JEMS

Pavle V. M. Blagojević · Benjamin Matschke · Günter M. Ziegler

Optimal bounds for the colored Tverberg problem

Abstract. We prove a "Tverberg type" multiple intersection theorem. It strengthens the prime case of the original Tverberg theorem from 1966, as well as the topological Tverberg theorem of Bárány et al. (1980), by adding color constraints. It also provides an improved bound for the (topological) colored Tverberg problem of Bárány & Larman (1992) that is tight in the prime case and asymptotically optimal in the general case. The proof is based on relative equivariant obstruction theory.

Keywords. Optimal colored Tverberg theorem, Bárány–Larman conjecture, equivariant obstruction theory, chessboard complexes

1. Introduction

Tverberg's theorem from 1966 [18], [13, Sect. 8.3] states that any family of $(d+1)(r-1) + 1$ points in $\mathbb{R}^d$ can be partitioned into r sets whose convex hulls intersect; a look at the codimensions of intersections shows that the number $(d+1)(r-1) + 1$ of points is minimal for this.

In their 1990 study of halving lines and halving planes, Bárány, Füredi & Lovász [2] observed, "we need a colored version of Tverberg's theorem", and provided a first case, for three triangles in the plane. In response to this, Bárány & Larman [3] in 1992 formulated the following general problem and solved it for the planar case.

P. V. M. Blagojević: Matematički Institut SANU, Knez Mihailova 36. 11001 Beograd, Serbia, and Institut für Mathematik, FU Berlin, Arnimallee 2, 14195 Berlin, Germany; e-mail: pavleb@mi.sanu.ac.rs; blagojevic@math.fu-berlin.de

B. Matschke: Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany; e-mail: matschke@mpim-bonn.mpg.de

G. M. Ziegler: Institut für Mathematik, FU Berlin, Arnimallee 2, 14195 Berlin, Germany; e-mail: ziegler@math.fu-berlin.de

The colored Tverberg problem. *Determine the smallest number $t = t(d, r)$ such that for every collection $\mathcal{C} = C_0 \uplus \cdots \uplus C_d$ of points in $\mathbb{R}^d$ with $|C_i| \geq t$, there are r disjoint subcollections $F_1, \ldots, F_r$ of $\mathcal{C}$ satisfying*

(A) $|C_i \cap F_j| \leq 1$ *for every $i \in \{0, \ldots, d\}$, $j \in \{1, \ldots, r\}$, and*
(B) $\mathrm{conv}\,(F_1) \cap \cdots \cap \mathrm{conv}\,(F_r) \neq \emptyset$.

A family of disjoint subcollections $F_1, \ldots, F_r$ of $\mathcal{C}$ satisfying condition (A), i.e., that contain at most one point from each *color class C_i*, is called a *colored r-partition*. (We do not require $F_1 \cup \cdots \cup F_r = \mathcal{C}$ for this.) We allow color classes to be multisets of points in $\mathbb{R}^d$; in this case the cardinalities have to account for these. This convention is compatible with the phrasing of the colored Tverberg problem and its topological generalization, where one replaces the collection of points $\mathcal{C}$ in $\mathbb{R}^d$ by the images of the vertices of a $(|\mathcal{C}| - 1)$-simplex $\Delta_{|\mathcal{C}|-1}$ under an (affine resp. continuous) map to $\mathbb{R}^d$.

A colored r-partition $F_1, \ldots, F_r$ having in addition property (B) is a *colored Tverberg r-partition*.

A trivial lower bound is $t(d, r) \geq r$: Collections $\mathcal{C}$ of only $(r - 1)(d + 1)$ points in general position do not admit an intersecting r-partition, again for codimension reasons.

Bárány and Larman showed that the trivial lower bound is tight in the cases $t(1, r) = r$ and $t(2, r) = r$, presented a proof by Lovász for $t(d, 2) = 2$, and conjectured the following equality that is the main content of the colored Tverberg problem.

The Bárány–Larman conjecture. $t(d, r) = r$ *for all $r \geq 2$ and $d \geq 1$.*

Still in 1992, Živaljević & Vrećica [24] established for r prime the upper bound $t(d, r) \leq 2r - 1$. The same bound holds for prime powers according to Živaljević [23]. The bound for primes also yields bounds for arbitrary r: For example, one gets $t(d, r) \leq 4r - 3$, since there is a prime p (and certainly a prime power!) between r and $2r$.

As in the case of Tverberg's classical theorem, one can consider a topological version of the colored Tverberg problem.

The topological Tverberg theorem ([4], [14, Sect. 6.4]). *Let $r \geq 2$ be a prime power, $d \geq 1$, and $N = (d + 1)(r - 1)$. Then for every continuous map f of an N-simplex Δ_N to $\mathbb{R}^d$ there are r disjoint faces $F_1, \ldots, F_r$ of Δ_N whose images under f intersect in $\mathbb{R}^d$.*

The topological colored Tverberg problem. *Determine the smallest number $t = tt(d, r)$ such that for every simplex Δ with $(d + 1)$-colored vertex set $\mathcal{C} = C_0 \uplus \cdots \uplus C_d$, with $|C_i| \geq t$ for all i, and for every continuous map $f : \Delta \rightarrow \mathbb{R}^d$, there are r disjoint faces $F_1, \ldots, F_r$ of Δ satisfying*

(A) $|C_i \cap F_j| \leq 1$ *for every $i \in \{0, \ldots, d\}$, $j \in \{1, \ldots, r\}$, and*
(B) $f(F_1) \cap \cdots \cap f(F_r) \neq \emptyset$.

A family of faces $F_1, \ldots, F_r$ satisfying both conditions (A) and (B) is called a *topological colored Tverberg r-partition*.

The argument from [24] and [23] gives the same upper bound $tt(d, r) \leq 2r - 1$ for r a prime power, and consequently the upper bound $tt(d, r) \leq 4r - 3$ for arbitrary r. Notice that $t(d, r) \leq tt(d, r)$.

The topological Bárány–Larman conjecture. $tt(d, r) = r$ for all $r \geq 2$ and $d \geq 1$.

The Lovász proof for $t(d, 2) = 2$ presented in [3] is topological and thus also valid for the topological Bárány–Larman conjecture. Therefore $tt(d, 2) = 2$.

The general case of the topological Bárány–Larman conjecture would classically be approached via a study of the existence of an $\mathfrak{S}_r$-equivariant map

$$\Delta_{r,|C_0|} * \cdots * \Delta_{r,|C_d|} \xrightarrow[\mathfrak{S}_r]{} S(W_r^{\oplus(d+1)}) = S^{(r-1)(d+1)-1}, \qquad (1)$$

where W_r is the $(r-1)$-dimensional real standard representation of $\mathfrak{S}_r$ obtained by restricting the coordinate permutation action on $\mathbb{R}^r$ to $\{(\xi_1, \ldots, \xi_r) \in \mathbb{R}^r : \xi_1 + \cdots + \xi_r = 0\}$ and $\Delta_{r,n}$ denotes the $r \times n$ chessboard complex $([r])_{\Delta(2)}^{*n}$ (cf. [14, Remark after Thm. 6.8.2]). However, we will establish in Proposition 4.1 that this approach fails when applied to the colored Tverberg problem directly, due to the fact that the square chessboard complexes $\Delta_{r,r}$ admit $\mathfrak{S}_r$-equivariant collapses that reduce the dimension.

In the following, we circumvent this problem by a different, particular choice of parameters, which produces chessboard complexes $\Delta_{r,r-1}$ that are closed pseudomanifolds and thus do not admit collapses.

2. Statement of the main results

Our main result is the following strengthening of (the prime case of) the topological Tverberg theorem.

