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Counting arithmetic subgroups and subgroup growth of virtually free groups

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Abstract. Let K be a p-adic field, and let $H = PSL_2(K)$ endowed with the Haar measure determined by giving a maximal compact subgroup measure 1. Let $AL_H(x)$ denote the number of conjugacy classes of arithmetic lattices in H with co-volume bounded by x. We show that under the assumption that K does not contain the element $\zeta + \zeta^{-1}$, where ζ denotes the p-th root of unity over $\mathbb{Q}_p$, we have

$$\lim_{x\to\infty} \frac{\log AL_H(x)}{x \log x} = q - 1$$

where q denotes the order of the residue field of K.

The main innovation of this paper is the proof of a sharp bound on subgroup growth of lattices in H as above.

Keywords. Arithmetic subgroups, counting lattices, subgroup growth, virtually free groups

1. Introduction

Let H denote a simple non-compact Lie group endowed with a fixed Haar measure μ. Recall that a *lattice* $\Gamma < H$ is a discrete subgroup such that $\mu(\Gamma \setminus H) < \infty$. A theorem of Wang [W] asserts that if H is not locally isomorphic to $PSL_2(\mathbb{R})$ or $PSL_2(\mathbb{C})$, then for every $0 < x \in \mathbb{R}$ the number $L_H(x)$ of conjugacy classes of lattices in H of co-volume at most x is finite. In the last decade there have been several results towards a quantitative version of Wang's theorem in several cases (see the references in [BGLS]). If $H = PSL_2(\mathbb{R})$ or $H = PSL_2(\mathbb{C})$, then $L_H(x)$ is not finite. In fact, in the case of $PSL_2(\mathbb{R})$ it is well known that there are uncountably many conjugacy classes of lattices of bounded co-volume. Nevertheless, it has been shown by Borel [B1] that the number $AL_H(x)$ of conjugacy classes of *arithmetic* lattices with co-volume bounded by x in H is finite for these groups as well. A very accurate asymptotic formula for $AL_H(x)$ has been proven recently for $H = PSL_2(\mathbb{R})$ (see [BGLS]).

The theme of this paper is the estimation of $AL_H(x)$ for $H = PSL_2(K)$, where K is a p-adic field. Note that here also there are uncountably many conjugacy classes of lattices

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of bounded co-volume (see [L2]). The main result is an asymptotic formula for a large family of p-adic fields:

Theorem 1.1. *Let K be a p-adic field (i.e. a finite extension field of some $\mathbb{Q}_p$) such that $\zeta+\zeta^{-1} \notin K$ where ζ is the p-th root of unity in $\overline{\mathbb{Q}_p}$. Let $H = PSL_2(K)$ be endowed with the Haar measure that gives a maximal compact subgroup measure* 1*. Then*

$$\lim_{x\to\infty} \frac{\log AL_H(x)}{x \log x} = q - 1$$

where $AL_H(x)$ denotes the number of conjugacy classes of arithmetic lattices Γ in H of co-volume at most x, and q denotes the order of the residue field of K.

Note that in particular the theorem is valid for $PSL_2(\mathbb{Q}_p)$ for $p \neq 2, 3$. It is very likely that the result remains true in general.

The proof of this result resembles the proof of the following theorem in several aspects and differs in others.

Theorem 1.2 ([BGLS]). *Let $H = PSL_2(\mathbb{R})$ endowed with the Haar measure induced from the Riemannian measure of the hyperbolic plane $\mathbb{H}^2 = PSL_2(\mathbb{R})/PSO(2)$. Then*

$$\lim_{x\to\infty} \frac{\log AL_H(x)}{x \log x} = \frac{1}{2\pi}.$$

In the proof of both results one is interested in showing that there are some constants $r, s > 0$ such that

$$(rx)^{a(H)x} \leq AL_H(x) \leq (sx)^{a(H)x}$$

where $a(H) = 1/(2\pi)$ if $H = PSL_2(\mathbb{R})$ and $a(H) = q - 1$ if $H = PSL_2(K)$ and K is a p-adic field with residue field of order q. It turns out that the proof of the upper bound is the more difficult part in both cases.

The proof of the upper bound consists of two stages. First, it is shown that the rate of growth of the number of conjugacy classes of maximal arithmetic subgroups of co-volume $< x$ is relatively slow. More precisely, we have the following:

Theorem 1.3. *Let $H = PGL_2(\mathbb{R})^a \times PGL_2(\mathbb{C})^b \times \prod_{i=1}^r PGL_2(K_i)$ where K_i are non-archimedean local fields of characteristic zero. Let $MAL_H(x)$ denote the number of conjugacy classes of maximal arithmetic lattices of co-volume at most x in H. Then, for every $\epsilon > 0$, there exists a positive constant $\beta = \beta(\epsilon)$ such that $MAL_H(x) \leq x^{\beta(\log x)^\epsilon}$ for $x \gg 0$.*

In particular the conclusion is true for $H = PGL_2(K)$ for any local field K of characteristic zero.

Denote by $S_n(\Gamma)$ the number of subgroups of index at most n in Γ. We prove the following theorem:

Theorem 1.4. *Let K be either $\mathbb{R}$ or a p-adic field such that $\zeta + \zeta^{-1} \notin K$ (ζ is as in Theorem* 1.1*), and $H = PSL_2(K)$. Then there exists a constant $c > 0$, depending only on H, such that $s_n(\Gamma) \leq (cn)^{-\chi(\Gamma)n}$ for every lattice $\Gamma < H$ and every $n \in \mathbb{N}$.*

Here $\chi(\Gamma)$ is the Euler characteristic of Γ.

Once Theorems 1.3 and 1.4 are proven, Theorem 1.1 is deduced just as Theorem 1.2 has been deduced in [BGLS]. For the convenience of the reader we sketch the argument:

One can write $AL_H(x) \leq MAL_H(x) \cdot \max\{s_{x/y}(\Gamma) : y \leq x, \operatorname{covol}(\Gamma) = y\}$ where Γ runs over the arithmetic lattices of co-volume at most x (and $MAL_H(x)$ is as above). Using Theorems 1.3 and 1.4 one gets

$$AL_H(x) \leq x^{b \log x} \cdot \left(c\frac{x}{y}\right)^{-\chi(\Gamma)\frac{-a(H)x}{\chi(\Gamma)}} \leq (sx)^{a(H)x}$$

for some constant s. Here we use the fact that $y = \operatorname{covol}(\Gamma) = -2\pi\chi(\Gamma)$ in the real case (the Gauss–Bonnet formula), and $y = \operatorname{covol}(\Gamma) = -\frac{1}{q-1}\chi(\Gamma)$ in the p-adic case for our normalization (see [S2, p. 134]).

The lower bound is achieved by counting the subgroups of finite index of a single arithmetic lattice Γ. Choose Γ such that $S_n(\Gamma) \geq (cn)^{-\chi(\Gamma)n}$ for some constant c. (For all Fuchsian groups Γ one has $s_n(\Gamma) \geq n^{-\chi(\Gamma)n}$; see [BGLS, Proposition 4.11].) In the p-adic case one can take a lattice which is a free group (see [LS, Chapter 2]). Thus, one just has to check that there are not too many subgroups of Γ conjugate to each other in H. To prove this, one uses an upper bound on the subgroup growth of congruence subgroups of an arithmetic group, proven in [L1]. This enables one to say there are a lot (that is, of order $(c_1 n)^{-\chi(\Gamma)n}$) of subgroups with index bounded by n, with the same closure in the congruence topology. One then shows that just a few (of order $c_2 n$) of these subgroups with the same congruence closure can be conjugate. For the details see [BGLS].

Thus, in order to prove Theorem 1.1 we will have to prove Theorems 1.3 and 1.4 in the p-adic case. The proof of Theorem 1.3 is in fact similar to the proof given in [BGLS] for $H = PGL_2(\mathbb{R})$. The only work here is to check that several results used along the proof of the theorem remain valid in the case where there are also p-adic factors in the product of the groups PSL_2 over local fields.

These are mainly results by Borel on the distribution of arithmetic lattices within a commensurability class, and a lemma by Chinburg and Friedman (which has a close connection with Borel's work). These results where originally proven for products of the form $PGL_2(\mathbb{R})^a \times PGL_2(\mathbb{C})^b$, but it turns out that they are valid in our case as well with no substantial change in the arguments.

The proof of Theorem 1.4 in the p-adic case on the other hand is very different from the proof of the analogous theorem for the real case in [BGLS]. The proof there is based mainly on the explicit presentation of Fuchsian groups and on the theory of characters of the symmetric group, which is used in order to count permutation representations of Fuchsian groups. We count the number of permutation representations of a lattice Γ in PSL_2 over a p-adic field using the presentation of such a group as the fundamental group of a graph of groups. Note that in this case the proof is restricted to p-adic fields K as in the formulation of Theorem 1.1. On the other hand we remark that the proof of Theorem 1.4 is also valid in the case of uniform lattices in PSL_2 over a local field of positive characteristic. It is very probable that one can give a theorem analogous to Theorem 1.1 for the number of uniform arithmetic lattices in these groups. For this one has to check if the results needed for the proof of Theorem 1.3 can be adapted to the case of positive characteristic, and so can be used to prove the analogue to this theorem in the positive characteristic case.

The main bulk of this paper will deal with the proof of Theorem 1.4. We start in Section 2 with the discussion of a theorem proven recently by J. C. Schlage-Puchta [SP, Theorem 3]. We have the following invariant:

Definition 1.1. Let Γ be some finitely generated group. Denote by $a_n(\Gamma)$ the number of subgroups of index n in Γ, and define

$$\nu(\Gamma) = \limsup_{n\to\infty} \frac{\log a_n(\Gamma)}{n \log n}.$$

Schlage-Puchta's theorem asserts that for finitely generated virtually free groups, the value of $\nu(\Gamma)$ is the solution of a linear optimization problem with rational coefficients.

In the following sections we will make use of his theorem (and its proof) in order to prove the following:

Theorem 1.5. *Let Γ be a lattice in $PSL_2(K)$, where K is a p-adic field (i.e. a finite extension field of some $\mathbb{Q}_p$). Assume there are no elements of order p in Γ. Then $\nu(\Gamma) = -\chi(\Gamma)$.*

(Here again $\chi(\Gamma)$ stands for the Euler–Poincaré characteristic of Γ.) Note that the condition on the p-adic field K in Theorem 1.1 that $\zeta + \zeta^{-1} \notin K$ is equivalent to the condition that $PSL_2(K)$ does not contain elements of order p, so in particular Theorem 1.5 is valid for all lattices in $PSL_2(K)$ as in Theorem 1.1.

In Section 3 we will see, by means of two simple counterexamples, that the equality provided in Theorem 1.5 is far from being the rule for general virtually free finitely generated groups, and need not hold even for co-compact tree lattices.

The proof of Theorem 1.5 consists of two parts. First, in Section 4 we give a sufficient condition for a finitely generated virtually free group Γ to have $\nu(\Gamma) = -\chi(\Gamma)$, using the linear optimization problem constructed in [SP], and an elementary lemma from the theory of linear optimization.

The proof of Theorem 1.5 is completed in Section 5. Lattices in p-adic PSL_2 are finitely generated virtually free groups. We show that for lattices as in Theorem 1.5 the sufficient condition described in Section 4 is indeed fulfilled.

From Theorem 1.5 one can deduce the inequality $s_n(\Gamma) \leq (cn)^{-\chi(\Gamma)n}$ for some constant c, but this constant may depend on Γ. We shall see nonetheless that from the proof of Schlage-Puchta's theorem one can easily deduce that the constant can be taken to be independent of the lattice Γ chosen (although it may depend on the field K). This will be shown in Section 6 and will thus finish the proof of Theorem 1.4.

Section 7 will deal with Theorem 1.3. We will mainly state the results of Borel and the lemma of Chinburg and Friedman in a version adapted to our needs. Thus their results will be stated in the setting of a product of PGL_2 groups that may contain PGL_2 factors over p-adic fields. We will not however care to state precise numerical values corresponding to a given normalization of the Haar measure, as this is not needed for the proof of Theorem 1.3. As a rule, the proofs of the theorems stated are similar to the original proofs. Using the same proof as in [BGLS, Section 3] one can thus prove Theorem 1.3.

