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Quantization commutes with reduction in the non-compact setting: the case of holomorphic discrete series

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Abstract. In this paper we show that the multiplicities of holomorphic discrete series representations relative to reductive subgroups satisfy the credo "quantization commutes with reduction".

Keywords. Holomorphic discrete series, moment map, reduction, geometric quantization, transversally elliptic symbol

1. Introduction

The orbit method, introduced by Kirillov in the 1960s, proposes a correspondence between the irreducible unitary representations of a Lie group G and its orbits in the coadjoint representation: the representation $\pi^G_{\mathcal{O}}$ should be the geometric quantization of the Hamiltonian action of G on the coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$. The important feature of this correspondence is the functoriality relative to inclusions $G' \hookrightarrow G$ of closed subgroups. This means that if we start with representations $\pi^G_{\mathcal{O}}$ and $\pi^{G'}_{\mathcal{O}'}$ attached to coadjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$ and $\mathcal{O}' \subset (\mathfrak{g}')^*$, one expects that the multiplicity of $\pi^{G'}_{\mathcal{O}'}$ in the restriction $\pi^G_{\mathcal{O}}|_{G'}$ can be computed in terms of the space

$$\mathcal{O} \cap \pi^{-1}_{\mathfrak{g}',\mathfrak{g}}(\mathcal{O}')/G', \tag{1.1}$$

where $\pi_{\mathfrak{g}',\mathfrak{g}} : \mathfrak{g}^* \to (\mathfrak{g}')^*$ denotes the canonical projection. Symplectic geometers recognise that (1.1) is a symplectic reduced space in the sense of Marsden–Weinstein, since $\pi_{\mathfrak{g}',\mathfrak{g}} : \mathcal{O} \to (\mathfrak{g}')^*$ is the moment map relative to the Hamiltonian action of G' on $\mathcal{O}$. Let us give some examples where this theory is known to be valid.

For simply connected nilpotent Lie groups, Kirillov [Kir62] described the correspondence $\mathcal{O} \mapsto \pi^G_{\mathcal{O}}$, and Corwin–Greenleaf [CG88] proved its functoriality relative to subgroups: the multiplicity appearing in the direct integral decomposition of $\pi^G_{\mathcal{O}}|_{G'}$ is the cardinality of the reduced space (1.1).

For compact connected Lie groups, Heckman [Hec82] proved that the multiplicity is asymptotically given by the volume of the reduced space (1.1). Just after, Guillemin and

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Sternberg [GS82b] replaced this functoriality principle in a more geometric framework and proposed another version of this rule for a good quantization process: *quantization should commute with reduction*. This means that if $\mathcal{Q}_{G'}(M)$ is the geometric quantization of a Hamiltonian action of a compact Lie group G' on a symplectic manifold M, then the multiplicity of the representation $\pi^{G'}_{\mathcal{O}'}$ in $\mathcal{Q}_{G'}(M)$ should be the (dimension of the) geometric quantization of the reduced space $(\Phi^{G'}_M)^{-1}(\mathcal{O}')/G'$. Here $\Phi^{G'}_M : M \to (\mathfrak{g}')^*$ denotes the moment map.

A good quantization process for compact Lie group actions on compact symplectic manifolds turns out to be the equivariant index of a Dolbeault–Dirac operator [Sja96, Ver02]. In the late 1990s, Meinrenken and Meinrenken–Sjamaar proved that the principle of Guillemin–Sternberg works in this setting [Mei98, MS99]. Afterwards, this quantization procedure was extended to non-compact manifolds with a proper moment map by Ma–Zhang and the author [Par09, MZ09, Par11, MZ14]. See also the work of Duflo–Vergne on the multiplicities of tempered representations relative to compact subgroups [DV11].

The purpose of this article is to show that the "quantization commutes with reduction" principle holds in a case where the symmetry group is a real reductive Lie group. Loosely speaking, we prove that if $\pi^G_{\mathcal{O}}$ and $\pi^{G'}_{\mathcal{O}'}$ are holomorphic discrete series representations of real reductive Lie groups $G' \subset G$, then the multiplicity of $\pi^{G'}_{\mathcal{O}'}$ in the restriction $\pi^G_{\mathcal{O}}|_{G'}$ is equal to the quantization of the reduced space (1.1).

We now turn to a description of the contents of the consecutive sections, highlighting the main features.

In Section 2, we clarify previous work of Weinstein [Wei01] and Duflo–Vargas [DVa07, DVa10] concerning the Hamiltonian action of a connected reductive real Lie group G on a symplectic manifold M. The main point is that if the action of G on M is *proper* and the moment map $\Phi^G_M : M \to \mathfrak{g}^*$ relative to this action is *proper*, then the image of Φ^G_M is contained in the open subset

$$\mathfrak{g}^*_{\rm se} := \{\xi \in \mathfrak{g}^* \mid \text{the stabilizer subgroup } G_\xi \text{ is compact}\},$$

of strongly elliptic elements and the manifold has a decomposition

$$M \simeq G \times_K Y. \tag{1.2}$$

Here K is a maximal compact subgroup of G, and Y is a closed K-invariant symplectic submanifold of M. Thanks to (1.2), we remark that the reduced space $(\Phi^G_M)^{-1}(\mathcal{O})/G$ is connected for any coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$; this is a notable difference with the nilpotent case where the reduced space (1.1) can be disconnected.

The decomposition (1.2) will be the main ingredient of this paper to prove some "quantization commutes with reduction" phenomenon. Note that Hochs [Hoc09] already used this idea when the manifold Y is compact to get a "quantization commutes with reduction" theorem in the setting of KK-theory. He applied some induction process, while we will use (1.2) to prove some functoriality relative to a restriction procedure.

In this context, it is natural to look at the induced action of a reductive subgroup $G' \subset G$ on M, and we have another decomposition $M \simeq G' \times_{K'} Y'$ if the moment map $\Phi_M^{G'}$ is proper. In Section 2.3, we give a criterion that ensures the properness of $\Phi_M^{G'}$.

In Section 3, we turn to a close study of holomorphic discrete series representations of a reductive Lie group G. Recall that the parametrization of those representations depends on the choice of an element z in the centre of the Lie algebra of K such that the adjoint map $\mathrm{ad}(z)$ defines a complex structure on $\mathfrak{g}/\mathfrak{k}$. Let T be a maximal torus in K, with Lie algebra $\mathfrak{t}$. The existence of an element z forces $\mathfrak{t}$ to be a Cartan subalgebra of $\mathfrak{g}$, and it defines a closed cone $\mathcal{C}_{\mathrm{hol}}^{\rho}(z) \subset \mathfrak{t}^*$ (see (3.2)). If $\Lambda^* \subset \mathfrak{t}^*$ is the weight lattice, we consider the subset

$$\widehat{G}_{\mathrm{hol}}(z) := \mathcal{C}_{\mathrm{hol}}^{\rho}(z) \cap \Lambda_+^*$$

where Λ_+^* is the set of dominant weights. The work of Harish-Chandra tells us that we can attach a holomorphic discrete series representation V_λ^G to any $\lambda \in \widehat{G}_{\mathrm{hol}}(z)$.

In Section 4, we look at formal geometric quantization procedures attached to the Hamiltonian action of G on a symplectic manifold M. We suppose that the properness assumptions are satisfied and that

$$\mathrm{Image}(\Phi_M^G) \subset G \cdot \mathcal{C}_{\mathrm{hol}}^{\rho}(z). \tag{1.3}$$

We define the formal geometric quantization of the G-action on M as the formal sum

$$\mathcal{Q}_G^{-\infty}(M) := \sum_{\lambda \in \widehat{G}_{\mathrm{hol}}(z)} \mathcal{Q}(M_{\lambda,G}) V_\lambda^G, \tag{1.4}$$

where $\mathcal{Q}(M_{\lambda,G}) \in \mathbb{Z}$ is the geometric quantization of the *compact* symplectic reduced space $M_{\lambda,G} := (\Phi_M^G)^{-1}(G \cdot \lambda)/G$.

Since the moment map Φ_M^K is also proper, we define similarly the formal geometric quantization of the K-action on M as

$$\mathcal{Q}_K^{-\infty}(M) := \sum_{\mu \in \Lambda_+^*} \mathcal{Q}(M_{\mu,K}) V_\mu^K, \tag{1.5}$$

where $M_{\mu,K} := (\Phi_M^K)^{-1}(K \cdot \mu)/K$, and V_μ^K denotes the irreducible representation of K with highest weight μ. The formal quantization procedure $\mathcal{Q}_K^{-\infty}$, together with its functorial properties, has been studied by Ma–Zhang and the author [Par09, MZ09, Par11, MZ14].

Let $R^{-\infty}(G, z)$ be the $\mathbb{Z}$-module formed by the infinite sums $\sum_{\lambda \in \widehat{G}_{\mathrm{hol}}(z)} m_\lambda V_\lambda^G$ with $m_\lambda \in \mathbb{Z}$. We also consider the $\mathbb{Z}$-module $R^{-\infty}(K)$ formed by the infinite sums $\sum_{\mu \in \Lambda_+^*} n_\mu V_\mu^K$ with $n_\mu \in \mathbb{Z}$. The following basic result will be an important tool in our paper (see Lemma 3.11).

Lemma A. *The restriction morphism* $\mathbf{r}_{K,G} : R^{-\infty}(G, z) \to R^{-\infty}(K)$ *is injective.*

We shall need to work under one of the following hypotheses:

A1. The set of critical points of the function $\|\Phi_M^G\|^2$ is compact,
A2. The map $\langle \Phi_M^G, z \rangle : M \to \mathbb{R}$ is proper.

We will see in Lemma 4.12 that **A2** is automatically satisfied when $\mathfrak{g}$ is simple. We can now state state one of our main results (see Theorem 4.10).

Theorem B. *Under Assumption* **A1** *or* **A2***,*

$$\mathbf{r}_{K,G}(\mathcal{Q}_G^{-\infty}(M)) = \mathcal{Q}_K^{-\infty}(M), \qquad \mathcal{Q}_K^{-\infty}(M) = \mathcal{Q}_K^{-\infty}(Y) \otimes S^\bullet(\mathfrak{p}).$$

Here $\mathcal{Q}_K^{-\infty}(Y)$ is the formal geometric quantization of the slice Y, and $S^\bullet(\mathfrak{p})$ is the symmetric algebra of the complex K-module $\mathfrak{p} := (\mathfrak{g}/\mathfrak{k}, \mathrm{ad}(z))$.

We will apply Theorem B to $M = G \cdot \lambda$ for $\lambda \in \widehat{G}_{\mathrm{hol}}(z)$. Here $\mathcal{Q}_G^{-\infty}(G \cdot \lambda) = V_\lambda^G$ by definition, but we will restrict the action to a reductive subgroup $G' \subset G$ for which $z \in \mathfrak{g}'$. It is not difficult to see that the moment map $\Phi_{G\cdot\lambda}^{G'}$ is proper, and we prove in Proposition 3.13 that the inclusion (1.3) holds. The term $\mathcal{Q}_{G'}^{-\infty}(G \cdot \lambda) \in R^{-\infty}(G', z)$ is then well defined.

It is well known [Mar75, JV79, Kob98] that the representation V_λ^G admits an admissible restriction to G': the restriction $V_\lambda^G|_{G'}$ is a discrete sum formed by holomorphic discrete series representations $V_\mu^{G'}$ with $\mu \in \widehat{G'}_{\mathrm{hol}}(z)$. We can now state the major result of this paper (see Theorem 4.11).

Theorem C. *Let $\lambda \in \widehat{G}_{\mathrm{hol}}(z)$. Then*

$$V_\lambda^G|_{G'} = \mathcal{Q}_{G'}^{-\infty}(G \cdot \lambda).$$

This means that for any $\mu \in \widehat{G'}_{\mathrm{hol}}(z)$, the multiplicity of the representation $V_\mu^{G'}$ in $V_\lambda^G|_{G'}$ is equal to the geometric quantization $\mathcal{Q}((G\cdot\lambda)_{\mu,G'}) \in \mathbb{Z}$ of the (compact) reduced space $(G \cdot \lambda)_{\mu,G'}$.

In [JV79], Jakobsen–Vergne proposed another formula for the multiplicity of $V_\mu^{G'}$ in $V_\lambda^G|_{G'}$. In Section 4.4, we explain how to recover their result from Theorem C.

Section 5 is devoted to the proofs of the main results of this paper. We use previous work of the author on localization techniques in the setting of transversally elliptic operators.

Notation. In this paper, G denotes a connected real reductive Lie group; we follow the convention of Knapp [Kna04]. We have a Cartan involution Θ on G such that the fixed point set $K := G^\Theta$ is a connected maximal compact subgroup. We have Cartan decompositions: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ at the level of Lie algebras and $G \simeq K \times \exp(\mathfrak{p})$ at the level of groups. We denote by b a G-invariant non-degenerate bilinear form on $\mathfrak{g}$ that is equal to the Killing form on $[\mathfrak{g}, \mathfrak{g}]$, and that defines a K-invariant scalar product $(X, Y) := -b(X, \Theta(Y))$. We will use the K-equivariant identification $\xi \mapsto \tilde{\xi}$, $\mathfrak{g}^* \simeq \mathfrak{g}$ defined by $(\tilde{\xi}, X) := \langle \xi, X \rangle$ for $\xi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$.

When V and V' are two representations of a group H, the multiplicity of V in V' will be denoted $[V : V']$.

2. Hamiltonian actions of real reductive Lie groups

This section is mainly a synthesis of previous work by Weinstein [Wei01], Duflo–Vargas [DVa07, DVa10] and Hochs [Hoc09], except the criterion that we obtain in Section 2.3.

Let G be a connected real reductive Lie group. We consider a Hamiltonian action of G on a connected symplectic manifold (M, Ω_M). The corresponding moment map $\Phi_M^G : M \to \mathfrak{g}^*$ is defined (modulo a constant) by the relations

$$\iota(X_M)\Omega_M = -d\langle \Phi_M^G, X\rangle, \quad \forall X \in \mathfrak{g}, \tag{2.1}$$

where $X_M(m) := \frac{d}{ds} e^{-sX} \cdot m|_{s=0}$ is the vector field generated by $X \in \mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. Let $K \subset G$ be the maximal compact subgroup with Lie algebra $\mathfrak{k}$. Thus we have a decomposition

$$\Phi_M^G = \Phi_M^K \oplus \Phi_M^{\mathfrak{p}}$$

where $\Phi_M^K : M \to \mathfrak{k}^*$ is the moment map relative to the action of K on (M, Ω_M), and $\Phi_M^{\mathfrak{p}} : M \to \mathfrak{p}^*$ is K-equivariant.

We will denote by κ^G, κ^K and $\kappa^{\mathfrak{p}}$ the Hamiltonian vectors fields of the K-invariant functions $\frac{-1}{2}\|\Phi_M^G\|^2$, $\frac{-1}{2}\|\Phi_M^K\|^2$, and $\frac{-1}{2}\|\Phi_M^{\mathfrak{p}}\|^2$ respectively. Relations (2.1) give

$$\kappa^-(m) = \left[\widetilde{\Phi_M^-(m)}\right]_M(m), \quad \forall m \in M, \tag{2.2}$$

for $- \in \{G, K, \mathfrak{p}\}$.

2.1. Proper actions

In this section we suppose

C1. The action of G on M is proper.

We then have the fundamental fact:

Lemma 2.1. • *The map $\Phi_M^{\mathfrak{p}} : M \to \mathfrak{p}^*$ is a K-equivariant submersion, so for any $a \in \mathfrak{p}^*$, the fibre $Y_a := (\Phi_M^{\mathfrak{p}})^{-1}(a)$ is either empty or a submanifold of M.*

• *The set of critical points of $\|\Phi_M^{\mathfrak{p}}\|^2 : M \to \mathbb{R}$ is $Y_0 := (\Phi_M^{\mathfrak{p}})^{-1}(0)$.*

Proof. Let us prove the first point. Let $m \in M$. Since the tangent map $\mathbf{T}\Phi_M^{\mathfrak{p}}(m) : \mathbf{T}_m M \to \mathfrak{p}^*$ satisfies

$$\langle \mathbf{T}\Phi_M^{\mathfrak{p}}(m), X\rangle = -\iota(X_M)\Omega_M|_m, \quad \forall X \in \mathfrak{p}, \tag{2.3}$$

the orthogonal of the image of $\mathbf{T}\Phi_M^{\mathfrak{p}}(m)$ is $\mathfrak{p}_m := \{X \in \mathfrak{p} \mid X_M(m) = 0\}$. As the action of G on M is proper, the stabilizer subgroup G_m is compact. This forces $\mathfrak{p}_m = \mathrm{Lie}(G_m) \cap \mathfrak{p}$ to be $\{0\}$. Thus $\mathbf{T}\Phi_M^{\mathfrak{p}}(m)$ is onto and the first point is proved.

Let $m \in M$ be a critical point of $\|\Phi_M^{\mathfrak{p}}\|^2$. The Hamiltonian vector field $\kappa^{\mathfrak{p}}$ vanishes at m, and (2.2) tells us that $\widetilde{\Phi_M^{\mathfrak{p}}(m)} \in \mathfrak{p}_m = \{0\}$. The second point is proved. □

For the remaining part of this section, we consider the K-invariant submanifold $Y := Y_0 \subset M$, which we suppose to be non-empty. Let us consider the restriction Ω_Y of the 2-form Ω_M to Y. For $y \in Y$, let $\mathfrak{p} \cdot y = \{X_M(y) \mid X \in \mathfrak{p}\} \subset \mathbf{T}_y M$. The tangent space $\mathbf{T}_y Y$ is by definition the kernel of $\mathbf{T}\Phi_M^{\mathfrak{p}}(y)$. Relations (2.3) show that

$$\mathbf{T}_y Y = (\mathfrak{p} \cdot y)^{\perp} \tag{2.4}$$

where the orthogonal is taken relative to the symplectic form. Hence the kernel of $\Omega_Y|_y$ is equal to $(\mathfrak{p} \cdot y)^{\perp} \cap \mathfrak{p} \cdot y$. For $X, X' \in \mathfrak{p}$ and $y \in Y$, we have

$$\Omega_M(X_M(y), X'_M(y)) = \langle \Phi_M^G(y), [X, X'] \rangle = \langle \Phi_M^K(y), [X, X'] \rangle.$$

Hence $(\mathfrak{p} \cdot y)^{\perp} \cap \mathfrak{p} \cdot y \simeq \mathfrak{g}_\xi \cap \mathfrak{p}$ for $\xi = \Phi_M^K(y)$. Note that for $\xi \in \mathfrak{k}$, we have $\mathfrak{g}_\xi = \mathfrak{g}_\xi \cap \mathfrak{k} \oplus \mathfrak{g}_\xi \cap \mathfrak{p}$. We have thus proved

Lemma 2.2. *Let $y \in Y$. The 2-form $\Omega_Y|_y$ is non-degenerate if and only if $\mathfrak{g}_\xi \subset \mathfrak{k}$ for $\xi = \Phi_M^K(y)$.*

We have a canonical G-equivariant map $\pi : G \times_K Y \to M$ that sends $[g, y]$ to $g \cdot y$. Following Weinstein [Wei01], we consider the G-invariant open subset

$$\mathfrak{g}^*_{\mathrm{se}} = \{\xi \in \mathfrak{g}^* \mid G_\xi \text{ is compact}\} \tag{2.5}$$

of strongly elliptic elements. It is non-empty if and only if the groups G and K have the same rank; real reductive Lie groups with this property are the ones admitting discrete series. We note that $\mathfrak{k}^*_{\mathrm{se}} := \mathfrak{g}^*_{\mathrm{se}} \cap \mathfrak{k}^*$ is equal to $\{\xi \in \mathfrak{k}^* \mid G_\xi \subset K\}$, and

$$\mathfrak{g}^*_{\mathrm{se}} = \mathrm{Ad}^*(G) \cdot \mathfrak{k}^*_{\mathrm{se}}. \tag{2.6}$$

Let us consider the invariant open subsets $M_{\mathrm{se}} = (\Phi_M^G)^{-1}(\mathfrak{g}^*_{\mathrm{se}}) \subset M$ and $Y_{\mathrm{se}} := Y \cap M_{\mathrm{se}} \subset Y$.