Theorem 2.1. *Let $r \geq 2$ be prime, $d \geq 1$, and $N := (r-1)(d+1)$. Let Δ_N be an N-dimensional simplex with a partition of its vertex set into $m+1$ parts ("color classes")*

$$\mathcal{C} = C_0 \uplus \cdots \uplus C_m,$$

with $|C_i| \leq r - 1$ for all i. Then for every continuous map $f : \Delta_N \to \mathbb{R}^d$, there is a colored r-partition given by disjoint faces $F_1, \ldots, F_r$ of Δ_N whose images under f intersect, that is,

(A) $|C_i \cap F_j| \leq 1$ *for every $i \in \{0, \ldots, m\}$, $j \in \{1, \ldots, r\}$, and*
(B) $f(F_1) \cap \cdots \cap f(F_r) \neq \emptyset$.

The requirement $|C_i| \leq r - 1$ forces there to be at least $d + 2$ nonempty color classes. Theorem 2.1 is tight in the sense that there would exist counterexamples f if $|C_0| = r$ and $|C_1| = \cdots = |C_m| = 1$.

Our first step will be to reduce Theorem 2.1 to the following special case.

Theorem 2.2. *Let $r \geq 2$ be prime, $d \geq 1$, and $N := (r-1)(d+1)$. Let Δ_N be an N-dimensional simplex with a partition of its vertex set into $d + 2$ parts,*

$$\mathcal{C} = C_0 \uplus \cdots \uplus C_d \uplus C_{d+1},$$

with $|C_i| = r - 1$ for all $i \leq d$ and $|C_{d+1}| = 1$. Then for every continuous map $f : \Delta_N \to \mathbb{R}^d$, there are r disjoint faces $F_1, \ldots, F_r$ of Δ_N satisfying

(A) $|C_i \cap F_j| \leq 1$ *for every $i \in \{0, \ldots, d+1\}$, $j \in \{1, \ldots, r\}$, and*
(B) $f(F_1) \cap \cdots \cap f(F_r) \neq \emptyset$.

Reduction of Theorem 2.1 to Theorem 2.2. Suppose that we are given a continuous map $f : \Delta_N \to \mathbb{R}^d$ and a coloring $C_0 \uplus \cdots \uplus C_m$ of the vertex set of Δ_N. Let $N' := (r-1)(m+1)$ and $C_{m+1} := \emptyset$. We enlarge the color classes C_i by adding $N' - N = (r-1)(m-d)$ new vertices and obtain new color classes $C_0', \ldots, C_{m+1}'$ such that $C_i \subseteq C_i'$ for all i, $|C_0'| = \cdots = |C_m'| = r-1$ and $|C_{m+1}'| = 1$. Using f, we construct a new map $f' : \Delta_{N'} \to \mathbb{R}^m$, as follows: We regard $\mathbb{R}^d$ as the subspace of $\mathbb{R}^m$ where the last $m-d$ coordinates are zero. So we let f' be the same as f on the N-dimensional front face of $\Delta_{N'}$. We assemble the further $N' - N$ vertices into $m-d$ groups $V_1, \ldots, V_{m-d}$ of $r-1$ vertices each. The vertices in V_i shall be mapped to e_{d+i}, the $(d+i)$th standard basis vector of $\mathbb{R}^m$. We extend this map using barycentric coordinates to all of $\Delta_{N'}$ in order to obtain f'. We apply Theorem 2.2 to f' and the coloring $C_0', \ldots, C_{m+1}'$ and obtain disjoint faces $F_1', \ldots, F_r'$ of $\Delta_{N'}$. Let $F_i := F_i' \cap \Delta_N$ be the intersection of F_i' with the N-dimensional front face of $\Delta_{N'}$. By construction of f', the nonempty intersection $f'(F_1') \cap \cdots \cap f'(F_r')$ lies in $\mathbb{R}^d$. Therefore, already $F_1, \ldots, F_r$ is a topological colored Tverberg r-partition for f', and hence it is also a topological colored Tverberg r-partition for f: we have $f(F_1) \cap \cdots \cap f(F_r) \neq \emptyset$.   □

Such a reduction previously appeared in Sarkaria's proof for the prime power Tverberg theorem [17, (2.7.3)]; see also de Longueville's exposition [11, Prop. 2.5].

Either of our Theorems 2.1 and 2.2 immediately implies the topological Tverberg theorem for the case when r is a prime, as any colored Tverberg partition, as provided by Theorems 2.1 and 2.2, is, in particular, a Tverberg partition (if one ignores the color constraints). Thus Theorems 2.1 and 2.2 are "constrained" Tverberg theorems as recently discussed by Hell [9].

More importantly, however, Theorem 2.2 implies the topological Bárány–Larman conjecture for the case when $r+1$ is a prime, as follows.

Corollary 2.3. *If $r+1$ is prime, then $t(d,r) = tt(d,r) = r$.*

Proof. We prove that if $r \geq 3$ is prime, then $tt(d, r-1) \leq r-1$. For this, let Δ_{N-1} be a simplex where $N = (r-1)(d+1)$ and with vertex set $\mathcal{C} = C_0 \uplus \cdots \uplus C_d$, $|C_i| = r-1$ for all i, and let $f : \Delta_{N-1} \to \mathbb{R}^d$ be continuous. Extend this to a map $\Delta_N \to \mathbb{R}^d$, where Δ_N has an extra vertex v_N, and set $C_{d+1} := \{v_N\}$. Then Theorem 2.2 can be applied, and yields a topological colored Tverberg r-partition. Ignore the part that contains v_N.   □

Using estimates on prime numbers one can derive from this tight bounds for the colored Tverberg problem also in the general case. The classical Bertrand postulate ("For every r there is a prime p with $r+1 \leq p < 2r$") can be used here, but there are also much stronger estimates available, such as the existence of a prime p between r and $r + r^{6/11+\varepsilon}$ for arbitrary $\varepsilon > 0$ if r is large enough, according to Lou & Yao [12].

Corollary 2.4. (i) $r \leq t(d,r) \leq tt(d,r) \leq 2r-2$ *for all $d \geq 1$ and $r \geq 2$.*
(ii) $r \leq t(d,r) \leq tt(d,r) \leq (1+o(1))r$ *for $d \geq 1$ and $r \to \infty$.*

Proof. The first, explicit estimate is obtained from Bertrand's postulate: For any given r there is a prime p with $r+1 \leq p < 2r$. We use $|C_i| \geq 2r-2 \geq p-1$ to derive the

existence of a topological colored Tverberg $(p-1)$-partition, which in particular yields an r-partition since $p-1 \geq r$.

The second, asymptotic estimate uses the Lou & Yao bound instead.   □

Remark 2.5. The colored Tverberg problem as originally posed by Bárány & Larman [3] in 1992 was different from the version we have given above (following Bárány, Füredi & Lovász [2] and Živaljević & Vrećica [24]): Bárány and Larman had asked for an upper bound $N(d,r)$ on the cardinality of the union $|\mathcal{C}|$ that together with $|C_i| \geq r$ would force the existence of a colored Tverberg r-partition. This original formulation has two major disadvantages: One is that the Živaljević–Vrećica result does not apply to it. A second one is that it does not lend itself to estimates for the general case in terms of the prime case.

However, our Corollary 2.3 also solves the original version for the case when $r+1$ is a prime.

The colored Tverberg problem originally arose as a tool to obtain complexity bounds in computational geometry. As a consequence, our new bounds can be applied to improve these bounds, as follows. Note that in some of these results $t(d, d+1)^d$ appears in the exponent, so even slightly improved estimates on $t(d, d+1)$ have considerable effect. For surveys see [1], [13, Sect. 9.2], and [22, Sect. 11.4.2].

Let $S \subseteq \mathbb{R}^d$ be a set of size n in general position, that is, such that no $d+1$ points of S lie on a hyperplane. Let $h_d(n)$ denote the number of hyperplanes that bisect the set S and are spanned by the elements of the set S. According to Bárány [1, p. 239],

$$h_d(n) = O(n^{d-\varepsilon_d}) \quad \text{with} \quad \varepsilon_d = t(d, d+1)^{-(d+1)}.$$

Thus we obtain the following bound and equality.

Corollary 2.6. *If $d+2$ is a prime then*

$$h_d(n) = O(n^{d-\varepsilon_d}) \quad \text{with} \quad \varepsilon_d = (d+1)^{-(d+1)}.$$

For general d, we obtain e.g. $\varepsilon_d \geq (d+1)^{-(d+1)-O(\log d)}$.

