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## Calculus of variations with differential forms

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**Abstract.** We study integrals of the form  $\int_{\Omega} f(d\omega)$ , where  $1 \leq k \leq n$ ,  $f : \Lambda^k \rightarrow \mathbb{R}$  is continuous and  $\omega$  is a  $(k - 1)$ -form. We introduce the appropriate notions of convexity, namely ext. one convexity, ext. quasiconvexity and ext. polyconvexity. We study their relations, give several examples and counterexamples. We conclude with an application to a minimization problem.

**Keywords.** Calculus of variations, differential forms, quasiconvexity, polyconvexity and ext. one convexity

### 1. Introduction

In this article, we study integrals of the form

$$\int_{\Omega} f(d\omega),$$

where  $1 \leq k \leq n$  are integers,  $f : \Lambda^k \rightarrow \mathbb{R}$  is a continuous function,  $\Omega \subset \mathbb{R}^n$  is open, bounded and  $\omega$  is a  $(k - 1)$ -form. When  $k = 1$ , by abuse of notation identifying  $\Lambda^1$  with  $\mathbb{R}^n$  and the operator  $d$  with the gradient, this is the classical problem of the calculus of variations where one studies integrals of the form

$$\int_{\Omega} f(\nabla\omega).$$

This is a *scalar* problem in the sense that there is only one function  $\omega$ . It is well known that in this last case the *convexity* of  $f$  plays a crucial role. As soon as  $k \geq 2$ , the problem is more of a *vectorial* nature, since then  $\omega$  has several components. However, it has some special features that a general vectorial problem does not have. Before going further, one should have two examples in mind.

1) If  $k = 2$ , with our usual abuse of notation,

$$\omega : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \text{and} \quad d\omega = \text{curl } \omega.$$

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2) If  $k = n$ , by abuse of notation and up to some changes of signs,

$$\omega : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \text{and} \quad d\omega = \operatorname{div} \omega.$$

So let us now discuss some specific features of our problem.

- The first important point is the lack of *coercivity*. Indeed, even if the function  $f$  grows at infinity as the norm to a certain power, this does not imply control on the full gradient but only on some combination of it, namely  $d\omega$ . So when dealing with minimization problems, this fact requires special attention (see Theorem 5.1).

- From the point of view of convexity, the situation is, in some cases, simpler than in the general vectorial problem. Indeed, consider the above two examples (with  $n = 3$  for the first one). Although the problems are vectorial, they behave as if they were scalar (cf. Theorem 2.8).

- One peculiarity (cf. Theorem 3.3) that particularly stands out is how the problem changes its behaviour with a change in the order of the form. When  $k$  is odd, or when  $2k > n$  (in particular  $k = n$ ), there is no nonlinear function which is ext. quasilinear, and therefore the problem behaves as if it were scalar. However, the situation changes significantly when  $k \leq 2n$  is even. It turns out that we have an ample supply of nonlinear functions that are ext. quasilinear in this case. For example, the nonlinear function

$$f(\xi) = \langle c; \xi \wedge \xi \rangle,$$

where  $c \in \Lambda^{2k}$ , is ext. quasilinear. See Theorem 3.3 for a complete characterization of ext. quasilinear functions, which, in turn, determines all weakly continuous functions with respect to the  $d$ -operator (see Bandyopadhyay–Sil [4] for details).

Because of the special nature of our problem, we are led to introduce the following terminology: *ext. one convexity*, *ext. quasiconvexity* and *ext. polyconvexity*, which are the counterparts of the classical notions of the vectorial calculus of variations (see, in particular, Dacorogna [8]), namely rank one convexity, quasiconvexity and polyconvexity. The relations between these notions (cf. Theorem 2.8) as well as their manifestations on the minimization problem are the subject of the present paper. Examples and counterexamples are also discussed in detail, notably the case of ext. quasilinear functions (see Theorem 3.3), the quadratic case (see Theorem 4.5) and a fundamental counterexample (see Theorem 4.8) similar to the famous example of Šverák [20].

Some of what has been done in this article may also be seen through classical vectorial calculus of variations. This connection is elaborated and pursued in detail in a forthcoming article (see Bandyopadhyay–Sil [3]). However, the case of differential forms in the context of calculus of variations deserves a separate and independent treatment because of its special algebraic structure which renders much of the calculation intrinsic, natural and coordinate free.

We conclude this introduction by pointing out that the results discussed in this introduction may be interpreted very broadly in terms of the theory of compensated compactness introduced by Murat and Tartar [14], [21] (see also Dacorogna [7], Robbin–Rogers–Temple [15]). In particular, our notion of ext. one convexity is related to the so called

convexity in the directions of the wave cone  $\Lambda$ . Our definition of ext. quasiconvexity is related to those of A and A-B quasiconvexity introduced by Dacorogna (cf. [6] and [7]; see also Fonseca–Müller [10]).

## 2. Definitions and main properties

### 2.1. Definitions

We start with various notions of convexity and affinity.

**Definition 2.1.** Let  $1 \leq k \leq n$  and  $f : \Lambda^k \rightarrow \mathbb{R}$ .

(i) We say that  $f$  is *ext. one convex* if the function

$$g : t \mapsto g(t) = f(\xi + t\alpha \wedge \beta)$$

is convex for every  $\xi \in \Lambda^k, \alpha \in \Lambda^{k-1}$  and  $\beta \in \Lambda^1$ . If the function  $g$  is affine, we say that  $f$  is *ext. one affine*.

(ii)  $f$  is said to be *ext. quasiconvex* if  $f$  is Borel measurable, locally bounded and

$$\int_{\Omega} f(\xi + d\omega) \geq f(\xi) \text{ meas } \Omega$$

for every bounded open set  $\Omega \subset \mathbb{R}^n, \xi \in \Lambda^k$  and  $\omega \in W_0^{1,\infty}(\Omega; \Lambda^{k-1})$ . If equality holds, we say that  $f$  is *ext. quas affine*.

(iii) We say that  $f$  is *ext. polyconvex* if there exists a convex function

$$F : \Lambda^k \times \Lambda^{2k} \times \dots \times \Lambda^{[n/k]k} \rightarrow \mathbb{R}$$

such that

$$f(\xi) = F(\xi, \xi^2, \dots, \xi^{[n/k]}) \quad \text{for all } \xi \in \Lambda^k.$$

If  $F$  is affine, we say that  $f$  is *ext. polyaffine*.

**Remark 2.2.** (i) The “ext.” stands for “exterior product” in the first and third items, and for “exterior derivative” in the second one.

(ii) When  $k$  is odd (since then  $\xi^s = 0$  for every  $s \geq 2$ ) or when  $2k > n$  (in particular, when  $k = n$  or  $k = n - 1$ ), then ext. polyconvexity is equivalent to ordinary convexity (see Proposition 2.14).

(iii) When  $k = 1$ , all the above notions are equivalent to the classical notion of convexity (cf. Theorem 2.8).

(iv) As in [8, Proposition 5.11], it can easily be shown that if the inequality of ext. quasiconvexity holds for a given bounded open set  $\Omega$ , it holds for any bounded open set.

(v) The definition of ext. quasiconvexity is equivalent (as in [8, Proposition 5.13]) to the following. Let  $D = (0, 1)^n$ . Then

$$\int_D f(\xi + d\omega) \geq f(\xi)$$

for every  $\xi \in \Lambda^k$  and every  $\omega \in W_{\text{per}}^{1,\infty}(D; \Lambda^{k-1})$ , where

$$W_{\text{per}}^{1,\infty}(D; \Lambda^{k-1}) = \{\omega \in W^{1,\infty}(D; \Lambda^{k-1}) : \omega \text{ is 1-periodic in each variable}\}.$$

**Definition 2.3.** Let  $0 \leq k \leq n$  and  $f : \Lambda^k \rightarrow \mathbb{R}$ . The *Hodge transform* of  $f$  is the function  $f_* : \Lambda^{n-k} \rightarrow \mathbb{R}$  defined as

$$f_*(\xi) = f(*\xi) \quad \text{for all } \xi \in \Lambda^{n-k}.$$

The notion of Hodge transform allows us to extend the notions of convexity with respect to the interior product and the  $\delta$ -operator as follows.

**Definition 2.4.** Let  $0 \leq k \leq n - 1$  and  $f : \Lambda^k \rightarrow \mathbb{R}$ . We say that

- (i)  $f$  is *int. one convex* if  $f_*$  is ext. one convex.
- (ii)  $f$  is *int. quasiconvex* if  $f_*$  is ext. quasiconvex.
- (iii)  $f$  is *int. polyconvex* if  $f_*$  is ext. polyconvex.

**Remark 2.5.** (i) Statements similar to those in Remark 2.2 hold for int. convexity as well.  
(ii) It is easy to check that  $f$  is int. one convex if and only if the function

$$g : t \mapsto g(t) = f(\xi + t\beta \lrcorner \alpha)$$

is convex for every  $\xi \in \Lambda^k$ ,  $\beta \in \Lambda^1$  and  $\alpha \in \Lambda^{k+1}$ . Furthermore,  $f$  is int. quasiconvex if and only if  $f$  is Borel measurable, locally bounded and

$$\int_{\Omega} f(\xi + \delta\omega) \geq f(\xi) \text{ meas } \Omega$$

for every bounded open set  $\Omega \subset \mathbb{R}^n$ ,  $\xi \in \Lambda^k$  and  $\omega \in W_0^{1,\infty}(\Omega; \Lambda^{k+1})$ . The case of int. polyconvexity is however a little more involved and we leave out the details.

In what follows, we will discuss the case of ext. convexity only. Int. convexity can be handled analogously.

## 2.2. Preliminary lemmas

In this subsection, we state two lemmas; see [17] for their proofs. We start with the following problem of prescribed differentials. Let us recall that  $\alpha \in \Lambda^k$  is said to be *1-divisible* if  $\alpha = a \wedge b$  for some  $a \in \Lambda^{k-1}$  and  $b \in \Lambda^1$ .

**Lemma 2.6.** Let  $1 \leq k \leq n$  and let  $\omega_1, \omega_2 \in \Lambda^k$ . Then there exists  $\omega \in W^{1,\infty}(\Omega; \Lambda^{k-1})$  satisfying

$$d\omega \in \{\omega_1, \omega_2\} \quad \text{a.e. in } \Omega$$

(and taking both values) if and only if  $\omega_1 - \omega_2$  is 1-divisible.

Using Lemma 2.6, one can deduce the following approximation lemma for  $k$ -forms. See [8, Lemma 3.11] for the case of the gradient.

**Lemma 2.7.** *Let  $1 \leq k \leq n$ ,  $t \in [0, 1]$  and let  $\alpha, \beta \in \Lambda^k$  be such that  $\alpha \neq \beta$  and  $\alpha - \beta$  is 1-divisible. Let  $\Omega \subset \mathbb{R}^n$  be open, bounded and let  $\omega : \overline{\Omega} \rightarrow \Lambda^{k-1}$  satisfy*

$$d\omega = t\alpha + (1 - t)\beta \quad \text{in } \overline{\Omega}.$$

*Then, for every  $\epsilon > 0$ , there exist  $\omega_\epsilon \in \text{Aff}_{\text{piece}}(\overline{\Omega}; \Lambda^{k-1})$  and disjoint open sets  $\Omega_\alpha, \Omega_\beta \subset \Omega$  such that*

1.  $|\text{meas}(\Omega_\alpha) - t \text{meas}(\Omega)| \leq \epsilon$  and  $|\text{meas}(\Omega_\beta) - (1 - t) \text{meas}(\Omega)| \leq \epsilon$ ,
2.  $\omega_\epsilon = \omega$  in a neighbourhood of  $\partial\Omega$ ,
3.  $\|\omega_\epsilon - \omega\|_{L^\infty(\overline{\Omega})} \leq \epsilon$ ,
4.  $d\omega_\epsilon(x) = \begin{cases} \alpha & \text{if } x \in \Omega_\alpha, \\ \beta & \text{if } x \in \Omega_\beta, \end{cases}$
5.  $\text{dist}(d\omega_\epsilon(x); \{t\alpha + (1 - t)\beta : t \in [0, 1]\}) \leq \epsilon$  for a.e.  $x \in \Omega$ .

2.3. Main properties

The different notions of convexity are related as follows.

