 and the second part of Corollary 7.4, and Theorem 7.6 implies that $\prod_{\gamma \in Cl_{Q(R)}} (\Psi_{I_\gamma})^{*}$ is a ring isomorphism. □

Finally, let us study the inner structure of semigroup C*-algebras of $ax+b$-semigroups over Dedekind domains. We start with two definitions:

Definition 8.2.2. A C*-algebra A is *purely infinite* if A has no non-zero abelian quotients and for every pair of positive elements a and b in A with $b \in \overline{AaA}$, there exists a sequence $(x_n)_n$ in A such that $\lim_{n \to \infty} x_n^* a x_n = b$.

The reader may consult [Rør], [Pas-Rør] or [Kir-Rør1] for more details.

Definition 8.2.3. A C*-algebra has the *ideal property* if projections separate ideals.

Further explanations can be found in [Pas-Rør].

Our final goal is to prove

Theorem 8.2.4. *For every Dedekind domain which has infinitely many pairwise distinct prime ideals, the semigroup C*-algebra $C_r^*(R \rtimes R^{\times})$ is purely infinite and has the ideal property.*

For us, the following result of C. Pasnicu and M. Rørdam [Pas-Rør, Proposition 2.11] is important:

A C*-algebra is purely infinite and has the ideal property if and only if every non-zero hereditary sub-C*-algebra in any quotient contains an infinite projection.

Actually, we will only need the implication "$\Leftarrow$". Our goal is to prove that for every ideal $\mathfrak{I}$ of $C_r^*(R \rtimes R^{\times})$, every non-zero hereditary sub-C*-algebra of $C_r^*(R \rtimes R^{\times})/\mathfrak{I}$ contains an infinite projection.

Let us start with a general observation. Let D be a unital C*-algebra with an action α of a semigroup P by injective endomorphisms. Form the semigroup crossed product $D \overset{e}{\rtimes}_{\alpha} P$ in the sense of [La] or [Li1, §A1]. Recall that $D \overset{e}{\rtimes}_{\alpha} P$ is a unital C*-algebra which comes by definition with a unital homomorphism $i_D : D \to D \overset{e}{\rtimes}_{\alpha} P$ and a semigroup homomorphism $v : P \to \mathrm{Isom}(D \overset{e}{\rtimes}_{\alpha} P)$, $p \mapsto v_p$, such that $v_p i_D(d) v_p^* = i_D(\alpha_p(d))$ for all $p \in P$ and $d \in D$. The triple $(D \overset{e}{\rtimes}_{\alpha} P, i_D, v)$ has the universal property that given a unital C*-algebra T, a unital homomorphism $j_D : D \to T$ and a semigroup homomorphism $w : P \to \mathrm{Isom}(T)$ such that $w_p j_D(d) w_p^* = j_D(\alpha_p(d))$ for all $p \in P$ and $d \in D$, there is a unique homomorphism $j_D \overset{e}{\rtimes} w : D \overset{e}{\rtimes}_\alpha P \to T$ satisfying $(j_D \overset{e}{\rtimes} w) \circ i_D = j_D$ and $(j_D \overset{e}{\rtimes} w) \circ v = w$.

Lemma 8.2.5. *Assume that P is a left Ore semigroup and the enveloping group G of P is amenable. Moreover, assume that*

$$v_p^* i_D(d) v_p \in i_D(D) \quad \text{for all } d \in D. \tag{22}$$

Then there exists a faithful conditional expectation $E : D \overset{e}{\rtimes}_\alpha P \to D$ which is uniquely determined by

$$E(v_q^* i_D(d) v_p) = \delta_{q,p} i_D^{-1}(v_p^* i_D(d) v_p). \tag{23}$$

Proof. In the situation of the lemma, we have

$$D \overset{e}{\rtimes}_\alpha P = \overline{\operatorname{span}}\{v_q^* i_D(d) v_p : p, q \in P,\ d \in D\}$$

by [La, Remark 1.3.1]. This explains why (23) completely determines E.

To prove existence of E, let $(D_\infty, G, \alpha^{(\infty)})$ be the minimal automorphic dilation in the sense of [La, Definition 2.1.2]. By [La, Theorem 2.2.1], we have canonical embeddings $i : D \hookrightarrow D_\infty$, $i^{(\rtimes)} : D \overset{e}{\rtimes}_\alpha P \hookrightarrow D_\infty \rtimes_{\alpha^{(\infty)}} G$ and $D_\infty \overset{\subseteq}{\hookrightarrow} D_\infty \rtimes_{\alpha^{(\infty)}} G$ such that the diagram

$$\begin{array}{ccc} D \overset{e}{\rtimes}_\alpha P & \xrightarrow{i^{(\rtimes)}} & D_\infty \rtimes_{\alpha^{(\infty)}} G \\ \uparrow{\scriptstyle i_D} & & \uparrow{\scriptstyle \subseteq} \\ D & \xrightarrow{i} & D_\infty \end{array}$$

commutes. It follows that i_D is injective. Moreover, Theorem 2.2.1 in [La] tells us that $\operatorname{Im}(i^{(\rtimes)}) = i(1_D)(D_\infty \rtimes_{\alpha^{(\infty)}} G) i(1_D)$.

Now, as G is amenable, $D_\infty \rtimes_{\alpha^{(\infty)}} G \cong D_\infty \rtimes_{\alpha^{(\infty)},r} G$. And since G is also discrete, Lemma 2.5.3 implies that there is a faithful conditional expectation $E_\infty : D_\infty \rtimes_{\alpha^{(\infty)}} G \to D_\infty$ determined by

$$E_\infty(d_\infty u_g) = \delta_{g,e} d_\infty.$$

Here u_g are the canonical unitaries in the multiplier algebra of $D_\infty \rtimes_{\alpha^{(\infty)}} G$ which implement $\alpha^{(\infty)}$. As $E_\infty(i(1_D)) = i(1_D)$, the composition

$$D \overset{e}{\rtimes}_\alpha P \xrightarrow{i^{(\rtimes)}} D_\infty \rtimes_{\alpha^{(\infty)}} G \xrightarrow{E_\infty} D_\infty$$

has image in $i(1_D)(D_\infty \rtimes_{\alpha^{(\infty)}} G) i(1_D) = \operatorname{Im}(i^{(\rtimes)})$, so that we can form

$$E' := (i^{(\rtimes)})^{-1} \circ (E_\infty \circ i^{(\rtimes)})|^{\operatorname{Im}(i^{(\rtimes)})}.$$

