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Witten's Conjecture for many four-manifolds of simple type

Abstract. We prove that Witten's Conjecture [40] on the relationship between the Donaldson and Seiberg–Witten series for a four-manifold of Seiberg–Witten simple type with $b_1 = 0$ and odd $b_2^+ \geq 3$ follows from our SO(3)-monopole cobordism formula [6] when the four-manifold has $c_1^2 \geq \chi_h - 3$ or is abundant.

1. Introduction

1.1. Main results

Throughout this article, we shall assume that X is a *standard* four-manifold by which we mean that X is closed, connected, oriented, and smooth with $b_1(X) = 0$ and odd $b^+(X) \geq 3$. For such manifolds, we define (by analogy with their values when X is a complex surface),

$$c_1^2(X) := 2\chi + 3\sigma \quad \text{and} \quad \chi_h(X) := \tfrac{1}{4}(\chi + \sigma), \tag{1.1}$$

where χ and σ are the Euler characteristic and signature of X.

For standard four-manifolds, the Seiberg–Witten (SW) invariants [29], [34], [40] comprise a function with finite support, $SW_X : \mathrm{Spin}^c(X) \to \mathbb{Z}$, where $\mathrm{Spin}^c(X)$ is the set of isomorphism classes of spin^c structures on X. The set of Seiberg–Witten (SW) basic classes, $B(X)$, is the image under a map $c_1 : \mathrm{Spin}^c(X) \to H^2(X; \mathbb{Z})$ of the support of SW_X [40]. A standard four-manifold X has *Seiberg–Witten simple type* if $c_1^2(\mathfrak{s}) = c_1^2(X)$ for all $c_1(\mathfrak{s}) \in B(X)$ and is *abundant* if $B(X)^\perp \subset H^2(X; \mathbb{Z})$ contains a hyperbolic summand, where $B(X)^\perp$ denotes the orthogonal complement of $B(X)$ with respect to the intersection form Q_X on $H^2(X; \mathbb{Z})$. We extend Q_X from $H_2(X; \mathbb{Z})$ to $H_2(X; \mathbb{R})$ by linearity.

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We refer to [23], or §2.2 in this article, for the definitions of the Donaldson series, $\mathbf{D}_X^w(h)$, Kronheimer–Mrowka (KM) basic classes, and four-manifolds of Kronheimer–Mrowka (KM) simple type.

Conjecture 1.1 (Witten's Conjecture [40]). *Let X be a standard four-manifold with Seiberg–Witten simple type. The four-manifold X then has Kronheimer–Mrowka simple type, and the Kronheimer–Mrowka and Seiberg–Witten basic classes coincide. For any $w \in H^2(X; \mathbb{Z})$ and $h \in H_2(X; \mathbb{R})$, one has*

$$\mathbf{D}_X^w(h) = 2^{2-(\chi_h-c_1^2)} e^{Q_X(h)/2} \sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s}) e^{\langle c_1(\mathfrak{s}), h\rangle}. \quad (1.2)$$

E. Witten derived formula (1.2) using arguments from quantum field theory which, as far as the authors can tell, have no direct, mathematically rigorous justification. Consequently, the challenge ever since the publication of [40] has been to provide a mathematically rigorous proof of (1.2).

In [6], we proved that a formula (restated in this article in Theorem 3.2) relating Donaldson and Seiberg–Witten invariants followed from certain properties, described in Remark 3.3, of the gluing map for SO(3) monopoles constructed in [5]. A proof of the required SO(3)-monopole gluing-map properties is currently being developed by the authors. The formula in Theorem 3.2 involves polynomials with unknown coefficients depending on topological data and thus lacks the elegance and simplicity of the formula in Conjecture 1.1; moreover, it appears extremely difficult, if not impossible, to compute these coefficients directly by the method of proof of Theorem 3.2. However, in this article, we use a family of manifolds constructed by R. Fintushel, J. Park, and R. J. Stern [14] to determine sufficiently many of these coefficients to prove

Main Theorem 1.2. *Let X be a standard four-manifold with Seiberg–Witten simple type which is abundant or has $c_1^2(X) \geq \chi_h(X) - 3$. Then the SO(3)-monopole cobordism formula (Theorem 3.2) implies that Conjecture 1.1 holds for X.*

The quantum field theory argument giving Witten's formula (1.2) for standard four-manifolds has been extended by G. Moore and E. Witten [28] to allow $b^+(X) \geq 1$, and $b_1(X) \geq 0$, and four-manifolds X of non-simple type. The SO(3)-monopole cobordism gives a relation between the Donaldson and Seiberg–Witten invariants for these manifolds as well and so should also lead to a proof of Moore and Witten's more general conjecture. However, the methods of this article do not extend to the more general case because of the lack of examples of four-manifolds not of simple type.

A proof of Witten's Conjecture, also assuming Theorem 3.2, for a more restricted class of manifolds has appeared previously in [24, Corollary 7]. Conjecture 1.1 is known to hold, by direct calculation of both sides of equation (1.2), for elliptic surfaces by work of R. Fintushel and R. J. Stern [17]. Conjecture 1.1 also holds for all simply-connected, minimal surfaces of general type. Indeed, Theorem 1.2 implies that Witten's Conjecture holds for all abundant four-manifolds, and this includes both elliptic surfaces and surfaces of general type by [10, Corollary A.3]; by the discussion in [10, §A.2], this includes all

simply-connected, closed, complex surfaces with $b^+ \geq 3$. In Remark 4.9, we explain why the arguments used in §4 of our proof of Theorem 1.2 do not appear, by themselves, sufficient to allow us to remove the restriction that X be abundant or have $c_1^2(X) \geq \chi_h(X) - 3$.

For a complex projective surface X, Mochizuki [27] proved a formula (see [22, Theorem 4.1]) expressing the Donaldson invariants in a form similar to that given by the SO(3)-monopole cobordism formula (our Theorem 3.2), but the coefficients are given as the residues of a generating function for integrals of $\mathbb{C}^*$-equivariant cohomology classes over the product of Hilbert schemes of points on X. In [22, p. 309], L. Göttsche, H. Nakajima, and K. Yoshioka suggest that the coefficients in Mochizuki's formula (which remain valid for a standard four-manifold) and in our SO(3)-monopole cobordism formula are the same. They prove an explicit formula for complex projective surfaces relating Donaldson invariants and Seiberg–Witten invariants of four-manifolds of simple type using Nekrasov's deformed partition function for the $N = 2$ SUSY gauge theory with a single fundamental matter and from this formula deduce Witten's Conjecture. In [22, p. 323], they discuss the relationship between their approach, Mochizuki's formula, and our SO(3)-monopole cobordism formula. See also [21, pp. 344–347] for a related discussion concerning their wall-crossing formula for the Donaldson invariants of a four-manifold with $b^+ = 1$.

1.2. Outline of the article

In [6], we proved that any Donaldson invariant of a four-manifold X can be expressed as a polynomial p_X in the intersection form of X, namely Q_X, the Seiberg–Witten basic classes of X and an additional cohomology class $\Lambda \in H^2(X; \mathbb{Z})$ which does not appear in equation (1.2). If X has SW-simple type, then the coefficients of p_X depend only on the degree of the Donaldson invariant, Λ^2, $\chi_h(X)$, $c_1^2(X)$, and $c_1(\mathfrak{s}) \cdot \Lambda$ for an SW-basic class $c_1(\mathfrak{s})$. We prove Theorem 1.2 by using examples of manifolds known to satisfy Conjecture 1.1 to determine sufficiently many of these coefficients.

In §2, we review the definitions of the Donaldson series, the Seiberg–Witten invariants, and results on the surgical operations of blowing up and blowing down which preserve equation (1.2). In §3, we summarize the background material from [6] required to state our SO(3)-monopole cobordism formula (Theorem 3.2). We give the proof of Theorem 1.2 in §4.

2. Preliminaries

We begin by reviewing the relevant properties of the Donaldson and Seiberg–Witten invariants.

2.1. Seiberg–Witten invariants

As stated in the introduction, the *Seiberg–Witten invariants* defined in [40] (see also [29, 33, 34]) define a map with finite support,

$$SW_X : \mathrm{Spin}^c(X) \to \mathbb{Z},$$

where $\mathrm{Spin}^c(X)$ denotes the set of spin^c structures on X. For a spin^c structure $\mathfrak{s} = (W^\pm, \rho)$, where $W^\pm \to X$ are complex rank-two Hermitian vector bundles and ρ is a Clifford multiplication map, define $c_1 : \mathrm{Spin}^c(X) \to H^2(X; \mathbb{Z})$ by $c_1(\mathfrak{s}) = c_1(W^+)$. For all $\mathfrak{s} \in \mathrm{Spin}^c(X)$, the cohomology class $c_1(\mathfrak{s})$ is characteristic.

The invariant $SW_X(\mathfrak{s})$ is defined by the homology class of $M_{\mathfrak{s}}$, the moduli space of Seiberg–Witten monopoles. One calls $c_1(\mathfrak{s})$ a *Seiberg–Witten (SW) basic class* if $SW_X(\mathfrak{s}) \neq 0$. Define

$$B(X) = \{c_1(\mathfrak{s}) : SW_X(\mathfrak{s}) \neq 0\}. \tag{2.1}$$

If $H^2(X; \mathbb{Z})$ has 2-torsion, then $c_1 : \mathrm{Spin}^c(X) \to H^2(X; \mathbb{Z})$ is not injective; moreover, the formulas in this article often involve (real) homology and cohomology, so we define

$$SW'_X : H^2(X; \mathbb{Z}) \to \mathbb{Z}, \quad K \mapsto \sum_{\mathfrak{s} \in c_1^{-1}(K)} SW_X(\mathfrak{s}), \tag{2.2}$$

and set $SW_X(K) = 0$ if K is not characteristic. With this definition, Witten's formula (1.2) is equivalent to

$$\mathbf{D}_X^w(h) = 2^{2-(\chi_h-c_1^2)} e^{Q_X(h)/2} \sum_{K \in B(X)} (-1)^{\frac{1}{2}(w^2+K\cdot w)} SW'_X(K) e^{\langle K, h \rangle}. \tag{2.3}$$

One says that a four-manifold, X, has *Seiberg–Witten (SW) simple type* if $SW_X(\mathfrak{s}) \neq 0$ implies that $c_1^2(\mathfrak{s}) = c_1^2(X)$.