Lemma 2.3. • *The 2-form Ω_Y is non-degenerate on Y_{se}.*
- *The action of the group K on $(Y_{\mathrm{se}}, \Omega_{Y_{\mathrm{se}}})$ is Hamiltonian, with moment map the restriction $\Phi^K_{Y_{\mathrm{se}}}$ of Φ_M^G to Y_{se}.*
- *The map π induces a G-equivariant diffeomorphism $\pi_{\mathrm{se}} : G \times_K Y_{\mathrm{se}} \to M_{\mathrm{se}}$.*

Proof. The first point is a direct consequence of Lemma 2.2. The second point is immediate. Let us check the last point.

Relation (2.6) shows that π_{se} is onto. Let $[g_0, y_0], [g_1, y_1]$ be such that $g_0 \cdot y_0 = g_1 \cdot y_1$. Then by taking the image by the moment map, we obtain $\mathrm{Ad}^*(g_0)\xi_0 = \mathrm{Ad}^*(g_1)\xi_1$ where the $\xi_k = \Phi_M^K(y_k)$ belong to $\mathfrak{k}^*_{\mathrm{se}}$. Let $h = g_1^{-1} g_0 \in G$. We have $\mathrm{Ad}^*(h)\xi_0 = \xi_1$, and $\mathrm{Ad}^*(\Theta(h))\xi_0 = \xi_1$ by taking the Cartan involution. Finally, $h^{-1}\Theta(h) \in G_{\xi_0}$. Since $G_{\xi_0} \subset K$, we find that $h \in K$, and finally $[g_0, y_0] = [g_1, y_1]$ in $G \times_K Y_{\mathrm{se}}$. □

Let us denote by $\Omega_{M_{\mathrm{se}}}$ the restriction of the symplectic form Ω_M to the open subset M_{se}. We will finish this section by giving a simple expression for the pull-back $\pi^*_{\mathrm{se}}(\Omega_{M_{\mathrm{se}}}) \in \mathcal{A}^2(G \times_K Y_{\mathrm{se}})$.

Let $\theta^G \in \mathcal{A}^1(G) \otimes \mathfrak{g}$ be the canonical connection 1-form relative to the G-action by right translations: $\iota(X^r)\theta^G = X$ for all $X \in \mathfrak{g}$, where $X^r(g) = \frac{d}{dt}(ge^{tX})\big|_0$. Let $\theta^K \in \mathcal{A}^1(G) \otimes \mathfrak{k}$ be the composition of θ^G with the orthogonal projection $X \to X_{\mathfrak{k}}$ from $\mathfrak{g}$ to $\mathfrak{k}$. We will use the $G \times K$-invariant 1-form on $G \times Y_{\mathrm{se}}$ defined by $\langle \Phi^K_{Y_{\mathrm{se}}}, \theta^K \rangle$.

Note that the space of differential forms on $G \times_K Y_{\mathrm{se}}$ admits a canonical identification with the space of K-basic differential forms on $G \times Y_{\mathrm{se}}$.

Proposition 2.4. *The* 2*-form* $\pi^*_{\mathrm{se}}(\Omega_{M_{\mathrm{se}}})$ *is equal to the* K*-basic,* G*-invariant,* 2*-form* $\Omega_{Y_{\mathrm{se}}} - d\langle \Phi^K_{Y_{\mathrm{se}}}, \theta^K \rangle$.

Proof. Let $\pi_1 : G \times Y_{\mathrm{se}} \to M_{\mathrm{se}}$ be the map that factorizes π_{se}. By G-invariance, we need only show that $\pi_1^*(\Omega_{M_{\mathrm{se}}}) = \Omega_{Y_{\mathrm{se}}} - d\langle \Phi^K_{Y_{\mathrm{se}}}, \theta^K \rangle$ at the point $(1, y) \in G \times Y_{\mathrm{se}}$. Let $(X', v'), (X, v) \in \mathfrak{g} \times \mathbf{T}_y Y_{\mathrm{se}} = \mathbf{T}_{(1,y)}(G \times Y_{\mathrm{se}})$. We have

$$\begin{aligned}
\pi_1^*(\Omega_{M_{\mathrm{se}}})((X', v'), (X, v)) &= \Omega_M(-X'_M(y) + v', -X_M(y) + v) \\
&= \Omega_M(v', v) + \Omega_M(X'_M(y), X_M(y)) - \Omega_M(X'_M(y), v) + \Omega_M(X_M(y), v') \\
&= \Omega_{Y_{\mathrm{se}}}(v', v) + \underbrace{\langle \Phi^K_{Y_{\mathrm{se}}}(y), [X', X]_{\mathfrak{k}} \rangle}_{\mathbf{A}} + \underbrace{d\langle \Phi^K_{Y_{\mathrm{se}}}, X'_{\mathfrak{k}} \rangle|_y(v) - d\langle \Phi^K_{Y_{\mathrm{se}}}, X_{\mathfrak{k}} \rangle|_y(v')}_{\mathbf{B}} \\
&= \Omega_{Y_{\mathrm{se}}}(v', v) - d\langle \Phi^K_{Y_{\mathrm{se}}}, \theta^K \rangle((X', v'), (X, v)).
\end{aligned}$$

The last equality is due to the fact that $\mathbf{A} = -\langle \Phi^K_{Y_{\mathrm{se}}}(y), d\theta^K|_1 \rangle(X', X)$ since $d\theta^K((X')^r, (X)^r) = -[X', X]_{\mathfrak{k}}$ and $\mathbf{B} = -\langle d\Phi^K_{Y_{\mathrm{se}}}, \theta^K \rangle((X', v'), (X, v))$. □

2.2. *Proper moment map*

In this section we consider "proper[2]" Hamiltonian G-manifolds: the Hamiltonian action of the real reductive group G on a symplectic manifold (M, Ω_M) satisfies the following conditions:

C1. The action of G on M is proper.
C2. The moment map $\Phi^G_M : M \to \mathfrak{g}^*$ is a proper map.

Condition **C2** implies that the image of Φ^G_M is a closed subset of $\mathfrak{g}^*$. Let $\tilde{A}$ be a compact subset of Image(Φ^G_M), and let $A = (\Phi^G_M)^{-1}(\tilde{A})$ be the corresponding compact subset of M. We then see that, for all $g \in G$,

$$g \cdot A \cap A \neq \emptyset \;\Leftrightarrow\; g \cdot \tilde{A} \cap \tilde{A} \neq \emptyset.$$

Condition **C1** tells us that $\{g \in G \mid g \cdot A \cap A \neq \emptyset\}$ is compact, so $\{g \in G \mid g \cdot \tilde{A} \cap \tilde{A} \neq \emptyset\}$ is compact for any compact set $\tilde{A}$ in the image of Φ^G_M. By taking $\tilde{A}$ equal to a point, we get

Lemma 2.5. *Under* **C1** *and* **C2***, the image of* Φ^G_M *is contained in the open subset* $\mathfrak{g}^*_{\mathrm{se}}$ *of strongly elliptic elements* (*see* (2.5))*. In particular,* $0 \notin \mathrm{Image}(\Phi^G_M)$.

The previous lemma gives a strong condition on the reductive Lie group G: it may act in a Hamiltonian fashion on a symplectic manifold, *properly* and with a *proper* moment map only if $\mathfrak{g}^*_{se} \neq \emptyset$.

If we use the last section we see that $M = M_{se}$. We summarize with

Proposition 2.6.
- *The set Y is a K-invariant symplectic submanifold of M, with proper moment map Φ^K_Y equal to the restriction of Φ^G_M to Y.*
- *The manifold $G \times_K Y$ carries an induced symplectic structure $\Omega_Y - d\langle \Phi^K_Y, \theta^K\rangle$. The corresponding moment map is $[g, y] \mapsto g \cdot \Phi^K_Y(y)$.*
- *The map $\pi : G \times_K Y \to M$ is a G-equivariant diffeomorphism of Hamiltonian G-manifolds.*
- *The manifold Y is connected.*

Proof. Thanks to the Cartan decomposition, the third point implies that $\mathfrak{p} \times Y \simeq M$ and then the last point follows. □

Let $\mathfrak{t}$ be the Lie algebra of a maximal torus T in K. We know that $\mathfrak{g}^*_{se} \neq \emptyset$ if and only if $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{k}^*_{se} = \mathfrak{g}^*_{se} \cap \mathfrak{k}^*$ and $\mathfrak{t}^*_{se} = \mathfrak{g}^*_{se} \cap \mathfrak{t}^*$. We have $\mathfrak{g}^*_{se} = \mathrm{Ad}^*(G) \cdot \mathfrak{k}^*_{se} = \mathrm{Ad}^*(G) \cdot \mathfrak{t}^*_{se}$.

Let $\Lambda^* \subset \mathfrak{t}^*$ be the weight lattice: $\alpha \in \Lambda^*$ if $i\alpha$ is the differential of a character of T. Let $\mathfrak{R} \subset \Lambda^*$ be the set of roots for the action of T on $\mathfrak{g}\otimes\mathbb{C}$. We have $\mathfrak{R} = \mathfrak{R}_c \cup \mathfrak{R}_n$ where $\mathfrak{R}_c$ and $\mathfrak{R}_n$ are respectively the set of roots for the action of T on $\mathfrak{k} \otimes \mathbb{C}$ and $\mathfrak{p} \otimes \mathbb{C}$. We fix a system $\mathfrak{R}^+_c$ of positive roots in $\mathfrak{R}_c$; let $\mathfrak{t}^*_{\geq 0} \subset \mathfrak{t}^*$ be the corresponding Weyl chamber. Let $W = W(K, T)$ be the Weyl group. We then have

$$\mathfrak{t}^*_{se} = W \cdot (\mathfrak{t}^*_{se} \cap \mathfrak{t}^*_{\geq 0}) = W \cdot \{\xi \in \mathfrak{t}^*_{\geq 0} \mid (\xi, \alpha) \neq 0,\ \forall\alpha \in \mathfrak{R}_n\} = W \cdot (\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_N),$$

where each $\mathcal{C}_j$ is an open cone of the Weyl chamber.

We recover the following result due to Weinstein [Wei01].

Theorem 2.7.
- *The Kirwan set $\Delta_K(Y) := \mathrm{Image}(\Phi^K_Y) \cap \mathfrak{t}^*_{\geq 0}$ is a closed convex locally polyhedral subset contained in one cone $\mathcal{C}_j$.*
- $\mathrm{Image}(\Phi^G_M)/\mathrm{Ad}^*(G) \simeq \Delta_K(Y)$.

Proof. Since Φ^K_Y is proper and Y is connected, the Convexity Theorem [Ati82, GS82a, Kir84, LMTW98] tells us that $\Delta_K(Y)$ is a closed, convex, locally polyhedral subset of the Weyl chamber. On the other hand, the image of Φ^K_Y is contained in $\mathfrak{k}^*_{se}$. Thus $\Delta_K(Y) \subset \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_N$, but since $\Delta_K(Y)$ is connected we have $\Delta_K(Y) \subset \mathcal{C}_j$ for a unique cone $\mathcal{C}_j$. The last point is obvious since the isomorphism $\pi : G \times_K Y \to M$ satisfies $\Phi^G_M \circ \pi([g, y]) = g \cdot \Phi^K_Y(y)$. □

We finish this section with

Theorem 2.8. *Let (M, Ω_M, Φ^G_M) be a Hamiltonian G-manifold.*

- *If the G-action on M is* proper, *then Φ^G_M is* proper *if and only if Φ^K_M is* proper.
- *Under conditions* **C1** *and* **C2**, *we have*

$$\emptyset \neq \mathrm{Cr}(\|\Phi^G_M\|^2) = \mathrm{Cr}(\|\Phi^K_M\|^2) = \mathrm{Cr}(\|\Phi^K_Y\|^2) \subset Y.$$

Proof. Let us prove the first point. As $\|\Phi^G_M\| \geq \|\Phi^K_M\|$ one implication trivially holds. Suppose now that Φ^G_M is proper. Thanks to Propositions 2.6 and 2.7 we know that $M = G\times_K Y$ where Y is a K-Hamiltonian manifold, with proper moment map Φ^K_Y, and with Kirwan set $\Delta_K(Y)$ being a closed set in $\mathfrak{t}^*_{\rm se}$. Let $R>0$. We consider

- $M_{\leq R} = \{m\in M \mid \|\Phi^K_M(m)\|^2 \leq R\}$,
- $Y_{\leq R} = \{y\in Y \mid \|\Phi^K_Y(y)\|^2 \leq R\}$, which is a compact subset of Y,
- $\mathcal{K} = \Delta_K(Y)\cap\{\xi\in\mathfrak{t}^* \mid \|\xi\|^2\leq R\}$, which is a compact subset of $\mathfrak{t}^*_{\rm se}$,
- $c(\mathcal{K}) = \inf_{\xi\in\mathcal{K},\,\alpha\in\mathfrak{R}_n}\frac{|(\alpha,\xi)|^2}{2\|\xi\|}$, which is strictly positive.

We have to show that $M_{\leq R}$ is a compact subset of M. Take $m=[ke^X, y]$ with $k\in K$ and $X\in\mathfrak{p}$. Since $\Phi^G_M(m) = ke^X\cdot\Phi^K_Y(y)$, we have

$$\|\Phi^K_M(m)\|^2 \geq -b(\Phi^G_M(m),\Phi^G_M(m)) = \|\Phi^K_Y(y)\|^2, \qquad \|\Phi^K_M(m)\|^2 = \|[e^X\cdot\Phi^K_Y(y)]_{\mathfrak{k}^*}\|^2.$$

Hence if $m=[ke^X,y]\in M_{\leq R}$, we have $y\in Y_{\leq R}$ and $\Phi^K_Y(y) = k_o\cdot\xi_o$ for some $k_o\in K$ and $\xi_o\in\mathcal{K}$. Hence, for $X' = k_o^{-1}\cdot X\in\mathfrak{p}$,

$$\begin{aligned}\|\Phi^K_M(m)\| = \|[e^{X'}\cdot\xi_o]_{\mathfrak{k}^*}\| &\geq \frac{1}{\|\xi_o\|}(e^{X'}\cdot\xi_o,\xi_o) = \frac{1}{\|\xi_o\|}\sum_{n\in\mathbb{N}}\frac{1}{2n!}\|\mathrm{ad}^*(X')^n\xi_o\|^2\\ &\geq \frac{1}{2\|\xi_o\|}\|\mathrm{ad}^*(X')\xi_o\|^2 \geq c(\mathcal{K})\|X\|^2.\end{aligned}$$

Thus if $m=[ke^X,y]\in M_{\leq R}$, the vector X is bounded and y belongs to the compact subset $Y_{\leq R}$. This proves that $M_{\leq R}$ is compact.

Let us turn to the last point. First we note that since the map $\|\Phi^G_M\|^2$ is proper, then its infimum is attained, and so $\mathrm{Cr}(\|\Phi^G_M\|^2)\neq\emptyset$. Let $-\in\{G,K\}$. Thanks to (2.2),

$$m\in\mathrm{Cr}(\|\Phi^-_M\|^2) \Leftrightarrow \kappa^-(m)=0 \Leftrightarrow \widetilde{\Phi^-_M(m)}\in\mathfrak{g}_m.$$

Since $\mathfrak{g}_m\subset\mathfrak{g}_\xi$ with $\Phi^G_M(m)=\xi=\xi_{\mathfrak{k}}\oplus\xi_{\mathfrak{p}}$, we have $m\in\mathrm{Cr}(\|\Phi^-_M\|^2)$ only if $[\tilde{\xi_{\mathfrak{p}}},\tilde{\xi}]=0$. Since ξ is strongly elliptic the last condition implies that $\xi_{\mathfrak{p}}=0$. We have proved that $\mathrm{Cr}(\|\Phi^K_M\|^2)$ and $\mathrm{Cr}(\|\Phi^G_M\|^2)$ are both contained in $\{\Phi^{\mathfrak{p}}_M=0\}=Y$. We have $\kappa^G=\kappa^K+\kappa^{\mathfrak{p}}$ and the vector field $\kappa^{\mathfrak{p}}$ vanishes on Y. Finally,

$$\mathrm{Cr}(\|\Phi^G_M\|^2) = \mathrm{Cr}(\|\Phi^K_M\|^2) = \{y\in Y \mid [\widetilde{\Phi^K_M(y)}]_M(y)=0\} = \mathrm{Cr}(\|\Phi^K_Y\|^2).$$

The last equality is due to the fact that Φ^K_Y is the restriction of Φ^K_M to Y. □

2.3. *Criterion*

We have seen in Theorem 2.8 a situation where the properness of the moment map Φ^G_M is equivalent to the properness of Φ^K_M. In this section, we start with a symplectic manifold (M,Ω_M) admitting a Hamiltonian action of a compact connected Lie group K. We suppose that the moment map Φ^K_M is *proper*. Let $K' \subset K$ be a closed subgroup. The aim of the section is to give a criterion under which the induced moment map $\Phi^{K'}_M$ is still proper. We start by recalling basic facts concerning the notion of asymptotic cone.

For any non-empty subset C of a real vector space E, we define its *asymptotic cone* $\mathrm{As}(C) \subset E$ as the set of all limits $y = \lim_{k\to\infty} t_k y_k$ where (t_k) is a sequence of non-negative reals converging to 0 and $y_k \in C$. Note that $\mathrm{As}(C) = \{0\}$ if and only if C is compact.

We recall the following basic fact.

Lemma 2.9. *Let C_i, $i = 0, 1$, be* closed convex *subsets of E. Then:*

- $C_i + \mathrm{As}(C_i) \subset C_i$.
- *If $C_0 \cap C_1$ is non-empty, we have* $\mathrm{As}(C_0) \cap \mathrm{As}(C_1) = \mathrm{As}(C_0 \cap C_1)$.
- *If $C_0 \cap C_1$ is non-empty and compact, we have* $\mathrm{As}(C_0) \cap \mathrm{As}(C_1) = \{0\}$.

Proof. Let us check the first point. Take $z \in C_i$ and $y = \lim_{k\to\infty} t_k y_k \in \mathrm{As}(C_i)$. Then $z + y = \lim_{k\to\infty}((1-t_k)z + t_k y_k)$. Since $(1-t_k)z + t_k y_k \in C_i$ if $t_k \leq 1$, and C_i is closed, the term $z + y$ belongs to C_i.