E' is a faithful conditional expectation determined by

$$E'(v_q^* i_D(d) v_p) = \delta_{q,p} v_p^* i_D(d) v_p. \tag{24}$$

By our assumption (22), $v_p^* i_D(d) v_p$ lies in $i_D(D)$ for all $p \in P$ and $d \in D$. Thus $\mathrm{Im}(E') = i_D(D)$, and since i_D is injective, we may set

$$E := i_D^{-1} \circ (E'|^{i_D(D)}).$$

This is the desired faithful conditional expectation. It satisfies (23) because of (24). □

Lemma 8.2.6. *In the situation of the previous lemma, let $\mathfrak{I}$ be an ideal of $D \overset{e}{\rtimes}_\alpha P$. Then*

$$\mathfrak{I}_D := i_D^{-1}(i_D(D) \cap \mathfrak{I})$$

is an ideal of D such that for every $p \in P$, the endomorphism

$$\dot{\alpha}_p : D/\mathfrak{I}_D \to D/\mathfrak{I}_D, \qquad d + \mathfrak{I}_D \mapsto \alpha_p(d) + \mathfrak{I}_D,$$

is well-defined and injective. Denote the corresponding P-action on $D/\mathfrak{I}_D$ by $\dot{\alpha}$ and the associated semigroup crossed product by $((D/\mathfrak{I}_D) \overset{e}{\rtimes}_{\dot{\alpha}} P, i_{D/\mathfrak{I}_D}, \dot{v})$. Let π be the canonical projection $D \twoheadrightarrow D/\mathfrak{I}_D$. By the universal property of $(D \overset{e}{\rtimes}_\alpha P, i_D, v)$, there exists a homomorphism $\pi \overset{e}{\rtimes} P : D \overset{e}{\rtimes}_\alpha P \to (D/\mathfrak{I}_D) \overset{e}{\rtimes}_{\dot{\alpha}} P$ determined by

$$(\pi \overset{e}{\rtimes} P)(v_q^* i_D(d) v_p) = \dot{v}_q^* i_{D/\mathfrak{I}_D}(d + \mathfrak{I}_D)\dot{v}_p.$$

This homomorphism $\pi \overset{e}{\rtimes} P$ induces an isomorphism

$$(\pi \overset{e}{\rtimes} P)^{\cdot} : D \overset{e}{\rtimes}_\alpha P/\langle i_D(\mathfrak{I}_D)\rangle \overset{\cong}{\longrightarrow} (D/\mathfrak{I}_D) \overset{e}{\rtimes}_{\dot{\alpha}} P$$

determined by

$$(\pi \overset{e}{\rtimes} P)^{\cdot}(v_q^* i_D(d) v_p + \langle i_D(\mathfrak{I}_D)\rangle) = \dot{v}_q^* i_{D/\mathfrak{I}_D}(d + \mathfrak{I}_D)\dot{v}_p. \tag{25}$$

Here $\langle i_D(\mathfrak{I}_D)\rangle$ is the ideal of $D \overset{e}{\rtimes}_\alpha P$ generated by $i_D(\mathfrak{I}_D)$.

Proof. If d lies in $\mathfrak{I}_D$, then $i_D(\alpha_p(d)) = v_p(i_D(d))v_p^*$ lies in $\mathfrak{I}$ as $i_D(d)$ lies in $\mathfrak{I}$. At the same time, $v_p(i_D(d))v_p^* = i_D(\alpha_p(d))$ lies in $i_D(D)$. Thus $i_D(\alpha_p(d))$ lies in $i_D(D) \cap \mathfrak{I}$, and hence $\alpha_p(d)$ lies in $i_D^{-1}(i_D(D) \cap \mathfrak{I}) = \mathfrak{I}_D$. Therefore $\dot{\alpha}_p$ is well-defined. To see injectivity of α_p, observe that for $d \in D$, $\alpha_p(d) \in \mathfrak{I}_D$ implies

$$d = i_D^{-1} i_D(d) = i_D^{-1}(v_p^* v_p i_D(d) v_p^* v_p) = i_D^{-1}(v_p^* i_D(\alpha_p(d)) v_p).$$

Now $v_p^* i_D(\alpha_p(d)) v_p$ lies in $\mathfrak{I}$ as $i_D(\alpha_p(d))$ lies in $i_D(\mathfrak{I}_D) \subseteq \mathfrak{I}$, and $v_p^* i_D(\alpha_p(d)) v_p$ lies in $i_D(D)$ by (22). Hence $v_p^* i_D(\alpha_p(d)) v_p$ lies in $i_D(D) \cap \mathfrak{I}$, and thus $d = i_D^{-1}(v_p^* i_D(\alpha_p(d)) v_p)$ lies in $i_D^{-1}(i_D(D) \cap \mathfrak{I}) = \mathfrak{I}_D$. So far, we have proven the first part of the lemma.

Finally, the homomorphism $\pi \overset{e}{\rtimes} P : D \overset{e}{\rtimes}_\alpha P \to (D/\mathfrak{I}_D) \overset{e}{\rtimes}_{\dot{\alpha}} P$ determined by

$$(\pi \overset{e}{\rtimes} P)(v_q^* i_D(d) v_p) = \dot{v}_q^* i_{D/\mathfrak{I}_D}(d + \mathfrak{I}_D)\dot{v}_p$$

vanishes on $i_D(\mathfrak{I}_D)$, hence on $\langle i_D(\mathfrak{I}_D)\rangle$. Therefore $\pi \overset{e}{\rtimes} P$ factorizes through the quotient $D \overset{e}{\rtimes}_\alpha P/\langle i_D(\mathfrak{I}_D)\rangle$. This gives rise to the desired homomorphism $(\pi \overset{e}{\rtimes} P)^{\cdot}$. To prove that it is an isomorphism, we use the universal property of $((D/\mathfrak{I}_D) \overset{e}{\rtimes}_{\dot{\alpha}} P, i_{D/\mathfrak{I}_D}, \dot{v})$ to construct an inverse. The composition

$$D \xrightarrow{i_D} D \overset{e}{\rtimes}_\alpha P \overset{\pi^{(\rtimes)}}{\twoheadrightarrow} D \overset{e}{\rtimes}_\alpha P/\langle i_D(\mathfrak{I}_D)\rangle$$

