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Quantitative stability for sumsets in \mathbb{R}^n

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Abstract. Given a measurable set $A \subset \mathbb{R}^n$ of positive measure, it is not difficult to show that $|A + A| = |2A|$ if and only if A is equal to its convex hull minus a set of measure zero. We investigate the stability of this statement: If $(|A + A| - |2A|)/|A|$ is small, is A close to its convex hull? Our main result is an explicit control, in arbitrary dimension, on the measure of the difference between A and its convex hull in terms of $(|A + A| - |2A|)/|A|$.

Keywords. Quantitative stability, sumsets, Freiman's theorem

1. Introduction

Let A and B be measurable subsets of \mathbb{R}^n , and let $c > 0$. Define the set sum and scalar multiple by

$$A + B := \{a + b : a \in A, b \in B\}, \quad cA := \{ca : a \in A\}. \quad (1.1)$$

Let $|A|$ denote the Lebesgue measure of A , and assume that $|A| > 0$. It is clear that $\frac{1}{2}(A + A) \supset A$, so, in particular, $|\frac{1}{2}(A + A)| \geq |A|$, and it is not difficult to show that $|\frac{1}{2}(A + A)| = |A|$ implies that A is equal to its convex hull minus a set of measure zero. (Notice that, even if A is measurable, $A + A$ may not be. In this case, $|A + A|$ denotes the outer Lebesgue measure of $A + A$.)

The goal of this paper is to investigate whether this statement is stable. Let us define the *deficit* of A as

$$\delta(A) := \frac{|\frac{1}{2}(A + A)|}{|A|} - 1 = \frac{|A + A|}{|2A|} - 1.$$

The question we address is whether small deficit implies that A is close to its convex hull.

This question has already been extensively investigated in the one-dimensional case. If one approximates sets in \mathbb{R} with finite unions of intervals, then one can translate the

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problem to \mathbb{Z} , and in the discrete setting the question becomes a well studied problem in additive combinatorics. There are many results on this topic, usually called Freiman-type theorems; we refer to the book [12] for a comprehensive list of references. Our problem can be seen as a very particular case.

The precise statement in one dimension is the following.

Theorem 1.1. *Let $A \subset \mathbb{R}$ be a measurable set, and denote by $\text{co}(A)$ its convex hull. Then*

$$|A + A| - 2|A| \geq \min\{|\text{co}(A) \setminus A|, |A|\},$$

or, equivalently, if $|A| > 0$ then

$$\delta(A) \geq \frac{1}{2} \min\left\{\frac{|\text{co}(A) \setminus A|}{|A|}, 1\right\}.$$

This theorem can be obtained as a corollary of a result of G. Freiman [8] about the structure of additive subsets of \mathbb{Z} . (See [9] or [12, Theorem 5.11] for a statement and a proof.) However, it turns out that to prove Theorem 1.1 one only needs weaker results. For the convenience of the reader, instead of relying on deep and intricate combinatorial results, we will give an elementary, completely self-contained proof of Theorem 1.1. Our proof is based on the simple observation that a subset of \mathbb{R} can be discretized to a subset of \mathbb{Z} starting at 0 and ending at a prime number p . This may look strange from an analytic point of view, but it considerably simplifies the combinatorial aspects.

The main result of this paper is a quantitative stability result in arbitrary dimension, showing that a power of $\delta(A)$ dominates the measure of the difference between A and its convex hull $\text{co}(A)$.

Theorem 1.2. *Let $n \geq 2$. There exist computable dimensional constants $\delta_n, c_n > 0$ such that if $A \subset \mathbb{R}^n$ is a measurable set of positive measure with $\delta(A) \leq \delta_n$, then*

$$\delta(A)^{\alpha_n} \geq c_n \frac{|\text{co}(A) \setminus A|}{|A|}, \quad \alpha_n := \frac{1}{8 \cdot 16^{n-2} n!(n-1)!}.$$

Concerning this higher dimensional case, M. Christ [2, 3] proved that if $|A + B|^{1/n} - |A|^{1/n} - |B|^{1/n} \rightarrow 0$, then A and B are both close to some dilation of the same convex set. In particular, as a corollary one finds that if $\delta(A) \rightarrow 0$ then $|\text{co}(A) \setminus A| \rightarrow 0$. Although Christ's result does not imply any quantitative estimate for our problem, in another direction it is more general, since it represents a qualitative stability for the Brunn–Minkowski inequality. Furthermore, if we restrict A and B to the class of convex sets, a quantitative stability estimate for the Brunn–Minkowski inequality, with the sharp power law dependence on the deficit, was proved in [6, 7].

Much of the difficulty in Christ's work arises from the fact that he is dealing with different sets A and B . In Appendix B, we show how his methods yield a relatively quick proof of qualitative stability when $A = B$ is bounded. Our purpose here is to provide a quantitative stability estimate, and since the argument is very involved, we decided to focus on the case $A = B$.

Although in a few places our arguments may resemble those of Christ, our strategy and most elements of our proof are very different, and his techniques and ours can be seen as complementary. Indeed, as shown in a sequel to this paper [5], a combination of them with the results from [6, 7] (and several new ideas) makes it possible to prove that if $|A + B|^{1/n} - |A|^{1/n} - |B|^{1/n}$ is small relative to the measure of A and B , then both A and B are close, in a quantitative way, to dilations of the same convex set, yielding a proof of quantitative stability of the Brunn–Minkowski inequality for measurable sets.

The paper is structured as follows. In the next section we prove Theorem 1.1. Then in Section 3 we prove Theorem 1.2 by induction on dimension. The strategy is outlined at the beginning of Section 3. Some of the technical results used in the proof of Theorem 1.2 are collected in Appendix A.

2. The 1d case: Proof of Theorem 1.1

To prove Theorem 1.1, we first collect some preliminary results.

Definition 2.1. Let A, B be nonempty finite subsets of an additive group Z , and let $e \in A - B$. Then we define the e -transform of A and B as

$$A_{(e)} := A \cup (B + e), \quad B_{(e)} := B \cap (A - e).$$

The following are three simple properties of the e -transform (in this discrete context, $|A|$ denotes the cardinality of A):

$$A_{(e)} + B_{(e)} \subset A + B, \quad (2.1)$$

$$|A_{(e)}| + |B_{(e)}| = |A| + |B|, \quad (2.2)$$

$$|A_{(e)}| \geq |A|, \quad |B_{(e)}| \leq |B|. \quad (2.3)$$

The following is a classical result about sum of sets in \mathbb{Z}_p [1, 4], but for completeness we include its simple proof. A key fact used in the proof is that \mathbb{Z}_p has no nontrivial subgroups.

Lemma 2.2 (Cauchy–Davenport inequality). *If $Z = \mathbb{Z}_p$ with p prime, then*

$$|A + B| \geq \min\{|A| + |B| - 1, p\}.$$

Proof. The proof is by induction on the size of $|B|$, the case $|B| = 1$ being trivial. For the induction step, we consider two cases:

Case 1: There exists $e \in A - B$ such that $|B_e| < |B|$. Then by the inductive step

$$|A_{(e)} + B_{(e)}| \geq \min\{|A_{(e)}| + |B_{(e)}| - 1, p\},$$

and we conclude by (2.1) and (2.2).

Case 2: $|B_e| = |B|$ for any $e \in A - B$. This means that $B_{(e)} = B$ for any $e \in A - B$, which implies that $B + e \subset A$ for any $e \in A - B$, that is,

$$A + B - B \subset A.$$

Thus $B - B$ is contained inside $\text{Sym}_1(A) := \{h \in \mathbb{Z}_p : A + h = A\}$. (Notice that, since $|A + h| = |A|$, the inclusion $A + h \subset A$ is equivalent to the equality $A + h = A$.)

It is a general fact (easy to check) that $\text{Sym}_1(A)$ is a subgroup. Since in our case the only subgroups are $\{0\}$ and \mathbb{Z}_p , and we are assuming that $|B| > 1$, this means that $\text{Sym}_1(A) = \mathbb{Z}_p$, so $A = \mathbb{Z}_p$ and the result is true since $|A + B| \geq |A| = p$. \square

We can now prove the following important result, which is just a special case of Freiman’s “ $3k - 3$ Theorem” [8, 9].

Proposition 2.3. *Let A be a finite nonempty subset of \mathbb{Z} with $\min(A) = 0$ and $\max(A) = p$, with p prime. Assume that $|A + A| < 3|A| - 3$. Then*

$$|\{0, \dots, p\} \setminus A| \leq |A + A| - 2|A| + 1.$$

Proof. Since the cases $|A| = 1, 2$ are trivial, we can assume $|A| \geq 3$. We want to show that $p \leq |A + A| - |A|$.

Let $\phi_p : \mathbb{Z} \rightarrow \mathbb{Z}_p$ denote the canonical quotient map. We claim that

$$|\phi_p(A + A)| \leq |A + A| - |A|. \tag{2.4}$$

Indeed, $A + A$ can be written as the disjoint union of the three sets

$$(A + A) \cap [0, p - 1], \quad A + p, \quad ((A + A) \cap [p, 2p - 1]) \setminus (A + p),$$

hence

$$|A + A| - |A| \geq |(A + A) \cap [0, p - 1]| + |((A + A) \cap [p, 2p - 1]) \setminus (A + p)|. \tag{2.5}$$

In addition, since $\phi_p(A + p) = \phi_p(A) \subset \phi_p((A + A) \cap [0, p - 1])$ (because $0 \in A$), we have

$$\phi_p(A + A) = \phi_p((A + A) \cap [0, p - 1]) \cup \phi_p(((A + A) \cap [p, 2p - 1]) \setminus (A + p)),$$

which implies that

$$|\phi_p(A + A)| \leq |(A + A) \cap [0, p - 1]| + |((A + A) \cap [p, 2p - 1]) \setminus (A + p)|. \tag{2.6}$$

Combining (2.5) and (2.6), we obtain (2.4).

By (2.4) and the hypothesis $|A + A| < 3|A| - 3$, we get (observe that $|\phi_p(A)| = |A| - 1$)

$$|\phi_p(A + A)| < 2|A| - 3 = 2|\phi_p(A)| - 1,$$

so by the Cauchy–Davenport inequality (Lemma 2.2) we deduce that $|\phi_p(A + A)| = p$. Using (2.4) again, this gives $|A + A| - |A| \geq p$, concluding the proof. \square

We can now prove Theorem 1.1.

Proof of Theorem 1.1. We may assume that $|A + A| - 2|A| < |A|$, otherwise there is nothing to prove. After dilation and translation, we can also assume $(0, 1) \subset \text{co}(A) \subset [0, 1]$. We prove the result in two steps.

Step 1: A is compact. Since A is compact, so is $\text{co}(A)$, and so the inclusion $(0, 1) \subset \text{co}(A) \subset [0, 1]$ implies that $\text{co}(A) = [0, 1]$.