As discussed in [29, §6.8], there is an involution on $\mathrm{Spin}^c(X)$, $\mathfrak{s} \mapsto \bar{\mathfrak{s}}$, with $c_1(\bar{\mathfrak{s}}) = -c_1(\mathfrak{s})$, defined essentially by taking the complex conjugate bundles. By [29, Corollary 6.8.4], one has $SW_X(\bar{\mathfrak{s}}) = (-1)^{\chi_h(X)} SW_X(\mathfrak{s})$ and so $B(X)$ is closed under the action of $\{\pm 1\}$ on $H^2(X; \mathbb{Z})$.

Let $\widetilde{X} = X \# \overline{\mathbb{CP}}^2$ be the blow-up of X. For every $n \in \mathbb{Z}$, there is a unique $\mathfrak{s}_n \in \mathrm{Spin}^c(\overline{\mathbb{CP}}^2)$ with $c_1(\mathfrak{s}_n) = (2n+1)e^*$, where $e^* \in H^2(\widetilde{X}; \mathbb{Z})$ is the Poincaré dual of the exceptional curve. By [33, §4.6.2], there is a bijection,

$$\mathrm{Spin}^c(X) \times \mathbb{Z} \to \mathrm{Spin}^c(\widetilde{X}), \quad (\mathfrak{s}_X, n) \mapsto \mathfrak{s}_X \# \mathfrak{s}_n,$$

given by a connected-sum construction with $c_1(\mathfrak{s}_X \# \mathfrak{s}_n) = c_1(\mathfrak{s}_X) + (2n+1)e^*$. Versions of the following result have appeared in [16], [33, Theorem 4.6.7], and [19, Theorem 14.1.1].

Theorem 2.1 (Blow-up formula for Seiberg–Witten invariants [19, Theorem 14.1.1]). *Let X be a standard four-manifold and let $\widetilde{X} = X \# \overline{\mathbb{CP}}^2$ be its blow-up. Then $\widetilde{X}$ has SW-simple type if and only if that is true for X. If X has simple type, then*

$$B(\widetilde{X}) = \{K \pm e^* : K \in B(X)\}, \tag{2.4}$$

and if $K \in B(X)$, then $SW'_{\widetilde{X}}(K \pm e^*) = SW'_X(K)$.

2.2. Donaldson invariants

2.2.1. Definitions and the structure theorem. We now recall the definition [23, §2] of the Donaldson series for standard four-manifolds. For any choice of $w \in H^2(X; \mathbb{Z})$, the *Donaldson invariant* is a linear function

$$D_X^w : \mathbb{A}(X) \to \mathbb{R},$$

where $\mathbb{A}(X)$ is the symmetric algebra,

$$\mathbb{A}(X) = \mathrm{Sym}(H_{\mathrm{even}}(X; \mathbb{R})).$$

For $h \in H_2(X; \mathbb{R})$ and a generator $x \in H_0(X; \mathbb{Z})$, we define $D_X^w(h^{\delta-2m}x^m) = 0$ unless

$$\delta \equiv -w^2 - 3\chi_h(X) \ (\mathrm{mod}\ 4). \tag{2.5}$$

If (2.5) holds, then $D_X^w(h^{\delta-2m}x^m)$ is defined by pairing cohomology classes corresponding to elements of $\mathbb{A}(X)$ with the Uhlenbeck compactification of a moduli space of anti-self-dual SO(3) connections [1], [2], [18], [23].

A four-manifold has *Kronheimer–Mrowka (KM) simple type* if for all $w \in H^2(X; \mathbb{Z})$ and all $z \in \mathbb{A}(X)$ one has

$$D_X^w(x^2 z) = 4 D_X^w(z). \tag{2.6}$$

The *Donaldson series* is a formal power series,

$$\mathbf{D}_X^w(h) = D_X^w\big((1 + \tfrac{1}{2}x)e^h\big), \quad h \in H_2(X; \mathbb{R}), \tag{2.7}$$

which determines all Donaldson invariants for standard manifolds of KM-simple type. The Donaldson series of a manifold with KM-simple type has the following description (see also [15, Theorems 5.9 and 5.13] for a proof by a different method):

Theorem 2.2 (Structure of Donaldson invariants [23, Theorem 1.7(a)]). *Let X be a standard four-manifold with KM-simple type. Suppose that some Donaldson invariant of X is non-zero. Then there is a function,*

$$\beta_X : H^2(X; \mathbb{Z}) \to \mathbb{Q}, \tag{2.8}$$

such that $\beta_X(K) \neq 0$ for at least one and at most finitely many classes, K, which are integral lifts of $w_2(X) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$ (the KM-basic classes*), and for any $w \in H^2(X; \mathbb{Z})$, one has the following equality of analytic functions of $h \in H_2(X; \mathbb{R})$:*

$$\mathbf{D}_X^w(h) = e^{Q_X(h)/2} \sum_{K \in H^2(X;\mathbb{Z})} (-1)^{(w^2 + K \cdot w)/2} \beta_X(K) e^{\langle K, h \rangle}. \tag{2.9}$$

The following lemma reduces the proof of Conjecture 1.1 to proving that equation (1.2) holds.

Lemma 2.3. *Assume the hypotheses of Theorem 2.2. If equation (1.2) holds for X, then the KM-basic classes and SW-basic classes coincide.*

Proof. The result follows by comparing equation (2.3) (which is equivalent to (1.2)) and (2.9) and by exploiting the linear independence of the functions $e^{r_i t}$ for different values of r_i. $\qquad \square$

2.2.2. Independence from w. We now discuss the role of w. Proofs that the condition (2.6) is independent of w appear, in varying degrees of generality, in [20], [23], [32], [39]:

Theorem 2.4 ([23], [32, Theorem 2]). *Let X be a standard four-manifold. If equation (2.6) holds for one $w \in H^2(X; \mathbb{Z})$, then it holds for all w.*

The following proposition allows us to work with a specific w:

Proposition 2.5. *Let X be a standard four-manifold of SW-simple type. If Witten's Conjecture 1.1 holds for one $w \in H^2(X; \mathbb{Z})$, then it holds for all $w \in H^2(X; \mathbb{Z})$.*

Proof. Assume that Conjecture 1.1 and hence (2.3) holds for some $w_0 \in H^2(X; \mathbb{Z})$,

$$
e^{\frac{1}{2}Q_X(h)} \sum_{K \in B(X)} (-1)^{\frac{1}{2}(w_0^2 + K \cdot w_0)} \beta_X(K) e^{\langle K, h \rangle}
$$
$$
= 2^{2-(\chi_h - c_1^2)} e^{\frac{1}{2}Q_X(h)} \sum_{K \in B(X)} (-1)^{\frac{1}{2}(w_0^2 + K \cdot w_0)} SW_X'(K) e^{\langle K, h \rangle}. \quad (2.10)
$$

We shall denote the SW-basic classes by K_i, for $1 \leq i \leq s$, so $B(X) = \{K_1, \ldots, K_s\}$. Because Q_X is indefinite, the following subset of $H_2(X; \mathbb{R})$ is non-empty:

$$
U = Q_X^{-1}(0) \setminus \left(\bigcup_{i < j} (K_i - K_j)^{-1}(0) \right) \subset H_2(X; \mathbb{R}).
$$

If $r_i = \langle K_i, h_0 \rangle$ for some fixed $h_0 \in U$, then $r_i \neq r_j$ for $i \neq j$. Replacing h by th_0 where $t \in \mathbb{R}$ in (2.10) gives

$$
\sum_{i=1}^{s} (-1)^{\frac{1}{2}(w_0^2 + K_i \cdot w_0)} \beta_X(K_i) e^{r_i t} = 2^{2-(\chi_h - c_1^2)} \sum_{i=1}^{s} (-1)^{\frac{1}{2}(w_0^2 + K_i \cdot w_0)} SW_X'(K_i) e^{r_i t}.
$$

The preceding identity and linear independence of the functions $e^{r_1 t}, \ldots, e^{r_s t}$ imply that

$$
\beta_X(K) = 2^{2-(\chi_h - c_1^2)} SW_X'(K). \quad (2.11)
$$

Let w be any other element of $H^2(X; \mathbb{Z})$. Since X has KM-simple type for w_0 (by our hypothesis that Conjecture 1.1 holds for some w_0), Theorem 2.4 implies that X has KM-simple type for w. The conclusion now follows from (2.9) and (2.11).   □

2.2.3. Behavior under blow-ups. We note that the KM-simple type condition (2.6) is invariant under blow-ups.

Proposition 2.6. *A standard four-manifold X has KM-simple type if and only if its blow-up $\widetilde{X}$ has KM-simple type.*

Proof. Assume $\widetilde{X}$ has KM-simple type. The blow-up formula $D_X^w(z) = D_{\widetilde{X}}^w(z)$ provided by [18, Theorem III.8.4] implies that, for any $z \in \mathbb{A}(X)$,

$$D_X^w(x^2 z) = D_{\widetilde{X}}^w(x^2 z) = 4 D_{\widetilde{X}}^w(z) = D_X^w(z),$$

and thus X has KM-simple type. The converse implication follows from [23, Proposition 1.9]. $\square$

We also note the behavior of Witten's formula (1.2) under blow-up.

Theorem 2.7 ([17, Theorem 8.9]). *Let X be a standard four-manifold. Then Witten's formula* (1.2) *holds for X if and only if it holds for the blow-up $\widetilde{X}$.*

2.2.4. Donaldson invariants determined by Witten's formula. Theorem 2.2 gives the following values for Donaldson invariants of four-manifolds satisfying Conjecture 1.1. For a standard four-manifold, X, we define

$$c(X) := \chi_h(X) - c_1^2(X), \tag{2.12}$$

where $\chi_h(X)$ and $c_1^2(X)$ are given in (1.1).

Lemma 2.8. *Let X be a standard four-manifold. Then Witten's formula* (1.2) *holds, and X has KM-simple type if and only if the Donaldson invariants of X satisfy $D_X^w(h^{\delta-2m} x^m)$ $= 0$ when δ does not obey* (2.5)*, and when δ obeys* (2.5)*, then*

$$
\begin{aligned}
&D_X^w(h^{\delta-2m} x^m) \\
&\quad = \sum_{i+2k=\delta-2m} \sum_{K \in B(X)} (-1)^{\varepsilon(w,K)} \frac{SW_X'(K)(\delta-2m)!}{2^{k+c(X)-2-m} k! i!} \langle K, h \rangle^i Q_X^k(h),
\end{aligned}
\tag{2.13}
$$

where $\varepsilon(w, K) := \frac{1}{2}(w^2 + w \cdot K)$.