($\pi^{(\rtimes)}$ is the canonical projection) obviously vanishes on $\mathfrak{I}_D$, giving a homomorphism $(i_D)^{\cdot} : D/\mathfrak{I}_D \to D \overset{e}{\rtimes}_\alpha P/\langle i_D(\mathfrak{I}_D)\rangle$. It is straightforward to see that $(i_D)^{\cdot}$ and $P \ni p \mapsto \pi^{(\rtimes)}(v_p) \in \mathrm{Isom}(D \overset{e}{\rtimes}_\alpha P/\langle i_D(\mathfrak{I}_D)\rangle)$ satisfy the covariance relation $\mathrm{Ad}(\pi^{(\rtimes)}(v_p)) \circ (i_D)^{\cdot} = (i_D)^{\cdot} \circ \dot{\alpha}_p$. Therefore, by the universal property of $((D/\mathfrak{I}_D) \overset{e}{\rtimes}_{\dot{\alpha}} P, i_{D/\mathfrak{I}_D}, \dot{v})$, there exists a homomorphism

$$(i_D)^{\cdot} \overset{e}{\rtimes} P : (D/\mathfrak{I}_D) \overset{e}{\rtimes}_{\dot{\alpha}} P \to D \overset{e}{\rtimes}_\alpha P/\langle i_D(\mathfrak{I}_D)\rangle$$

determined by

$$((i_D)^{\cdot} \overset{e}{\rtimes} P)(\dot{v}_q^* i_{D/\mathfrak{I}_D}(d + \mathfrak{I}_D)\dot{v}_p) = v_q^* i_D(d) v_p + \langle i_D(\mathfrak{I}_D)\rangle. \tag{26}$$

Comparing (25) and (26), we see that $(i_D)^{\cdot} \overset{e}{\rtimes} P$ is the inverse of $(\pi \overset{e}{\rtimes} P)^{\cdot}$. □

Corollary 8.2.7. *In the situation of the previous two lemmas, let $E : D \overset{e}{\rtimes}_\alpha P \to D$ be the faithful conditional expectation from Lemma 8.2.5. Let $\mathfrak{I}$ be an ideal of $D \overset{e}{\rtimes}_\alpha P$ as in Lemma 8.2.6. Then for all $x \in (D \overset{e}{\rtimes}_\alpha P)_+$, $E(x) \in \mathfrak{I}$ implies $x \in \mathfrak{I}$.*

Proof. Condition (22) is satisfied for $i_{D/\mathfrak{I}_D}$ and $\dot{v}$ from Lemma 8.2.6 since

$$\dot{v}_p^* i_{D/\mathfrak{I}_D}(d + \mathfrak{I}_D)\dot{v}_p = (\pi \overset{e}{\rtimes} P)^{\cdot}(\underbrace{v_p^* i_D(d) v_p}_{\substack{\in i_D(D) \\ \text{by (22)}}}) \in (\pi \overset{e}{\rtimes} P)^{\cdot}(i_D(D)) = i_{D/\mathfrak{I}_D}(D/\mathfrak{I}_D).$$

Thus, by Lemma 8.2.5 applied to the C*-dynamical semisystem $(D/\mathfrak{I}_D, P, \dot{\alpha})$, there exists a faithful conditional expectation

$$\dot{E} : (D/\mathfrak{I}_D) \overset{e}{\rtimes}_{\dot{\alpha}} P \to D/\mathfrak{I}_D$$

which is determined by

$$\dot{E}(\dot{v}_q^* i_{D/\mathfrak{I}_D}(d + \mathfrak{I}_D)\dot{v}_p) = \delta_{q,p}(i_{D/\mathfrak{I}_D})^{-1}(\dot{v}_p^* i_{D/\mathfrak{I}_D}(d + \mathfrak{I}_D)\dot{v}_p). \tag{27}$$

Comparing (23) and (27) and using the homomorphism $\pi \overset{e}{\rtimes} P$ from Lemma 8.2.6, we see that the diagram

$$\begin{array}{ccc} D \overset{e}{\rtimes}_\alpha P & \xrightarrow{\pi \overset{e}{\rtimes} P} & (D/\mathfrak{I}_D) \overset{e}{\rtimes}_{\dot{\alpha}} P \\ {\scriptstyle E}\downarrow & & \downarrow{\scriptstyle \dot{E}} \\ D & \xrightarrow{\pi} & D/\mathfrak{I}_D \end{array} \tag{28}$$

commutes.

Now take $x \in (D \overset{e}{\rtimes}_\alpha P)_+$ with $E(x) \in \mathfrak{I}$. This means that

$$0 + \mathfrak{I}_D = \pi(E(x)) \overset{(28)}{=} \dot{E}((\pi \overset{e}{\rtimes} P)(x)).$$

As $\dot{E}$ is faithful, we conclude that $(\pi \overset{e}{\rtimes} P)(x) = 0$ in $(D/\mathfrak{I}_D) \overset{e}{\rtimes}_{\dot{\alpha}} P$. From Lemma 8.2.6, it follows directly that the kernel of $\pi \overset{e}{\rtimes} P$ is $\langle i_D(\mathfrak{I}_D)\rangle$. Thus $x \in \langle i_D(\mathfrak{I}_D)\rangle \subseteq \mathfrak{I}$. □

Let us now return to the situation of interest. Let R be a Dedekind domain, and let $R \rtimes R^\times$ be the $ax+b$-semigroup over R. As explained in [Li2, second half of §2.4], the family $\mathcal{J}$ of right ideals of $R \rtimes R^\times$ is

$$\mathcal{J} = \{(r + I) \times I^\times : r \in R, (0) \neq I \triangleleft R\} \cup \{\emptyset\}. \tag{29}$$

By [Li2, Proposition 3.13] and because the constructible right ideals of $R \rtimes R^\times$ are independent by Lemma 2.29 in [Li2], the left regular representation

$$\lambda : C^*(R \rtimes R^\times) \to C^*_r(R \rtimes R^\times)$$

from [Li2, §2.1] is an isomorphism. Using λ, we will from now on always identify the full and the reduced semigroup C*-algebras of $R \rtimes R^\times$ (i.e. we may write v_p for V_p and e_X for E_X using the notation from [Li2]). Moreover, for a subset X of $R \rtimes R^\times$, let $e_X \in \mathcal{L}(\ell^2(R \rtimes R^\times))$ be the orthogonal projection onto $\ell^2(X) \subseteq \ell^2(R \rtimes R^\times)$. This is consistent with the notation from [Li2]. We sometimes write $e_{[X]}$ for e_X if the expression for X is rather long.

