J. Eur. Math. Soc. 17, 689–709

DOI 10.4171/JEMS/514

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Poincaré inequalities and rigidity for actions on Banach spaces

Received July 4, 2011

Abstract. The aim of this paper is to extend the framework of the spectral method for proving property (T) to the class of reflexive Banach spaces and present a condition implying that every affine isometric action of a given group G on a reflexive Banach space X has a fixed point. This last property is a strong version of Kazhdan's property (T) and is equivalent to $H^1(G,\pi)$ being zero for every isometric representation π of G on X. The condition is expressed in terms of p-Poincaré constants and we provide examples of groups which satisfy such conditions and for which $H^1(G,\pi)$ vanishes for every isometric representation π on an L_p space for some $p > 2$. Our methods allow estimating such a p explicitly and yield several interesting applications. In particular, we obtain quantitative estimates for vanishing of 1-cohomology with coefficients in uniformly bounded representations on a Hilbert space. We also give lower bounds on the conformal dimension of the boundary of a hyperbolic group in the Gromov density model.

Keywords. Poincaré inequality, Kazhdan's property (T), affine isometric action, 1-cohomology

1. Introduction

Kazhdan's property (T) is a powerful rigidity property of groups with numerous applications and several characterizations. In this article we focus on the following description of property (T): *a group G has property* (T) *if and only if every affine isometric action of G on the Hilbert space has a fixed point*. This characterization can be rephrased as a cohomological condition: $H^1(G,\pi) = 0$ for every unitary representation π of G. A generalization of property (T) to other Banach spaces is then straightforward: we are interested in conditions implying that every affine isometric action of a given group on a given Banach space has a fixed point. Such rigidity properties for actions on Banach spaces, as well as other generalizations of property (T), and their applications, were studied earlier in [2, 12, 8, 21].

One very successful method of proving property (T) is through spectral conditions on links of vertices of complexes acted upon by a group. Variations of such conditions were studied in [3, 9, 10, 14, 17, 19, 33, 38, 39, 35, 36] in the context of Hilbert spaces

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Mathematics Subject Classification (2010): Primary 22D12; Secondary 46B03

and non-positively curved spaces. Given a group G acting on a 2-dimensional simplicial complex, one considers the link of a vertex. This link is a finite graph. If for every vertex, the first positive eigenvalue of the discrete Laplacian is strictly larger than $1/2$, then G has property (T).

The main purpose of this work is to extend the framework of the spectral method, and some of the rigidity results, beyond Hilbert spaces. Our main result provides such a framework for the class of reflexive Banach spaces. The difficulty lies in the fact that in the Hilbert space case the spectral method heavily relies on orthogonality, in particular self-duality of Hilbert spaces. When passing to other Banach spaces, dual spaces of certain Banach spaces and of their subspaces have to be identified, and this is often a difficult task. We show that when the representation is isometric, such computations are possible and we can use duality effectively.

We focus on link graphs constructed using generating sets of a group, as in [39]. For a finite, symmetric generating set S not containing the identity element, the vertices of the link graph $\mathcal{L}(S)$ are the elements of S; generators s and t are connected by an edge if $s^{-1}t$ is a generator. We will also assume that the graph is equipped with a weight ω on the edges.

Let X be a Banach space and denote by $\kappa_p(S, X)$ be the optimal constant in the p-Poincaré inequality for the link graph $\mathcal{L}(S)$ of G and the norm of X,

$$\sum_{s\in S} \|f(s) - Af\|_X^p \deg_\omega(s) \leq \kappa_p^p \sum_{s\sim t} \|f(s) - f(t)\|_X^p \omega(s,t),$$

where Af is the mean value of f. When $X = L_2$, the constant $\kappa_2(S, L_2) = \kappa_2(S, \mathbb{R})$ can be expressed in terms of the first eigenvalue of the discrete Laplacian.

Our main result shows that sufficiently small constants in Poincaré inequalities for the graph $\mathcal{L}(S)$ imply the desired cohomological vanishing. Given a number $1 < p < \infty$ we denote by p^* its adjoint index, satisfying $1/p + 1/p^* = 1$.

Theorem 1.1. *Let X be a reflexive Banach space and let G be a group generated by a finite symmetric set S not containing the identity element. If the link graph $\mathcal{L}(S)$ is connected and for some $1 < p < \infty$ the associated Poincaré constants satisfy*

$$\max\{2^{-1/p}\kappa_p(S, X), 2^{-1/p^*}\kappa_{p^*}(S, X^*)\} < 1,$$

then $H^1(G, \pi) = 0$ for every isometric representation π of G on X.

Clearly, by reflexivity, the same conclusion holds for actions on X^*. Interestingly, the roles of the two constants in the proof of the above theorem are not symmetric.

We apply Theorem 1.1 to L_p spaces. The interesting case is $p > 2$. Indeed, when $1 < p \leq 2$, affine isometric actions exhibit the same behavior as for the Hilbert space: G has property (T) if and only if any affine action on an L_p space for $1 < p \leq 2$ has a fixed point [2]. Also, G admits a metrically proper affine isometric action on the Hilbert space (i.e., is a-T-menable) if and only if it admits such an action on any $L_p[0, 1]$ for $1 < p \leq 2$ [28] (see corrected version [29]). This last property is a strong negation of the existence of a fixed point.

Fixed point properties for groups acting on L_p spaces for $p > 2$ are difficult to prove and only a handful of results are known:

1. higher rank algebraic groups and their lattices have fixed points for every affine isometric action on L_p spaces for all $p > 1$ [2];
2. in [23] it was proved that $\mathrm{SL}_n(\mathbb{Z}[x_1, \ldots, x_k])$ has fixed points for every affine isometric action on L_p for every $p > 1$ and $n \geq 4$;
3. Naor and Silberman [24] showed that Gromov's random groups, containing (in a certain weak sense) expanders in their Cayley graphs, have a fixed point for affine isometric actions on any L_p for $p > 1$;
4. a general argument due to Fisher and Margulis (see the proof in [2]) shows that for every property (T) group G there exists a constant $\varepsilon = \varepsilon(G) > 0$ such that any affine isometric action on L_p for $p \in [2, 2+\varepsilon)$ has a fixed point. However, their argument does not give any control over ε.

On the other hand, there are also groups which have property (T) but act without fixed points on L_p spaces. One example is furnished by $\mathrm{Sp}(n, 1)$, which has property (T) but has non-vanishing L_p-cohomology for $p > 4n + 2$, by a result of Pansu [32]. It is also known that there exist hyperbolic groups which have property (T). Nevertheless, Bourdon and Pajot [5] showed that for every hyperbolic group G and sufficiently large $p > 2$ there is an affine isometric action on $\ell_p(G)$ whose linear part is the regular representation and which does not have a fixed point. Moreover, Yu [37] showed that every hyperbolic group admits a proper, affine isometric action on $\ell_p(G \times G)$ for all sufficiently large $p > 2$ (see also [27] for another construction). We refer to [30] for a recent survey.

The techniques we use to establish the fixed point properties are different from the ones used previously for general Banach spaces. In particular, we do not need the Howe–Moore property. This representation-theoretic property was necessary in [2, 23]. The outcome is also slightly different, as our methods are not expected to give fixed points on L_p for all $p > 1$. One reason is that the p-Poincaré constants usually increase above $2^{1/p}$ as p grows to infinity. The second reason is that the main result applies to random hyperbolic groups, which, as remarked earlier, act without fixed points on L_p spaces for $p > 2$ sufficiently large. Using our approach we obtain the vanishing of $H^1(G, \pi)$ for isometric representations π on L_p spaces with $p \in [2, 2+c)$, where the value of c depends on the group and can be explicitly estimated. Finally, we point out that our techniques and the Poincaré inequalities we use are all linear, in contrast to the non-linear approach used e.g. in [17, 35]. Linearity allows us to use interpolation methods effectively and also to obtain additional information about the structure of cohomology in the presence of spectral gaps.

To apply Theorem 1.1 we need to estimate p-Poincaré constants for $p > 2$. Even in classical settings, such as convex domains in $\mathbb{R}^n$, estimates exist but exact values of p-Poincaré constants are not known, except for a few special cases. The situation is even worse for finite graphs, where very few estimates are known for cases other than $p = 1, 2$. Here we consider the family of $\widetilde{A}_2$-groups, indexed by powers of primes. These groups were introduced and studied in [6, 7]. For every q, the group G_q has a generating set whose link graph is the incidence graph of the finite projective plane over the field $\mathbb{F}_q$. Spectra of such graphs were computed in [11] and give, in particular, the exact value of the Poincaré constant $\kappa_2(S, \mathbb{R})$. We use this fact to estimate $\kappa_p(S, L_p)$ for these graphs, which allows us to obtain for each q a number c_q such that any affine isometric action of G on any L_p has a fixed point for $p \in [2, 2 + c_q)$. Explicit estimates of c_q are given in Theorem 5.1.