Take a sequence $\{p_k\}_{k \in \mathbb{N}}$ of prime numbers tending to infinity, and for each k define the family of closed intervals

$$I_{k,j} := \left[\frac{j}{p_k + 1}, \frac{j + 1}{p_k + 1} \right], \quad j = 0, \dots, p_k.$$

Consider now the sets

$$A_k := \bigcup_{j: I_{k,j} \cap A \neq \emptyset} I_{k,j}.$$

Observe that $A_k \supset A$ by construction. In addition, since A is compact (and so also $A + A$ is compact), one can easily check that

$$\bigcap_{k=1}^{\infty} \bigcup_{\ell=k}^{\infty} A_{\ell} = A, \quad \bigcap_{k=1}^{\infty} \bigcup_{\ell=k}^{\infty} (A_{\ell} + A_{\ell}) = A + A,$$

from which it follows that

$$\left| \bigcup_{\ell=k}^{\infty} A_{\ell} \right| \rightarrow |A|, \quad \left| \bigcup_{\ell=k}^{\infty} (A_{\ell} + A_{\ell}) \right| \rightarrow |A + A| \quad \text{as } k \rightarrow \infty.$$

In particular, since $A_k \supset A$ for any k , this implies

$$|A_k| \rightarrow |A|, \quad |A_k + A_k| \rightarrow |A + A| \quad \text{as } k \rightarrow \infty. \tag{2.7}$$

Since $|A + A| - 2|A| < |A|$ and $1/p_k \rightarrow 0$, it follows from (2.7) that

$$|A_k + A_k| - 2|A_k| < |A_k| - \frac{3}{p_k + 1} \tag{2.8}$$

for k sufficiently large.

Let us consider the sets

$$B_k := \{j \in \mathbb{Z} : I_{k,j} \subset A_k\}.$$

Recalling that $\text{co}(A) = [0, 1]$, it is easy to check that $\min(B_k) = 0$ and $\max(B_k) = p_k$. In addition, it follows immediately from (2.8) that $|B_k + B_k| < 3|B_k| - 3$. Hence we can apply Proposition 2.3 to deduce that

$$|\{0, \dots, p_k\} \setminus B_k| \leq |B_k + B_k| - 2|B_k| + 1,$$

which expressed in terms of A_k becomes

$$|[0, 1] \setminus A_k| \leq |A_k + A_k| - 2|A_k| + \frac{1}{p_k + 1}.$$

Letting $k \rightarrow \infty$ and using (2.7) proves the result when A is compact.

Step 2: A is a measurable set. This case will follow easily by inner approximation. Let $\{A_k\}_{k \in \mathbb{N}}$ be an increasing sequence of compact sets contained in A such that $|A_k| \rightarrow |A|$.

Also, let $\{a_k, b_k\}_{k \in \mathbb{N}} \subset A$ be sequences of points such that $[a_k, b_k] \rightarrow \text{co}(A)$ as $k \rightarrow \infty$. Then, up to replacing A_k with $A_k \cup \bigcup_{1 \leq j \leq k} \{a_j, b_j\}$, we can assume that $\text{co}(A_k) \supset [a_k, b_k]$, so that in particular

$$|\text{co}(A) \setminus \text{co}(A_k)| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

In addition, since $A_k \subset A$, we have $|A_k + A_k| \leq |A + A|$. Therefore, $|A + A| - 2|A| < |A|$ implies that $|A_k + A_k| - 2|A_k| < |A_k|$ for k sufficiently large. So, by Step 1,

$$|\text{co}(A_k) \setminus A_k| \leq |A_k + A_k| - 2|A_k| \leq |A + A| - 2|A_k|,$$

and letting $k \rightarrow \infty$ concludes the proof. □

3. The induction step: Proof of Theorem 1.2

Let \mathcal{H}^k denote the k -dimensional Hausdorff measure on \mathbb{R}^n . Denote by $(y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ a point in \mathbb{R}^n , and by $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ the canonical projection $\pi(y, t) := y$. Given $E \subset \mathbb{R}^n$ and $y \in \mathbb{R}^{n-1}$, we use the notation

$$E_y := E \cap \pi^{-1}(y). \tag{3.1}$$

We say that E is t -convex if E_y is a segment for every $y \in \pi(E)$.

Throughout the proof, C will denote a generic constant depending only on the dimension, which may change from line to line. The proof of the inductive step is long and involved, so we will divide it into several steps and substeps. First, we outline the strategy.

Notice that, as in the proof of Theorem 1.1, it suffices to consider only the case when A is compact, since the general case follows easily by inner approximation.

In Step 1 we replace A by a t -convex set A^* as follows. By applying Theorem 1.1 to the sets $\{A_y\}_{y \in \pi(A)}$, we deduce that most of these subsets of \mathbb{R} are close to their convex hull. Also, by Theorem 1.2 applied with $n - 1$, we can find a small constant $s_0 > 0$ such that the set $E_{s_0} := \{y : \mathcal{H}^1(A_y) > s_0\}$ is close to a convex set. This allows us to construct a t -convex set A^* whose sections consist of “vertical” segments $\text{co}(A_y) = \{y\} \times [a(y), b(y)]$ at most points y of E_{s_0} . We then show that A^* is close to A and has several other nice geometric properties. These properties lead, in Step 2, to the fact that the midpoints $c(y) := (a(y) + b(y))/2$ of the sections of A^* have bounded second differences as a function of y .

In Step 3, we show that, after an affine transformation of determinant 1, A^* can be assumed to be bounded. Observe that such transformations preserve the Lebesgue measure $|A|$, the deficit $\delta(A)$, and, of course, the property of convexity. To carry out the third step, the geometric properties of A^* once again play a crucial role. Near convexity of E_{s_0} immediately shows that there is a linear transformation in y that makes E_{s_0} bounded. Showing that one can subtract a linear function from $c(y)$ so that it is bounded is more complicated. We prove first that this is the case on a significant fraction of E_{s_0} using the

bound on second differences of c . Then, using this bound again allows us to control c at every point. A similar estimate already appeared in [2, 3], although here we need to use a different strategy to obtain quantitative bounds.

In Step 4 we show that A^* is close to a convex set. The proof relies not only on the geometric properties of A^* established in the preceding parts, but also on a further application of Theorem 1.2 with $n - 1$ to the level sets of the functions $a(y)$ and $b(y)$.

Finally, in the last step we show that such a convex set can be assumed to be the convex hull of A . This will conclude the proof.

Step 1: A is close to a t -convex set

We assume that we have already proved Theorem 1.2 through $n - 1$, and we want to show its validity for n . So, let $A \subset \mathbb{R}^n$ be a compact set (recall that, by inner approximation, it is sufficient to consider this case), and assume without loss of generality that $|A| = 1$. We show in this step that, if $\delta(A)$ is sufficiently small (the smallness depending only on the dimension), there exists a t -convex set $A^* \subset \mathbb{R}^n$ such that

$$|A^* \Delta A| \leq C\delta(A)^{1/2}.$$

(Here and below, $E \Delta F$ denotes the symmetric difference between E and F , that is, $E \Delta F = (E \setminus F) \cup (F \setminus E)$.)

Step 1a: Most of the sections A_y are close to their convex hull. Since $\delta(A)$ is finite, it follows from Lemma A.1 (applied with $k = n - 1$, $\pi = \pi_k$) that $\sup_{y \in \mathbb{R}^{n-1}} \mathcal{H}^1(A_y) < \infty$ (see (3.1)). Hence, up to a linear transformation of the form

$$(y, t) \mapsto (\lambda y, \lambda^{1-n}t), \quad \lambda > 0,$$

we can assume that

$$\sup_{y \in \mathbb{R}^{n-1}} \mathcal{H}^1(A_y) = 1. \quad (3.2)$$

(Observe that, since the transformation has determinant one, both $|A|$ and $\delta(A)$ are unchanged.) With this renormalization, using Lemma A.1 again, we deduce that

$$\mathcal{H}^{n-1}(\pi(A)) \leq 2^{n+1} \quad (3.3)$$

provided $\delta(A)$ is sufficiently small.

Let us write $\pi(A)$ as $F_1 \cup F_2$, where

$$F_1 := \{y \in \pi(A) : \mathcal{H}^1(A_y + A_y) - 2\mathcal{H}^1(A_y) < \mathcal{H}^1(A_y)\}, \quad F_2 := \pi(A) \setminus F_1. \quad (3.4)$$

Let us notice that, by definition, $A_y \subset \{y\} \times \mathbb{R} \subset \mathbb{R}^n$ (see (3.1)), so

$$2A_y \subset A_y + A_y \subset (A + A)_{2y}. \quad (3.5)$$

(Here and below we use $2A_y$ to denote the set $2(A_y)$, which by definition (1.1) is a subset

of $\{2y\} \times \mathbb{R}$.) Since by Fubini's Theorem

$$\int_{\mathbb{R}^{n-1}} \mathcal{H}^1((A + A)_{2y} \setminus 2A_y) dy = |A + A| - |2A| = 2^n \delta(A), \tag{3.6}$$

by (3.4), (3.5), and Theorem 1.1 applied to each set $A_y \subset \mathbb{R}$, we deduce that

$$\int_{F_1} \mathcal{H}^1(\text{co}(A_y) \setminus A_y) dy + \int_{F_2} \mathcal{H}^1(A_y) dy \leq 2^n \delta(A). \tag{3.7}$$

Let $F'_1 \subset F_1$ denote the set of $y \in F_1$ such that

$$\mathcal{H}^1(\text{co}(A_y) \setminus A_y) + \mathcal{H}^1((A + A)_{2y} \setminus 2A_y) \leq \delta(A)^{1/2}, \tag{3.8}$$

and notice that, by (3.6), (3.7), and Chebyshev's inequality,

$$\mathcal{H}^{n-1}(F_1 \setminus F'_1) \leq C \delta(A)^{1/2}. \tag{3.9}$$

Step 1b: Most of the levels sets $\{\mathcal{H}^1(A_y) > s\}$ are close to their convex hull. Here we apply the inductive step to the function $y \mapsto \mathcal{H}^1(A_y)$ to deduce that most of its level sets are almost convex. More precisely, let us define

$$E_s := \{y \in \mathbb{R}^{n-1} : \mathcal{H}^1(A_y) > s\}, \quad s > 0.$$

Observe that, because of (3.2), E_s is empty for $s > 1$. In addition, recalling (3.5), it is immediate to check that

$$E_s + E_s \subset \{y \in \mathbb{R}^{n-1} : \mathcal{H}^1((A + A)_{2y}) > 2s\},$$

so, by Fubini's Theorem and (3.6),

$$2 \int_0^1 (\mathcal{H}^{n-1}(E_s + E_s) - \mathcal{H}^{n-1}(2E_s)) ds \leq |A + A| - |2A| = 2^n \delta(A). \tag{3.10}$$