2. The linear optimization problem

To prove Theorem 1.5, we have to understand the explicit construction of the linear optimization problem that appears in the theorem of Schlage-Puchta formulated below. To some extent it is also necessary to understand the proof of the theorem in order to later derive Theorem 1.4. We shall therefore describe the proof and the linear optimization problem. We shall supply details where this is important for our needs, or where we think it helps to clarify some issues in [SP]. For full details see [SP, Section 2].

We first recall a few basic elements of the Bass–Serre theory of graphs of groups. For the details see [S1]. A *graph of groups* is a pair (Γ, A) where Γ is an assignment of a group Γ_v for all $v \in V$, and a group Γ_e for all $e \in E$, such that $\Gamma_e = \Gamma_{\bar{e}}$ (where $\bar{e}$ denotes the opposite edge of e), and monomorphisms $\psi_e : \Gamma_e \to \Gamma_{\partial_0(e)}$ (where $\partial_0(e)$ denotes the initial vertex of e).

The *fundamental group* $\pi_1(\Gamma, A, T)$ of a graph of groups (Γ, A) at a maximal subtree T of A is the group generated by the vertex groups $\{\Gamma_v : v \in V\}$ and by elements $\{g_e\}_{e \in E}$, subject to the relations:

- $g_{\bar{e}} = g_e^{-1}$,
- $g_e = 1$ for all $e \in E(T)$,
- $g_e \psi_{\bar{e}}(a) g_e^{-1} = \psi_e(a)$ for all $e \in E$ and $a \in \Gamma_e$.

(This is independent of the choice of T up to isomorphism.)

A *finite* graph of finite groups is just a graph of groups for which the underlying graph A is finite, and all vertex groups are finite. Recall that a group Γ is called *virtually free* if it has a finite index free subgroup. One can show that if Γ is the fundamental group of a finite graph of finite groups, then Γ is virtually free (see [S1, II.2.6]; Γ is obviously also finitely generated).

It is a non-trivial fact (proven by Karras and Solitar) that the converse is also true, that is, every finitely generated virtually free group is the fundamental group of a finite graph of finite groups (see the reference in [S1]).

Let Γ be a group acting without inversion on a connected non-empty graph X. Then one can construct a graph of groups (Γ, A) in a natural way so that $A = X/\Gamma$, and the vertex and edge groups correspond respectively to stabilizers in Γ of vertices and edges in a fundamental domain for the action of Γ on X. The monomorphisms from edge to vertex groups are given by conjugation by elements in Γ (possibly trivial). One has the following theorem [S1, Theorem 13]: If X is a tree, then $\Gamma \cong \pi_1(\Gamma, A, T)$.

We have defined above the *growth coefficient*

$$\nu(\Gamma) := \limsup_{n \to \infty} \frac{\log a_n(\Gamma)}{n \log n}.$$

This quantity has been computed for a variety of cases (cf. [SP] and the references there).

In a recent paper [SP] J. C. Schlage-Puchta has shown the following:

Theorem 2.1. *Let Γ be a finitely generated virtually free group, represented by a finite graph of finite groups (Γ, A). Then the growth coefficient $\nu(\Gamma)$ is the solution of a linear optimization problem with rational coefficients which can be effectively computed from (Γ, A).*

From this he concluded that $\nu(\Gamma)$ is always a rational number. In addition he showed that every positive rational number can be obtained as the growth coefficient of some virtually free group, and thus he concluded that the spectrum of the growth coefficient for virtually free groups is exactly $\mathbb{Q}^+$.

Note that $H = PSL_2(K)$ for a K a non-archimedean local field acts on the Bruhat–Tits tree X which is a $q+1$-regular tree (q being again the order of the residue field). If $\Gamma < H$ is a co-compact lattice then it acts on X with a finite fundamental domain (by co-compactness) and with finite stabilizers (by discreteness). If K is p-adic, then all lattices are co-compact. (See [S1, II] for all these facts.) Thus one can use the above theorem in the case that interests us. (We actually do not even need the Karras–Solitar theorem, as Schlage-Puchta's theorem is proven purely in terms of the finite graph of finite groups corresponding to the group.)

In order to prove the theorem one notes first that counting subgroups of index n is essentially the same as counting homomorphisms to S_n and hence for a finitely generated group Γ we have

$$\nu(\Gamma) = \limsup_{n\to\infty} \frac{\log|\mathrm{Hom}(\Gamma, S_n)|}{n\log n} - 1$$

(see [SP, Lemma 1]).

We assume that Γ is the fundamental group of a finite graph of finite groups (relative to a maximal subtree). We thus want to count permutation representations of the fundamental group Γ. Such a permutation representation is determined by giving homomorphisms $\phi_v : \Gamma_v \to S_n$ for each $v \in V(A)$, and choosing images for all the elements $g_e,\ e \in E(A) \setminus E(T)$, so that they satisfy the conditions imposed by the defining relations of Γ. Thus, given a set $\{\phi_v : \Gamma_v \to S_n\}_{v\in V}$ of representations, they lift to some representation of Γ if and only if the representations induced by $\Gamma_{\partial_0(e)}$ and $\Gamma_{\partial_1(e)}$ on Γ_e are equivalent for all $e \in E(A)$, and actually coincide for $e \in E(T)$.

For each $v \in V$, let $\rho_{v,1}, \ldots, \rho_{v,m_v}$ be a complete list of transitive permutation representations of Γ_v (up to equivalence). We will denote by $n_{v,i}$ the degree of $\rho_{v,i}$ (that is, the order of the set on which Γ_v acts by $\rho_{v,i}$).

Any permutation representation of a group is a direct sum of transitive representations. This implies that a representation $\rho : \Gamma_v \to S_n$ is determined up to equivalence by the multiplicities $\xi_{v,i}$ of $\rho_{v,i}$ $(1 \le i \le m_v)$ in ρ.

In a similar manner we denote by $\rho_{e,1}, \ldots, \rho_{e,m_e}$ the list of transitive representations of Γ_e. (Here again $n_{e,i}$ will denote the order of $\rho_{e,i}$.) Let $v = \partial_0(e)$ or $v = \partial_1(e)$. Given a representation of Γ_v, determined by $\xi_{v,i}\ 1 \le i \le m_v$, the multiplicity of $\rho_{e,j}$ in the representation induced on Γ_e is given by a linear functional $l^v_{e,j}(\xi_{v,1}, \ldots, \xi_{v,m_v})$ with integral coefficients, where the coefficient of $\xi_{v,i}$ is the multiplicity of $\rho_{e,j}$ in the representation induced by $\rho_{v,i}$.

Now, given a representation tuple $(\phi_v)_{v\in V}$, where $\phi_v : \Gamma_v \to S_n$, if $(\xi_{v,i})_{v\in V,\,1\le i\le m_v}$ are the multiplicities corresponding to $(\phi_v)_{v\in V}$, then we call the tuple $(\xi_{v,i})_{v\in V, 1\le i\le m_v}$ the *type* of $(\phi_v)_{v\in V}$.

We will call a type *admissible* if it is the type of a representation tuple of the vertex groups induced from a representation of Γ. Every tuple $(\xi_{v,i})_{v\in V,1\le i\le m_v}$ of non-negative integers such that $\sum_{i=1}^{m_v} n_{v,i}\xi_{v,i} = n$ for all $v \in V$ defines a type of a representation tuple, and from the discussion above it turns out that it is an admissible type if and only if $l^v_{e,j}(\xi_{v,1},\dots,\xi_{v,m_v}) = l^u_{e,j}(\xi_{u,1},\dots,\xi_{u,m_u})$ for all $e \in E$, $e = [u, v]$, and for all $1 \le j \le m_e$.

The number of types grows polynomially in n (see [SP]). One thus concludes that in order to compute the growth coefficient $\nu(\Gamma)$, one just has to find the admissible type giving the largest number of representations, and calculate the number of these.

The number of permutation representations of a given type can be computed in a quite straightforward manner. First, one counts the number of permutation representations of one Γ_v of type $(\xi_{v,i})_{i=1}^{m_v}$. This is standard combinatorics, and the result is that there are $n!/\prod_{i=1}^{m_v}(\xi_{v,i})!n_{v,i}^{\xi_{v,i}}$ such representations.

Next, one counts the number of representations 'along the maximal tree', that is, without yet taking into account the number of possible images of the elements g_e with $e \in E(A) \setminus E(T)$.

Starting with some vertex v, we have $n!/\prod_{i=1}^{m_v}(\xi_{v,i})!n_{v,i}^{\xi_{v,i}}$ permutation representations for this vertex. Continuing to a neighbouring vertex u, we now have the restricting condition that every permutation representation induced on Γ_e, where $e = [u, v]$, from a representation of Γ_u has to coincide with a representation induced on Γ_e from Γ_v. Thus, if we want to count the number of possible permutation representations, we have to fix a representation of Γ_e and count in how many ways one can extend it to a permutation representation of Γ_u. This is just the total number of representations of Γ_u of type $(\xi_{u,i})_{i=1}^{m_u}$, divided by the number of representations of Γ_e of the induced type $(l^u_{e,i}(\xi_{u,1},\dots,\xi_{u,m_u}))_{i=1}^{m_e}$. These are computed by the formula above. Continuing for the next vertices along the maximal tree in a similar fashion one finds that the number of representations 'along the tree' is

$$\prod_{v\in V}\frac{n!}{\prod_{i=1}^{m_v}(\xi_{v,i})!n_{v,i}^{\xi_{v,i}}}\cdot\prod_{[u,v]=e\in E^g(T)}\left(\frac{n!}{\prod_{i=1}^{m_e}(l^v_{e,i}(\xi_{v,1},\dots,\xi_{v,m_v}))!n_{e,i}^{l^v_{e,i}(\xi_{v,1},\dots,\xi_{v,m_v})}}\right)^{-1}.$$

Here $E^g(T)$ denotes the geometric edges of T, thus we take in the product only one edge e of any pair of opposite edges $e, \bar{e}$. Note that by admissibility we could have taken above

$$\frac{n!}{\prod_{i=1}^{m_e}(l^u_{e,i}(\xi_{u,1},\dots,\xi_{u,m_u}))!n_{e,i}^{l^u_{e,i}(\xi_{u,1},\dots,\xi_{u,m_u})}}$$

instead of

$$\frac{n!}{\prod_{i=1}^{m_e}(l^v_{e,i}(\xi_{v,1},\dots,\xi_{v,m_v}))!n_{e,i}^{l^v_{e,i}(\xi_{v,1},\dots,\xi_{v,m_v})}}.$$

Finally, we count the total number of representations by adding to the account the possible images of the g_e. If $[u, v] = e \in E(A) \setminus E(T)$ then the image of g_e has to conjugate the representation induced on Γ_e by Γ_v to the representation induced on Γ_e by Γ_u, and this is the only restriction on the image of g_e. Thus the set of possible images is a coset of the centralizer of the representation of Γ_e induced by Γ_v. As S_n acts transitively by conjugation on the representations of Γ_e of a given type, the order of the centralizer of a representation is just

$$\frac{|S_n|}{|\{\phi : \Gamma_e \to S_n\}|} = \prod_{i=1}^{m_e} (l^v_{e,i}(\xi_{v,1}, \ldots, \xi_{v,m_v}))! n_{e,i}^{l^v_{e,i}(\xi_{v,1},\ldots,\xi_{v,m_v})}$$

where the denominator on the left side denotes the number of permutation representations of Γ_e. All in all, the number of permutation representations of Γ of type $(\xi_{v,i})_{v\in V,\, 1\le i\le m_v}$ is

$$n!^{|(E(A)\setminus E(T))^g|} \prod_{v\in V} \frac{n!}{\prod_{i=1}^{m_v} (\xi_{v,i})! n_{v,i}^{\xi_{v,i}}} \times \prod_{e=[u,v]\in E(A)^g} \left(\frac{n!}{\prod_{i=1}^{m_e} l^v_{e,i}(\xi_{v,1}, \ldots, \xi_{v,m_v})! n_{e,i}^{l^v_{e,i}(\xi_{v,1},\ldots,\xi_{v,m_v})}} \right)^{-1}.$$

Here again E^g denotes the geometric edges , and as above we take only one representative of each pair $e, \bar{e}$. As above, for each edge $e = [u, v]$ one chooses only one of the products

$$\prod_{i=1}^{m_e} (l^v_{e,i}(\xi_{v,1}, \ldots, \xi_{v,m_v}))! n_{e,i}^{l^v_{e,i}(\xi_{v,1},\ldots,\xi_{v,m_v})}.$$

Now one takes the logarithm of this expression, and uses the estimate $m! = n!^{m/n} e^{O(m)}$, following essentially from Stirling's formula, in order to replace each term $\log((\xi_{v,i})! n_{v,i}^{\xi_{v,i}})$ by $(\xi_{v,i}/n) \log n! + O(n)$, and $\log(l^v_{e,i}(\xi_{v,1}, \ldots, \xi_{v,m_v})! n_{e,i}^{l^v_{e,i}(\xi_{v,1},\ldots,\xi_{v,m_v})})$ by $(l^v_{e,i}(\xi_{v,1}, \ldots, \xi_{v,m_v})/n) \log n! + O(n)$. One also uses the fact that $|E^g(T)| = |V| - 1$ and concludes that the logarithm of the number of representations is

$$n \log n \Big(1 - \sum_{v\in V} \sum_{i=1}^{m_v} \xi_{v,i}/n + \sum_{[u,v]\in E^g} \sum_{i=1}^{m_e} l^v_{e,i}(\xi_{v,1}, \ldots, \xi_{v,m_v})/n\Big) + O(n).$$

In the above formula and in the following, $E^g := E^g(A)$.