Let $\mathcal{C} \subseteq \mathbb{R}^d$ be a finite set. A $\mathcal{C}$-*simplex* is the convex hull of some collection of $d+1$ points of $\mathcal{C}$. The second selection lemma [13, Thm. 9.2.1] claims that for an n-point set $\mathcal{C} \subseteq \mathbb{R}^d$ and the family $\mathcal{F}$ of $\alpha \binom{n}{d+1}$ $\mathcal{C}$-simplices with $\alpha \in (0, 1]$ there exists a point contained in at least $c \cdot \alpha^{s_d} \binom{n}{d+1}$ $\mathcal{C}$-simplices of $\mathcal{F}$. Here $c = c(d) > 0$ and s_d are constants. For dimensions $d > 2$, the presently known proof gives $s_d \approx t(d, d+1)^{d+1}$. Again, Corollary 2.4 yields the following, much better bounds for the constant s_d.

Corollary 2.7. *If $d+2 > 4$ is a prime then the second selection lemma holds for $s_d = (d+1)^{d+1}$, and in general e.g. for $s_d = (2d+2)^{d+1}$.*

Let $X \subset \mathbb{R}^d$ be an n-element set. A k-*facet* of X is an oriented $(d-1)$-simplex conv$\{x_1, \ldots, x_d\}$ spanned by elements of X such that there are exactly k points of X on its strictly positive side. When $n - d$ is even, $(\frac{n-d}{2})$-facets of X are called *halving facets*. From [13, Thm. 11.3.3] we have a new, better estimate for the number of halving facets.

Corollary 2.8. *For $d > 2$ and $n - d$ even, the number of halving facets of an n-set $X \subset \mathbb{R}^d$ is $O(n^{d-1/(2d)^d})$.*

3. The Configuration Space/Test Map scheme

According to the "deleted joins" version of the general "Configuration Space/Test Map" (CS/TM) scheme for multiple intersection problems, as pioneered by Sarkaria, Vrećica & Živaljević, and others, formalized by Živaljević, and beautifully presented by Matoušek [14, Chap. 6], we proceed as follows.

Assume that we want to prove the existence of a topological colored Tverberg r-partition for an arbitrary colored point set $\mathcal{C} = C_0 \uplus C_1 \uplus \cdots \uplus C_k$ in $\mathbb{R}^d$ with $|C_i| = t_i$. So we have to rule out the existence of a (continuous or affine) map

$$f : C_0 * C_1 * \cdots * C_k \to \mathbb{R}^d$$

for which any r images of disjoint simplices from the simplicial complex (join of discrete sets) $C_0 * C_1 * \cdots * C_k$ have empty intersection in $\mathbb{R}^d$ (cf. Živaljević [22, Sect. 11.4.2]).

The "deleted joins" configuration space/test map scheme now suggests taking an r-fold deleted join of this map f, where one has to take an r-fold 2-wise deleted join in the domain and an r-fold r-wise deleted join in the range (cf. [14, Sect. 6.3]):

$$f_{\Delta(2)}^{*r} : (C_0 * C_1 * \cdots * C_k)_{\Delta(2)}^{*r} \xrightarrow[\mathfrak{S}_r]{} (\mathbb{R}^d)_{\Delta}^{*r}.$$

As the join and deleted join operations for simplicial complexes commute [14, Lemma 6.5.3], we get the sequence of isomorphisms of simplicial complexes

$$(C_0 * C_1 * \cdots * C_k)_{\Delta(2)}^{*r} \cong (C_0)_{\Delta(2)}^{*r} * (C_1)_{\Delta(2)}^{*r} * \cdots * (C_k)_{\Delta(2)}^{*r}$$
$$\cong \Delta_{|C_0|,r} * \Delta_{|C_1|,r} * \cdots * \Delta_{|C_k|,r}, \tag{2}$$

where $\Delta_{r,|C_i|} = (C_i)_{\Delta(2)}^{*r}$ is the chessboard complex on r rows and $|C_i|$ columns, on which $\mathfrak{S}_r$ acts by permuting the r rows. Thus we arrive at an $\mathfrak{S}_r$-equivariant map

$$f_{\Delta(2)}^{*r} : \Delta_{r,|C_0|} * \Delta_{r,|C_1|} * \cdots * \Delta_{r,|C_k|} \xrightarrow[\mathfrak{S}_r]{} (\mathbb{R}^d)_{\Delta}^{*r} \subset \mathbb{R}^{r \times (d+1)} \backslash T \simeq S(W_r^{\oplus(d+1)}). \tag{3}$$

Here make the following observations.

(i) The simplicial complex $X := (C_0 * C_1 * \cdots * C_k)_{\Delta(2)}^{*r} \cong \Delta_{r,|C_0|} * \Delta_{r,|C_1|} * \cdots * \Delta_{r,|C_k|}$ on the left hand side is an $\mathfrak{S}_r$-simplicial complex on $r(|C_0| + |C_1| + \cdots + |C_k|)$ vertices, of dimension $|C_0| + |C_1| + \cdots + |C_k| - 1$ if $|C_i| \leq r$ for every i, and of dimension $\min\{|C_0|, r\} + \min\{|C_1|, r\} + \cdots + \min\{|C_k|, r\} - 1$ in general.

Points in X can be represented as convex combinations $\lambda_1 x_1 + \cdots + \lambda_r x_r$, where x_i is a point in (a simplex of) the i-th "join component" of the iterated deleted join $(C_0 * C_1 * \cdots * C_k)_{\Delta(2)}^{*r}$, with $\lambda_i \geq 0$ for all i and $\sum_i \lambda_i = 1$.

(ii) $(\mathbb{R}^d)_{\Delta}^{*r} := \{\alpha_1 y_1 + \cdots + \alpha_r y_r \in (\mathbb{R}^d)^{*r} : \alpha_i \geq 0, \sum_i \alpha_i = 1\} \backslash \{\frac{1}{r} y + \cdots + \frac{1}{r} y : y \in \mathbb{R}^d\}$ is a deleted join, which $\mathfrak{S}_r$-equivariantly embeds into the space of all real $r \times (d+1)$-matrices for which not all rows are equal, and where $\mathfrak{S}_r$ acts by permuting the rows. The diagonal T is the $(d+1)$-dimensional subspace of all matrices for which all rows are equal. To project on the orthogonal complement of the diagonal T we subtract from each row the average of all the rows. This operation yields an $\mathfrak{S}_r$-equivariant

orthogonal projection to $W_r^{\oplus(d+1)}\backslash\{0\}$, the space of all real $r \times (d+1)$-matrices with column sums equal to zero but for which not all rows are zero, and where $\mathfrak{S}_r$ still acts by permuting the rows. This in turn is homotopy equivalent to the sphere $S(W_r^{\oplus(d+1)}) = (S^{r-2})^{*(d+1)} = S^{(r-1)(d+1)-1} = S^{N-1}$, where $\pi \in \mathfrak{S}_r$ reverses the orientation exactly if $(\operatorname{sgn}\pi)^{d+1}$ is negative.

(iii) The action of $\mathfrak{S}_r$ is nonfree exactly on the subcomplex $A \subset X = (C_0 * C_1 * \cdots * C_k)_{\Delta(2)}^{*r}$ given by all the points $\lambda_1 x_1 + \cdots + \lambda_r x_r \in (C_0 * C_1 * \cdots * C_k)_{\Delta(2)}^{*r}$ such that $\lambda_i = \lambda_j = 0$ for two distinct indices $i < j$. These lie in simplices that have no vertices in the i-th and j-th "join component" of the iterated deleted join $(C_0 * C_1 * \cdots * C_k)_{\Delta(2)}^{*r}$, so the transposition $\pi_{ij} := (ij) = \left(\begin{smallmatrix}\cdots i \cdots j \cdots\\ \cdots j \cdots i \cdots\end{smallmatrix}\right)$ fixes these simplices pointwise.