As mentioned earlier, our results apply to random hyperbolic groups, more precisely, to random groups in the Gromov density model with densities $1/3 < d < 1/2$, and yield important consequences. These groups are hyperbolic and have Kazhdan's property (T) with overwhelming probability [39, 20]. We give lower bounds on p for which fixed points exists for all isometric actions on any L_p space. A connection with the conformal dimension arises through the work of Bourdon and Pajot [5] and allows us to give a lower bound on the conformal dimension of a boundary of a random hyperbolic group, using an associated link graph (see Section 6). The problem of estimating the conformal dimension of random hyperbolic groups was posed by Gromov [16, 9.B (g)] and Pansu [31, IV.b].

Our methods also apply to affine actions whose linear part is a uniformly bounded representation on a Hilbert space. More precisely, we show that $H^1(G, \pi) = 0$ whenever π is a uniformly bounded representation with norms of all operators bounded by a constant which depends on the group but is close to $\sqrt{2}$ in many cases (see Theorems 5.5 and 6.3). The question of extending property (T) in the form of cohomological vanishing from unitary to uniformly bounded representations is a well-known open problem. In particular, Shalom conjectures that for every hyperbolic group there exists a uniformly bounded representation with a proper cocycle. The case of $\mathrm{Sp}(n, 1)$ is an unpublished result of Shalom.

Finally, we present other applications. We improve the differentiability class of diffeomorphic actions on the circle in the rigidity theorem of [25, 26] and estimate eigenvalues of the discrete p-Laplacian on finite quotients of groups using Kazhdan-type constants.

2. Actions on Banach spaces

2.1. Generating sets and link graphs

Let G denote a discrete group generated by a finite symmetric set $S = S^{-1}$ not containing the identity. Let $\mathcal{L}(S)$ denote the following graph, called the *link graph* of S. The vertices are given by $\mathcal{V} = S$. Two vertices $s, t \in S$ are connected by an edge, denoted $s \sim t$, if and only if $s^{-1}t \in S$ and $t^{-1}s \in S$.

We define

$$E = \{(s, t) \in S \times S : s^{-1}t \in S\}.$$

Note that E can be viewed as the set of oriented edges and in E every edge is counted twice.

A *weight* on $\mathcal{L}(S)$ is a function $\omega : E \to (0, \infty)$ such that $\omega(s, t) = \omega(t, s)$ for all $s, t \in S$. Given a weight on the link graph, the associated *degree* of a vertex $s \in S$ is defined to be

$$\deg_\omega(s) = \sum_{t : t \sim s} \omega(t, s).$$

A weight ω on a link graph $\mathcal{L}(S)$ is *admissible* if it satisfies

- $\deg_\omega(s) = \deg_\omega(s^{-1})$, and
- $\deg_\omega(r) = \sum_{(s,t):\, s^{-1}t=r} \omega(s,t)$,

for all $r, s, t \in S$. Note that

$$\omega(E) := \sum_{(s,t)\in E} \omega(s,t) = \sum_{s\in S} \deg_\omega(s).$$

Throughout the article we consider only admissible weights on link graphs of generating sets.

2.2. *Isometric representations and associated Banach spaces*

Let X be a Banach space with a norm $\|\cdot\|_X$. We assume throughout that X is reflexive and that $\pi : G \to B(X)$ is a representation of G into the bounded invertible operators on X. Let X^* denote the continuous dual of X, with its standard norm. X^* is naturally equipped with the adjoint representation of $G, \overline{\pi} : G \to B(X^*)$,

$$\overline{\pi}_g = \pi^*_{g^{-1}}.$$

Throughout we fix $1 < p < \infty$. The value of p will be chosen later depending on the context. We denote by p^* the adjoint index, satisfying $1/p + 1/p^* = 1$, and by L_p the space $L_p(\mu)$ for any measure μ (our results apply with no assumptions on the measure). We also use $\simeq$ to denote isomorphism and $\cong$ to denote isometric isomorphism of Banach spaces.

Define the Banach space $C^{(0,p)}(G,\pi)$ to be the linear space X with the norm

$$\|v\|_{(0,p)} = \omega(E)^{1/p}\|v\|_X.$$

Let $\langle\cdot,\cdot\rangle_X$ denote the natural pairing between X and X^*. The pairing between $C^{(0,p)}(G,\pi)$ and $C^{(0,p^*)}(G,\overline{\pi})$ is given by

$$\langle v, w\rangle_0 = \omega(E)\langle v, w\rangle_X.$$

Then $C^{(0,p^*)}(G,\overline{\pi})$ is the dual space of $C^{(0,p)}(G,\pi)$.

We define $C^{(1,p)}(G,\pi)$ to be the finite direct sum $\bigoplus_{s\in S} X$ with the norm given by

$$\|f\|_{(1,p)} = \Big(\sum_{s\in S} \|f(s)\|_X^p \deg_\omega(s)\Big)^{1/p}.$$

The dual of $C^{(1,p)}(G,\pi)$ is $C^{(1,p^*)}(G,\overline{\pi})$, via the pairing

$$\langle f, \phi\rangle_1 = \sum_{s\in S}\langle f(s), \phi(s)\rangle_X \deg_\omega(s)$$

for $f \in C^{(1,p)}(G,\pi)$ and $\phi \in C^{(1,p)}(G,\pi)^*$.

Define an operator Q_π on $C^{(1,p)}(G,\pi)$, by

$$Q_\pi f(s) = \pi_s f(s^{-1}).$$

A similar operator $Q_{\overline{\pi}}$ is defined on $C^{(1,p^*)}(G,\overline{\pi})$. The following is straightforward to verify.

Lemma 2.1. *The operator Q_π is an involution satisfying $Q_\pi^* = Q_{\overline{\pi}}$.*

Consider the following subspaces of $C^{(1,p)}(G,\pi)$, defined as eigenspaces of Q_π:

$$C_+^{(1,p)}(G,\pi) = \{f \in C^{(1,p)}(G,\pi) : f = Q_\pi f\},$$
$$C_-^{(1,p)}(G,\pi) = \{f \in C^{(1,p)}(G,\pi) : f = -Q_\pi f\}.$$

Lemma 2.2. *For any $1 < p < \infty$ we have $C^{(1,p)}(G,\pi) = C_+^{(1,p)}(G,\pi) \oplus C_-^{(1,p)}(G,\pi)$.*

Proof. We define two bounded operators: $P_\pi^{\pm} : C^{(1,p)}(G,\pi) \to C_{\pm}^{(1,p)}(G,\pi)$,

$$P_\pi^+ = \frac{I + Q_\pi}{2}, \qquad P_\pi^- = \frac{I - Q_\pi}{2}.$$

Clearly $P_\pi^+ + P_\pi^- = I$. Moreover, $C_+^{(1,p)}(G,\pi) = \ker P_\pi^- = \operatorname{im} P_\pi^+$ and $C_-^{(1,p)}(G,\pi) = \ker P_\pi^+ = \operatorname{im} P_\pi^-$. Indeed, we have

$$\pi_{s^{-1}}(P_\pi^+ f(s)) = \frac{\pi_{s^{-1}} f(s) + f(s^{-1})}{2} = P_\pi^+ f(s^{-1}).$$

Finally, P_π^+ restricted to $C_+^{(1,p)}(G,\pi)$ and P_π^- restricted to $C_-^{(1,p)}(G,\pi)$ are the identity operators, so that P_π^+ and P_π^- are projections onto the respective subspaces. □

We now analyze the structure of $C^{(1,p)}(G,\pi)$ in relation to the one of $C^{(1,p^*)}(G,\overline{\pi})$.

2.3. Duality for $C_-^{(1,p)}(G,\pi)$

The dual of $C^{(1,p)}(G,\pi)$ is $C^{(1,p^*)}(G,\overline{\pi})$. Let $P_{\overline{\pi}}^+ : C^{(1,p^*)}(G,\overline{\pi}) \to C_+^{(1,p^*)}(G,\overline{\pi})$ and $P_{\overline{\pi}}^- : C^{(1,p^*)}(G,\overline{\pi}) \to C_-^{(1,p^*)}(G,\overline{\pi})$ denote projections as above at the dual level.