Proof. Assume that Witten's formula (1.2), and hence (2.3), holds and that X has KM-simple type. By definition, the Donaldson invariant $D_X^w(h^{\delta-2m} x^m)$ will vanish unless δ obeys (2.5). Then equation (2.3) holds for X if and only if

$$
\begin{aligned}
2^{c(X)-2} \sum_{d=0}^{\infty} &\left(\frac{1}{d!} D_X^w(h^d) + \frac{1}{d!} D_X^w(h^d x) \right) \\
&= \left(\sum_{k=0}^{\infty} \frac{1}{2^k k!} Q_X^k(h) \right) \left(\sum_{i=0}^{\infty} \frac{1}{i!} \sum_{K \in B(X)} (-1)^{\varepsilon(w,K)} SW_X'(K) \langle K, h \rangle^i \right) \\
&= \sum_{d=0}^{\infty} \sum_{i+2k=d} \sum_{K \in B(X)} (-1)^{\varepsilon(w,K)} \frac{SW_X'(K)}{2^k k! i!} \langle K, h \rangle^i Q_X^k(h).
\end{aligned}
$$

The parity restriction (2.5) implies that, for $d \not\equiv -w^2 - 3\chi_h \pmod 2$, one has

$$D_X^w(h^d) + \tfrac{1}{2} D_X^w(h^d x) = 0,$$

while for $d \equiv -w^2 - 3\chi_h \pmod{2}$, equation (2.3) holds for X if and only if

$$2^{c(X)-2}\big(D_X^w(h^d) + \tfrac{1}{2}D_X^w(h^d x)\big)$$
$$= \sum_{i+2k=d} \sum_{K \in B(X)} (-1)^{\varepsilon(w,K)} \frac{SW_X'(K)d!}{2^k k! i!} \langle K, h \rangle^i Q_X^k(h).$$

We can now read off the value of $D_X^w(h^{\delta-2m}x^m)$ from the preceding equation as follows. If $\delta \equiv -w^2 - 3\chi_h \pmod{4}$ and m is even, then $\delta - 2m \equiv -w^2 - 3\chi_h \pmod{4}$, so, by the KM-simple type condition (2.6) and the vanishing condition (2.5) (which implies that the term $D_X^w(h^{\delta-2m}x)$ below is zero),

$$D_X^w(h^{\delta-2m}x^m) = 2^m\big(D_X^w(h^{\delta-2m}) + \tfrac{1}{2}D_X^w(h^{\delta-2m}x)\big)$$
$$= \sum_{i+2k=\delta-2m} \sum_{K \in B(X)} (-1)^{\varepsilon(w,K)} \frac{SW_X'(K)(\delta - 2m)!}{2^{k+c(X)-2-m}k! i!} \langle K, h \rangle^i Q_X^k(h).$$

Similarly, if $\delta \equiv -w^2 - 3\chi_h \pmod{4}$ and m is odd, then $\delta - 2m + 2 \equiv -w^2 - 3\chi_h \pmod{4}$, so, by the KM-simple type condition and the vanishing condition (2.5),

$$D_X^w(h^{\delta-2m}x^m) = 2^{m-1}D_X^w(h^{\delta-2m}x) = 2^m\big(D_X^w(h^{\delta-2m}) + \tfrac{1}{2}D_X^w(h^{\delta-2m}x)\big)$$
$$= \sum_{i+2k=\delta-2m} \sum_{K \in B(X)} (-1)^{\varepsilon(w,K)} \frac{SW_X'(K)(\delta - 2m)!}{2^{k+c(X)-2-m}k! i!} \langle K, h \rangle^i Q_X^k(h),$$

as required.

Conversely, if the Donaldson invariants satisfy equation (2.13) then the KM-simple type condition (2.6) follows immediately. The fact that Witten's formula (1.2) holds for X follows by reversing the preceding arguments. □

3. The SO(3)-monopole cobordism formula

In this section, we review the SO(3)-monopole cobordism formula. More detailed expositions appear in [6, 8, 10, 11, 12].

Recall that we denote spin^c structures on X by $\mathfrak{s} = (W^{\pm}, \rho)$, so $W = W^+ \oplus W^- \to X$ is a rank-four, complex Hermitian vector bundle and ρ is a Clifford multiplication map. We call $\mathfrak{t} = (W \otimes E, \rho \otimes \mathrm{id}_E)$ a spin^u *structure* if (W, ρ) is a spin^c structure and $E \to X$ is a rank-two complex Hermitian vector bundle. A spin^u structure, $\mathfrak{t}$, defines an associated bundle, $\mathfrak{g}_\mathfrak{t} = \mathfrak{su}(E)$, and characteristic classes

$$c_1(\mathfrak{t}) = c_1(W^+) + c_1(E) \quad \text{and} \quad p_1(\mathfrak{t}) = p_1(\mathfrak{g}_\mathfrak{t}).$$

We denote

$$\Lambda := c_1(\mathfrak{t}), \quad \kappa := -\tfrac{1}{4}\langle p_1(\mathfrak{t}), [X] \rangle, \quad \text{and} \quad w = c_1(E). \tag{3.1}$$

We let $\mathcal{M}_{\mathfrak{t}}$ denote the moduli space of SO(3) monopoles for the spin^u structure $\mathfrak{t}$, as defined in [10, equation (2.33)]. We use the class w to provide an orientation for $\mathcal{M}_{\mathfrak{t}}$. The moduli space $\mathcal{M}_{\mathfrak{t}}$ admits an S^1 action with fixed point subspaces given by M_κ^w, the moduli space of anti-self-dual connections on the bundle $\mathfrak{g}_{\mathfrak{t}}$, and by Seiberg–Witten moduli spaces, $M_{\mathfrak{s}}$, where $E = L_1 \oplus L_2$ and $\mathfrak{s} = W \otimes L_1$. For a spin^c structure, $\mathfrak{s}$, with $M_{\mathfrak{s}} \subset \mathcal{M}_{\mathfrak{t}}$, we have $(c_1(\mathfrak{s}) - \Lambda)^2 = p_1(\mathfrak{t})$.

The dimension of M_κ^w is given by 2δ, where

$$\delta = -p_1(\mathfrak{t}) - 3\chi_h.$$

The dimension of $\mathcal{M}_{\mathfrak{t}}$ is $2\delta + 2n_a(\mathfrak{t})$, where $n_a(\mathfrak{t})$ is the complex index of a Dirac operator defined by $\mathfrak{t}$, and $n_a(\mathfrak{t}) = (I(\Lambda) - \delta)/4$ with

$$I(\Lambda) = \Lambda^2 - \tfrac{1}{4}(3\chi(X) + 7\sigma(X)) = \Lambda^2 + 5\chi_h(X) - c_1^2(X). \tag{3.2}$$

Thus, M_κ^w has positive codimension in $\mathcal{M}_{\mathfrak{t}}$ if and only if $I(\Lambda) > \delta$. Note also that because $n_a(\mathfrak{t})$ is an integer, $I(\Lambda) \equiv \delta \pmod 4$, so, recalling that $c(X) = \chi_h(X) - c_1^2(X)$,

$$\Lambda^2 + c(X) \equiv \delta \pmod 4, \tag{3.3}$$

where we used the fact that $I(\Lambda) = \Lambda^2 + c(X) + 4\chi_h(X)$ from (3.2).

The moduli space $\mathcal{M}_{\mathfrak{t}}$ is not compact but admits a type of Uhlenbeck compactification,

$$\bar{\mathcal{M}}_{\mathfrak{t}} \subset \bigcup_{\ell=0}^{N} \mathcal{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^\ell(X),$$

where $\mathfrak{t}(\ell)$ is the spin^u structure satisfying $c_1(\mathfrak{t}(\ell)) = c_1(\mathfrak{t})$ and $p_1(\mathfrak{t}(\ell)) = p_1(\mathfrak{t}) + 4\ell$ [9, Theorem 4.20]. The S^1 action extends continuously over $\bar{\mathcal{M}}_{\mathfrak{t}}$. The closure of M_κ^w in $\bar{\mathcal{M}}_{\mathfrak{t}}$ is the usual Uhlenbeck compactification, $\bar{M}_\kappa^w$, of M_κ^w [2]. There are additional fixed points of the S^1 action in $\bar{\mathcal{M}}_{\mathfrak{t}}$ of the form $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$. If $\bar{\mathbf{L}}_{\mathfrak{t},\kappa}^w$ and $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ are the links of $\bar{M}_\kappa^w$ and $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$, respectively, in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$, then $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ defines a compact, orientable cobordism between $\bar{\mathbf{L}}_{\mathfrak{t},\kappa}^w$ and the union, over $\mathfrak{s} \in \mathrm{Spin}^c(X)$, of the links $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$. If $I(\Lambda) > \delta$, then pairing certain cohomology classes with the link $\bar{\mathbf{L}}_{\mathfrak{t},\kappa}^w$ gives a multiple of the Donaldson invariant (see [11, Proposition 3.29]). As these cohomology classes are defined on the complement of the fixed point set in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$, the cobordism gives an equality between this multiple of the Donaldson invariant and the pairing of these cohomology classes with the union, over $\mathfrak{s} \in \mathrm{Spin}^c(X)$, of the links $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$. In [6], we computed an expression for this pairing, giving a cobordism formula.

Hypothesis 3.1 (Properties of local SO(3)-monopole gluing maps). The local gluing map, constructed in [5], gives a continuous parametrization of a neighborhood of $M_{\mathfrak{s}} \times \Sigma$ in $\bar{\mathcal{M}}_{\mathfrak{t}}$ for each smooth stratum $\Sigma \subset \mathrm{Sym}^\ell(X)$.

Hypothesis 3.1 is recorded in greater detail in [6]. The question of how to assemble the *local* gluing maps for neighborhoods of $M_{\mathfrak{s}} \times \Sigma$ in $\bar{\mathcal{M}}_{\mathfrak{t}}$, as Σ ranges over all smooth strata of $\mathrm{Sym}^\ell(X)$, into a *global* gluing map for a neighborhood of $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t}}$ is itself difficult—involving the so-called 'overlap problem' described in [12]—but one which we do solve in [6]. See Remark 3.3 for a further discussion of this point.

Theorem 3.2 (SO(3)-monopole cobordism formula [6]). *Let X be a standard four-man-ifold of Seiberg–Witten simple type. Assume that Hypothesis* 3.1 *holds. Assume further that* $w, \Lambda \in H^2(X; \mathbb{Z})$ *and* $\delta, m \in \mathbb{N}$ *satisfy:*

1. $w - \Lambda \equiv w_2(X) \pmod{2}$.
2. $I(\Lambda) > \delta$, *where* $I(\Lambda)$ *is defined in* (3.2).
3. $\delta \equiv -w^2 - 3\chi_h \pmod{4}$.
4. $\delta - 2m \geq 0$.