Now set

$$D(R \rtimes R^\times) = C^*(\{e_{(r+I)\times I^\times} : r \in R,\ (0) \neq I \triangleleft R\}),$$

and let τ be the action of $R \rtimes R^\times$ on $D(R \rtimes R^\times)$ given by $\tau_p = \mathrm{Ad}(v_p)$ for all p in $R \rtimes R^\times$. By [Li2, Lemma 2.14], we can canonically identify $C^*(R \rtimes R^\times)$ (hence also $C^*_r(R \rtimes R^\times)$) with $D(R \rtimes R^\times) \overset{e}{\rtimes}_\tau (R \rtimes R^\times)$. As τ_p is given by conjugation with an isometry, it is injective. Moreover, we have already seen in the proof of Theorem 8.2.1 that $R \rtimes R^\times$ is a left Ore semigroup whose enveloping group is amenable. As (22) is also satisfied for $(D(R \rtimes R^\times), R \rtimes R^\times, \tau)$ by [Li2, Corollary 2.9], we can apply Lemma 8.2.5 to $(D(R \rtimes R^\times), R \rtimes R^\times, \tau)$. Using the canonical identification of $C^*_r(R \rtimes R^\times)$ with $D(R \rtimes R^\times) \overset{e}{\rtimes}_\tau (R \rtimes R^\times)$, we obtain a faithful conditional expectation $E : C^*_r(R \rtimes R^\times) \to D(R \rtimes R^\times)$ which is determined by

$$E(v_q^* e_{(r+I)\times I^\times} v_p) = \delta_{q,p} v_p^* e_{(r+I)\times I^\times} v_p.$$

Corollary 8.2.7 then tells us the following:

Corollary 8.2.8. *Let $\mathfrak{I}$ be an ideal of $C^*_r(R \rtimes R^\times)$. For a positive element y in $C^*_r(R \rtimes R^\times)$, $E(y) \in \mathfrak{I}$ implies $y \in \mathfrak{I}$.*

Our first goal is to prove the following variation of [C-D-L, Lemma 4.12]:

Lemma 8.2.9. *Let R be a Dedekind domain with infinitely many pairwise distinct prime ideals, let $\mathfrak{I}$ be an ideal of $C_r^*(R \rtimes R^\times)$, and let y be a positive element in the *-algebra generated by the isometries v_p, $p \in R \rtimes R^\times$, i.e.*

$$y \in ({}^*\text{-alg}(\{v_p : p \in R \rtimes R^\times\}))_+.$$

If y does not lie in $\mathfrak{I}$, then there is a projection δ in $C_r^*(R \rtimes R^\times)$ of the form

$$\delta = e_{[(r+I)\times I^\times \setminus \bigcup_{k=1}^n (s_k+J_k)\times J_k^\times]} \tag{30}$$

with $r, s_1, \dots, s_n \in R$ and non-zero ideals $I, J_1, \dots, J_n$ of R such that

1. *δ does not lie in $\mathfrak{I}$,*
2. *$\delta y \delta = (\|E(y) + \mathfrak{I}\|_{C_r^*(R\rtimes R^\times)/\mathfrak{I}})\delta$.*

Note that in (30), the case $n = 0$ is possible; it corresponds to $\delta = e_{[(r+I)\times I^\times]}$.

Proof. As y lies in *-alg$(\{v_p : p \in R \rtimes R^\times\})$, it is of the form

$$y = d + \sum_{i=1}^m v_{q_i}^* d_i v_{p_i} \tag{31}$$

with $p_1, \dots, p_m, q_1, \dots, q_m$ in $R \rtimes R^\times$ such that $q_i \neq p_i$ for all $1 \le i \le m$ and where d and d_i $(1 \le i \le m)$ are finite linear combinations of the projections e_X, $X \in \mathcal{J}$. The condition $q_i \neq p_i$ for all $1 \le i \le m$ implies $E(y) = d$. Moreover, we can write d as a finite sum $d = \sum_X \lambda_X e_X$. Now we can orthogonalize the projections e_X which appear in this finite sum, and we obtain pairwise orthogonal projections e_Y and a presentation $d = \sum_Y \mu_Y e_Y$. By Corollary 8.2.8, $y \notin \mathfrak{I}$ implies $d = E(y) \notin \mathfrak{I}$. Thus $0 < \|E(y) + \mathfrak{I}\|_{C_r^*(R\rtimes R^\times)/\mathfrak{I}} = \sup(\{\mu_Y\})$ where the supremum is taken over all coefficients μ_Y corresponding to $e_Y \notin \mathfrak{I}$ appearing in the sum above which represents d. Since this sum is finite and the $\{e_Y\}$ are pairwise orthogonal, there exists a projection e_Y such that the corresponding coefficient precisely coincides with $\|E(y) + \mathfrak{I}\|_{C_r^*(R\rtimes R^\times)/\mathfrak{I}}$. This implies that this projection satisfies

$$e_Y d e_Y = (\|E(y) + \mathfrak{I}\|_{C_r^*(R\rtimes R^\times)/\mathfrak{I}}) e_Y. \tag{32}$$

Moreover, since the projections $\{e_Y\}$ were obtained by orthogonalizing the commuting projections $\{e_X\}$, the subset Y of $R \rtimes R^\times$ must be of the form

$$Y = (\tilde{r} + \tilde{I}) \times \tilde{I}^\times \setminus \bigcup_{k=1}^n (\tilde{s}_k + \tilde{J}_k) \times \tilde{J}_k^\times \tag{33}$$

with $\tilde{r}, \tilde{s}_1, \dots, \tilde{s}_n \in R$ and non-zero ideals $\tilde{I}, \tilde{J}_1, \dots, \tilde{J}_n$ of R such that $\tilde{J}_k \subseteq \tilde{I}$ for all $1 \le k \le n$. The case $n = 0$ is allowed; it corresponds to $Y = (\tilde{r} + \tilde{I}) \times \tilde{I}^\times$. That Y is of this form follows from (29).