Lemma 2.3. *We have $P_{\overline{\pi}}^+ = (P_\pi^+)^*$ and $P_{\overline{\pi}}^- = (P_\pi^-)^*$.*

Proof. Let $f \in C^{(1,p)}(G,\pi)$ and $\phi \in C^{(1,p^*)}(G,\overline{\pi})$. Then

$$(P_\pi^-)^* = \tfrac{1}{2}(I - Q_\pi)^* = \tfrac{1}{2}(I - Q_{\overline{\pi}}) = P_{\overline{\pi}}^-.$$

Similarly for P_π^+. □

Lemma 2.4. *We have the isomorphisms $C_-^{(1,p)}(G,\pi)^* \simeq C_-^{(1,p^*)}(G,\overline{\pi})$ and $C_+^{(1,p)}(G,\pi)^* \simeq C_+^{(1,p^*)}(G,\overline{\pi})$.*

Proof. Let $f \in C_-^{(1,p)}(G,\pi)$ and $\phi \in C^{(1,p^*)}(G,\overline{\pi})$. Then

$$\langle f, \phi \rangle_1 = \langle -Q_\pi f, \phi \rangle_1 = \langle f, -Q_{\overline{\pi}} \phi \rangle_1.$$

Therefore, $2\langle f, P_{\overline{\pi}}^+ \phi \rangle_1 = 0$, which shows that $C_+^{(1,p^*)}(G,\overline{\pi})$ annihilates $C_-^{(1,p)}(G,\pi)$.

Conversely, if $\phi \in C^{(1,p^*)}(G,\overline{\pi})$ annihilates $C_-^{(1,p)}(G,\pi)$, then

$$\langle P_\pi^- f, \phi\rangle_1 = \langle f, P_{\overline{\pi}}^- \phi\rangle = 0$$

for every $f \in C^{(1,p)}(G,\pi)$. Consequently, $P_{\overline{\pi}}^-\phi = 0$ and $\phi = P_{\overline{\pi}}^+\phi$, which means ϕ belongs to $C_+^{(1,p^*)}(G,\overline{\pi})$. Thus,

$$C_-^{(1,p)}(G,\pi)^* \cong C^{(1,p)}(G,\overline{\pi})/C_+^{(1,p^*)}(G,\overline{\pi}) \simeq C_-^{(1,p^*)}(G,\overline{\pi}).$$

Other cases are proved similarly. □

In order to identify the dual of $C_-^{(1,p)}(G,\pi)$, isomorphism is not sufficient: we need isometric isomorphism instead. For a representation π, $C_-^{(1,p)}(G,\pi)^*$ is not in general isometrically isomorphic to $C_-^{(1,p^*)}(G,\overline{\pi})$. However, it turns out that this does hold when the representation π is isometric.

Theorem 2.5. *Assume that π_s is an isometry for every $s \in S$. Then we have the isometric isomorphisms $C_-^{(1,p)}(G,\pi)^* \cong C_-^{(1,p^*)}(G,\overline{\pi})$ and $C_+^{(1,p)}(G,\pi)^* \cong C_+^{(1,p^*)}(G,\overline{\pi})$.*

Proof. Consider $C^{(1,p^*)}(G,\overline{\pi})/C_+^{(1,p^*)}(G,\overline{\pi})$, which consists of cosets $[\phi] = C_+^{(1,p^*)}(G,\overline{\pi}) + \phi$ for $\phi \in C^{(1,p^*)}(G,\overline{\pi})$. We need to show that for each such coset N, $\inf\{\|\phi\| : N = [\phi]\}$ is attained when $\phi \in C_-^{(1,p^*)}(G,\overline{\pi})$.

For $\phi \in C_-^{(1,p^*)}(G,\overline{\pi})$ and $\psi \in C_+^{(1,p^*)}(G,\overline{\pi})$, we have

$$\|\phi + \psi\|_{(1,p^*)} = \|-Q_{\overline{\pi}}\phi + Q_{\overline{\pi}}\psi\|_{(1,p^*)} = \|\phi - \psi\|_{(1,p^*)},$$

since the involution $Q_{\overline{\pi}}$ is an isometry whenever π, or equivalently $\overline{\pi}$, is an isometric representation. Now consider the coset $[\phi]$ for $\phi \in C_-^{(1,p^*)}(G,\overline{\pi})$ and consider another element, $\zeta \in C^{(1,p^*)}(G,\overline{\pi})$, such that $\zeta - \phi \in C_+^{(1,p^*)}(G,\overline{\pi})$, so that $\zeta = \phi + \psi$ for some $\psi \in C_+^{(1,p^*)}(G,\overline{\pi})$. This implies

$$\|\phi\|_{(1,p^*)} \leq \frac{\|\phi - \psi\|_{(1,p^*)} + \|\phi + \psi\|_{(1,p^*)}}{2} = \|\zeta\|_{(1,p^*)},$$

which proves the claim. □

This last statement allows us to identify $C_-^{(1,p)}(G,\pi)^*$ with $C_-^{(1,p^*)}(G,\overline{\pi})$ for isometric representations and is crucial in the proof of the main theorem.

2.4. The operator δ

We define an operator $\delta_\pi : C^{(0,p)}(G,\pi) \to C_-^{(1,p)}(G,\pi)$ by the formula

$$\delta_\pi v(s) = v - \pi_s v.$$

Theorem 2.5 allows us to express the adjoint of δ_π in a way which is convenient for calculations. We have the following explicit formula for δ_π^*.

Lemma 2.6. *The operator* $\delta^*_\pi : C^{(1,p^*)}_-(G,\overline{\pi}) \to C^{(0,p^*)}(G,\overline{\pi})$ *is given by*

$$\delta^*_\pi \phi = 2\sum_{s\in S}\phi(s)\frac{\deg_\omega(s)}{\omega(E)}. \tag{2.1}$$

Proof. We have

$$\begin{aligned}\langle \delta_\pi v, \phi\rangle_1 &= \sum_{s\in S}\langle v-\pi_s v, \phi(s)\rangle_X \deg_\omega(s) = \sum_{s\in S}\big(\langle v,\phi(s)\rangle_X - \langle v, \overline{\pi}_{s^{-1}}\phi(s)\rangle_X\big)\deg_\omega(s)\\ &= \sum_{s\in S}\big(\langle v,\phi(s)\rangle_X + \langle v,\phi(s^{-1})\rangle_X\big)\deg_\omega(s) = \left\langle v, 2\sum_{s\in S}\phi(s)\frac{\deg_\omega(s)}{\omega(E)}\right\rangle_0. \qquad \square\end{aligned}$$

It is now clear that δ^*_π admits a continuous extension to the space $C^{(1,p^*)}(G,\overline{\pi})$, defined by the right hand side of (2.1).

2.5. The operators D, L, and d

We define

$$C^{(2,p)}(G,\pi) = \Big\{\eta \in \bigoplus_{(s,t)\in E} X : \eta(s,t) = -\eta(t,s)\Big\}.$$

It is a Banach space when equipped with the norm

$$\|\eta\|_{(2,p)} = \Big(\sum_{(s,t)\in E}\|\eta(s,t)\|^p_X\,\omega(s,t)\Big)^{1/p}.$$

We also define operators $D, L_\pi, d_\pi : C^{(1,p)}_-(G,\pi) \to C^{(2,p)}(G,\pi)$ by the formulas

$$Df(s,t) = f(t) - f(s), \qquad L_\pi f(s,t) = \pi_s f(s^{-1}t), \qquad d_\pi = L_\pi - D.$$

Similarly, we define $\overline{D}$, $L_{\overline{\pi}}$, and $d_{\overline{\pi}}$ for the adjoint representation.

Lemma 2.7. *Let* π *be an isometric representation. The operator* L_π *is an isometry onto its image. Consequently, so is* D *when restricted to* $\ker d_\pi$. (*The same holds for* $L_{\overline{\pi}}$ *and* $\overline{D}$ *restricted to* $\ker d_{\overline{\pi}}$.)

Proof. By direct calculation,

$$\|L_\pi f\|^p_{(2,p)} = \sum_{(s,t)\in E}\|\pi_s f(s^{-1}t)\|^p_X\,\omega(s,t) = \sum_{s\in S}\|f(s)\|^p_X \deg_\omega(s) = \|f\|^p_{(1,p)}. \qquad \square$$

The kernel of $\overline{D}$ consists of the constant functions on S, which is a complemented subspace of $C^{(1,p)}(G,\pi)$. The projection onto this subspace is given by

$$\overline{M}\phi(s) = \sum_{s\in S}\phi(s)\frac{\deg_\omega(s)}{\omega(E)}.$$

Note that for $\phi \in C^{(1,p^*)}_-(G,\overline{\pi})$ we have

$$\overline{M}\phi(s) = \tfrac{1}{2}\delta^*_\pi\phi$$

for every $s \in S$.

Lemma 2.8. *Let* $\phi \in C_{-}^{(1,p^*)}(G, \overline{\pi})$. *Then* $\|\overline{M}\phi\|_{(1,p^*)} = \frac{1}{2}\|\delta_\pi^*\phi\|_{(0,p^*)}$.