(iv) The map $f_{\Delta(2)}^{*r} : X \to \mathbb{R}^{r \times (d+1)}$ suggested by the "deleted joins" scheme takes the point $\lambda_1 x_1 + \cdots + \lambda_r x_r$ and maps it to the $r \times (d+1)$-matrix in $\mathbb{R}^{r \times (d+1)}$ whose ℓ-th row is $(\lambda_\ell, \lambda_\ell f(x_\ell))$. For an arbitrary map f, the image of A under $f_{\Delta(2)}^{*r}$ does not intersect the diagonal T: if $\lambda_i = \lambda_j = 0$, then not all rows $(\lambda_\ell, \lambda_\ell f(x_\ell))$ can be equal, since $\sum_\ell \lambda_\ell = 1$.

However, for the following we replace $f_{\Delta(2)}^{*r}$ by the map $F_0 : X \to \mathbb{R}^{r \times (d+1)}$ that maps $\lambda_1 x_1 + \cdots + \lambda_r x_r$ to the $r \times (d+1)$-matrix whose ℓ-th row is $(\lambda_\ell, (\prod_{h=1}^r \lambda_h) f(x_\ell))$. The two maps $f_{\Delta(2)}^{*r}$ and F_0 are homotopic as maps $A \to \mathbb{R}^{r \times (d+1)} \setminus T$ by a linear homotopy, so the resulting extension problems are equivalent by [8, Prop. 3.15(ii)]. The advantage of the map F_0 is that its restriction to A is independent of f. Indeed, for $\lambda_1 x_1 + \cdots + \lambda_r x_r \in A$ and any map f the corresponding F_0-image is the $r \times (d+1)$-matrix whose ℓ-th row is $(\lambda_\ell, 0)$.

Thus we have established the following.

Proposition 3.1 (CS/TM scheme for the generalized topological colored Tverberg problem). *If for some parameters $(d, r, k; t_0, \ldots, t_k)$ no $\mathfrak{S}_r$-equivariant extension (3) of the map $F_0|_A : A \to \mathbb{R}^{r \times (d+1)}\backslash T$ exists, then for each continuous $f : C_0 * C_1 * \cdots * C_k \to \mathbb{R}^d$ with $|C_i| \geq t_i$ for all i there exists a topological colored Tverberg r-partition.*

Živaljević & Vrećica [24] achieved this for $(d, r, d; 2r-1, \ldots, 2r-1)$ and prime r by applying a Borsuk–Ulam type theorem to the action of the cyclic subgroup $\mathbb{Z}_r \subset \mathfrak{S}_r$, which acts freely on the join of chessboard complexes if r is a prime. However, they lose a factor of 2 from the fact that the chessboard complexes $\Delta_{r,t}$, for $r \leq t$, of dimension $r-1$ are homologically $(r-2)$-connected only if $t \geq 2r-1$ (compare [5], [20], and [16]).

Our Theorem 2.2 claims this for $(d, r, d+1; r-1, \ldots, r-1, 1)$. To prove it, we will use relative equivariant obstruction theory, as presented by tom Dieck in [8, Sect. II.3].

4. Proof of Theorem 2.2

First we establish that the scheme of Proposition 3.1 fails when applied to the colored Tverberg problem $(d, r, d; r, \ldots, r)$ associated with the Bárány–Larman conjecture directly.

Proposition 4.1. *For all $r \geq 2$ and $d \geq 1$, with $N = (r - 1)(d + 1)$, there exists an $\mathfrak{S}_r$-equivariant map*

$$F : (\Delta_{r,r})^{*(d+1)} \xrightarrow[\mathfrak{S}_r]{} W_r^{\oplus(d+1)} \setminus \{0\} \simeq S^{N-1}.$$

Proof. Let $M := r(d + 1) - 1$, and let Δ_M be an M-dimensional simplex whose vertex set $\mathcal{C} = C_0 \uplus C_1 \uplus \cdots \uplus C_d$ is colored by $d + 1$ colors such that $|C_i| = r$ for every i. For an arbitrary continuous map $f : \Delta_M \to \mathbb{R}^d$ the "deleted join" configuration space/test map scheme, as in (3), induces an $\mathfrak{S}_r$-equivariant map F:

$$(C_0 * C_1 * \cdots * C_d)_{\Delta(2)}^{*r} \cong (\Delta_{r,r})^{*(d+1)} \xrightarrow[\mathfrak{S}_r]{} \mathbb{R}^{r \times (d+1)} \xrightarrow[\mathfrak{S}_r]{} W_r^{\oplus(d+1)},$$

where the second map is the projection on the orthogonal complement of the diagonal.

The $\mathfrak{S}_r$-action on the configuration space $(\Delta_{r,r})^{*(d+1)}$ is not free; let A denote the subcomplex of $(\Delta_{r,r})^{*(d+1)}$ on which $\mathfrak{S}_r$ does not act freely. As we have seen in Section 3, item (iv), the F-image of A avoids the origin in $W_r^{\oplus(d+1)}$.

For any facet of the $(r - 1)$-dimensional chessboard complex $\Delta_{r,r}$ there is an elementary collapse which removes the facet together with its subfacet (ridge of the chessboard complex) obtained by deleting the vertex in the r-th column. Performing these collapses simultaneously, we see that $\Delta_{r,r}$ collapses $\mathfrak{S}_r$-equivariantly to an $(r - 2)$-dimensional subcomplex of $\Delta_{r,r}$, and thus $(\Delta_{r,r})^{*(d+1)}$ equivariantly retracts to a subcomplex $X \subset (\Delta_{r,r})^{*(d+1)}$ whose dimension is only $(d + 1)(r - 1) - 1 = N - 1$. Now it is enough to construct an $\mathfrak{S}_r$-equivariant map $X \to_{\mathfrak{S}_r} S(W_r^{\oplus(d+1)}) = S^{N-1}$.

The $\mathfrak{S}_r$-action on X is also not free. The subcomplex of X on which $\mathfrak{S}_r$ does not act freely is $X \cap A$. Since $\dim X = \dim S^{N-1}$, S^{N-1} is $(N - 1)$-simple and S^{N-1} is $(N - 2)$-connected, by relative equivariant obstruction theory, there is no obstruction to the existence of an $\mathfrak{S}_r$-equivariant map $X \to_{\mathfrak{S}_r} S^{N-1}$ provided that an $\mathfrak{S}_r$-equivariant map $X \cap A \to_{\mathfrak{S}_r} S^{N-1}$ on the nonfree part of the domain can be exhibited.

Since the F-image of A avoids the origin in $W_r^{\oplus(d+1)}$, the restriction $F|_{X \cap A}$ composed with the $\mathfrak{S}_r$-equivariant radial projection to the sphere induces the required $\mathfrak{S}_r$-equivariant map $X \cap A \to_{\mathfrak{S}_r} W_r^{\oplus(d+1)} \setminus \{0\} \to_{\mathfrak{S}_r} S^{N-1}$.   □

We now specialize the general scheme of Proposition 3.1 to the situation of Theorem 2.2. Let $[n] := \{1, \ldots, n\}$ denote the 0-dimensional simplicial complex on n vertices. Then we have to show the following.

Proposition 4.2. *Let $r \geq 2$ and $d \geq 1$ be integers, and $N = (r - 1)(d + 1)$. There exists an $\mathfrak{S}_r$-equivariant map*

$$F : (\Delta_{r,r-1})^{*d} * \Delta_{r,r-1} * [r] \xrightarrow[\mathfrak{S}_r]{} W_r^{\oplus(d+1)} \setminus \{0\}$$

that extends the equivariant map $F_0|_A$ from Section 3, item (iv), if and only if

$$r \mid (r - 1)!^d.$$
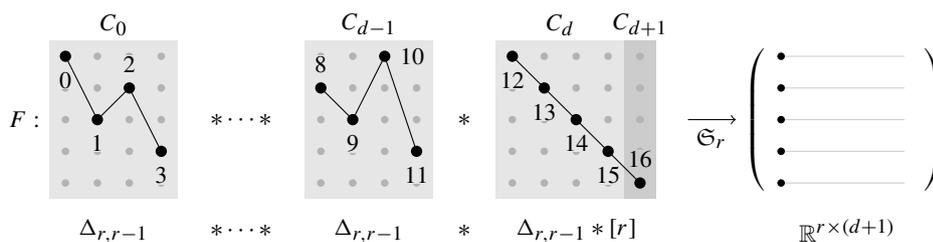
Fig. 1. The vertex set, and one facet in Φ of the combinatorial configuration space for $r = 5$.