Dividing by $n \log n$ and subtracting 1 one gets

$$\frac{\log |\mathrm{Hom}(\Gamma, S_n)|}{n \log n} - 1 = -\sum_{v\in V} \sum_{i=1}^{m_v} \xi_{v,i}/n + \sum_{e\in E^g} \sum_{i=1}^{m_e} l^v_{e,i}(\xi_{v,1}, \ldots, \xi_{v,m_v})/n + O\left(\frac{1}{\log n}\right)$$

for a choice of a type $(\xi_{v,i})_{v\in V, 1\le i\le m_v}$ maximizing the expression on the right.

Remark. Note that the error term is the sum of the error terms corresponding to each of the terms $\log((\xi_{v,i})! n_{v,i}^{\xi_{v,i}})$ and $\log(l^v_{e,i}(\xi_{v,1}, \ldots, \xi_{v,m_v})! n_{e,i}^{l^v_{e,i}(\xi_{v,1},\ldots,\xi_{v,m_v})})$ in the product above. These error terms do not depend on Γ. As there are all in all $\sum_{v\in V} m_v + \sum_{e\in E^g} m_e$ such terms we conclude that we can write the error term above as $(\sum_{v\in V} m_v + \sum_{e\in E^g} m_e) O(1/\log n)$ where $O(1/\log n)$ is independent of Γ. This fact will be used later on in Section 6.

Now one considers the following optimization problem: Maximize the linear form

$$-\sum_{v\in V}\sum_{i=1}^{m_v} x_{v,i} + \sum_{e=[u,v]\in E^g}\sum_{i=1}^{m_e} l_{e,i}^{v}(x_{v,1},\dots,x_{v,m_v})$$

(with the same convention for the second sum as in the formulae above) subject to the following constraints:

1. $x_{v,i} \geq 0$ for all $v \in V$, $1 \leq i \leq m_v$.
2. $\sum_{i=1}^{m_v} n_{v,i}x_{v,i} = 1$ for all $v \in V$ (recall $n_{v,i}$ is the degree of $\rho_{v,i}$).
3. $l_{e,j}^{v}(x_{v,1},\dots,x_{v,m_v}) = l_{e,j}^{u}(x_{u,1},\dots,x_{u,m_u})$ for all $e \in E$, $e = [u,v]$, and all $1 \leq j \leq m_e$.

This is a linear program with rational coefficients. Let $(x_{v,i})_{v\in V,\,1\leq i\leq m_v}$ denote the optimal solution, and let $\nu_0(\Gamma)$ denote the value of the functional on $(x_{v,i})_{v\in V,\,1\leq i\leq m_v}$, that is, the optimal value. These are both rational because all the coefficients of the linear program are rational. Let m be the l.c.m. of the denominators of all the $x_{v,i}$. Whenever $m \mid n$ we obtain a set of positive integers $\xi_{v,i} = nx_{v,i}$ defining an admissible type, for which the number of representations is (up to the error term) ν_0 and thus $\nu(\Gamma) \geq \nu_0(\Gamma)$. On the other hand, from every admissible type we get a solution to the constraints of the linear program (by dividing by n), so the number of representations (up to the error term) is bounded by $\nu_0(\Gamma)$. Thus $\nu(\Gamma) = \nu_0(\Gamma)$ is the solution of the above linear optimization problem.

3. Two counterexamples

Before proving Theorem 1.5 we add a few remarks, and give two examples in order to put the theorem in some context. A measure of subgroup growth related to $\nu(\Gamma)$ is

$$\nu_{\mathrm{f}}(\Gamma) = \limsup_{n\to\infty}\frac{\log a_n^{\mathrm{f}}(\Gamma)}{n\log n},$$

where $a_n^{\mathrm{f}}(\Gamma)$ denotes the number of free subgroups of Γ of index at most n.

From results of Müller [Mü] that give the asymptotic behavior of $a_n^{\mathrm{f}}(\Gamma)$ it immediately turns out that for every virtually free group Γ, one has $\nu_{\mathrm{f}}(\Gamma) = -\chi(\Gamma)$, where $\chi(\Gamma)$ denotes the Euler characteristic. Clearly $\nu_{\mathrm{f}}(\Gamma) \leq \nu(\Gamma)$. Thus, for finitely generated virtually free groups one has $\nu(\Gamma) \geq -\chi(\Gamma)$.

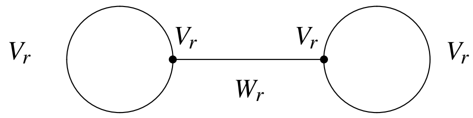
There are many cases where one has equality $\nu(\Gamma) = -\chi(\Gamma)$, e.g. for free groups and finite free products of cyclic groups. There are however cases where there is strict inequality. In the following we give two examples of finitely generated virtually free groups for which the inequality is strict.

We first give a somewhat trivial example. Take Γ to be the product of a free group F_r on r generators and a finite non-trivial group S. An elementary result from the theory of subgroup growth states that if G is a group and $H < G$ is a subgroup of finite index m in G, then $s_n(G) \leq (mn)^{[\log m]}s_n(H)$ (see [LS, p. 14]), thus clearly $\nu(\Gamma) \leq \nu(F_r)$. On the other hand, obviously $s_n(F_r) \leq s_n(\Gamma)$ as F_r is a quotient of Γ. So $\nu(\Gamma) = \nu(F_r)$.

Furthermore we have $(1/|S|)\chi(F_r) = \chi(F_r \times S)$. Moreover, taking the group S to be larger and larger we also conclude that $\nu(\Gamma)/\chi(\Gamma)$ is unbounded for virtually free groups.

The next example is from the family of uniform tree lattices, i.e. co-compact lattices in $\mathrm{Aut}(X_k)$, where X_k denotes the k-regular tree. The family of uniform tree lattices consists of virtually free groups, and contains also the lattices in $PSL_2(K)$ over a p-adic field (the examples we give are of course not in $PSL_2(K)$) . Here again we claim that there is a family of co-compact lattices $\Gamma_r < \mathrm{Aut}(X_k)$ (with k fixed, at least for $k \geq 4$) such that the set $\{\nu(\Gamma_r)/\chi(\Gamma_r)\}$ is unbounded.

This can be seen as follows: Suppose we can find a sequence of lattices $\Gamma_r < \mathrm{Aut}(X_k)$ such that on the one hand $\chi(\Gamma_r) \to 0$, and on the other hand all Γ_r map onto the free group F_2 on two generators. Then $\nu(\Gamma_r) \geq \nu(F_2) = 1$ for all r, and clearly the values of $\nu(\Gamma_r)/\chi(\Gamma_r)$ are unbounded. To find such sequences one can slightly modify examples given in [BK, Section 7]. Thus we look at the following graph of groups:



Here $V_r = M^{(\mathbb{Z}/r\mathbb{Z})}$ where M is a group of order $m \geq 2$ and $W_r = \{x \in V_r \mid x(0) = 1 \in M\}$. The mappings from the edge group W_r to the vertex groups V_r are the identity mappings on W_r. There are two mappings from an edge group V_r to a single vertex group V_r. Take one of them to be the identity mapping, and the other to be $\alpha_r \in \mathrm{Aut}(V_r)$ defined by $\alpha_r(x)(i) = x(i+1)$ (see also [BK, Example 7.4]). The sequence of graphs of groups indexed by r gives rise to the sequence $\{\Gamma_r\}$ of the fundamental groups of these graphs of groups. By an elementary result of Bass (see [Ba, p. 11]) the fundamental groups Γ_r act on the $m+2$-regular tree, and the action is co-compact by construction. These groups act faithfully on the tree and thus embed into $\mathrm{Aut}(X_{m+2})$. This results from the same considerations as in [BK, Section 7] showing that, in the terminology of [Ba], the graph of groups is effective (see [Ba, p. 14]). One sees easily that all the Γ_r map onto the free group F_2, by sending the vertex groups to 1, and the elements corresponding to the two loops to the generator of F_2. On the other hand we have $\chi(\Gamma_r) = -1/m^{r-1} \to 0$ as required (see [S2, Proposition 14]). This proves the claim for all X_k where $k \geq 4$.

In the following sections we would like to show that for lattices in $PSL_2(K)$ where K is a p-adic field as in Theorem 1.1 (or more generally for all uniform lattices in PSL_2 over a non-archimedean field with no elements of order p) we have the equality $\nu(\Gamma) = -\chi(\Gamma)$.

4. A sufficient condition for $\nu(\Gamma) = -\chi(\Gamma)$

The first stage in the proof of Theorem 1.5 will be the proof of Lemma 4.2 below that provides a sufficient condition for the fundamental group of a finite graph of finite groups Γ to have $\nu(\Gamma) = -\chi(\Gamma)$. One should note though that this is certainly not a necessary condition. (Indeed, there are examples of p-adic fields K such that $\zeta + \zeta^{-1} \in K$, where ζ is as in Theorem 1.1, such that there are lattices $\Gamma < PSL_2(K)$ where $\nu(\Gamma) = -\chi(\Gamma)$ whereas the conditions of Lemma 4.2 are not satisfied.)

Let us denote by ρ_{m_v}, respectively ρ_{m_e}, the regular representation of Γ_v, respectively Γ_e, for all $v \in V$ and $e \in E$.

If we set

$$x_{v,m_v} = \frac{1}{|\Gamma_v|} \quad \forall v \in V, \qquad x_{v,i} = 0 \quad \text{otherwise}, \tag{4.1}$$

then one sees that this is a so called feasible solution of the linear optimization problem, that is, it is a solution of the system of constraints. The only condition that we have to check and which is not altogether trivial is condition 3.