Proof. We have

$$\|\overline{M}\phi\|_{(1,p^*)}^{p^*} = \sum_{s\in S}\left\|\frac{\delta_\pi^*\phi}{2}\right\|_X^{p^*}\deg_\omega(s) = \frac{1}{2^{p^*}}\|\delta_\pi^*\phi\|_X^{p^*}\Big(\sum_{s\in S}\deg_\omega(s)\Big) = \frac{1}{2^{p^*}}\|\delta_\pi^*\phi\|_0^{p^*}.$$

□

2.6. Sufficient conditions for vanishing of cohomology

Given a group G and a representation π, 1-*cocycles* associated to π are functions $b : G \to X$ satisfying the cocycle condition

$$b_{gh} = \pi_g b_h + b_g$$

for all $g, h \in G$. *Coboundaries* are those cocycles which are of the form

$$b_g = v - \pi_g v$$

for some $v \in X$ and all $g \in G$. The first cohomology of G with coefficients in π is defined to be $H^1(G, \pi) = $ cocycles/coboundaries.

An *affine* action of G on X is defined as

$$A_g v = \pi_g v + b_g,$$

where π is called the *linear part* of the action and b is a cocycle for π. The vanishing of $H^1(G, \pi)$ is equivalent to the existence of a fixed point for any affine action with linear part π. We refer to [4] for background on cohomology and affine actions.

The reader can easily verify the following lemma.

Lemma 2.9. $\operatorname{im}\delta_\pi \subseteq \ker d_\pi$.

This fact allows us to formulate the following sufficient condition for the fixed point property for affine actions on X.

Proposition 2.10. *If* $\operatorname{im}\delta_\pi = \ker d_\pi$, *then* $H^1(G, \pi) = 0$.

Proof. Let $b : G \to X$ be a 1-cocycle for π and let b' denote the restriction of b to the generating set S. The cocycle condition implies that $b' \in C_{-}^{(1,p)}(G, \pi)$, and furthermore that $b' \in \ker d_\pi$. If δ_π is onto $\ker d_\pi$, then $b' = \delta_\pi v$ for some $v \in X$. Since b is trivial on the generators, we conclude that b is trivial. □

It is important to remark that the technical details here are slightly different than in [39], where the original condition in terms of almost invariant vectors is deduced, and one needs the Delorme–Guichardet theorem to obtain cohomological vanishing. The above argument allows us to bypass the use of that theorem and obtain the vanishing of cohomology directly.

Note that the image of δ_π is always properly contained in $C_{-}^{(1,p)}(G, \pi)$. By the open mapping theorem we also have the following

Corollary 2.11. *Assume π does not have invariant vectors. If δ_π is onto* $\ker d_\pi$ *then there is a constant $K > 0$ such that*

$$\sup_{s\in S} \|v - \pi_s v\|_X \geq K\|v\|_X \quad \text{for every } v \in X.$$

The constant K in the above statement can be viewed as a version of the Kazhdan constant for isometric representations of G on X.

3. Poincaré inequalities associated to norms

Consider a weighted, finite graph $\Gamma = (\mathcal{V}, \mathcal{E})$, a number $p \geq 1$ and a Banach space X. The *p-Poincaré inequality* for Γ and for the norm of X is the inequality

$$\Big(\sum_{x\in\mathcal{V}} \|f(x) - Af\|_X^p \deg_\omega(x)\Big)^{1/p} \leq \kappa_p \Big(\sum_{x\sim y} \|f(x) - f(y)\|_X^p \omega(x, y)\Big)^{1/p}, \tag{3.1}$$

for all functions $f : \mathcal{V} \to X$, where $Af = \frac{1}{2\omega(\mathcal{E})} \sum_{x\in\mathcal{V}} f(x) \deg_\omega(x)$. On a finite graph, the inequality (3.1) is always satisfied for some $\kappa_p > 0$.

Definition 3.1. Let $\mathcal{L}(S)$ be the link graph of a generating set S, with weight ω. For a Banach space X and a number $1 < p < \infty$ we define the constant $\kappa_p(S, X)$ of $\mathcal{L}(S)$ by setting

$$\kappa_p(S, X) = \inf \kappa_p,$$

where the infimum is taken over all κ_p for which inequality (3.1) holds.

We will omit the reference to ω in the notation for κ.

Hilbert spaces. When $X = L_2$ is the Hilbert space, this constant is related to the smallest positive eigenvalue λ_1 of the Laplacian on the graph:

$$\kappa_2(S, L_2) = \sqrt{\lambda_1^{-1}},$$

since the latter can be defined via the variational expression and the Rayleigh quotient.

L_p spaces, $1 \leq p < \infty$. Let (Y, μ) be any measure space and for $X = \mathbb{R}$ consider a p-Poincaré inequality

$$\sum_{x\in\mathcal{V}} |f(x) - Af|^p \deg_\omega(x) \leq \kappa_p^p \sum_{x\sim y} |f(x) - f(y)|^p \omega(x, y) \tag{3.2}$$

on a finite graph. By integrating over Y with respect to μ we obtain

$$\sum_{x\in\mathcal{V}} \|f(x) - Af\|_{L_p}^p \deg_\omega(x) \leq \kappa_p^p \sum_{x\sim y} \|f(x) - f(y)\|_{L_p}^p \omega(x, y)$$

for any $f : \mathcal{V} \to L_p$. This gives

$$\|f - Af\|_{(1,p)} \leq \kappa_p \|\nabla f\|_{(2,p)},$$

so that $\kappa_p(S, L_p)$ is equal to $\kappa_p(S, \mathbb{R})$ in the inequality (3.2). Here $\nabla f(x, y) = f(y) - f(x)$ for an oriented edge (x, y) and an arbitrary orientation on Γ.

Direct sums. More generally, consider an ℓ_p-direct sum $X = (\bigoplus_{i\in I} X_i)_p$ of Banach spaces $\{X_i\}_{i\in I}$. A similar argument to that above shows that $\kappa_p(S, X) \le \sup_{i\in I} \kappa_p(S, X_i)$.

The case $p = \infty$. Consider $s, s^{-1} \in S$ and choose $x \in X$ such that $\|x\|_X = 1$. Let d_S denote the path metric on $\mathcal{L}(S)$. Define $f : \Gamma \to \mathbb{R}$ by the formula

$$f(t) = \begin{cases} \left(1 - 2\dfrac{d_S(s,t)}{d_S(s,s^{-1})}\right)x & \text{if } d_S(s,t) \le d_S(s,s^{-1})/2, \\ \left(-1 + 2\dfrac{d_S(s,t)}{d_S(s,s^{-1})}\right)x & \text{if } d_S(s^{-1},t) \le d_S(s,s^{-1})/2, \\ 0 & \text{if } d_S(s,t) > d_S(s,s^{-1})/2 \\ & \text{and } d_S(s^{-1},t) > d_S(s,s^{-1})/2. \end{cases}$$

For such f we have $\|f\|_{(1,\infty)} = 1$ and $Af = 0$, however $\|Df\|_{(2,\infty)} = 1/d_S(s, s^{-1})$. Thus we have

$$\kappa_\infty(G,\pi) \ge \max_{s\in S} d_S(s,s^{-1})$$

and for sufficiently large S, the above Poincaré constant is at least 1. Additionally, for any $\varepsilon > 0$ there exists a sufficiently large $p < \infty$ such that the norms $\|f\|_{(1,p)}$ and $\|f\|_{(1,\infty)}$ are ε-close. For a sufficiently small $\varepsilon > 0$ and the corresponding p as above, we also have $2^{-1/p}\kappa_p(S, X) \ge 1$.

Behavior under isomorphisms. Let $T : X \to Y$ be an isomorphism of Banach spaces X, Y, satisfying $\|x\|_X \le \|Tx\|_y \le L\|x\|_X$ for every $x \in X$. Then

$$\kappa_p(G,\pi) \le L\kappa_p(G,Y). \tag{3.3}$$

4. Vanishing of cohomology

4.1. An inequality for κ_p and δ^*_π

Note that since in E each edge of $\mathcal{L}(S)$ is counted twice, we have $\|Df\|_{(2,p)} = 2^{1/p}\|\nabla f\|_{\ell_p(S,X)}$ and $Mf = Af$. The following result describes the relation between Poincaré constants and the operator δ^*_π.