Let $\delta_{n-1} > 0$ be given by Theorem 1.2 with $n - 1$ in place of n , and let us partition $[0, 1]$ as $G_1 \cup G_2$, where

$$G_1 := \{s \in [0, 1] : \mathcal{H}^{n-1}(E_s + E_s) - 2^{n-1} \mathcal{H}^{n-1}(E_s) < 2^{n-1} \delta_{n-1} \mathcal{H}^{n-1}(E_s)\},$$

$$G_2 := [0, 1] \setminus G_1.$$

Then by Theorem 1.2 applied with $n - 1$ we get

$$c_{n-1}^{1/\alpha_{n-1}} \int_{G_1} \frac{\mathcal{H}^{n-1}(\text{co}(E_s) \setminus E_s)^{1/\alpha_{n-1}}}{\mathcal{H}^{n-1}(E_s)^{1/\alpha_{n-1}-1}} ds + \delta_{n-1} \int_{G_2} \mathcal{H}^{n-1}(E_s) ds \leq \delta(A). \tag{3.11}$$

Since $\mathcal{H}^{n-1}(E_s) \leq 2^{n+1}$ for any $s > 0$ (by (3.3)), $\int_0^1 \mathcal{H}^{n-1}(E_s) ds = |A| = 1$, and $s \mapsto \mathcal{H}^{n-1}(E_s)$ is a decreasing function, we easily deduce that

$$1/2 \leq \mathcal{H}^{n-1}(E_s) \leq 2^{n+1} \quad \forall s \in (0, 2^{-n-2}). \tag{3.12}$$

Thus, by (3.11),

$$\mathcal{H}^1(G_2 \cap (0, 2^{-n-2})) < C\delta(A).$$

Hence

$$\mathcal{H}^1(G_1 \cap [10\delta(A)^{1/2}, 20\delta(A)^{1/2}]) \geq 9\delta(A)^{1/2}$$

for $\delta(A)$ small enough, and it follows from (3.10)–(3.12) and Chebyshev's inequality that we can find a level $s_0 \in [10\delta(A)^{1/2}, 20\delta(A)^{1/2}]$ such that

$$\mathcal{H}^{n-1}((E_{s_0} + E_{s_0}) \setminus 2E_{s_0}) \leq C\delta(A)^{1/2}, \quad (3.13)$$

and

$$\mathcal{H}^{n-1}(\text{co}(E_{s_0}) \setminus E_{s_0})^{1/\alpha_{n-1}} \leq C\delta(A)^{1/2},$$

or equivalently

$$\mathcal{H}^{n-1}(\text{co}(E_{s_0}) \setminus E_{s_0}) \leq C\delta(A)^{\alpha_{n-1}/2}. \quad (3.14)$$

Let us define F_1'' to be a compact subset of $F_1' \cap E_{s_0}$ which satisfies

$$\mathcal{H}^{n-1}((F_1' \cap E_{s_0}) \setminus F_1'') \leq \delta(A)^{1/2}, \quad (3.15)$$

where E_{s_0} is as in (3.14). Notice that $\mathcal{H}^1(A_y) \geq 10\delta(A)^{1/2}$ for $y \in E_{s_0}$, so by (3.7) we get

$$10\mathcal{H}^{n-1}(F_2 \cap E_{s_0})\delta(A)^{1/2} \leq \int_{F_2 \cap E_{s_0}} \mathcal{H}^1(A_y) dy \leq 2^n \delta(A),$$

hence

$$\mathcal{H}^{n-1}(F_2 \cap E_{s_0}) \leq C\delta(A)^{1/2}. \quad (3.16)$$

Since

$$F_1'' \subset E_{s_0} \quad \text{and} \quad E_{s_0} \setminus F_1'' \subset (F_1 \setminus F_1') \cup ((F_1' \cap E_{s_0}) \setminus F_1'') \cup (F_2 \cap E_{s_0}),$$

combining (3.9), (3.14), (3.15), and (3.16), we obtain (observing that $\alpha_{n-1} \leq 1$)

$$\mathcal{H}^{n-1}(\text{co}(F_1'') \setminus F_1'') \leq \mathcal{H}^{n-1}(\text{co}(E_{s_0}) \setminus E_{s_0}) + \mathcal{H}^{n-1}(E_{s_0} \setminus F_1'') \leq C\delta(A)^{\alpha_{n-1}/2}. \quad (3.17)$$

Moreover, (3.9), (3.13), (3.15), and (3.16) give

$$\begin{aligned} \mathcal{H}^{n-1}((F_1'' + F_1'') \setminus 2F_1'') &\leq \mathcal{H}^{n-1}((E_{s_0} + E_{s_0}) \setminus 2E_{s_0}) + 2^{n-1}\mathcal{H}^{n-1}(E_{s_0} \setminus F_1'') \\ &\leq C\delta(A)^{1/2}. \end{aligned} \quad (3.18)$$

Step 1c: Construction of A^* . Let us define

$$A^* := \bigcup_{y \in F_1''} \text{co}(A_y) \subset \mathbb{R}^n.$$

Observe that, since $\mathcal{H}^1(A_y) \leq 20\delta(A)^{1/2}$ for $y \in F_1' \setminus E_{s_0}$, by (3.7), (3.2), (3.3), (3.9), and (3.15), we get

$$\begin{aligned} |A^* \Delta A| &\leq \int_{F_1''} \mathcal{H}^1(\text{co}(A_y) \setminus A_y) dy + \int_{F_2} \mathcal{H}^1(A_y) dy \\ &\quad + \int_{F_1' \setminus E_{s_0}} \mathcal{H}^1(A_y) dy + \int_{F_1 \setminus F_1'} \mathcal{H}^1(A_y) dy + \int_{(F_1' \cap E_{s_0}) \setminus F_1''} \mathcal{H}^1(A_y) dy \\ &\leq 2^n \delta(A) + 20 \delta(A)^{1/2} \mathcal{H}^{n-1}(\pi(A)) + \mathcal{H}^{n-1}(F_1 \setminus F_1') + \mathcal{H}^{n-1}((F_1' \cap E_{s_0}) \setminus F_1'') \\ &\leq C\delta(A)^{1/2}. \end{aligned} \tag{3.19}$$

Also, since $\mathcal{H}^1(A_y^*) = \mathcal{H}^1(\text{co}(A_y)) \leq 1 + \delta(A)^{1/2}$ for all $y \in F_1''$ (see (3.2) and (3.8)), it follows from (3.19) that

$$1 - C\delta(A)^{1/2} \leq |A^*| = \int_{F_1''} \mathcal{H}^1(A_y^*) dy \leq (1 + \delta(A)^{1/2}) \mathcal{H}^{n-1}(F_1''),$$

which implies in particular

$$\mathcal{H}^{n-1}(F_1'') \geq 1/2. \tag{3.20}$$

Step 2: The sections of A^* have controlled barycenter

Here we show that, if we write A_y^* as $\{y\} \times [a(y), b(y)]$ (recall that $A_y^* = \text{co}(A_y)$ is a segment) and we define $c : F_1'' \rightarrow \mathbb{R}$ to be the barycenter of A_y^* , that is,

$$c(y) := \frac{a(y) + b(y)}{2} \quad \forall y \in F_1'',$$

then

$$|c(y') + c(y'') - 2c(y)| \leq 6 \quad \forall y, y', y'' \in F_1'', y = (y' + y'')/2. \tag{3.21}$$

Step 2a: Some geometric properties of A^* . First of all, since

$$(A + A)_{2y} = \bigcup_{2y=y'+y''} (A_{y'} + A_{y''}),$$

by (3.8) we get

$$\mathcal{H}^1\left(\left(\bigcup_{2y=y'+y''} (A_{y'} + A_{y''})\right) \setminus 2A_y\right) \leq \delta(A)^{1/2} \quad \forall y \in F_1''. \tag{3.22}$$

Also, if we define the characteristic functions

$$\chi_y(t) := \begin{cases} 1 & \text{if } (y, t) \in A_y, \\ 0 & \text{otherwise,} \end{cases} \quad \chi_y^*(t) := \chi_{[a(y), b(y)]}(t) = \begin{cases} 1 & \text{if } (y, t) \in A_y^*, \\ 0 & \text{otherwise,} \end{cases}$$

then by (3.8) we have the following estimate on the convolution of the functions χ_y and χ_y^* :

$$\begin{aligned} \|\chi_{y'}^* * \chi_{y''}^* - \chi_{y'} * \chi_{y''}\|_{L^\infty(\mathbb{R})} &\leq \|\chi_{y''}^* - \chi_{y''}\|_{L^1(\mathbb{R})} + \|\chi_{y'}^* - \chi_{y'}\|_{L^1(\mathbb{R})} \\ &\leq \mathcal{H}^1(\text{co}(A_{y''}) \setminus A_{y''}) + \mathcal{H}^1(\text{co}(A_{y'}) \setminus A_{y'}) \\ &< 3\delta(A)^{1/2} \quad \forall y', y'' \in F_1''. \end{aligned} \tag{3.23}$$

Let us define $\bar{\pi} : \mathbb{R}^n \rightarrow \mathbb{R}$ to be the orthogonal projection onto the last component, that is, $\bar{\pi}(y, t) := t$, and denote by $[a, b]$ the interval $\bar{\pi}(A_{y'}^* + A_{y''}^*)$. Notice that, since by construction $\mathcal{H}^1(A_z) \geq 10\delta(A)^{1/2}$ for any $z \in F_1''$, this interval has length greater than $20\delta(A)^{1/2}$. Also, it is easy to check that the function $\chi_{y'}^* * \chi_{y''}^*$ is supported on $[a, b]$, has slope 1 (resp. -1) inside $[a, a + 3\delta(A)^{1/2}]$ (resp. $[b - 3\delta(A)^{1/2}, b]$), and it is greater than $3\delta(A)^{1/2}$ inside $[a + 3\delta(A)^{1/2}, b - 3\delta(A)^{1/2}]$. Since $\bar{\pi}(A_{y'} + A_{y''}) \supset \{\chi_{y'} * \chi_{y''} > 0\}$, by (3.23) we deduce that

$$\bar{\pi}(A_{y'} + A_{y''}) \supset [a + 3\delta(A)^{1/2}, b - 3\delta(A)^{1/2}]. \tag{3.24}$$

We claim that if $2y = y' + y''$ and $y, y', y'' \in F_1''$, then

$$[a(y') + a(y''), b(y') + b(y'')] \subset [2a(y) - 16\delta(A)^{1/2}, 2b(y) + 16\delta(A)^{1/2}]. \tag{3.25}$$

Indeed, if this were false, since $[a(y') + a(y''), b(y') + b(y'')] = \bar{\pi}(A_{y'}^* + A_{y''}^*) =: [a, b]$ is an interval of length at least $20\delta(A)^{1/2} \geq 16\delta(A)^{1/2}$, it follows that

$$\mathcal{H}^1([a, b] \setminus [2a(y), 2b(y)]) \geq 16\delta(A)^{1/2}.$$

This implies that

$$\mathcal{H}^1([a + 3\delta(A)^{1/2}, b - 3\delta(A)^{1/2}] \setminus [2a(y), 2b(y)]) \geq 10\delta(A)^{1/2},$$

so by (3.24) we get (recall that $[2a(y), 2b(y)] = \bar{\pi}(2A_y^*)$)

$$\mathcal{H}^1((A_{y'} + A_{y''}) \setminus 2A_y^*) \geq 10\delta(A)^{1/2}. \tag{3.26}$$

However, since $A_y^* = \text{co}(A_y) \supset A_y$, this contradicts (3.22), proving the claim (3.25).