This is easily seen to be fulfilled by noting that the regular representation of Γ_v induces on Γ_e the representation which is the direct sum of $[\Gamma_v : \Gamma_e]$ times the regular representation of Γ_e. So, the coefficient of x_{v,m_v} in $l^v_{e,i}$ is $[\Gamma_v : \Gamma_e]$ for $i = m_e$, and 0 otherwise. Thus for the proposed solution (4.1) we have

$$\begin{aligned} l^v_{e,i}\left(0,\dots,0,\frac{1}{|\Gamma_v|}\right) &= 0 \quad \text{for all } v \in V,\ i \neq m_e, \\ l^v_{e,m_e}\left(0,\dots,0,\frac{1}{|\Gamma_v|}\right) &= [\Gamma_v : \Gamma_e]\cdot\frac{1}{|\Gamma_v|} = \frac{1}{|\Gamma_e|} = [\Gamma_u : \Gamma_e]\cdot\frac{1}{|\Gamma_u|} \\ &= l^u_{e,m_e}\left(0,\dots,0,\frac{1}{|\Gamma_u|}\right). \end{aligned}$$

Now suppose we could show that (4.1) is an optimal solution. Then we could easily compute the optimal value of the linear form, using the same argument used above to prove that condition 3 is fulfilled, and we get

$$\nu(\Gamma) = -\sum_{v\in V}\frac{1}{|\Gamma_v|} + \sum_{e\in E}[\Gamma_v : \Gamma_e]\cdot\frac{1}{|\Gamma_v|} = -\sum_{v\in V}\frac{1}{|\Gamma_v|} + \sum_{e\in E}\frac{1}{|\Gamma_e|}.$$

Now recall that $\operatorname{covol}(\Gamma) = \frac{1}{2}\sum_{v\in V}\frac{1}{|\Gamma_v|} = \frac{1}{q+1}\sum_{e\in E^g}\frac{1}{|\Gamma_e|}$ (see [S1, p. 84]), and so

$$-\sum_{v\in V}\frac{1}{|\Gamma_v|} + \sum_{e\in E}\frac{1}{|\Gamma_e|} = -2\operatorname{covol}(\Gamma) + (q+1)\operatorname{covol}(\Gamma) = (q-1)\operatorname{covol}(\Gamma) = -\chi(\Gamma)$$

as required. In the last equality we have used the fact that in our normalization $(q-1)\operatorname{covol}(\Gamma) = -\chi(\Gamma)$ (see [S2, p. 134]). One could actually derive the last equation also directly from the expression on the left e.g. by [S2, Prop. 14].

So it would suffice to find a condition for (4.1) to be an optimal solution.

Remark. Note that the existence of this particular solution confirms again that $\nu(\Gamma) \geq -\chi(\Gamma)$. Moreover, using the computations in the proof of Schlage-Puchta's theorem (see Section 2) we get a new proof of the fact resulting from Müller's work that $\nu_f(\Gamma) = -\chi(\Gamma)$ (see Section 3). Indeed, let m be the g.c.d. of the orders of the vertex groups Γ_v. If $m \mid n$ then taking $\xi_{v,i} = n x_{v,i}$ where the $x_{v,i}$ are given by the feasible solution (4.1) defined above, we get a representation type which is the unique type for which all the vertex groups act regularly (with no fixed points in $\{1,\dots,n\}$). There is a correspondence between transitive permutation representations of Γ and subgroups of Γ (the correspondence is $(n-1)!$ to 1), and the transitive representations of this type correspond precisely to the free subgroups of Γ. To see this, recall that this correspondence is given by taking for a transitive permutation representation $\phi : \Gamma \to S_n$ the stabilizer H of $1 \in \{1, \ldots, n\}$ (see [LS, p. 12]). Thus for a representation of the above type the intersections of H with all the conjugates of the Γ_v are trivial, and thus H is free (see also [S1, p. 122]). Conversely, if H is a free subgroup, then it is easy to see from similar considerations that the Γ_v have to act regularly. Note also that in case m does not divide n, there are no representations where Γ_v acts regularly for all $v \in V$, and there are also no free subgroups of index n. Denote by $\mathrm{Hom}^{\mathrm{r}}(\Gamma, S_n)$ the set of homomorphisms $\phi : \Gamma \to S_n$ such that the Γ_v act regularly under ϕ. By the same arguments as in [SP, proof of Lemma 1] we have

$$\nu_{\mathrm{f}}(\Gamma) = \limsup_{n\to\infty} \frac{\log |\mathrm{Hom}^{\mathrm{r}}(\Gamma, S_n)|}{n \log n} - 1.$$

From the computations in the proof of Schlage-Puchta's theorem we now conclude that

$$\frac{\log |\mathrm{Hom}^{\mathrm{r}}(\Gamma, S_n)|}{n \log n} - 1 = -\chi(\Gamma) + \frac{1}{\log n}$$

(for $m \mid n$), and so $\nu_{\mathrm{f}}(\Gamma) = -\chi(\Gamma)$.

We start by rewriting the linear form in the formulation of our linear program as

$$-\sum_{v\in V}\sum_{i=1}^{m_v} x_{v,i} + \frac{1}{2}\sum_{v\in V}\sum_{e\in \mathrm{st}(v)}\sum_{i=1}^{m_e} l^v_{e,i}(x_{v,1}, \ldots, x_{v,m_v}). \tag{4.2}$$

Here $e \in \mathrm{st}(v)$ means that $\partial_0(e) = v$. This gives the same value as the original functional on the solutions of the system of constraints.

In order to find a sufficient condition for (4.1) to be an optimal solution, we shall actually find a condition for (4.1) to be the solution of the linear program given by the linear functional (4.2) and the constraints of types 1 and 2, disregarding the constraints of the third type.

First, we recall some basic notions, and an elementary lemma from the theory of linear optimization. We use [P] as a reference.

One can write our linear optimization problem in the form $\min\{cx : Ax = b,\ x \geq 0\}$, where A is some $m \times n$ matrix with full row rank, x is the vector of variables, with n entries, $c \in \mathbb{R}^n$ is a row vector, $b \in \mathbb{R}^m$ is a column vector, and $x \geq 0$ means every coordinate is ≥ 0. (We take min instead of max just to keep with the convention in [P].) This is called a *linear program in standard form*. We call $x \in \mathbb{R}^n$ a *feasible solution* if it satisfies the constraints $Ax = b$, $x \geq 0$, and we call it an *optimal solution* if it minimizes cx on the set defined by the constraints.

Choosing any m columns of A one gets a submatrix B. Call such a matrix a *base* if $\mathrm{rank}(B) = m$. Suppose B is a base, and $B^{-1}b \geq 0$. Suppose also that B was obtained by choosing from A the columns indexed by $i_1, \ldots, i_m$. Define now the vector x by $x_{i_k} = (B^{-1}b)_k$ and $x_i = 0$ if $i \notin \{i_1, \ldots, i_m\}$. Then x is a feasible solution. We call it the *solution defined by* B. Let $c_B \in \mathbb{R}^m$ be a row vector defined by $(c_B)_k = c_{i_k}$. We have the following elementary lemma (for a proof see e.g. [P, p. 43]):

Lemma 4.1. *If $c - c_B B^{-1} A \geq 0$ then the solution x defined by B is optimal.*

Here again, saying a vector is ≥ 0 means every coordinate is ≥ 0.

Let us now turn to our case. Let $V = \{v_1, \dots, v_k\}$ denote the vertices of the graph of groups of Γ, in particular let $|V| = k$. In our case, A is a $k \times \sum_{v \in V} m_v$ matrix, given by

$$\begin{pmatrix} n_{v_1,1} \dots n_{v_1,m_{v_1}} & & & \\ & n_{v_2,1} \dots n_{v_2,m_{v_2}} & & \\ & & \ddots & \\ & & & n_{v_k,1} \dots n_{v_k,m_{v_k}} \end{pmatrix}.$$

The vector $b \in \mathbb{R}^k$ is just

$$b = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Let us write $l^{v_j}_{e,i}(x_{v_j,1}, \dots, x_{v_j,m_{v_j}}) = \sum_{r=1}^{m_{v_j}} \eta^r_{i,j} x_{v_j,r}$, that is, $\eta^r_{i,j}$ is the multiplicity of $\rho_{e,i}$ in the representation induced on Γ_e from the representation $\rho_{v_j,r}$ of Γ_{v_j}.

Now, the vector c in our linear program is given as follows. We have, for $1 \le r \le m_{v_1}$,

$$c_r = 1 - \frac{1}{2} \sum_{e \in \mathrm{st}(v_1)} \sum_{i=1}^{m_e} \eta^r_{i,1}.$$

Next we have

$$c_{m_{v_1}+r} = 1 - \frac{1}{2} \sum_{e \in \mathrm{st}(v_2)} \sum_{i=1}^{m_e} \eta^r_{i,2}$$

for $1 \le r \le m_{v_2}$ and so on; that is, in general,

$$c_{m_{v_1}+\dots+m_{v_{j-1}}+r} = 1 - \frac{1}{2} \sum_{e \in \mathrm{st}(v_j)} \sum_{i=1}^{m_e} \eta^r_{i,j}$$

where $1 \le r \le m_{v_j}$. (Note we have changed the sign. As mentioned before, this is done just in order to keep the convention of [P].)

Now take the base B to be the $k \times k$ matrix defined by the columns $m_{v_1}, \dots, m_{v_k}$:

$$B = \begin{pmatrix} |\Gamma_{v_1}| & & & \\ & |\Gamma_{v_2}| & & \\ & & \ddots & \\ & & & |\Gamma_{v_k}| \end{pmatrix}.$$

So,

$$B^{-1}A = \begin{pmatrix} \frac{n_{v_1,1}}{|\Gamma_{v_1}|} \dots \frac{n_{v_1,m_{v_1}}}{|\Gamma_{v_1}|} & & & \\ & \frac{n_{v_2,1}}{|\Gamma_{v_2}|} \dots \frac{n_{v_2,m_{v_2}}}{|\Gamma_{v_2}|} & & \\ & & \ddots & \\ & & & \frac{n_{v_k,1}}{|\Gamma_{v_k}|} \dots \frac{n_{v_k,m_{v_k}}}{|\Gamma_{v_k}|} \end{pmatrix}.$$

Moreover,

$$c_B = \left(1 - \frac{1}{2} \sum_{e \in \mathrm{st}(v_1)} \sum_{i=1}^{m_e} \eta^{m_{v_1}}_{i,1}, \dots, 1 - \frac{1}{2} \sum_{e \in \mathrm{st}(v_k)} \sum_{i=1}^{m_e} \eta^{m_{v_k}}_{i,k}\right).$$

We have

$$\eta_{i,j}^{m_{v_j}} = \begin{cases} [\Gamma_{v_j} : \Gamma_e] & \text{if } i = m_e, \\ 0 & \text{otherwise,} \end{cases}$$

where we remind the reader that for $e \in \mathrm{st}(v_j)$, $\eta_{i,j}^{m_{v_j}}$ denotes the multiplicity of the i-th transitive representation of Γ_e in the representation induced on Γ_e from the regular representation of Γ_{v_j}, and that the m_e-th representation of Γ_e is the regular representation of this group. Consequently,

$$c_B = \Bigg(1 - \frac{1}{2} \sum_{e \in \mathrm{st}(v_1)} [\Gamma_{v_1} : \Gamma_e], \dots, 1 - \frac{1}{2} \sum_{e \in \mathrm{st}(v_k)} [\Gamma_{v_k} : \Gamma_e]\Bigg).$$

We can now compute $\bar{c} = c - c_B B^{-1} A$. As $\bar{c}$ is a bit to messy to write down as a row vector, we just say that the coordinate of $\bar{c}$ corresponding to the r-th representation of the j-th vertex v_j is given by

$$1 - \frac{n_{v_j,r}}{|\Gamma_v|} + \frac{1}{2} \sum_{e \in \mathrm{st}(v)} \left(\frac{n_{v_j,r}}{|\Gamma_e|} - \sum_{i=1}^{m_e} \eta_{i,j}^r \right).$$

Now, the solution defined by the base B is just the one given by (4.1), so by Lemma 4.1 we get a sufficient condition for the solution defined in (4.1) to be an optimal solution, stated in the following lemma.

Lemma 4.2. *A sufficient condition for* $\nu(\Gamma) = -\chi(\Gamma)$ *is*

$$1 - \frac{n_{v,r}}{|\Gamma_v|} + \frac{1}{2} \sum_{e \in \mathrm{st}(v)} \left(\frac{n_{v,r}}{|\Gamma_e|} - \sum_{i=1}^{m_e} \eta_i^r \right) \geq 0$$

for every $v \in V$ *and* $1 \leq r \leq m_v$.

(We now no longer need the subscript j; the notation in the lemma should therefore be clear.) Another simple lemma makes the computation easier:

Lemma 4.3. *If* $\Gamma_e = 1$ *then* $n_{v,r}/|\Gamma_e| - \sum_{i=1}^{m_e} \eta_i^r = 0$.