The vertex set of the join $(\Delta_{r,r-1})^{*d} * \Delta_{r,r-1} * [r]$ may be represented by a rectangular array of size $r \times ((r-1)(d+1)+1)$, which carries the $d+1$ chessboard complexes $\Delta_{r,r-1}$ lined up from left to right, and in the last column has the chessboard complex $\Delta_{r,1} = [r]$, which is just a discrete set. (See Figure 1.)

The join of chessboard complexes $(\Delta_{r,r-1})^{*d} * \Delta_{r,r-1} * [r]$ has dimension $(r-1)(d+1) = N$, while the target sphere has dimension $N-1$. On both of them, $\mathfrak{S}_r$ acts by permuting the rows.

While the chessboard complexes $\Delta_{r,r}$ collapse equivariantly to lower-dimensional complexes, the chessboard complexes $\Delta_{r,r-1}$ are closed oriented pseudomanifolds of dimension $r-2$ and thus do not collapse; for example, $\Delta_{3,2}$ is a circle and $\Delta_{4,3}$ is a torus. We will read the maximal simplices of such a complex from left to right, which yields the orientation cycle in a special form with few signs that will be very convenient.

Lemma 4.3 (cf. [5], [16], [10, p. 145]). *For $r > 2$, the chessboard complex $\Delta_{r,r-1}$ is a connected, orientable pseudomanifold of dimension $r - 2$. Therefore*

$$H_{r-2}(\Delta_{r,r-1}; \mathbb{Z}) = \mathbb{Z}$$

and an orientation cycle is

$$z_{r,r-1} = \sum_{\pi \in \mathfrak{S}_r} (\operatorname{sgn} \pi) \langle (\pi(1), 1), \ldots, (\pi(r-1), r-1) \rangle. \tag{4}$$

$\mathfrak{S}_r$ *acts on* $\Delta_{r,r-1}$ *by permuting the rows; this affects the orientation according to* $\pi \cdot z_{r,r-1} = (\operatorname{sgn} \pi) z_{r,r-1}$.

Proof of Proposition 4.2. For $r = 2$, since $2 \nmid 1$, we have to prove that there is no equivariant map $S^N \to S^{N-1}$, where both spheres are equipped with the antipodal action: This is the Borsuk–Ulam theorem (and the Lovász proof). Thus we may now assume that $r \geq 3$.

Let $X := (\Delta_{r,r-1})^{*(d+1)} * [r]$ be our combinatorial configuration space, $A \subset X$ the nonfree subcomplex, and $F_0 : A \to_{\mathfrak{S}_r} S(W_r^{\oplus(d+1)})$ the prescribed map that we are to extend $\mathfrak{S}_r$-equivariantly to X.

Since

- $\dim X = N$ and $\dim S(W_r^{\oplus(d+1)}) = N - 1$, with
- $\operatorname{conn} S(W_r^{\oplus(r+1)}) = N - 2$ and
- $S(W_r^{\oplus(r+1)})$ being $(N-2)$-simple,

by [8, Sect. II.3] the existence of an $\mathfrak{S}_r$-equivariant extension of the map $F_0 : A \to_{\mathfrak{S}_r} S(W_r^{\oplus(d+1)})$ to a $\mathfrak{S}_r$-equivariant map $X \to_{\mathfrak{S}_r} S(W_r^{\oplus(d+1)})$ is equivalent to the vanishing of the primary obstruction

$$\mathfrak{o} \in H^N_{\mathfrak{S}_r}\big(X, A; \pi_{N-1}(S(W_r^{\oplus(d+1)}))\big).$$

The Hurewicz isomorphism gives an isomorphism of the coefficient $\mathfrak{S}_r$-module with a homology group,

$$\pi_{N-1}(S(W_r^{\oplus(r+1)})) \;\cong\; H_{N-1}(S(W_r^{\oplus(r+1)}); \mathbb{Z}) =: \mathcal{Z}.$$

As an abelian group, this module $\mathcal{Z} = \langle \zeta \rangle$ is isomorphic to $\mathbb{Z}$. The action of the permutation $\pi \in \mathfrak{S}_r$ on the module $\mathcal{Z}$ is given by

$$\pi \cdot \zeta = (\operatorname{sgn} \pi)^{d+1} \zeta.$$

Computing the obstruction cocycle. We will now compute an obstruction cocycle $\mathfrak{c}_f$ in the cochain group $C^N_{\mathfrak{S}_r}(X, A; \mathcal{Z})$. Then we show that the cocycle $\mathfrak{c}_f$ is not a coboundary (that is, it does not vanish when passing to $\mathfrak{o} = [\mathfrak{c}_f]$ in the cohomology group $H^N_{\mathfrak{S}_r}(X, A; \mathcal{Z})$) if and only if $r \nmid (r-1)!^d$.

For this, we use a specific general position map $f : \Delta_N \to \mathbb{R}^d$, which induces a map $F : X \to \mathbb{R}^{r \times (d+1)}$; the value of the obstruction cocycle $\mathfrak{c}_f$ on an oriented maximal simplex σ of X is then given by the signed intersection number of $F(\sigma)$ with the test space, the diagonal T, or in other words by the mapping degree $\deg(F|_{\partial\sigma} : \partial\sigma \to S(W_r^{\oplus(d+1)}))$ (cf. [8] and [6]).

Let $e_1, \ldots, e_d$ be the standard basis vectors of $\mathbb{R}^d$, set $e_0 := 0 \in \mathbb{R}^d$, and denote by $v_0, \ldots, v_N$ the set of vertices of the N-simplex Δ_N in the given order, that is, such that $C_i = \{v_{i(r-1)}, \ldots, v_{(i+1)(r-1)-1}\}$ for $i \le d$ and $C_{d+1} = \{v_{(d+1)(r-1)}\}$. Let $f : \Delta_N \to \mathbb{R}^d$ be the affine map defined on the vertices by

$$v_i \overset{f}{\longmapsto} e_{\lfloor i/(r-1) \rfloor} \quad \text{for } 0 \le i \le N-1, \qquad v_N \overset{f}{\longmapsto} \frac{1}{d+1} \sum_{i=0}^{d} e_i,$$

that is, the vertices in C_i are mapped to the vertex e_i of the standard d-simplex for $i \le d$, while $v_N \in C_{d+1}$ is mapped to the center of this simplex.

This induces an affine map $f : C_0 * C_1 * \cdots * C_{d+1} \to \mathbb{R}^d$ and thus an equivariant map $F : X \to \mathbb{R}^{r \times (d+1)}$, taking $\lambda_1 x_1 + \cdots + \lambda_r x_r$ to the $r \times (d+1)$-matrix whose ℓ-th row is $(\lambda_\ell, (\prod_{h=1}^{r} \lambda_h) x_\ell)$, which extends the prescribed map $F_0 : A \to \mathbb{R}^{r \times (d+1)} \setminus T$. The intersection points of the image of F with the diagonal T correspond to the topological colored Tverberg r-partitions of the configuration $\mathcal{C} = C_0 \uplus \cdots \uplus C_{d+1}$ in $\mathbb{R}^d$. Since $\lambda_1 = \cdots = \lambda_r = 1/r$ at all these intersection points, we find that F is in general position with respect to T.