The inclusion $\mathrm{As}(C_0 \cap C_1) \subset \mathrm{As}(C_0) \cap \mathrm{As}(C_1)$ follows from $C_0 \cap C_1 \subset C_i$. Let $z \in C_0 \cap C_1$ and $y \in \mathrm{As}(C_0) \cap \mathrm{As}(C_1)$. Thanks to the first point we know that $z + \mathbb{R}^{\geq 0} y \subset C_0 \cap C_1$. Then $y = \lim_{t\to 0^+} t(z + t^{-1}y) \in \mathrm{As}(C_0 \cap C_1)$. The second point is thus proved, and the last point follows easily. □

The following proposition is a useful tool for finding a proper moment map. For a closed subgroup K' of K, we denote by $\pi_{\mathfrak{k}',\mathfrak{k}} : \mathfrak{k}^* \to (\mathfrak{k}')^*$ the projection which is dual to the inclusion $\mathfrak{k}' \hookrightarrow \mathfrak{k}$. Its kernel $\pi^{-1}_{\mathfrak{k}',\mathfrak{k}}(0)$ is denoted $(\mathfrak{k}')^\perp$.

Proposition 2.10. *Let (M, Ω_M) be a Hamiltonian K-manifold with a* proper *moment map Φ^K_M. Let $\Delta_K(M)$ be its Kirwan polyhedron. Let $K' \subset K$ be a closed subgroup. The following statements are equivalent:*

(a) *the moment map $\Phi^{K'}_M = \pi_{\mathfrak{k}',\mathfrak{k}} \circ \Phi^K_M$ is* proper,
(b) $\mathrm{As}(\Delta_K(M)) \cap K \cdot (\mathfrak{k}')^\perp = \{0\}$,
(c) *there exists $\varepsilon > 0$ such that $\|\Phi^{K'}_M\| \geq \varepsilon\|\Phi^K_M\| - \varepsilon^{-1}$ on M.*

Proof. If (c) does not hold we have a sequence $m_i \in M$ such that $\|\Phi^{K'}_M(m_i)\| \leq i^{-1}\|\Phi^K_M(m_i)\| - i$ for all $i \geq 1$. Then $\|\Phi^K_M(m_i)\| \to \infty$ and $\|\Phi^{K'}_M(m_i)\|/\|\Phi^K_M(m_i)\| \to 0$. We write $\Phi^K_M(m_i) = k_i \cdot y_i$ with $k_i \in K$ and $y_i \in \Delta_K(M)$. The sequence $\pi_{\mathfrak{k}',\mathfrak{k}}(k_i \cdot y_i/\|y_i\|)$ converges to 0. Here we can assume that $k_i \to k \in K$ and $y_i/\|y_i\| \to y \in \mathrm{As}(\Delta_K(M))$. Then $\pi_{\mathfrak{k}',\mathfrak{k}}(k \cdot y) = 0$. In other words, y is a non-zero element in $\mathrm{As}(\Delta_K(M)) \cap K \cdot (\mathfrak{k}')^\perp$. We have proved (b)⇒(c).

The implication (c)⇒(a) is obvious. Let us prove (a)⇒(b). First, the properness of $\Phi^{K'}_M$ implies that the projection $\pi_{\mathfrak{k}',\mathfrak{k}}$ is proper when restricted to the closed subset $\mathrm{Image}(\Phi^K_M) = K \cdot \Delta_K(M)$. Let $k \in K$ and $\xi_o \in k \cdot \Delta_K(M)$. Then

$$k \cdot \Delta_K(M) \cap (\xi_o + (\mathfrak{k}')^\perp) \subset \mathrm{Image}(\Phi^K_M) \cap \pi^{-1}_{\mathfrak{k}',\mathfrak{k}}(\pi_{\mathfrak{k}',\mathfrak{k}}(\xi_o))$$

is non-empty and compact. If we apply the last point of Lemma 2.9 to the closed convex sets $k \cdot \Delta_K(M)$ and $\xi_o + (\mathfrak{k}')^\perp$ we find that

$$\mathrm{As}(k \cdot \Delta_K(M)) \cap \mathrm{As}(\xi_o + (\mathfrak{k}')^\perp) = k \cdot \mathrm{As}(\Delta_K(M)) \cap (\mathfrak{k}')^\perp$$

is $\{0\}$. Hence $\mathrm{As}(\Delta_K(M)) \cap k \cdot (\mathfrak{k}')^\perp = \{0\}$ for any $k \in K$. □

2.4. *Kostant–Souriau line bundle*

In the Kostant–Souriau framework, a Hamiltonian G-manifold (M, Ω_M, Φ_M^G) is *prequantized* if there is an equivariant Hermitian line bundle L_M with an invariant Hermitian connection ∇_M such that

$$\mathcal{L}(X) - \iota(X_M)\nabla_M = i\langle \Phi_M^G, X\rangle \quad \text{and} \quad (\nabla_M)^2 = -i\Omega_M, \tag{2.7}$$

for every $X \in \mathfrak{g}$. The data (L_M, ∇_M) is called a *Kostant–Souriau line bundle*.

We now suppose that conditions **C1** and **C2** hold. Then $M \simeq G \times_K Y$ where $Y \subset M$ is the K-invariant symplectic submanifold defined in Section 2.2. Let (L_M, ∇_M) be a Kostant–Souriau line bundle on M. We denote by L_Y the restriction of L_M over Y. The connection ∇_M induces a K-invariant connection ∇_Y on $L_Y \to Y$, and we check easily that (L_Y, ∇_Y) is a Kostant–Souriau line bundle on Y.

Conversely, if (L_Y, ∇_Y) is a Kostant–Souriau line bundle on (Y, Ω_Y, Φ_Y^K), we define on M the line bundle $L_M := (G \times L_Y)/K$ equipped with the connection

$$\nabla_M := \nabla_Y + d^G + i\langle \Phi_Y^K, \theta^K\rangle,$$

where d^G is the de Rham differential on G. Since $\Omega_M = \Omega_Y - d\langle \Phi_Y^K, \theta^K\rangle$, we check easily that (L_M, ∇_M) is a G-equivariant Kostant–Souriau line bundle on (M, Ω_M, Φ_M^G).

2.5. *The case of elliptic orbits*

Let G be a connected real reductive Lie group, with maximal compact subgroup K. Let $T \subset K$ be a maximal torus.

In this section, we consider the examples given by *elliptic* coadjoint orbits $G \cdot \lambda$ for some $\lambda \in \mathfrak{t}^*$. The Kirillov–Kostant–Souriau symplectic structure $\Omega_{G\cdot\lambda}$ is defined by

$$\Omega_{G\cdot\lambda}|_m(X_{G\cdot\lambda}|_m, Y_{G\cdot\lambda}|_m) = \langle m, [X, Y]\rangle$$

for $m \in G \cdot \lambda$ and $X, Y \in \mathfrak{g}$. The moment map relative to the G-action on $G \cdot \lambda$ is the inclusion $\Phi_{G\cdot\lambda}^G : G \cdot \lambda \hookrightarrow \mathfrak{g}^*$. We have the following well known fact [DHV84].

Lemma 2.11. *The moment maps $\Phi_{G\cdot\lambda}^G$ and $\Phi_{G\cdot\lambda}^K$ are* proper.

The Convexity Theorem tells us that the Kirwan polyhedron $\Delta_K(G\cdot\lambda) := \mathrm{Image}(\Phi_{G\cdot\lambda}^K) \cap \mathfrak{t}^*_{\geq 0}$ is a closed convex locally polyhedral subset of $\mathfrak{t}^*$. Duflo–Heckman–Vergne [DHV84] show that $\Delta_K(G\cdot\lambda)$ is defined by a finite number of inequalities (at least when λ is regular). In general $\Delta_K(G\cdot\lambda)$ is not known, but we can use at least the following observation.

Let $\mathfrak{R}_n(\lambda)$ be the set of non-compact roots α such that $(\alpha, \lambda) > 0$. Consider the following cone in $\mathfrak{t}^*$:

$$\mathcal{C}(\lambda) := \sum_{\alpha\in\mathfrak{R}_n(\lambda)} \mathbb{R}^{\geq 0}\alpha.$$

Lemma 2.12. *The Kirwan polyhedron $\Delta_K(G\cdot\lambda)$ is contained in $\lambda + \mathcal{C}(\lambda)$.*

Proof. The Lie algebra of the stabilizer subgroup G_λ has a decomposition $\mathfrak{k}_\lambda \oplus \mathfrak{p}_\lambda$. Let C_λ be the cone tangent to $\Delta_K(G\cdot\lambda)$ at λ:

$$C_\lambda = \mathbb{R}^{\geq 0}\cdot\{\xi - \lambda \mid \xi \in \Delta_K(G\cdot\lambda)\} \subset \mathfrak{t}^*.$$

We have to show that C_λ is contained in $\mathcal{C}(\lambda)$. Thanks to a result of Sjamaar [Sja98], the cone C_λ is determined by a local Hamiltonian model near $K\cdot\lambda \subset G\cdot\lambda$.

The maximal torus T of K is still a maximal torus for the stabilizer subgroup K_λ; let $\mathfrak{t}^*_{\lambda,\geq 0}$ be a Weyl chamber for (K_λ, T) which contains $\mathfrak{t}^*_{\geq 0}$. Here, we consider the vector space $\mathfrak{p}/\mathfrak{p}_\lambda$ equipped with the linear symplectic structure $\Omega_\lambda(X,Y) := \langle\lambda, [X,Y]\rangle$. The group K_λ acts in a Hamiltonian fashion on $(\mathfrak{p}/\mathfrak{p}_\lambda, \Omega_\lambda)$. Let $\Delta_{K_\lambda}(\mathfrak{p}/\mathfrak{p}_\lambda) \subset \mathfrak{t}^*_{\lambda,\geq 0}$ be the corresponding Kirwan polytope (which is a rational cone). Since the K-stabilizer of the point $\lambda \in G\cdot\lambda$ coincides with the stabilizer subgroup K_λ of its image by the moment map $\Phi^K_{G\cdot\lambda}$, the local form of Marle and Guillemin–Sternberg tells us that $G\cdot\lambda$ is symplectomorphic to $K\times_{K_\lambda}(\mathfrak{p}/\mathfrak{p}_\lambda)$ in a neighbourhood of $K\cdot\lambda$. Theorem 6.5 of [Sja98] tells us then that $C_\lambda = \Delta_{K_\lambda}(\mathfrak{p}/\mathfrak{p}_\lambda)$.

Consider the Hamiltonian action of the torus T on $(\mathfrak{p}/\mathfrak{p}_\lambda, \Omega_\lambda)$. Let J_λ be an invariant complex structure on $\mathfrak{p}/\mathfrak{p}_\lambda$ which is compatible with Ω_λ; the weights of the T-action on $(\mathfrak{p}/\mathfrak{p}_\lambda, J_\lambda)$ are $-\alpha$, for $\alpha \in \mathfrak{R}_n(\lambda)$. Hence $\Delta_T(\mathfrak{p}/\mathfrak{p}_\lambda)$ is equal to the cone generated by the weights $\alpha \in \mathfrak{R}_n(\lambda)$. Finally, the proof is completed since $C_\lambda = \Delta_{K_\lambda}(\mathfrak{p}/\mathfrak{p}_\lambda)$ is contained in $\Delta_T(\mathfrak{p}/\mathfrak{p}_\lambda) = \mathcal{C}(\lambda)$. □

3. Holomorphic discrete series

Let G be a connected real reductive Lie group and let K be a maximal connected compact subgroup. Let $\mathfrak{c}_\mathfrak{k}$, $\mathfrak{c}_\mathfrak{g}$ be the centres of $\mathfrak{k}$ and $\mathfrak{g}$ respectively. In what follows we assume that

$$Z_\mathfrak{g}(\mathfrak{c}_\mathfrak{k}) = \mathfrak{k}, \tag{3.1}$$

i.e. the centralizer of $\mathfrak{c}_\mathfrak{k}$ in $\mathfrak{g}$ coincides with $\mathfrak{k}$. Hence $\mathfrak{c}_\mathfrak{g} \subset \mathfrak{c}_\mathfrak{k} \subset \mathfrak{k}$.

Remark 3.1. The non-compact simple real Lie groups satisfying (3.1) are $\mathrm{Sp}(\mathbb{R}^{2n})$, $\mathrm{SO}^*(2n)$, $\mathrm{SO}_o(2,n)$, $\mathrm{SU}(p,q)$, $E_{6(-14)}$ and $E_{7(-25)}$.

We choose a maximal torus T in K with Lie algebra $\mathfrak{t}$. Note that (3.1) forces $\mathfrak{t}$ to be a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{R} = \mathfrak{R}_c \cup \mathfrak{R}_n$ be the set of roots. We fix a system $\mathfrak{R}_c^+$ of positive roots in $\mathfrak{R}_c$. Condition (3.1) implies the existence of elements $z \in \mathfrak{c}_\mathfrak{k} \cap [\mathfrak{g},\mathfrak{g}]$ such that $\mathrm{ad}(z)$ defines a complex structure on $\mathfrak{p}$ (see [Kna04, Section 9]).

Remark 3.2. If G is the product of N simple real Lie groups belonging to the list of Remark 3.1, then there are 2^N choices for the element z.

For such z, we define

$$\mathfrak{R}_n(z) := \{\alpha \in \mathfrak{R}_n \mid \langle \alpha, z \rangle = 1\}.$$

which is invariant relative to the action of the Weyl group $W = W(K, T)$. The union $\mathfrak{R}_c^+ \cup \mathfrak{R}_n(z)$ defines a system of positive roots in $\mathfrak{R}$. We will be interested in several closed W-invariant cones in $\mathfrak{t}^*$:

$$\begin{aligned} \mathcal{C}_{\text{hol}}(z) &:= \left\{\xi \in \mathfrak{t}^* \mid (\beta, \xi) \geq 0,\ \forall \beta \in \mathfrak{R}_n(z)\right\}, \\ \mathcal{C}^{\rho}_{\text{hol}}(z) &:= 2\rho_n(z) + \mathcal{C}_{\text{hol}}(z), \\ \mathcal{C}(z) &:= \sum_{\beta \in \mathfrak{R}_n(z)} \mathbb{R}^{\geq 0}\beta. \end{aligned} \tag{3.2}$$

Here $2\rho_n(z) = \sum_{\beta \in \mathfrak{R}_n(z)} \beta$ is W-invariant. We recall the following basic facts.

Lemma 3.3. *We have*

$$\mathcal{C}(z) \subset \mathcal{C}_{\text{hol}}(z) \subset \{\xi \in \mathfrak{t}^* \mid \langle \xi, z \rangle \geq 0\} \quad \textit{and} \quad \mathcal{C}^{\rho}_{\text{hol}}(z) \subset \mathcal{C}_{\text{hol}}(z). \tag{3.3}$$

Proof. Since $(\beta_0, \beta_1) \geq 0$ for any $\beta_0, \beta_1 \in \mathfrak{R}_n(z)$, we see that $\mathcal{C}(z) \subset \mathcal{C}_{\text{hol}}(z)$ and $\rho_n(z) \in \mathcal{C}_{\text{hol}}(z)$, so the inclusion $\mathcal{C}^{\rho}_{\text{hol}}(z) \subset \mathcal{C}_{\text{hol}}(z)$ follows. On the other hand, we can check that $(X, z) := -\text{Tr}(\text{ad}(X)\text{ad}(z))$ is equal to $2\langle \rho_n(z), X \rangle$ for any $X \in \mathfrak{t}$. Hence $2\widetilde{\rho_n(z)} = z$ and

$$\langle \xi, z \rangle = 2 \sum_{\beta \in \mathfrak{R}_n(z)} (\xi, \beta), \quad \forall \xi \in \mathfrak{t}^*.$$

This proves that $\mathcal{C}_{\text{hol}}(z) \subset \{\xi \in \mathfrak{t}^* \mid \langle \xi, z \rangle \geq 0\}$. □

3.1. Holomorphic coadjoint orbits

The *holomorphic* coadjoint orbits are $G \cdot \lambda$ with λ in the interior of $\mathcal{C}_{\text{hol}}(z)$. These symplectic manifolds possess a G-invariant (integrable) complex structure J_λ which is compatible with the symplectic structure $\Omega_{G\cdot\lambda}$ (see [Par08]). Hence $(G \cdot \lambda, \Omega_{G\cdot\lambda}, J_\lambda)$ is a Kähler manifold when $\lambda \in \text{Interior}(\mathcal{C}_{\text{hol}}(z))$.

The real K-module $\mathfrak{p}$ is equipped with the invariant linear symplectic structure $\Omega_{\mathfrak{p}}(A, B) := -b(z, [A, B])$. We have two families of Hamiltonian K-manifolds: $K \cdot \lambda \times \mathfrak{p}$ and $G \cdot \lambda$ for $\lambda \in \mathcal{C}_{\text{hol}}(z)$. We start with a fundamental fact.

Proposition 3.4. *Let* $\lambda \in \text{Interior}(\mathcal{C}_{\text{hol}}(z))$. *We have*

(a) $\Delta_K(G \cdot \lambda) \subset \lambda + \mathcal{C}(z) \subset \mathcal{C}_{\text{hol}}(z)$,
(b) $\Delta_K(G \cdot \lambda) = \Delta_K(K \cdot \lambda \times \mathfrak{p})$,
(c) $\text{As}(\Delta_K(G \cdot \lambda)) = \Delta_K(\mathfrak{p})$.

Proof. Point (a) is the translation of Lemma 2.12 since the cone $\mathcal{C}(\lambda)$ is equal to $\mathcal{C}(z)$. Point (b) is proved in [Par08]. Another proof is given by Deltour [Del13], by showing the stronger result that the Hamiltonian K-manifolds $G\cdot\lambda$ and $K\cdot\lambda\times\mathfrak{p}$ are symplectomorphic. Point (c) follows easily from (b). □

Remark 3.5. The generators of the cone $\Delta_K(\mathfrak{p})$ can be defined in term of *strongly orthogonal roots* (see Section 5 in [Par08]). Note also that Deltour [Del10, Del12] has completely described the facet of the polytopes $\Delta_K(G\cdot\lambda)$ when $\lambda\in \text{Interior}(\mathcal{C}_{\text{hol}}(z))$.

Let $S^\bullet(\mathfrak{p})$ be the symmetric algebra of the complex K-module $(\mathfrak{p}, \text{ad}(z))$; it is an admissible representation of K. Let K' be a closed connected subgroup of K. We denote by $\Phi^{K'}_{G\cdot\lambda}$ and $\Phi^{K'}_{\mathfrak{p}}$ the corresponding moment maps.

Proposition 3.6. *Let* $\lambda\in \text{Interior}(\mathcal{C}_{\text{hol}}(z))$. *The following assertions are equivalent:*

(a) $\Phi^{K'}_{G\cdot\lambda}: G\cdot\lambda\to(\mathfrak{k}')^*$ *is a proper map,*
(b) $\Delta_K(\mathfrak{p})\cap K\cdot(\mathfrak{k}')^\perp=\{0\}$,
(c) $\Phi^{K'}_{\mathfrak{p}}:\mathfrak{p}\to(\mathfrak{k}')^*$ *is a proper map,*
(d) $\{\Phi^{K'}_{\mathfrak{p}}=0\}$ *reduces to* $\{0\}$,
(e) $[S^\bullet(\mathfrak{p})]^{K'}=\mathbb{C}$.