Indeed, if $\Gamma_e = 1$ then $n_{v,r}/|\Gamma_e| - \sum_{i=1}^{m_e} \eta_i^r = n_{v,r} - n_{v,r} = 0$.

Thus in Lemma 4.2 we have only to consider the edges $e \in \mathrm{st}(v)$ such that $\Gamma_e \neq 1$. We summarize all this in the following proposition:

Proposition 4.1. *Let* Γ *be a finitely generated virtually free group given as the fundamental group of a graph of groups* (Γ, A). *A sufficient condition for* $\nu(\Gamma) = -\chi(\Gamma)$ *is*

$$1 - \frac{n_{v,r}}{|\Gamma_v|} + \frac{1}{2} \sum_{e \in \mathrm{st}(v),\, \Gamma_e \neq 1} \left(\frac{n_{v,r}}{|\Gamma_e|} - \sum_{i=1}^{m_e} \eta_i^r \right) \geq 0 \tag{4.3}$$

for any $v \in V(A)$ *and* $1 \leq r \leq m_v$, *where* $n_{v,1}, \dots, n_{v,m_v}$ *are the degrees of the transitive permutation representations* $\rho_{v,1}, \dots, \rho_{v_{m_v}}$ *of* Γ_v, *and* $\eta_1^r, \dots, \eta_{m_e}^r$ *denote the multiplicities of the transitive permutation representations* $\rho_{e,1}, \dots, \rho_{e_{m_e}}$ *of* Γ_e *in the representation induced on* Γ_e *from* $\rho_{v,r}$.

5. Proof of Theorem 1.5

We wish to prove that for a lattice Γ in $H = PSL_2(K)$, where K is a p-adic field, such that $\zeta + \zeta^{-1} \notin K$, we have $\nu(\Gamma) = -\chi(\Gamma)$. Note again that the restricting condition is equivalent to the condition that there are no elements of order p in $PSL_2(K)$, and this is the fact we will use in the following. Thus, we actually show that if a lattice $\Gamma < PSL_2(K)$ (where K is any p-adic field) does not contain an element of order p, then $\nu(\Gamma) = -\chi(\Gamma)$. As already mentioned, these lattices act on the $q+1$-regular Bruhat–Tits tree X, where q is the order of the residue field F. As a vertex stabilizer in $PSL_2(K)$ is compact, vertex stabilizers in a lattice $\Gamma < PSL_2(K)$ are finite. The only possible finite subgroups of Γ (and therefore the only groups that can appear as vertex stabilizers) are the dihedral groups D_{2n} and the cyclic group C_n, where $n \mid \frac{q+1}{2}$ or $n \mid \frac{q-1}{2}$, the alternating groups A_4, A_5, and the symmetric group S_4. This can be seen as follows: Each finite subgroup of $PSL_2(K)$ is contained in a vertex stabilizer, which is isomorphic to $PSL_2(\mathcal{O})$, where $\mathcal{O}$ denotes the ring of integers of K. As there are no elements of order p, each such subgroup maps isomorphically to $PSL_2(F)$, since $\mathrm{Ker}(PSL_2(\mathcal{O}) \to PSL_2(F))$ is a pro-p group. The claim above results from the fact that the groups mentioned above are the only subgroups of $PSL_2(F)$ that do not contain elements of order p (cf. [H, p. 213], see also [LW, Section 3]).

Recall that the graph of groups corresponding to Γ is obtained by taking $Y := X/\Gamma$ as the underlying graph. Then one takes a maximal subtree Z of Y, and a lift of Z to X. Finally, one extends the lift to a section $V(Y) \cup E(Y) \to X$. The vertex and edge groups of the graph of groups then correspond to the vertex and edge stabilizers of the lifted vertices and edges. As a consequence, if v is a vertex in the graph of groups of Γ that lifts to $\tilde{v}$, then there is an obvious bijection between the orbits of $\Gamma_v := \Gamma_{\tilde{v}}$ on $\mathrm{st}(\tilde{v})$ and the edges in $\mathrm{st}(v)$.

Thus, given a vertex $\tilde{v} \in X$ and a group $\Gamma_v < G_{\tilde{v}}$ from the list above, we need to analyze the action of Γ_v on $\mathrm{st}(\tilde{v})$. This means in our case finding the various orbits of edges in $\mathrm{st}(\tilde{v})$ with non-trivial stabilizers, and finding those stabilizers (up to conjugation).

Γ_v is mapped isomorphically to $PSL_2(F)$, and its action on $\mathrm{st}(v)$ is given by the action of its image in $PSL_2(F)$ on $P^1(F)$. Thus we might as well think of Γ_v as a subgroup of $PSL_2(F)$, and consider its action on $P^1(F)$.

Note that an element $g \in PSL_2(F)$ whose order (denoted by $o(g)$) is not divisible by p, is either contained in a non-split torus and has no fixed points in $P^1(F)$, or is contained in a split torus and has two fixed points in $PSL_2(F)$. The first case occurs when $o(g) \mid \frac{q+1}{2}$, and the second when $o(g) \mid \frac{q-1}{2}$ (see e.g. [H, II.8]). This also implies that two elements fix the same edge if and only if they commute (which is the same as being in the same split torus). This makes the analysis above quite easy, and these facts will be repeatedly used in the following discussion.

Next we have to find the transitive permutation representations of Γ_v, and for each $e \in \mathrm{st}(v)$ the representation induced on Γ_e by each transitive representation of Γ_v.

Finally we just have to calculate the expression on the left hand side of (4.3) for all transitive representations of the groups above, and hope the result will always be ≥ 0 (which is fortunately the case).

So in the following we just analyze case by case, first in more detail, in order to explain the method, and then just stating the results.

5.1. D_{2n}

As the transitive permutation representations of a group G correspond to conjugacy classes of its subgroups (through the action on cosets), we need to have a list of conjugacy classes of subgroups of D_{2n}. The subgroups of D_{2n} are all isomorphic to D_{2m} or C_m, for $m \mid n$ (we allow also $m = 1$). We have the following simple

Lemma 5.1. *Let $m \mid n$. If n/m is even, then there are two conjugacy classes of subgroups isomorphic to D_{2m} in D_{2n}. If n/m is odd, then there is just one conjugacy class.*

Proof. Let H be the cyclic subgroup of index 2 in D_{2n}. A subgroup isomorphic to D_{2m} is generated by the unique cyclic subgroup H_m of H of order m, and some reflection. Under conjugation, H_m goes to itself, and reflections go to reflections. So two subgroups isomorphic to D_{2m} are conjugate iff they contain conjugate reflections.

If n is odd, then all reflections are conjugate, so in this case the assertion of the lemma is true.

If n is even, then there are two conjugacy classes of reflections. They are easily described as follows: If y is a generator of H then the conjugacy classes are $\{xy^{2i}\}$, $\{xy^{2i+1}\}$ for some (any) reflection x. Now H_m is generated by $y^{n/m}$. If n/m is even, then all the reflections in a subgroup isomorphic to D_{2m} are in the same conjugacy class. From this it is clear that there are two conjugacy classes of subgroups isomorphic to D_{2m} (corresponding to the two conjugacy classes of reflections).

If on the other hand n/m is odd, then a subgroup isomorphic to D_{2m} contains reflections of both conjugacy classes, and so all such subgroups are conjugate.

We now turn to checking if the condition we described in Proposition 4.1 is fulfilled in the case $\Gamma_v = D_{2n}$. We divide into four cases:

(a) $2 \mid \frac{q+1}{2}$ and $n \mid \frac{q+1}{2}$.
(b) $2 \mid \frac{q+1}{2}$ and $n \mid \frac{q-1}{2}$.
(c) $2 \mid \frac{q-1}{2}$ and $n \mid \frac{q+1}{2}$.
(d) $2 \mid \frac{q-1}{2}$ and $n \mid \frac{q-1}{2}$.

In case (a) the cyclic index 2 subgroup of Γ_v acts with no fixed points on $\mathrm{st}(v)$, as its image under the projection to $PSL_2(F)$ is contained in a non-split torus. The reflections also act with no fixed points as they also project to (other) non-split tori. So there is no edge in $\mathrm{st}(v)$ with non-trivial edge stabilizer. So in this case (4.3) clearly holds.

In case (b), reflections in Γ_v act freely, and the cyclic index 2 subgroup has two fixed edges in $\mathrm{st}(\tilde{v})$, which are interchanged by the reflections. So there is one edge $e \in \mathrm{st}(v)$ with a non-trivial edge group $C_n < D_{2n} = \Gamma_v$. Note that in this case n is odd.

In case (c), each reflection stabilizes two edges, and the cyclic index 2 subgroup acts freely on $\mathrm{st}(\tilde{v})$. Note that here again n is odd. The orbit of an edge stabilized by a reflection is of order n, so there are two such orbits. As n is odd in this case, all reflections are conjugate. In particular, each of the orbits above contains exactly one edge stabilized by a given reflection, and all edge stabilizers are conjugate.

Finally in case (d), the cyclic index 2 subgroup stabilizes two edges, which are again interchanged by the reflections. Here n is either odd or even. If n is odd, then in (c), we have two orbits of edges stabilized by reflections, with all the edge stabilizers of both orbits conjugate. If n is even, then there are again two orbits of edges stabilized by reflections, but this time, the stabilizers of edges in one orbit are not conjugate to those of the second orbit. (That is, each orbit contains the edges stabilized by the reflections of one conjugacy class.) All in all we thus have in this case one edge in $\mathrm{st}(v)$ stabilized by C_n, and two edges stabilized by reflections. If n is even, then the edge groups of the latter two edges are the (groups generated by) non-conjugate reflections in D_{2n}. (If n is odd then of course the edge groups are conjugate.)

We now introduce a table showing the various transitive actions (= coset actions) of D_{2n}, and the actions induced on the possible edge groups. In the case of an edge group generated by a reflection, the only possibilities are the trivial action, that is, a fixed point, or a transitive action. In the case of $C_n < D_{2n}$ the actions are given by the coset actions of C_n on C_n/C_m for $m \mid n$.

In the table, τ represents a reflection, while if n is even, τ' represents another reflection, which is not conjugate to τ. If n is odd, one should disregard τ', as all reflections are conjugate.

In case both n and n/m are even, there are two different coset representations corresponding to the two conjugacy classes of D_{2m}. We denote by D_{2m} a subgroup containing reflections conjugate to τ, and by D'_{2m} a subgroup containing reflections conjugate to τ'. The box for the action of $\langle\tau'\rangle$ is left empty when the action is similar to that of $\langle\tau\rangle$.

Coset action	Order	Action of $\langle\tau\rangle$	Action of $\langle\tau'\rangle$	Action of C_n
D_{2n}/D_{2m} (n/m odd)	n/m	$1\times$ fixed point, $(n/m-1)/2\times$ reg. action		$1\times$ action of C_n on C_n/C_m
D_{2n}/D_{2m} (n/m even)	n/m	$2\times$ fixed point, $(n/m-2)/2\times$ reg. action	$n/(2m)\times$ reg. action	$1\times$ action of C_n on C_n/C_m
D_{2n}/D'_{2m} (n/m even)	n/m	$n/(2m)\times$ reg. action	$2\times$ fixed point, $(n/m-2)/2\times$ reg. action	$1\times$ action of C_n on C_n/C_m
D_{2n}/C_m	$2n/m$	$n/m\times$ reg. action		$2\times$ action of C_n on C_n/C_m

We explain the table briefly. A coset gD_{2m} is a fixed point of τ iff $\tau \in D_{2m}^g$. It is clear that there is precisely one conjugate of D_{2m} containing τ (because τ generates with the unique subgroup C_m a group isomorphic to D_{2m}); we can assume that $\tau \in D_{2m}$. Suppose first that n/m is odd. So gD_{2m} is a fixed point of τ iff $g \in N_{D_{2n}}(D_{2m})$. But as each of the n reflections belongs to exactly one conjugate of D_{2m}, and as each such conjugate contains m reflections, it turns out that there are n/m such conjugates, and so D_{2m} must be self-normalizing. So the only fixed point of τ is D_{2m}.