We now choose $(b, a) \in R \rtimes R^\times$ satisfying

$1_{b,a}$. $v_{(b,a)} e_Y = e_Y v_{(b,a)}$,
$2_{b,a}$. $v_{(b,a)} v_{(b,a)}^* v_{q_i}^* d_i v_{p_i} v_{(b,a)} v_{(b,a)}^* = 0$ for all $1 \le i \le m$.

Let $q_i = (b'_i, a'_i) \in R \rtimes R^\times$ and $p_i = (b_i, a_i) \in R \rtimes R^\times$. Then

$$\begin{aligned} & v_{(b,a)} v^*_{(b,a)} v^*_{q_i} d_i v_{p_i} v_{(b,a)} v^*_{(b,a)} \\ & \qquad = v^*_{q_i} v_{q_i} e_{(b+aR)\times(aR)^\times} v^*_{q_i} d_i v_{p_i} e_{(b+aR)\times(aR)^\times} v^*_{p_i} v_{p_i} \\ & \qquad = v^*_{q_i} d_i e_{[(b'_i+a'_i b+a'_i aR)\times(a'_i aR)^\times]} e_{[(b_i+a_i b+a_i aR)\times(a_i aR)^\times]} v_{p_i} \\ & \qquad = v^*_{q_i} d_i e_{[((b'_i+a'_i b+a'_i aR)\times(a'_i aR)^\times)\cap((b_i+a_i b+a_i aR)\times(a_i aR)^\times)]} v_{p_i} \end{aligned}$$

vanishes if

$$(b'_i + a'_i b + a'_i aR) \cap (b_i + a_i b + a_i aR) = \emptyset \tag{34}$$

for all $1 \le i \le m$. If for all $1 \le i \le m$, we have $b'_i + a'_i b - (b_i + a_i b) = (b'_i - b_i) + (a'_i - a_i)b \notin aR$, then certainly (34) holds.

We claim that we can choose $b \in R$ such that

1_b. $b \in \tilde{J}_1 \cap \cdots \cap \tilde{J}_n$ (or $b \in \tilde{I}$ if $n = 0$),
2_b. $(b'_i - b_i) + (a'_i - a_i)b \neq 0$ for all $1 \le i \le m$.

The reason is that we have by assumption $(b'_i, a'_i) \neq (b_i, a_i)$ for all $1 \le i \le m$. This implies that for all $1 \le i \le m$, either $a'_i = a_i \wedge b'_i \neq b_i$ or $a'_i \neq a_i$. If $a'_i = a_i \wedge b'_i \neq b_i$, then $(b'_i - b_i) + (a'_i - a_i)b = b'_i - b_i \neq 0$ for all $b \in R$, and if $a'_i \neq a_i$, then $(b'_i - b_i) + (a'_i - a_i)b \neq 0$ for all $b \in R$ with $b \neq -(a'_i - a_i)^{-1}(b'_i - b_i)$. This shows that there are only finitely many ring elements which do not satisfy 2_b. On the other hand, by our assumption that R is a Dedekind domain with infinitely many pairwise distinct prime ideals, $\tilde{J}_1 \cap \cdots \cap \tilde{J}_n$ (or $\tilde{I}$ if $n = 0$) is an infinite set. Thus we can find b in R satisfying 1_b and 2_b at the same time. Let us fix such a choice for $b \in R$.

As a next step, we claim that we can choose $a \in R^\times$ such that

1_a. $a \in 1 + \tilde{J}_1 \cap \cdots \cap \tilde{J}_n$ (or $a \in 1 + \tilde{I}$ if $n = 0$).
2_a. $(b'_i - b_i) + (a'_i - a_i)b \notin aR$ for all $1 \le i \le m$.

To see this, first note that if $\prod_{i=1}^m ((b'_i - b_i) + (a'_i - a_i)b)$ does not lie in aR, then 2_a follows. By 2_b, the element $\prod_{i=1}^m ((b'_i - b_i) + (a'_i - a_i)b)$ is not zero. Thus, by our assumption that R is a Dedekind domain with infinitely many pairwise distinct prime ideals, there exists a prime ideal P of R such that $\prod_{i=1}^m ((b'_i - b_i) + (a'_i - a_i)b)$ does not lie in P and also $\tilde{J}_1 \cap \cdots \cap \tilde{J}_n \nsubseteq P$ (or $\tilde{I} \nsubseteq P$ if $n = 0$). By the Chinese Remainder Theorem (see for example [Neu, Chapter I, Theorem (3.6)]), there exists a non-zero element a of the prime ideal P such that $a \in 1 + \tilde{J}_1 \cap \cdots \cap \tilde{J}_n$ (or $a \in 1 + \tilde{I}$ if $n = 0$). This a obviously satisfies 1_a and 2_a.

Finally, we claim that this choice for $(b, a) \in R \rtimes R^\times$ satisfies $1_{b,a}$ and $2_{b,a}$. First, by our observations, 2_a implies (34), hence $2_{b,a}$. To prove $1_{b,a}$, note that 1_a implies that aR is coprime to each of the ideals $\tilde{I}, \tilde{J}_1, \ldots, \tilde{J}_n$, so that

$$\begin{aligned} & (a\tilde{r} + a\tilde{I}) \times (a\tilde{I})^\times \setminus \bigcup_{k=1}^n (a\tilde{s}_k + a\tilde{J}_k) \times (a\tilde{J}_k)^\times \\ & \qquad = \Big((a\tilde{r} + \tilde{I}) \times \tilde{I}^\times \setminus \bigcup_{k=1}^n (a\tilde{s}_k + \tilde{J}_k) \times \tilde{J}_k^\times \Big) \cap (aR \times (aR)^\times). \end{aligned} \tag{35}$$

Thus

$$\begin{aligned}
v_{(0,a)} e_Y v_{(0,a)}^* &\overset{(33)}{=} e_{[(a\tilde{r}+a\tilde{I})\times(a\tilde{I})^\times\setminus\bigcup_{k=1}^n(a\tilde{s}_k+a\tilde{J}_k)\times(a\tilde{J}_k)^\times]} \\
&\overset{(35)}{=} e_{[(a\tilde{r}+\tilde{I})\times\tilde{I}^\times\setminus\bigcup_{k=1}^n(a\tilde{s}_k+\tilde{J}_k)\times\tilde{J}_k^\times]} e_{aR\times(aR)^\times} \\
&= e_{[(a\tilde{r}+\tilde{I})\times\tilde{I}^\times\setminus\bigcup_{k=1}^n(a\tilde{s}_k+\tilde{J}_k)\times\tilde{J}_k^\times]} v_{(0,a)} v_{(0,a)}^* \\
&\overset{1_a}{=} e_{[(\tilde{r}+\tilde{I})\times\tilde{I}^\times\setminus\bigcup_{k=1}^n(\tilde{s}_k+\tilde{J}_k)\times\tilde{J}_k^\times]} v_{(0,a)} v_{(0,a)}^* \\
&= e_Y v_{(0,a)} v_{(0,a)}^*.
\end{aligned}$$