Then, for any $h \in H_2(X; \mathbb{R})$ *and generator* $x \in H_0(X; \mathbb{Z})$, *we have*

$$
D_X^w(h^{\delta-2m}x^m)
$$
$$
= \sum_{K \in B(X)} (-1)^{\frac{1}{2}(w^2-\sigma)+\frac{1}{2}(w^2+(w-\Lambda)\cdot K)} SW'_X(K) f_{\delta,m}(\chi_h, c_1^2, K, \Lambda)(h), \quad (3.4)
$$

where the map

$$
f_{\delta,m}(h) : \mathbb{Z} \times \mathbb{Z} \times H^2(X; \mathbb{Z}) \times H^2(X; \mathbb{Z}) \to \mathbb{R}[h],
$$

taking values in the ring of polynomials in the variable h with real coefficients, is univer-sal (*independent of X*) *and given by*

$$
f_{\delta,m}(\chi_h, c_1^2, K, \Lambda)(h)
$$
$$
:= \sum_{i+j+2k=\delta-2m} a_{i,j,k}(\chi_h, c_1^2, K \cdot \Lambda, \Lambda^2, m)\langle K, h\rangle^i \langle \Lambda, h\rangle^j Q_X^k(h), \quad (3.5)
$$

and, for each triple of non-negative integers $i, j, k \in \mathbb{N}$, *the coefficients*

$$
a_{i,j,k} : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} \to \mathbb{R}
$$

are real analytic (*independent of X*) *in the variables* χ_h, c_1^2, $c_1(\mathfrak{s}) \cdot \Lambda$, Λ^2, *and m with real coefficients.*

Remark 3.3. The proof of Theorem 3.2 in [6] assumes Hypothesis 3.1. The local gluing maps are the analogues for SO(3) monopoles of the local gluing maps for anti-self-dual SO(3) connections constructed by Taubes [35, 36, 37] and Donaldson and Kronheimer [2, §7.2]; see also [30, 31]. We have established the existence of local gluing maps in [5] and expect that a proof of the continuity for the local gluing maps with respect to Uhlenbeck limits should be similar to our proof in [4] of this property for the local gluing maps for anti-self-dual SO(3) connections. The remaining properties of local gluing maps assumed in [6] are that they are injective and also surjective in the sense that elements of $\bar{\mathcal{M}}_{\mathfrak{t}}$ sufficiently close (in the Uhlenbeck topology) to $M_{\mathfrak{s}} \times \Sigma$ are in the image of at least one of the local gluing maps. In special cases, proofs of these properties for the local gluing maps for anti-self-dual SO(3) connections (namely, continuity with respect to Uhlenbeck limits, injectivity, and surjectivity) have been given in [2, §7.2.5, 7.2.6], [35, 36, 37]. The authors are currently developing a proof of the required properties for the local gluing maps for SO(3) monopoles. Our proof will also yield the analogous properties for the local gluing maps for anti-self-dual SO(3) connections.

Remark 3.4. In [24], Kronheimer and Mrowka show that Theorem 3.2, together with their work on the structure of the Donaldson invariants for manifolds of simple type [23], can be used to prove that Witten's Conjecture 1.1 holds for a suitably restricted class of standard four-manifolds [24, Corollary 7] and hence prove the Property P conjecture for knots. Kronheimer and Mrowka also gave a proof of Property P which did not rely on Theorem 3.2—see [25, Corollary 7.23].

4. Determining the coefficients

In this section, we prove that a standard four-manifold X of Seiberg–Witten simple type satisfying Witten's Conjecture can determine sufficiently many of the coefficients of the polynomial $f_{\delta,m}(\chi_h, c_1^2, c_1(\mathfrak{s}), \Lambda)$ appearing in (3.4) with $\chi_h = \chi_h(X)$ and $c_1^2 = c_1^2(X)$ to prove Conjecture 1.1, provided X is abundant or has $c_1^2(X) \geq \chi_h(X) - 3$.

4.1. Algebraic preliminaries

We begin with a generalization of [18, Lemma VI.2.4], which we shall later use to determine the coefficients in equation (3.4).

Lemma 4.1. *Let V be a finite-dimensional real vector space. Let $T_1, \ldots, T_n$ be linearly independent elements of the dual space V^*. Let Q be a quadratic form on V which is non-zero on $\bigcap_{i=1}^n \operatorname{Ker}(T_i)$. Then $T_1, \ldots, T_n, Q$ are algebraically independent in the sense that if $F(z_0, \ldots, z_n) \in \mathbb{R}[z_0, \ldots, z_n]$ and $F(Q, T_1, \ldots, T_n) : V \to \mathbb{R}$ is the zero map, then $F(z_0, \ldots, z_n)$ is the zero element of $\mathbb{R}[z_0, \ldots, z_n]$.*

Proof. We use induction on n. For $n = 1$, the result follows from [18, Lemma VI.2.4].

Assume that there is a polynomial $F(z_0, \ldots, z_n)$ such that $F(Q, T_1, \ldots, T_n) : V \to \mathbb{R}$ is the zero map. Assigning z_0 degree two and z_i degree one for $i > 0$, we can assume that F is homogeneous of degree d. Write $F(z_0, \ldots, z_n) = z_n^r G(z_0, \ldots, z_n)$, where z_n does not divide $G(z_0, \ldots, z_n)$. Because $T_n^r G(Q, T_1, \ldots, T_n)$ vanishes on V, the polynomial $G(Q, T_1, \ldots, T_n)$ must vanish on the dense set $T_n^{-1}(\mathbb{R}^*)$ and hence on V. We now write $G(z_0, \ldots, z_n) = \sum_{i=0}^m G_i(z_0, \ldots, z_{n-1}) z_n^{m-i}$. Since z_n does not divide $G(z_0, \ldots, z_n)$, if $G(z_0, \ldots, z_n)$ is not the zero polynomial, then $G_m(z_0, \ldots, z_{n-1})$ is not zero. However, as $G(Q, T_1, \ldots, T_n)$ is the zero map, the function $G_m(Q, T_1, \ldots, T_{n-1})$ vanishes on $\operatorname{Ker}(T_n)$. If there are scalars $c_1, \ldots, c_{n-1} \in \mathbb{R}$ such that the restriction of $c_1 T_1 + \cdots + c_{n-1} T_{n-1}$ to $\operatorname{Ker}(T_n)$ vanishes, then there is a scalar $c_n \in \mathbb{R}$ such that $c_1 T_1 + \cdots + c_{n-1} T_{n-1} = c_n T_n$. Consequently, the linear independence of $T_1, \ldots, T_n$ implies that $c_1 = \cdots = c_n = 0$. Hence, the restrictions of $T_1, \ldots, T_{n-1}$ to $\operatorname{Ker}(T_n)$ are linearly independent. Induction then implies that $G_m(z_0, \ldots, z_{n-1}) = 0$, a contradiction to $G(z_0, \ldots, z_n)$ being non-zero. Hence, F must be the zero polynomial. $\square$

Being closed under the action of $\{\pm 1\}$, the set $B(X)$ is not linearly independent over $\mathbb{R}$. Thus, in order to apply Lemma 4.1 to determine the coefficients $a_{i,j,k}$ in (3.5) from examples of manifolds satisfying Witten's formula (1.2), we rewrite the sums over $B(X)$ in (2.13) and (3.4) as sums over a smaller set of basic classes.

Let $B'(X)$ be a fundamental domain for the action of $\{\pm 1\}$ on $B(X)$, so the projection map $B'(X) \to B(X)/\{\pm 1\}$ is a bijection. Lemma 2.8 can then be rephrased as follows.

Lemma 4.2. *Let X be a standard four-manifold. Then Witten's formula (1.2) holds and X has KM-simple type if and only if the Donaldson invariants of X satisfy*

$$D_X^w(h^{\delta-2m}x^m) = 0$$

when $\delta \not\equiv -w^2 - 3\chi_h \pmod 4$, and when $\delta \equiv -w^2 - 3\chi_h \pmod 4$, they satisfy

$$
\begin{aligned}
&D_X^w(h^{\delta-2m}x^m) \\
&= \sum_{i+2k=\delta-2m} \sum_{K \in B'(X)} (-1)^{\varepsilon(w,K)} n(K) \frac{SW_X'(K)(\delta-2m)!}{2^{k+c(X)-3-m}k!i!} \langle K, h \rangle^i Q_X^k(h), \quad (4.1)
\end{aligned}
$$

where $\varepsilon(w, K)$ is as defined in Lemma 2.8 and

$$n(K) := \begin{cases} 1/2 & \text{if } K = 0, \\ 1 & \text{if } K \neq 0. \end{cases} \tag{4.2}$$

Proof. We will show that equation (2.13) holds if and only if (4.1) holds and so the conclusion will follow from Lemma 2.8.

Recall from §2.1 that $K \in B(X)$ if and only if $-K \in B(X)$. We rewrite the sum in (2.13) as a sum over $B'(X)$ by combining the K and $-K$ terms as follows. These two terms differ only in their factors of $(-1)^{\varepsilon(w,K)}$, and $SW_X'(K)$, and $\langle K, h \rangle^i$. Because K is characteristic, we see that

$$\tfrac{1}{2}(w^2 + w \cdot K) - \tfrac{1}{2}(w^2 - w \cdot K) \equiv w \cdot K \equiv w^2 \pmod 2.$$

From [29, Corollary 6.8.4], we have $SW_X'(-K) = (-1)^{\chi_h} SW_X'(K)$, so we can combine the distinct K and $-K$ terms in (2.13) using the identity

$$
\begin{aligned}
&(-1)^{\varepsilon(w,-K)} SW_X'(-K)\langle -K, h \rangle^i + (-1)^{\varepsilon(w,K)} SW_X'(K)\langle K, h \rangle^i \\
&\qquad\qquad = \left((-1)^{\chi_h+w^2+i} + 1\right)(-1)^{\varepsilon(w,K)} SW_X'(K)\langle K, h \rangle^i. \quad (4.3)
\end{aligned}
$$

In the sum appearing in (2.13), where $i + 2k = \delta - 2m$, we have $i \equiv \delta \pmod 2$. By the parity condition (2.5), we have $\delta + w^2 \equiv \chi_h \pmod 4$ and so $\chi_h + w^2 + i \equiv \chi_h + w^2 + \delta \equiv 0 \pmod 2$. Thus, if $K \neq 0$, the K and $-K$ terms will combine as in (4.3) to give the factor of two in (4.1). When $K = 0$, the K and $-K$ terms are the same and so we must offset this factor of two using the expression for $n(K)$ given in (4.2). $\qquad\square$

We now perform a similar reduction for the sum appearing in (3.4). For each triple of non-negative integers $i, j, k \in \mathbb{N}$, we define a universal polynomial map

$$b_{i,j,k} : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} \to \mathbb{R}$$

by setting

$$
\begin{aligned}
b_{i,j,k}&(\chi_h, c_1^2, K \cdot \Lambda, \Lambda^2, m) \\
&:= (-1)^{c(X)+i} a_{i,j,k}(\chi_h, c_1^2, -K \cdot \Lambda, \Lambda^2, m) + a_{i,j,k}(\chi_h, c_1^2, K \cdot \Lambda, \Lambda^2, m), \quad (4.4)
\end{aligned}
$$

where the $a_{i,j,k}$ are the universal, real coefficients appearing in the expression (3.5). Definition (4.4) implies that

$$
b_{i,j,k}(\chi_h, c_1^2, -K \cdot \Lambda, \Lambda^2, m) = (-1)^{c(X)+i} b_{i,j,k}(\chi_h, c_1^2, K \cdot \Lambda, \Lambda^2, m). \quad (4.5)
$$

We also define

$$
\tilde{\varepsilon}(w, \Lambda, K) := \tfrac{1}{2}(w^2 - \sigma) + \tfrac{1}{2}(w^2 + (w - \Lambda) \cdot K). \quad (4.6)
$$

We can now state the desired reduction.