**Theorem 2.8.** *Let  $1 \leq k \leq n$  and  $f : \Lambda^k \rightarrow \mathbb{R}$ .*

(i) *The following implications hold:*

$$f \text{ convex} \Rightarrow f \text{ ext. polyconvex} \Rightarrow f \text{ ext. quasiconvex} \Rightarrow f \text{ ext. one convex.}$$

(ii) *If  $k = 1, n - 1, n$  or  $k = n - 2$  is odd, then*

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

*Moreover, if  $k$  is odd or  $2k > n$ , then*

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex.}$$

(iii) *If either  $2 \leq k \leq n - 3$  or  $k = n - 2 \geq 2$  is even, then*

$$f \text{ ext. polyconvex} \begin{matrix} \Rightarrow \\ \not\Leftrightarrow \end{matrix} f \text{ ext. quasiconvex,}$$

*while if  $2 \leq k \leq n - 3$  (and thus  $n \geq k + 3 \geq 5$ ), then*

$$f \text{ ext. quasiconvex} \begin{matrix} \Rightarrow \\ \not\Leftrightarrow \end{matrix} f \text{ ext. one convex.}$$

**Remark 2.9.** (i) The last equivalence in (ii) is false for  $k$  even and  $n \geq 2k$ , as the following simple example shows. Let  $f : \Lambda^2(\mathbb{R}^4) \rightarrow \mathbb{R}$  be defined by

$$f(\xi) = \langle e^1 \wedge e^2 \wedge e^3 \wedge e^4; \xi \wedge \xi \rangle.$$

Then  $f$  is clearly ext. polyconvex but not convex.

(ii) The study of the implications and counter implications for ext. one convexity, ext. quasiconvexity and ext. polyconvexity is therefore complete, except for the last implication, namely

$$f \text{ ext. quasiconvex} \begin{matrix} \Rightarrow \\ \not\Leftarrow \end{matrix} f \text{ ext. one convex},$$

only for the case  $k = n - 2 \geq 2$  even (including  $k = 2$  and  $n = 4$ ), which remains open.

(iii) It is interesting to read the theorem when  $k = 2$ :

- If  $n = 2$  or  $n = 3$ , then

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex}.$$

- If  $n \geq 4$ , then

$$f \text{ convex} \begin{matrix} \Rightarrow \\ \not\Leftarrow \end{matrix} f \text{ ext. polyconvex} \begin{matrix} \Rightarrow \\ \not\Leftarrow \end{matrix} f \text{ ext. quasiconvex}.$$

- If  $n \geq 5$ , then

$$f \text{ ext. quasiconvex} \begin{matrix} \Rightarrow \\ \not\Leftarrow \end{matrix} f \text{ ext. one convex},$$

while the case  $n = 4$  remains open.

*Proof of Theorem 2.8.* (i) *Step 1.* Obviously,  $f \text{ convex} \Rightarrow f \text{ ext. polyconvex}$ .

*Step 2.* The statement  $f \text{ ext. polyconvex} \Rightarrow f \text{ ext. quasiconvex}$  is proved as follows. We first show that if  $\xi \in \Lambda^k$  and  $\omega \in W_0^{1,\infty}(\Omega; \Lambda^{k-1})$ , then

$$\int_{\Omega} (\xi + d\omega)^s = \xi^s \text{ meas } \Omega \quad \text{for every integer } s. \tag{1}$$

We proceed by induction on  $s$ . The case  $s = 1$  is trivial, so we assume that the result has already been established for  $s - 1$ . Note that

$$\begin{aligned} (\xi + d\omega)^s &= \xi \wedge (\xi + d\omega)^{s-1} + d\omega \wedge (\xi + d\omega)^{s-1} \\ &= \xi \wedge (\xi + d\omega)^{s-1} + d[\omega \wedge (\xi + d\omega)^{s-1}]. \end{aligned}$$

Integrating, using the inductive assumption for the first integral and the fact that  $\omega = 0$  on  $\partial\Omega$  for the second one, we indeed obtain (1).

We can now conclude. Since  $f$  is ext. polyconvex, we can find a convex function  $F : \Lambda^k \times \Lambda^{2k} \times \dots \times \Lambda^{[n/k]k} \rightarrow \mathbb{R}$  such that

$$f(\xi) = F(\xi, \xi^2, \dots, \xi^{[n/k]}).$$

Using the Jensen inequality we find

$$\frac{1}{\text{meas } \Omega} \int_{\Omega} f(\xi + d\omega) \geq F\left(\frac{1}{\text{meas } \Omega} \int_{\Omega} (\xi + d\omega), \dots, \frac{1}{\text{meas } \Omega} \int_{\Omega} (\xi + d\omega)^{[n/k]}\right).$$

Invoking (1), we obtain

$$\int_{\Omega} f(\xi + d\omega) \geq f(\xi) \text{ meas } \Omega,$$

and the proof of Step 2 is complete.

*Step 3.* It follows from Lemma 2.7 that  $f$  ext. quasiconvex  $\Rightarrow$   $f$  ext. one convex. With Lemma 2.7 at our disposal, the proof is very similar to that of the case of the gradient (cf. [8, Theorem 5.3]) and is omitted. See [17] for the details. This concludes the proof of (i).

(ii) In all the cases under consideration any  $\xi \in \Lambda^k$  is 1-divisible (cf. [5, Proposition 2.43]). Hence,

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

The extra statement  $f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex}$  (i.e. when  $k$  is odd or  $2k > n$ ) is proved in Remark 2.2(ii) and Proposition 2.14.

(iii) The statement that

$$f \text{ ext. polyconvex} \begin{matrix} \Rightarrow \\ \not\Leftrightarrow \end{matrix} f \text{ ext. quasiconvex}$$

when  $3 \leq k \leq n - 3$  or  $k = n - 2 \geq 4$  is even follows from Theorem 4.5(v) and from Proposition 4.11 when  $k = 2$  and  $n \geq 4$  (for  $k = 2$  and  $n \geq 6$ , we can also apply Theorem 4.5(ii)).

The statement that if  $2 \leq k \leq n - 3$  (and thus  $n \geq k + 3 \geq 5$ ), then

$$f \text{ ext. quasiconvex} \begin{matrix} \Rightarrow \\ \not\Leftrightarrow \end{matrix} f \text{ ext. one convex,}$$

follows from Theorem 4.8. □

We also have the following elementary properties.

**Proposition 2.10.** *Let  $1 \leq k \leq n$  and  $f : \Lambda^k \rightarrow \mathbb{R}$ .*

- (i) *Any ext. one convex function is locally Lipschitz.*
- (ii) *If  $f$  is ext. one convex and  $C^2$ , then for every  $\xi \in \Lambda^k$ ,  $\alpha \in \Lambda^{k-1}$  and  $\beta \in \Lambda^1$ ,*

$$\sum_{I, J \in \mathcal{T}_k^n} \frac{\partial^2 f(\xi)}{\partial \xi_I \partial \xi_J} (\alpha \wedge \beta)_I (\alpha \wedge \beta)_J \geq 0.$$

*Proof.* (i) The fact that  $f$  is locally Lipschitz follows from the observation that any ext. one convex function is in fact separately convex. These last functions are known to be locally Lipschitz (cf. [8, Theorem 2.31]).

(ii) We next assume that  $f$  is  $C^2$ . By definition the function

$$g : t \mapsto g(t) = f(\xi + t\alpha \wedge \beta)$$

is convex for every  $\xi \in \Lambda^k$ ,  $\alpha \in \Lambda^{k-1}$  and  $\beta \in \Lambda^1$ . Since  $f$  is  $C^2$ , our claim follows from the fact that  $g''(0) \geq 0$ . □

We now give an equivalent formulation of ext. quasiconvexity, but first we need the following notation.

**Notation 2.11.** Let  $\Omega \subset \mathbb{R}^n$  be a smooth open set. We define

$$W_{\delta,T}^{1,\infty}(\Omega; \Lambda^k) = \{\omega \in W^{1,\infty}(\Omega; \Lambda^k) : \delta\omega = 0 \text{ in } \Omega \text{ and } \nu \wedge \omega = 0 \text{ on } \partial\Omega\},$$

where  $\nu$  is the outward unit normal to  $\partial\Omega$ .  $C_{\delta,T}^\infty(\overline{\Omega}; \Lambda^k)$  is defined analogously.

**Proposition 2.12.** Let  $f : \Lambda^k \rightarrow \mathbb{R}$  be continuous. The following statements are equivalent:

- (i)  $f$  is ext. quasiconvex.
- (ii) For every bounded smooth open set  $\Omega \subset \mathbb{R}^n$ ,  $\psi \in W_{\delta,T}^{1,\infty}(\Omega; \Lambda^{k-1})$  and  $\xi \in \Lambda^k$ ,

$$\int_{\Omega} f(\xi + d\psi) \geq f(\xi) \text{ meas } \Omega.$$

**Remark 2.13.** Given a function  $f : \Lambda^k \rightarrow \mathbb{R}$ , its ext. quasiconvex envelope, which is the largest ext. quasiconvex function below  $f$ , is given by (as in [8, Theorem 6.9])

$$\begin{aligned} Q_{\text{ext}}f(\xi) &= \inf \left\{ \frac{1}{\text{meas } \Omega} \int_{\Omega} f(\xi + d\omega) : \omega \in W_0^{1,\infty}(\Omega; \Lambda^{k-1}) \right\} \\ &= \inf \left\{ \frac{1}{\text{meas } \Omega} \int_{\Omega} f(\xi + d\psi) : \psi \in W_{\delta,T}^{1,\infty}(\Omega; \Lambda^{k-1}) \right\}. \end{aligned}$$

*Proof of Proposition 2.12.* (ii)  $\Rightarrow$  (i): Let  $\Omega \subset \mathbb{R}^n$  be a bounded smooth open set,  $\xi \in \Lambda^k$  and  $\omega \in W_0^{1,\infty}(\Omega; \Lambda^{k-1})$ . Using density, we find  $\omega_\epsilon \in C_0^\infty(\Omega; \Lambda^{k-1})$  such that

$$\sup_{\epsilon > 0} \|\nabla \omega_\epsilon\|_{L^\infty} < \infty \quad \text{and} \quad \omega_\epsilon \rightarrow \omega \text{ in } W^{1,2}(\Omega; \Lambda^{k-1}). \tag{2}$$

Appealing to [5, Theorem 7.2], we now find  $\psi_\epsilon \in C_{\delta,T}^\infty(\overline{\Omega}; \Lambda^{k-1})$  such that

$$\begin{cases} d\psi_\epsilon = d\omega_\epsilon & \text{in } \Omega, \\ \delta\psi_\epsilon = 0 & \text{in } \Omega, \\ \nu \wedge \psi_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

We use (2) to apply the dominated convergence theorem to obtain

$$\int_{\Omega} f(\xi + d\omega) = \lim_{\epsilon \rightarrow 0} \int_{\Omega} f(\xi + d\omega_\epsilon) = \lim_{\epsilon \rightarrow 0} \int_{\Omega} f(\xi + d\psi_\epsilon) \geq f(\xi) \text{ meas } \Omega,$$

where we have used (ii) in the last step. Therefore,  $f$  is ext. quasiconvex.

(i)  $\Rightarrow$  (ii): Let  $\psi \in W_{\delta,T}^{1,\infty}(\Omega; \Lambda^{k-1})$ . Then, by [5, Theorem 8.16], we can find  $\omega$  in  $W_0^{1,2}(\Omega; \Lambda^{k-1})$  such that

$$\begin{cases} d\omega = d\psi & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega. \end{cases}$$

With a similar argument to the one above, we infer that

$$\int_{\Omega} f(\xi + d\psi) = \int_{\Omega} f(\xi + d\omega) = \lim_{\epsilon \rightarrow 0} \int_{\Omega} f(\xi + d\omega_\epsilon) \geq f(\xi) \text{ meas } \Omega. \quad \square$$

We finally also have another formulation of ext. polyconvexity.