Proof. The equivalences (a)⇔(b) and (b)⇔(c) follow from Propositions 2.10 and 3.4. The equivalences (c)⇔(d)⇔(e) are well known (for proofs, see for example [Par09, Section 5]). □

Let us consider the moment map $\Phi^K_{\mathfrak{p}}:\mathfrak{p}\to\mathfrak{k}^*$. Via the identification $\mathfrak{k}^*\simeq\mathfrak{k}$, the moment map $\Phi^K_{\mathfrak{p}}$ is defined by $\Phi^K_{\mathfrak{p}}(X)=-[X,[z,X]]$ for $X\in\mathfrak{p}$. Hence we see that $\langle\Phi^K_{\mathfrak{p}},z\rangle:\mathfrak{p}\to\mathbb{R}$ is a *proper map*. This simple fact together with Proposition 3.6 gives

Corollary 3.7. *Let* G' *be a connected reductive subgroup of* G, *and let* $\lambda\in \text{Interior}(\mathcal{C}_{\text{hol}}(z))$. *The moment map* $\Phi^{G'}_{G\cdot\lambda}$ *is proper when* $\mathbb{R}z\subset\mathfrak{g}'$.

Example 3.8. The condition $\mathbb{R}z\subset\mathfrak{g}'$ is fulfilled in the following cases:

1. $G'=\text{SO}_o(2,p)\subset G=\text{SO}_o(2,n)$ for $0\leq p\leq n$.
2. G' is the identity component of G^σ, where σ is an involution of G such that $\sigma(z)=z$. For example $G=\text{U}(p,q)$ and $G'=\text{U}(i,j)\times\text{U}(p-i,q-j)$.
3. G' is the diagonal in $G:=G'\times\cdots\times G'$.

3.2. Holomorphic discrete series

Let $\Lambda^*\subset\mathfrak{t}^*$ be the lattice of characters of T. The set $\Lambda^*_+:=\Lambda^*\cap\mathfrak{t}^*_{\geq 0}$ parametrizes the set $\widehat{K}$ of irreducible representations of K: for any $\mu\in\Lambda^*_+$, we denote by V^K_μ the irreducible representation of K with highest weight μ. We will be interested in the following subset of dominant weights:

$$\widehat{G}_{\text{hol}}(z):=\Lambda^*_+\cap\mathcal{C}^\rho_{\text{hol}}(z),$$

where the cone $\mathcal{C}^\rho_{\text{hol}}(z)$ is defined in (3.2).

Theorem 3.9 (Harish-Chandra). *For any* $\lambda\in\widehat{G}_{\text{hol}}(z)$, *there exists an irreducible unitary representation of* G, *denoted* V^G_λ, *such that the vector space of* K*-finite vectors is* $V^G_\lambda|_K:=V^K_\lambda\otimes S^\bullet(\mathfrak{p})$.

Moreover the K-type of $V_\lambda^G|_K$ satisfies the following well known relations:

$$[V_\mu^K : V_\lambda^G|_K] = 1 \quad \text{if } \mu = \lambda, \qquad (3.4)$$
$$[V_\mu^K : V_\lambda^G|_K] \neq 0 \Rightarrow \mu \in \lambda + \mathcal{C}(z) \subset \mathcal{C}_{\text{hol}}^\rho(z).$$

Note that for $\mu \in \lambda + \mathcal{C}(z)$ we have $\|\mu\| > \|\lambda\|$ unless $\mu = \lambda$. Finally, the condition $\left[V_\mu^K : V_\lambda^G|_K\right] \neq 0$ implies that $\|\mu\| > \|\lambda\|$ or $\mu = \lambda$.

We will see in Section 4.2 that (3.4) is a consequence of the "quantization commutes with reduction" principle.

3.3. *Restriction*

We now consider a connected reductive subgroup $G' \subset G$ such that $\mathbb{R}z \subset \mathfrak{g}'$. The group G' satisfies (3.1). Let $K' \subset K$ be the maximal compact subgroup in G', and let $T' \subset T$ be a maximal torus in K'. Let $\mathcal{C}_{\text{hol}}(z), \mathcal{C}_{\text{hol}}^\rho(z) \subset \mathfrak{t}^*$ and $\mathcal{C}'_{\text{hol}}(z), \mathcal{C}'^\rho_{\text{hol}}(z) \subset (\mathfrak{t}')^*$ be the corresponding convex cones. Recall that the set $\widehat{G}'_{\text{hol}}(z)$ parametrizes a subset of the holomorphic discrete series of G'.

3.3.1. *Restriction to K*

Definition 3.10. We denote by $\widehat{K}_{\text{hol}}(z) \subset \widehat{K}$ the subset $\Lambda_+^* \cap \mathcal{C}_{\text{hol}}^\rho(z)$.

We see that $\widehat{K}_{\text{hol}}(z)$ and $\widehat{G}_{\text{hol}}(z)$ are the same set but they parametrize representations of different groups (K and G respectively). Let us denote by

$$R^{-\infty}(G, z) \qquad (3.5)$$

the $\mathbb{Z}$-module formed by the infinite sums $\sum_{\lambda \in \widehat{G}_{\text{hol}}(z)} m_\lambda V_\lambda^G$ with $m_\lambda \in \mathbb{Z}$. Similarly, we define $R^{-\infty}(K, z) \subset R^{-\infty}(K)$ as the submodule formed by the infinite sums $\sum_{\mu \in \widehat{K}_{\text{hol}}(z)} m_\mu V_\mu^K$. The following basic result will be used in the next sections.

Lemma 3.11. • *The restriction to K defines an* injective *morphism*

$$\mathbf{r}_{K,G} : R^{-\infty}(G, z) \to R^{-\infty}(K, z). \qquad (3.6)$$

• *The product by $S^\bullet(\mathfrak{p})$ defines a map from $R^{-\infty}(K, z)$ into itself.*

Proof. Let us prove the first point. Thanks to (3.4), we have $V_\lambda^G|_K = \sum_{\mu \in \Lambda_+^*} n_\lambda^\mu V_\mu^K$ where $n_\lambda^\mu \neq 0$ only if $\mu \in \lambda + \mathcal{C}(z)$; this condition implies that $\mu \in \widehat{K}_{\text{hol}}(z)$ with $\|\mu\| \geq \|\lambda\|$. Let $A = \sum_{\lambda \in \widehat{G}_{\text{hol}}(z)} a(\lambda) V_\lambda^G$ be an element of $R^{-\infty}(G, z)$. Then

$$\mathbf{r}_{K,G}(A) := \sum_{\lambda \in \widehat{G}_{\text{hol}}(z)} a(\lambda) V_\lambda^G|_K = \sum_{\mu \in \widehat{K}_{\text{hol}}(z)} \Big(\sum_{\lambda \in \widehat{G}_{\text{hol}}(z)} a(\lambda) n_\lambda^\mu \Big) V_\mu^K \in R^{-\infty}(K, z),$$

where each sum $r(\mu) := \sum_\lambda a(\lambda) n_\lambda^\mu$ has a finite number of non-zero terms since $n_\lambda^\mu = 0$ if $\|\lambda\| > \|\mu\|$. Suppose that A is non-zero, and let $\lambda_o \in \widehat{G}_{\text{hol}}(z)$ be such that $\|\lambda_o\|$

is minimal in the set $\{\|\lambda\| \mid a(\lambda) \neq 0\}$. Let $\mathbf{r}_{K,G}(A) = \sum_\mu r(\mu) V_\mu^K$. Then $r(\lambda_o) = a(\lambda_o) + \sum_{\lambda \neq \lambda_o} a(\lambda) n_\lambda^{\lambda_o}$. But $n_\lambda^{\lambda_o} = 0$ if $\lambda \neq \lambda_o$ and $\|\lambda\| \geq \|\lambda_o\|$. And the term $a(\lambda)$ is zero if $\|\lambda\| < \|\lambda_o\|$. We have checked that $r(\lambda_o) = a(\lambda_o) \neq 0$ and so $\mathbf{r}_{K,G}(A) \neq 0$.

The second point follows from the first. Let $A_K = \sum_{\mu \in \widehat{K}_{\mathrm{hol}}(z)} a(\mu) V_\mu^K \in R^{-\infty}(K, z)$. Take $A_G = \sum_{\mu \in \widehat{G}_{\mathrm{hol}}(z)} a(\mu) V_\mu^G$; then $A_K \otimes S^\bullet(\mathfrak{p}) = \mathbf{r}_{K,G}(A_G)$ is well defined. □

3.3.2. Restriction: the algebraic part. Let $\lambda \in \widehat{G}_{\mathrm{hol}}(z)$. Since the representation V_λ^G is discretely admissible relative to the circle group $\exp(\mathbb{R}z)$, it is also discretely admissible relative to G'. We can be more precise [Mar75, JV79, Kob98]:

Proposition 3.12. *We have a Hilbertian direct sum*

$$V_\lambda^G|_{G'} = \bigoplus_{\mu \in \widehat{G'}_{\mathrm{hol}}(z)} m_\lambda(\mu) V_\mu^{G'},$$

with $m_\lambda(\mu)$ finite for any μ.

3.3.3. Restriction: the geometric part. For $\mathfrak{l} \in \{\mathfrak{t}, \mathfrak{k}, \mathfrak{g}\}$, we denote by $\pi_{\mathfrak{l}',\mathfrak{l}} : \mathfrak{l}^* \to (\mathfrak{l}')^*$ the canonical projection. We have the following important fact.

Proposition 3.13. *We have*

(a) $\pi_{\mathfrak{t}',\mathfrak{t}}(\mathcal{C}_{\mathrm{hol}}(z)) \subset \mathcal{C}'_{\mathrm{hol}}(z)$,
(b) $\pi_{\mathfrak{k}',\mathfrak{k}}(K \cdot \mathcal{C}_{\mathrm{hol}}(z)) \subset K' \cdot \mathcal{C}'_{\mathrm{hol}}(z)$,
(c) $\pi_{\mathfrak{k}',\mathfrak{k}}(K \cdot \mathcal{C}^\rho_{\mathrm{hol}}(z)) \subset K' \cdot \mathcal{C}'^\rho_{\mathrm{hol}}(z)$,
(d) $\pi_{\mathfrak{g}',\mathfrak{g}}(G \cdot \mathcal{C}^\rho_{\mathrm{hol}}(z)) \subset G' \cdot \mathcal{C}'^\rho_{\mathrm{hol}}(z)$.

Proof. Let $\alpha \in \mathfrak{t}^*$ be a non-compact root of $(\mathfrak{g}, \mathfrak{t})$. Let $\mathfrak{g}_\alpha \subset \mathfrak{p} \otimes \mathbb{C}$ be the corresponding 1-dimensional weight space. Then there exists $h_\alpha \in i[\mathfrak{g}_\alpha, \overline{\mathfrak{g}_\alpha}] \cap \mathfrak{t}$ such that $\alpha = -b(h_\alpha, \cdot)$. Note that the half-line $\mathbb{R}^{>0} h_\alpha$ does not depend on the bilinear form b, and the condition $(\alpha, \xi) \geq 0$ is equivalent to $\langle \xi, h_\alpha \rangle \geq 0$ for any $\xi \in \mathfrak{t}^*$.

Let $\alpha \in \mathfrak{t}^*$ be a non-compact root of $(\mathfrak{g}, \mathfrak{t})$ whose restriction $\alpha' = \pi_{\mathfrak{t}',\mathfrak{t}}(\alpha)$ is a non-compact root of $(\mathfrak{g}', \mathfrak{t}')$. Since the 1-dimensional weight spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}'_{\alpha'}$ coincide, we have $\mathbb{R}^{>0} h_\alpha = \mathbb{R}^{>0} h_{\alpha'} \subset \mathfrak{t}'$. Then $\langle \xi, h_\alpha \rangle \geq 0$ is equivalent to $\langle \pi_{\mathfrak{t}',\mathfrak{t}}(\xi), h_{\alpha'} \rangle \geq 0$. Finally, we have proved (a): if $\langle \xi, h_\alpha \rangle \geq 0$ for any positive non-compact root α of $(\mathfrak{g}, \mathfrak{t})$, then $\langle \pi_{\mathfrak{t}',\mathfrak{t}}(\xi), h_{\alpha'} \rangle \geq 0$ for any positive non-compact root α' of $(\mathfrak{g}', \mathfrak{t}')$.

Let $\xi \in \mathcal{C}_{\mathrm{hol}}(z)$ and $\xi' \in \pi_{\mathfrak{k}',\mathfrak{k}}(K \cdot \xi) \cap (\mathfrak{t}')^*$. Then $\xi' \in \pi_{\mathfrak{t}',\mathfrak{t}} \circ \pi_{\mathfrak{t},\mathfrak{k}}(K \cdot \xi)$. By the Convexity Theorem [Ati82, GS82a, Kir84, LMTW98], the projection $\pi_{\mathfrak{t},\mathfrak{k}}(K \cdot \xi)$ is equal to the convex hull of $W\xi$. But ξ belongs to the W-invariant convex cone $\mathcal{C}_{\mathrm{hol}}(z)$, and so $\pi_{\mathfrak{t},\mathfrak{k}}(K \cdot \xi) \subset \mathcal{C}_{\mathrm{hol}}(z)$. Finally, $\xi' \in \pi_{\mathfrak{t}',\mathfrak{t}}(\mathcal{C}_{\mathrm{hol}}(z)) \subset \mathcal{C}'_{\mathrm{hol}}(z)$ thanks to (a).

Let $\xi \in \mathcal{C}_{\mathrm{hol}}(z)$. Since $2\rho_n(z)$ is K-invariant, we have $K \cdot (2\rho_n(z) + \xi) = 2\rho_n(z) + K \cdot \xi$. Thanks to (b),

$$\pi_{\mathfrak{k}',\mathfrak{k}}(K \cdot (2\rho_n(z) + \xi)) = \pi_{\mathfrak{k}',\mathfrak{k}}(2\rho_n(z)) + \pi_{\mathfrak{k}',\mathfrak{k}}(K \cdot \xi) \subset K' \cdot \big(\pi_{\mathfrak{k}',\mathfrak{k}}(2\rho_n(z)) + \mathcal{C}'_{\mathrm{hol}}(z)\big).$$

The K'-invariant term $\pi_{\mathfrak{k}',\mathfrak{k}}(2\rho_n(z))$ belongs to $(\mathfrak{t}')^*$ and is equal to $2\rho'_n(z) + \pi_{\mathfrak{t}',\mathfrak{t}}(A)$ where A is the sum of the positive non-compact roots α such that $\mathfrak{g}_\alpha \not\subset \mathfrak{p}' \otimes \mathbb{C}$. Hence $A \in \mathcal{C}_{\mathrm{hol}}(z)$ and thanks to (a) the projection $\pi_{\mathfrak{t}',\mathfrak{t}}(A)$ belongs to $\mathcal{C}'_{\mathrm{hol}}(z)$. Thus (c) is proved.

Let $\lambda \in \mathcal{C}^\rho_{\mathrm{hol}}(z)$. The coadjoint orbit $G \cdot \lambda$ is contained in $\mathfrak{g}^*_{\mathrm{se}}$, and the moment map $\Phi^{G'}_{G\cdot\lambda}$ is proper since $z \in \mathfrak{g}'$ (see Corollary 3.7). Then, $\pi_{\mathfrak{g}',\mathfrak{g}}(G \cdot \lambda) = \mathrm{Image}(\Phi^{G'}_{G\cdot\lambda}) = G' \cdot (\pi_{\mathfrak{g}',\mathfrak{g}}(G \cdot \lambda) \cap (\mathfrak{k}')^*)$ and

$$\begin{aligned}\pi_{\mathfrak{g}',\mathfrak{g}}(G \cdot \lambda) \cap (\mathfrak{k}')^* &\subset \pi_{\mathfrak{k}',\mathfrak{k}} \circ \pi_{\mathfrak{k},\mathfrak{g}}(G \cdot \lambda) \subset \pi_{\mathfrak{k}',\mathfrak{k}}(K \cdot \Delta_K(G \cdot \lambda)) \\ &\subset \pi_{\mathfrak{k}',\mathfrak{k}}(K \cdot \mathcal{C}^\rho_{\mathrm{hol}}(z)) \subset K' \cdot \mathcal{C}'^{\rho}_{\mathrm{hol}}(z).\end{aligned}$$

The last but one inclusion is due to the fact that $\Delta_K(G \cdot \lambda) \subset \lambda + \mathcal{C}(z) \subset \mathcal{C}^\rho_{\mathrm{hol}}(z)$ when $\lambda \in \mathcal{C}^\rho_{\mathrm{hol}}(z)$ (see Lemma 3.4); and the last one corresponds to (c). □

Remark 3.14. When the Lie algebra $\mathfrak{g}$ is simple, the set $G \cdot \mathcal{C}_{\mathrm{hol}}(z) \subset \mathfrak{g}^*_{\mathrm{se}}$ is a maximal closed convex G-invariant cone. See [Vin80, Pan83].

4. Quantization commutes with reduction

Let us first recall the definition of the geometric quantization of a smooth and compact Hamiltonian manifold. Then we show how to extend the notion of geometric quantization to the case of a *non-compact* Hamiltonian manifold.

4.1. Formal geometric quantization

Let K be a compact connected Lie group. Let (M, Ω_M, Φ^K_M) be a Hamiltonian K-manifold which is pre-quantized by the Hermitian line bundle L_M (see Section 2.4).

Let us recall the notion of geometric quantization when M is *compact*. Choose a K-invariant almost complex structure J on M which is compatible with Ω_M in the sense that the symmetric bilinear form $\Omega_M(\cdot, J\,\cdot)$ is a Riemannian metric. Let $\overline{\partial}_{L_M}$ be the Dolbeault operator with coefficients in L, and let $\overline{\partial}^*_{L_M}$ be its (formal) adjoint. The *Dolbeault–Dirac operator* on M with coefficients in L_M is $D_{L_M} = \sqrt{2}(\overline{\partial}_{L_M} + \overline{\partial}^*_{L_M})$, considered as an elliptic operator from $\mathcal{A}^{0,\mathrm{even}}(M, L_M)$ to $\mathcal{A}^{0,\mathrm{odd}}(M, L_M)$. Let $R(K)$ be the representation ring of K.

Definition 4.1. The *geometric quantization* of a *compact* Hamiltonian K-manifold (M, Ω_M, Φ^K_M) is the element $\mathcal{Q}_K(M) \in R(K)$ defined as the equivariant index of the Dolbeault–Dirac operator D_{L_M}.

Let us consider the case of a *proper* pre-quantized Hamiltonian K-manifold M: the manifold is perhaps *non-compact* but the moment map $\Phi^K_M : M \to \mathfrak{k}^*$ is supposed to be proper. In this setting, we have two ways of extending the geometric quantization procedure.

First way: $\mathcal{Q}^{-\infty}_K$. One defines the *formal geometric quantization* of M as an element $\mathcal{Q}^{-\infty}_K(M)$ that belongs to $R^{-\infty}(K) := \hom_{\mathbb{Z}}(R(K), \mathbb{Z})$ [Weits01, Par09, MZ09, Par11, MZ14]. Let us recall the definition.

For any $\mu \in \widehat{K}$ which is a regular value of the moment map Φ, the reduced space[1] (or symplectic quotient)

$$M_\mu := (\Phi_M^K)^{-1}(K \cdot \mu)/K \tag{4.1}$$

is a *compact* orbifold equipped with a symplectic structure Ω_μ. Moreover

$$L_\mu := (L|_{(\Phi_M^K)^{-1}(\mu)} \otimes \mathbb{C}_{-\mu})/K_\mu$$

is a Kostant–Souriau line orbibundle over (M_μ, Ω_μ). The definition of the index of the Dolbeault–Dirac operator carries over to the orbifold case, hence $\mathcal{Q}(M_\mu) \in \mathbb{Z}$ is defined. This notion of geometric quantization extends further to the case of singular symplectic quotients [MS99, Par01]. So the integer $\mathcal{Q}(M_\mu) \in \mathbb{Z}$ is well defined for every $\mu \in \widehat{K}$; in particular, $\mathcal{Q}(M_\mu) = 0$ if μ is not in the Kirwan polytope $\Delta_K(M)$.