Lemma 4.3. *Assume the hypotheses of Theorem* 3.2. *Denote the coefficients in* (4.4) *more concisely by*

$$
b_{i,j,k}(K \cdot \Lambda) := b_{i,j,k}(\chi_h, c_1^2, K \cdot \Lambda, \Lambda^2, m).
$$

Then

$$
\begin{aligned}
D_X^w(h^{\delta-2m}x^m) = \sum_{i+j+2k=\delta-2m} \sum_{K \in B'(X)} n(K)(-1)^{\tilde{\varepsilon}(w,\Lambda,K)} SW_X'(K) \\
\times\, b_{i,j,k}(K \cdot \Lambda)\langle K, h\rangle^i \langle\Lambda, h\rangle^j Q_X^k(h), \quad (4.7)
\end{aligned}
$$

where $n(K)$ *is defined by* (4.2).

Proof. Because the class $w - \Lambda$ is characteristic and $K^2 = c_1^2(X)$, we have

$$
\tilde{\varepsilon}(w, \Lambda, -K) = \tilde{\varepsilon}(w, \Lambda, K) - (w - \Lambda) \cdot K) \equiv \tilde{\varepsilon}(w, \Lambda, K) + c_1^2(X) \pmod 2.
$$

For $K \neq 0$, we can combine the distinct K and $-K$ terms in the sum appearing in (3.4) as in the identity (4.3) to obtain the expression (4.4) for the coefficients $b_{i,j,k}$. For $K = 0$, the factor of $n(K) = 1/2$ is necessary because the addition of the two identical terms in (4.4) would correspond to counting the term for $K = -K = 0$ in (3.4) twice. $\qquad\square$

4.2. The example manifolds

A four-manifold with the properties described in Definition 4.4 can be used with Lemmas 4.1–4.3 to determine many of the coefficients $b_{i,j,k}$ in (4.7).

Definition 4.4 (Useful four-manifolds). We call a standard four-manifold, X, *useful* if:

1. X has SW-simple type and $|B'(X)| = 1$.
2. X satisfies Witten's equation (4.1).
3. There are cohomology classes, $f_1, f_2 \in B(X)^\perp$, with $f_i^2 = 0$ and $f_1 \cdot f_2 = 1$ such that $\{f_1, f_2\} \cup B'(X)$ is linearly independent over $\mathbb{R}$.
4. If f_1, f_2 are the cohomology classes in the previous condition, then the restriction of Q_X to $(\bigcap_{i=1}^2 \mathrm{Ker}(f_i)) \cap (\bigcap_{K \in B'(X)} \mathrm{Ker}(K))$ is non-zero.

We prove the existence of a family of useful four-manifolds in the following lemma.

Lemma 4.5 (Existence of useful four-manifolds). *For every integer $h = 2, 3, 4, \ldots,$ there is a useful four-manifold Y_h with $\chi_h(Y_h) = h$, $c_1^2(Y_h) = h - 3$, and $c(Y_h) = 3$.*

Proof. In [14, Proposition 3.5], R. Fintushel, J. Park, and R. Stern construct examples of standard four-manifolds X_p and X_p' for integer $p \geq 4$ with $c_1^2(X_p) = 2p - 7$ and $c_1^2(X_p') = 2p - 8$ and both satisfying $c_1^2 = \chi_h - 3$. In addition, $|B(X_p)/\{\pm 1\}| = |B(X_p')/\{\pm 1\}| = 1$. The four-manifolds constructed in [14] define a ray in the (χ_h, c_1^2) plane but the restrictions on p mean that they do not include the point $\chi_h = 2$ and $c_1^2 = -1$. We will write Y_h for the member of this family of manifolds with $\chi_h(Y_h) = h$ and set $Y_2 := K3 \# \bar{\mathbb{CP}}^2$, where '$K3$' denotes the K3 surface. We further note that $Y_3 = E(3)$ by the construction in [17, §3], where one notes that the operation of rationally blowing down the empty configuration C_1 is trivial [13]. Because $B(K3) = \{0\}$ by [17], the blow-up formula in Theorem 2.1 implies that $|B'(Y_2)| = 1$.

As shown in the discussion following Lemma 3.4 in [14], for $p \geq 4$, the four-manifolds X_p and X_p' are rational blow-downs of the elliptic surfaces $E(2p - 4)$ and $E(2p - 5)$, respectively, along taut configurations (in the sense of [17, §7]) of embedded spheres. These elliptic surfaces have SW-simple type and satisfy Conjecture 1.1 (see, for example, [17, Theorem 8.7]). By [17, Theorem 8.9], these properties (having SW-simple type and satisfying Conjecture 1.1) are preserved under rational blow-down and hence also hold for Y_h for $h > 2$. For $Y_2 = K3 \# \bar{\mathbb{CP}}^2$, these two properties hold because they hold for $K3 = E(2)$, by [23] and [17], and because these properties are preserved under blow-ups by Theorem 2.7.

Recall that a four-manifold X is abundant if there are cohomology classes $f_1, f_2 \in B(X)^\perp \subset H^2(X; \mathbb{Z})$ with $f_i^2 = 0$ and $f_1 \cdot f_2 = 1$. By [10, Corollary A.3], if X is simply connected and the SW-basic classes are all multiples of a single cohomology class, then X is abundant. This result, together with the fact that $|B'(Y_h)| = 1$ for all $h \geq 2$, implies that our four-manifolds Y_h are abundant.

We now show that the cohomology-class linear independence property holds for the four-manifolds Y_h. If the cohomology classes $f_1, f_2 \in B(Y_h)^\perp$ are as described in the Definition 4.4 of a useful four-manifold and $K \in B(Y_h)$ and $af_1 + bf_2 + cK = 0$ for some $a, b, c \in \mathbb{R}$, then

$$a = f_2 \cdot (af_1 + bf_2 + cK) = 0 \quad \text{and} \quad b = f_1 \cdot (af_1 + bf_2 + cK) = 0,$$

and thus $cK = 0$. If $K \neq 0$, then $c = 0$ and the set $\{K, f_1, f_2\}$ is linearly independent. If $K = 0$, then because the four-manifolds Y_h have SW-simple type, we would have $0 = K^2 = c_1^2(Y_h)$, which is only true if $h = 3$ and $Y_3 = E(3)$. For $h = 3$, we have $B'(Y_3) = \{F\}$, where F is the Poincaré dual of a generic fiber of the elliptic fibration on Y_3 by [17] and $F \neq 0$. Hence, $K \neq 0$ for all our manifolds Y_h, so the set $\{K, f_1, f_2\}$ is linearly independent over $\mathbb{R}$.

To prove that our manifolds Y_h satisfy the fourth condition in the Definition 4.4 of a useful four-manifold, we identify the kernels of the cohomology classes K, f_1, and f_2 with their orthogonal complements in $H^2(Y_h; \mathbb{Z})$ by Poincaré duality, and show that

the restriction of Q_{Y_h} to this orthogonal complement is non-zero. If $K^2 \neq 0$, then the determinant of the restriction of Q_{Y_h} to the span of $\{K, f_1, f_2\}$ is non-zero. Hence, the determinant of the restriction of Q_{Y_h} (and thus the restriction of Q_{Y_h}) to the orthogonal complement of this span is also non-zero. As in the preceding paragraph, if $K^2 = 0$, then $h = 3$ and $Y_3 = E(3)$. If $F \in H^2(E(3); \mathbb{Z})$ is the Poincaré dual of a generic fiber of the elliptic fibration and $\sigma \in H^2(E(3); \mathbb{Z})$ is the Poincaré dual of a section, then $1 = F \cdot \sigma \equiv \sigma^2 \pmod 2$, so $Q_{E(3)}$ is odd and there is an isomorphism of quadratic forms,

$$\left(H^2(E(3); \mathbb{Z}), Q_{E(3)} \right) \cong \left(\bigoplus_{i=1}^{5} \mathbb{Z}e_i \right) \oplus \left(\bigoplus_{j=1}^{29} \mathbb{Z}g_j \right),$$

where $e_i^2 = 1$ and $g_j^2 = -1$. Following the argument of [10, Lemma A.4], we define

$$L := 3e_1 + 3e_2 + 3e_3 + e_4 + e_5 + \sum_{j=1}^{29} g_j,$$

$$\tilde{f}_1 := e_5 + g_2, \quad \tilde{f}_2 := e_5 + g_3, \quad P := e_1 - e_2.$$

Then L is primitive and characteristic with $L^2 = 0$, while $\{\tilde{f}_1, \tilde{f}_2\}$ span a hyperbolic summand orthogonal to L. The class P is orthogonal to the span of $\{L, \tilde{f}_1, \tilde{f}_2\}$, and $P^2 \neq 0$. Thus, $Q_{E(3)}$ is non-zero on the orthogonal complement of that span. Because $\sigma \cdot F = 1$, then F is primitive as well as characteristic with $F^2 = 0$. As observed in [10, Lemma A.4], a result of Wall (see [38, Proposition 1.2.28]) implies that the orthogonal group of $(H^2(E(3); \mathbb{Z}), Q_{E(3)})$ acts transitively on the primitive characteristic elements with a given square. Hence, there is an isometry of $(H^2(E(3); \mathbb{Z}), Q_{E(3)})$ mapping L to F. If we take f_i to be the image of $\tilde{f}_i$ under this isometry, then we see that $Q_{E(3)}$ is non-zero on the orthogonal complement of the span of $\{F, f_1, f_2\}$, as desired. □

4.3. The blow-up formulas

To determine the coefficients $b_{i,j,k}$ for a sufficiently wide range of values of χ_h, c_1^2, Λ^2, and $K \cdot \Lambda$, we will need to work with the blow-ups of the useful four-manifolds described in Lemma 4.5. Thus, let $\widetilde{X}(n)$ be the blow-up of X at n points, where X is one of the useful four-manifolds described in Lemma 4.5. For non-negative integers $m \leq n$, we will consider $H^2(\widetilde{X}(m); \mathbb{Z})$ as a subspace of $H^2(\widetilde{X}(n); \mathbb{Z})$ using the inclusion defined by the pullback of the blow-down map. Let $\{e_1, \ldots, e_n\} \subset H_2(\widetilde{X}(n); \mathbb{Z})$ be the homology classes of the exceptional curves and let $e_u^* := \mathrm{PD}[e_u]$ for $u = 1, \ldots, n$.