**Proposition 2.14.** *Let  $f : \Lambda^k \rightarrow \mathbb{R}$ . The following statements are then equivalent:*

- (i)  $f$  is ext. polyconvex.
- (ii) For every  $\xi \in \Lambda^k$ , there exist  $c_s = c_s(\xi) \in \Lambda^{ks}$ ,  $1 \leq s \leq [n/k]$ , such that

$$f(\eta) \geq f(\xi) + \sum_{s=1}^{[n/k]} \langle c_s(\xi); \eta^s - \xi^s \rangle \quad \text{for every } \eta \in \Lambda^k.$$

(iii) Let

$$N = \sum_{s=1}^{[n/k]} \binom{n}{sk}.$$

For all  $t_i \geq 0$  with  $\sum_{i=1}^{N+1} t_i = 1$  and all  $\xi_i \in \Lambda^k$  such that

$$\sum_{i=1}^{N+1} t_i \xi_i^s = \left( \sum_{i=1}^{N+1} t_i \xi_i \right)^s \quad \text{for every } 1 \leq s \leq [n/k],$$

we have

$$f\left(\sum_{i=1}^{N+1} t_i \xi_i\right) \leq \sum_{i=1}^{N+1} t_i f(\xi_i).$$

*Proof.* (i) $\Rightarrow$ (ii): Since  $f$  is ext. polyconvex, there exists a convex function  $F$  such that

$$f(\xi) = F(\xi, \xi^2, \dots, \xi^{[n/k]}).$$

$F$  being convex, for every  $\xi \in \Lambda^k$  there exist  $c_s = c_s(\xi) \in \Lambda^{ks}$ ,  $1 \leq s \leq [n/k]$ , such that, for all  $\eta \in \Lambda^k$ ,

$$f(\eta) - f(\xi) = F(\eta, \dots, \eta^{[n/k]}) - F(\xi, \dots, \xi^{[n/k]}) \geq \sum_{s=1}^{[n/k]} \langle c_s; \eta^s - \xi^s \rangle.$$

(ii) $\Rightarrow$ (i): Conversely, assume that the inequality is valid and, for  $\theta = (\theta_1, \dots, \theta_{[n/k]}) \in \Lambda^k \times \dots \times \Lambda^{[n/k]k}$ , we define

$$F(\theta) = \sup_{\xi \in \Lambda^k} \left\{ f(\xi) + \sum_{s=1}^{[n/k]} \langle c_s(\xi); \theta_s - \xi^s \rangle \right\}.$$

Clearly  $F$  is convex as a supremum of affine functions. Moreover if  $\theta = (\eta, \dots, \eta^{[n/k]})$ , then, in view of the inequality, the supremum is attained by  $\xi = \eta$ , i.e.  $f(\eta) = F(\eta, \dots, \eta^{[n/k]})$ , and thus  $f$  is ext. polyconvex.

(i) $\Rightarrow$ (iii): Since  $f$  is ext. polyconvex, there exists a convex function  $F$  such that

$$f(\xi) = F(\xi, \xi^2, \dots, \xi^{[n/k]}).$$

The convexity of  $F$  implies that

$$\begin{aligned} f\left(\sum_{i=1}^{N+1} t_i \xi_i\right) &= F\left(\left(\sum_{i=1}^{N+1} t_i \xi_i\right), \dots, \left(\sum_{i=1}^{N+1} t_i \xi_i\right)^{[n/k]}\right) \\ &= F\left(\sum_{i=1}^{N+1} t_i (\xi_i, \dots, \xi_i^{[n/k]})\right) \leq \sum_{i=1}^{N+1} t_i F(\xi_i, \dots, \xi_i^{[n/k]}) = \sum_{i=1}^{N+1} t_i f(\xi_i). \end{aligned}$$

(iii) $\Rightarrow$ (i): The proof is based on Carathéodory's theorem and runs exactly as in [8, Theorem 5.6].  $\square$

### 3. The quasilinear case

#### 3.1. Some preliminary results

We start with two elementary results.

**Lemma 3.1.** *Let  $f : \Lambda^k \rightarrow \mathbb{R}$  be ext. one affine with  $1 \leq k \leq n$ . Then*

$$f\left(\xi + \sum_{i=1}^N t_i \alpha_i \wedge a\right) = f(\xi) + \sum_{i=1}^N t_i [f(\xi + \alpha_i \wedge a) - f(\xi)]$$

for every  $t_i \in \mathbb{R}$ ,  $\xi \in \Lambda^k$ ,  $\alpha_i \in \Lambda^{k-1}$ ,  $a \in \Lambda^1$ .

*Proof.* *Step 1.* Since  $f$  is ext. one affine,

$$f(\xi + t\alpha \wedge a) = f(\xi) + t[f(\xi + \alpha \wedge a) - f(\xi)].$$

*Step 2.* Let us first prove that

$$f(\xi + \alpha \wedge a + \beta \wedge a) + f(\xi) = f(\xi + \alpha \wedge a) + f(\xi + \beta \wedge a).$$

First assume that  $s \neq 0$ . Using Step 1, we have

$$\begin{aligned} f(\xi + s\alpha \wedge a + \beta \wedge a) &= f\left(\xi + s\left(\alpha + \frac{1}{s}\beta\right) \wedge a\right) \\ &= f(\xi) + s\left[f\left(\xi + \left(\alpha + \frac{1}{s}\beta\right) \wedge a\right) - f(\xi)\right], \end{aligned}$$

and hence, using Step 1 again,

$$\begin{aligned} f(\xi + s\alpha \wedge a + \beta \wedge a) &= f(\xi) + s\left\{f(\xi + \alpha \wedge a) + \frac{1}{s}[f(\xi + \alpha \wedge a + \beta \wedge a) - f(\xi + \alpha \wedge a)] - f(\xi)\right\} \\ &= f(\xi) + s[f(\xi + \alpha \wedge a) - f(\xi)] + [f(\xi + \alpha \wedge a + \beta \wedge a) - f(\xi + \alpha \wedge a)]. \end{aligned}$$

Since  $f$  is continuous, we have the result by letting  $s \rightarrow 0$ .

*Step 3.* We now prove the claim. We proceed by induction. The case  $N = 1$  is just Step 1. We first use the induction hypothesis to write

$$\begin{aligned} f\left(\xi + \sum_{i=1}^N t_i \alpha_i \wedge a\right) &= f\left(\xi + t_N \alpha_N \wedge a + \sum_{i=1}^{N-1} t_i \alpha_i \wedge a\right) \\ &= f(\xi + t_N \alpha_N \wedge a) + \sum_{i=1}^{N-1} t_i [f(\xi + t_N \alpha_N \wedge a + \alpha_i \wedge a) - f(\xi + t_N \alpha_N \wedge a)]. \end{aligned}$$

We then appeal to Step 1 to get

$$\begin{aligned} f\left(\xi + \sum_{i=1}^N t_i \alpha_i \wedge a\right) &= f(\xi) + t_N [f(\xi + \alpha_N \wedge a) - f(\xi)] \\ &\quad + \sum_{i=1}^{N-1} t_i \left\{ f(\xi + \alpha_i \wedge a) + t_N [f(\xi + \alpha_i \wedge a + \alpha_N \wedge a) - f(\xi + \alpha_i \wedge a)] \right. \\ &\quad \left. - f(\xi) - t_N [f(\xi + \alpha_N \wedge a) - f(\xi)] \right\}, \end{aligned}$$

and thus

$$\begin{aligned} f\left(\xi + \sum_{i=1}^N t_i \alpha_i \wedge a\right) &= f(\xi) + \sum_{i=1}^N t_i [f(\xi + \alpha_i \wedge a) - f(\xi)] \\ &\quad + t_N \sum_{i=1}^{N-1} t_i \{ f(\xi + \alpha_i \wedge a + \alpha_N \wedge a) - f(\xi + \alpha_i \wedge a) - f(\xi + \alpha_N \wedge a) + f(\xi) \}. \end{aligned}$$

By Step 2, the last term vanishes, completing the induction reasoning. □

The following result is an immediate consequence.

**Corollary 3.2.** *Let  $f : \Lambda^k \rightarrow \mathbb{R}$  be ext. one affine with  $1 \leq k \leq n$ . Then*

$$\begin{aligned} [f(\xi + \alpha \wedge a + \beta \wedge b) - f(\xi)] &+ [f(\xi + \beta \wedge a + \alpha \wedge b) - f(\xi)] \\ &= [f(\xi + \alpha \wedge a) - f(\xi)] + [f(\xi + \beta \wedge a) - f(\xi)] \\ &\quad + [f(\xi + \alpha \wedge b) - f(\xi)] + [f(\xi + \beta \wedge b) - f(\xi)] \end{aligned}$$

for every  $\xi \in \Lambda^k$ ,  $\alpha, \beta \in \Lambda^{k-1}$ ,  $a, b \in \Lambda^1$ .

### 3.2. The main theorem

**Theorem 3.3.** *Let  $1 \leq k \leq n$  and  $f : \Lambda^k \rightarrow \mathbb{R}$ . The following statements are equivalent:*

- (i)  *$f$  is ext. polyaffine.*
- (ii)  *$f$  is ext. quasilinear.*
- (iii)  *$f$  is ext. one affine.*
- (iv) *For every  $0 \leq s \leq [n/k]$ , there exists  $c_s \in \Lambda^{ks}$  such that*

$$f(\xi) = \sum_{s=0}^{[n/k]} \langle c_s; \xi^s \rangle \quad \text{for every } \xi \in \Lambda^k.$$

**Remark 3.4.** When  $k$  is odd (since then  $\xi^s = 0$  for every  $s \geq 2$ ) or when  $2k > n$  (in particular when  $k = n$  or  $k = n - 1$ ), all the statements are equivalent to  $f$  being affine.

*Proof of Theorem 3.3.* The statements (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follow at once from Theorem 2.8. The statement (iv) $\Rightarrow$ (i) is a direct consequence of the definition of ext. polyconvexity. So it only remains to prove (iii) $\Rightarrow$ (iv). We divide the proof into three steps.

*Step 1.* We first prove that  $f$  is a polynomial of degree at most  $n$  and is of the form

$$f(\xi) = \sum_{s=0}^n f_s(\xi), \tag{3}$$

where, for each  $s = 0, \dots, n$ ,  $f_s$  is a homogeneous polynomial of degree  $s$  and is ext. one affine. To prove (3), let us proceed by induction on  $n$ . The case  $n = 1$  is trivial. For each  $\xi \in \Lambda^k$ , we write

$$\xi = \sum_{I \in \mathcal{T}_k^n, 1 \in I} \xi_I e^I + \xi_N, \quad \text{where} \quad \xi_N = \sum_{I \in \mathcal{T}_k^n, 1 \notin I} \xi_I e^I.$$

Note that  $\xi_N \in \Lambda^k(\{e^1\}^\perp)$ . Invoking Lemma 3.1, we obtain

$$f(\xi) = f(\xi_N) + \sum_{I \in \mathcal{T}_k^n, 1 \in I} \xi_I [f(\xi_N + e^I) - f(\xi_N)].$$

Since  $f$  is ext. one affine on  $\Lambda^k(\{e^1\}^\perp)$ , applying the induction hypothesis to  $f|_{\Lambda^k(\{e^1\}^\perp)}$  and  $f(e^I + \cdot)|_{\Lambda^k(\{e^1\}^\perp)}$ , we deduce that both are polynomials of degree at most  $n - 1$ . Hence,  $f$  is a polynomial of degree at most  $n$ . This proves the claim by induction. That each of  $f_s$  is ext. one affine follows from the fact that each  $f_s$  has a different degree of homogeneity.

*Step 2.* We now show that  $f$  is, in fact, a polynomial of degree at most  $\lfloor n/k \rfloor$ , which is equivalent to proving that each  $f_s$  in (3) is a polynomial of degree at most  $\lfloor n/k \rfloor$ . Since  $f_s$  is a homogeneous polynomial of degree  $s$ , we can write

$$f_s(\xi) = \sum_{I^1, \dots, I^s \in \mathcal{T}_k^n} d_{I^1 \dots I^s} \xi_{I^1} \cdots \xi_{I^s}, \tag{4}$$

where  $d_{I^1 \dots I^s} \in \mathbb{R}$ . It is enough to prove that, for some  $I^1, \dots, I^s \in \mathcal{T}_k^n$ , whenever  $d_{I^1 \dots I^s} \neq 0$  we have

$$I^p \cap I^q = \emptyset \quad \text{for all } p, q = 1, \dots, s, p \neq q.$$

Suppose to the contrary that  $I^p \cap I^q \neq \emptyset$  for some  $p \neq q$ . For  $t \in \mathbb{R}$ , define

$$\xi(t) = t(e^{I^p} + e^{I^q}) + \sum_{\substack{a=1 \\ a \neq p, q}}^s e^{I^a}.$$

According to (4),

$$f_s(\xi(t)) = t^2 d_{I^1 \dots I^s} \quad \text{for all } t \in \mathbb{R}.$$

On the other hand, using Lemma 3.1, it follows that

$$f_s(\xi(t)) = f_s\left(\sum_{\substack{a=1 \\ a \neq p,q}}^s e^{I^a} + t(e^{I^p} + e^{I^q})\right) = f_s(\xi(0)) + t[f_s(\xi(1)) - f_s(\xi(0))],$$

which is an affine function of  $t$ . This proves the claim by contradiction.