Definition 4.2. Let (M, Ω_M, Φ_M^K) be a *proper* Hamiltonian K-manifold which is pre-quantized by a Kostant–Souriau line bundle L. The formal quantization of (M, Ω_M, Φ_M^K) is the element of $R^{-\infty}(K)$ defined by

$$\mathcal{Q}_K^{-\infty}(M) = \sum_{\mu \in \widehat{K}} \mathcal{Q}(M_\mu) V_\mu^K.$$

When M is compact, the fact that

$$\mathcal{Q}_K(M) = \mathcal{Q}_K^{-\infty}(M) \tag{4.2}$$

is known as the "quantization commutes with reduction" theorem. This was conjectured by Guillemin–Sternberg in [GS82b] and was first proved by Meinrenken [Mei98] and Meinrenken–Sjamaar [MS99]. Other proofs of (4.2) were also given by Tian–Zhang [TZ98] and the author [Par01]. For complete references on the subject the reader should consult [Sja96, Ver02]. One of the main features of the formal geometric quantization $\mathcal{Q}^{-\infty}$ is summarized in

Theorem 4.3 (Restriction to subgroups [Par09]). *Let M be a pre-quantized Hamiltonian K-manifold which is* proper. *Let $H \subset K$ be a closed connected Lie subgroup such that M is still* proper *as a Hamiltonian H-manifold. Then $\mathcal{Q}_K^{-\infty}(M)$ is H-admissible and $\mathcal{Q}_K^{-\infty}(M)|_H = \mathcal{Q}_H^{-\infty}(M)$ in $R^{-\infty}(H)$.*

Second way: $\mathcal{Q}_K^{\Phi}$. When M is a proper pre-quantized Hamiltonian K-manifold, we can define another *formal geometric quantization* of M through a non-abelian localization procedure à la Witten [Wit92]. In [MZ09, Par11, MZ14], it is proved that an element

$$\mathcal{Q}_K^{\Phi}(M) \in R^{-\infty}(K) \tag{4.3}$$

is well defined by localizing the index of the Dolbeault–Dirac operator D_{L_M} on the set $\mathrm{Cr}(\|\Phi_M^K\|^2)$ of critical points of the square of the moment map (see Section 5.3).

The crucial result is that these two procedures coincide [MZ09, Par11, MZ14].

[1] The symplectic quotient will be denoted $M_{\mu,K}$ when we need more precise notation.

Theorem 4.4 (Ma–Zhang, Paradan). *Let M be a proper pre-quantized Hamiltonian K-manifold. Then*

$$\mathcal{Q}_K^{-\infty}(M) = \mathcal{Q}_K^{\Phi}(M) \quad \text{in } R^{-\infty}(K). \tag{4.4}$$

4.2. Formal geometric quantization of holomorphic orbits

Let us come back to the holomorphic discrete representation V_λ^G. Consider a coadjoint orbit $G \cdot \lambda$ for $\lambda \in \Lambda_+^*$ in the interior of the chamber $\mathcal{C}_{\mathrm{hol}}(z)$, so that λ is strongly elliptic. The action of G on $G \cdot \lambda$ is Hamiltonian, and the line bundle

$$L := G \times_{K_\lambda} \mathbb{C}_\lambda$$

is a Kostant–Souriau line bundle over $G \cdot \lambda \simeq G/K_\lambda$. Here $\mathbb{C}_\lambda$ denotes the 1-dimensional representation of the stabilizer subgroup K_λ that can be attached to the weight λ.

By Lemma 2.11, the moment map $\Phi_{G\cdot\lambda}^K$ relative to the action of K on $G \cdot \lambda$ is proper. Hence the reduced spaces

$$(G \cdot \lambda)_\mu := (\Phi_{G\cdot\lambda}^K)^{-1}(K \cdot \mu)/K$$

are compact for any $\mu \in \Lambda_+^*$, and the generalized character $\mathcal{Q}_K^{\Phi}(G \cdot \lambda) \in R^{-\infty}(K)$ is well defined. We have proved in [Par03, Par08] the following

Theorem 4.5. *Let $\lambda \in \Lambda_+^* \cap \mathrm{Interior}(\mathcal{C}_{\mathrm{hol}}(z))$. Then*

$$\mathcal{Q}_K^{\Phi}(G \cdot \lambda) = V_\lambda^K \otimes S^\bullet(\mathfrak{p}) \quad \text{in } R^{-\infty}(K).$$

This result will be generalized in Theorem 4.10. When $\lambda \in \mathcal{C}_{\mathrm{hol}}^{\rho}(z)$, the generalized character $\mathcal{Q}_K^{\Phi}(G \cdot \lambda)$ coincides with the vector space of K-finite vectors of the holomorphic discrete representation V_λ^G. Theorems 4.5 and 4.4 give the following information concerning the K-multiplicities of $V_\lambda^K \otimes S^\bullet(\mathfrak{p})$.

Corollary 4.6. *Let $\lambda \in \Lambda_+^* \cap \mathrm{Interior}(\mathcal{C}_{\mathrm{hol}}(z))$, and $\mu \in \Lambda_+^*$. Then:*

- $\left[V_\mu^K : V_\lambda^K \otimes S^\bullet(\mathfrak{p})\right] = \mathcal{Q}((G \cdot \lambda)_\mu)$,
- $\left[V_\lambda^K : V_\lambda^K \otimes S^\bullet(\mathfrak{p})\right] = 1$,
- $[V_\mu^K : V_\lambda^K \otimes S^\bullet(\mathfrak{p})] \neq 0 \to \mu \in \lambda + \mathcal{C}(z) \subset \mathcal{C}_{\mathrm{hol}}(z)$.

Proof. The first point follows from $\mathcal{Q}_K^{\Phi}(G \cdot \lambda) = \mathcal{Q}_K^{-\infty}(G \cdot \lambda)$. The second point is due to the fact that the reduced space $(G \cdot \lambda)_\lambda$ reduces to a point [Par11, Section 2.4]. Hence if $[V_\mu^K : V_\lambda^K \otimes S^\bullet(\mathfrak{p})] \neq 0$, the weight μ belongs to the Kirwan polytope $\Delta_K(G \cdot \lambda)$, and $\Delta_K(G \cdot \lambda) \subset \lambda + \mathcal{C}(z)$ thanks to Lemma 2.12. □

4.3. Formal geometric quantization of G-actions

In this section we consider the Hamiltonian action of a connected real reductive Lie group G on a symplectic manifold (M, Ω_M). We suppose that the action of G on M is *proper* and that the moment map $\Phi_M^G : M \to \mathfrak{g}^*$ is *proper*. We have proved in Section 2.2 that we have a global slice $Y \subset M$ such that $M \simeq G \times_K Y$, and that the G-orbits in the image of Φ_M^G are parametrized by the Kirwan polytope $\Delta_K(Y)$.

Suppose that there exists of a G-equivariant pre-quantum line bundle $L_M \to M$. Note that L_M is completely determined by its restriction $L_Y \to Y$ to the submanifold Y; here L_Y is a K-equivariant pre-quantum line bundle over (Y, Ω_Y). For any dominant weight μ, the reduced space $M_{\mu,G} := (\Phi_M^G)^{-1}(G \cdot \mu)/G$ coincides with $Y_{\mu,K} := (\Phi_M^K)^{-1}(K \cdot \mu)/K$. Hence its quantization

$$\mathcal{Q}(M_{\mu,G}) := \mathcal{Q}(Y_{\mu,K}) \in \mathbb{Z}$$

is well defined (see Section 4.1).

We also suppose that G satisfies (3.1), and we fix a complex structure $\mathrm{ad}(z)$ on $\mathfrak{p}$. Let $\mathcal{C}^{\rho}_{\mathrm{hol}}(z) \subset \mathfrak{t}^*$ be the corresponding cone.

Lemma 4.7. *Let (M, Ω_M, Φ_M^G) be a pre-quantized proper[2] Hamiltonian manifold such that the image of Φ_M^G is contained in $G \cdot \mathcal{C}^{\rho}_{\mathrm{hol}}(z)$. Then:*

(a) *the Kirwan polytopes $\Delta_K(Y) \subset \Delta_K(M)$ are contained in $\mathcal{C}^{\rho}_{\mathrm{hol}}(z)$,*
(b) *the functions $\langle \Phi_Y^K, z\rangle$ and $\langle \Phi_M^K, z\rangle$ take strictly positive values.*

Proof. We have

$$\Delta_K(M) \subset \pi_{\mathfrak{k},\mathfrak{g}}(G \cdot \mathcal{C}^{\rho}_{\mathrm{hol}}(z)) \cap \mathfrak{t}^*_{\geq 0} = \bigcup_{\lambda \in \mathcal{C}^{\rho}_{\mathrm{hol}}(z)} \Delta_K(G \cdot \lambda) \subset \mathcal{C}^{\rho}_{\mathrm{hol}}(z),$$

where the last inclusion follows from Proposition 3.4(a). The first point is proved, and the second follows from the first. □

We will use the following notion of formal geometric quantization that extends the case of compact Lie group actions.

Definition 4.8. Let (M, Ω_M, Φ_M^G) be a pre-quantized proper[2] Hamiltonian manifold such that the image of Φ_M^G is contained in $G \cdot \mathcal{C}^{\rho}_{\mathrm{hol}}(z)$. We define the formal geometric quantization of M as the following element of $R^{-\infty}(G, z)$:

$$\mathcal{Q}_G^{-\infty}(M) := \sum_{\mu \in \widehat{G}_{\mathrm{hol}}(z)} \mathcal{Q}(M_{\mu,G}) V_\mu^G.$$

In the setting of Definition 4.8, the moment maps Φ_M^K and Φ_Y^K are proper (see Theorem 2.8). Then the formal geometric quantization of M and Y relative to the K-action is well defined, and by Lemma 4.7(a) the generalized characters $\mathcal{Q}_K^{-\infty}(M)$ and $\mathcal{Q}_K^{-\infty}(Y)$ belong to $R^{-\infty}(K, z)$.

We have proved in Theorem 2.8 that the sets of critical points of the functions $\|\Phi_M^G\|^2$, $\|\Phi_M^K\|^2$ and $\|\Phi_Y^K\|^2$ are equal. We will need to work under one of the following hypotheses:

Assumption 4.9.

A1. The set $\mathrm{Cr}(\|\Phi_M^G\|^2)$ is compact.
A2. The map $\langle \Phi_M^G, z\rangle : M \to \mathbb{R}$ is proper.

Let $\mathbf{r}_{K,G} : R^{-\infty}(G,z) \to R^{-\infty}(K,z)$ be the restriction morphism defined in Lemma 3.11. We can now state the main result of this section.

Theorem 4.10. *If Assumption* **A1** *or* **A2** *is satisfied, then:*

(a) $\mathcal{Q}_K^{-\infty}(M) = \mathcal{Q}_K^{-\infty}(Y) \otimes S^\bullet(\mathfrak{p})$,
(b) $\mathbf{r}_{K,G}(\mathcal{Q}_G^{-\infty}(M)) = \mathcal{Q}_K^{-\infty}(M)$.

Proof. Point (a) will be proved in Sections 5.4 and 5.6. We can compute

$$\begin{aligned}\mathbf{r}_{K,G}(\mathcal{Q}_G^{-\infty}(M)) &= \sum_{\mu\in\widehat{G}_{\mathrm{hol}}(z)} \mathcal{Q}(M_{\mu,G}) V_\mu^G|_K \\ &= \Big(\sum_{\mu\in\widehat{K}_{\mathrm{hol}}(z)} \mathcal{Q}(Y_{\mu,K}) V_\mu^K\Big) \otimes S^\bullet(\mathfrak{p}) \\ &= \mathcal{Q}_K^{-\infty}(Y) \otimes S^\bullet(\mathfrak{p}),\end{aligned}$$

hence (b) follows from (a). Note that the products are well defined thanks to Lemma 3.11. In the last equality we use the fact that $\mathcal{Q}_K^{-\infty}(Y) = \sum_{\mu\in\widehat{K}_{\mathrm{hol}}(z)} \mathcal{Q}(Y_{\mu,K}) V_\mu^K$ since the Kirwan polytope $\Delta_K(Y)$ is contained in $\mathcal{C}_{\mathrm{hol}}^\rho(z)$ (see Lemma 4.7). □

We now consider a connected reductive subgroup $G' \subset G$ for which $z \in \mathfrak{g}'$. The coadjoint orbit $G\cdot\lambda$ is pre-quantized when $\lambda \in \widehat{G}_{\mathrm{hol}}(z)$, and obviously

$$\mathcal{Q}_G^{-\infty}(G\cdot\lambda) = V_\lambda^G.$$

The moment map $\Phi_{G\cdot\lambda}^{G'} : G\cdot\lambda \to (\mathfrak{g}')^*$ relative to the G'-action on $G\cdot\lambda$ is proper. In fact we have more: the map $\langle\Phi_{G\cdot\lambda}^{G'}, z\rangle = \langle\Phi_{G\cdot\lambda}^{G}, z\rangle$ is proper, thus Assumption **A2** holds.

We are interested in the compact reduced spaces

$$(G\cdot\lambda)_{\mu,G'} := (\Phi_{G\cdot\lambda}^{G'})^{-1}(G'\cdot\mu)/G'$$

for $\mu \in \widehat{G}'_{\mathrm{hol}}(z)$. We are now able to prove

Theorem 4.11. *Let* $\lambda \in \widehat{G}_{\mathrm{hol}}(z)$. *Then*

$$V_\lambda^G|_{G'} = \mathcal{Q}_{G'}^{-\infty}(G\cdot\lambda) \quad \text{in } R^{-\infty}(G',z).$$

This means that for any $\mu \in \widehat{G}'_{\mathrm{hol}}(z)$, *the multiplicity of the representation* $V_\mu^{G'}$ *in* $V_\lambda^G|_{G'}$ *is equal to the geometric quantization* $\mathcal{Q}((G\cdot\lambda)_{\mu,G'}) \in \mathbb{Z}$ *of the (compact) reduced space* $(G\cdot\lambda)_{\mu,G'}$.

Proof. Since the restriction morphism $\mathbf{r}_{K',G'} : R^{-\infty}(G',z) \to R^{-\infty}(K',z)$ is injective (see Lemma 3.11), it suffices to prove that

$$\mathbf{r}_{K',G'}(V_\lambda^G|_{G'}) = \mathbf{r}_{K',G'}(\mathcal{Q}_{G'}^{-\infty}(G\cdot\lambda)). \tag{4.5}$$

But the left hand side of (4.5) is equal to $V_\lambda^G|_{K'}$, while the right hand side is $\mathcal{Q}_{K'}^{-\infty}(G\cdot\lambda)$ thanks to Theorem 4.10. Theorem 4.5 tells us that $\mathcal{Q}_K^{-\infty}(G\cdot\lambda) = V_\lambda^G|_K$, and the functoriality of the quantization process $\mathcal{Q}^{-\infty}$ (see Theorem 4.3) ensures that the restriction $V_\lambda^G|_{K'} = \mathcal{Q}_K^{-\infty}(G\cdot\lambda)|_{K'}$ is equal to $\mathcal{Q}_{K'}^{-\infty}(G\cdot\lambda)$. □

We finish this section by exhibiting cases where Assumption **A1** or **A2** is satisfied.

Lemma 4.12. • *Suppose that we are in the algebraic setting: the manifold M is real algebraic and Φ_M^G is a proper algebraic map. Then $\mathrm{Cr}(\|\Phi_M^G\|^2)$ is compact.*

• *Suppose that the Lie algebra $\mathfrak{g}$ is simple. Then, in the context of Definition 4.8, the map $\langle\Phi_M^G, z\rangle : M \to \mathbb{R}$ is proper.*

Proof. Let us prove the first point. The map $\varphi := \|\Phi_M^G\|^2 : M \to \mathbb{R}$ is a real algebraic map on a real algebraic manifold. Thus $\mathrm{Cr}(\varphi)$ is an algebraic variety, and by a standard theorem of Whitney, it has a finite number of connected components $C_1, \ldots, C_p$. Each C_i is contained in $\varphi^{-1}(\varphi(C_i))$, which is compact since φ is proper. The proof is complete.

For the second point we use Proposition 2.10 and the facts that, since $\mathfrak{g}$ is simple, $[\mathfrak{p},\mathfrak{p}] = \mathfrak{k}$ and the centre $\mathfrak{c}_\mathfrak{k}$ of $\mathfrak{k}$ reduces to $\mathbb{R}z$.

The function $\langle\Phi_M^G, z\rangle$, which is the moment map for the S^1-action, is proper if and only if $\mathrm{As}(\Delta_K(M)) \cap (\mathbb{R}z)^\perp = \{0\}$. Since $\Delta_K(M) \subset \mathcal{C}_{\mathrm{hol}}^\rho(z)$ (see Lemma 4.7), it is sufficient to prove that $\mathcal{C}_{\mathrm{hol}}(z) \cap (\mathbb{R}z)^\perp = \{0\}$. Let $\xi \in \mathcal{C}_{\mathrm{hol}}(z)$. We have $\langle\xi, z\rangle = 2\sum_{\beta\in\mathfrak{R}_n(z)}(\xi,\beta)$. If $\langle\xi, z\rangle = 0$, we must have $(\beta,\xi) = 0$ for all $\beta \in \mathfrak{R}_n(z)$, or equivalently $[\tilde{\xi},\mathfrak{p}] = 0$. Then $\tilde{\xi}$ commutes with all elements in $[\mathfrak{p},\mathfrak{p}] = \mathfrak{k}$, i.e. $\tilde{\xi} \in \mathfrak{c}_\mathfrak{k} = \mathbb{R}z$. Thus, we have proved that $\xi \in (\mathbb{R}z)^\perp$ and $\tilde{\xi} \in \mathbb{R}z$, hence $\xi = 0$. □

4.4. Geometric quantization of the slice Y

Let $\lambda \in \widehat{G}_{\mathrm{hol}}(z)$. Consider the coadjoint orbit $G\cdot\lambda$ associated to the holomorphic discrete series representation V_λ^G. Let G' be a reductive subgroup of G such that $\mathbb{R}z \subset \mathfrak{g}'$. We have a geometric decomposition $G\cdot\lambda \simeq G' \times_{K'} Y$ where $Y \subset G'\cdot\lambda$ is a closed K'-invariant symplectic submanifold.

In [JV79], Jakobsen and Vergne proved that the multiplicity $m_\lambda(\mu) := [V_\mu^{G'} : V_\lambda^G|_{G'}]$ is equal to the multiplicity of the representation $V_\mu^{K'}$ in $S^\bullet(\mathfrak{p}/\mathfrak{p}') \otimes V_\lambda^K|_{K'}$. On the other hand, Theorem 4.11 tells us that

$$m_\lambda(\mu) = \mathcal{Q}((G\cdot\lambda)_{\mu,G'}) = \mathcal{Q}(Y_{\mu,K'}).$$

We would like to understand a priori why $\mathcal{Q}(Y_{\mu,K'}) = [V_\mu^{K'} : S^\bullet(\mathfrak{p}/\mathfrak{p}') \otimes V_\lambda^K|_{K'}]$ for any $\mu \in \widehat{K'}_{\mathrm{hol}}(z)$, or equivalently why

$$\mathcal{Q}_{K'}^{-\infty}(Y) = S^\bullet(\mathfrak{p}/\mathfrak{p}') \otimes V_\lambda^K|_{K'}. \tag{4.6}$$

Note that Assumption **A2** holds in this setting: the map $\langle\Phi_{G\cdot\lambda}^{G'}, z\rangle$ is proper.