We now describe $B(\widetilde{X}(n))$ in more detail. Let $\pi_u : (\mathbb{Z}/2\mathbb{Z})^n \to \mathbb{Z}/2\mathbb{Z}$ be projection onto the u-th factor. For $K \in B(X)$ and $\varphi \in (\mathbb{Z}/2\mathbb{Z})^n$, define

$$K_\varphi := K + \sum_{u=1}^{n} (-1)^{\pi_u(\varphi)} e_u^* \quad \text{and} \quad K_0 := K + \sum_{u=1}^{n} e_u^*. \tag{4.8}$$

If $0 \notin B(X)$, then the Seiberg–Witten blow-up formula (2.4) implies that

$$B'(\widetilde{X}(n)) = \{K_\varphi : K \in B'(X) \text{ and } \varphi \in (\mathbb{Z}/2\mathbb{Z})^n\}.$$

Even if the set $B'(X)$ of SW-basic classes is linearly independent, the set $B'(\widetilde{X}(n))$ will not be linearly independent for $n \geq 2$.

To rewrite Lemma 4.3 in terms of linearly independent SW-basic classes, we will require a result from combinatorics. For a function $f : \mathbb{Z} \to \mathbb{R}$ and $p, q \in \mathbb{Z}$, define

$$(\nabla_p^q f)(x) := f(x) + (-1)^q f(x + p), \quad \forall x \in \mathbb{Z}, \tag{4.9}$$

and for $a \in \mathbb{Z}/2\mathbb{Z}$ and $p \in \mathbb{Z}$, define

$$pa := -\tfrac{1}{2}(-1 + (-1)^a)p. \tag{4.10}$$

We then have

Lemma 4.6. *Let $f : \mathbb{Z} \to \mathbb{R}$ be a function and $n \geq 1$ an integer. Then, for all $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ in $\mathbb{Z}^n$, one has*

$$\sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^n} (-1)^{\sum_{u=1}^n q_u \pi_u(\varphi)} f\left(x + \sum_{u=1}^n p_u \pi_u(\varphi)\right) = (\nabla_{p_1}^{q_1} \nabla_{p_2}^{q_2} \ldots \nabla_{p_n}^{q_n} f)(x),$$

and if C is the constant function, then

$$(\nabla_{p_n}^{q_n} \nabla_{p_{n-1}}^{q_{n-1}} \ldots \nabla_{p_1}^{q_1} C) = \begin{cases} 0 & \text{if } q_u \equiv 1 \ (\mathrm{mod}\, 2) \text{ for some } u \in \{1, \ldots, n\}, \\ 2^n C & \text{if } q_u \equiv 0 \ (\mathrm{mod}\, 2) \text{ for all } u \in \{1, \ldots, n\}. \end{cases} \tag{4.11}$$

Proof. The proof uses induction on n. For $n = 1$, the statement is trivial. Define

$$(L_{p_1, \ldots, p_n}^{q_1, \ldots, q_n} f)(x) := \sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^n} (-1)^{\sum_{u=1}^n q_u \pi_u(\varphi)} f\left(x + \sum_{u=1}^n p_u \pi_u(\varphi)\right).$$

For $n \geq 2$, the preceding expression can be expanded as

$$\sum_{\varphi \in \pi_n^{-1}(0)} (-1)^{\sum_{u=1}^{n-1} q_u \pi_u(\varphi)} f\left(x + \sum_{u=1}^{n-1} p_u \pi_u(\varphi)\right)$$

$$+ (-1)^{q_n} \sum_{\varphi \in \pi_n^{-1}(1)} (-1)^{\sum_{u=1}^{n-1} q_u \pi_u(\varphi)} f\left(x + p_n + \sum_{u=1}^{n-1} p_u \pi_u(\varphi)\right)$$

$$= (L_{p_1, \ldots, p_{n-1}}^{q_1, \ldots, q_{n-1}} f)(x) + (-1)^{q_n}(L_{p_1, \ldots, p_{n-1}}^{q_1, \ldots, q_{n-1}} f)(x + p_n) = (\nabla_{p_n}^{q_n}(L_{p_1, \ldots, p_{n-1}}^{q_1, \ldots, q_{n-1}} f))(x),$$

where in the penultimate step we have identified $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ with $\pi_n^{-1}(0)$ and $\pi_n^{-1}(1)$ as sets. The first assertion in the lemma now follows by induction.

The identity (4.11) follows from the fact that

$$\nabla_p^q C = C + (-1)^q C = \begin{cases} 0 & \text{if } q \equiv 1 \ (\mathrm{mod}\, 2), \\ 2C & \text{if } q \equiv 0 \ (\mathrm{mod}\, 2), \end{cases}$$

and induction on n. $\qquad\square$

If X is a four-manifold with blow-up $\widetilde{X}(n)$ for some integer $n \geq 1$ and $w \in H^2(X; \mathbb{Z}) \subset H^2(\widetilde{X}(n); \mathbb{Z})$, we denote

$$\tilde{w} := w + \sum_{u=1}^{n} w_u e_u^*. \qquad (4.12)$$

We can now rewrite Lemmas 4.2 and 4.3 in terms of linearly independent SW-basic classes.

Lemma 4.7. *Continue the notation of the preceding paragraphs and Definition 4.4. Let X be a useful four-manifold and $n \geq 1$ an integer. For $w \in H^2(X; \mathbb{Z}) \subset H^2(\widetilde{X}(n); \mathbb{Z})$, let $\tilde{w}$ be as in (4.12). Let $\Lambda \in H^2(\widetilde{X}(n); \mathbb{Z})$ satisfy $I(\Lambda) > \delta$ and $\Lambda - \tilde{w} \equiv w_2(\widetilde{X}(n))$ (mod 2). Define*

$$b_{i,j,k}(K_\varphi \cdot \Lambda) := b_{i,j,k}\big(\chi_h(\widetilde{X}(n)), c_1^2(\widetilde{X}(n)), K_\varphi \cdot \Lambda, \Lambda^2, m\big).$$

Then, for $\delta - 2m \geq 0$,

$$(-1)^{\varepsilon(\tilde{w}, K_0)} \sum_{i_0 + \cdots + i_n + 2k = \delta - 2m} \binom{i_0 + \cdots + i_n}{i_0, \ldots, i_n} \frac{SW_X'(K)(\delta - 2m)!}{2^{k+c+n-3-m} k! i!}$$

$$\times p^{\tilde{w}}(i_1, \ldots, i_n) \langle K, h \rangle^{i_0} \prod_{u=1}^{n} \langle e_u^*, h \rangle^{i_u} Q_X^k(h)$$

$$= (-1)^{\tilde{\varepsilon}(\tilde{w}, \Lambda, K_0)} \sum_{i_0 + \cdots + i_n + j + 2k = \delta - 2m} \binom{i_0 + \cdots + i_n}{i_0, \ldots, i_n} SW_X'(K)$$

$$\times \sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^n} (-1)^{\sum_{u=1}^{n}(1+i_u)\pi_u(\varphi)} b_{i,j,k}(K_\varphi \cdot \Lambda)$$

$$\times \langle K, h \rangle^{i_0} \prod_{u=1}^{n} \langle e_u^*, h \rangle^{i_u} \langle \Lambda, h \rangle^{j} Q_X^k(h), \qquad (4.13)$$

where $c = c(X) = \chi_h(X) - c_1^2(X)$, as in (2.12), and

$$p^{\tilde{w}}(i_1, \ldots, i_n) := \begin{cases} 0 & \text{if } w_q + i_q \equiv 1 \ (\text{mod } 2) \text{ for some } q \in \{1, \ldots n\}, \\ 2^n & \text{if } w_q + i_q \equiv 0 \ (\text{mod } 2) \text{ for all } q \in \{1, \ldots, q\}. \end{cases} \qquad (4.14)$$

Proof. Comparing (4.1) and (4.7) yields, for $\varepsilon(\tilde{w}, \varphi) = \frac{1}{2}(\tilde{w}^2 + \tilde{w} \cdot K_\varphi)$,

$$\sum_{i+2k=\delta-2m} \sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^n} (-1)^{\varepsilon(\tilde{w}, \varphi)} \frac{SW_X'(K)(\delta - 2m)!}{2^{k+c+n-3-m} k! i!} \langle K_\varphi, h \rangle^{i} Q_X^k(h)$$

$$= \sum_{i+j+2k=\delta-2m} \sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^n} (-1)^{\tilde{\varepsilon}(w, \Lambda, K_\varphi)} SW_X'(K)$$

$$\times b_{i,j,k}(K_\varphi \cdot \Lambda) \langle K_\varphi, h \rangle^{i} \langle \Lambda, h \rangle^{j} Q_X^k(h). \qquad (4.15)$$

For $\varphi \in (\mathbb{Z}/2\mathbb{Z})^n$, we have

$$\varepsilon(\tilde{w}, \varphi) \equiv \tfrac{1}{2}(\tilde{w}^2 + \tilde{w} \cdot K_\varphi) \equiv \tfrac{1}{2}(\tilde{w}^2 + \tilde{w} \cdot K_0) + \sum_{u=1}^{n} w_u \pi_u(\varphi) \pmod{2}. \qquad (4.16)$$