*Step 3.* Henceforth, to avoid any ambiguity, let us choose the order in which the multi-indices  $I^1, \dots, I^s$  appear in (4) so that

$$i_1^1 < \dots < i_1^s,$$

where  $i_1^j$  is the first element of  $I^j$ , for all  $j = 1, \dots, s$ . With this, we rearrange (4) to have

$$f(\xi) = \sum_{s=0}^{[n/k]} f_s(\xi), \quad \text{where} \quad f_s(\xi) = \sum_{I^1, \dots, I^s} c_{I^1 \dots I^s} \xi_{I^1} \dots \xi_{I^s}, \quad (5)$$

with  $c_{I^1 \dots I^s} \in \mathbb{R} \setminus \{0\}$ , and the ordered multiindices  $I^1, \dots, I^s \in \mathcal{T}_k^n$  with  $i_1^1 < \dots < i_1^s$ . Note that the theorem is proved once we show that  $f_s(\xi) = \langle c_s; \xi^s \rangle$ , which is equivalent to proving that

$$c_{J^1 \dots J^s} = \text{sgn}(\sigma) c_{I^1 \dots I^s} \quad (6)$$

for all  $I^1, \dots, I^s, J^1, \dots, J^s \in \mathcal{T}_k^n$  satisfying  $J^1 \cup \dots \cup J^s = I^1 \cup \dots \cup I^s$  and  $\sigma(J^1 \dots J^s) = (I^1 \dots I^s)$ , where  $\sigma \in \mathcal{S}^{sk}$  is a permutation of indices that respects the aforementioned order.

*Step 3.1.* Observe that, for all  $s = 2, \dots, n$ ,

$$f_s\left(\sum_{i=1}^{s-1} t_i \alpha_i\right) = 0,$$

where  $t_i \in \mathbb{R}$  and  $\alpha_i$  is an element of the standard basis of  $\Lambda^k$ . This is a direct consequence of the fact that  $f_s$  is homogeneous of degree  $s$  and that  $\xi = \sum_{i=1}^{s-1} t_i \alpha_i$  has at most  $s - 1$  nonzero coefficients.

*Step 3.2.* We finally establish (6). Let  $I^1, \dots, I^s, J^1, \dots, J^s \in \mathcal{T}_k^n$  satisfy

$$J^1 \cup \dots \cup J^s = I^1 \cup \dots \cup I^s \quad \text{and} \quad \sigma(J^1 \dots J^s) = (I^1 \dots I^s),$$

where  $\sigma \in \mathcal{S}^{sk}$  is a permutation that respects the order. Since any permutation that respects the order is a product of permutations each of which effects an exchange of a single index between two multiindices (i.e. each of the two multiindices interchanges one of its indices with one index from the other one) while respecting the order, it is enough to prove (6) for such a permutation  $\sigma$ . Then (6) reads

$$c_{J^1 \dots J^s} = -c_{J^1 \dots J^s}. \quad (7)$$

Let us write

$$I^1 = (i_1^1, \dots, i_k^1), \dots, I^s = (i_1^s, \dots, i_k^s),$$

$$J^1 = (j_1^1, \dots, j_k^1), \dots, J^s = (j_1^s, \dots, j_k^s).$$

Then  $i_1^1 < \dots < i_1^s$  and  $j_1^1 < \dots < j_1^s$ . Since  $\sigma$  respects the order, it flips two indices  $i_{r_1}^{q_1}$  and  $i_{r_2}^{q_2}$  with  $q_1 \neq q_2$  and leaves the others fixed leaves the others fixed up to reordering within the multiindices. Note that, from (5),

$$c_{I^1 \dots I^s} = f_s \left( \sum_{m=1}^s e^{I^m} \right) = f_s \left( \sum_{m=1}^s e^{i_1^m} \wedge \dots \wedge e^{i_k^m} \right), \tag{8}$$

$$c_{J^1 \dots J^s} = f_s \left( \sum_{m=1}^s e^{J^m} \right) = f_s \left( \sum_{m=1}^s e^{j_1^m} \wedge \dots \wedge e^{j_k^m} \right). \tag{9}$$

Since  $f_s$  is ext. one affine, setting

$$a = e^{i_{r_1}^{q_1}}, \quad b = e^{i_{r_2}^{q_2}}, \quad \xi = \sum_{\substack{m=1 \\ m \neq q_1, q_2}}^s e^{i_1^m} \wedge \dots \wedge e^{i_k^m} = \sum_{\substack{m=1 \\ m \neq q_1, q_2}}^s e^{I^m},$$

$$\alpha = \pm e^{i_1^{q_1}} \wedge \dots \wedge \widehat{e^{i_{r_1}^{q_1}}} \wedge \dots \wedge e^{i_k^{q_1}} \quad \text{and} \quad \beta = \pm e^{i_1^{q_2}} \wedge \dots \wedge \widehat{e^{i_{r_2}^{q_2}}} \wedge \dots \wedge e^{i_k^{q_2}},$$

with signs chosen appropriately so that

$$\alpha \wedge a = e^{I^{q_1}} = e^{i_1^{q_1}} \wedge \dots \wedge e^{i_k^{q_1}} \quad \text{and} \quad \beta \wedge b = e^{I^{q_2}} = e^{i_1^{q_2}} \wedge \dots \wedge e^{i_k^{q_2}},$$

we can apply Corollary 3.2 to  $f_s$  to obtain

$$\begin{aligned} & [f_s(\xi + \alpha \wedge a + \beta \wedge b) - f_s(\xi)] + [f_s(\xi + \beta \wedge a + \alpha \wedge b) - f_s(\xi)] \\ &= [f_s(\xi + \alpha \wedge a) - f_s(\xi)] + [f_s(\xi + \beta \wedge b) - f_s(\xi)] \\ &+ [f_s(\xi + \beta \wedge a) - f_s(\xi)] + [f_s(\xi + \alpha \wedge b) - f_s(\xi)]. \end{aligned}$$

By Step 3.1, all terms except  $f_s(\xi + \alpha \wedge a + \beta \wedge b)$  and  $f_s(\xi + \beta \wedge a + \alpha \wedge b)$  are 0. Hence,

$$f_s(\xi + \alpha \wedge a + \beta \wedge b) = -f_s(\xi + \beta \wedge a + \alpha \wedge b),$$

which together with (8) and (9) proves (7). This concludes the proof of Step 3.2 and thus of the theorem.  $\square$

### 4. Some examples

#### 4.1. The quadratic case

4.1.1. *Some preliminary results.* Before stating the main theorem on quadratic forms, we need a lemma whose proof is straightforward.

**Lemma 4.1.** *Let  $1 \leq k \leq n$ , let  $M : \Lambda^k \rightarrow \Lambda^k$  be a symmetric linear operator and define  $f : \Lambda^k \rightarrow \mathbb{R}$  by  $f(\xi) = \langle M\xi; \xi \rangle$  for every  $\xi \in \Lambda^k$ . Then:*

- (i)  *$f$  is ext. polyconvex if and only if there exists  $\beta \in \Lambda^{2k}$  such that  $f(\xi) \geq \langle \beta; \xi \wedge \xi \rangle$  for every  $\xi \in \Lambda^k$ .*
- (ii)  *$f$  is ext. quasiconvex if and only if  $\int_{\Omega} f(d\omega) \geq 0$  for every bounded open set  $\Omega \subset \mathbb{R}^n$  and  $\omega \in W_0^{1,\infty}(\Omega; \Lambda^{k-1})$ .*
- (iii)  *$f$  is ext. one convex if and only if  $f(a \wedge b) \geq 0$  for every  $a \in \Lambda^{k-1}$  and  $b \in \Lambda^1$ .*

4.1.2. *Some examples.* We start with the following example that will be used in Theorem 4.5 below.

**Proposition 4.2.** *Let  $2 \leq k \leq n - 2$ . Suppose  $\alpha \in \Lambda^k$  is not 1-divisible. Then there exists  $c > 0$  such that the function*

$$f(\xi) = |\xi|^2 - c\langle \alpha; \xi \rangle^2$$

*is ext. quasiconvex but not convex. If in addition  $\alpha \wedge \alpha = 0$ , then for an appropriate  $c$ , the above  $f$  is ext. quasiconvex but not ext. polyconvex.*

**Remark 4.3.** (i) It is easy to see that  $\alpha$  is not 1-divisible if and only if

$$\text{rank}[*\alpha] = n.$$

This results from [5, Remark 2.44(iv) (with the help of Proposition 2.33)]. Such an  $\alpha$  always exists if either of the following holds (see [5, Propositions 2.37(ii) and 2.43]):

- $k = n - 2 \geq 2$  is even,
- $2 \leq k \leq n - 3$ .

For example,

$$\alpha = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6 \in \Lambda^3(\mathbb{R}^6)$$

is not 1-divisible.

(ii) Note that when  $k = 2$ , every form  $\alpha$  such that  $\alpha \wedge \alpha = 0$  is necessarily 1-divisible, while when  $k$  is even and  $4 \leq k \leq n - 2$ , there exists  $\alpha$  that is not 1-divisible, but  $\alpha \wedge \alpha = 0$ : for example, when  $k = 4$ , take

$$\alpha = e^1 \wedge e^2 \wedge e^3 \wedge e^4 + e^1 \wedge e^2 \wedge e^5 \wedge e^6 + e^3 \wedge e^4 \wedge e^5 \wedge e^6 \in \Lambda^4(\mathbb{R}^6).$$

*Proof of Proposition 4.2.* Since the function is quadratic, its ext. one convexity and ext. quasiconvexity are equivalent (see Theorem 4.5(i) below). We therefore only need to discuss the ext. one convexity. We divide the proof into two steps.

*Step 1.* We first show that if

$$1/c = \sup\{\langle \alpha; a \wedge b \rangle^2 : a \in \Lambda^{k-1}, b \in \Lambda^1, |a \wedge b| = 1\},$$

then

$$1/c < |\alpha|^2.$$

We prove this as follows. Let  $a_s \in \Lambda^{k-1}$ ,  $b_s \in \Lambda^1$  be a maximizing sequence. Up to a subsequence that we do not relabel, there exists  $\lambda \in \Lambda^k$  such that

$$a_s \wedge b_s \rightarrow \lambda \quad \text{with} \quad |\lambda| = 1.$$

Similarly, up to a subsequence that we do not relabel, there exists  $\bar{b} \in \Lambda^1$  such that

$$b_s/|b_s| \rightarrow \bar{b}.$$

Since

$$a_s \wedge b_s \wedge b_s/|b_s| = 0,$$

we deduce that

$$\lambda \wedge \bar{b} = 0.$$

The Cartan lemma (see [5, Theorem 2.42]) implies that there exists  $\bar{a} \in \Lambda^{k-1}$  such that

$$\lambda = \bar{a} \wedge \bar{b} \quad \text{with} \quad |\bar{a} \wedge \bar{b}| = 1.$$

We therefore have found that

$$1/c = \langle \alpha; \bar{a} \wedge \bar{b} \rangle^2.$$

Note that  $1/c < |\alpha|^2$ , otherwise  $\bar{a} \wedge \bar{b}$  would be parallel to  $\alpha$  and thus  $\alpha$  would be 1-divisible, which contradicts the hypothesis.

*Step 2.* So let  $f(\xi) = |\xi|^2 - c\langle \alpha; \xi \rangle^2$ . Observe that  $f$  is not convex since  $c|\alpha|^2 > 1$  (by Step 1). Indeed,

$$f\left(\frac{1}{2}\alpha + \frac{1}{2}(-\alpha)\right) = f(0) = 0 > |\alpha|^2(1 - c|\alpha|^2) = f(\alpha) = \frac{1}{2}f(\alpha) + \frac{1}{2}f(-\alpha).$$

However,  $f$  is ext. one convex (and thus, by Theorem 4.5, ext. quasiconvex). Indeed, let

$$g(t) = f(\xi + ta \wedge b) = |\xi + ta \wedge b|^2 - c\langle \alpha; \xi + ta \wedge b \rangle^2.$$

Note that

$$g''(t) = 2[|a \wedge b|^2 - c\langle \alpha; a \wedge b \rangle^2],$$

which is nonnegative by Step 1. Thus  $g$  is convex.