Step 2b: Estimating the second differences of c . Because of (3.2) and (3.8), each set A_y^* is an interval of length at most 2. Hence (3.21) follows easily from (3.25).

Step 3: After a volume-preserving affine transformation, A^* is universally bounded

We show that there exist linear maps $T : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ and $L : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, with $\det(T) = 1$, and a point $(y_0, t_0) \in \mathbb{R}^n$, such that the image of A^* under the affine transformation

$$\mathbb{R}^{n-1} \times \mathbb{R} \ni (y, t) \mapsto (Ty, t - Ly) + (y_0, t_0) \tag{3.27}$$

is universally bounded. Notice that such a transformation has unit determinant.

Step 3a: After a volume-preserving affine transformation, $\pi(A^*)$ is universally bounded. We claim that, after an affine transformation of the form

$$\mathbb{R}^{n-1} \times \mathbb{R} \ni (y, t) \mapsto (Ty, t) + (y_0, 0)$$

with $T : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ linear and $\det(T) = 1$, we can assume that

$$B_r \subset \text{co}(F_1'') \subset B_{(n-1)r}, \tag{3.28}$$

where $B_r \subset \mathbb{R}^{n-1}$ is the $(n - 1)$ -dimensional ball of radius r centered at the origin, and $1/C_n < r < C_n$ for a constant C_n depending only on the dimension. Since $\pi(A^*) = F_1'' \subset \text{co}(F_1'')$, this proves in particular the boundedness of $\pi(A^*)$.

Indeed, by John’s Lemma [10] applied to the convex set $\text{co}(F_1'')$, there exist a linear map $T : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ with $\det(T) = 1$, and a point $y_0 \in \mathbb{R}^{n-1}$, such that

$$B_r \subset T(\text{co}(F_1'')) + y_0 \subset B_{(n-1)r}.$$

Since $\mathcal{H}^{n-1}(T(\text{co}(F_1''))) = \mathcal{H}^{n-1}(\text{co}(F_1'')) \geq \mathcal{H}^{n-1}(F_1'') \geq 1/2$ (see (3.20)), we deduce that r is universally bounded from below. For the upper bound note that, by (3.17) and (3.3), the volume of $\text{co}(F_1'')$ is bounded above, proving the claim.

Step 3b: Selecting some “good” points y_1, \dots, y_n inside F_1'' . We claim there exists a dimensional constant $c_n > 0$ such that the following holds: we can find n points $y_1, \dots, y_n \in B_r$, with $r > 0$ as in (3.28), such that the following conditions are satisfied:

- (a) All points y_1, \dots, y_n and $(y_1 + y_2)/2$ belong to F_1'' .
- (b) Let $\Sigma_i, i = 1, \dots, n$, denote the $(i - 1)$ -dimensional simplex generated by y_1, \dots, y_i , and define $\Sigma'_i := \frac{1}{2}(\Sigma_i + y_{i+1}), i = 1, \dots, n - 1$. Then

- (i) $\mathcal{H}^{i-1}(\Sigma_i) \geq c_n, \quad \frac{\mathcal{H}^{i-1}(\Sigma_i \cap F_1'')}{\mathcal{H}^{i-1}(\Sigma_i)} \geq 1 - \delta(A)^{\alpha_{n-1}/3}, \quad \forall i = 2, \dots, n;$
- (ii) $\frac{\mathcal{H}^{i-1}(\Sigma'_i \cap F_1'')}{\mathcal{H}^{i-1}(\Sigma'_i)} \geq 1 - \delta(A)^{\alpha_{n-1}/3} \quad \forall i = 2, \dots, n - 1.$

To see this, observe that $\mathcal{H}^{n-1}(B_r \setminus F_1'') \leq C\delta(A)^{\alpha_{n-1}/2}$ (by (3.17) and (3.28)), which implies that, for any point $y \in B_r \cap F_1''$,

$$\mathcal{H}^{n-1}(B_r \cap F_1'' \cap (2(B_r \cap F_1'') - y)) \geq \mathcal{H}^{n-1}(B_r) - 3C\delta(A)^{\alpha_{n-1}/2}.$$

Hence, for $\{y_i\}_{i=1,\dots,n}$ to satisfy (a) above, we only need to ensure that

$$y_i \in F_1'', \quad y_2 \in F_1'' \cap (2F_1'' - y_1),$$

while property (b) means that each simplex Σ_i has substantial measure (i.e., the points $\{y_i\}_{i=1,\dots,n}$ are not too close to each other), and most of the points in Σ_i and Σ_i' belong to F_1'' . Since $\Sigma_i' = \frac{1}{2}(\Sigma_i + y_{i+1})$, this is equivalent to the fact that Σ_i intersect $F_1'' \cap (2F_1'' - y_{i+1})$ in a large fraction.

Since all sets $\{F_1'' \cap (2F_1'' - y_i)\}_{i=1,\dots,n}$ cover B_r up to a set of measure $C\delta(A)^{\alpha_{n-1}/2} \ll \delta(A)^{\alpha_{n-1}/3}$, by a simple Fubini argument we can choose the points $\{y_i\}_{i=1,\dots,n}$ so that both (a) and (b) are satisfied (we leave the details to the reader).

Step 3c: A second volume-preserving affine transformation. Let y_1, \dots, y_n be the points constructed in Step 3b. We may apply an affine transformation of the form

$$\mathbb{R}^{n-1} \times \mathbb{R} \ni (y, t) \mapsto (y, t - Ly) + (0, t_0)$$

where $L : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ is linear and $t_0 \in \mathbb{R}$, in order to assume that

$$(y_k, 0) \in A^*, \quad k = 1, \dots, n.$$

We now prove that A^* is universally bounded.

Step 3d: A nontrivial fraction of A^* is bounded. We start by iteratively applying Lemma A.2(ii): Because $(y_i, 0) \in A^*$ for all $i = 1, \dots, n$ and A_y^* has length at most 2 for any $y \in F_1''$ (see (3.2) and (3.8)), we deduce that

$$|c(y_i)| \leq 1 \quad \forall i = 1, \dots, n.$$

(recall that $c(y_i)$ is the barycenter of $A_{y_i}^*$). Also, by (a) in Step 3b we know that $\frac{1}{2}(y_1 + y_2) \in F_1''$, and by (b)(i) most of the points in the segment $\Sigma_2 = [y_1, y_2]$ belong to F_1'' . So, thanks to (3.21), we can apply Lemma A.2(ii) to the function $[0, 1] \ni \tau \mapsto \frac{1}{3}c(\tau y_1 + (1 - \tau)y_2)$ and deduce that c is universally bounded on $\Sigma_2 \cap F_1''$.

We now use both (b)(i) and (b)(ii) to iterate this construction: since c is universally bounded on $\Sigma_2 \cap F_1''$ and at y_3 , for any point $z \in \Sigma_2 \cap F_1''$ such that $\frac{1}{2}(z + y_3) \in \Sigma_2' \cap F_1''$ (these are most of the points) we can apply again Lemma A.2(ii) to the function $[0, 1] \ni \tau \mapsto \frac{1}{3}c(\tau z + (1 - \tau)y_3)$ to deduce that c is universally bounded on the set $[z, y_3] \cap F_1''$. Hence, we have proved that c is universally bounded on the set

$$\Sigma_3'' := \bigcup_{z \in (\Sigma_2 \cap F_1'') \cap (2(\Sigma_2' \cap F_1'') - y_3)} [z, y_3] \cap F_1''.$$

Notice that, thanks to (b) above, $\mathcal{H}^2(\Sigma_3 \setminus \Sigma_3'') \leq C\delta(A)^{\alpha_{n-1}/3}$.

Continue to iterate this construction by picking a point $z \in \Sigma_3''$ such that $\frac{1}{2}(z + y_4) \in \Sigma_3'$, applying again Lemma A.2(ii) to the segments $[z, y_4]$, and so on. After $n - 1$ steps we finally obtain a set $\Sigma_n'' \subset F_1''$ such that $\mathcal{H}^{n-1}(\Sigma_n \setminus \Sigma_n'') \leq C\delta(A)^{\alpha_{n-1}/3}$ and c is universally bounded on Σ_n'' . Thanks to (a), this implies in particular that $\mathcal{H}^{n-1}(\Sigma_n'') \geq c_n/2$ provided $\delta(A)$ is sufficiently small.

Step 3e: A^* is bounded. Since A_y^* is a segment of length at most 2 for any $y \in F_1''$, we only need to prove that $c(y)$ is universally bounded for any $y \in F_1''$.

Fix $\bar{y} \in F_1''$. Since F_1'' is almost of full measure inside its convex hull (see (3.17)), $\text{co}(F_1'')$ is universally bounded (see Step 3a), and Σ_n'' is a simplex inside $\text{co}(F_1'')$ of nontrivial measure, by a simple Fubini argument we can find a point $\bar{y}' \in F_1'' \cap (2F_1'' - \bar{y})$ such that most of the points on the segment $[\bar{y}, \bar{y}']$ belong to F_1'' , and in addition $\mathcal{H}^1([\bar{y}, \bar{y}'] \cap \Sigma_n'') \geq c'_n$ for some dimensional constant $c'_n > 0$.

By applying Lemma A.2(i) to the function $[0, 1] \ni \tau \mapsto \frac{1}{3}c(\tau\bar{y} + (1 - \tau)\bar{y}')$, we deduce that $|c - \ell| \leq 3M$ on $[\bar{y}, \bar{y}'] \cap F_1''$ for some linear function ℓ . However, we already know that c is universally bounded on $[\bar{y}, \bar{y}'] \cap \Sigma_n''$, so ℓ is universally bounded there. Since this set has nontrivial measure, this implies that ℓ has to be universally bounded on the whole segment $[\bar{y}, \bar{y}']$ (since ℓ is a linear function). Hence c is universally bounded on $[\bar{y}, \bar{y}'] \cap F_1''$ as well, and this provides a universal bound for $c(\bar{y})$, concluding the proof.

Remark 3.1. From the boundedness of A^* we can easily prove that, after the affine transformation described above, A is bounded as well. More precisely, let $R > 0$ be such that $A^* \subset B_R$. We claim that (for $\delta(A)$ sufficiently small)

$$A \subset B_{3R}. \tag{3.29}$$

Indeed, we deduce from $|A \setminus A^*| \leq C\delta(A)^{1/2}$ (see (3.19)) that $|A \setminus B_R| \leq C\delta(A)^{1/2}$. Hence, if there is a point $x \in A \setminus B_{3R}$, since

$$\frac{A + A}{2} \supset (A \cap B_R) \cup \left(\frac{(A \cap B_R) + x}{2} \right),$$

and the two sets on the right hand side are disjoint, we get

$$\left| \frac{A + A}{2} \right| \geq (1 + 2^{-n})(1 - C\delta(A)^{1/2})|A|,$$

which implies $\delta(A) \geq 2^{-n-1}$, a contradiction.