Let us consider a more general situation. Let $(M, \Omega_M, \Phi_M^{G'})$ be a pre-quantized proper[2] Hamiltonian G'-manifold. We suppose that the moment map $\Phi_M^{G'}$ takes values in $G' \cdot \mathcal{C}'^{\rho}_{\mathrm{hol}}(z)$. We suppose furthermore that Assumption **A2** holds. Let $Y \subset M$ be the symplectic slice. The aim of this section is to compute $\mathcal{Q}^{-\infty}_{K'}(Y)$ in a way similar to (4.6).

Let $\mathcal{X}$ be a connected component of the fixed point submanifold $Y^z = \mathrm{Cr}(\langle \Phi^{G'}_M, z\rangle)$; the submanifold $\mathcal{X}$ is compact since $\langle \Phi^{G'}_M, z\rangle$ is proper. Fix a K'-invariant almost complex structure on $\mathcal{X}$ which is compatible with the symplectic structure. Let $\mathrm{RR}^{K'}(\mathcal{X}, -)$ be the corresponding Riemann–Roch character (see Section 5.2). Recall that if $L_{\mathcal{X}}$ denotes the restriction of the Kostant–Souriau line bundle L_M over $\mathcal{X}$, then $\mathcal{Q}_{K'}(\mathcal{X}) = \mathrm{RR}^{K'}(\mathcal{X}, L_{\mathcal{X}})$.

Let $\mathcal{N}_{\mathcal{X}} \to \mathcal{X}$ be the normal bundle of $\mathcal{X}$ in Y; it inherits a complex structure $J_{\mathcal{X}}$ and a linear endomorphism $\mathcal{L}(z)$ on the fibres. We have a decomposition $\mathcal{N}_{\mathcal{X}} = \sum_{a\in\mathbb{R}} \mathcal{N}^a_{\mathcal{X}}$ where $\mathcal{N}^a_{\mathcal{X}} = \{v \in \mathcal{N}_{\mathcal{X}} \mid \mathcal{L}(z)v = aJ_{\mathcal{X}}(v)\}$ is a subbundle of $\mathcal{N}_{\mathcal{X}}$. We define the vector bundle $\mathcal{N}^{\pm,z}_{\mathcal{X}} := \sum_{\pm a>0} \mathcal{N}_{\mathcal{X}}$ and

$$|\mathcal{N}_{\mathcal{X}}|^z = \mathcal{N}^{+,z}_{\mathcal{X}} \oplus \overline{\mathcal{N}^{-,z}_{\mathcal{X}}}.$$

The following theorem will be proved in Section 5.5.

Theorem 4.13. *We have*

$$\mathcal{Q}^{-\infty}_{K'}(Y) = \sum_{\mathcal{X}} (-1)^{r_{\mathcal{X}}} \mathrm{RR}^{K'}\big(\mathcal{X}, L_{\mathcal{X}} \otimes \det(\mathcal{N}^{+,z}_{\mathcal{X}}) \otimes S^{\bullet}(|\mathcal{N}_{\mathcal{X}}|^z)\big) \quad \textit{in } R^{-\infty}(K'),$$

where $r_{\mathcal{X}}$ is the complex rank of $\mathcal{N}^{+,z}_{\mathcal{X}}$, and the sum runs over the connected components of the fixed point submanifold Y^z.

Let us explain how the formulas of Jakobsen–Vergne can be recovered from Theorem 4.13. When $M = G\cdot\lambda$, the submanifolds Y^z and M^z are both equal to $K\cdot\lambda$. The restriction of the Kostant–Souriau line bundle $L_M \to M$ over Y^z is $[\mathbb{C}_\lambda] := K \times_{K_\lambda} \mathbb{C}_\lambda \to K\cdot\lambda$. Relation (2.4) tells us that the normal bundle $\mathcal{N}_1$ of Y in M is equal to the trivial bundle $\mathfrak{p}' \times Y$, and the normal bundle $\mathcal{N}_2$ of Y^z in M is $Y^z \times \mathfrak{p}$. Hence the normal bundle of Y^z in Y is

$$\mathcal{N} = \mathcal{N}_2/(\mathcal{N}_1|_{Y^z}) = Y^z \times (\mathfrak{p}/\mathfrak{p}').$$

We check that $\mathcal{N}^{+,z} = 0$: this is due to the fact that the function $\langle \Phi^{G'}_{G\cdot\lambda}, z\rangle$ takes its minimal value on Y^z (see [Par01, Lemma 7.3]). So $|\mathcal{N}|^z = \overline{\mathcal{N}}$ is the trivial complex bundle with fibre $(\mathfrak{p}/\mathfrak{p}', \mathrm{ad}(z))$. Theorem 4.13 gives

$$\begin{aligned}\mathcal{Q}^{-\infty}_{K'}(Y) &= \mathrm{RR}^{K'}(K\cdot\lambda, [\mathbb{C}_\lambda] \otimes S^{\bullet}(\mathfrak{p}/\mathfrak{p}')) = \mathrm{RR}^{K}(K\cdot\lambda, [\mathbb{C}_\lambda])|_{K'} \otimes S^{\bullet}(\mathfrak{p}/\mathfrak{p}')\\ &= V^K_\lambda|_{K'} \otimes S^{\bullet}(\mathfrak{p}/\mathfrak{p}').\end{aligned}$$

In the last equality, we use $\mathrm{RR}^K(K\cdot\lambda, [\mathbb{C}_\lambda]) = V^K_\lambda$ thanks to the Borel–Weil theorem.

5. Transversally elliptic operators

The aim of this section is to prove Theorems 4.10 and 4.13. In the first subsection, we briefly introduce the material we need from the theory of transversally elliptic operators. And in Section 5.3 we recall the definition of the geometric quantization process $\mathcal{Q}^{\Phi}$. In what follows, K denotes a connected compact Lie group.

5.1. Transversally elliptic operators

Here we give the basic definitions from the theory of transversally elliptic symbols (or operators) defined by Atiyah–Singer [Ati74]. For an axiomatic treatment of the index morphism see Berline–Vergne [BV96a, BV96b] and Paradan–Vergne [PV09]. For a short introduction see [Par01].

Let $\mathcal{X}$ be a *compact* K-manifold. Let $p : \mathbf{T}\mathcal{X} \to \mathcal{X}$ be the projection, and let $(-,-)_{\mathcal{X}}$ be a K-invariant Riemannian metric. If E^0, E^1 are K-equivariant complex vector bundles over $\mathcal{X}$, a K-equivariant morphism

$$\sigma \in \Gamma(\mathbf{T}\mathcal{X}, \hom(p^*E^0, p^*E^1))$$

is called a *symbol* on $\mathcal{X}$. The subset of all $(x, v) \in \mathbf{T}\mathcal{X}$ where[2] $\sigma(x, v) : E^0_x \to E^1_x$ is not invertible is called the *characteristic set* of σ, and is denoted by $\mathrm{Char}(\sigma)$.

In the following, the product of a symbol σ by a complex vector bundle $F \to M$ is the symbol $\sigma \otimes F$ defined by $\sigma \otimes F(x, v) = \sigma(x, v) \otimes \mathrm{Id}_{F_x}$ from $E^0_x \otimes F_x$ to $E^1_x \otimes F_x$. Note that $\mathrm{Char}(\sigma \otimes F) = \mathrm{Char}(\sigma)$.

Let

$$\mathbf{T}_K\mathcal{X} = \{(x, v) \in \mathbf{T}\mathcal{X} \mid (v, X_{\mathcal{X}}(x))_{\mathcal{X}} = 0 \text{ for all } X \in \mathfrak{k}\}.$$

A symbol σ is *elliptic* if σ is invertible outside a compact subset of $\mathbf{T}\mathcal{X}$ (i.e. $\mathrm{Char}(\sigma)$ is compact), and is K-*transversally elliptic* if the restriction of σ to $\mathbf{T}_K\mathcal{X}$ is invertible outside a compact subset of $\mathbf{T}_K\mathcal{X}$ (i.e. $\mathrm{Char}(\sigma) \cap \mathbf{T}_K\mathcal{X}$ is compact). An elliptic symbol σ defines an element in the equivariant $\mathbf{K}^0$-theory of $\mathbf{T}\mathcal{X}$ with compact support, which is denoted by $\mathbf{K}^0_K(\mathbf{T}\mathcal{X})$, and the index of σ is a virtual finite-dimensional representation of K, denoted by $\mathrm{Index}^K_{\mathcal{X}}(\sigma) \in R(K)$ [ASe68, AS68a, AS68b, AS71].

A K-*transversally elliptic* symbol σ defines an element of $\mathbf{K}^0_K(\mathbf{T}_K\mathcal{X})$, and the index of σ is defined as a trace class virtual representation of K, which we still denote $\mathrm{Index}^K_{\mathcal{X}}(\sigma) \in R^{-\infty}(K)$ [Ati74].

Using the *excision property*, one can show that the index map $\mathrm{Index}^K_{\mathcal{U}} : \mathbf{K}^0_K(\mathbf{T}_K\mathcal{U}) \to R^{-\infty}(K)$ is still defined when $\mathcal{U}$ is a K-invariant relatively compact open subset of a K-manifold (see [Par01, Section 3.1]).

Suppose now that the group K is a product $K_1 \times K_2$. An intermediate notion between the "ellipticity" and "$K_1 \times K_2$-transversal ellipticity" is the "K_1-transversal ellipticity". When a $K_1 \times K_2$-equivariant symbol σ is K_1-*transversally elliptic*, its index $\mathrm{Index}^{K_1\times K_2}_{\mathcal{X}}(\sigma) \in R^{-\infty}(K_1 \times K_2)$, viewed as a generalized function on $K_1 \times K_2$, is *smooth* relative to the variable in K_2 [Ati74, BV96b, PV09]. This implies that:

- $\mathrm{Index}^{K_1\times K_2}_{\mathcal{X}}(\sigma) = \sum_{\lambda \in \widehat{K_1}} \theta_\lambda \otimes V^{K_1}_\lambda$ with $\theta_\lambda \in R(K_2)$,
- we can restrict $\mathrm{Index}^{K_1\times K_2}_{\mathcal{X}}(\sigma)$ to the subgroup K_1 and

$$\mathrm{Index}^{K_1\times K_2}_{\mathcal{X}}(\sigma)|_{K_1} = \sum_{\lambda \in \widehat{K_1}} \dim(\theta_\lambda) V^{K_1}_\lambda = \mathrm{Index}^{K_1}_{\mathcal{X}}(\sigma). \tag{5.1}$$

Here $\dim : R(K_2) \to \mathbb{Z}$ is the morphism induced by restriction to $1 \in K_2$.

[2] The map $\sigma(x, v)$ will also be denoted $\sigma|_x(v)$.

Let us recall the multiplicative property of the index map for the product of manifolds proved by Atiyah–Singer [Ati74]. Consider a compact Lie group K_2 acting on two manifolds $\mathcal{X}_1$ and $\mathcal{X}_2$, and assume that another compact Lie group K_1 acts on $\mathcal{X}_1$ commuting with the action of K_2. The external product of complexes on $\mathbf{T}\mathcal{X}_1$ and $\mathbf{T}\mathcal{X}_2$ induces a multiplication (see [Ati74])

$$\odot : \mathbf{K}^0_{K_1\times K_2}(\mathbf{T}_{K_1}\mathcal{X}_1) \times \mathbf{K}^0_{K_2}(\mathbf{T}_{K_2}\mathcal{X}_2) \to \mathbf{K}^0_{K_1\times K_2}(\mathbf{T}_{K_1\times K_2}(\mathcal{X}_1 \times \mathcal{X}_2)).$$

Let us recall the definition of this external product. For $k = 1, 2$, we consider equivariant morphisms[3] $\sigma_k : \mathcal{E}_k^+ \to \mathcal{E}_k^-$ on $\mathbf{T}\mathcal{X}_k$. We consider the equivariant morphism on $\mathbf{T}(\mathcal{X}_1 \times \mathcal{X}_2)$,

$$\sigma_1 \odot \sigma_2 : \mathcal{E}_1^+ \otimes \mathcal{E}_2^+ \oplus \mathcal{E}_1^- \otimes \mathcal{E}_2^- \to \mathcal{E}_1^- \otimes \mathcal{E}_2^+ \oplus \mathcal{E}_1^+ \otimes \mathcal{E}_2^-,$$

defined by

$$\sigma_1 \odot \sigma_2 = \begin{pmatrix} \sigma_1 \otimes \mathrm{Id} & -\mathrm{Id} \otimes \sigma_2^* \\ \mathrm{Id} \otimes \sigma_2 & \sigma_1^* \otimes \mathrm{Id} \end{pmatrix}. \tag{5.2}$$

We see that the set $\mathrm{Char}(\sigma_1 \odot \sigma_2) \subset \mathbf{T}\mathcal{X}_1 \times \mathbf{T}\mathcal{X}_2$ is equal to $\mathrm{Char}(\sigma_1) \times \mathrm{Char}(\sigma_2)$. We now suppose that the morphisms σ_k are K_k-transversally elliptic. As $\mathbf{T}_{K_1\times K_2}(\mathcal{X}_1 \times \mathcal{X}_2) \neq \mathbf{T}_{K_1}\mathcal{X}_1 \times \mathbf{T}_{K_2}\mathcal{X}_2$, the morphism $\sigma_1 \odot \sigma_2$ is not necessarily $K_1 \times K_2$-transversally elliptic, but it is so if σ_2 is taken *almost homogeneous* (see [PV09]). So the exterior product $a_1 \odot a_2$ is the $\mathbf{K}$-theory class defined by $\sigma_1 \odot \sigma_2$, where $a_k = [\sigma_k]$ and σ_2 is almost homogeneous.

The following property is a useful tool (see [Ati74, Lecture 3] and [PV09]).

Theorem 5.1 (Multiplicative property). *For any* $[\sigma_1] \in \mathbf{K}^0_{K_1\times K_2}(\mathbf{T}_{K_1}\mathcal{X}_1)$ *and any* $[\sigma_2] \in \mathbf{K}^0_{K_2}(\mathbf{T}_{K_2}\mathcal{X}_2)$ *we have*

$$\mathrm{Index}^{K_1\times K_2}_{\mathcal{X}_1\times\mathcal{X}_2}([\sigma_1] \odot [\sigma_2]) = \mathrm{Index}^{K_1\times K_2}_{\mathcal{X}_1}([\sigma_1]) \otimes \mathrm{Index}^{K_2}_{\mathcal{X}_2}([\sigma_2]).$$

5.2. Riemann–Roch character

Let M be a compact K-manifold equipped with an invariant almost complex structure J. Let $p : \mathbf{T}M \to M$ be the projection. The complex vector bundle $(\mathbf{T}^*M)^{0,1}$ is K-equivariantly identified with the tangent bundle $\mathbf{T}M$ equipped with the complex structure J. Let h_M be an Hermitian structure on $(\mathbf{T}M, J)$. The symbol

$$\mathrm{Thom}(M, J) \in \Gamma(\mathbf{T}M, \hom(p^*(\textstyle\bigwedge_{\mathbb{C}}^{\mathrm{even}} \mathbf{T}M),\ p^*(\bigwedge_{\mathbb{C}}^{\mathrm{odd}} \mathbf{T}M)))$$

at $(m, v) \in \mathbf{T}M$ is equal to the Clifford map

$$\mathbf{c}_m(v) : \textstyle\bigwedge_{\mathbb{C}}^{\mathrm{even}} \mathbf{T}_m M \to \bigwedge_{\mathbb{C}}^{\mathrm{odd}} \mathbf{T}_m M, \tag{5.3}$$

where $\mathbf{c}_m(v).w = v \wedge w - \iota(v)w$ for $w \in \bigwedge_{\mathbb{C}}^{\bullet} \mathbf{T}_m M$. Here $\iota(v) : \bigwedge_{\mathbb{C}}^{\bullet} \mathbf{T}_m M \to \bigwedge_{\mathbb{C}}^{\bullet-1} \mathbf{T}_m M$ denotes the contraction map relative to h_M. Since $\mathbf{c}_m(v)^2 = -\|v\|^2\,\mathrm{Id}$, the map $\mathbf{c}_m(v)$ is invertible for all $v \neq 0$. Hence the characteristic set of $\mathrm{Thom}(M, J)$ corresponds to the 0-section of $\mathbf{T}M$.

[3] To simplify the notation, we do not distinguish between vector bundles on $\mathbf{T}\mathcal{X}$ and on $\mathcal{X}$.

Definition 5.2. To any K-equivariant complex vector bundle $E \to M$, we associate its *Riemann–Roch character*

$$\mathrm{RR}^K(M, E) := \mathrm{Index}^K_M(\mathrm{Thom}(M, J) \otimes E) \in R(K).$$

Remark 5.3. The character $\mathrm{RR}^K(M, E)$ is equal to the equivariant index of the Dolbeault–Dirac operator $\mathcal{D}_E := \sqrt{2}(\overline{\partial}_E + \overline{\partial}^*_E)$, since $\mathrm{Thom}(M, J) \otimes E$ corresponds to the principal symbol of $\mathcal{D}_E$ (see [BGV91, Proposition 3.67]).

5.3. *Definition of $\mathcal{Q}^\Phi$*

Let (M, Ω_M, Φ^K_M) be a compact Hamiltonian K-manifold pre-quantized by an equivariant line bundle L_M. Let J be an invariant almost complex structure compatible with Ω. Let $\mathrm{RR}^K(M, -)$ be the corresponding Riemann–Roch character. The topological index of $\mathrm{Thom}(M, J) \otimes L_M \in \mathbf{K}^0_K(\mathbf{T}M)$ is equal to the analytical index of the Dolbeault–Dirac operator $\sqrt{2}(\overline{\partial}_{L_M} + \overline{\partial}^*_{L_M})$:

$$\mathcal{Q}_K(M) = \mathrm{RR}^K(M, L_M). \tag{5.4}$$

When M is not compact, the topological index of $\mathrm{Thom}(M, J) \otimes L_M$ is not defined. In order to extend the notion of geometric quantization to this setting, we deform the symbol $\mathrm{Thom}(M, J) \otimes L$ in the "Witten" way [Par01, Par03, MZ09, MZ14]. Consider the identification $\xi \mapsto \widetilde{\xi}$, $\mathfrak{k}^* \to \mathfrak{k}$ defined by a K-invariant scalar product on $\mathfrak{k}^*$. We define the *Kirwan vector field* on M:

$$\kappa_m = \left(\widetilde{\Phi^K_M(m)}\right)_M(m), \quad m \in M. \tag{5.5}$$

Definition 5.4. The symbol $\mathrm{Thom}(M, J) \otimes L$ *pushed by the vector field* κ is the symbol $\mathbf{c}^\kappa$ defined by setting

$$\mathbf{c}^\kappa|_m(v) = \mathrm{Thom}(M, J) \otimes L|_m(v - \kappa_m)$$

for any $(m, v) \in \mathbf{T}M$. More generally, if $E \to M$ is an equivariant complex vector bundle, one defines $\mathbf{c}^\kappa_E$ by the same relation (with E in place of L).