Let now  $\alpha \wedge \alpha = 0$  and assume, for the sake of contradiction, that  $f$  is ext. polyconvex. Then there should exist (cf. Lemma 4.1)  $\beta \in \Lambda^{2k}$  such that  $f(\xi) \geq \langle \beta; \xi \wedge \xi \rangle$  for every  $\xi \in \Lambda^k$ . This is clearly impossible, in view of the fact that  $c|\alpha|^2 > 1$ , since choosing  $\xi = \alpha$ , we get

$$f(\alpha) = |\alpha|^2(1 - c|\alpha|^2) < 0 = \langle \beta; \alpha \wedge \alpha \rangle. \quad \square$$

We conclude with another example.

**Proposition 4.4.** *Let  $1 \leq k \leq n$ ,  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a symmetric linear operator and  $T^* : \Lambda^k \rightarrow \Lambda^k$  be the pullback of  $T$ . Let  $f : \Lambda^k \rightarrow \mathbb{R}$  be defined as*

$$f(\xi) = \langle T^*(\xi); \xi \rangle \quad \text{for } \xi \in \Lambda^k.$$

*Then  $f$  is ext. one convex if and only if  $f$  is convex.*



*Proof.* Since convexity implies ext. one convexity, we only have to prove the other implication.

*Step 1.* Since  $T$  is symmetric, we can find eigenvalues  $\{\lambda_1, \dots, \lambda_n\}$  (not necessarily distinct) of  $T$  with a corresponding set  $\{\varepsilon^1, \dots, \varepsilon^n\}$  of orthonormal eigenvectors. Let  $\{e^1, \dots, e^n\}$  be the standard basis of  $\mathbb{R}^n$ , let  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$  and let  $Q$  be the orthogonal matrix so that

$$Q^*(e^i) = e^i \quad \text{for } i = 1, \dots, n.$$

In terms of matrices what we have written just means that  $T = Q\Lambda Q^t$ . Observe that, for every  $i = 1, \dots, n$ ,

$$T^*(\varepsilon^i) = (Q\Lambda Q^t)^*(\varepsilon^i) = (Q^t)^*(\Lambda^*(Q^*(\varepsilon^i))) = (Q^t)^*(\Lambda^*(e^i)) = \lambda_i \varepsilon^i.$$

This implies, for every  $1 \leq k \leq n$  and  $I \in \mathcal{T}_k^n$ ,

$$T^*(\varepsilon^I) = T^*(\varepsilon^{i_1} \wedge \dots \wedge \varepsilon^{i_k}) = T^*(\varepsilon^{i_1}) \wedge \dots \wedge T^*(\varepsilon^{i_k}) = \left(\prod_{j=1}^k \lambda_{i_j}\right) \varepsilon^I.$$

*Step 2.* Since  $f$  is ext. one convex and in view of Lemma 4.1(iii), we have

$$f(\varepsilon^I) = \langle T^*(\varepsilon^I); \varepsilon^I \rangle \geq 0$$

and thus

$$\prod_{j=1}^k \lambda_{i_j} = \prod_{i \in I} \lambda_i \geq 0. \tag{10}$$

Writing  $\xi$  in the basis  $\{\varepsilon^1, \dots, \varepsilon^n\}$ , we get

$$f(\xi) = \langle T^*(\xi); \xi \rangle = \left\langle T^*\left(\sum_{I \in \mathcal{T}_k^n} \xi_I \varepsilon^I\right); \sum_{I \in \mathcal{T}_k^n} \xi_I \varepsilon^I \right\rangle = \sum_{I \in \mathcal{T}_k^n} \left(\prod_{i \in I} \lambda_i\right) (\xi_I)^2,$$

which according to (10) is nonnegative. This shows that  $f$  is convex as wished. □

**4.1.3. The main result.** We now turn to the main theorem.

**Theorem 4.5.** *Let  $1 \leq k \leq n$ ,  $M : \Lambda^k \rightarrow \Lambda^k$  be a symmetric linear operator and  $f : \Lambda^k \rightarrow \mathbb{R}$  be such that  $f(\xi) = \langle M\xi; \xi \rangle$  for every  $\xi \in \Lambda^k$ .*

(i) *In all cases*

$$f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

(ii) *Let  $k = 2$ . If  $n = 2$  or  $n = 3$ , then*

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

*If  $n = 4$ , then*

$$f \text{ convex} \begin{matrix} \Rightarrow \\ \not\Leftrightarrow \end{matrix} f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex,}$$

*while if  $n \geq 6$ , then*

$$f \text{ ext. polyconvex} \begin{matrix} \Rightarrow \\ \not\Leftrightarrow \end{matrix} f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

(iii) If  $k$  is odd or if  $2k > n$ , then

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex.}$$

(iv) If  $k$  is even and  $2k \leq n$ , then

$$f \text{ convex} \begin{matrix} \Rightarrow \\ \neq \end{matrix} f \text{ ext. polyconvex.}$$

(v) If either  $3 \leq k \leq n - 3$  or  $k = n - 2 \geq 4$  is even, then

$$f \text{ ext. polyconvex} \begin{matrix} \Rightarrow \\ \neq \end{matrix} f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

**Remark 4.6.** (i) We recall that when  $k = 1$ , all notions of convexity are equivalent.

(ii) When  $k = 2$  and  $n = 5$ , the equivalence between ext. polyconvexity and ext. quasiconvexity remains open.

*Proof of Theorem 4.5.* (i) The result follows from Lemma 4.1 and the Plancherel formula. The proof is similar to that of the case of the gradient (cf. [8, Theorem 5.25 and Lemma 5.28]).

(ii) If  $n = 2$  or  $n = 3$ , the result follows from Theorem 2.8(ii). If  $n \geq 6$ , see Theorem 4.7. So we now assume that  $n = 4$  (for the reverse implication see (iv) below). We only have to prove that

$$f \text{ ext. one convex} \Rightarrow f \text{ ext. polyconvex.}$$

We know (by ext. one convexity) that, for every  $a, b \in \Lambda^1(\mathbb{R}^4)$ ,

$$f(a \wedge b) \geq 0,$$

and we wish to show (cf. Lemma 4.1) that we can find  $\alpha \in \Lambda^4(\mathbb{R}^4)$  such that

$$f(\xi) \geq \langle \alpha; \xi \wedge \xi \rangle.$$

*Step 1.* Let us change the notation slightly and write  $\xi \in \Lambda^2(\mathbb{R}^4)$  as a vector of  $\mathbb{R}^6$  in the following manner:

$$\xi = (\xi_{12}, \xi_{13}, \xi_{14}, \xi_{23}, \xi_{24}, \xi_{34});$$

then  $f$  can be seen as a quadratic form over  $\mathbb{R}^6$  which is nonnegative whenever the (indefinite) quadratic form

$$g(\xi) = \langle e^1 \wedge e^2 \wedge e^3 \wedge e^4; \xi \wedge \xi \rangle = 2(\xi_{12}\xi_{34} - \xi_{13}\xi_{24} + \xi_{14}\xi_{23})$$

vanishes. Indeed, note that, by [5, Proposition 2.37],

$$g(\xi) = 0 \Leftrightarrow \xi \wedge \xi = 0 \Leftrightarrow \text{rank}[\xi] \in \{0, 2\}.$$

By [5, Proposition 2.43], this last condition is equivalent to the existence of  $a, b \in \Lambda^1(\mathbb{R}^4)$  such that  $\xi = a \wedge b$ , and by ext. one convexity we know that  $f(a \wedge b) \geq 0$ .

Step 2. We now invoke [12, Theorem 2] to find  $\lambda \in \mathbb{R}$  such that

$$f(\xi) - \lambda g(\xi) \geq 0.$$

But this is exactly what we had to prove.

(iii) This is a general fact (see Remark 2.2(ii) and Theorem 2.8).

(iv) The counterexample is just  $f(\xi) = \langle \alpha; \xi \wedge \xi \rangle$  for any  $\alpha \in \Lambda^{2k}$ ,  $\alpha \neq 0$ .

(v) This is just Proposition 4.2 and the remark following it. Indeed, we consider the following two cases.

If  $k$  is odd (and since  $3 \leq k \leq n - 3$ , then  $n \geq 6$ ), we know from (iii) that  $f$  is ext. polyconvex if and only if  $f$  is convex, and we also know that there exists an exterior  $k$ -form which is not 1-divisible. Proposition 4.2 therefore gives the result.

If  $k$  is even and  $4 \leq k \leq n - 2$  (which implies again  $n \geq 6$ ), then there exists an exterior  $k$ -form  $\alpha$  which is not 1-divisible, but  $\alpha \wedge \alpha = 0$ . The result thus follows again from Proposition 4.2.  $\square$

4.1.4. A counterexample for  $k = 2$ . We now turn to a counterexample that has been mentioned in Theorem 4.5.

**Theorem 4.7.** *Let  $n \geq 6$ . Then there exists a quadratic form  $f : \Lambda^2 \rightarrow \mathbb{R}$  which is ext. one convex but not ext. polyconvex.*

*Proof.* It is enough to establish the theorem for  $n = 6$ . Our counterexample is inspired by Serre [16] and Terpstra [22] (see [8, Theorem 5.25(iii)]). It is more convenient to write here  $\xi \in \Lambda^2(\mathbb{R}^6)$  as

$$\xi = \sum_{1 \leq i < j \leq 6} \xi_j^i e^i \wedge e^j.$$

So let

$$g(\xi) = (\xi_2^1)^2 + (\xi_3^1)^2 + (\xi_3^2)^2 + (\xi_5^4)^2 + (\xi_6^4)^2 + (\xi_6^5)^2 + h(\xi),$$

where

$$h(\xi) = (\xi_4^1 - \xi_5^3 - \xi_6^2)^2 + (\xi_5^1 - \xi_4^3 + \xi_6^1)^2 + (\xi_4^2 - \xi_4^3 - \xi_6^1)^2 + (\xi_5^2)^2 + (\xi_6^3)^2.$$

Note that  $g \geq 0$ . We claim that there exists  $\gamma > 0$  such that

$$f(\xi) = g(\xi) - \gamma |\xi|^2$$

is ext. one convex (Step 1 below) but not ext. polyconvex (Step 2).

Step 1. Define

$$\gamma := \inf\{g(a \wedge b) : a, b \in \Lambda^1(\mathbb{R}^6), |a \wedge b| = 1\}. \tag{11}$$

Note that  $\gamma \geq 0$ , and it follows from Lemma 4.1 that  $f$  is ext. one convex.

Let us show that, in fact,  $\gamma > 0$ , which will imply in Step 2 that  $f$  is not ext. polyconvex. Assume that  $\gamma = 0$ . Then we can find  $a, b \in \Lambda^1(\mathbb{R}^6)$  with  $|a \wedge b| = 1$  such that

$$\begin{cases} a^1 b_2 - a^2 b_1 = 0, \\ a^1 b_3 - a^3 b_1 = 0, \\ a^2 b_3 - a^3 b_2 = 0, \end{cases} \quad \begin{cases} a^4 b_5 - a^5 b_4 = 0, \\ a^4 b_6 - a^6 b_4 = 0, \\ a^5 b_6 - a^6 b_5 = 0, \end{cases} \quad \begin{cases} a^2 b_5 - a^5 b_2 = 0, \\ a^3 b_6 - a^6 b_3 = 0, \end{cases}$$

$$\begin{cases} (a^1 b_4 - a^4 b_1) - (a^3 b_5 - a^5 b_3) - (a^2 b_6 - a^6 b_2) = 0, \\ (a^1 b_5 - a^5 b_1) - (a^3 b_4 - a^4 b_3) + (a^1 b_6 - a^6 b_1) = 0, \\ (a^2 b_4 - a^4 b_2) - (a^3 b_4 - a^4 b_3) - (a^1 b_6 - a^6 b_1) = 0. \end{cases}$$

Write

$$\underline{a} = \begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \bar{a} = \begin{pmatrix} a^4 \\ a^5 \\ a^6 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} b_4 \\ b_5 \\ b_6 \end{pmatrix}.$$

Then the first and second sets of equations lead to

$$\underline{a} \parallel \underline{b} \quad \text{and} \quad \bar{a} \parallel \bar{b}.$$

We consider two cases, starting with the generic case.