In the case that n/m is even, τ has no fixed points acting on the cosets of D'_{2m} as it does not belong to any conjugate of it. On the other hand, D_{2m} has now $n/(2m)$ conjugates. So

D_{2m} is an index 2 subgroup of its normalizer. As here again, gD_{2m} is a fixed point for τ if and only if $g \in N_{D_{2n}}(D_{2m})$, it clearly follows that τ has two fixed points, D_{2m} and gD_{2m} for some $g \in N_{D_{2n}}(D_{2m}) \setminus D_{2m}$.

It is clear that $\langle\tau\rangle$ acts with no fixed points on the cosets of $C_m < D_{2n}$ (as $\tau \notin C_m^g = C_m$).

As to the action of C_n, one can see that if $C_n = \langle y\rangle$, then $D_{2m}, yD_{2m}, \dots, y^{n/m-1}D_{2m}$ is a complete list of cosets of D_{2m}, because $y^i \notin D_{2m}$ for $1 \leq i < n/m$. From this it is clear that it acts as stated. Similar considerations apply to the action on cosets of C_m, where a list of cosets may be given by $C_m, yC_m, \dots, y^{n/m-1}C_m, \tau C_m, y\tau C_m, \dots, y^{n/m-1}\tau C_m$.

Now, all that is left is to check (4.3) for cases (b)–(d), for each transitive representation of D_{2n}.

As a consequence of the following simple lemma it will suffice to check the condition only for case (d), and this will imply all other cases:

Lemma 5.2. $n_{v,r}/|\Gamma_e| \leq \sum_{i=1}^{m_e} \eta_i^r$.

Recall that $n_{v,r}$ denotes the degree of the r-th permutation representation $\rho_{v,r}$ of Γ_v, and η_i^r denotes the multiplicity of the transitive permutation representation $\rho_{e,i}$ of Γ_e induced from $\rho_{v,r}$. The proof of the lemma is immediate: $n_{v,r}/|\Gamma_e| = |\Gamma_e|^{-1}\sum_{i=1}^{m_e} \eta_i^r n_{e,i} \leq \sum_{i=1}^{m_e} \eta_i^r$ (where $n_{e,i}$ is the degree of $\rho_{e,i}$). As a consequence, all the summands on the right hand side of (4.3) are ≤ 0, and thus we only have to check case (d).

Having all the information at hand, this is just a simple computation. If n is odd, we calculate, for the action on the cosets of D_{2m},

$$1 - \frac{1}{2m} + \frac{1}{2}\left[2 \cdot \left(\frac{n}{2m} - 1 - \frac{n/m-1}{2}\right) + \frac{1}{m} - 1\right] = 0.$$

For the action on the cosets of C_m we get

$$1 - \frac{1}{m} + \frac{1}{2}\left[2 \cdot \left(\frac{n}{m} - \frac{n}{m}\right) + \frac{2}{m} - 2\right] = 0.$$

If n is even, then, when n/m is odd, the calculation for the action on the cosets of D_{2m} is similar to the calculation for n odd. If n/m is even, then for the action on the cosets of D_{2m} we get

$$1 - \frac{1}{2m} + \frac{1}{2}\left[\frac{n}{2m} - 2 - \frac{n/m-2}{2} + \frac{n}{2m} - \frac{n}{2m} + \frac{1}{m} - 1\right] = 0.$$

The calculation of the action on the cosets of D'_{2m} is similar. The calculation of the action on the cosets of C_m for n is even is the same as when n is odd.

5.2. C_n

We divide into two cases. If $n \mid \frac{q+1}{2}$ then C_n acts freely on $\mathrm{st}(\tilde{v})$, and so (4.3) is clearly true.

If $n \mid \frac{q-1}{2}$ then C_n will have two fixed edges in $\mathrm{st}(\tilde{v})$. In this case for every transitive representation $\rho_{v,r}$ of C_n, (4.3) is $1 - \frac{n_{v,r}}{n} + \frac{n_{v,r}}{n} - 1 = 0$ (as the restriction of an action of C_n to itself is just the same action).

5.3. A_4

We divide again into different cases:

(a) Neither 2 nor 3 divides $\frac{q-1}{2}$.
(b) $3 \mid \frac{q-1}{2}$ but 2 does not.
(c) $2 \mid \frac{q-1}{2}$ but 3 does not.
(d) $2, 3 \mid \frac{q-1}{2}$.

In case (a) there is no non-trivial element of A_4 fixing an edge, so (4.3) is true. In case (b), every order 3 cyclic subgroup of A_4 has two fixed edges in $\mathrm{st}(\tilde{v})$. The order of the orbit is 4. As there are four cyclic groups of order 3 in A_4, each with two fixed points, we conclude that A_4 has two orbits of edges with stabilizers of order 3. So, in the graph of groups we have two edges with non-trivial edge groups, both isomorphic to C_3.

In case (c), every order 2 subgroup has two fixed edges. There are three such subgroups, and the order of the orbit is 6, so all edges with non-trivial stabilizer belong to the same orbit of A_4. We conclude that in the graph of groups there is a unique edge with a non-trivial edge group, isomorphic to C_2.

Case (d) is just a combination of (b) and (c), and in the graph of groups one gets two edges with stabilizers C_3, and one edge with stabilizer C_2.

We now present a table, showing the various transitive actions of A_4, and the action induced on subgroups isomorphic to C_2 and C_3. Note that it does not depend on the specific group chosen, as all groups isomorphic to C_3 are conjugate, as are all groups isomorphic to C_2 (f.p. stands for fixed points, reg. action stands for regular action).

Coset action	Order	Action of C_2	Action of C_3
A_4/A_4	1	$1\times$ f.p.	$1\times$ f.p.
A_4/C_3	4	$2\times$ reg. action	$1\times$ f.p., $1\times$ reg. action
$A_4/\mathbb{Z}_2 \times \mathbb{Z}_2$	3	$3\times$ f.p.	$1\times$ reg. action
A_4/C_2	6	$2\times$ f.p., $2\times$ reg. action	$2\times$ reg. action
$A_4/1$	12	$6\times$ reg. action	$4\times$ reg. action

Once again we have to check (4.3) in cases (b)–(d), for all transitive actions of A_4. Here again by Lemma 5.2 it is enough to check case (d). It turns out (4.3) is always ≥ 0. (Actually, in case (d), it is always 0. A similar phenomenon occurs for all the vertex stabilizers below for the case with maximal number of edges with non-trivial edge stabilizer.)

5.4. S_4

We note that S_4 can appear as a vertex stabilizer only when $q \equiv \pm 1 \pmod 8$ (see [H], or [LW] as above). In these cases, if $2 \mid \frac{q-1}{2}$ then also $4 \mid \frac{q-1}{2}$.

Keeping this fact in mind, we divide again into four different cases:

(a) Neither 2 nor 3 divides $\frac{q-1}{2}$.
(b) $3 \mid \frac{q-1}{2}$ but 2 does not.
(c) $4 \mid \frac{q-1}{2}$ but 3 does not.
(d) $3, 4 \mid \frac{q-1}{2}$.

Case (a) is again trivial. In case (b), using the same kind of reasoning as the one used for A_4, and noting we have four subgroups isomorphic to C_3 which are all isomorphic, we see that we have one orbit of edges with stabilizer isomorphic to C_3, and so in the graph of groups there is one edge with a non-trivial edge group isomorphic to C_3.

In case (c) we have two kinds of cyclic groups that are the edge stabilizers of edges in $\mathrm{st}(\tilde{v})$: subgroups isomorphic to C_4, and subgroups isomorphic to C_2. Again using arguments similar to those employed in the study of A_4, we see that in the graph of groups we have just one edge with edge group C_4 and one edge with edge group C_2 in $\mathrm{st}(v)$. Note that there are three subgroups isomorphic to C_4 that are all conjugate, as are the six edge groups isomorphic to C_2 (the latter are generated by transpositions—all other elements of order 2 in S_4 belong to groups isomorphic to C_4).

Finally, case (d) is again a combination of (b) and (c), and we get one edge with edge group C_2, one with edge group C_3 and one with C_4.

We now present a table for S_4. Note that C_4 has three transitive actions: trivial, regular, and its action on the cosets of $C_2 < C_4$.

Coset action	Order	Action of C_2	Action of C_3	Action of C_4
S_4/S_4	1	1× f.p.	1× f.p.	1× f.p.
S_4/A_4	2	1× reg. action	2× f.p.	1× action on C_4/C_2
S_4/D_8	3	1× f.p., 1× reg. action	1× reg. action	1× f.p., 1× action on C_4/C_2
S_4/S_3	4	2× f.p., 1× reg. action	1× f.p., 1× reg. action	1× reg. action
$S_4/\mathbb{Z}_2 \times \mathbb{Z}_2$ (1)	6	3× reg. action	2× reg. action	3× action on C_4/C_2
$S_4/\mathbb{Z}_2 \times \mathbb{Z}_2$ (2)	6	2× f.p., 2× reg. action	2× reg. action	1× action on C_4/C_2, 1× reg. action
S_4/C_4	6	3× reg. action	2× reg. action	2× f.p., 1× reg. action
S_4/C_3	8	4× reg. action	2× f.p., 2× reg. action	2× reg. action
S_4/C_2 (1)	12	2× f.p., 5× reg. action	4× reg. action	3× reg. action
S_4/C_2 (2)	12	6× reg. action	4× reg. action	2× action on C_4/C_2, 2× reg. action
$S_4/1$	24	12× reg. action	8× reg. action	6× reg. action

A few comments are in order. D_8 is the 2-Sylow subgroup of S_4. The first $\mathbb{Z}_2 \times \mathbb{Z}_2$ denotes the group $\langle (12)(34), (14)(23) \rangle$, the second is (conjugate to) $\langle (12), (34) \rangle$. The first C_2 is (conjugate to) $\langle(12)\rangle$, while the second is (conjugate to) $\langle(12)(34)\rangle$. Filling the table up is just a matter of routine reasoning, similar to that used for the previous cases. We only note that the action of $C_4 = \langle y\rangle$ on the orbit of some coset is the one on C_4/C_2 if y^2 has a fixed point, but y does not.

Here again one has to perform the now quite tedious work of checking that (4.3) is ≥ 0 for all transitive representations of S_4, for cases (b)–(d). Again, by Lemma 5.2 it is enough to check case (d).

5.5. A_5

Here we divide again into different cases:

(a) Neither 2 nor 3 nor 5 divides $\frac{q-1}{2}$.
(b) $2 \mid \frac{q-1}{2}$ but 3, 5 do not.
(c) $3 \mid \frac{q-1}{2}$ but 2, 5 do not.
(d) $5 \mid \frac{q-1}{2}$ but 2, 3 do not.
(e) $2, 3 \mid \frac{q-1}{2}$ but 5 does not.
(f) $2, 5 \mid \frac{q-1}{2}$ but 3 does not.
(g) $3, 5 \mid \frac{q-1}{2}$ but 2 does not.
(h) $2, 3, 5 \mid \frac{q-1}{2}$.

One can check that for each $p \in \{2, 3, 5\}$ that divides $(q-1)/2$ we have exactly one edge in the corresponding graph of groups with edge group isomorphic to C_p, and if p does not divide $(q-1)/2$ then there is no such edge group. Note that all subgroups isomorphic to C_p ($p \in \{2, 3, 5\}$) are conjugate.

Following is the table for A_5. The maximal subgroups of A_5 are known to be isomorphic to A_4, D_{10} or S_3, and all subgroups isomorphic to each of them are conjugate. (The coset actions on them are known to be the primitive actions of A_5.) So we can take them and their subgroups to give a full list of a transitive actions of A_5.

Coset Action	Order	Action of C_2	Action of C_3	Action of C_5
A_5/A_5	1	1× f.p.	1× f.p.	1× f.p.
A_5/A_4	5	1× f.p., 2× reg. action	2× f.p., 1× reg. action	1× reg. action
A_5/D_{10}	6	2× f.p., 2× reg. action	2× reg. action	1× f.p., 1× reg. action
A_5/S_3	10	2× f.p., 4× reg. action	1× f.p., 3× reg. action	2× reg. action
A_5/C_5	12	6× reg. action	4× reg. action	2× f.p., 2× reg. action
$A_5/\mathbb{Z}_2 \times \mathbb{Z}_2$	15	3× f.p., 6× reg. action	5× reg. action	3× reg. action
A_5/C_3	20	10× reg. action	2× f.p., 6× reg. action	4× reg. action
A_5/C_2	30	2× f.p., 14× reg. action	10× reg. action	6× reg. action
$A_5/1$	60	30× reg. action	20× reg. action	12× reg. action

Once again, one checks that (4.3) is ≥ 0 for all transitive actions of A_5. Here (by Lemma 5.2) it is enough to check case (h).