The only Tverberg r-partitions of the point configuration $\mathcal{C}$ (even ignoring colors) are given by $r-1$ d-simplices with its vertices at $e_0, e_1, \ldots, e_d$, together with one singleton point (0-simplex) at the center. Clearly there are $(r-1)!^d$ such partitions.
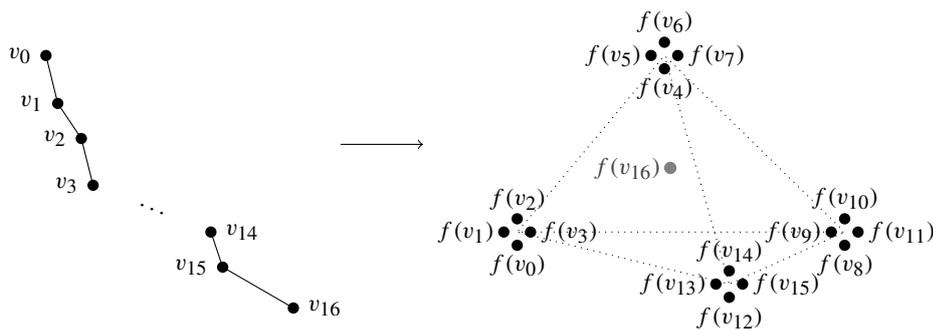
Fig. 2. The map $f : \Delta^{16} \to \mathbb{R}^3$ in the case $d = 3$ and $r = 5$.

We take representatives for the $\mathfrak{S}_r$-orbits of maximal simplices of X such that from the last $\Delta_{r,r-1}$ factor, the vertices $(1, 1), \ldots, (r - 1, r - 1)$ are taken.

On the simplices of X we use the orientation that is induced by ordering all vertices left-to-right on the array of Figure 1. This orientation is $\mathfrak{S}_r$-invariant, as permutation of the rows does not affect the left-to-right ordering.

The obstruction cocycle evaluated on subcomplexes of X. Let us consider the following chains of dimensions N resp. $N - 1$ (illustrated in Figure 3), where $z_{r,r-1}$ denotes the orientation cycle for the chessboard complex $\Delta_{r,r-1}$, as given by Lemma 4.3:

$$\Phi = (z_{r,r-1})^{*d} * \langle (1, 1), \ldots, \ldots, \ldots, (r - 1, r - 1), (r, r) \rangle,$$

$$\Omega_j = (z_{r,r-1})^{*d} * \langle (1, 1), \ldots, \ldots, \ldots, (r - 1, r - 1), (j, r) \rangle \qquad (1 \le j < r),$$

$$\Theta_i = (z_{r,r-1})^{*d} * \langle (1, 1), \ldots, \widehat{(i, i)}, \ldots, (r - 1, r - 1), (r, r) \rangle \qquad (1 \le i \le r),$$

$$\Theta_{i,j} = (z_{r,r-1})^{*d} * \langle (1, 1), \ldots, \widehat{(i, i)}, \ldots, (r - 1, r - 1), (j, r) \rangle \qquad (1 \le i, j < r).$$

Here we use the usual notation $\langle w_0, \ldots, \widehat{w_i}, \ldots, w_k \rangle$ for an oriented simplex with ordered vertex set $(w_0, \ldots, \widehat{w_i}, \ldots, w_k)$ where the vertex w_i is omitted.

Explicitly the signs in these chains are as follows. If σ denotes the facet $\langle (1, 1), \ldots, (r - 1, r - 1) \rangle$ of $\Delta_{r,r-1}$ such that $\pi\sigma = \langle (\pi(1), 1), \ldots, (\pi(r - 1), r - 1) \rangle$, then Φ is given by

$$\Phi = \sum_{\pi_1, \ldots, \pi_d \in \mathfrak{S}_r} (\text{sgn}\,\pi_1) \cdots (\text{sgn}\,\pi_d)\, \pi_1\sigma * \cdots * \pi_d\sigma * \langle (1, 1), \ldots, (r - 1, r - 1), (r, r) \rangle$$

and similarly for Ω_j, Θ_i, and $\Theta_{i,j}$.

The evaluation of $\mathfrak{c}_f$ on Φ picks out the facets that correspond to topological colored Tverberg r-partitions: Since the last part of the partition must be the singleton vertex v_N, we find that the last rows of the chessboard complex $\Delta_{r,r-1}$ factors are not used. We may define the orientation on $S(W_r^{\oplus(d+1)})$ such that

$$\mathfrak{c}_f(\sigma * \cdots * \sigma * \langle (1, 1), \ldots, (r - 1, r - 1), (r, r) \rangle) = +\zeta.$$

$$\Phi = (\Delta_{r,r-1})^{*d} \quad * \quad$$

$$\Omega_j = (\Delta_{r,r-1})^{*d} \quad * \quad \longleftarrow j$$

$$\Theta_i = (\Delta_{r,r-1})^{*d} \quad * \quad \longleftarrow i$$

$$\Theta_{i,j} = (\Delta_{r,r-1})^{*d} \quad * \quad \longleftarrow j \atop \longleftarrow i$$

$$\Theta_{j,j} = (\Delta_{r,r-1})^{*d} \quad * \quad \longleftarrow j$$
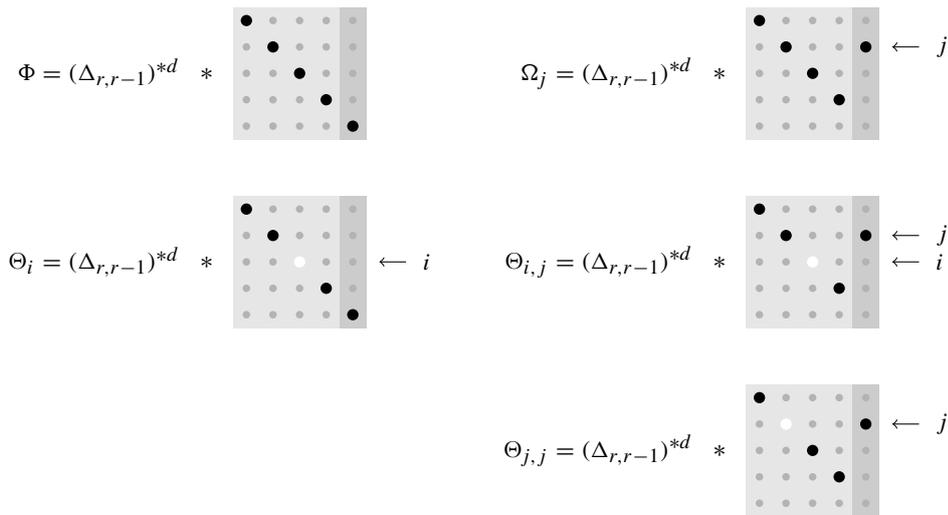
Fig. 3. Schemes for the combinatorics of the chains Φ, Ω_j, Θ_i, and $\Theta_{i,j}$.

Then we get

$$\mathfrak{c}_f\big(\pi_1\sigma * \cdots * \pi_d\sigma * \langle(1,1), \ldots, (r-1, r-1), (r,r)\rangle\big)$$
$$= \begin{cases} (\operatorname{sgn} \pi_1) \cdots (\operatorname{sgn} \pi_d)\zeta & \text{if } \pi_1(r) = \cdots = \pi_d(r) = r, \\ 0 & \text{otherwise.} \end{cases}$$

The sign $(\operatorname{sgn} \pi_1) \cdots (\operatorname{sgn} \pi_d)$ comes from the fact that F maps $\sigma * \cdots * \sigma * \langle(1,1), \ldots, (r-1, r-1), (r,r)\rangle$ and $\pi_1\sigma * \cdots * \pi_d\sigma * \langle(1,1), \ldots, (r-1, r-1), (r,r)\rangle$ to the same simplex in $W_r^{\oplus(d+1)}$, however with a different order of the vertices. Thus,

$$\mathfrak{c}_f(\Phi) = (r-1)!^d\, \zeta.$$

Furthermore, for any topological colored Tverberg r-partition in our configuration the last point v_N has to be a singleton, while the facets of Ω_j correspond to colored r-partitions where the j-th face pairs v_N with a point in C_d. Thus the cochains Ω_j do not capture any Tverberg partitions, and we get

$$\mathfrak{c}_f(\Omega_j) = 0 \quad \text{for } 1 \le j < r.$$

Is the cocycle $\mathfrak{c}_f$ a coboundary? Let us assume that $\mathfrak{c}_f$ is a coboundary. Then there is an equivariant cochain $\mathfrak{h} \in C_{\mathfrak{S}_r}^{N-1}(X, A; \mathcal{Z})$ such that $\mathfrak{c}_f = \delta\mathfrak{h}$, where δ is the coboundary operator.