*Case 1:* There exist  $\lambda, \mu \in \mathbb{R}$  such that

$$\underline{a} = \lambda \underline{b} \quad \text{and} \quad \bar{a} = \mu \bar{b}.$$

(The same reasoning applies to all other cases, for example  $\underline{b} = \lambda \underline{a}$  and  $\bar{b} = \mu \bar{a}$ .) Note that  $\lambda \neq \mu$ , otherwise we would have  $a = \lambda b$  and thus  $a \wedge b = 0$ , contradicting the fact that  $|a \wedge b| = 1$ . Inserting this in the third and fourth sets of equations we get

$$\begin{cases} (\lambda - \mu)b_2 b_5 = 0, \\ (\lambda - \mu)b_3 b_6 = 0, \end{cases} \quad \begin{cases} (\lambda - \mu)[b_1 b_4 - b_3 b_5 - b_2 b_6] = 0, \\ (\lambda - \mu)[b_1 b_5 - b_3 b_4 + b_1 b_6] = 0, \\ (\lambda - \mu)[b_2 b_4 - b_3 b_4 - b_1 b_6] = 0, \end{cases}$$

and thus, since  $\lambda \neq \mu$ ,

$$b_2 b_5 = b_3 b_6 = b_1 b_4 - b_3 b_5 - b_2 b_6 = b_1 b_5 - b_3 b_4 + b_1 b_6 = b_2 b_4 - b_3 b_4 - b_1 b_6 = 0.$$

We have to consider separately the cases  $b_2 = b_3 = 0, b_5 = b_6 = 0, b_2 = b_6 = 0$  and  $b_3 = b_5 = 0$ . We do only the first case. Other cases can be handled similarly. So assume that  $b_2 = b_3 = 0$ . Then

$$b_1 b_4 = b_1 b_5 + b_1 b_6 = b_1 b_6 = 0.$$

So either  $b_1 = 0$  and thus  $\underline{b} = 0$  and hence  $\underline{a} = 0$ , and again this implies that  $a = \mu b$ , which contradicts  $|a \wedge b| = 1$ ; or  $b_4 = b_5 = b_6 = 0$  and thus  $\bar{b} = \bar{a} = 0$ , which as before contradicts  $|a \wedge b| = 1$ .

*Case 2:*  $\underline{b} = 0$  and  $\bar{a} = 0$  (or  $\underline{a} = 0$  and  $\bar{b} = 0$ , which is handled similarly). This means that  $a^4 = a^5 = a^6 = 0$  and  $b_1 = b_2 = b_3 = 0$ . Therefore,

$$a^2 b_5 = a^3 b_6 = a^1 b_4 - a^3 b_5 - a^2 b_6 = a^1 b_5 - a^3 b_4 + a^1 b_6 = a^2 b_4 - a^3 b_4 - a^1 b_6 = 0.$$

Four cases can happen:  $a^2 = a^3 = 0$ ,  $a^2 = b_6 = 0$ ,  $a^3 = b_5 = 0$  or  $b_5 = b_6 = 0$ . Again, we handle only the first case. Assuming that  $a^2 = a^3 = 0$ , we have

$$a^1 b_4 = a^1 b_5 + a^1 b_6 = a^1 b_6 = 0.$$

So either  $a^1 = 0$  and thus  $a = 0$ , which is impossible; or  $b_4 = b_5 = b_6 = 0$  and thus  $b = 0$ , which again cannot happen. Hence, we have proved that  $\gamma$  defined in (11) is positive.

*Step 2.* We now show that  $f$  is not ext. polyconvex. In view of Lemma 4.1(i), it is sufficient to show that for every  $\alpha \in \Lambda^4(\mathbb{R}^6)$ , there exists  $\xi \in \Lambda^2(\mathbb{R}^6)$  such that

$$f(\xi) + \frac{1}{2} \langle \alpha; \xi \wedge \xi \rangle < 0.$$

We prove that the above inequality holds for  $\xi$  of the following form:

$$\begin{aligned} \xi &= (b + d)e^1 \wedge e^4 + (c - a)e^1 \wedge e^5 + ae^1 \wedge e^6 \\ &\quad + (c + a)e^2 \wedge e^4 + be^2 \wedge e^6 + ce^3 \wedge e^4 + de^3 \wedge e^5. \end{aligned}$$

Note that

$$\begin{aligned} \frac{1}{2} \xi \wedge \xi &= (c^2 - a^2)e^1 \wedge e^2 \wedge e^4 \wedge e^5 + (ac + a^2 - b^2 - bd)e^1 \wedge e^2 \wedge e^4 \wedge e^6 \\ &\quad + (ab - bc)e^1 \wedge e^2 \wedge e^5 \wedge e^6 + (c^2 - ac - bd - d^2)e^1 \wedge e^3 \wedge e^4 \wedge e^5 \\ &\quad + ace^1 \wedge e^3 \wedge e^4 \wedge e^6 + ade^1 \wedge e^3 \wedge e^5 \wedge e^6 \\ &\quad + (-cd - ad)e^2 \wedge e^3 \wedge e^4 \wedge e^5 + bce^2 \wedge e^3 \wedge e^4 \wedge e^6 \\ &\quad + bde^2 \wedge e^3 \wedge e^5 \wedge e^6. \end{aligned}$$

For such forms we have  $g(\xi) = 0$  and therefore

$$f(\xi) = -\gamma |\xi|^2 = -\gamma [(b + d)^2 + (c - a)^2 + a^2 + (c + a)^2 + b^2 + c^2 + d^2].$$

Moreover,

$$\begin{aligned} \frac{1}{2} \langle \alpha; \xi \wedge \xi \rangle &= \alpha_{1245}(c^2 - a^2) + \alpha_{1246}(ac + a^2 - b^2 - bd) \\ &\quad + \alpha_{1256}(ab - bc) + \alpha_{1345}(c^2 - ac - bd - d^2) + \alpha_{1346}ac \\ &\quad + \alpha_{1356}ad + \alpha_{2345}(-cd - ad) + \alpha_{2346}bc + \alpha_{2356}bd. \end{aligned}$$

We consider three cases.

*Case 1.* If  $\alpha_{1246} > 0$ , then take  $a = c = d = 0$  and  $b \neq 0$  to get

$$f(\xi) + \frac{1}{2} \langle \alpha; \xi \wedge \xi \rangle = -\gamma(2b^2) - \alpha_{1246}b^2 < 0.$$

*Case 2.* If  $\alpha_{1345} > 0$ , then take  $a = b = c = 0$  and  $d \neq 0$  to get

$$f(\xi) + \frac{1}{2} \langle \alpha; \xi \wedge \xi \rangle = -\gamma(2d^2) - \alpha_{1345}d^2 < 0.$$

We can therefore assume that  $\alpha_{1246} \leq 0$  and  $\alpha_{1345} \leq 0$ .

Case 3. If  $\alpha_{1245} + \alpha_{1345} < 0$  (and  $\alpha_{1246} \leq 0, \alpha_{1345} \leq 0$ ), then take  $a = b = d = 0$  and  $c \neq 0$  to get

$$f(\xi) + \frac{1}{2}\langle \alpha; \xi \wedge \xi \rangle = -\gamma(3c^2) + (\alpha_{1245} + \alpha_{1345})c^2 < 0.$$

We therefore assume  $\alpha_{1246} \leq 0, \alpha_{1345} \leq 0$  and  $\alpha_{1245} + \alpha_{1345} \geq 0$ . Hence we deduce that  $\alpha_{1246} - \alpha_{1245} \leq 0$ , and then taking  $b = c = d = 0$  and  $a \neq 0$ , we get

$$f(\xi) + \frac{1}{2}\langle \alpha; \xi \wedge \xi \rangle = -\gamma(3a^2) + (\alpha_{1246} - \alpha_{1245})a^2 < 0.$$

This concludes the proof of the theorem. □

4.2. Ext. one convexity does not imply ext. quasiconvexity

We now give an important counterexample for any  $k \geq 2$ . It is an adaptation of the fundamental result of Šverák [20] (see also [8, Theorem 5.50]).

**Theorem 4.8.** *Let  $2 \leq k \leq n - 3$ . Then there exists  $f : \Lambda^k \rightarrow \mathbb{R}$  that is ext. one convex but not ext. quasiconvex.*

**Remark 4.9.** We know that when  $k = 1, n - 1, n$  or  $k = n - 2$  is odd, then

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

Therefore only the case  $k = n - 2 \geq 2$  even (including  $k = 2$  and  $n = 4$ ) remains open.

The main algebraic tool in order to adapt Šverák’s example is given in the following lemma.

**Lemma 4.10.** *Let  $k \geq 2$  and  $n = k + 3$ . There exist*

$$\alpha, \beta, \gamma \in \text{span}\{e^{i_1} \wedge \dots \wedge e^{i_{k-1}} : 3 \leq i_1 < \dots < i_{k-1} \leq k + 3\} \subset \Lambda^{k-1}(\mathbb{R}^{k+3})$$

such that if

$$L = \text{span}\{e^1 \wedge \alpha, e^2 \wedge \beta, (e^1 + e^2) \wedge \gamma\},$$

then any 1-divisible  $\xi = (x, y, z) = xe^1 \wedge \alpha + ye^2 \wedge \beta + z(e^1 + e^2) \wedge \gamma \in L$  satisfies

$$xy = xz = yz = 0.$$

*Proof. Step 1.* We choose (recall that  $n = k + 3$ )

$$\alpha = \begin{cases} \sum_{i=2}^{l+1} \widehat{e^{2i}} \wedge \widehat{e^{2i+1}} & \text{if } k = 2l, \\ \sum_{i=2}^{l+2} \widehat{e^{2i-1}} \wedge \widehat{e^{2i}} & \text{if } k = 2l + 1, \end{cases}$$

$$\beta = \begin{cases} \widehat{e^3} \wedge \widehat{e^{2l+3}} & \text{if } k = 2l, \\ \widehat{e^3} \wedge \widehat{e^5} + \widehat{e^4} \wedge \widehat{e^6} & \text{if } k = 3, \\ \sum_{i=2}^l \widehat{e^{2i-1}} \wedge \widehat{e^{2i}} & \text{if } k = 2l + 1 \text{ and } k \geq 5, \end{cases}$$

$$\gamma = \begin{cases} \sum_{i=2}^{l+1} \widehat{e^{2i-1}} \wedge \widehat{e^{2i}} & \text{if } k = 2l, \\ \widehat{e^{2l+1}} \wedge \widehat{e^{2l+4}} + \widehat{e^{2l+2}} \wedge \widehat{e^{2l+3}} & \text{if } k = 2l + 1, \end{cases}$$

where we write, by abuse of notation, for  $3 \leq i < j \leq k + 3$ ,

$$\widehat{e^i} \wedge \widehat{e^j} = e^3 \wedge \dots \wedge \widehat{e^i} \wedge \dots \wedge \widehat{e^j} \wedge \dots \wedge e^{k+3}.$$

Observe that  $\{\alpha, \beta, \gamma\}$  are linearly independent.