Multiplication on the right with $v_{(0,a)}$ implies

$$v_{(0,a)} e_Y = e_Y v_{(0,a)}.$$

Furthermore, 1_b implies

$$v_{(b,1)} e_Y = e_Y v_{(b,1)}.$$

Thus we obtain

$$v_{(b,a)} e_Y = v_{(b,1)} v_{(0,a)} e_Y = v_{(b,1)} e_Y v_{(0,a)} = e_Y v_{(b,1)} v_{(0,a)} = e_Y v_{(b,a)}.$$

This proves $1_{b,a}$. So we have seen that (b, a) satisfies $1_{b,a}$ and $2_{b,a}$.

Now we set $\delta := v_{(b,a)} e_Y v_{(b,a)}^*$. Then δ is a projection in $C_r^*(R \rtimes R^\times)$ of the desired form as in (30) by construction. In addition, δ does not lie in $\mathfrak{I}$ because e_Y does not lie in $\mathfrak{I}$. And finally, we have

$$\delta = v_{(b,a)} e_Y v_{(b,a)}^* \overset{1_{b,a}}{=} e_Y v_{(b,a)} v_{(b,a)}^* = v_{(b,a)} v_{(b,a)}^* e_Y \tag{36}$$

and hence

$$\begin{aligned}
\delta y \delta &\overset{(31)}{=} \delta d \delta + \sum_{i=1}^m \delta v_{q_i}^* d_i v_{p_i} \delta \\
&\overset{(36)}{=} v_{(b,a)} v_{(b,a)}^* e_Y d e_Y v_{(b,a)} v_{(b,a)}^* + \sum_{i=1}^m e_Y \underbrace{v_{(b,a)} v_{(b,a)}^* v_{q_i}^* d_i v_{p_i} v_{(b,a)} v_{(b,a)}^*}_{=0 \text{ by } 2_{(b,a)}} e_Y \\
&\overset{(32)}{=} (\|E(y) + \mathfrak{I}\|_{C_r^*(R\rtimes R^\times)/\mathfrak{I}}) e_Y v_{(b,a)} v_{(b,a)}^* \\
&\overset{(36)}{=} (\|E(y) + \mathfrak{I}\|_{C_r^*(R\rtimes R^\times)/\mathfrak{I}}) \delta.
\end{aligned}$$

Therefore this projection δ has the desired properties and satisfies conditions 1 and 2 of the lemma. □

To proceed, we need

Lemma 8.2.10. *In the situation of the previous lemmma, the projection δ gives rise to a properly infinite projection $\delta + \mathfrak{I}$ in the quotient $C_r^*(R \rtimes R^\times)/\mathfrak{I}$.*

Proof. The projection δ is of the form $\delta = e_{[(r+I)\times I^\times \setminus \bigcup_{k=1}^n (s_k+J_k)\times J_k^\times]}$ by (30). Again, $n = 0$ is allowed. Now choose $c \in R^\times$ and $r_1, r_2 \in R$ such that

($*_c$) c is not invertible and c lies in $1 + \bigcap_{k=1}^n J_k$ (or $1 + I$ if $n = 0$),
($*_r$) r_1, r_2 lie in $\bigcap_{k=1}^n J_k$ (or I if $n = 0$) but $r_1 + cR \neq r_2 + cR$.

This is possible because by our assumption that R is a Dedekind domain with infinitely many pairwise distinct prime ideals, we can first find an element $c \in R^\times$ satisfying ($*_c$) using strong approximation (compare [Bour, Chapitre VII, §2.4, Proposition 2]). Then, as ($*_c$) implies that cR and $\bigcap_{k=1}^n J_k$ (or I for $n = 0$) are coprime, the Chinese Remainder Theorem tells us that we can find elements r_1 and r_2 in R satisfying ($*_r$). Then, by analogous computations to those in the proof of the previous lemma, ($*_c$) and ($*_r$) imply

$$v_{(r_i,c)}\delta = \delta v_{(r_i,c)} \quad \text{for } i = 1, 2. \tag{37}$$

Set $\delta_i = v_{(r_i,c)}\delta v^*_{(r_i,c)}$ for $i = 1, 2$. Then we certainly have $\delta_i \sim \delta$ for $i = 1, 2$, where $\sim$ stands for "Murray–von Neumann equivalent". Moreover, for $i = 1, 2$, we obtain

$$\delta_i = v_{(r_i,c)}\delta v^*_{(r_i,c)} \overset{(37)}{=} \delta v_{(r_i,c)} v^*_{(r_i,c)} \leq \delta.$$

And finally,

$$\begin{aligned}\delta_1\delta_2 &= v_{(r_1,c)}\delta v^*_{(r_1,d)} v_{(r_2,c)}\delta v^*_{(r_2,c)} \overset{(37)}{=} \delta v_{(r_1,c)} v^*_{(r_1,d)} v_{(r_2,c)} v^*_{(r_2,c)}\delta \\ &= \delta e_{(r_1+cR)\times(cR)^\times} e_{(r_2+cR)\times(cR)^\times}\delta \\ &= \delta e_{[((r_1+cR)\times(cR)^\times)\cap((r_2+cR)\times(cR)^\times)]}\delta = 0\end{aligned}$$

as $r_1 + cR \neq r_2 + cR$ by ($*_r$). As $\delta + \mathfrak{I}$ is a non-zero projection in $C_r^*(R \rtimes R^\times)/\mathfrak{I}$ by condition 1 in Lemma 8.2.9, this proves our claim. □

With these preparations, we are ready for

Proof of Theorem 8.2.4. Let R be a Dedekind domain with infinitely many pairwise distinct prime ideals. By [Pas-Rør, Proposition 2.11], we have to prove that every non-zero hereditary sub-C*-algebra in any quotient of $C_r^*(R \rtimes R^\times)$ contains an infinite projection. Let $\mathfrak{I}$ be an ideal of $C_r^*(R \rtimes R^\times)$. It suffices to show that every hereditary sub-C*-algebra of $C_r^*(R \rtimes R^\times)/\mathfrak{I}$ of the form

$$\overline{(z + \mathfrak{I})(C_r^*(R \rtimes R^\times)/\mathfrak{I})(z + \mathfrak{I})}$$

for some $z \in C_r^*(R \rtimes R^\times)_+ \setminus \mathfrak{I}$ contains an infinite projection because every non-zero hereditary sub-C*-algebra of $C_r^*(R \rtimes R^\times)/\mathfrak{I}$ contains a subalgebra of that form.