Note that $\mathbf{c}^\kappa|_m(v)$ is invertible unless $v = \kappa_m$. If furthermore v belongs to the subset $\mathbf{T}_K M$ of tangent vectors orthogonal to the K-orbits, then $v = 0$ and $\kappa_m = 0$. Indeed, κ_m is tangent to $K \cdot m$ while v is orthogonal.

Since κ is the Hamiltonian vector field of the function $\frac{-1}{2}\|\Phi^K_M\|^2$, the set of zeros of κ coincides with the set of critical points of $\|\Phi^K_M\|^2$. Finally, we have

$$\mathrm{Char}(\mathbf{c}^\kappa) \cap \mathbf{T}_K M \simeq \mathrm{Cr}(\|\Phi^K_M\|^2).$$

In general $\mathrm{Cr}(\|\Phi^K_M\|^2)$ is not compact, so $\mathbf{c}^\kappa$ does not define a transversally elliptic symbol on M. In order to define a kind of index of $\mathbf{c}^\kappa$, we proceed as follows. For any invariant open relatively compact subset $U \subset M$ the set $\mathrm{Char}(\mathbf{c}^\kappa|_U) \cap \mathbf{T}_K U \simeq \mathrm{Cr}(\|\Phi\|^2) \cap U$ is compact whenever

$$\partial U \cap \mathrm{Cr}(\|\Phi\|^2) = \emptyset. \tag{5.6}$$

When (5.6) holds we denote by

$$\mathcal{Q}_K^\Phi(U) := \mathrm{Index}_U^K(\mathbf{c}^\kappa|_U) \in R^{-\infty}(K) \tag{5.7}$$

the equivariant index of the transversally elliptic symbol $\mathbf{c}^\kappa|_U$.

Let us recall the description of the critical points of $\|\Phi_M^K\|^2$ when the moment map Φ_M^K is proper. We know that $m \in \mathrm{Cr}(\|\Phi_M^K\|^2)$ if and only if $\widetilde{\beta}_M(m) = 0$ for $\beta = \Phi_M^K(m)$. Hence the set $\mathrm{Cr}(\|\Phi_M^K\|^2)$ has the decomposition

$$\mathrm{Cr}(\|\Phi_M^K\|^2) = \bigcup_{\beta \in \mathfrak{t}^*} M^{\widetilde{\beta}} \cap (\Phi_M^K)^{-1}(\beta) = \bigcup_{\beta \in \mathcal{B}} \underbrace{K \cdot (M^{\widetilde{\beta}} \cap (\Phi_M^K)^{-1}(\beta))}_{Z_\beta},$$

where $\mathcal{B}$ is a subset of the Weyl chamber $\mathfrak{t}^*_{\geq 0}$. We denote by $B_r \subset \mathfrak{t}^*$ the open ball $\{\xi \in \mathfrak{t}^* \mid \|\xi\| < r\}$. The following proposition is proved in [Par11].

Proposition 5.5. • *For any $r > 0$, the set $\mathcal{B} \cap B_r$ is finite.*
- *The set of singular values of $\|\Phi_M^K\|^2 : M \to \mathbb{R}$ is a sequence $0 \leq r_1 < r_2 < \cdots$ which is finite if and only if $\mathrm{Cr}(\|\Phi_M^K\|^2)$ is compact. In the other case $\lim_{k\to\infty} r_k = \infty$.*

For any $\beta \in \mathcal{B}$, we consider a relatively compact open invariant neighbourhood $\mathcal{U}_\beta$ of Z_β such that $\mathrm{Cr}(\|\Phi_M^K\|^2) \cap \overline{\mathcal{U}_\beta} = Z_\beta$. The excision property tells us that the generalized character $\mathcal{Q}_K^\Phi(\mathcal{U}_\beta) = \mathrm{Index}_{\mathcal{U}_\beta}^K(\mathbf{c}^\kappa|_{\mathcal{U}_\beta})$ does not depend of the choice of $\mathcal{U}_\beta$. In order to simplify the notation we make the following

Definition 5.6. • We denote by $\mathcal{Q}_K^\beta(M) \in R^{-\infty}(K)$ the equivariant index[4] of the transversally elliptic symbol $\mathbf{c}^\kappa|_{\mathcal{U}_\beta}$.
- When $E \to M$ is an equivariant complex vector bundle, we denote by $\mathrm{RR}_\beta^K(M, E)$ the equivariant index of the transversally elliptic symbol $\mathbf{c}_E^\kappa|_{\mathcal{U}_\beta}$.

The following crucial property is proved in [MZ09, Par11, MZ14].

Theorem 5.7. *A representation V_λ^K occurs in the generalized character $\mathcal{Q}_K^\beta(M) \in R^{-\infty}(K)$ only if $\|\lambda\| \geq \|\beta\|$.*

Definition 5.8. The generalized character $\mathcal{Q}_K^\Phi(M) \in R^{-\infty}(K)$ is defined by

$$\mathcal{Q}_K^\Phi(M) = \sum_{\beta \in \mathcal{B}} \mathcal{Q}_K^\beta(M). \tag{5.8}$$

[4] The index of $\mathbf{c}^\kappa|_{\mathcal{U}_\beta}$ was denoted $RR_\beta^K(M, L)$ in [Par01].

The sum (5.8) converges in $R^{-\infty}(K)$ since by Theorem 5.7 the multiplicity of V_λ^K in $\mathcal{Q}_K^\beta(M)$ is zero when $\|\beta\| > \|\lambda\|$.

We finish this section by recalling a result that will be needed in Section 5.5. Suppose that $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ where $[\mathfrak{k}_1, \mathfrak{k}_2] = 0$ and the $\mathfrak{k}_i$ are the Lie algebras of closed connected subgroups K_i. We assume that the moment map $\Phi_M^{K_1} : M \to \mathfrak{k}_1^*$ relative to the K_1-action is *proper*. Let us explain how we can use the K-invariant proper map $\|\Phi_M^{K_1}\|^2$ instead of $\|\Phi_M^K\|^2$ in order to define the generalized character $\mathcal{Q}_K^\Phi(M)$.

Choose $\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$ such that $\mathfrak{t}_i \subset \mathfrak{k}_i$ is a maximal abelian subalgebra. We start from a decomposition

$$\mathrm{Cr}(\|\Phi_M^{K_1}\|^2) = \bigcup_{\beta \in \mathcal{B}_1} \underbrace{K \cdot (M^{\widetilde{\beta}} \cap (\Phi_M^{K_1})^{-1}(\beta))}_{Z_\beta^1} \tag{5.9}$$

with $\mathcal{B}_1 \subset \mathfrak{t}_1^*$.

Let κ_1 be the Hamiltonian vector field of $\frac{-1}{2}\|\Phi_M^{K_1}\|^2$, and let $\mathbf{c}^{\kappa_1}$ be the corresponding pushed symbol. For any $\beta \in \mathcal{B}_1$, we consider a relatively compact open K-invariant neighbourhood $\mathcal{U}_\beta^1$ of Z_β^1 such that $\mathrm{Cr}(\|\Phi_M^{K_1}\|^2) \cap \overline{\mathcal{U}_\beta^1} = Z_\beta^1$. We denote by $\mathcal{Q}_K^{\beta,1}(M) \in R^{-\infty}(K)$ the equivariant index of the K_1-transversally elliptic symbol $\mathbf{c}^{\kappa_1}|_{\mathcal{U}_\beta^1}$. Theorem 5.7 admits the following extension:

Theorem 5.9. *A representation V_λ^K occurs in the generalized character $\mathcal{Q}_K^{\beta,1}(M)$ only if $\|\lambda_1\| \geq \|\beta\|$. Here $\lambda \in \Lambda^* \subset \mathfrak{t}^*$ decomposes into $\lambda = \lambda_1 \oplus \lambda_2$ with $\lambda_i \in \mathfrak{t}_i^*$.*

As in Definition 5.8, we can define the generalized character

$$\mathcal{Q}_K^{\Phi_1}(M) = \sum_{\beta \in \mathcal{B}_1} \mathcal{Q}_K^{\beta,1}(M). \tag{5.10}$$

In [Par11, Section 4.1], we prove

Theorem 5.10. *Let (M, Ω_M, Φ_M^K) be a proper Hamiltonian K-manifold that is pre-quantized. If the moment map $\Phi_M^{K_1} : M \to \mathfrak{k}_1^*$ is proper, then*

$$\mathcal{Q}_K^\Phi(M) = \mathcal{Q}_K^{\Phi_1}(M) \quad \textit{in } R^{-\infty}(K).$$

5.4. Proof of Theorem 4.10 under Assumption **A1**

In this section we consider the manifold $M = G \times_K Y$, where (Y, Ω_Y, Φ_Y^K) is a Hamiltonian K-manifold pre-quantized by a line bundle L_Y. We suppose that the moment map Φ_Y^K is proper, and that the Kirwan polytope $\Delta_K(Y)$ is contained in the cone $\mathcal{C}_{\mathrm{hol}}^\rho(z) \subset \mathfrak{t}_{\mathrm{se}}^*$.

Then on M we have an induced G-invariant symplectic form Ω_M and a moment map $\Phi_M^G : M \to \mathfrak{g}^*$ defined by $\Phi_M^G([g, y]) = g \cdot \Phi_M^K(y)$. The line bundle $L_M = (G \times L_Y)/K$ pre-quantizes the Hamiltonian manifold (M, Ω_M, Φ_M^G). Let us consider the K-action on M; the moment map Φ_M^K is also proper.

We are then in a setting where the formal geometric quantizations of M and Y relative to the K-action are well defined: $\mathcal{Q}^{\Phi}_K(M), \mathcal{Q}^{\Phi}_K(Y) \in R^{-\infty}(K)$. The aim of this section is to prove that

$$\mathcal{Q}^{\Phi}_K(M) = \mathcal{Q}^{\Phi}_K(Y) \otimes S^{\bullet}(\mathfrak{p}) \tag{5.11}$$

whenever (M, Ω_M, Φ^G_M) satisfies Assumption **A1**. The set

$$\mathrm{Cr}(\|\Phi^G_M\|^2) = \mathrm{Cr}(\|\Phi^K_M\|^2) = \mathrm{Cr}(\|\Phi^K_Y\|^2) = \bigcup_{\beta \in \mathcal{B}} \underbrace{K \cdot (Y^{\tilde{\beta}} \cap (\Phi^K_Y)^{-1}(\beta))}_{Z_\beta}$$

is compact: the parametrizing set $\mathcal{B}$ is *finite* (see Theorem 2.8). So we have $\mathcal{Q}^{\Phi}_K(M) = \sum_{\beta \in \mathcal{B}} \mathcal{Q}^{\beta}_K(M)$ and $\mathcal{Q}^{\Phi}_K(Y) = \sum_{\beta \in \mathcal{B}} \mathcal{Q}^{\beta}_K(Y)$, and we are reduced to proving

Theorem 5.11. *For any $\beta \in \mathcal{B}$,*

$$\mathcal{Q}^{\beta}_K(M) = \mathcal{Q}^{\beta}_K(Y) \otimes S^{\bullet}(\mathfrak{p}) \quad \textit{in } R^{-\infty}(K). \tag{5.12}$$

Proof. Let κ_M be the Kirwan vector field on M associated to the moment map Φ^K_M. Let J_M be a K-invariant almost complex structure compatible with Ω_M, and let $\mathcal{U}_\beta$ be a (small) neighbourhood of Z_β in M.

The symbol $\mathrm{Thom}(M, J_M) \otimes L_M$ pushed by the vector field κ_M is denoted $\mathbf{c}^{\kappa}_M$. By definition $\mathcal{Q}^{\beta}_K(M)$ is the equivariant index of the K-transversally elliptic symbol $\mathbf{c}^{\kappa}_M|_{\mathcal{U}_\beta}$. Note that $\mathcal{Q}^{\beta}_K(M)$ does not depend on the choice of the neighbourhood $\mathcal{U}_\beta$ or of the almost complex structure on $\mathcal{U}_\beta$.

We use the K-diffeomorphism $\varphi : \mathfrak{p} \times Y \simeq M$ defined by $\varphi(X, y) = [e^X, y]$. The Kirwan vector field $\kappa_{\mathfrak{p}\times Y} := \varphi^*(\kappa_M)$ is defined by $\kappa_{\mathfrak{p}\times Y}(X, y) = (\kappa_1(X, y), \kappa_2(X, y)) \in \mathbf{T}(\mathfrak{p} \times Y)$ where

$$\kappa_2(X, y) = A_Y(y), \quad \kappa_1(X, y) = -[A, X], \qquad A = [e^X \cdot \widetilde{\Phi^K_Y(y)}]_{\mathfrak{k}}.$$

Here $[Z]_{\mathfrak{k}}, [X]_{\mathfrak{p}}$ are respectively the $\mathfrak{k}$ and $\mathfrak{p}$ components of $Z \in \mathfrak{g}$.

The Kostant–Souriau line bundle $\varphi^*(L_M)$ is K-diffeomorphic to L_Y since Y is a deformation retract of $\mathfrak{p}\times Y$. Let us compute the pull-back of the symplectic form $\Omega_{\mathfrak{p}\times Y} = \varphi^*(\Omega_M)$ at $(0, y)$. For $v, v' \in \mathbf{T}_y Y$ and $\eta, \eta' \in \mathbf{T}_0\mathfrak{p} = \mathfrak{p}$, we have

$$\Omega_{\mathfrak{p}\times Y}(\eta \oplus v, \eta' \oplus v') = \Omega_M(v \oplus \eta \cdot y, v' \oplus \eta' \cdot y) = \Omega_Y(v, v') + \langle \Phi^K_Y(y), [\eta, \eta'] \rangle.$$

Lemma 5.12. $\langle \xi, [\eta, \mathrm{ad}(z)\eta] \rangle = -([\tilde{\xi}, \eta], [z, \eta]) < 0$ *for any* $\xi \in K \cdot \mathcal{C}^{\rho}_{\mathrm{hol}}(z)$ *and any* $\eta \in \mathfrak{p} \setminus \{0\}$.

Proof. Recall that the scalar product on $\mathfrak{g}$ is defined by $(X, Y) = -b(X, \Theta(Y))$. Hence

$$\langle \xi, [\eta, \mathrm{ad}(z)\eta] \rangle = -b(\tilde{\xi}, \Theta([\eta, \mathrm{ad}(z)\eta])) = (\mathrm{ad}(z)\mathrm{ad}(\tilde{\xi})\eta, \eta) = (\mathrm{ad}(z)\mathrm{ad}(\tilde{\xi}')\eta', \eta')$$

where $\xi = k \cdot \xi'$ with $\xi' \in \mathcal{C}^{\rho}_{\mathrm{hol}}(z)$ and $\eta = k \cdot \eta'$ for some $k \in K$. We can then check that the symmetric endomorphism $\mathrm{ad}(z)\mathrm{ad}(\tilde{\xi}') : \mathfrak{p} \to \mathfrak{p}$ is negative definite when $\xi' \in \mathcal{C}^{\rho}_{\mathrm{hol}}(z)$; the lemma is proved. □

If J_Y is a K-invariant almost complex structure on Y compatible with Ω_Y, the last lemma tells us that $(-\mathrm{ad}(z), J_Y)$ is a K-invariant almost complex structure on $\mathfrak{p} \times Y$ compatible with $\Omega_{\mathfrak{p}\times Y}$ in a neighbourhood of Y.

Fix $\mathcal{U}_\beta$ such that $\varphi^{-1}(\mathcal{U}_\beta) = B_r \times \mathcal{V}_\beta$ where $\mathcal{V}_\beta$ is a neighbourhood of Z_β in Y and $B_r := \{X \in \mathfrak{p} \mid \|X\| < r\}$. The almost complex structure J_M on $\mathcal{U}_\beta$ defined by $\varphi^*(J_M) = (-\mathrm{ad}(z), J_Y)$ is compatible with Ω_M if $\mathcal{V}_\beta$ and B_r are small enough. Finally, the symbol $\varphi^*(\mathbf{c}^\kappa_M|_{\mathcal{U}_\beta})$ is equal to the product $\sigma_1 \odot \sigma_2|_{B_r\times\mathcal{V}_\beta}$, where

$$\sigma_2(X, y; \eta, v) = \mathbf{c}(v - \kappa_2(X, y)), \quad (X, y; \eta, v) \in \mathbf{T}(\mathfrak{p} \times Y),$$

acts on $\bigwedge^*_{\mathbb{C}} \mathbf{T}_y Y \otimes L_Y$, and

$$\sigma_1(X, y; \eta, v) = \mathbf{c}(\eta - \kappa_1(X, y)), \quad (X, y; \eta, v) \in \mathbf{T}(\mathfrak{p} \times Y),$$

acts on $\bigwedge^*_{\mathbb{C}} \mathfrak{p}^-$ (here $\mathfrak{p}^-$ denotes the complex K-module $(\mathfrak{p}, -\mathrm{ad}(z))$).

Let κ_Y be the Kirwan vector field on Y associated to the moment map Φ^K_Y. We denoted by $\mathbf{c}^\kappa_Y$ the symbol $\mathrm{Thom}(Y, J_Y) \otimes L_Y$ pushed by the vector field κ_Y. By definition $\mathcal{Q}^\beta_K(Y)$ is the equivariant index of the K-transversally elliptic symbol $\mathbf{c}^\kappa_Y|_{\mathcal{V}_\beta}$.

The Atiyah symbol $\mathrm{At}_\mathfrak{p}$ on $\mathfrak{p}$ is defined by setting, for $(X, \eta) \in \mathbf{T}\mathfrak{p}$,

$$\mathrm{At}_\mathfrak{p}(X, \eta) := \mathbf{c}(\eta + [z, X]) : \textstyle\bigwedge^{\mathrm{even}}_{\mathbb{C}} \mathfrak{p}^- \to \bigwedge^{\mathrm{odd}}_{\mathbb{C}} \mathfrak{p}^-. \tag{5.13}$$

Lemma 5.13. *The symbols* $\sigma_1 \odot \sigma_2|_{B_r\times\mathcal{V}_\beta}$ *and* $\mathrm{At}_\mathfrak{p} \odot \mathbf{c}^\kappa_Y|_{B_r\times\mathcal{V}_\beta}$ *define the same class in* $\mathbf{K}^0_K(\mathbf{T}_K(B_r \times \mathcal{V}_\beta))$.

Proof. We consider the paths $[0, 1] \ni s \mapsto A^s := [e^{sX} \cdot \widetilde{\Phi^K_Y(y)}]_\mathfrak{k}$, $\kappa^s_2(X, y) = A^s_Y(y)$, and $\kappa^s_1(X, y) = -[A^s, X]$. We then define the paths at the level of symbols, σ^s_1 and σ^s_2. We check that

$$\mathrm{Char}(\sigma^s_1 \odot \sigma^s_2) \cap \mathbf{T}_K(Y \times \mathfrak{p}) = \{(X, y; v, \eta) \mid v = A^s_Y(y) = 0 \text{ and } \eta = [A^s, X] = 0\}.$$

But since $\Phi^K_Y(y) \in \mathfrak{k}^*_{\mathrm{se}}$, the condition $[A^s, X] = [e^X \cdot \widetilde{\Phi^K_Y(y)}, X]_\mathfrak{p} = 0$ forces X to be zero. Hence

$$\mathrm{Char}(\sigma^s_1 \odot \sigma^s_2) \cap \mathbf{T}_K(Y \times \mathfrak{p}) \simeq \mathrm{Cr}(\|\Phi^K_Y\|^2) \times \{0\}, \quad \forall s \in [0, 1].$$

We have proved that $[0, 1] \ni s \mapsto \sigma^s_1 \odot \sigma^s_2|_{B_r\times\mathcal{V}_\beta}$ is a homotopy of transversally elliptic symbols, so $\sigma_1 \odot \sigma_2$ and $\sigma^0_1 \odot \sigma^0_2$ define the same class in $\mathbf{K}^0_K(\mathbf{T}_K(B_r \times \mathcal{V}_\beta))$.