*Step 2.* We now prove the statement, namely that if  $\xi = (x, y, z) \in L$  is 1-divisible (i.e.  $\xi = b \wedge a$  for  $a \in \Lambda^1$  and  $b \in \Lambda^{k-1}$ ), then necessarily  $xy = xz = yz = 0$ . Assume that  $\xi \neq 0$  (otherwise the result is trivial) and thus  $a \neq 0$ . Note that if  $\xi = b \wedge a$ , then  $a \wedge \xi = 0$ . We write

$$a = \sum_{i=1}^{k+3} a_i e^i \neq 0.$$

*Step 2.1.* Since  $a \wedge \xi = 0$  we deduce that the term involving  $e^1 \wedge e^2$  must be 0 and thus

$$-a_2 x \alpha + a_1 y \beta + (a_1 - a_2) z \gamma = 0.$$

Since  $\{\alpha, \beta, \gamma\}$  are linearly independent, we deduce that

$$a_2 x = a_1 y = (a_1 - a_2) z = 0.$$

From this we infer that  $xy = xz = yz = 0$  as soon as either  $a_1 \neq 0$  or  $a_2 \neq 0$ . So it is enough to consider  $a$  of the form

$$a = \sum_{i=3}^{k+3} a_i e^i \neq 0.$$

We then have

$$\sum_{i=3}^{k+3} a_i e^i \wedge [e^1 \wedge (x\alpha + z\gamma) + e^2 \wedge (y\beta + z\gamma)] = 0,$$

which implies that

$$\begin{cases} a \wedge (x\alpha + z\gamma) = \sum_{i=3}^{k+3} a_i e^i \wedge (x\alpha + z\gamma) = 0, \\ a \wedge (y\beta + z\gamma) = \sum_{i=3}^{k+3} a_i e^i \wedge (y\beta + z\gamma) = 0. \end{cases} \tag{12}$$

We continue the discussion considering separately the cases of  $k$  even,  $k = 3$  and  $k \geq 5$  odd. They are all treated in the same way, so we handle only the even case.

*Step 2.2:  $k = 2l \geq 2$ .* We have to prove that if  $a = \sum_{i=3}^{2l+3} a_i e^i \neq 0$  satisfies (12), then necessarily  $xy = xz = yz = 0$ .

We find (up to a + or - sign but here it is immaterial)

$$\begin{aligned} a \wedge \alpha &= \sum_{i=2}^{l+1} a_{2i+1} \widehat{e^{2i}} + \sum_{i=2}^l a_{2i} \widehat{e^{2i+1}} + a_{2l+2} \widehat{e^{2l+3}}, \\ a \wedge \beta &= a_{2l+3} \widehat{e^3} + a_3 \widehat{e^{2l+3}}, \\ a \wedge \gamma &= a_4 \widehat{e^3} + \sum_{i=2}^{l+1} a_{2i-1} \widehat{e^{2i}} + \sum_{i=2}^l a_{2i+2} \widehat{e^{2i+1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} a \wedge (x\alpha + z\gamma) &= za_4\widehat{e^3} + \sum_{i=2}^{l+1} (xa_{2i+1} + za_{2i-1})\widehat{e^{2i}} \\ &\quad + \sum_{i=2}^l (xa_{2i} + za_{2i+2})\widehat{e^{2i+1}} + xa_{2l+2}\widehat{e^{2l+3}}, \\ a \wedge (y\beta + z\gamma) &= (ya_{2l+3} + za_4)\widehat{e^3} + z \left\{ \sum_{i=2}^{l+1} a_{2i-1}\widehat{e^{2i}} + \sum_{i=2}^l a_{2i+2}\widehat{e^{2i+1}} \right\} + ya_3\widehat{e^{2l+3}}. \end{aligned}$$

*Case 1:*  $x = z = 0$ . This is our claim.

*Case 2:*  $z = 0$  and  $x \neq 0$ . We can also assume that  $y \neq 0$ , as otherwise we have the claim  $y = z = 0$ . From the first equation we obtain

$$a_{2i} = a_{2i+1} = 0, \quad i = 2, \dots, l+1.$$

So only  $a_3$  might be nonzero. However since  $y \neq 0$  we deduce from the second equation that  $a_3 = 0$  and thus  $a = 0$ , which is not the case.

*Case 3:*  $x = 0$  and  $z \neq 0$ . We can also assume that  $y \neq 0$ , as otherwise we have the claim  $x = y = 0$ . From the first equation we obtain

$$a_{2i} = a_{2i-1} = 0, \quad i = 2, \dots, l+1.$$

So only  $a_{2l+3}$  might be nonzero. However since  $y \neq 0$  we deduce, appealing to the second equation, that  $a_{2l+3} = 0$  and thus  $a = 0$ , contrary to assumption.

*Case 4:*  $xz \neq 0$ . From the first equation we deduce that

$$a_{2i} = 0, \quad i = 2, \dots, l+1.$$

Inserting this in the second equation we get

$$a \wedge (y\beta + z\gamma) = ya_{2l+3}\widehat{e^3} + z \sum_{i=2}^{l+1} (a_{2i-1}\widehat{e^{2i}}) + ya_3\widehat{e^{2l+3}}.$$

Since  $z \neq 0$ , we infer that

$$a_{2i-1} = 0, \quad i = 2, \dots, l+1.$$

So only  $a_{2l+3}$  might be nonzero. However returning to the first equation we have  $xa_{2l+3} = 0$ . But since  $x \neq 0$ , we deduce that  $a_{2l+3} = 0$  and thus  $a = 0$ , a contradiction again.  $\square$

We can now prove Theorem 4.8. Once the above lemma is established, the proof is almost identical to the proof of Šverák.

*Proof of Theorem 4.8.* It is enough to prove the theorem for  $n = k + 3$ .



*Step 1.* We start with some notation. Let  $L$  be as in Lemma 4.10. An element  $\xi$  of  $L$  is, when convenient, denoted by  $\xi = (x, y, z) \in L$ . Recall that if  $\xi = (x, y, z) \in L$  is 1-divisible, meaning that  $\xi = b \wedge a$  for some  $a \in \Lambda^1$  and  $b \in \Lambda^{k-1}$ , then necessarily

$$xy = xz = yz = 0.$$

We next let  $P : \Lambda^k(\mathbb{R}^{k+3}) \rightarrow L$  be the projection map; in particular  $P(\xi) = \xi$  if  $\xi \in L$ .

*Step 2.* Let  $g : \Lambda^k(\mathbb{R}^{k+3}) \supset L \rightarrow \mathbb{R}$  be defined by

$$g(\xi) = -xyz.$$

Observe that  $g$  is ext. one affine when restricted to  $L$ . Indeed, if  $\xi = (x, y, z) \in L$  and  $\eta = (a, b, c) \in L$  is 1-divisible (which implies that  $ab = ac = bc = 0$ ), then

$$g(\xi + t\eta) = -(x + ta)(y + tb)(z + tc) = -xyz - t[xyz + xzb + yza].$$

Therefore, for every  $\xi, \eta \in L$  with  $\eta$  1-divisible,

$$L_g(\xi, \eta) = \left. \frac{d^2}{dt^2} g(\xi + t\eta) \right|_{t=0} = 0.$$

*Step 3.* Let  $\omega \in C_{\text{per}}^\infty((0, 2\pi)^{k+3}; \Lambda^{k-1})$  be defined by

$$\omega = (\sin x_1)\alpha + (\sin x_2)\beta + (\sin(x_1 + x_2))\gamma,$$

so that

$$d\omega = (\cos x_1)dx^1 \wedge \alpha + (\cos x_2)dx^2 \wedge \beta + (\cos(x_1 + x_2))(dx^1 + dx^2) \wedge \gamma,$$

and hence  $d\omega \in L$ . Note that

$$\int_0^{2\pi} \int_0^{2\pi} g(d\omega) dx_1 dx_2 = - \int_0^{2\pi} \int_0^{2\pi} (\cos x_1)^2 (\cos x_2)^2 dx_1 dx_2 < 0.$$

*Step 4.* Assume (cf. Step 5 below) that we have shown that for every  $\epsilon > 0$ , we can find  $\gamma = \gamma(\epsilon) > 0$  such that

$$f_\epsilon(\xi) = g(P(\xi)) + \epsilon|\xi|^2 + \epsilon|\xi|^4 + \gamma|\xi - P(\xi)|^2$$

is ext. one convex. Then noting that

$$f_\epsilon(d\omega) = g(d\omega) + \epsilon|d\omega|^2 + \epsilon|d\omega|^4,$$

we deduce from Step 3 that, for  $\epsilon > 0$  small enough,

$$\int_{(0, 2\pi)^{k+3}} f_\epsilon(d\omega) dx < 0.$$

This shows that  $f_\epsilon$  is not ext. quasiconvex. The proposition is therefore proved.

*Step 5.* It remains to prove that for every  $\epsilon > 0$  we can find  $\gamma = \gamma(\epsilon) > 0$  such that the function

$$f_\epsilon(\xi) = g(P(\xi)) + \epsilon|\xi|^2 + \epsilon|\xi|^4 + \gamma|\xi - P(\xi)|^2$$

is ext. one convex. This is equivalent to showing that, for every  $\xi, \eta \in \Lambda^k$  with  $\eta$  1-divisible,

$$\begin{aligned} L_{f_\epsilon}(\xi, \eta) &= \left. \frac{d^2}{dt^2} f_\epsilon(\xi + t\eta) \right|_{t=0} \\ &= L_g(P(\xi), P(\eta)) + 2\epsilon|\eta|^2 + 4\epsilon|\xi|^2|\eta|^2 + 8\epsilon(\langle \xi; \eta \rangle)^2 + 2\gamma|\eta - P(\eta)|^2 \\ &\geq 0. \end{aligned}$$

The proof is standard; see [8] and [17] for details. □

### 4.3. Some further examples

We here give another counterexample for  $k = 2$ .

**Proposition 4.11.** *Let  $n \geq 4$ . Then there is an ext. quasiconvex function  $f : \Lambda^2(\mathbb{R}^n) \rightarrow \mathbb{R}$  which is not ext. polyconvex.*

**Remark 4.12.** This example is mostly interesting when  $n = 4$  or  $5$ , since when  $n \geq 6$ , we already have such a counterexample (cf. Theorem 4.7).

*Proof of Proposition 4.11.* As in previous theorems, it is easy to see that it is enough to establish the theorem for  $n = 4$ . Let  $1 < p < 2$ ,  $\alpha = e^1 \wedge e^2 + e^3 \wedge e^4$  and  $g : \Lambda^2(\mathbb{R}^4) \rightarrow \mathbb{R}$  be given by

$$g(\xi) = (|\xi|^2 - 2|\langle \alpha; \xi \rangle| + |\alpha|^2)^{p/2} = \min\{|\xi - \alpha|^p, |\xi + \alpha|^p\}.$$

We claim that  $f = Q_{\text{ext}}g$  has all the desired properties (the proof is inspired by the one of Šverák [19], see also [8, Theorem 5.54]). Indeed,  $f$  is by construction ext. quasiconvex and if we can show (see Step 2 below) that  $f$  is not convex (note that  $f$  is subquadratic and using Proposition 2.14, any subquadratic ext. polyconvex function is convex), we will have established the proposition.

*Step 1.* First observe that a direct computation gives

$$|\xi|^2 - 2|\langle \alpha; \xi \rangle| + |\alpha|^2 = \min\{|\xi - \alpha|^2, |\xi + \alpha|^2\} \geq \frac{1}{2}[|\xi|^2 - \frac{1}{2}\langle \alpha \wedge \alpha; \xi \wedge \xi \rangle] \geq 0.$$

Therefore there exists a constant  $c_1 > 0$  such that

$$\begin{aligned} g(\xi) &\geq \left(\frac{1}{2}\right)^{p/2} [|\xi|^2 - \frac{1}{2}\langle \alpha \wedge \alpha; \xi \wedge \xi \rangle]^{p/2} \\ &\geq c_1[|\xi_{12} - \xi_{34}|^p + |\xi_{13} + \xi_{24}|^p + |\xi_{14} - \xi_{23}|^p] =: h. \end{aligned}$$

*Step 2.* For contradiction, suppose that  $f$  is convex. This implies that  $f(0) = 0$ , because

$$0 \leq f(0) = f\left(\frac{1}{2}\alpha + \frac{1}{2}(-\alpha)\right) \leq \frac{1}{2}f(\alpha) + \frac{1}{2}f(-\alpha) = 0.$$

Use Remark 2.13 to find a sequence  $\omega_s \in W_{\delta,T}^{1,\infty}(\Omega; \Lambda^1)$  (we can choose an  $\Omega$  with smooth boundary and by density we can also assume that  $\omega_s \in C_{\delta,T}^\infty(\bar{\Omega}; \Lambda^1)$ ) such that

$$0 \leq \frac{1}{\text{meas } \Omega} \int_{\Omega} g(d\omega_s) \leq Q_{\text{ext}}g(0) + \frac{1}{s} = f(0) + \frac{1}{s} = \frac{1}{s},$$

which implies that

$$\lim_{s \rightarrow \infty} \frac{1}{\text{meas } \Omega} \int_{\Omega} g(d\omega_s) = 0. \tag{13}$$

On the other hand, from Step 1, we deduce that

$$0 \leq \int_{\Omega} h(d\omega_s) \leq \frac{\text{meas } \Omega}{s} \rightarrow 0.$$

We now invoke Step 3 below: there is a constant  $c_2 > 0$  such that

$$c_2 \|\nabla \omega_s\|_{L^p}^p \leq \int_{\Omega} h(d\omega_s).$$

Thus  $\|d\omega_s\|_{L^p} \rightarrow 0$  and hence, up to the extraction of a subsequence,

$$\frac{1}{\text{meas } \Omega} \int_{\Omega} g(d\omega_s) \rightarrow g(0) = |\alpha|^p \neq 0,$$

which contradicts (13). Therefore,  $f$  is not convex.