Step 4: A^* is close to a convex set

We show that there exists a convex set $\mathcal{K} \subset \mathbb{R}^n$ such that

$$|\mathcal{K} \Delta A^*| \leq C\delta(A)^{\frac{\alpha_{n-1}\sigma_n}{8(n-1)}}. \tag{3.30}$$

Before beginning the proof, let us recall some of the main properties of A^* that we proved so far, and which will be used in the argument below.

First of all, A^* is a t -convex set of the form

$$A^* = \bigcup_{y \in F_1''} \{y\} \times [a(y), b(y)],$$

where F_1'' is compact (see (3.15)), it is close to its convex hull $\text{co}(F_1'')$ (see (3.17)), and $F_1'' + F_1''$ is even closer to $2F_1''$ (see (3.18)). In addition, by Step 3, up to an affine transformation as in (3.27) we can assume that $\text{co}(F_1'')$ is comparable to a ball whose radius is bounded from above and below by two dimensional constants (see (3.28)), and that $A^* \subset B_{(n-1)r} \times [-M, M]$ for some $M > 0$ universal. Finally, $a(y)$ and $b(y)$ satisfy (3.25).

In order to simplify the notation, we denote $\Omega := \text{co}(F_1'')$ and $F := F_1''$. Hence, by what we just said,

$$A^* = \bigcup_{y \in F} \{y\} \times [a(y), b(y)], \quad F \text{ compact},$$

$$\Omega = \text{co}(F), \quad \mathcal{H}^{n-1}(\Omega \setminus F) \leq C\delta(A)^{\alpha_{n-1}/2}, \tag{3.31}$$

$$\mathcal{H}^{n-1}((F + F) \setminus 2F) \leq C\delta(A)^{1/2}, \tag{3.32}$$

$$B_r \subset \Omega \subset B_{(n-1)r}, \quad 1/C_n < r < C_n, \tag{3.33}$$

$$-M \leq a(y) \leq b(y) \leq M \quad \forall y \in F, \tag{3.34}$$

and

$$a\left(\frac{y' + y''}{2}\right) - 8\delta(A)^{1/2} \leq \frac{a(y') + a(y'')}{2} \leq \frac{b(y') + b(y'')}{2} \leq b\left(\frac{y' + y''}{2}\right) + 8\delta(A)^{1/2} \tag{3.35}$$

whenever $y', y'', (y' + y'')/2 \in F$.

Our goal is to show that b (resp. a) is L^1 -close to a concave (resp. convex) function defined on Ω . The argument being completely symmetric, we focus just on b .

Step 4a: Making b uniformly concave at points that are well separated. Let $\beta \in (0, 1/6]$ to be fixed later, and define $\varphi : \Omega \rightarrow \mathbb{R}$ as

$$\varphi(y) := \begin{cases} b(y) + 2M - 20\delta(A)^\beta |y|^2, & y \in F, \\ 0, & y \in \Omega \setminus F. \end{cases} \tag{3.36}$$

Notice that, because of (3.34) and (3.35), we have $0 \leq \varphi \leq 3M$ and

$$\frac{\varphi(y') + \varphi(y'')}{2} \leq \varphi\left(\frac{y' + y''}{2}\right) + 8\delta(A)^{1/2} - 5\delta(A)^\beta |y' - y''|^2 \quad \forall y', y'', (y' + y'')/2 \in F,$$

which implies in particular that

$$\frac{\varphi(y') + \varphi(y'')}{2} \leq \varphi\left(\frac{y' + y''}{2}\right) + 8\delta(A)^{1/2} \quad \forall y', y'', (y' + y'')/2 \in F, \tag{3.37}$$

and (since $\beta \leq 1/6$)

$$\frac{\varphi(y') + \varphi(y'')}{2} < \varphi\left(\frac{y' + y''}{2}\right) - \delta(A)^\beta |y' - y''|^2$$

$$\forall y', y'', (y' + y'')/2 \in F, |y' - y''| \geq 2\delta(A)^\beta, \quad (3.38)$$

that is, φ is uniformly concave on points of F that are at least $2\delta(A)^\beta$ apart.

Step 4b: Constructing a concave function that should be close to φ . Let us take $\gamma \in (0, 1/4]$ to be fixed later, and define

$$\bar{\varphi}(y) := \min\{\varphi(y), h\},$$

where $h \in [0, 3M]$ is given by

$$h := \inf\{t > 0 : \mathcal{H}^{n-1}(\{\varphi > t\}) \leq \delta(A)^\gamma\}. \quad (3.39)$$

Since $0 \leq \varphi \leq 3M$, we get

$$\int_{\Omega} [\varphi(y) - \bar{\varphi}(y)] dy = \int_h^{3M} \mathcal{H}^{n-1}(\{\varphi > s\}) ds \leq 3M\delta(A)^\gamma. \quad (3.40)$$

Notice that $\bar{\varphi}$ still satisfies (3.37), and it also satisfies (3.38) whenever $\varphi((y' + y'')/2) = \bar{\varphi}((y' + y'')/2) < h$.

Finally, we define $\Phi : \Omega \rightarrow [0, h]$ to be the concave envelope of $\bar{\varphi}$, that is, the infimum among all linear functions that are above $\bar{\varphi}$ in Ω . Our goal is to show that Φ is L^1 -close to $\bar{\varphi}$ (and hence to φ).

Step 4c: The geometry of contact sets of supporting hyperplanes. Let y belong to the interior of Ω , and let L be the linear function representing the supporting hyperplane for Φ at y , that is, $L \geq \Phi$ in Ω , and $L(y) = \Phi(y)$.

Let $X := \{\Phi = L\} \cap \Omega$. Observe that X is a convex compact set (since Ω is convex and compact, being the convex hull of the compact set F) and $y \in X$. Since Φ is the concave envelope of $\bar{\varphi}$, by Carathéodory's theorem [11, Theorem 1.1.4] there are m points $y_1, \dots, y_m \in X$, with $m \leq n$, such that $y \in \text{co}(\{y_1, \dots, y_m\})$ and all y_j 's are contact points:

$$\Phi(y_j) = L(y_j) = \bar{\varphi}(y_j), \quad j = 1, \dots, m.$$

Observe that, because of (3.34) and (3.36), $\varphi > 0$ on F , and $\varphi = 0$ on $\Omega \setminus F$. We show next that $y_j \in F$ for all j .

Fix j . Since $\Omega = \text{co}(F)$ and F is compact, we can apply Carathéodory's theorem again to find ℓ points $z_1, \dots, z_\ell \in F$, with $\ell \leq n$, such that $y_j \in \text{co}(\{z_1, \dots, z_\ell\})$. But $z_i \in F$ implies $\Phi(z_i) \geq \bar{\varphi}(z_i) > 0$, and hence by concavity $\Phi(y_j) > 0$. It follows that $\bar{\varphi}(y_j) = \Phi(y_j) > 0$, and therefore $y_j \in F$.

In summary, every point in the interior of Ω belongs to a simplex S such that

$$S := \text{co}(\{y_1, \dots, y_m\}), \quad y_j \in \{\Phi = L = \bar{\varphi}\} \cap F, \quad m \leq n.$$

Step 4d: The set $\{\Phi = \bar{\varphi}\}$ is $K\delta(A)^\beta$ dense in $\Omega \setminus \text{co}(\{\bar{\varphi} > h - K\delta(A)^\beta\})$. Let $\beta \in (0, 1/6]$ be as in Step 4b. We claim that there exists a dimensional constant $K > 0$ such that the following hold, provided β is sufficiently small (the smallness depending only on the dimension): For any $y \in \Omega$ either

- there is $x \in \{\Phi = \bar{\varphi}\} \cap \Omega$ with $|y - x| \leq K\delta(A)^\beta$, or
- y belongs to the convex hull of the set $\{\bar{\varphi} > h - K\delta(A)^\beta\}$.

To prove this, we define

$$\Omega_\beta := \{y \in \Omega : \text{dist}(y, \partial\Omega) \geq \delta(A)^\beta\}.$$

Of course, up to enlarging the value of K , it suffices to consider the case when $y \in \Omega_\beta$.

So, let us fix $y \in \Omega_\beta$. Since Ω is a convex set comparable to a ball of unit size (see (3.33)) and Φ is a nonnegative concave function bounded by $3M$ inside Ω , there exists a dimensional constant C' such that, for every linear function $L \geq \Phi$ satisfying $L(y) = \Phi(y)$, we have

$$|\nabla L| \leq \frac{C'}{\delta(A)^\beta}. \tag{3.41}$$

By Step 4c, there are $m \leq n$ points $y_1, \dots, y_m \in F$ such that $y \in S := \text{co}(\{y_1, \dots, y_m\})$, and all y_j 's are contact points:

$$\Phi(y_j) = L(y_j) = \bar{\varphi}(y_j), \quad j = 1, \dots, m.$$

If the diameter of S is less than $K\delta(A)^\beta$, then its vertices are contact points within $K\delta(A)^\beta$ of y and we are done.

Hence, let us assume that the diameter of S is at least $K\delta(A)^\beta$. We claim that

$$\bar{\varphi}(y_i) > h - K\delta(A)^\beta \quad \forall i = 1, \dots, m. \tag{3.42}$$

Observe that, if we can prove (3.42), then

$$y \in S \subset \text{co}(\{\bar{\varphi} > h - K\delta(A)^\beta\}),$$

and we are done again.

It remains only to prove (3.42). To begin the proof, given $i \in \{1, \dots, m\}$, take $j \in \{1, \dots, m\}$ such that $|y_i - y_j| \geq K\delta(A)^\beta/2$ (such a j always exists because of the assumption on the diameter of S). We rename $i = 1$ and $j = 2$.

Let $N \in \mathbb{N}$ to be chosen later, and for $x \in \Omega$ define

$$H_N(x) := F \cap (2F - x) \cap (4F - 3x) \cap \dots \cap (2^N F - (2^N - 1)x) = \bigcap_{k=0}^N (2^k F - (2^k - 1)x).$$

Observe that, since Ω is convex,

$$\begin{aligned} \mathcal{H}^{n-1}(\Omega \setminus (2^k F - (2^k - 1)x)) &= 2^{k(n-1)} \mathcal{H}^{n-1}((2^{-k}\Omega + (1 - 2^{-k})x) \setminus F) \\ &\leq 2^{k(n-1)} \mathcal{H}^{n-1}(\Omega \setminus F), \end{aligned}$$

so, by (3.31),

$$\mathcal{H}^{n-1}(\Omega \setminus H_N(x)) \leq \sum_{k=0}^N 2^{k(n-1)} \mathcal{H}^{n-1}(\Omega \setminus F) \leq C 2^{N(n-1)} \delta(A)^{\alpha_{n-1}/2}. \quad (3.43)$$

Let $w_0 \in H_N(y_2)$, and define $w_k := \frac{1}{2}(y_2 + w_{k-1})$, $k = 1, \dots, N$. Since $w_0 \in H_N(y_2)$ we have

$$w_k = (1 - 2^{-k})y_2 + 2^{-k}w_0 \in F \quad \forall k = 0, \dots, N. \quad (3.44)$$

Then, since $y_2 \in F$ and by (3.44), we can apply iteratively (3.37) to get (recall that $0 \leq \bar{\varphi} \leq 3M$)

$$\begin{aligned} \bar{\varphi}(w_N) &\geq \bar{\varphi}(y_2)/2 + \bar{\varphi}(w_{N-1})/2 - 8\delta(A)^{1/2} \\ &\geq (1 - 1/4)\bar{\varphi}(y_2) + \bar{\varphi}(w_{N-2})/4 - (1 + 1/2)8\delta(A)^{1/2} \geq \dots \\ &\geq (1 - 2^{-N})\bar{\varphi}(y_2) + 2^{-N}\bar{\varphi}(w_0) - 16\delta(A)^{1/2} \\ &\geq (1 - 2^{-N})\bar{\varphi}(y_2) - 16\delta(A)^{1/2} \geq \bar{\varphi}(y_2) - C(2^{-N} + \delta(A)^{1/2}). \end{aligned} \quad (3.45)$$

In addition, since the diameter of F is bounded (see (3.31) and (3.33)), $|w_N - y_2| \leq C 2^{-N}$.