Theorem 4.1. *For every $\phi \in \ker d_{\overline{\pi}}$,*

$$2(1 - 2^{-1/p^*}\kappa_{p^*}(S,X^*))\|\phi\|_{(1,p^*)} \le \|\delta^*_\pi\phi\|_{(0,p^*)}. \tag{4.1}$$

Proof. Let $\phi : S \to X^*$. Then

$$\begin{aligned}\kappa_{p^*}(S,X^*)\,\|\overline{D}\phi\|_{(2,p^*)} &= \kappa_{p^*}(S,X^*)\,2^{1/p^*}\|\overline{\nabla}\phi\|_{\ell_{p^*}(\mathcal{E},X^*)} \ge 2^{1/p^*}\|\phi - \overline{M}\phi\|_{(1,p^*)} \\ &\ge 2^{1/p^*}(\|\phi\|_{(1,p^*)} - \|\overline{M}\phi\|_{(1,p^*)}).\end{aligned}$$

Since $\overline{D}$ is an isometry on $\ker d_{\overline{\pi}}$,

$$2^{1/p^*}\|\phi\|_{(1,p^*)} - \kappa_{p^*}(S,X^*)\|\phi\|_{(1,p^*)} \le 2^{1/p^*}\|\overline{M}\phi\|_{(1,p^*)},$$

which, by Lemma 2.8, becomes

$$(1 - 2^{-1/p^*}\kappa_{p^*}(S,X^*))\|\phi\|_{(1,p^*)} \le \tfrac{1}{2}\|\delta^*_\pi\phi\|_{(0,p^*)}. \qquad \square$$

Remark 4.2. The above inequality does not reduce to the one in [39] in the case $X = L_2$ and $p = 2$, even though in both cases the constant is non-zero if $\kappa_2(S, \mathbb{R}) < \sqrt{2}$. For $X = L_2$ and $p = 2$, Theorem 4.1 gives a strictly smaller lower estimate for the norm of the operator δ_π^*. Indeed, in that case the estimate obtained using spectral methods is

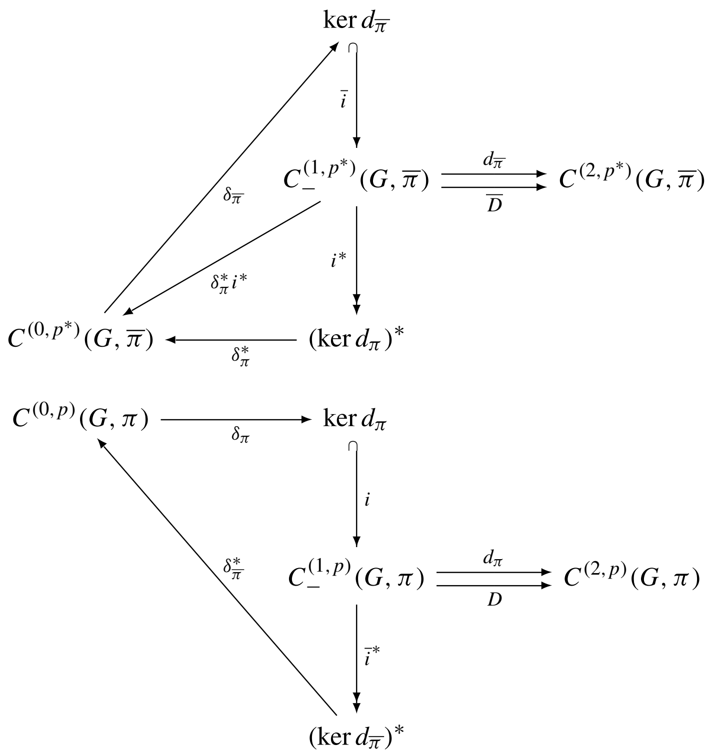
$$\sqrt{2(2 - \kappa_2(S, L_2)^2)}\, \|\phi\|_{(1,2)} \leq \|\delta_\pi^* \phi\|_{(2,2)}.$$

This difference is a consequence of the fact that for $p = 2$ and $X = L_2$ we can apply the Pythagorean theorem instead of the triangle inequality in the first sequence of inequalities in the above proof.

A similar inequality to the one in Theorem 4.1 holds for $\kappa_p(S, X)$ and the norm of $\delta_{\overline{\pi}}^*$. The above inequality can now be used to show that sufficiently small constants in Poincaré inequalities on the link graph imply fixed point properties.

4.2. Proof of the main theorem

To prove Theorem 1.1, we consider the following dual diagrams:



For the purposes of this proof we will view δ_π, abusing the notation, as an operator $C^{(0,p)}(G,\pi) \to \ker d_\pi$. The natural injection of $\ker d_\pi$ into $C_-^{(1,p)}(G,\pi)$ will be denoted by i. Consequently, (2.1) expresses the composition $\delta_\pi^* \circ i^*$, as in the diagram.

Similar notation is used at the dual level, with $\bar{i}$ denoting the inclusion of $\ker d_{\overline{\pi}}$ into $C^{(1,p^*)}(G,\overline{\pi})$.

By Theorem 4.1, if $2^{-1/p}\kappa_p(S,X) < 1$ we conclude that $\delta^*_{\overline{\pi}} \circ \bar{i}^*$ is injective with closed image when restricted to $\ker d_\pi$, that is, $\delta^*_{\overline{\pi}} \circ \bar{i}^* \circ i$ is injective with closed image. In particular, $\bar{i}^* \circ i$ is injective with closed image, and thus its dual, $i^* \circ \bar{i} : \ker d_{\overline{\pi}} \to (\ker d_\pi)^*$, is surjective.

A similar argument applied to $2^{-1/p^*}\kappa_{p^*}(S,X^*) < 1$ shows that $\delta^*_\pi \circ i^* \circ \bar{i}$ is also injective with closed image, which implies that δ^*_π is injective with closed image on the image of $i^* \circ \bar{i}$. Since the latter is surjective, δ^*_π is injective with closed image on $(\ker d_\pi)^*$. This on the other hand implies that δ_π is onto, which proves the theorem by Proposition 2.10. □

Remark 4.3. Note that under the assumptions of Theorem 1.1, $(\ker d_\pi)^*$ and $\ker d_{\overline{\pi}}$ are isomorphic (a similar fact holds for $\ker d_\pi$ and $(\ker d_{\overline{\pi}})^*$).

It is an interesting question for which isometric representations $i^* \circ \bar{i}$ is automatically an isomorphism, or at least is surjective. This property would eliminate, for such representations, the need to use the inequality $2^{-1/p}\kappa_p < 1$, which is necessary in the above proof. A special case is discussed and applied in Section 7.2.

Remark 4.4. Note that it is not clear whether the above method can be extended to subspaces $Y \subseteq X$. This would require estimating $\kappa_p(S,Y)$ for some p, together with $\kappa_{p^*}(S,Y^*)$, where Y^* is a quotient of X^*.

5. $\widetilde{A}_2$-groups

In this section we apply Theorem 1.1 to specific groups and Banach spaces. In [7] the authors studied a family $\{G_q\}$ of groups called $\widetilde{A}_2$-groups. These groups were introduced in [6]; see also [4] for a detailed discussion. The group G_q has a presentation whose associated link graph $\mathcal{L}(S)$ is the incidence graph of the finite projective plane $\mathbf{P}^2(\mathbb{F}_q)$ (here, q is a power of a prime number). Spectra of such graphs, with weight $\omega \equiv 1$, were computed by Feit and Higman [11] (see also [4, 39]). It follows that

$$\kappa_2(S,\mathbb{R}) = \left(1 - \frac{\sqrt{q}}{q+1}\right)^{-1/2}.$$

In general, any estimates of p-Poincaré constants are difficult to obtain. In our case, the link graphs are finite graphs and we can use norm inequalities and a version of (3.3) to give the necessary estimates.

Theorem 5.1. *For each $q = k^n$ for some $n \in \mathbb{N}$ and prime k we have $H^1(G_q,\pi) = 0$ for all*

$$2 \le p < \frac{\ln(q^2+q+1) + \ln(q+1)}{\frac{1}{2}\ln(2(q^2+q+1)(q+1)) - \ln(2) - \ln\left(\sqrt{1-\frac{\sqrt{q}}{q+1}}\right)}$$

and for any isometric representation π of G_q on $L_p(Y,\mu)$ for any measure space.