We see that $\sigma^0_2 = \mathbf{c}^\kappa_Y$ and

$$\sigma^0_1(X, y; \eta, v) = \mathbf{c}(\eta + [\widetilde{\Phi^K_Y(y)}, X]).$$

We consider another path of symbols,

$$\tau^t(X, y; \eta, v) = \mathbf{c}(\eta + [t\widetilde{\Phi^K_Y(y)} + (1 - t)z, X]), \quad t \in [0, 1].$$

We check that if $(X, y; \eta, v) \in \mathrm{Char}(\tau^t \odot \mathbf{c}_Y^\kappa) \cap \mathbf{T}_K(\mathfrak{p} \times Y)$ then the vector $\eta \oplus v \in \mathbf{T}_{(X,y)}(\mathfrak{p} \times Y)$ is orthogonal to the vector field generated by $\widetilde{\Phi_Y^K(y)}$, and moreover $v = \kappa_Y(y)$ and $\eta = -[t\widetilde{\Phi_Y^K(y)} + (1-t)z, X]$. Thus

$$\begin{aligned} 0 &= \|\kappa_Y(y)\|^2 + \big([t\tilde{\xi} + (1-t)z, X], [\tilde{\xi}, X]\big) \\ &= \|\kappa_Y(y)\|^2 + \underbrace{t\,\|[\tilde{\xi}, X]\|^2 + (1-t)\big([z, X], [\tilde{\xi}, X]\big)}_{\delta} \end{aligned}$$

where $\xi = \Phi_Y^K(y) \in K \cdot \mathcal{C}_{\mathrm{hol}}^\rho(z)$. Since ξ is strongly elliptic, and by Lemma 5.12 the term δ is strictly positive if $X \neq 0$, we have $\kappa_Y(y) = 0$ and $X = 0$.

We have proved that $[0,1] \ni t \mapsto \tau^t \odot \mathbf{c}_Y^\kappa|_{B_r \times \mathcal{V}_\beta}$ is a homotopy of K-transversally elliptic symbols, so that $\sigma_1^0 \odot \sigma_2^0$ and $\mathrm{At}_\mathfrak{p} \odot \mathbf{c}_Y^\kappa$ define the same class in the group $\mathbf{K}_K^0(\mathbf{T}_K(B_r \times \mathcal{V}_\beta))$. □

At this stage, $\mathcal{Q}_K^\beta(M) = \mathrm{Index}_{B_r \times \mathcal{V}_\beta}^K(\mathrm{At}_\mathfrak{p}|_{B_r} \odot \mathbf{c}_Y^\kappa|_{\mathcal{V}_\beta})$. Since $\mathbf{c}_Y^\kappa \odot \mathrm{At}_\mathfrak{p}$ is also K-transversally elliptic on $\mathcal{V}_\beta \times \mathfrak{p}$, the excision property also gives $\mathcal{Q}_K^\beta(M) = \mathrm{Index}_{\mathfrak{p} \times \mathcal{V}_\beta}^K(\mathrm{At}_\mathfrak{p} \odot \mathbf{c}_Y^\kappa|_{\mathcal{V}_\beta})$.

Let S^1 be the circle subgroup of K with Lie algebra $\mathbb{R}z$. We can consider $\mathfrak{p}$ as an $S^1 \times K$-manifold. We note that the Atiyah symbol $\mathrm{At}_\mathfrak{p}$ is $S^1 \times K$-equivariant and S^1-transversally elliptic. Its index is computed in [Ati74] (see also [Par01, Section 5]):

$$\mathrm{Index}_\mathfrak{p}^{S^1 \times K}(\mathrm{At}_\mathfrak{p}) = S^\bullet(\mathfrak{p}) \quad \text{in } R^{-\infty}(S^1 \times K).$$

Consider the classes $\mathrm{At}_\mathfrak{p} \in \mathbf{K}_{S^1 \times K}^0(\mathbf{T}_{S^1}\mathfrak{p})$ and $\mathbf{c}_Y^\kappa|_{\mathcal{V}_\beta} \in \mathbf{K}_K^0(\mathbf{T}_K \mathcal{V}_\beta)$. By the *multiplicative property* (see Theorem 5.1), the product $\mathrm{At}_\mathfrak{p} \odot \mathbf{c}_Y^\kappa$ has the following $S^1 \times K$-equivariant index:

$$\begin{aligned} \mathrm{Index}_{\mathfrak{p} \times \mathcal{V}_\beta}^{S^1 \times K}(\mathrm{At}_\mathfrak{p} \odot \mathbf{c}_Y^\kappa|_{\mathcal{V}_\beta}) &= \mathrm{Index}_\mathfrak{p}^{S_1 \times K}(\mathrm{At}_\mathfrak{p}) \otimes \mathrm{Index}_{\mathcal{V}_\beta}^K(\mathbf{c}_Y^\kappa|_{\mathcal{V}_\beta}) \\ &= S^\bullet(\mathfrak{p}) \otimes \mathrm{Index}_{\mathcal{V}_\beta}^K(\mathbf{c}_Y^\kappa|_{\mathcal{V}_\beta}) = S^\bullet(\mathfrak{p}) \otimes \mathcal{Q}_K^\beta(Y) \in R^{-\infty}(S^1 \times K). \end{aligned}$$

Finally, by the *restriction property* (5.1), the term

$$\mathcal{Q}_K^\beta(M) = \mathrm{Index}_{\mathfrak{p} \times \mathcal{V}_\beta}^K(\mathrm{At}_\mathfrak{p} \odot \mathbf{c}_Y^\kappa|_{\mathcal{V}_\beta}) \in R^{-\infty}(K)$$

is equal to the restriction of

$$\mathrm{Index}_{\mathfrak{p} \times \mathcal{V}_\beta}^{S^1 \times K}(\mathrm{At}_\mathfrak{p} \odot \mathbf{c}_Y^\kappa|_{\mathcal{V}_\beta}) = S^\bullet(\mathfrak{p}) \otimes \mathcal{Q}_K^\beta(Y) \in R^{-\infty}(S^1 \times K)$$

to the subgroup $K \hookrightarrow S^1 \times K$. The theorem is thus proved. □

5.5. Proof of Theorem 4.13

Here we work with a pre-quantized Hamiltonian K-manifold (P, Ω_P, Φ_P^K), and we assume that the map $\langle \Phi_P^K, z \rangle$ is proper. Here $\mathbb{R}z$ is the Lie algebra of a circle subgroup $S^1 \subset K$ contained in the centre of K.

We are in the context of Theorem 5.10. We have a decomposition $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ where $\mathfrak{k}_1 := \mathbb{R}z$ and $\mathfrak{k}_2$ are ideals of $\mathfrak{k}$ and the moment map $\langle \Phi^K_P, z\rangle$ relative to the S^1-action is proper. Then

$$\mathcal{Q}^{-\infty}_K(P) = \mathcal{Q}^{\Phi}_K(P) = \mathcal{Q}^{\langle\Phi,z\rangle}_K(P) \in R^{-\infty}(K) \tag{5.14}$$

where the right hand side is computed via a localization procedure on the set $\mathrm{Cr}(\varphi_P)$ of critical points of the proper map $\varphi_P := \langle \Phi^K_P, z\rangle^2$. We note that

$$\mathrm{Cr}(\varphi_P) = \varphi_P^{-1}(0) \cup P^z.$$

We are interested in the following cases:

1. P is a proper Hamiltonian G-manifold (M, Ω_M, Φ^G_M) with a moment map taking values in $G \cdot \mathcal{C}^{\rho}_{\mathrm{hol}}(z)$, and which satisfies Assumption **A2**.
2. P is the symplectic slice Y of the former case $M := G \times_K Y$.

By Lemma 4.7, the map φ_P is strictly positive in the two cases described above: hence $\varphi_P^{-1}(0) = \emptyset$. Let us compute the generalized character $\mathcal{Q}^{\langle\Phi,z\rangle}_K(P)$ in this case.

Let κ_φ be the Hamiltonian vector field of $\frac{-1}{2}\varphi_P$. The symbol $\mathrm{Thom}(P, J_P) \otimes L_P$ pushed by the vector field κ_φ is denoted $\mathbf{c}^{\varphi}_P$. Let $\mathcal{B}_P$ be the set of connected components of P^z. For any $\mathcal{X} \in \mathcal{B}_P$, we consider a relatively compact open K-invariant neighbourhood $\mathcal{U}_{\mathcal{X}}$ of $\mathcal{X}$ such that $\mathrm{Cr}(\varphi_P) \cap \overline{\mathcal{U}_{\mathcal{X}}} = \mathcal{X}$. We denote by $\mathcal{Q}^{\mathcal{X}}_K(P) \in R^{-\infty}(K)$ the equivariant index of the S^1-transversally elliptic symbol $\mathbf{c}^{\varphi}_P|_{\mathcal{U}_{\mathcal{X}}}$.

When $\varphi_P^{-1}(0) = \emptyset$, the generalized character $\mathcal{Q}^{\langle\Phi,z\rangle}_K(P)$ is defined by

$$\mathcal{Q}^{\langle\Phi,z\rangle}_K(P) = \sum_{\mathcal{X}\in\mathcal{B}_P} \mathcal{Q}^{\mathcal{X}}_K(P) \in R^{-\infty}(K). \tag{5.15}$$

For $\mathcal{X} \in \mathcal{B}_P$, we denote by

- $L_{\mathcal{X}}$ the restriction of the Kostant–Souriau line bundle L_P on $\mathcal{X}$,
- $\mathcal{N}_{\mathcal{X}}$ the normal bundle of $\mathcal{X}$ in P, and $|\mathcal{N}_{\mathcal{X}}|^z, \mathcal{N}^{+,z}_{\mathcal{X}}$ its z-polarized versions (see Section 4.4).

If we use (5.14) and (5.15), the proof of Theorem 4.13 is reduced to

Proposition 5.14. *We have*

$$\mathcal{Q}^{\mathcal{X}}_K(P) = (-1)^{r_{\mathcal{X}}} \mathrm{RR}^K\big(\mathcal{X}, L_{\mathcal{X}} \otimes \det(\mathcal{N}^{+,z}_{\mathcal{X}}) \otimes S^{\bullet}(|\mathcal{N}_{\mathcal{X}}|^z)\big) \quad \textit{in } R^{-\infty}(K), \tag{5.16}$$

where $r_{\mathcal{X}}$ is the complex rank of $\mathcal{N}^{+,z}_{\mathcal{X}}$.

Proof. Relations (2.1) show that $\kappa_\varphi = \langle \Phi^K_P, z\rangle z_P$. Since $\langle \Phi^K_P, z\rangle > 0$ in a neighbourhood of $\mathcal{U}_{\mathcal{X}}$, we can replace κ_φ by the vector field z_P without changing the index of the corresponding transversally elliptic operator. This means that $\mathcal{Q}^{\mathcal{X}}_K(M)$ is equal to the index of $\sigma^z|_{\mathcal{U}_{\mathcal{X}}}$, where the symbol σ^z is defined by setting, for $(m, v) \in \mathbf{T}P$,

$$\sigma^z(m, v) := \mathbf{c}(v - z_P(m)) : \textstyle\bigwedge^{\mathrm{even}}_{\mathbb{C}} \mathbf{T}_m P \otimes L_P|_m \to \bigwedge^{\mathrm{odd}}_{\mathbb{C}} \mathbf{T}_m P \otimes L_P|_m. \tag{5.17}$$

We have proved in [Par01, Theorem 5.8] that the index of $\sigma^z|_{\mathcal{U}_\mathcal{X}}$ is equal to the right hand side of (5.16). □

We now want to clarify the convergence of the sum that appears in (5.15) when P^z is non-compact. Let T be a maximal torus in K; it contains the circle subgroup S^1. Let $\Lambda \subset \mathfrak{t}$ be the lattice which is the kernel of $\exp : \mathfrak{t} \to T$. Let $z_o \in \mathbb{R}^{>0}z \cap \Lambda$ generate the sublattice $\mathbb{R}z \cap \Lambda$; the torus S^1 acts on an irreducible representation V_μ^K through the character $t \mapsto t^n$ with $n = \langle \mu, z_o \rangle/(2\pi) \in \mathbb{Z}$. We then have a gradation $R(K) = \sum_{n\in\mathbb{Z}} R_n(K)$ where $R_n(K)$ is the group generated by the representations V_μ^K such that $\langle \mu, z_o \rangle/(2\pi) = n$. We see that $R_n(K) \cdot R_m(K) \subset R_{n+m}(K)$.

For any $n \in \mathbb{Z}$, we denote by $R_{\geq n}(K)$ (resp. $R^{-\infty}_{\geq n}(K)$) the subgroup formed by the finite (resp. infinite) sums $\sum_{l\geq n} E_l$ where $E_l \in R_l(K)$. We have the following basic lemma (the proof is left to the reader).

Lemma 5.15.
- *If $A \in R^{-\infty}_{\geq n}(K)$ and $B \in R^{-\infty}_{\geq m}(K)$, then the product $A \cdot B$ is well defined and belongs to $R^{-\infty}_{\geq n+m}(K)$.*
- *An infinite sum $\sum_{n\geq 0} A_n$ with $A_n \in R^{-\infty}_{\geq n}(K)$ converges in $R^{-\infty}_{\geq 0}(K)$.*

For $\mathcal{X} \in \mathcal{B}_P$, the action of S^1 is trivial on $\mathcal{X}$, and (2.7) shows that S^1 acts on the fibres of the Kostant–Souriau line bundle $L_\mathcal{X}$ through the character $t \mapsto t^{n(\mathcal{X})}$, where $n(\mathcal{X}) = \langle \Phi_P^K(\mathcal{X}), z_o \rangle/(2\pi)$ is a strictly positive integer.

Proposition 5.16.
- *The generalized character $\mathcal{Q}_K^\mathcal{X}(P)$ belongs to $R^{-\infty}_{\geq n(\mathcal{X})}(K)$.*
- *The sum $\sum_{\mathcal{X}\in\mathcal{B}_P} \mathcal{Q}_K^\mathcal{X}(P)$ converges in $R^{-\infty}_{\geq 0}(K)$.*

Proof. The generalized character $\mathcal{Q}_K^\mathcal{X}(P)$ is equal to $(-1)^{r(\mathcal{X})} \sum_{p\geq 0} E_p$ with $E_p = \mathrm{RR}^K(\mathcal{X}, L_\mathcal{X} \otimes \det(\mathcal{N}_\mathcal{X}^{+,z}) \otimes S^p(|\mathcal{N}_\mathcal{X}|^z)) \in R(K)$. Since S^1 acts on the fibres of the polarized bundles $\mathcal{N}_\mathcal{X}^{+,z}$ and $|\mathcal{N}_\mathcal{X}|^z$ through the characters t^n with $n > 0$, we see that $E_p \in R_{\geq n(\mathcal{X})+p}(K)$. Hence $\mathcal{Q}_K^\mathcal{X}(P) = (-1)^{r(\mathcal{X})} \sum_{p\geq 0} E_p$ converges in $R^{-\infty}_{\geq n(\mathcal{X})}(K)$.

For the second point we see that $\sum_{\mathcal{X}\in\mathcal{B}_P} \mathcal{Q}_K^\mathcal{X}(P) = \sum_{n\geq 0} A_n$ with

$$A_n = \sum_{n(\mathcal{X})=n} \mathcal{Q}_K^\mathcal{X}(P) \in R^{-\infty}_{\geq n}(K).$$

The former sum is finite (and so well defined) because the map $\langle \Phi_P^K, z_o \rangle$ is proper: for any $C > 0$, we have only a finite number of $\mathcal{X} \in \mathcal{B}_P$ such that $\langle \Phi_P^K(\mathcal{X}), z_o \rangle \leq C$. The second point is thus proved. □

5.6. Proof of Theorem 4.10 under Assumption **A2**

If Assumption **A2** is satisfied, the conclusion of Theorem 4.10 follows directly from the results of Section 5.5 applied to the following two cases:

1. (M, Ω_M, Φ_M^G) is a proper Hamiltonian G-manifold with a moment map taking values in $G \cdot \mathcal{C}^\rho_{\mathrm{hol}}(z)$, and which satisfies Assumption **A2**.
2. Y is the symplectic slice of the former case.

Note that the fixed point sets M^z and Y^z coincide. For a connected component $\mathcal{X}$ of Y^z, let $\mathcal{N}_{\mathcal{X}}$ (resp. $\mathcal{N}'_{\mathcal{X}}$) be the normal bundle of $\mathcal{X}$ in M (resp. Y). Since the normal bundle of Y in M is the trivial bundle $Y \times \mathfrak{p}$, we have $\mathcal{N}_{\mathcal{X}} = \mathcal{N}'_{\mathcal{X}} \oplus \mathfrak{p}$. A small computation shows that

$$\mathcal{N}_{\mathcal{X}}^{+,z} = (\mathcal{N}'_{\mathcal{X}})^{+,z} \quad \text{and} \quad |\mathcal{N}_{\mathcal{X}}|^{+,z} = |\mathcal{N}'_{\mathcal{X}}|^{+,z} \oplus (\mathfrak{p}, \mathrm{ad}(z)).$$

Finally, (5.14) and Proposition 5.16 give

$$\begin{aligned}
\mathcal{Q}_K^{-\infty}(M) &= \sum_{\mathcal{X}} (-1)^{r_{\mathcal{X}}} \mathrm{RR}^K\big(\mathcal{X}, L_{\mathcal{X}} \otimes \det(\mathcal{N}_{\mathcal{X}}^{+,z}) \otimes S^\bullet(|\mathcal{N}_{\mathcal{X}}|^z)\big) \\
&= \sum_{\mathcal{X}} (-1)^{r_{\mathcal{X}}} \mathrm{RR}^K\big(\mathcal{X}, L_{\mathcal{X}} \otimes \det(\mathcal{N}'_{\mathcal{X}})^{+,z} \otimes S^\bullet(|\mathcal{N}'_{\mathcal{X}}|^z) \otimes S^\bullet(\mathfrak{p})\big) \\
&= \Big(\sum_{\mathcal{X}} (-1)^{r_{\mathcal{X}}} \mathrm{RR}^K\big(\mathcal{X}, L_{\mathcal{X}} \otimes \det(\mathcal{N}'_{\mathcal{X}})^{+,z} \otimes S^\bullet(|\mathcal{N}'_{\mathcal{X}}|^z)\big)\Big) \otimes S^\bullet(\mathfrak{p}) \\
&= \mathcal{Q}_K^{-\infty}(Y) \otimes S^\bullet(\mathfrak{p}).
\end{aligned}$$

By Proposition 5.16, the term $\sum_{\mathcal{X}} (-1)^{r_{\mathcal{X}}} \mathrm{RR}^K(\mathcal{X}, L_{\mathcal{X}} \otimes \det(\mathcal{N}'_{\mathcal{X}})^{+,z} \otimes S^\bullet(|\mathcal{N}'_{\mathcal{X}}|^z))$ belongs to $R^{-\infty}_{\geq 0}(K)$. We see also that $S^\bullet(\mathfrak{p}) \in R^{-\infty}_{\geq 0}(K)$. Hence their product is well defined (see Lemma 5.15).

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