Let us choose N such that $2^N = c' \delta(A)^{-\frac{\alpha_{n-1}}{2(n-1)}}$ for some small dimensional constant $c' > 0$. In this way, from (3.33) and (3.43) we get $\mathcal{H}^{n-1}(H_N(y_2)) \geq c_n/2$, which implies

$$\mathcal{H}^{n-1}((1 - 2^{-N})y_2 + 2^{-N}H_N(y_2)) \geq 2^{-(n-1)N} c_n/2 = \frac{c_n}{2(c')^{n-1}} \delta(A)^{\alpha_{n-1}/2}.$$

Hence, since by convexity of Ω and (3.31),

$$\mathcal{H}^{n-1}(\Omega \setminus (F \cap (2F - y_1))) \leq 3\mathcal{H}^{n-1}(\Omega \setminus F) \leq 3C\delta(A)^{\alpha_{n-1}/2},$$

we see that the set $F \cap (2F - y_1) \cap ((1 - 2^{-N})y_2 + 2^{-N}H_N(y_2))$ is nonempty provided c' is sufficiently small. So, let x_2 be an arbitrary point inside this set. Observe that, with this choice,

$$|x_2 - y_2| \leq C\delta(A)^{\frac{\alpha_{n-1}}{2(n-1)}}, \quad z_1 := \frac{y_1 + x_2}{2} \in F.$$

We now prove (3.42): Since L has gradient of order at most $C\delta(A)^{-\beta}$ (see (3.41)), we have

$$|L(z_1) - L((y_1 + y_2)/2)| \leq C\delta(A)^{-\beta} |x_2 - y_2| \leq C\delta(A)^{\frac{\alpha_{n-1}}{2(n-1)} - \beta}.$$

Hence, since y_1 and y_2 are contact points and $L \geq \bar{\varphi}$, using (3.45) we get

$$\begin{aligned} \frac{\bar{\varphi}(y_1) + \bar{\varphi}(x_2)}{2} &\geq \frac{\bar{\varphi}(y_1) + \bar{\varphi}(y_2)}{2} - C\delta(A)^{\frac{\alpha_{n-1}}{2(n-1)}} = L((y_1 + y_2)/2) - C\delta(A)^{\frac{\alpha_{n-1}}{2(n-1)}} \\ &\geq L(z_1) - C\delta(A)^{\frac{\alpha_{n-1}}{2(n-1)} - \beta} \geq \bar{\varphi}(z_1) - C\delta(A)^{\frac{\alpha_{n-1}}{2(n-1)} - \beta}. \end{aligned} \quad (3.46)$$

We now claim that, for some suitable choice of $K > 0$ and $\beta \in (0, \frac{\alpha_{n-1}}{4(n-1)}]$, we can infer that $\bar{\varphi}(z_1) = h$. Observe that, if we can do so, then since $\bar{\varphi} \leq h$ it follows immediately from (3.46) that both $\bar{\varphi}(y_1)$ and $\bar{\varphi}(x_2)$ have to be greater than $h - C\delta(A)^{\frac{\alpha_{n-1}}{2(n-1)} - \beta} \geq h - C\delta(A)^\beta$, proving (3.42).

So, let us show that $\bar{\varphi}(z_1) = h$. If not, we could apply (3.38) with $y' = y_1, y'' = x_2$, and $\varphi = \bar{\varphi}$, to get

$$\bar{\varphi}(z_1) \geq \frac{1}{2}(\bar{\varphi}(y_1) + \bar{\varphi}(x_2)) + \delta(A)^\beta |y_1 - x_2|^2.$$

Since $|y_1 - x_2| \geq |y_1 - y_2|/2 \geq K\delta(A)^\beta/4$, this implies that

$$\bar{\varphi}(z_1) \geq \frac{1}{2}(\bar{\varphi}(y_1) + \bar{\varphi}(x_2)) + \frac{1}{16}K^2\delta(A)^{3\beta},$$

which contradicts (3.46) provided we choose $\beta = \frac{\alpha_{n-1}}{8(n-1)}$ and K sufficiently large.

This concludes the proof with the choice

$$\beta := \frac{\alpha_{n-1}}{8(n-1)}. \tag{3.47}$$

Step 4e: Most of the level sets of $\bar{\varphi}$ are close to their convex hull. Let us now define the nonnegative function $\psi : \Omega \rightarrow \mathbb{R}$ as

$$\psi(y) := \sup_{y'+y''=2y, y', y'' \in \Omega} \min\{\bar{\varphi}(y'), \bar{\varphi}(y'')\}.$$

Notice that:

- $0 \leq \bar{\varphi} \leq \psi$ (just pick $y' = y'' = y$);
- $\psi \leq \bar{\varphi} + 8\delta(A)^{1/2}$ on F , and $\psi = 0$ outside $(F + F)/2$ (since by (3.36) we have $\min\{\bar{\varphi}(y'), \bar{\varphi}(y'')\} = 0$ unless both y' and y'' belong to F , and then use (3.37));
- $\psi \leq h$ (since $\bar{\varphi} \leq h$).

Thus, thanks to (3.32) we get

$$\begin{aligned} \int_{\Omega} \bar{\varphi}(y) dy &\leq \int_{\Omega} \psi(y) dy = \int_{(F+F)/2} \psi(y) dy \\ &\leq \int_F \bar{\varphi}(y) dy + 8\delta(A)^{1/2} \mathcal{H}^{n-1}(F) + h \mathcal{H}^{n-1}((F + F) \setminus 2F) \\ &\leq \int_{\Omega} \bar{\varphi}(y) dy + C\delta(A)^{1/2}. \end{aligned} \tag{3.48}$$

Since $\{\psi > s\} \supset (\{\bar{\varphi} > s\} + \{\bar{\varphi} > s\})/2$, we can apply Theorem 1.2 with $n - 1$ to the level sets of $\bar{\varphi}$: if we define

$$\begin{aligned} H_1 &:= \{s : \mathcal{H}^{n-1}(\{\psi > s\}) - \mathcal{H}^{n-1}(\{\bar{\varphi} > s\}) < \delta_{n-1} \mathcal{H}^{n-1}(\{\bar{\varphi} > s\})\}, \\ H_2 &:= [0, h] \setminus H_1, \end{aligned}$$

by (3.48) and Fubini’s Theorem we get

$$c_{n-1}^{1/\alpha_{n-1}} \int_{H_1} \frac{\mathcal{H}^{n-1}(\text{co}(\{\bar{\varphi} > s\}) \setminus \{\bar{\varphi} > s\})^{1/\alpha_{n-1}}}{\mathcal{H}^{n-1}(\{\bar{\varphi} > s\})^{1/\alpha_{n-1}-1}} ds + \delta_{n-1} \int_{H_2} \mathcal{H}^{n-1}(\{\bar{\varphi} > s\}) ds \leq C\delta(A)^{1/2},$$

Recalling the definition of h (see (3.39)), we have

$$\delta(A)^\gamma \leq \mathcal{H}^{n-1}(\{\bar{\varphi} > s\}) \leq \mathcal{H}^{n-1}(\Omega) \leq C, \quad 0 \leq s < h.$$

Thus

$$\mathcal{H}^1(H_2) \leq C\delta(A)^{1/2-\gamma} \leq C\delta(A)^{1/4} \tag{3.49}$$

(recall that, by assumption, $\gamma \leq 1/4$), and

$$\int_{H_1} \mathcal{H}^{n-1}(\text{co}(\{\bar{\varphi} > s\}) \setminus \{\bar{\varphi} > s\})^{1/\alpha_{n-1}} ds \leq C\delta(A)^{1/2},$$

so by Hölder’s inequality (notice that $1/\alpha_{n-1} \geq 1$)

$$\int_{H_1} \mathcal{H}^{n-1}(\text{co}(\{\bar{\varphi} > s\}) \setminus \{\bar{\varphi} > s\}) ds \leq C\delta(A)^{\alpha_{n-1}/2}. \tag{3.50}$$

Step 4f: b is L^1 -close to a concave function. Notice that, since the sets $\{\bar{\varphi} > s\}$ are decreasing in s , so are their convex hulls $\text{co}(\{\bar{\varphi} > s\})$. Hence, we can define a new function $\xi : \Omega \rightarrow \mathbb{R}$ with convex level sets given by

$$\{\xi > s\} := \begin{cases} \text{co}(\{\bar{\varphi} > s\}) & \text{if } s \in H_1, \\ \bigcap_{\tau \in H_1, \tau < s} \text{co}(\{\bar{\varphi} > \tau\}) & \text{if } s \in H_2, \end{cases}$$

Then by (3.49) and (3.50) we see that ξ satisfies

$$0 \leq \bar{\varphi} \leq \xi, \quad \int_{\Omega} |\xi - \bar{\varphi}| \leq C\delta(A)^{\alpha_{n-1}/2}, \quad \xi \leq \Phi \text{ (the convex envelope of } \bar{\varphi}\text{)}. \tag{3.51}$$

Also, because of (3.39), we see that

$$\mathcal{H}^{n-1}(\{\xi > s\}) \geq \delta(A)^\gamma \quad \forall 0 \leq s < h. \tag{3.52}$$

Since by Step 4d the contact set $\{\Phi = \bar{\varphi}\}$ is $K\delta(A)^\beta$ -dense outside the set

$$\text{co}(\{\bar{\varphi} > h - K\delta(A)^\beta\}) \subset \{\xi > h - K\delta(A)^\beta\},$$

the same is true for the contact set $\{\Phi = \xi\}$.

We claim that there exist dimensional constants $K', \eta > 0$ such that, for any $0 \leq s < h - K\delta(A)^\beta$, each level set $\{\Phi > s\}$ is contained in a $K'\delta(A)^\eta$ -neighborhood of $\{\xi > s\}$.

Indeed, if this were not the case, we could find a point $y \in \{\Phi > s\}$ such that $\text{dist}(y, \{\xi > s\}) > K'\delta(A)^\eta$. We now distinguish between the cases $n = 2$ and $n \geq 3$.