By the multinomial theorem, for $\varphi \in (\mathbb{Z}/2\mathbb{Z})^n$ we can expand the factor $\langle K_\varphi, h \rangle^i$ as

$$\langle K_\varphi, h \rangle^i = \sum_{i_0 + \cdots + i_n = i} \binom{i}{i_0, \ldots, i_n} (-1)^{\sum_{u=1}^n \pi_u(\varphi) i_u} \langle K, h \rangle^{i_0} \prod_{u=1}^{n} \langle e_u^*, h \rangle^{i_u}, \qquad (4.17)$$

where, for $i = i_0 + \cdots + i_n$,

$$\binom{i}{i_0, \ldots, i_n} = \frac{i!}{i_0! \ldots i_n!}.$$

The identities (4.16) and (4.17) imply that we can rewrite the left-hand side of (4.15) as

$$\sum_{i+2k=\delta-2m} \sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^n} (-1)^{\varepsilon(\tilde{w}, \varphi)} \frac{SW_X'(K)(\delta - 2m)!}{2^{k+c+n-3-m} k! i!} \langle K_\varphi, h \rangle^i Q_X^k(h)$$

$$= (-1)^{\varepsilon(\tilde{w}, K_0)} \sum_{i_0 + \cdots + i_n + 2k = \delta - 2m} \binom{i_0 + \cdots + i_n}{i_0, \ldots, i_n} \frac{SW_X'(K)(\delta - 2m)!}{2^{k+c+n-3-m} k! i!}$$

$$\times \sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^n} (-1)^{\sum_{u=1}^n \pi_u(\varphi)(w_u + i_u)} \langle K, h \rangle^{i_0} \prod_{u=1}^{n} \langle e_u^*, h \rangle^{i_u} Q_X^k(h). \qquad (4.18)$$

By applying Lemma 4.6, we write the sum over $\varphi \in (\mathbb{Z}/2\mathbb{Z})^n$ in (4.18) as

$$\sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^n} (-1)^{\sum_{u=1}^n \pi_u(\varphi)(w_u + i_u)} = \nabla_0^{w_1 + i_1} \cdots \nabla_0^{w_n + i_n} 1.$$

Equation (4.11) shows that the preceding expression is equal to $p^{\tilde{w}}(i_1, \ldots, i_n)$, as defined in (4.14). Therefore, (4.18) implies that the left-hand side of (4.15) equals the left-hand side of (4.13).

We now rewrite the right-hand side of (4.15). The discussion is essentially the same as that for the left-hand side. However, note that

$$\tilde{\varepsilon}(\tilde{w}, \Lambda, K_\varphi) - \tilde{\varepsilon}(\tilde{w}, \Lambda, K_0) = \tfrac{1}{2}(\Lambda - \tilde{w}) \cdot (K_\varphi - K_0).$$

Because

$$K_\varphi - K_0 = \sum_{u=1}^{n} ((-1)^{\pi_u(\varphi)} - 1) e_u^* = -2 \sum_{u=1}^{n} \pi_u(\varphi) e_u^*,$$

and since $\Lambda - \tilde{w}$ is characteristic, we have

$$\tfrac{1}{2}(\Lambda - \tilde{w}) \cdot (K_\varphi - K_0) \equiv \sum_{u=1}^{n} \pi_u(\varphi) \pmod{2}.$$

The preceding identity replaces the orientation sign-change factor computed in (4.16), and we can conclude that the right-hand side of (4.15) is equal to the right-hand side of (4.13). $\qquad \square$

4.4. Determining the coefficients $b_{i,j,k}$

We now apply Lemmas 4.1 and 4.7 to the manifolds discussed in Lemma 4.5 to determine the coefficients $b_{i,j,k}$ with $i \geq c(X) - 3 > 0$.

Proposition 4.8. *For any integers x, y, for any integers $m \geq 0$, $n > 0$, and $\chi_h \geq 2$, and for any non-negative integers i, j, k satisfying $i + j + 2k = \delta - 2m$, $i \geq n$, and $2y > \delta - 4\chi_h - 3 - n$, the coefficients $b_{i,j,k}(\chi_h, c_1^2, K \cdot \Lambda, \Lambda^2, m)$ defined in (4.4) satisfy*

$$b_{i,j,k}(\chi_h, \chi_h - 3 - n, 2x, 2y, m) = \begin{cases} (-1)^{x+y} \dfrac{(\delta - 2m)!}{k!i!} 2^{m-k-n} & \text{if } j = 0, \\ 0 & \text{if } j > 0. \end{cases}$$

Proof. For one of the useful four-manifolds, X, described in Lemma 4.5, let $\widetilde{X}(n)$ be the blow-up of X at n points. We apply Lemma 4.7 with

$$\Lambda = (y + 2x^2)f_1 + f_2 + 2xe_1^*,$$

where $f_1, f_2 \in B(X)^\perp$ are the cohomology classes in Definition 4.4 satisfying $f_i^2 = 0$ and $f_1 \cdot f_2 = 1$. Thus,

$$\Lambda^2 = 2y \quad \text{and} \quad K_0 \cdot \Lambda = -2x.$$

The condition $2y > \delta - 4\chi_h - 3 - n$ implies that $I(\Lambda) > \delta$. Observe that

$$(K_\varphi - K_0) \cdot \Lambda = \begin{cases} 0 & \text{if } \pi_1(\varphi) = 0, \\ 4x & \text{if } \pi_1(\varphi) = 1. \end{cases}$$

If we write $\tilde{w} = w + \sum_{u=1}^n w_u e_u^*$, as in (4.12), then the requirement that $\Lambda - \tilde{w}$ is characteristic implies that $w_u \equiv 1 \pmod 2$ for all u. Hence, the coefficient of the term

$$K^{i_0}(e_1^*)^{i_1} \cdots (e_n^*)^{i_n} \Lambda^j Q_X^k \tag{4.19}$$

on the left-hand side of (4.13) will vanish if $j > 0$ while, if $j = 0$, the coefficient is equal to

$$(-1)^{\varepsilon(\tilde{w}, K_0)} \binom{i}{i_0, \ldots, i_n} \frac{SW_X'(K)(\delta - 2m)!}{2^{k+n-m}k!i!} p^{\tilde{w}}(i_i, \ldots, i_n), \tag{4.20}$$

where $i = i_0 + \cdots + i_n$.

The coefficient of the term (4.19) on the right-hand side of (4.13) is

$$(-1)^{\tilde{\varepsilon}(\tilde{w}, \Lambda, K_0)} \binom{i}{i_0, \ldots, i_n} SW_X'(K)\Big(b_{i,j,k}(-2x)\Big(\sum_{\varphi \in \pi_1^{-1}(0)} (-1)^{\sum_{u=1}^n (1+i_u)\pi_u(\varphi)} \Big).$$

$$+ b_{i,j,k}(2x)\Big(\sum_{\varphi \in \pi_1^{-1}(1)} (-1)^{\sum_{u=1}^n (1+i_u)\pi_u(\varphi)} \Big)\Big). \tag{4.21}$$

Equation (4.11) implies that, for $a = 0, 1$,

$$\sum_{\varphi \in \pi^{-1}(a)} (-1)^{\sum_{u=2}^{n}(1+i_u)\pi_u(\varphi)} = \begin{cases} 0 & \text{if } i_q \equiv 0 \ (\text{mod } 2) \text{ for some } q \in \{2, \ldots, n\}, \\ 2^{n-1} & \text{if } i_q \equiv 1 \ (\text{mod } 2) \text{ for all } q \in \{2, \ldots, n\}. \end{cases}$$

We define a map $p^1 : \mathbb{Z}^{n-1} \to \mathbb{Z}$ by setting $p^1(i_2, \ldots, i_n)$ equal to the right-hand side of the preceding expression. Hence,

$$\sum_{\varphi \in \pi_1^{-1}(0)} (-1)^{\sum_{u=1}^{n}(1+i_u)\pi_u(\varphi)} = p^1(i_2, \ldots, i_n),$$

$$\sum_{\varphi \in \pi_1^{-1}(1)} (-1)^{\sum_{u=1}^{n}(1+i_u)\pi_u(\varphi)} = (-1)^{1+i_1} p^1(i_2, \ldots, i_n).$$

The identity (4.5) and the identity $\Lambda^2 - \delta \equiv c(\widetilde{X}(n)) \ (\text{mod } 4)$ implied by (3.3) and our assumptions that $\Lambda^2 \equiv 0 \ (\text{mod } 2)$ and $\delta \equiv i + j \ (\text{mod } 2)$ yield

$$b_{i,j,k}(-2x) = (-1)^{\delta+i} b_{i,j,k}(2x) = (-1)^j b_{i,j,k}(2x).$$

Because $\Lambda - \tilde{w}$ is characteristic, we have $(\Lambda - \tilde{w})^2 \equiv \sigma \ (\text{mod } 8)$ and $\Lambda^2 \equiv \Lambda \cdot (\Lambda - \tilde{w})$ (mod 2), so $\Lambda \cdot \tilde{w} \equiv 0 \ (\text{mod } 2)$. Thus, $(\Lambda - \tilde{w})^2 \equiv \sigma \ (\text{mod } 8)$ implies that $\Lambda^2 + \tilde{w}^2 \equiv \sigma$ (mod 4) and so $\frac{1}{2}(\tilde{w}^2 - \sigma) \equiv \frac{1}{2}\Lambda^2 \ (\text{mod } 2)$. Therefore, by the definitions of $\varepsilon(\tilde{w}, K_0)$ and $\tilde{\varepsilon}(\tilde{w}, \Lambda, K_0)$, we have

$$\tilde{\varepsilon}(\tilde{w}, \Lambda, K_0) - \varepsilon(\tilde{w}, K_0) = \tfrac{1}{2}(\tilde{w}^2 - \sigma) - \tfrac{1}{2}K_0 \cdot \Lambda \equiv \tfrac{1}{2}(\Lambda^2 + K_0 \cdot \Lambda) \ (\text{mod } 2).$$

By the preceding analysis, we can rewrite the coefficient (4.21) as

$$(-1)^{\varepsilon(\tilde{w}, K_0)+x+y} \binom{i}{i_0, \ldots, i_n} SW'_X(K) b_{i,j,k}(2x)$$
$$\times p^1(i_2, \ldots, i_n)\big((-1)^j - (-1)^{i_1}\big). \quad (4.22)$$

Lemma 4.1 implies that the coefficients (4.20) and (4.22) must be equal. For this to be a non-trivial relation, $p^1(i_2, \ldots, i_n)$ must be non-zero and consequently we must have $i_u \equiv 1 \ (\text{mod } 2)$ for $u = 2, \ldots, n$. For j even, take $i_1 = \cdots = i_n = 1$ and $i_0 = i - n$, while for j odd, take $i_1 = 0, i_2 = \cdots = i_n = 1$, and $i_0 = i - n + 1$ to get the desired equalities. $\qquad\square$

Remark 4.9. Proposition 4.8 only determines the coefficients $b_{i,j,k}(\chi_h, c_1^2, K \cdot \Lambda, \Lambda^2, m)$ for $i \geq \chi_h - c_1^2 - 3$. An early manuscript version [7] of this article failed to note that because $p^1(i_2, \ldots, i_n)$ vanishes for low values of i (since $i = i_0 + i_1 + \cdots + i_n$ and so i small implies that each i_q is small) the resulting relations were trivial and gave no information about the coefficients $b_{i,j,k}$.