Proof. We proceed by estimating $\kappa_p(S, L_p)$ and applying Theorem 1.1. Recall that for $p \geq 2$ and Ω finite, the following norm inequalities hold:

$$\|f\|_{\ell_p(\Omega)} \leq \|f\|_{\ell_2(\Omega)} \leq (\#\Omega)^{1/2-1/p}\|f\|_{\ell_p(\Omega)},$$

where $\|f\|_{\ell_p(\Omega)} = (\sum_{x\in\Omega}|f(x)|^p)^{1/p}$. Since the degree of the incidence graphs of finite projective planes is constant and equal to $q+1$, for $f : S \to \mathbb{R}$ satisfying $Mf = 0$ we obtain

$$\begin{aligned}\|f\|_{(1,p)} &= (q+1)^{1/p}\|f\|_{\ell_p(S,X)} \leq (q+1)^{1/p-1/2}\|f\|_{(1,2)}\\ &\leq (q+1)^{1/p-1/2}\kappa_2(S, L_2)\|\nabla f\|_{\ell_2(\mathcal{E},X)}\\ &\leq (q+1)^{1/p-1/2}\kappa_2(S, L_2)(\omega(E)/2)^{1/2-1/p}\|\nabla f\|_{\ell_p(\mathcal{E},X)}.\end{aligned}$$

For each q we have $\omega(E) = 2(q^2+q+1)(q+1)$, which gives the inequality

$$\begin{aligned}2^{-1/p}\kappa_2(S, L_p) &\leq 2^{-1/p}(q+1)^{1/p-1/2}\kappa_2(S, L_2)(\omega(E)/2)^{1/2-1/p}\\ &= 2^{-1/p}\left(\sqrt{1-\frac{\sqrt{q}}{q+1}}\right)^{-1}(q^2+q+1)^{1/2-1/p}.\end{aligned}$$

Bounding the above quantity by 1 from above gives

$$p < \frac{2\ln(2(q^2+q+1))}{\ln(2(q^2+q+1)) - \ln\big(2\big(1-\frac{\sqrt{q}}{q+1}\big)\big)}.$$

A similar norm estimate for $p^* \leq 2$, by virtue of the inequality

$$\|f\|_{\ell_2(\Omega)} \leq \|f\|_{\ell_{p^*}(\Omega)} \leq (\#\Omega)^{1/p-1/2}\|f\|_{\ell_2(\Omega)},$$

yields

$$p^* > \frac{2\ln((q+1)(q^2+q+1))}{\ln(2(q^2+q+1)(q+1)) + \ln\big(2\big(1-\frac{\sqrt{q}}{q+1}\big)\big)}.$$

Simplifying and comparing p and $p^*/(p^*-1)$ we obtain the claim. ◻

Remark 5.2. The same argument gives a similar conclusion for the Banach space $X = (\bigoplus X_i)_p$, the ℓ_p-sum in which X_i is finite-dimensional with a norm sufficiently, and uniformly in i, close to the Euclidean norm. We leave the details to the reader.

Remark 5.3. The largest value of p in Theorem 5.1 is approximately 2.106, attained for $q = 13$. As q increases to infinity, the values of p for which cohomology vanishes converge to 2 from above.

Remark 5.4. Although our estimate of the constant in the p-Poincaré inequality is not expected to be optimal, other interpolation methods do not seem to yield significantly better constants. For instance, Matoušek's interpolation method for p-Poincaré inequalities [22] gives a constant strictly greater than $2^{1/p}$ for any $p \geq 2$, since it emphasizes independence from dimension and is much better suited to dealing with sequences of graphs (e.g. expanders).

Recall that the Banach–Mazur distance $d_{BM}(x, y)$ between two Banach spaces is the infimum of the set of numbers L for which there exists an isomorphism $T : X \to Y$ satisfying $\|x\|_X \le \|T_x\|_Y \le L\|x\|_X$. Another consequence of Theorem 1.1 is that we obtain vanishing of cohomology for representations on Banach spaces whose Banach–Mazur distance from the Hilbert space is controlled. We phrase this property in terms of uniformly bounded representations.

Theorem 5.5. *Let G_q be an $\widetilde{A}_2$-group and π be a uniformly bounded representation of G_q on a Hilbert space H satisfying*

$$\sup_{g\in G} \|\pi_g\| < \sqrt{2\left(1 - \frac{\sqrt{q}}{q+1}\right)}.$$

Then $H^1(G, \pi) = 0$.

Proof. Let $\|v\|' = \sup_{g\in G} \|\pi_g v\|$. Then $\|\cdot\|'$ is a norm and π is an isometric representation on $X = (H, \|\cdot\|')$. The identity is an isomorphism $\mathrm{id} : X \to H$ with $L = \sqrt{2\big(1 - \frac{\sqrt{q}}{q+1}\big)}$, and $L\,\mathrm{id} : X^* \to H$ is an isomorphism with the same constant. The estimate now follows by letting $p = 2$ and using the relation between $\kappa_2(S, X)$, $\kappa_2(S, X^*)$ and $\kappa_2(S, H)$, described in (3.3). □

A similar fact (with appropriate constants) holds for L_p spaces, for the range of p as in the previous theorem.

6. Hyperbolic groups

In this section we discuss the consequences of Theorem 1.1 in the case of random hyperbolic groups. In [39] Żuk used spectral methods to show that many random groups have property (T) with overwhelming probability. A detailed account was recently provided in [20]. We sketch the strategy of the proof and generalize it to L_p spaces.

In [13] it was shown that for a certain random graph on n vertices of degree deg there exists a constant such that for any $\varepsilon > 1$ we have

$$\lim_{n\to\infty} \mathbb{P}\left(\kappa_2(S, \mathbb{R}) \le \left(1 - \left(\frac{\sqrt{2\deg}(\deg - 1)^{1/4}}{\deg} + \frac{\varepsilon}{\deg}\right)\right)\right) = 1. \tag{6.1}$$

In [39] a modified link graph, denoted $L'(S)$, with multiple edges was considered. $L'(S)$ decomposes into random graphs as above and it is shown, using the above estimate, that it has a spectral gap strictly larger than $1/2$ with probability 1. In our setting, the modified link graph $L'(S)$ can be viewed as a link graph with an admissible weight $\omega(s, t)$ which is defined to be the number of edges connecting s and t. Thus we can apply Theorem 1.1. Recall that in the Gromov model $\mathcal{G}(n, l, d)$ for random groups one chooses a density $0 < d < 1$ and considers a group given by a generating set S of cardinality n and $(2n - 1)^{ld}$ relations of length l, chosen at random, letting l increase to infinity.

Theorem 6.1 ([39]; see also [20] for a detailed proof). *Let G be a random group in the density model, where $1/3 < d < 1/2$. Then, with probability 1, G is hyperbolic and there exists a group Γ and a homomorphism $\phi : \Gamma \to G$ with the following properties:*

- *Γ has a generating set S whose link graph satisfies $2^{-1/2}\kappa_2(S, L_2) < 1$,*
- *$\phi(\Gamma)$ is of finite index in G.*

Given the above, we apply to the link graph of Γ similar norm inequalities to those in the case of $\widetilde{A}_2$-groups and, as before, obtain fixed point properties for affine isometric actions of the group Γ on L_p for certain $p > 2$. For any given $p > 1$, the property of having $H^1(G, \pi) = 0$ for *all* isometric representations π on L_p spaces passes to quotients and from finite index subgroups to the ambient group. We thus have

Theorem 6.2. *With the assumptions of the previous theorem, with probability 1, Theorem 1.1 applies to hyperbolic groups. More precisely, let G, Γ and ϕ be as above, and let $\mathcal{L}(S) = (\mathcal{V}, \mathcal{E})$ denote the link graph of Γ. Then $H^1(G, \pi) = 0$ for every isometric representation π of G on L_p for*

$$p < \min\{p_0, \overline{p}_0^*\},$$

where

$$p_0 = \frac{\ln \deg_\omega - \ln(2\#\mathcal{E})}{\frac{1}{2}\ln\left(\frac{\deg_\omega}{\#\mathcal{E}}\right) - \ln \kappa_2(S, \mathbb{R})} \quad \text{and} \quad \overline{p}_0 = \frac{\ln(\#\mathcal{V}\deg_\omega) - \ln 2}{\frac{1}{2}\ln(\#\mathcal{V}\deg_\omega) - \ln \kappa_2(S, \mathbb{R})}.$$

We also have an estimate for the norms of uniformly bounded representations to which cohomological vanishing can be extended for random hyperbolic groups.

Theorem 6.3. *Let G be a hyperbolic group in the Gromov model as above with $1/3 < d < 1/2$ and π be a uniformly bounded representation of G on a Hilbert space H satisfying*

$$\sup_{g\in G} \|\pi_g\| < \frac{\sqrt{2}}{\kappa_2(S, \mathbb{R})}.$$

Then $H^1(G, \pi) = 0$. In other words, $H^1(G, \pi)$ vanishes with probability 1 for representations bounded by $\sqrt{2}$.

We remark that in (6.1), $\kappa_2(S, \mathbb{R})$ tends to 1 as $\deg \to \infty$. Thus the above upper bound on the norm of the representation is $\sqrt{2}$ with probability 1. On the other hand, Shalom showed that $\mathrm{Sp}(n, 1)$ has non-trivial cohomology with respect to some uniformly bounded representations (unpublished). The same property for hyperbolic groups is conjectured by Shalom.

We also note that M. Cowling proposed to define a numerical invariant of a hyperbolic group by setting $\inf\{\sup_{g\in G} \|\pi_g\| \colon H^1(G, \pi) \neq 0\}$. Theorem 6.3 gives a uniform lower bound of $\sqrt{2}$ on such an invariant, with probability 1, for hyperbolic groups in the Gromov model.