If $n = 2$ the sets $\{\xi > s\}$ and $\{\Phi > s\}$ are both intervals, so we can find a point $z \in \{\Phi > s\} \setminus \{\xi > s\}$ such that $|y - z| \geq K'\delta(A)^\eta$ and the segment $[y, z]$ does not intersect $\{\xi > s\}$. Hence no contact points can be inside $[y, z]$, which contradicts the density of $\{\Phi = \bar{\varphi}\}$ provided $\eta \leq \beta$ and $K' > K$.

If $n \geq 3$, since $\{\xi > s\}$ is a (universally) bounded convex set in \mathbb{R}^{n-1} , by (3.52) we deduce that it contains an $(n - 1)$ -dimensional ball $B_\rho(y')$ with $\rho := c\delta(A)^\gamma$ for some dimensional constant $c > 0$. By convexity of $\{\Phi > s\}$, this implies that

$$C := \text{co}(\{y\} \cup B_\rho(y')) \subset \{\Phi > s\}.$$

Thanks to the fact that $\text{dist}(y, \{\xi > s\}) > K'\delta(A)^\eta$ and $\text{diam}(C) \leq C$, we can find an $(n - 1)$ -dimensional ball $B_r(z) \subset C$ such that $B_r(z) \cap \{\xi > s\} = \emptyset$, where $r := c'K'\delta(A)^{\eta+\gamma}$ and $c' > 0$ is a (small) dimensional constant. Hence no contact points can be inside $B_r(z)$, and this contradicts the density of $\{\Phi = \bar{\varphi}\}$ provided $\eta + \gamma \leq \beta$ and $c'K' > K$.

In conclusion, the claim holds with the choices

$$\eta := \begin{cases} \beta & \text{if } n = 2, \\ \beta - \gamma & \text{if } n \geq 3, \end{cases} \quad K' := \frac{2K}{c'}. \tag{3.53}$$

Since all level sets of ξ are (universally) bounded convex sets, as a consequence of the claim we deduce that

$$\mathcal{H}^{n-1}(\{\Phi > s\}) \leq \mathcal{H}^{n-1}(\{\xi > s\}) + C\delta(A)^\eta \quad \forall s \in [0, h - K\delta(A)^\beta].$$

In addition, since $\xi \leq \Phi \leq h$, we obviously have $|\Phi - \xi| \leq K\delta(A)^\beta$ inside the set $\{\xi > h - K\delta(A)^\beta\}$. Hence, by Fubini's Theorem,

$$\int_\Omega |\Phi - \xi| \leq C\delta(A)^\eta$$

(observe that, because of (3.53), $\eta \leq \beta$). Since $\eta \leq \alpha_{n-1}/2$ (see (3.47) and (3.53)), combining this estimate with (3.51) we get

$$\int_\Omega |\Phi - \bar{\varphi}| \leq C\delta(A)^\eta.$$

In addition, since by construction $|\varphi(y) - 2M - b(y)| \leq 20\delta(A)^\beta$ inside F (see (3.36)), by (3.40) we have

$$\begin{aligned} \int_F |\bar{\varphi}(y) - 2M - b(y)| dy &\leq \int_F |\bar{\varphi}(y) - \varphi(y)| dy + \int_F |\varphi(y) - 2M - b(y)| dy \\ &\leq C(\delta(A)^\gamma + \delta(A)^\beta). \end{aligned}$$

All in all, combining the two inequalities above, we see that

$$\int_F |\Phi(y) - 2M - b(y)| dy \leq C(\delta(A)^\gamma + \delta(A)^\eta).$$

We now finally fix γ and η : recalling (3.53) and (3.47), by choosing

$$\gamma = \eta := \sigma_n \beta \quad \text{with} \quad \sigma_n := \begin{cases} 1 & \text{if } n = 2, \\ 1/2 & \text{if } n \geq 3, \end{cases} \tag{3.54}$$

we obtain

$$\int_F |\Phi(y) - 2M - b(y)| dy \leq C\delta(A)^{\frac{\alpha_{n-1}\sigma_n}{8(n-1)}}. \tag{3.55}$$

Applying the symmetric argument to $a(y)$, we find a convex function $\Psi : \Omega \rightarrow [-3M, 0]$ such that

$$\int_F |\Psi(y) + 2M - a(y)| dy \leq C\delta(A)^{\frac{\alpha_{n-1}\sigma_n}{8(n-1)}}. \tag{3.56}$$

Step 4g: Conclusion of the argument. Let us define the convex set

$$\mathcal{K} := \{(y, t) \in \Omega \times \mathbb{R} : \Psi(y) + 2M \leq t \leq \Phi(y) - 2M\} \subset \Omega \times [-M, M].$$

Then, using (3.55), (3.56), (3.31), and (3.34), we get

$$\begin{aligned} |\mathcal{K} \Delta A^*| &\leq \int_F |\Phi(y) - 2M - b(y)| dy + \int_F |\Psi(y) + 2M - a(y)| dy + 2M\mathcal{H}^{n-1}(\Omega \setminus F) \\ &\leq C\delta(A)^{\frac{\alpha_{n-1}\sigma_n}{8(n-1)}}. \end{aligned}$$

This concludes Step 4.

Step 5: conclusion of the proof

Combining (3.19) and (3.30), we see that there exists a convex set \mathcal{K} such that $|\mathcal{K} \Delta A| \leq C\delta(A)^{\frac{\alpha_{n-1}\sigma_n}{8(n-1)}}$.

Observe that, by John’s Lemma [10], after replacing both \mathcal{K} and A by $L(\mathcal{K})$ and $L(A)$, where $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an affine transformation with $\det(L) = 1$, we can assume that $L(K) \subset B_R$ for some R depending only on the dimension.

Also, after replacing \mathcal{K} with $\mathcal{K} \cap \text{co}(A)$ (which decreases the measure of the symmetric difference between \mathcal{K} and A), we can assume that $\mathcal{K} \subset \text{co}(A)$.

Following the argument used in the proof of [2, Lemma 13.3], we now estimate $|\text{co}(A) \setminus \mathcal{K}|$. Indeed, let $x \in A \setminus \mathcal{K}$, denote by $x' \in \partial\mathcal{K}$ the closest point in \mathcal{K} to x , set $\rho := |x - x'| = \text{dist}(x, \mathcal{K})$, and let $v \in \mathbb{S}^{n-1}$ be the unit normal to a supporting hyperplane to \mathcal{K} at x' , that is,

$$(z - x') \cdot v \leq 0 \quad \forall z \in \mathcal{K}.$$

Let us define $\mathcal{K}_\rho := \{z \in \mathcal{K} : (z - x') \cdot v \geq -\rho\}$. Observe that, since \mathcal{K} is a bounded convex set with volume close to 1, $|\mathcal{K}_\rho| \geq c\rho^n$ for some dimensional constant $c > 0$. Since $x \in A$ we have

$$\frac{A + A}{2} \supset \frac{x + (\mathcal{K}_\rho \cap A)}{2} \cup (A \cap \mathcal{K}),$$

and the two sets on the right hand side are disjoint. This implies that

$$\delta(A) + |A| = \left| \frac{A + A}{2} \right| \geq c2^{-n} \rho^n - |\mathcal{K}_\rho \setminus A| + |A \cap \mathcal{K}| \geq c2^{-n} \rho^n + |A| - C\delta(A)^{\frac{\alpha_{n-1}\sigma_n}{8(n-1)}},$$

from which we deduce that

$$\rho \leq C\delta(A)^{\frac{\alpha_{n-1}\sigma_n}{8n(n-1)}}.$$

Since x is arbitrary, this implies that A is contained inside the $C\delta(A)^{\frac{\alpha_{n-1}\sigma_n}{8n(n-1)}}$ -neighborhood of \mathcal{K} . By convexity, also $\text{co}(A)$ has to be contained inside such a neighborhood, thus

$$|\text{co}(A) \setminus \mathcal{K}| \leq C\delta(A)^{\frac{\alpha_{n-1}\sigma_n}{8n(n-1)}}.$$

Combining all the estimates together, we conclude that

$$|\text{co}(A) \setminus A| \leq C\delta(A)^{\alpha_n}, \quad \alpha_n := \frac{\alpha_{n-1}\sigma_n}{8n(n-1)}.$$

Recalling the definition of σ_n in (3.54), since $\alpha_1 = 1$ (by Theorem 1.1) we deduce that $\alpha_n = \frac{1}{8 \cdot 16^{n-2} n!(n-1)!}$, as desired.

Remark 3.2. For use in the sequel [5], we observe that if $\delta(A) \leq \delta_n$, then there exists a convex set \mathcal{K} such that

$$\delta(A)^{n\alpha_n} \geq c_n \frac{|\mathcal{K} \Delta A|}{|A|}, \quad \alpha_n := \frac{1}{8 \cdot 16^{n-2} n!(n-1)!}. \tag{3.57}$$

In other words, if we only want to show that A is close to some convex set (which may be different from $\text{co}(A)$) the exponent in our stability estimate can be improved by a factor n . This is a direct consequence of Steps 1–4, although Step 5 is still essential to close the induction argument (for instance, in Step 4f, the fact that the sets $\{\bar{\varphi} > s\}$ are close to their convex hulls is used in a crucial way).

Appendix A. Technical results

Lemma A.1. *Let $A \subset \mathbb{R}^n$ be a nonempty measurable set, write $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, denote by $\pi_k : \mathbb{R}^n \rightarrow \mathbb{R}^k$ the canonical projection, and for $y \in \mathbb{R}^k$ set $A_y := A \cap \pi_k^{-1}(y)$. Then*

$$\left(\sup_{y \in \mathbb{R}^k} \mathcal{H}^{n-k}(A_y) \right) \mathcal{H}^k(\pi_k(A)) \leq 2^n (1 + \delta(A)) |A|.$$

Proof. Let $\{y_j\}_{j \in \mathbb{N}}$ be a sequence of points such that $\mathcal{H}^{n-k}(A_{y_j}) \rightarrow \sup_{y \in \mathbb{R}^k} \mathcal{H}^{n-k}(A_y)$. Then, since $A + A \supset (\{y_j\} \times A_{y_j}) + A$, we get

$$2^n (1 + \delta(A)) \geq |A + A| \geq \mathcal{H}^{n-k}(A_{y_j}) \mathcal{H}^k(\pi_k(A)),$$

and the estimate follows by letting $j \rightarrow \infty$. □

Lemma A.2. *Let $E \subset \mathbb{R}$, and let $f : E \rightarrow \mathbb{R}$ be a bounded measurable function such that*

$$\left| \frac{f(m') + f(m'')}{2} - f\left(\frac{m' + m''}{2}\right) \right| \leq 1 \quad \forall m', m'', (m' + m'')/2 \in E. \quad (\text{A.1})$$

Assume that there exist $m_1, m_2 \in \mathbb{R}$ such that $m_1, m_2, (m_1 + m_2)/2 \in E$, and $|E \cap [m_1, m_2]| \geq (1 - \varepsilon)|m_2 - m_1|$. Then the following hold provided ε is sufficiently small (the smallness being universal):

(i) *There exist a linear function $\ell : [m_1, m_2] \rightarrow \mathbb{R}$ and a universal constant M such that*

$$|f - \ell| \leq M \quad \text{in } E \cap [m_1, m_2].$$

(ii) *If in addition $|f(m_1)| + |f(m_2)| \leq K$ for some constant K , then $|f| \leq K + M$ inside E .*

Proof. Without loss of generality, we can assume $[m_1, m_2] = [-1, 1]$ and $E \subset [-1, 1]$. Given numbers $a \in \mathbb{R}$ and $b > 0$, we write $a = O(b)$ if $|a| \leq Cb$ for some universal constant C .