Remark 4.10 (Determining the remaining coefficients). We now describe some limitations on the ability of (4.13) to determine the coefficients $b_{i,j,k}$ using the four-manifolds X_h constructed in Lemma 4.5. For χ_h, c_1^2, Λ^2, and m fixed, define a function $c_{i,j,k} : \mathbb{Z} \to \mathbb{R}$ by setting $c_{i,j,k}(x) := b_{i,j,k}(\chi_h, c_1^2, x, \Lambda^2, m)$. If, in the notation of Proposition 4.8, one takes

$$\Lambda = y f_1 + f_2 + \sum_{u=1}^{n} \lambda_u e_u^*,$$

then Lemma 4.6 implies that the coefficient of the term (4.19) on the right-hand side of (4.13) would be

$$\nabla_{2\lambda_1}^{i_1+1} \ldots \nabla_{2\lambda_n}^{i_n+1} c_{i,j,k}(K_0 \cdot \Lambda).$$

Because $\nabla_{2\lambda_1}^1 \cdots \nabla_{2\lambda_n}^1 p(x) = 0$ for any polynomial $p(x)$ of degree $n-1$ or less, the arguments used in the proof of Proposition 4.8 using the four-manifolds X_h cannot determine the coefficients $b_{0,j,k}$. Arguing by induction on $i = v$ and by varying $i_1, \ldots, i_v$, one can show that these arguments determine $b_{i,j,k}$ only up to a polynomial of degree $n - i - 1$ in $\Lambda \cdot K$.

This failure of Proposition 4.8 to determine the coefficients $b_{i,j,k}$ using blow-ups of the manifolds X_h stems from the failure of the set $B'(\widetilde{X}_h(n))$ to be linearly independent. Further progress with our method would appear to rely on finding four-manifolds, Y, with $c(Y) > 3$ and $B'(Y)$ admitting few linear relations. The 'superconformal simple-type bound',

$$c_1^2(Y) \geq \chi_h(Y) - 2|B(Y)/\{\pm 1\}| - 1,$$

appearing in [26, Theorem 4.1] holds for all known standard four-manifolds and indicates that the number of basic classes increases as $c(Y)$ increases. Consequently, one would need to search for standard four-manifolds where the dimension of the span of $B'(Y)$ is large.

Proof of Theorem 1.2 for four-manifolds with $c_1^2 \geq \chi_h - 3$. Assume that Y is a standard four-manifold with $c_1^2(Y) \geq \chi_h(Y) - 3$. Let X_h be a useful four-manifold provided by Lemma 4.5 with $\chi_h(X_h) = \chi_h(Y)$. By Theorem 2.7 and by blowing up Y if necessary, we can assume that $c_1^2(Y) = c_1^2(X_h)$. Let $\widetilde{Y}$ and $\widetilde{X}_h$ be the blow-ups of Y and X_h, respectively, at a point. Let $e^* \in H^2(\widetilde{Y}; \mathbb{Z})$ be the Poincaré dual of the exceptional curve. For a characteristic $w \in H^2(Y; \mathbb{Z})$, define $\tilde{w} = w + e^* \in H^2(\widetilde{Y}; \mathbb{Z})$. Denoting $B'(Y) = \{K_1, \ldots, K_b\}$, there are cohomology classes $f_1, f_2 \in H^2(Y; \mathbb{Z})$ with $K_1 \cdot f_i = 0$ and $f_i^2 = 0$ for $i = 1, 2$, and $f_1 \cdot f_2 = 1$ by [10, Corollary A.3]. For a given δ, we can choose an integer a such that, for $\Lambda = 2(a f_1 + f_2) \in H^2(Y; \mathbb{Z}) \subset H^2(\widetilde{Y}; \mathbb{Z})$, we have $\Lambda^2 = 8a$ and $I(\Lambda) > \delta$. Because $I(\Lambda) > \delta$ and $\Lambda - \tilde{w}$ is characteristic, we can use this $\tilde{w}$ and Λ in Lemma 4.3 to compute the degree-δ Donaldson invariant of Y. Since $\Lambda^2 \equiv 0 \pmod{2}$ and K_i is characteristic, $K_i \cdot \Lambda \equiv 0 \pmod{2}$ for all $K_i \in B(\widetilde{Y})$. Proposition 4.8 then only gives an expression for the coefficients

$$b_{i,j,k}((K_i \pm e^*) \cdot \Lambda) = b_{i,j,k}\big(\chi_h(\widetilde{Y}), c_1^2(\widetilde{Y}), (K_i \pm e^*) \cdot \Lambda, 8a, m\big)$$

appearing in (4.7) for $i \geq 1$. We next show that we can ignore the terms in (4.7) with $i = 0$.

As $\tilde{w} - \Lambda$ is characteristic, we have

$$\tilde{\varepsilon}(\tilde{w}, \Lambda, K_i + e^*) \equiv \tilde{\varepsilon}(\tilde{w}, \Lambda, K_i - e^*) + (\tilde{w} - \Lambda) \cdot e^* \pmod 2$$
$$\equiv \tilde{\varepsilon}(\tilde{w}, \Lambda, K_i - e^*) + 1 \pmod 2.$$

Using the fact that $(K_i + e^*) \cdot \Lambda = (K_i - e^*) \cdot \Lambda$, we obtain

$$b_{i,j,k}((K_i + e^*) \cdot \Lambda) = b_{i,j,k}((K_i - e^*) \cdot \Lambda).$$

Finally, because $n(K_i \pm e^*) = 1$, the terms for $K_i + e^*$ and $K_i - e^*$ in (4.7) with $i = 0$ will cancel out. Thus, we may ignore the $i = 0$ terms.

Since $\tilde{w}$ is characteristic, the definition of $\tilde{\varepsilon}$ in (4.6) implies that

$$\tilde{\varepsilon}(\tilde{w}, \Lambda, K_i \pm e^*) + \tfrac{1}{2}\Lambda \cdot (K_i \pm e^*) \equiv \varepsilon(\tilde{w}, K_i \pm e^*) \pmod 2.$$

Therefore, the formula for the coefficients $b_{i,j,k}$ in Proposition 4.8 and the vanishing of the terms with $i = 0$ allow us to rewrite (4.7) as

$$D_{\tilde{Y}}^{\tilde{w}}(h^{\delta-2m}x^m)$$
$$= \sum_{i+j+2k=\delta-2m} \sum_{K\in B'(\tilde{Y})} (-1)^{\varepsilon(\tilde{w},K)}SW'_X(K)\frac{(\delta-2m)!}{k!i!}2^{m-k-1}\langle K,h\rangle^i Q_X^k(h). \quad (4.23)$$

Comparing (4.23) and (4.1), noting that $c(\tilde{Y}) = 4$, and applying Lemma 4.2 then shows that Witten's Conjecture 1.1 holds for $\tilde{Y}$ and thus for Y. $\qquad\square$

Before proceeding to the proof of Theorem 1.2 for abundant four-manifolds, we recall a vanishing result for abundant four-manifolds. If Y is a standard four-manifold, $w \in H^2(Y; \mathbb{Z})$, and $h \in H_2(Y; \mathbb{R})$, we define

$$SW_{Y,i}^w(h) := \sum_{K\in B(Y)} (-1)^{\varepsilon(w,K)}SW'_Y(K)\langle K,h\rangle^i.$$

We then recall

Theorem 4.11 ([3, Theorem 1.1]). *Theorem 3.2 implies that if Y is a standard and abundant four-manifold and w is characteristic, then $SW_{Y,i}^w$ vanishes for $i < c(Y) - 2$.*

Proof of Theorem 1.2 for abundant four-manifolds. We now show that Proposition 4.8 suffices to prove Witten's Conjecture 1.1 for abundant four-manifolds. By the argument in the proof of Lemma 4.2, for w characteristic (so $w^2 \equiv c_1^2(Y) \pmod 2$),

$$SW_{Y,i}^w(h) = (1 + (-1)^{c(X)+i}) \sum_{K\in B'(Y)} (-1)^{\varepsilon(w,K)}n(K)SW'_Y(K)\langle K,h\rangle^i. \quad (4.24)$$

By Theorem 2.7, it suffices to prove that Conjecture 1.1 holds for the blow-up of Y at any number of points. We can therefore assume that $c_1^2(Y) = \chi_h(Y) - 3 - n$ for $n \geq 1$. For any non-negative integers δ and m satisfying $\delta - 2m \geq 0$, choose an integer a such that $8a > \delta - 5\chi_h(Y) - c_1^2(Y)$. Let $f_1, f_2 \in B(Y)^\perp$ satisfy $f_1 \cdot f_2 = 1$ and $f_i^2 = 0$. Then

for $\Lambda = 2af_1 + 2f_2$, we have $I(\Lambda) > \delta$ as required in Lemma 4.3. Note that because $\Lambda \equiv 0 \pmod{2}$, for w characteristic, the class $\Lambda - w$ is also characteristic. Since w is characteristic and $\Lambda \in B(Y)^\perp$, we have

$$\tilde{\varepsilon}(w, \Lambda, K) \equiv \varepsilon(w, K) \pmod{2}.$$

For $\Lambda \in B'(Y)^\perp$, we have $b_{i,j,k} = 0$ unless $c(Y) + i \equiv 0 \pmod{2}$ by (4.5) and hence $1 + (-1)^{c(X)+i} = 2$ in (4.24). As $K \cdot \Lambda$ and hence the coefficients $b_{i,j,k} = b_{i,j,k}(K \cdot \Lambda)$ are independent of $K \in B'(Y)$, we can write the expression for the Donaldson invariant in Lemma 4.3 as

$$D_Y^w(h^{\delta-2m}x^m) = \sum_{i+j+2k=\delta-2m} \tfrac{1}{2}b_{i,j,k}SW_{Y,i}^w(h)\langle\Lambda, h\rangle^j Q_Y(h)^k. \qquad (4.25)$$

Theorem 4.11 allows us to ignore the coefficients $b_{i,j,k}$ in (4.25) with $i \le n = c(Y) - 3$. By Proposition 4.8, we can then rewrite (4.25) as

$$D_Y^w(h^{\delta-2m}x^m)$$
$$= \sum_{i+2k=\delta-2m}\sum_{K\in B'(Y)} (-1)^{\varepsilon(w,K)} \frac{(\delta-2m)!}{2^{n+k-m}k!i!} n(K)SW_Y'(K)\langle K, h\rangle^i Q_Y(h)^k.$$

Comparing this expression for $D_Y^w(h^{\delta-2m}x^m)$ with that in (4.1) completes the proof of the theorem. $\qquad \square$

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