Theorem 6.2 brings another interesting connection. Pansu [31] defined a quasi-isometry invariant of a hyperbolic group, called the *conformal dimension*, to be the number

$$\operatorname{confdim}(\partial G) = \inf\{\dim_{\mathrm{Haus}}(\partial G, d) : d \text{ is quasi-conformally equivalent to } d_v\},$$

where $\dim_{\mathrm{Haus}}$ denotes the Hausdorff dimension, ∂G denotes the Gromov boundary of the hyperbolic group G, and d_v denotes any visual metric on ∂G. We refer to [18] for a brief overview of conformal dimension of boundaries of hyperbolic groups. Bourdon and Pajot [5] showed that a hyperbolic group acts by affine isometries without fixed points on $L_p(G)$ for p greater than the conformal dimension of ∂G. Combining this with vanishing of cohomology as studied here we see that if $H^1(G,\pi)$ vanishes for all isometric representations on L_p then

$$p \leq \operatorname{confdim}(\partial G).$$

Gromov [16, 9.B (g)] and Pansu [31, IV.b] posed the question of estimating the conformal dimension of random hyperbolic groups. Using Theorem 1.1 we obtain such estimates.

Corollary 6.4. *With the assumptions and notation of Theorem 6.2,*

$$\operatorname{confdim}(\partial G) \geq \min\{p_0, \overline{p}_0^*\}.$$

Finally, as mentioned in the introduction, the above facts show that the method of Poincaré inequalities cannot in general give the vanishing of cohomology, as studied in this paper, for all $2 < p < \infty$. In addition, we have the following quantitative statement about Poincaré constants.

Corollary 6.5. *For any hyperbolic group G and any generating set S not containing the identity element, the Poincaré constants on the link graph associated to S satisfy*

$$\kappa_p(S, L_p) \geq 2^{1/p} \quad \text{or} \quad \kappa_{p^*}(S, L_{p^*}) \geq 2^{1/p^*},$$

for $p > \operatorname{confdim}(\partial G)$.

7. Other applications

7.1. Actions on the circle

Fixed point properties for the spaces L_p, $p > 2$, can be applied to studying actions on the circle, by applying the vanishing of cohomology to the L_p-Liouville cocycle. In [26] the following theorem was proved.

Theorem 7.1. *Let G be a discrete group such that $H^1(G,\pi) = 0$ for every isometric representation of G on L_p for some $p > 2$. Then for every $\alpha > 1/p$ every homomorphism $h : G \to \operatorname{Diff}_+^{1+\alpha}(S^1)$ has finite image.*

Combining this result with, for instance, Theorem 5.1 we obtain

Corollary 7.2. *Let q be a power of a prime number and G_q be the corresponding $\widetilde{A}_2$-group. Then every homomorphism $h : G_q \to \operatorname{Diff}_+^{1+\alpha}(S^1)$ has finite image for*

$$\alpha > \frac{\frac{1}{2}\ln(2(q^2+q+1)(q+1)) - \ln(2) - \ln\Big(\sqrt{1-\frac{\sqrt{q}}{q+1}}\Big)}{\ln(q^2+q+1)+\ln(q+1)}.$$

7.2. The finite-dimensional case and the p-Laplacian

Let $1 < p < \infty$. The p-Laplacian Δ_p is an operator $\Delta_p : \ell_p(V) \to \ell_p(V)$ defined by

$$\Delta_p f(x) = \sum_{x \sim y} (f(x) - f(y))^{[p]} \omega(x, y)$$

for $f : V \to \mathbb{R}$, where $a^{[p]} = |a|^{p-1} \operatorname{sign}(a)$. The p-Laplacian reduces to the standard discrete Laplacian for $p = 2$, and is non-linear when $p \neq 2$. The p-Laplacian is of great importance in the study of partial differential equations. Its discrete version was studied e.g. in [1, 34].

A real number λ is an *eigenvalue* of the p-Laplacian Δ_p if there exists a function $f : V \to \mathbb{R}$ satisfying

$$\Delta_p f = \lambda f^{[p]}.$$

The eigenvalues of the p-Laplacian are difficult to compute in the case $p \neq 2$, due to non-linearity of Δ_p; see [15] for explicit estimates. Define

$$\lambda_1^{(p)}(\Gamma) = \inf \left\{ \frac{\sum_{x \in V} \sum_{y \sim x} |f(x) - f(y)|^p \omega(x, y)}{\inf_{\alpha \in \mathbb{R}} \sum_{x \in V} |f(x) - \alpha|^p \deg_\omega(x)} \right\}, \tag{7.1}$$

with the infimum taken over all non-constant $f : V \to \mathbb{R}$. Then $\lambda_1^{(p)}$ is the smallest positive eigenvalue of the discrete p-Laplacian Δ_p or the *p-spectral gap*.

We now apply an estimate similar to the one in Corollary 2.11 to finite quotients of groups. Let G be a finitely generated group and consider a homomorphism $h : G \to H$, where H is a finite group. Let $p > 1$ and let $\ell_p^0(H)$ denote the subspace of $\ell_p(H)$ consisting of those functions which sum to 0.

We can identify the dual $\ell_p^0(H)^*$ with the space $\ell_{p^*}^0(H)$, with the norm

$$\|f\| = \inf_{\alpha \in \mathbb{R}} \|f - \alpha\|_{p^*}.$$

We will use our results to estimate the p^*-spectral gap for this norm on the Cayley graph of H.

Let $X = \ell_p^0(H)^*$ be equipped with the adjoint of the left regular representation λ on $\ell_p(H)$, restricted to $X^* = \ell_p^0(H)$. We have

$$\kappa_p(S, X^*) \leq \kappa_p(S, \ell_p(H)) = \kappa_p(S, \mathbb{R}).$$

Computing the Poincaré constant of the link graph for the norm of X is not straightforward. However, following the strategy outlined in Remark 4.3, we will show that we can bypass this condition. In order to do this we need to show that $i^* \circ \bar{i}$ is onto. In fact, a stronger statement is true.

Lemma 7.3. *Under the above assumptions, the map* $i^* \circ \bar{i} : \ker d_\pi \to \ker d_{\bar{\pi}}$ *is an isomorphism.*

Proof. We can view X and X^* as having the same underlying vector space (real-valued functions $f : H \to X$ with mean value 0), equipped with two different norms. Similarly, $C_-^{(1,p^*)}(G,\pi)$ and $C_-^{(1,p)}(G,\overline{\pi})$ also have the same underlying vector space, equipped with two different norms. The adjoint $\overline{\lambda}$ of the left regular representation λ coincides with λ. Hence $\ker d_\pi$ and $\ker d_{\overline{\pi}}$ are the same subspace. The claim follows from the fact that all the spaces involved are finite-dimensional and complemented. □

Now, since the representation of G on X does not have invariant vectors and δ_π is onto $\ker d_\pi$, we can conclude, by the Open Mapping Theorem, that δ_π in fact induces an isomorphism between $C^{(0,p^*)}(G,\pi)$ and $\ker d_\pi$. It follows from Theorem 4.1 that

$$2(1-2^{-1/p}\kappa_p(S,\mathbb{R}))\|f\|_{(0,p^*)} \le \|\delta_{\overline{\pi}} f\|_{(1,p^*)}.$$

Since $f \in \ell_p^0(H)$, this gives

$$(2(1-2^{-1/p}\kappa_p(S,\mathbb{R})))^{p^*}\|f\|_X^{p^*} \le \sum_{s\in S}\|f-\lambda_s f\|_X^{p^*}\frac{\deg_\omega(s)}{\omega(E)}.$$

Since $\|v\|_X \le \|v\|_{\ell_p(H)}$, this yields

$$\begin{aligned}(2(1-2^{-1/p}\kappa_p(S,\mathbb{R})))^{p^*} &\inf_{c\in\mathbb{R}}\sum_{h\in H}|f(h)-c|^{p^*}\deg_\omega(h)\\ &\le \sum_{h\in H}\sum_{g\sim h}|f(h)-f(g)|^{p^*}\frac{\deg_\omega(g^{-1}h)}{\omega(E)}.\end{aligned}$$

(Note that $\deg_\omega(g^{-1}h)$ refers to $\mathcal{L}(S)$, not to the Cayley graph of H.)

Corollary 7.4. *Let G be a group generated by a finite symmetric set S not containing the identity element. If the link graph $\mathcal{L}(S)$ is connected and for some $1<p<\infty$ the Poincaré constant satisfies*

$$2^{-1/p}\kappa_p(S,\mathbb{R}) < 1,$$

then

$$\lambda_1^{(p)} \ge 2(1-2^{-1/p}\kappa_p(S,\mathbb{R}))$$

on the Cayley graph of any finite quotient of G, for any weight $\omega(g,h) \ge \deg_\omega(g^{-1}h)/\omega(E)$.