This completes the proof of the theorem.

6. Proof of Theorem 1.4

We still have to show how to deduce Theorem 1.4 from Theorem 1.5. We have $a_n(\Gamma) = t_n(\Gamma)/(n-1)!$, where $t_n(\Gamma)$ denotes the number of transitive permutation representations. Let Γ be a lattice as in Theorem 1.4. By the discussion in Section 2 and Theorem 1.5 we see that:

$$\begin{aligned}\frac{\log a_n(\Gamma)}{n\log n} &\leq \frac{\log\left(\frac{|\mathrm{Hom}(\Gamma,S_n)|}{(n-1)!}\right)}{n\log n} = \frac{\log|\mathrm{Hom}(\Gamma,S_n)|}{n\log n} - \frac{\log((n-1)!)}{n\log n}\\ &\leq -\chi(\Gamma) + O_\Gamma\left(\frac{1}{\log n}\right) + \epsilon(n)\end{aligned}$$

where $\epsilon(n) = 1 - \frac{\log((n-1)!)}{n\log n} = O(1/n)$ by standard computations. From this it is easy to see that $a_n(\Gamma) \leq (cn)^{-\chi(\Gamma)n}$, but as the first error term might depend on Γ (as is suggested by the subscript), the constant may also depend on Γ.

We thus have to examine this error term more closely. From the proof of Schlage-Puchta's Theorem 2.1 one can see that the error term is a sum of error terms (each independent of Γ), one for each summand in the sum defining the linear optimization problem. One can thus write the error term as $(\sum_{v\in V} m_v + \sum_{e\in E^g} m_e)O(1/\log n)$ where the error term is independent of Γ (see the remark following the computations in Section 2).

As we saw above, in our case the vertex and edge stabilizers belong to a finite set of groups (up to isomorphism), thus the numbers m_v, m_e are bounded. We conclude that $O_\Gamma(1/\log n) = (|V|+|E|)O(1/\log n)$, where the error term on the right is independent of Γ.

Now recall that

$$\mathrm{covol}(\Gamma) = \frac{1}{2}\sum_{v\in V}\frac{1}{|\Gamma_v|} = \frac{1}{q+1}\sum_{e\in E^g}\frac{1}{|\Gamma_e|}$$

(see [S1, p. 84]). Using again the fact that we deal with a finite set of possible vertex and edge groups, we have $|\Gamma_v|, |\Gamma_e| < c'$ for some constant c'.

Thus, $|V| \leq 2c'\cdot\mathrm{covol}(\Gamma)$, $|E| \leq (q+1)c'\cdot\mathrm{covol}(\Gamma)$. So, the sum $|V|+|E|$ is linearly bounded by $-\chi(\Gamma)$, and we have $O_\Gamma(1/\log n) = -\chi(\Gamma)\cdot O(1/\log n)$ where the error term on the right is independent of Γ.

In conclusion we have

$$\frac{\log a_n(\Gamma)}{n\log n} \leq -\chi(\Gamma)\left(1 + O\left(\frac{1}{\log n}\right)\right) + O\left(\frac{1}{n}\right).$$

This gives $a_n(\Gamma) \leq (cn)^{-\chi(\Gamma)n}$ for a constant c independent of Γ, and consequently the same is true for $s_n(\Gamma)$.

7. Counting maximal arithmetic lattices

Here we would like to indicate how one proves Theorem 1.3, using the same arguments used in [BGLS].

Let $H = PGL_2(\mathbb{R})^a \times PGL_2(\mathbb{C})^b \times \prod_{i=1}^r PGL_2(K_i)$ where the K_i are p-adic fields. Arithmetic lattices in H are obtained as follows: Let k be a number field with exactly b complex places, and at least a real places. Suppose also that there are finite places $v_1, \ldots, v_r$ such that the completion of k at v_i is isomorphic to K_i. Let A be a quaternion algebra over k such that A splits at exactly a real places, and A splits also at $v_1, \ldots, v_r$. (Recall that A *splits* at a place v if $A(k_v) = M_2(k_v) \equiv M_2(\mathbb{R})$, while A *ramifies* at v—where v is not complex—if $A(v)$ is isomorphic to the unique division algebra over k_v.) Let $S = \{v_1, \ldots, v_r\}$ and let R_S be the ring of *S-integers* in k, i.e. of elements in k which are integral at all places not in S. Let $\mathcal{O}$ be an *S-order* in A, that is, a full rank finitely generated R_S-lattice which is also a ring with 1. Let $\mathcal{O}^*$ denote the invertible elements of $\mathcal{O}$. Then $\mathcal{O}^*$ embeds naturally in $GL_2(\mathbb{R})^a \times GL_2(\mathbb{C})^b \times \prod_{i=1}^r GL_2(K_i)$, and the projection to H of this image, which we denote by Γ, is discrete of finite co-volume. A subgroup of H that is commensurable to Γ is called an *arithmetic subgroup* of H. This definition coincides with the usual definition of (irreducible) arithmetic subgroups of a semisimple group. All arithmetic subgroups obtained from an R_S-order in a quaternion algebra A are commensurable. We note by $C_S(A)$ the commensurability class of all arithmetic lattices commensurable to an arithmetic subgroup obtained from an R_S-order in A. (For the above see e.g. [V, Chapter 4].)

Note that in the case interesting us in this paper, of $PSL_2(K)$ for a p-adic field K, the field k above has to be totally real, A has to ramify at all real places, k has a place v such that $k_v = K$ and A splits at v, and $S = \{v\}$.

In what follows we cite various results of Borel concerning maximal arithmetic subgroups in H. These results are presented in a setting which is more general than the original setting that concerns only groups of the form $PGL_2(\mathbb{R})^a \times PGL_2(\mathbb{C})^b$, but it turns out that one can prove the results in the general case (where H may contain p-adic factors) using the same arguments as in the original proofs. These proofs can be found either in Borel's paper [B1] or in [MR, Chapter 11]. We note that in some volume calculations we do not give a precise numerical value with respect to a fixed Haar measure (as is done in [B1] for the Haar measure induced from the hyperbolic measure on the product of hyperbolic planes and hyperbolic 3-spaces), but give an expression that is fixed up to scaling, where the scaling depends only on the choice of the Haar measure. This is sufficient for our needs.

In the following we fix a set S of places. We shall therefore omit the subscript S in the notation $C_S(A)$. We will consider R_S-orders in A for this fixed S, and we will call them just orders for short.

We start with a description of a subset of lattices in $C(A)$ that contains all maximal lattices.

Let $\mathcal{O}$ be a maximal order in A (with respect to inclusion). The completion $\mathcal{O}_v$ at the place v is then a maximal order for every $v \notin S$. Let $\mathrm{Ram}(A)$ denote the set of places where A ramifies. If $v \notin \mathrm{Ram}(A)$, then $A_v = M_2(k_v)$. If v is a finite place,

$v \notin S \cup \mathrm{Ram}(A)$, this implies that $\mathcal{O}_v$ corresponds to a vertex in the corresponding Bruhat–Tits lattice, that is, $\mathcal{O}_v = \mathrm{End}(L)$ for some lattice L in a 2-dimensional vector space over k_v.

An important tool in the construction below is the local-global principle for orders in quaternion algebras, which says the following:

Proposition 7.1. *Fix an R_S-order $\mathcal{D}$ of A. There is a bijection between the collection of R_S-orders $\mathcal{O}$ of A and the set $\{(\mathcal{O}_v)_{v \notin S} \mid \mathcal{O}_v$ an order of A_v, $\mathcal{O}_v = \mathcal{D}_v$ almost everywhere$\}$ given by $\mathcal{O} \mapsto (\mathcal{O}_v)_{v \notin S}$. The inverse is given by $(\mathcal{O}_v)_{v \notin S} \mapsto \mathcal{O} = \{x \in A \mid x \in \mathcal{O}_v \ \forall v \notin S\}$.*

(See [V, p. 83]. The proposition is actually valid for all ideals in A.) In particular note that in this correspondence maximal orders correspond to sequences of maximal local orders.

Now fix a maximal order $\mathcal{O}$ as above, and let T be a finite set of finite places such that $T \cap (S \cup \mathrm{Ram}(A)) = \emptyset$. For every $v \in T$ choose $(\mathcal{O}_v)' := \mathrm{End}(L')$ where $[L']$ is a neighbour of $[L] := \mathcal{O}_v$ in the Bruhat–Tits tree. Then $(\mathcal{O}_v)'$ is a maximal order in A_v. Let now $\mathcal{O}'$ be the maximal order of A defined by $(\mathcal{O}')_v = \mathcal{O}_v$ for $v \notin T \cup S$, and $(\mathcal{O}')_v = (\mathcal{O}_v)'$ for $v \in T$. This defines a maximal R_S-order by the above proposition. Let $\mathcal{E} = \mathcal{O} \cap \mathcal{O}'$. Now define

$$\Gamma_{T,\mathcal{O}} := P(N(\mathcal{E})).$$

Here $N(\mathcal{E})$ denotes the normalizer of $\mathcal{E}$ in A^*, and P denotes the projection modulo the centre.

The group $\Gamma_{T,\mathcal{O}}$ depends on the choices of the local maximal orders $(\mathcal{O}_v)'$ (that is, on the choice of a certain neighbour to the vertex stabilized by $\mathcal{O}_v$), only up to conjugation by an element $x \in \mathcal{O}^1$. (See [MR, p. 354]. One uses there a strong approximation theorem. For our needs one should use its formulation as in [V, p. 81].)

We note that every lattice in $C(A)$ can be taken to be a subgroup of $P(A^*)$ as it is contained in the commensurator of $\phi(\mathcal{O}^1)$ which is equal to $\phi(A^*)$ (see [V, p. 106]). Thus we think of $C(A)$ as a family of subgroups of $P(A^*)$ commensurable to $P(N(\mathcal{O}))$ (and to any other member of the family $\{\Gamma_{T,\mathcal{O}}\}$).

We say that an element $x \in PGL_2(k_v)$ is *odd* if in its action on the Bruhat–Tits tree it interchanges the two orbits of vertices under the action of $PSL_2(k_v)$. We will say that an element $x \in \Gamma$ ($\Gamma \in C(A)$) is *odd* at a prime v if it is odd as an element of $PGL_2(k_v)$ (which in this case means it inverts an edge). We now have the following theorem:

Theorem 7.1. *Let $\Gamma \in C(A)$. Let $T(\Gamma)$ denote the set of places v of k where Γ contains an element odd at v. Then Γ is conjugate to a subgroup of $\Gamma_{T(\Gamma),\mathcal{O}}$ for some maximal order $\mathcal{O}$.*

In particular, the collection of lattices $\Gamma_{T,\mathcal{O}}$, where T runs over finite sets of places not in $\mathrm{Ram}(A) \cup S$, and $\mathcal{O}$ runs over the maximal orders of A, contains the set of maximal arithmetic subgroups in $C(A)$.

Note that if $\mathcal{O}$ and $\mathcal{O}'$ are conjugate maximal orders, then $\Gamma_{T,\mathcal{O}}$ and $\Gamma_{T,\mathcal{O}'}$ are also conjugate. Thus, in the theorem above we can take the family of $\Gamma_{T,\mathcal{O}}$ where $\mathcal{O}$ runs just over representative maximal orders from each conjugacy class of orders. The number of conjugacy classes is called the *type number* of A, and it is finite. (In our case, as we are interested in R_S-orders, this would more precisely be called the S-type number, which is still finite.)