First of all, $z \notin \mathfrak{I}$ implies $E(z) \notin \mathfrak{I}$ by Corollary 8.2.8. As *-alg($\{v_p : p \in R \rtimes R^\times\}$) is dense in $C_r^*(R \rtimes R^\times)$, there exists a positive element y in *-alg($\{v_p : p \in R \rtimes R^\times\}$) such that

$$\|z - y\| < \tfrac{1}{3}\|E(z) + \mathfrak{I}\|_{C_r^*(R\rtimes R^\times)/\mathfrak{I}}.$$

It follows that

$$\begin{aligned}\|E(z) - E(y) + \mathfrak{I}\|_{C_r^*(R \rtimes R^\times)/\mathfrak{I}} &\leq \|E(z) - E(y)\| \leq \|z - y\| \\ &< \tfrac{1}{3}\|E(z) + \mathfrak{I}\|_{C_r^*(R \rtimes R^\times)/\mathfrak{I}},\end{aligned}$$

so that

$$\|E(y) + \mathfrak{I}\|_{C_r^*(R \rtimes R^\times)/\mathfrak{I}} > \tfrac{2}{3}\|E(z) + \mathfrak{I}\|_{C_r^*(R \rtimes R^\times)/\mathfrak{I}} > 0. \tag{38}$$

This implies that $E(y)$ does not lie in $\mathfrak{I}$ and hence, by Corollary 8.2.8, y does not lie in $\mathfrak{I}$. By Lemmas 8.2.9 and 8.2.10, there exists a projection δ in $C_r^*(R \rtimes R^\times)$ such that $\delta + \mathfrak{I}$ is a properly infinite projection in $C_r^*(R \rtimes R^\times)/\mathfrak{I}$ and

$$\delta y \delta = (\|E(y) + \mathfrak{I}\|_{C_r^*(R \rtimes R^\times)/\mathfrak{I}})\delta. \tag{39}$$

We get

$$\|\delta z \delta - \delta y \delta + \mathfrak{I}\|_{C_r^*(R \rtimes R^\times)/\mathfrak{I}} \leq \|\delta\| \, \|z - y\| \, \|\delta\| < \tfrac{1}{3}\|E(z) + \mathfrak{I}\|_{C_r^*(R \rtimes R^\times)/\mathfrak{I}}$$

and

$$\begin{aligned}&\big(\delta y \delta - \tfrac{1}{3}\|E(z) + \mathfrak{I}\|_{C_r^*(R \rtimes R^\times)/\mathfrak{I}}\big)_+ \\ &\overset{(39)}{=} \big((\|E(y) + \mathfrak{I}\|_{C_r^*(R \rtimes R^\times)/\mathfrak{I}})\delta - \tfrac{1}{3}\|E(z) + \mathfrak{I}\|_{C_r^*(R \rtimes R^\times)/\mathfrak{I}}\big)_+ \\ &= \underbrace{\big(\|E(y) + \mathfrak{I}\|_{C_r^*(R \rtimes R^\times)/\mathfrak{I}} - \tfrac{1}{3}\|E(z) + \mathfrak{I}\|_{C_r^*(R \rtimes R^\times)/\mathfrak{I}}\big)}_{=:C}\delta\end{aligned}$$

with $C > \frac{1}{3}\|E(z) + \mathfrak{I}\|_{C_r^*(R \rtimes R^\times)/\mathfrak{I}} > 0$ by (38). In this situation, by Lemma 2.2 in [Kir-Rør2] (applied to $A = C_r^*(R \rtimes R^\times)/\mathfrak{I}$, $a = \delta y \delta + \mathfrak{I}$, $b = \delta z \delta + \mathfrak{I}$ and $\varepsilon = \frac{1}{3}\|E(z) + \mathfrak{I}\|_{C_r^*(R \rtimes R^\times)/\mathfrak{I}}$), there exists $x' \in C_r^*(R \rtimes R^\times)/\mathfrak{I}$ with $C\delta + \mathfrak{I} = x'(\delta z \delta + \mathfrak{I})x'^*$. Now set $x := C^{-1/2}x'(\delta + \mathfrak{I})$. Then $\delta + \mathfrak{I} = x(z + \mathfrak{I})x^*$ is a properly infinite projection (see Lemma 8.2.10). We conclude that

$$(z + \mathfrak{I})^{1/2} x^* x (z + \mathfrak{I})^{1/2}$$

is a projection in

$$\overline{(z + \mathfrak{I})(C_r^*(R \rtimes R^\times)/\mathfrak{I})(z + \mathfrak{I})}$$

which is Murray–von Neumann equivalent to $\delta + \mathfrak{I}$, hence properly infinite itself. □

Combining Theorems 8.2.1 and 8.2.4 with the K-theoretic results for ring C*-algebras from [Cu-Li] and [Li-Lü], we obtain

Corollary 8.2.11. *For every ring of integers R in a number field, the semigroup C*-algebra $C_r^*(R \rtimes R^\times)$ is purely infinite, has the ideal property but does not have real rank zero.*