To prove (i), let us define

$$\ell(m) := \frac{f(1) - f(-1)}{2}m + \frac{f(1) + f(-1)}{2},$$

and set $F := f - \ell$. Observe that $F(-1) = F(1) = 0$, and F still satisfies (A.1). Hence, since by assumption $-1, 0, 1 \in E$, by (A.1) we get $|F(0)| \leq 1$. Let us extend F to the whole interval $[-1, 1]$ as $F(y) = 0$ if $y \notin E$, and set

$$M := \sup_{y \in E} |F(y)|.$$

We want to show that M is universally bounded.

Averaging (A.1) (applied to F in place of f) with respect to $m'' \in E$ and using $|E \cap [-1, 1]| \geq 2(1 - \varepsilon)$, we easily obtain the bound

$$F(m') = -\frac{1}{2} \int_{-1}^1 F(m) dm + 2 \int_{(m'-1)/2}^{(m'+1)/2} F(m) dm + O(1) + O(\varepsilon M).$$

Observe now that, since $|F(0)| \leq 1$, by (A.1) applied with $f = F$, $m' = m$, and $m'' = -m$, we get

$$|F(m) + F(-m)| \leq 4 \quad \forall m \in E \cap -E.$$

Since $|[-1, 1] \setminus (E \cap -E)| \leq 4\varepsilon$ and $|F| \leq M$, we deduce that

$$\int_{-1}^1 F(m) dm = \int_{E \cap -E} F(m) dm + \int_{[-1, 1] \setminus (E \cap -E)} F(m) dm = O(1) + O(\varepsilon M),$$

hence

$$F(m') = 2 \int_{(m'-1)/2}^{(m'+1)/2} F(m) dm + O(1) + O(\varepsilon M) \quad \forall m' \in E.$$

As a consequence, if $m' < m''$,

$$\begin{aligned} |F(m') - F(m'')| &\leq 2 \int_{(m'-1)/2}^{(m''-1)/2} |F(m)| dm + 2 \int_{(m'+1)/2}^{(m''+1)/2} |F(m)| dm + O(1) + O(\varepsilon M) \\ &= O(M|m' - m''|) + O(1) + O(\varepsilon M) \quad \forall m', m'' \in E. \end{aligned} \quad (\text{A.2})$$

Now pick a point $m_0 \in E$ such that

$$|F(m_0)| \geq M - 1. \quad (\text{A.3})$$

Since $|F|$ is already bounded by 1 at $-1, 0, 1$ we can assume that $m_0 \neq -1, 0, 1$ (otherwise there is nothing to prove). Without loss of generality we can suppose that $m_0 \in (-1, 0)$. Then, since $|[-1, 0] \setminus ((E-1)/2)| \leq \varepsilon$ and $|[-1, 0] \setminus E| \leq 2\varepsilon$, we can find a point

$$m_1 \in [-1, 0] \cap ((E-1)/2) \cap E \cap [m_0 - 4\varepsilon, m_0 + 4\varepsilon],$$

and by (A.2) and (A.3) we get

$$|F(m_1)| \geq M - O(1) - O(\varepsilon M).$$

Hence, applying (A.1) to $m' = -1$ and $m'' = 2m_1 + 1$ (observe that $m'' \in E$ because $m_1 \in (E-1)/2$), since $F(m') = 0$ and $|F(m'')| \leq M$, we get

$$M - O(1) - O(\varepsilon M) \leq |F(m_1)| \leq \left| F(m_1) - \frac{F(m') + F(m'')}{2} \right| + \frac{M}{2} \leq 1 + \frac{M}{2},$$

which proves that M is universally bounded provided ε is sufficiently small (the smallness being universal). This proves (i).

To prove (ii), it suffices to observe that if $|f(-1)| + |f(1)| \leq K$, then $|\ell| \leq K$ and we get

$$|f| \leq |\ell| + |F| \leq K + M. \quad \square$$

Appendix B. A qualitative version of Theorem 1.2

In [2, 3] M. Christ proved the following result: Let $A, B \subset \mathbb{R}^n$ be two measurable sets such that $|A|$ and $|B|$ are both uniformly bounded away from zero and infinity. For any $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$|A + B|^{1/n} \leq |A|^{1/n} + |B|^{1/n} + \delta,$$

then there exist a compact convex set \mathcal{K} , scalars $\alpha, \beta > 0$, and vectors $a, b \in \mathbb{R}^n$ such that

$$A \subset \alpha\mathcal{K} + a, \quad B \subset \beta\mathcal{K} + b, \quad |(\alpha\mathcal{K} + a) \setminus A| \leq \varepsilon, \quad |(\beta\mathcal{K} + b) \setminus B| \leq \varepsilon.$$

In the particular case when $A = B$ this result says that $|\text{co}(A) \setminus A|/|A| \rightarrow 0$ as $\delta(A) \rightarrow 0$. Here, following the ideas in [2], we show that this last result follows very easily once one

has proved that A is bounded (which amounts to Steps 1–3 in our proof). In particular, if one is only interested in a qualitative statement, the following simple argument allows one to skip Step 4.

Let $A \subset \mathbb{R}^n$ be a measurable set such that $|A| = 1$ and $A \subset B_R$ for some fixed $R > 0$. We want to show that $|\text{co}(A) \setminus A| \rightarrow 0$ as $\delta(A) \rightarrow 0$.

Given $k \in \{1, \dots, n\}$, let $\pi_k : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ denote the projection onto the hyperplane orthogonal to the k th axis:

$$\pi_k(x_1, \dots, x_n) := (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n),$$

and for $y \in \mathbb{R}^{n-1}$ define $A_y^k := A \cap \pi_k^{-1}(y)$.

As in Step 1a of Section 3, we can write $\pi_k(A)$ as $F_1^k \cup F_2^k$, where

$$F_1^k := \{y \in \pi_k(A) : \mathcal{H}^1(A_y^k + A_y^k) - 2\mathcal{H}^1(A_y^k) < \mathcal{H}^1(A_y^k)\}, \quad F_2^k := \pi_k(A) \setminus F_1^k,$$

and by Theorem 1.1 applied to each set $A_y^k \subset \mathbb{R}$ we deduce that

$$\int_{F_1^k} \mathcal{H}^1(\text{co}(A_y^k) \setminus A_y^k) dy + \int_{F_2^k} \mathcal{H}^1(A_y^k) dy \leq 2^n \delta(A) \tag{B.1}$$

(compare with (3.7)). Set

$$A_k^* := \bigcup_{y \in F_1^k} \text{co}(A_y^k).$$

Then by (B.1) and Fubini’s Theorem we get

$$|A \Delta A_k^*| \leq 2^n \delta(A). \tag{B.2}$$

We now follow the strategy in [2, Lemma 12.1] to show that A_k^* enjoys some fractional Sobolev regularity in the k th direction.

Observe that, since A_k^* is a union of intervals, if we write $f_k := \chi_{A_k^*}$ and $\text{co}(A_y^k) = \{y_1, \dots, y_{k-1}\} \times [a_y^k, b_y^k] \times \{y_k, \dots, y_{n-1}\}$, and we denote by \mathcal{F}_k the Fourier transform in the k th variable, that is,

$$\mathcal{F}_k[g](x_1, \dots, x_{k-1}, \xi_k, x_{k+1}, \dots, x_n) := \int_{\mathbb{R}} e^{ix_k \xi_k} g(x) dx_k,$$

then

$$\begin{aligned} \mathcal{F}_k[f_k](x_1, \dots, x_{k-1}, \xi_k, x_{k+1}, \dots, x_n) &= \frac{e^{i\xi_k b_y^k} - e^{i\xi_k a_y^k}}{\xi_k}, \\ y &:= (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n). \end{aligned}$$

Since $|b_y^k - a_y^k| \leq 2R$, we get

$$|\mathcal{F}_k[f_k]| \leq \frac{C(R)}{1 + |\xi_k|};$$

as the Fourier transform is an isometry in L^2 and $\mathcal{H}^{n-1}(\pi_k(A)) \leq C(n, R)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |\xi_k|^{1/2} |\hat{f}_k|^2(\xi) d\xi &= \int_{\pi_k(A)} |\xi_k|^{1/2} |\mathcal{F}_k[f_k]|^2 dx_1 \dots dx_{k-1} d\xi_k dx_{k+1} \dots dx_n \\ &\leq \mathcal{H}^{n-1}(\pi_k(A)) \int_{\mathbb{R}} \frac{|\xi_k|^{1/2}}{(1 + |\xi_k|)^2} \leq C(n, R). \end{aligned}$$

Using again the fact that the Fourier transform is an L^2 isometry, by (B.2) we get

$$\begin{aligned} &\int_{\mathbb{R}^n} \min\{|\xi_k|^{1/2}, \delta(A)^{-1}\} |\hat{\chi}_A|^2(\xi) d\xi \\ &\leq 2\delta(A)^{-1} \int_{\mathbb{R}^n} |\hat{\chi}_A - \hat{f}_k|^2(\xi) d\xi + 2 \int_{\mathbb{R}^n} |\xi_k|^{1/2} |\hat{f}_k|^2(\xi) d\xi \\ &= 2\delta(A)^{-1} \int_{\mathbb{R}^n} |\chi_A - \chi_{A_k^*}|^2(x) dx + 2 \int_{\mathbb{R}^n} |\xi_k|^{1/2} |\hat{f}_k|^2(\xi) d\xi \leq 2^{n+1} + C(n, R). \end{aligned}$$

Since $k \in \{1, \dots, n\}$ is arbitrary, this implies that

$$\int_{\mathbb{R}^n} \min\{|\xi|^{1/2}, \delta(A)^{-1}\} |\hat{\chi}_A|^2(\xi) d\xi \leq C(n, R).$$

It is a standard fact in Sobolev spaces theory that, thanks to this estimate, any sequence of sets $\{A_j\}_{j \in \mathbb{N}}$ with $\delta(A_j) \rightarrow 0$ is precompact in L^2 (see for instance the discussion in [2, Corollary 12.2]). Hence, up to a subsequence, χ_{A_j} converges in L^2 to some characteristic function χ_{A_∞} , and it is not difficult to check that $\delta(A_\infty) = 0$ (see for instance [2, Lemma 13.1]). By the characterization of the equality cases in the semi-sum inequality, we deduce that A_∞ is equal to its convex hull up to a set of measure zero, thus $|A_j \Delta \text{co}(A_\infty)| \rightarrow 0$. Arguing as in [2, Lemma 13.3] or as in Step 5 of Section 3, this actually implies that $|\text{co}(A_j) \setminus A_j| \rightarrow 0$, proving the result.

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