In order to simplify the notation, from now on we drop the join factor $(\Delta_{r,r-1})^{*d}$ from the notation of the subcomplexes Φ, Θ_i and Ω_i. Note that the join with this complex accounts for a global sign of $(-1)^{d(r-1)}$ in the boundary/coboundary operators, since in our vertex ordering the complex $(\Delta_{r,r-1})^{*d}$, whose facets have $d(r-1)$ vertices, comes first.

Thus we have

$$\partial \Phi = (-1)^{d(r-1)} \sum_{i=1}^{r} (-1)^{i-1} \Theta_i$$

and similarly for $1 \le j < r$,

$$\partial \Omega_j = (-1)^{d(r-1)} \left(\sum_{i=1}^{r-1} (-1)^{i-1} \Theta_{i,j} + (-1)^{r-1} \Theta_r \right).$$

Claim 1. *For $1 \le i, j < r$, $i \ne j$ we have $\mathfrak{h}(\Theta_{i,j}) = 0$.*

Proof. We consider the effect of the transposition $\pi_{ir} := (ir) = \left(\begin{smallmatrix} \dots i \dots r \dots \\ \dots r \dots i \dots \end{smallmatrix} \right)$. The simplex

$$\langle (1,1), \dots, \widehat{(i,i)}, \dots, (r-1, r-1), (j, r) \rangle$$

has no vertex in the i-th and in the r-th row, so it is fixed by π_{ir}. The d chessboard complexes in $\Theta_{i,j}$ are invariant but change orientation under the action of π_{ir}, so the effect on the chain $\Theta_{i,j}$ is $\pi_{ir} \cdot \Theta_{i,j} = (-1)^d \Theta_{i,j}$ and hence

$$\mathfrak{h}(\pi_{ir} \cdot \Theta_{i,j}) = \mathfrak{h}((-1)^d \Theta_{i,j}) = (-1)^d \mathfrak{h}(\Theta_{i,j}).$$

On the other hand, $\mathfrak{h}$ is equivariant, so

$$\mathfrak{h}(\pi_{ir} \cdot \Theta_{i,j}) = \pi_{ir} \cdot \mathfrak{h}(\Theta_{i,j}) = (-1)^{d+1} \mathfrak{h}(\Theta_{i,j})$$

since $\mathfrak{S}_r$ acts on $\mathcal{Z}$ by multiplication with $(\operatorname{sgn} \pi)^{d+1}$. Comparing the two evaluations of $\mathfrak{h}(\pi_{ir} \cdot \Theta_{i,j})$ yields $(-1)^d \mathfrak{h}(\Theta_{i,j}) = (-1)^{d+1} \mathfrak{h}(\Theta_{i,j})$. □

Claim 2. *For $1 \le j < r$ we have $\mathfrak{h}(\Theta_{j,j}) = -\mathfrak{h}(\Theta_j)$.*

Proof. The interchange of the j-th row with the r-th moves $\Theta_{j,j}$ to Θ_j, where we have to account for d orientation changes for the chessboard join factors.

Thus $\pi_{jr} \Theta_{j,j} = (-1)^d \Theta_j$, which yields

$$(-1)^d \mathfrak{h}(\Theta_j) = \mathfrak{h}((-1)^d \Theta_j) = \mathfrak{h}(\pi_{jr} \Theta_{j,j}) = \pi_{jr} \cdot \mathfrak{h}(\Theta_{j,j}) = (-1)^{d+1} \mathfrak{h}(\Theta_{j,j}). \quad \square$$

We now use the two claims to evaluate $\mathfrak{h}(\partial \Omega_j)$. Thus we obtain

$$0 = \mathfrak{c}_f(\Omega_j) = \delta\mathfrak{h}(\Omega_j) = \mathfrak{h}(\partial \Omega_j) = (-1)^{d(r-1)} \big((-1)^{j-1} \mathfrak{h}(\Theta_{j,j}) + (-1)^{r-1} \mathfrak{h}(\Theta_r) \big)$$

and hence

$$(-1)^j \mathfrak{h}(\Theta_j) = (-1)^r \mathfrak{h}(\Theta_r).$$

The final blow now comes from our earlier evaluation of the cochain $\mathfrak{c}_f$ on Φ:

$$(r-1)!^d \cdot \zeta = \mathfrak{c}_f(\Phi) = \delta\mathfrak{h}(\Phi) = \mathfrak{h}(\partial \Phi) = \mathfrak{h}((-1)^{d(r-1)} \sum_{j=1}^{r} (-1)^{j-1} \Theta_j)$$

$$= -(-1)^{d(r-1)} \sum_{j=1}^{r} (-1)^j \mathfrak{h}(\Theta_j) = -(-1)^{d(r-1)} \sum_{j=1}^{r} (-1)^r \mathfrak{h}(\Theta_r)$$

$$= (-1)^{(d+1)(r-1)} r \, \mathfrak{h}(\Theta_r).$$

Thus, the integer coefficient of $\mathfrak{h}(\Theta_r)$ should be equal to $\frac{(r-1)!^d}{r}\zeta$, up to a sign. Consequently, when $r \nmid (r-1)!^d$, the cocycle $\mathfrak{c}_f$ is not a coboundary, i.e., the cohomology class $\mathfrak{o} = [\mathfrak{c}_f]$ does not vanish and so there is no $\mathfrak{S}_r$-equivariant extension $X \to S(W_r^{\oplus(d+1)})$ of $F_0|_A$.

On the other hand, when $r \mid (r-1)!^d$ we can define

$$
\begin{aligned}
\mathfrak{h}(\Theta_j) &:= +(-1)^{(d+1)(r-1)+j+r}\tfrac{(r-1)!^d}{r}\cdot\zeta &&\text{for } 1 \le j \le r,\\
\mathfrak{h}(\Theta_{j,j}) &:= -(-1)^{(d+1)(r-1)+j+r}\tfrac{(r-1)!^d}{r}\cdot\zeta &&\text{for } 1 \le j < r, \qquad\qquad(5)\\
\mathfrak{h}(\Theta_{i,j}) &:= 0 &&\text{for } i \ne j,\ 1 \le i \le r,\ 1 \le j < r.
\end{aligned}
$$

Here we employ obstruction theory with respect to the filtration $(\Delta_{r,r-1})^{*d} * (\Delta_{r,r-1} * [r])^{(n)}$ of X, where $(\Delta_{r,r-1} * [r])^{(n)}$ denotes the n-skeleton of $\Delta_{r,r-1} * [r]$. The "cells" are of the form $(\Delta_{r,r-1})^{*d} * F$, where F ranges over the faces of $\Delta_{r,r-1} * [r]$. They are connected oriented pseudomanifolds with boundary, their boundary being the pseudomanifolds $(\Delta_{r,r-1})^{*d} * \partial F$. If $\dim(\Delta_{r,r-1})^{*d} * \partial F = N - 1 = \dim S(W_r^{\oplus(d+1)})$, then a map $(\Delta_{r,r-1})^{*d} * \partial F \to S(W_r^{\oplus(d+1)})$ can be *nonequivariantly* extended to $(\Delta_{r,r-1})^{*d} * F \to S(W_r^{\oplus(d+1)})$ if and only if its degree is zero; this uses a standard obstruction theory argument. Similarly, if $\dim(\Delta_{r,r-1})^{*d} * F = N - 1 = \dim S(W_r^{\oplus(d+1)})$, then the set of nonequivariant extensions of a map $(\Delta_{r,r-1})^{*d} * \partial F \to S(W_r^{\oplus(d+1)})$ to $(\Delta_{r,r-1})^{*d} * F \to S(W_r^{\oplus(d+1)})$ corresponds bijectively to the elements in $H^N((\Delta_{r,r-1})^{*d} * (F, \partial F); \mathbb{Z}) = \mathbb{Z}$; the bijection depends on the choice of one extension that should correspond to $0 \in \mathbb{Z}$. The obstruction cocycle $\mathfrak{c}_f$ can thus be regarded as an element in the simplicial cochain complex