The lattices $\Gamma_{\emptyset,\mathcal{O}}$ are maximal lattices; indeed, we will later see they are the lattices of minimal co-volume in $C(A)$. On the other hand, there may be other lattices $\Gamma_{T,\mathcal{O}}$ that are not maximal (and then they are contained in some $\Gamma_{T',\mathcal{O}}$ for some $T' \subset T$).

This theorem enables one to study the distribution of co-volumes of elements of $C(A)$. In particular one can show that all these co-volumes are integer multiples of a single number. For our needs we are just interested in finding the lattices of minimal co-volume, and calculating this volume (with respect to a given Haar measure). We first have the following theorem showing that the lattices of minimal co-volume in $C(A)$ are those of type $\Gamma_{\emptyset,\mathcal{O}}$. The index notation in the theorem is the generalized index:

$$[\Gamma_1 : \Gamma_2] = [\Gamma_1 : \Gamma_1 \cap \Gamma_2][\Gamma_2 : \Gamma_1 \cap \Gamma_2]^{-1}.$$

Theorem 7.2. *Let $\mathcal{O}$ be a maximal order in A. Then*

$$[\Gamma_{\emptyset,\mathcal{O}} : \Gamma_{T,\mathcal{O}}] = 2^{-m} \prod_{v \in T} (N(v) + 1)$$

for some $0 \leq m \leq |T|$. If $\mathcal{O}'$ is another maximal order, then $[\Gamma_{\emptyset,\mathcal{O}} : \Gamma_{\emptyset,\mathcal{O}'}] = 1$.

Here $N(v)$ denotes the norm of the prime v.

Next we give a formula for the co-volume of the image of $\mathcal{O}^1$, the group of elements in $\mathcal{O}$ with reduced norm 1, in H with respect to a fixed Haar measure. We will denote this image by $\Gamma_{\mathcal{O}^1}$. The formula will be given in terms of the arithmetic information on k, and of the information coming from A (essentially $\mathrm{Ram}(A)$ that determines A). It will be independent of the choice of the maximal order $\mathcal{O}$.

In [B1, Section 7] there is a precise calculation of $\mathrm{vol}(H/\Gamma_{\mathcal{O}^1})$ for the measure induced from the hyperbolic measure on the product of hyperbolic planes and hyperbolic 3-spaces. For our needs, however, it is enough to know that for a fixed Haar measure μ on H,

$$\mathrm{vol}(H/\Gamma_{\mathcal{O}^1}) = C \cdot \frac{\Delta_k^{3/2} \zeta_k(2) \prod_{v \in \mathrm{Ram}_f(A)} (N(v) - 1)}{(4\pi^2)^{|\mathrm{Ram}_\infty(A)|}}$$

where C is a fixed constant depending only on the choice of μ (and not on the maximal order $\mathcal{O}$), Δ_k is the discriminant of $k/\mathbb{Q}$, ζ_k is the Dedekind zeta function of k, and $\mathrm{Ram}_\infty(A)$ and $\mathrm{Ram}_f(A)$ denote the sets of archimedean, respectively finite places where A ramifies.

In particular, in the case that interests us, of an arithmetic lattice in $PSL_2(K)$, we have (up to a constant as above)

$$\mathrm{vol}(PGL_2(K)/\Gamma_{\mathcal{O}^1}) = \frac{\Delta_k^{3/2} \zeta_k(2) \prod_{v \in \mathrm{Ram}_f(A)} (N(v) - 1)}{(4\pi^2)^{d_k}}$$

where d_k is the degree of $k/\mathbb{Q}$.

For the case of lattices in $PGL_2(\mathbb{R})$ see e.g. [MR, p. 333], where the volume is normalized to be the one induced from the Riemannian measure on the hyperbolic plane.

Borel's formula for the minimal co-volume of a lattice in a commensurability class now follows from a calculation of the index $[\Gamma_{\emptyset,\mathcal{O}} : \Gamma_{\mathcal{O}^1}]$. This is done using the intermediary subgroup $\Gamma_{R_{\mathrm{f}}} = P(A_{R_{\mathrm{f}}})$ where $A_{R_{\mathrm{f}}} = \{\alpha \in N(\mathcal{O}) \mid n(\alpha) \in R_{\mathrm{f}}^*\}$ where R_{f} is the ring of elements of k which are integral at all finite places of k not in $\mathrm{Ram}_{\mathrm{f}}(A) \cup S$, and n denotes the reduced norm on A. Note that $\mathcal{O}^1 = \{\alpha \in N(\mathcal{O}) \mid n(\alpha) = 1\}$, and so $\Gamma_{\mathcal{O}^1} \subset \Gamma_{R_{\mathrm{f}}}$.

Let $R_{\mathrm{f},\infty}^*$ be the group of $\alpha \in (R_{\mathrm{f}})^*$ such that $\alpha > 0$ at all real places of k where A ramifies. (In particular, in the case of $PGL_2(K)$ over a p-adic field K, these are the totally real elements in $(R_{\mathrm{f}})^*$.) We have

Theorem 7.3. $\Gamma_{R_{\mathrm{f}}}/\Gamma_{\mathcal{O}^1} \cong R_{\mathrm{f},\infty}^*/(R_{\mathrm{f}}^*)^2$.

Note that in particular $[\Gamma_{R_{\mathrm{f}}} : \Gamma_{\mathcal{O}^1}] \leq 2^{r_1+r_2+r_{\mathrm{f}}}$, where r_1 (resp. r_2) is the number of real (resp. complex) places of k, and $r_{\mathrm{f}} = |S \cup \mathrm{Ram}_{\mathrm{f}}(A)|$, by the (generalized) Dirichlet unit theorem.

We now recall the following notation presented by Borel, and used also in [MR], with the slight adaptation to fit our case:

I_k = Group of fractional ideals of the ring R_S of <u>S-integers</u> in k.
P_k = Subgroup of principal fractional ideals.
$P_{k,\infty}$ = Subgroup of principal fractional ideals with a generator that is positive at all real ramified places of A.
M_1 = Subgroup of I_k generated by $P_{k,\infty}$ and the ideals $\mathcal{P} \in \mathrm{Ram}_{\mathrm{f}}(A)$, (that is, the ideals corresponding to the places in $\mathrm{Ram}_{\mathrm{f}}(A)$).
J_1 = I_k/M_1.
J_2 = Image of P_k in J_1.
${}_2J_1$ = Kernel of the mapping $y \mapsto y^2$ in J_1.

We then have the following result, which completes the calculation of the minimal co-volume of a lattice in $C(A)$:

Theorem 7.4. $[\Gamma_{\mathcal{O}} : \Gamma_{R_{\mathrm{f}}}] = [{}_2J_1 : J_2]$.

Note that the number $[{}_2J_1 : J_2]$ divides the order of the ideal class group of R_S and so also the order of the ordinary ideal class group of k (see [N, p. 75]).

In conclusion we get the following

Corollary 7.1. *The smallest co-volume of a group in the commensurability class $C(A)$ of an arithmetic group in H, for a fixed Haar measure μ on H, is (up to multiplication by a constant depending only on μ) equal to*

$$\frac{\Delta_k^{3/2}\zeta_k(2)\prod_{v\in\mathrm{Ram}_{\mathrm{f}}(A)}(N(v)-1)}{(4\pi^2)^{|\mathrm{Ram}_\infty(A)|}[R_{\mathrm{f},\infty}^* : (R_{\mathrm{f}}^*)^2][{}_2J_1 : J_2]}.$$

In particular in the case $G = PGL_2(K)$ for a p-adic field K the expression above reduces to

$$\frac{\Delta_k^{3/2}\zeta_k(2)\prod_{v\in\mathrm{Ram}_{\mathrm{f}}(A)}(N(v)-1)}{(4\pi^2)^{d_k}[R_{\mathrm{f},+}^* : (R_{\mathrm{f}}^*)^2][{}_2J_1 : J_2]} \tag{3.2}$$

(where $R_{\mathrm{f},+}$ is the subgroup of totally real units in R_{f}).

Finally, there is another result we need, due to Chinburg and Friedman [CF]. The result deals again with products of the form $PGL_2(\mathbb{R})^a \times PGL_2(\mathbb{C})^b$ and is based on Borel's work. Once more, their work generalizes easily to our case.

Proposition 7.2. *Let Γ be an arithmetic lattice in $C(A)$. Then*

$$\operatorname{covol}(\Gamma) > C_1 \exp\left(0.37 d_k - \frac{19.08}{h(k,2,A)}\right).$$

Here C_1 is a constant depending only on H and the Haar measure μ, and $h(k,2,A) := |J_1/J_2 \cdot J_1^2|$. In particular $h(k,2,A)$ is a positive integer. In [CF] a specific constant C is calculated corresponding to the Haar measure induced from the hyperbolic measure on the product of hyperbolic planes and hyperbolic 3-manifolds. For our needs the formulation above suffices as it yields the following lemma (see [BGLS, Lemma 3.1]):

Lemma 7.1. *There exist constants c_1, c_2 such that if $\Gamma \in C(A)$, where A is a quaternion algebra over the number field k, and $\operatorname{covol}(\Gamma) < x$ then $d_k \leq c_1 \log x + c_2$.*

There is only a slight change needed in [CF], which is the formulation of Lemma 7.2 below.

Lemma 7.2.

$$[R^*_{\mathrm{f},\infty} : (R^*_{\mathrm{f}})^2][_2J_1 : J_2] = 2^{r_{\mathrm{f}}+a+r_2}[K(A):k]$$

where r_2 is the number of complex places in k, $r_{\mathrm{f}} = |S \cup \mathrm{Ram}_{\mathrm{f}}(A)|$, and where $K(A)$ is the maximal abelian extension of k which is unramified at all finite places, whose Galois group is an elementary abelian 2-group and where the primes in $\mathrm{Ram}_{\mathrm{f}}(A)$ and S are completely decomposed.

(The change consists in the addition of S to the definition of r_{f} and $K(A)$.) The proof of this lemma remains essentially the same.

The rest of the proof in [CF] is based on Borel's results stated before, and on number-theoretic considerations which stay unaltered for the generalized case interesting us.

Having all the needed information at hand we can now prove Theorem 1.3 using the same proof as in [BGLS, Section 3]. Fix a number x. The proof is in three stages:

(1) One first bounds the number of possible fields k which can contribute a maximal arithmetic lattice of co-volume at most x. By this we mean that for k there exists a quaternion algebra A over k such that there exists a lattice of co-volume at most x which belongs to $C(A)$.i
(2) Given k, one bounds the number of quaternion algebras A over k that contribute maximal arithmetic lattices of co-volume at most x.
(3) Given A one finds a bound on the number of maximal arithmetic subgroups in $C(A)$ of co-volume at most x.

Combining these bounds one gets the required bound on the total number of maximal arithmetic subgroups of co-volume at most x.

We will describe how these bounds are obtained. We indicate the slight changes needed for the general case in all three stages. (The numbering (1)–(3) does not exist in the original).

In (1) one uses Borel's formula for the minimal co-volume. One has to use the estimate $[R^*_{\mathrm{f},\infty} : (R^*_{\mathrm{f}})^2][{}_2J_1 : J_2] \leq 2^{|\mathrm{Ram}_{\mathrm{f}}(A)|+|S|}h_k$ (adding S to the formula). In (2) the quaternion algebra A is ramified at $d_k - a$ real places. One uses the fact that a quaternion algebra over a number field k is completely determined by the places where it ramifies, and one bounds (polynomially in x) the number of sets of finite places in k where A may ramify if $C(A)$ contains a lattice with co-volume bounded by x. Here we only add that in order to bound the number of possible quaternion algebras one has to multiply this number by $\binom{d}{a}$; but using Stirling's formula and the lemma of Chinburg and Friedman one easily shows that $\binom{d}{a}$ is polynomially bounded by x and so also the number of possible quaternion algebras is bounded polynomially by x. In (3) one uses the fact that $\mathrm{covol}(\Gamma_{\emptyset,\mathcal{O}})$ is bounded from below by a fixed constant. For $PSL_2(\mathbb{R})$ this is a well known result by Siegel, and for $PSL_(\mathbb{C})$ this is a result of Kazhdan and Margulis. For PSL_2 over a p-adic field ($p \neq 2$) see [LW]. For the other cases this is a result of [B2].

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