Remark 7.5. A similar claim to Lemma 7.3 holds for any orthogonal representation which is also isometric on $\ell_p(H)$.

Acknowledgments. I am grateful to Florent Baudier, Pierre-Emmanuel Caprace, Valerio Capraro, Tadeusz Januszkiewicz and Juhani Koivisto for helpful comments and to Masato Mimura and Andrzej Żuk for enjoyable conversations on related topics. I am also greatly indebted to Antoine Gournay and Alain Valette, who suggested several improvements and simplifications, in particular the use of the idempotent Q_π.

The author was partially supported by NSF grant DMS-0900874 and by the Foundation for Polish Science.

References

[1] Amghibech, S.: Eigenvalues of the discrete p-Laplacian for graphs. Ars Combin. **67**, 283–302 (2003) Zbl 1080.31005 MR 1973245

[2] Bader, U., Furman, A., Gelander, T., Monod, N.: Property (T) and rigidity for actions on Banach spaces. Acta Math. **198**, 57–105 (2007) Zbl 1162.22005 MR 2316269

[3] Ballmann, W., Świątkowski, J.: On L^2-cohomology and property (T) for automorphism groups of polyhedral cell complexes. Geom. Funct. Anal. **7**, 615–645 (1997) Zbl 0897.22007 MR 1465598

[4] Bekka, B., de la Harpe, P., Valette, A.: Kazhdan's Property (T). New Math. Monogr. 11, Cambridge Univ. Press, Cambridge (2008) Zbl 1146.22009 MR 2415834

[5] Bourdon, M., Pajot, H.: Cohomologie l_p et espaces de Besov. J. Reine Angew. Math. **558**, 85–108 (2003) Zbl 1044.20026 MR 1979183

[6] Cartwright, D. I., Mantero, A. M., Steger, T., Zappa, A.: Groups acting simply transitively on the vertices of a building of type $\tilde{A}_2$. I. Geom. Dedicata **47**, 143–166 (1993) Zbl 0784.51010 MR 1232965

[7] Cartwright, D. I., Młotkowski, W., Steger, T.: Property (T) and $\tilde{A}_2$ groups. Ann. Inst. Fourier (Grenoble) **44**, 213–248 (1994) Zbl 0792.43002 MR 1262886

[8] Chatterji, I., Druţu, C., Haglund, F.: Kazhdan and Haagerup properties from the median viewpoint. Adv. Math. **225**, 882–921 (2010) Zbl 1271.20053 MR 2671183

[9] Dymara, J., Januszkiewicz, T.: Cohomology of buildings and their automorphism groups. Invent. Math. **150**, 579–627 (2002) Zbl 1140.20308 MR 1946553

[10] Ershov, M., Jaikin-Zapirain, A., Property (T) for noncommutative universal lattices. Invent. Math. **179**, 303–347 (2010) Zbl 1205.22003 MR 2570119

[11] Feit, W., Higman, G.: The nonexistence of certain generalized polygons. J. Algebra **1**, 114–131 (1964) Zbl 0126.05303 MR 0170955

[12] Fisher, D., Margulis, G.: Almost isometric actions, property (T), and local rigidity. Invent. Math. **162**, 19–80 (2005) Zbl 1076.22008 MR 2198325

[13] Friedman, J.: Relative expanders or weakly relatively Ramanujan graphs. Duke Math. J. **118**, 19–35 (2003) Zbl 1035.05058 MR 1978881

[14] Garland, H.: p-adic curvature and the cohomology of discrete subgroups of p-adic groups. Ann. of Math. (2) **97**, 375–423 (1973) Zbl 0262.22010 MR 0320180

[15] Grigorchuk, R. I., Nowak, P. W.: Diameters, distortion, and eigenvalues. Eur. J. Combin. **33**, 1574–1587 (2012) Zbl 1246.05046 MR 2923470

[16] Gromov, M.: Asymptotic invariants of infinite groups. In: Geometric Group Theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser. 182, Cambridge Univ. Press, Cambridge, 1–295 (1993) Zbl 0841.20039 MR 1253544

[17] Izeki, H., Nayatani, S.: Combinatorial harmonic maps and discrete-group actions on Hadamard spaces. Geom. Dedicata **114**, 147–188 (2005) Zbl 1108.58014 MR 2174098

[18] Kapovich, I., Benakli, N.: Boundaries of hyperbolic groups. In: Combinatorial and Geometric Group Theory (New York, 2000/Hoboken, NJ, 2001), Contemp. Math. 296, Amer. Math. Soc., Providence, RI, 39–93 (2002) Zbl 1044.20028

[19] Kassabov, M.: Subspace arrangements and property T. Groups Geom. Dynam. **5**, 445–477 (2011) Zbl 1244.20041 MR 2782180

[20] Kotowski, Marcin, Kotowski, Michał: Random groups and property (T): Żuk's theorem revisited. J. London Math. Soc. (2) **88**, 396–416 (2013) Zbl 06216290 MR 3106728

[21] Lafforgue, V.: Un renforcement de la propriété (T). Duke Math. J. **143**, 559–602 (2008) Zbl 1158.46049 MR 2423763

[22] Matoušek, J.: On embedding expanders into l_p spaces. Israel J. Math. **102**, 189–197 (1997) Zbl 0947.46007 MR 1489105

[23] Mimura, M.: Fixed point properties and second bounded cohomology of universal lattices on Banach spaces. J. Reine Angew. Math. **653**, 115–134 (2011) Zbl 1221.20032 MR 2794627

[24] Naor, A., Silberman, L.: Poincaré inequalities, embeddings, and wild groups. Compos. Math. **147**, 1546–1572 (2011) Zbl 1267.20057 MR 2834732

[25] Navas, A.: Actions de groupes de Kazhdan sur le cercle. Ann. Sci. École Norm. Sup. (4) **35**, 749–758 (2002) Zbl 1028.58010 MR 1951442

[26] Navas, A.: Reduction of cocycles and groups of diffeomorphisms of the circle. Bull. Belg. Math. Soc. Simon Stevin **13**, 193–205 (2006) Zbl 1131.37043 MR 2259900

[27] Nica, B.: Proper isometric actions of hyperbolic groups on L^p-spaces. Compos. Math. 149 (2013), 773–792. Zbl 1286.20055 MR 3069362

[28] Nowak, P. W.: Group actions on Banach spaces and a geometric characterization of a-T-menability. Topology Appl. **153**, 3409–3412 (2006) Zbl 1146.43002 MR 2270593

[29] Nowak, P. W.: Group actions on Banach spaces and a geometric characterization of a-T-menability (corrected version) arXiv:math/0404402v3 (2009)

[30] Nowak, P. W.: Group actions on Banach spaces. In: Handbook of Group Actions, L. Ji, A. Papadopoulos and S.-T. Yau (eds.), to appear

[31] Pansu, P.: Dimension conforme et sphère à l'infini des variétés à courbure négative. Ann. Acad. Sci. Fenn. Ser. A I Math. **14**, 177–212 (1989) Zbl 0722.53028 MR 1024425

[32] Pansu, P.: Cohomologie L_p: invariance sous quasiisométrie. Preprint (1995)

[33] Pansu, P.: Formules de Matsushima, de Garland et propriété (T) pour des groupes agissant sur des espaces symétriques ou des immeubles. Bull. Soc. Math. France **126**, 107–139 (1998) Zbl 0933.22009 MR 1651383

[34] Takeuchi, H.: The spectrum of the p-Laplacian and p-harmonic morphisms on graphs. Illinois J. Math. **47**, 939–955 (2003) Zbl 1122.31003 MR 2007245

[35] Wang, M.-T.: A fixed point theorem of discrete group actions on Riemannian manifolds. J. Differential Geom. **50**, 249–267 (1998) Zbl 0951.58020 MR 1684980

[36] Wang, M.-T.: Generalized harmonic maps and representations of discrete groups. Comm. Anal. Geom. **8**, 545–563 (2000) Zbl 0977.58018 MR 1775138

[37] Yu, G.: Hyperbolic groups admit proper affine isometric actions on l^p-spaces. Geom. Funct. Anal. **15**, 1144–1151 (2005) Zbl 1112.46054 MR 2221161

[38] Żuk, A.: La propriété (T) de Kazhdan pour les groupes agissant sur les polyèdres. C. R. Acad. Sci. Paris Sér. I Math. **323**, 453–458 (1996) Zbl 0858.22007 MR 1408975

[39] Żuk, A.: Property (T) and Kazhdan constants for discrete groups. Geom. Funct. Anal. **13**, 643–670 (2003) Zbl 1036.22004 MR 1995802