$$
C_{\mathfrak{S}_r}^{r-1}(\Delta_{r,r-1} * [r], B; \mathcal{Z} \otimes H_{(r-1)d-1}((\Delta_{r,r-1})^{*d}; \mathbb{Z})),
$$

where B denotes the subcomplex of $\Delta_{r,r-1} * [r]$ on which $\mathfrak{S}_r$ does not act freely. The coefficients are twisted with the top homology of $(\Delta_{r,r-1})^{*d}$ in order to account for the $\mathfrak{S}_r$-action on the orientation of the cells. The coboundary of $\mathfrak{h}$ as defined in (5) is $\mathfrak{c}_f$. Since $\mathfrak{h}$ is only nonzero on the cells Θ_j and $\Theta_{j,j}$, which are only invariant under $\mathrm{id} \in \mathfrak{S}_r$, we can solve the extension problem equivariantly. Note also that this map still coincides with F_0 on A.

Hence for $r \mid (r-1)!^d$ there exists an $\mathfrak{S}_r$-equivariant extension $X \to S(W_r^{\oplus(d+1)})$ of $F_0|_A$.   □

Remark 4.4 (February, 2013). We are happy that our work has attracted a lot of attention immediately after the first presentation in October 2009.

Soon after completion of the first version of the preprint for this paper we noticed (see [7, Sect. 2]) that the nonexistence part of Proposition 4.2 can also be phrased in more elementary terms using degrees rather than by using equivariant obstruction theory; this was also noticed by Vrećica and Živaljević [19].

We note that despite the condition $r \mid (r-1)!^d$ obtained from evaluation of the obstruction cocycle on a particular subcomplex, the correct value for the degree of the equivariant map in question is $(r-1)!^{d+1}$, so that the degree approach only yields the necessary condition $r \mid (r-1)!^{d+1}$ for the existence of the map.

We provide the degree formulation of the proof of nonexistence part of Proposition 4.2 in [7] as a special case of a Tverberg–Vrećica type transversal theorem, accompanied by much more complete cohomological index calculations, which also yield a second new proof that establishes Theorem 2.1 directly, without a reduction to Theorem 2.2. Matoušek, Tancer & Wagner [15] have presented an elementary version of the degree-based proof for the nonexistence part of Proposition 4.2.

The proof in terms of degrees, however, does not imply that the $\mathfrak{S}_r$-equivariant map proposed by the natural configuration space/test map scheme of Theorem 4.2 exists if r divides $(r-1)!^d$. Moreover, the nonexistence of an induced equivariant map in the case $d = 1$ and $r = 4$ can only be captured by the use of equivariant obstruction theory.

See [21] for an exposition of the history and context of this subject.

References

[1] Bárány, I.: Geometric and combinatorial applications of Borsuk's theorem. Chap. IX in: New Trends in Discrete and Combinatorial Geometry (J. Pach, ed.), Algorithms Combin. 10, Springer, Berlin, 235–249 (1993) Zbl 0788.52012 MR 1228045

[2] Bárány, I., Füredi, Z., Lovász, L.: On the number of halving planes. Combinatorica **10**, 175–183 (1990) Zbl 0718.52009 MR 1082647

[3] Bárány, I., Larman, D. G.: A colored version of Tverberg's theorem. J. London Math. Soc. **45**, 314–320 (1992) Zbl 0769.52008 MR 1171558

[4] Bárány, I., Shlosman, S. B., Szűcs, A.: On a topological generalization of a theorem of Tverberg. J. London Math. Soc. **23**, 158–164 (1981) Zbl 0453.55003 MR 0602247

[5] Björner, A., Lovász, L., Vrećica, S., Živaljević, R. T.: Chessboard complexes and matching complexes. J. London Math. Soc. **19**, 25–39 (1994) Zbl 0790.57014 MR 1253009

[6] Blagojević, P. V. M., Dimitrijević Blagojević, A.: Using equivariant obstruction theory in combinatorial geometry. Topology Appl. **4**, 2635–2655 (2007) Zbl 1158.55009 MR 2340948

[7] Blagojević, P. V. M., Matschke, B., Ziegler, G. M.: Optimal bounds for a colorful Tverberg–Vrećica type problem. Adv. Math. **226**, 5198–5215 (2011) Zbl 1213.52009 MR 2775898

[8] tom Dieck, T.: Transformation Groups. De Gruyter Stud. Math. 8, de Gruyter, Berlin (1987) Zbl 0611.57002 MR 0889050

[9] Hell, S.: Tverberg's theorem with constraints. J. Combin. Theory Ser. A **5**, 1402–1416 (2008) Zbl 1221.05024 MR 2455585

[10] Jonsson, J.: Simplicial Complexes of Graphs. Lecture Notes in Math. 1928, Springer, Heidelberg (2008) Zbl 1152.05001 MR 2368284

[11] de Longueville, M.: Erratum to: "Notes on the topological Tverberg theorem". Discrete Math. **247**, 271–297 (2002) Zbl 0994.52004 MR 1893037

[12] Lou, S. T., Yao, Q.: A Chebychev's type of prime number theorem in a short interval, II. Hardy-Ramanujan J. **15**, 1–33 (1992) Zbl 0780.11039 MR 1215589

[13] Matoušek, J.: Lectures on Discrete Geometry. Grad. Texts in Math. 212, Springer, New York (2002) Zbl 0999.52006 MR 1899299

[14] Matoušek, J.: Using the Borsuk–Ulam Theorem. Lectures on Topological Methods in Combinatorics and Geometry. Universitext, Springer, Heidelberg (2003); 2nd printing (2008) Zbl 1234.05002 MR 1988723

[15] Matoušek, J., Tancer, M., Wagner, U.: A geometric proof of the colored Tverberg theorem. Discrete Comput. Geom. **47**, 245–265 (2011) Zbl 1242.52009 MR 2872536

[16] Shareshian, J., Wachs, M. L.: Torsion in the matching complex and chessboard complex. Adv. Math. **212**, 525–570 (2007) Zbl 1117.05110 MR 2329312

[17] Sarkaria, K. S.: Tverberg partitions and Borsuk–Ulam theorems. Pacific J. Math. **196**, 231–241 (2000) Zbl 1045.55002 MR 1797243

[18] Tverberg, H.: A generalization of Radon's theorem. J. London Math. Soc. **41**, 123–128 (1966) Zbl 0131.20002 MR 0187147

[19] Vrećica, S., Živaljević, R. T.: Chessboard complexes indomitable. J. Combin. Theory Ser. A **118**, 2157–2166 (2011) Zbl 1250.05122 MR 2802193

[20] Ziegler, G. M.: Shellability of chessboard complexes. Israel J. Math. **87**, 97–110 (1994) Zbl 0828.57017 MR 1286818

[21] Ziegler, G. M.: $3N$ colored points in a plane. Notices Amer. Math. Soc. **58**, 550–557 (2011) Zbl 1222.51003 MR 2807521

[22] Živaljević, R. T.: Topological methods. Chap. 14 in: CRC Handbook on Discrete and Computational Geometry (J. E. Goodman and J. O'Rourke, eds.), 2nd ed., CRC Press, Boca Raton, FL, 305–329 (2004)

[23] Živaljević, R. T.: User's guide to equivariant methods in combinatorics II. Publ. Inst. Math. (Beograd) **64(78)**, 107–1132 (1998) Zbl 0999.52005 MR 1668705

[24] Živaljević, R. T., Vrećica, S.: The colored Tverberg's problem and complexes of injective functions. J. Combin. Theory Ser. A **61**, 309–318 (1992) Zbl 0782.52003 MR 1185000