Proof. Let R be the ring of integers in a number field. Comparing universal properties, it is clear that the ring C*-algebra $\mathfrak{A}[R]$ of R is a quotient of the semigroup C*-algebra $C^*(R \rtimes R^\times)$ of the $ax + b$-semigroup over R. Thus $\mathfrak{A}[R]$ is also a quotient of $C_r^*(R \rtimes R^\times)$. We have proven in [Cu-Li] and [Li-Lü] that $K_0(\mathfrak{A}[R])$ cannot be finitely generated, whereas it follows from Theorem 8.2.1 that $K_0(C_r^*(R \rtimes R^\times))$ is finitely generated. Hence the quotient map from $C_r^*(R \rtimes R^\times)$ to $\mathfrak{A}[R]$ cannot be surjective on K_0. In the language of [Pas-Rør], this means that $C_r^*(R \rtimes R^\times)$ is not K_0-liftable. As we have seen in Theorem 8.2.4 that $C_r^*(R \rtimes R^\times)$ is purely infinite, Theorem 4.2 in [Pas-Rør] implies that $C_r^*(R \rtimes R^\times)$ cannot have real rank zero. This shows the last part of our assertion. The first part follows from Theorem 8.2.4. □

Remark. For a cancellative semigroup, we do not only have the left regular representation, but also the right regular one. For groups, the C*-algebras generated by these representations are isomorphic due to invertibility of the group elements. But for a genuine (and let us say non-abelian) semigroup, the left and right regular representations generate in general different C*-algebras. For our present piece of work, the C*-algebra generated by the left regular representation of the $ax + b$-semigroup over the ring of integers of a number field was the motivating example. A natural question would be:

What about the C*-algebra generated by the right regular representation of such an $ax + b$-semigroup?

It turns out that although the C*-algebras for the left and right regular representations of such semigroups are quite different (the one for the right regular representation is not purely infinite), their K-theoretic invariants do coincide. In a forthcoming paper, the authors plan to discuss the C*-algebras of the right regular representations of such $ax + b$-semigroups in a general context.

Acknowledgments. Research supported by the Deutsche Forschungsgemeinschaft (SFB 878) and by the ERC through AdG 267079.

References

[Bla] Blackadar, B.: K-theory for Operator Algebras. 2nd ed., Math. Sci. Res. Inst. Publ. 5, Cambridge Univ. Press, Cambridge (1986) Zbl 0913.46054 MR 1656031

[Bour] Bourbaki, N.: Algèbre commutative. Chapitres 5 à 7. Springer, Berlin (2006) Zbl 1103.13002 MR 0194450(Chap. 5, 6) MR 0260715(Chap. 7)

[Br-Oz] Brown, N. P., Ozawa, N.: C*-algebras and Finite-Dimensional Approximations. Grad. Stud. Math. 88, Amer. Math. Soc., Providence, RI (2008) Zbl 1160.46001 MR 2391387

[C-E-O] Chabert, J., Echterhoff, S., Oyono-Oyono, H.: Going-down functors, the Künneth formula, and the Baum–Connes conjecture. Geom. Funct. Anal. **14**, 491–528 (2004) Zbl 1063.46056 MR 2100669

[Cl-Pr] Clifford, A. H., Preston, G. B.: The Algebraic Theory of Semigroups. Vol. I. Math. Surveys 7, Amer. Math. Soc., Providence, RI (1961) Zbl 0111.03403 MR 0132791

[C-D-L] Cuntz, J., Deninger, C., Laca, M.: C*-algebras of Toeplitz type associated with algebraic number fields. Math. Ann. **355**, 1383–1423 (2013) Zbl 1273.22008 MR 3037019

[Cu-Li] Cuntz, J., Li, X.: C*-algebras associated with integral domains and crossed products by actions on adele spaces. J. Noncommutative Geom. **5**, 1–37 (2011) Zbl 1229.46044 MR 2746649

[E-L-P-W] Echterhoff, S., Lück, W., Phillips, N. C., Walters, S.: The structure of crossed products of irrational rotation algebras by finite subgroups of $SL_2(\mathbb{Z})$. J. Reine Angew. Math. **639**, 173–221 (2010) Zbl 1202.46081 MR 2608195

[E-N-O] Echterhoff, S., R. Nest, Oyono-Oyono, H.: Fibrations with noncommutative fibres. J. Noncommutative Geometry **3**, 377–417 (2009) Zbl 1179.19003 MR 2511635

[H-K] Higson, N., Kasparov, G.: E-theory and KK-theory for groups which act properly and isometrically on Hilbert space. Invent. Math. **144**, 23–74 (2001) Zbl 0988.19003 MR 1821144

[Kas] Kasparov, G. G.: Equivariant KK-theory and the Novikov conjecture. Invent. Math. **91**, 147–201 (1988) Zbl 0647.46053 MR 0918241

[Kir-Rør1] Kirchberg, E., Rørdam, M.: Non-simple purely infinite C*-algebras. Amer. J. Math. **122**, 637–666 (2000) Zbl 0968.46042 MR 1759891

[Kir-Rør2] Kirchberg, E., Rørdam, M.: Infinite non-simple C*-algebras: absorbing the Cuntz algebra $\mathcal{O}_\infty$. Adv. Math. **167**, 195–264 (2002) Zbl 1030.46075 MR 1906257

[La] Laca, M.: From endomorphisms to automorphisms and back: dilations and full corners. J. London Math. Soc. **61**, 893–904 (2000) Zbl 0973.46066 MR 1766113

[Li1] Li, X.: Ring C*-algebras. Math. Ann. **348**, 859–898 (2010) Zbl 1207.46050 MR 2721644

[Li2] Li, X.: Semigroup C*-algebras and amenability of semigroups. J. Funct. Anal. **262**, 4302–4340 (2012) Zbl 1243.22006 MR 2900468

[Li-Lü] Li, X., Lück, W.: K-theory for ring C*-algebras—the case of number fields with higher roots of unity. J. Topol. Anal. **4**, 449–480 (2012) Zbl 1269.46050 MR 3021772

[Mey-Ne] Meyer, R., Nest, R.: The Baum–Connes conjecture via localisation of categories. Topology **45**, 209–259 (2006) Zbl 1092.19004 MR 2193334

[Neu] Neukirch, J., Algebraic Number Theory. Grundlehren Math. Wiss. 322, Springer, Berlin (1999) Zbl 0956.11021 MR 1697859

[Pas-Rør] Pasnicu, C., Rørdam, M.: Purely infinite C*-algebras of real rank zero. J. Reine Angew. Math. **642**, 51–73 (2007) Zbl 1162.46029 MR 2377129

[Rør] Rørdam, M.: Classification of nuclear, simple C^*-algebras. In: Classification of Nuclear C^*-Algebras. Entropy in Operator Algebras, Encyclopaedia Math. Sci. 126, Springer, Berlin, 1–145 (2002) Zbl 1016.46037 MR 1878882