*Step 3.* It remains to prove that there exists a constant  $\lambda > 0$  such that

$$\lambda \|\nabla \omega\|_{L^p}^p \leq \int_{\Omega} h(d\omega) = \|[h(d\omega)]^{1/p}\|_{L^p}^p \quad \text{for every } \omega \in C_{\delta,T}^\infty(\bar{\Omega}; \Lambda^1).$$

Let  $\omega \in C_{\delta,T}^\infty(\bar{\Omega}; \Lambda^1)$  and  $\alpha, \beta, \gamma \in C^\infty(\bar{\Omega})$  be such that

$$\begin{aligned} \alpha &= (d\omega)_{12} - (d\omega)_{34} = -\omega_{x_2}^1 + \omega_{x_1}^2 + \omega_{x_4}^3 - \omega_{x_3}^4, \\ \beta &= (d\omega)_{13} + (d\omega)_{24} = -\omega_{x_3}^1 + \omega_{x_1}^3 - \omega_{x_4}^2 + \omega_{x_2}^4, \\ \gamma &= (d\omega)_{14} - (d\omega)_{23} = -\omega_{x_4}^1 + \omega_{x_1}^4 + \omega_{x_3}^2 - \omega_{x_2}^3, \\ 0 &= \delta\omega = \omega_{x_1}^1 + \omega_{x_2}^2 + \omega_{x_3}^3 + \omega_{x_4}^4. \end{aligned}$$

Note that

$$h(d\omega) = c_1[|\alpha|^p + |\beta|^p + |\gamma|^p].$$

Differentiating appropriately the four equations we find

$$\begin{aligned} \Delta\omega^1 &= -\alpha_{x_2} - \beta_{x_3} - \gamma_{x_4}, & \Delta\omega^2 &= \alpha_{x_1} - \beta_{x_4} + \gamma_{x_3}, \\ \Delta\omega^3 &= \alpha_{x_4} + \beta_{x_1} - \gamma_{x_2}, & \Delta\omega^4 &= -\alpha_{x_3} + \beta_{x_2} + \gamma_{x_1}. \end{aligned}$$

Letting

$$\begin{aligned} \psi := & -(\alpha_{x_2} + \beta_{x_3} + \gamma_{x_4})dx^1 + (\alpha_{x_1} - \beta_{x_4} + \gamma_{x_3})dx^2 + (\alpha_{x_4} + \beta_{x_1} - \gamma_{x_2})dx^3 \\ & + (-\alpha_{x_3} + \beta_{x_2} + \gamma_{x_1})dx^4, \end{aligned}$$

we get

$$\begin{cases} \Delta \omega = \psi & \text{in } \Omega, \\ \nu \wedge \delta \omega = 0, \nu \wedge \omega = 0 & \text{on } \partial \Omega. \end{cases}$$

Using classical elliptic regularity theory (see, for example, [13, Theorem 6.3.7]), we deduce that

$$\|\omega\|_{W^{1,p}} \leq \lambda_2 \|\psi\|_{W^{-1,p}}.$$

In other words,

$$\|\nabla \omega\|_{L^p} \leq \lambda_2 \|\psi\|_{W^{-1,p}} \leq \lambda_3 \|(\alpha, \beta, \gamma)\|_{L^p} \leq \lambda_4 \| [h(d\omega)]^{1/p} \|_{L^p}.$$

This is exactly what had to be proved.  $\square$

## 5. Application to a minimization problem

**Theorem 5.1.** *Let  $1 \leq k \leq n$ ,  $p > 1$ ,  $\Omega \subset \mathbb{R}^n$  be a bounded smooth open set,  $\omega_0$  in  $W^{1,p}(\Omega; \Lambda^{k-1})$  and  $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$  be ext. quasiconvex satisfying*

$$c_1(|\xi|^p - 1) \leq f(\xi) \leq c_2(|\xi|^p + 1) \quad \text{for every } \xi \in \Lambda^k,$$

for some  $c_1, c_2 > 0$ . Let

$$(\mathcal{P}_0) \quad \inf \left\{ \int_{\Omega} f(d\omega) : \omega \in \omega_0 + W_0^{1,p}(\Omega; \Lambda^{k-1}) \right\} = m.$$

Then the problem  $(\mathcal{P}_0)$  has a minimizer.

**Remark 5.2.** (i) If

$$(\mathcal{P}_{\delta,T}) \quad \inf \left\{ \int_{\Omega} f(d\omega) : \omega \in \omega_0 + W_{\delta,T}^{1,p}(\Omega; \Lambda^{k-1}) \right\} = m_{\delta,T},$$

where  $\omega_0 + W_{\delta,T}^{1,p}(\Omega; \Lambda^{k-1})$  stands for the set of all  $\omega \in W^{1,p}(\Omega; \Lambda^{k-1})$  such that

$$\delta \omega = 0 \text{ in } \Omega \quad \text{and} \quad \nu \wedge \omega = \nu \wedge \omega_0 \text{ on } \partial \Omega,$$

the proof of the theorem will show that  $(\mathcal{P}_{\delta,T})$  also has a minimizer and  $m_{\delta,T} = m$ .

(ii) When the function  $f$  is not ext. quasiconvex, in general the problem will not have a solution. However, in many cases it does have one, but the argument is of a different nature and uses results on differential inclusions (see [1], [2] and [9]).

*Proof of Theorem 5.1. Step 1.* Using a variant of the classical result (see [3] and [17]), we note that if

$$\alpha_s \rightharpoonup \alpha \quad \text{in } W^{1,p}(\Omega; \Lambda^{k-1}),$$

then

$$\liminf_{s \rightarrow \infty} \int_{\Omega} f(d\alpha_s) \geq \int_{\Omega} f(d\alpha).$$

*Step 2.* Let  $\omega_s$  be a minimizing sequence of  $(\mathcal{P}_0)$ , i.e.

$$\int_{\Omega} f(d\omega_s) \rightarrow m.$$

In view of the coercivity condition, there exists a constant  $c_3 > 0$  such that

$$\|d\omega_s\|_{L^p} \leq c_3.$$

According to [5, Theorem 7.2] (when  $p \geq 2$ ) and [17] (when  $p > 1$ ), we can find  $\alpha_s \in \omega_0 + W_{\delta,T}^{1,p}(\Omega; \Lambda^{k-1})$  such that

$$\begin{cases} d\alpha_s = d\omega_s & \text{in } \Omega, \\ \delta\alpha_s = 0 & \text{in } \Omega, \\ \nu \wedge \alpha_s = \nu \wedge \omega_s = \nu \wedge \omega_0 & \text{on } \partial\Omega, \end{cases}$$

and there exist constants  $c_4, c_5 > 0$  such that

$$\|\alpha_s\|_{W^{1,p}} \leq c_4[\|d\omega_s\|_{L^p} + \|\omega_0\|_{W^{1,p}}] \leq c_5.$$

Therefore, up to the extraction of a subsequence that we do not relabel, there exists  $\alpha$  in  $\omega_0 + W_{\delta,T}^{1,p}(\Omega; \Lambda^{k-1})$  such that

$$\alpha_s \rightharpoonup \alpha \quad \text{in } W^{1,p}(\Omega; \Lambda^{k-1}).$$

We then use [5, Theorem 8.16] (when  $p \geq 2$ ) and [17] (when  $p > 1$ ) to find  $\omega$  in  $\omega_0 + W_0^{1,p}(\Omega; \Lambda^{k-1})$  such that

$$\begin{cases} d\omega = d\alpha & \text{in } \Omega, \\ \omega = \omega_0 & \text{on } \partial\Omega. \end{cases}$$

*Step 3.* We combine the two steps to get

$$m = \liminf_{s \rightarrow \infty} \int_{\Omega} f(d\omega_s) = \liminf_{s \rightarrow \infty} \int_{\Omega} f(d\alpha_s) \geq \int_{\Omega} f(d\alpha) = \int_{\Omega} f(d\omega) \geq m.$$

This concludes the proof of the theorem. □

**6. Notation**

We gather here the notation used throughout this article. For more details on exterior algebra and differential forms, see [5], and for the notions of convexity used in the calculus of variations, see [8].

1. Let  $k, n$  be integers.

- We write  $\Lambda^k(\mathbb{R}^n)$  (or simply  $\Lambda^k$ ) to denote the vector space of all alternating  $k$ -linear maps  $f : \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{k \text{ times}} \rightarrow \mathbb{R}$ . For  $k = 0$ , we set  $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$ . Note that

$$\Lambda^k(\mathbb{R}^n) = \{0\} \text{ for } k > n \text{ and, for } k \leq n, \dim(\Lambda^k(\mathbb{R}^n)) = \binom{n}{k}.$$

- $\wedge, \lrcorner, \langle ; \rangle$  and  $*$  denote the exterior product, the interior product, the scalar product and the Hodge star operator respectively.
- If  $\{e^1, \dots, e^n\}$  is a basis of  $\mathbb{R}^n$ , then, identifying  $\Lambda^1$  with  $\mathbb{R}^n$ ,

$$\{e^{i_1} \wedge \dots \wedge e^{i_k} : 1 \leq i_1 < \dots < i_k \leq n\}$$

is a basis of  $\Lambda^k$ . An element  $\xi \in \Lambda^k(\mathbb{R}^n)$  is therefore written as

$$\xi = \sum_{1 \leq i_1 < \dots < i_k \leq n} \xi_{i_1 \dots i_k} e^{i_1} \wedge \dots \wedge e^{i_k} = \sum_{I \in \mathcal{T}_k^n} \xi_I e^I,$$

where

$$\mathcal{T}_k^n = \{I = (i_1, \dots, i_k) \in \mathbb{N}^k : 1 \leq i_1 < \dots < i_k \leq n\}.$$

- We write

$$e^{i_1} \wedge \dots \wedge \widehat{e^{i_s}} \wedge \dots \wedge e^{i_k} = e^{i_1} \wedge \dots \wedge e^{i_{s-1}} \wedge e^{i_{s+1}} \wedge \dots \wedge e^{i_k}.$$

2. Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set.

- The spaces  $C^1(\Omega; \Lambda^k)$ ,  $W^{1,p}(\Omega; \Lambda^k)$  and  $W_0^{1,p}(\Omega; \Lambda^k)$ ,  $1 \leq p \leq \infty$ , are defined in the usual way.  $\text{Aff}_{\text{piece}}(\overline{\Omega}; \Lambda^k)$  denotes the space of piecewise affine functions on  $\overline{\Omega}$ , taking values in  $\Lambda^k$ .
- For  $\omega \in W^{1,p}(\Omega; \Lambda^k)$ , the exterior derivative  $d\omega$  belongs to  $L^p(\Omega; \Lambda^{k+1})$  and is defined by

$$(d\omega)_{i_1 \dots i_{k+1}} = \sum_{j=1}^{k+1} (-1)^{j+1} \frac{\partial \omega_{i_1 \dots i_{j-1} i_{j+1} \dots i_{k+1}}}{\partial x_{i_j}}$$

for  $1 \leq i_1 < \dots < i_{k+1} \leq n$ . If  $k = 0$ , then  $d\omega \simeq \text{grad } \omega$ . If  $k = 1$ , for  $1 \leq i < j \leq n$ ,

$$(d\omega)_{ij} = \frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j},$$

i.e.  $d\omega \simeq \text{curl } \omega$ .

- The interior derivative (or codifferential) of  $\omega \in W^{1,p}(\Omega; \Lambda^k)$ , denoted  $\delta\omega$ , belongs to  $L^p(\Omega; \Lambda^{k-1})$  and is defined as

$$\delta\omega = (-1)^{n(k-1)} *(d(*\omega)).$$

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